EXTREMAL PROBLEMS IN DIMENSION THEORY FOR PARTIALLY ORDERED SETS

by

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ABSTRACT

This thesis is concerned with answering the following question: "Given a finite poset, P, how many lists of the elements of P are needed to satisfy the condition that x is below y in P if and only if it is before y in each of the lists?" Chapter 1 gives a new proof of a theorem by Hiraguchi which states that for a poset containing $n \geq 4$ elements, at most $[\frac{n}{2}]$ lists are needed. In Chapter 2 we use the ideas of Chapter 1 to characterize those posets on $2n \geq 6$ elements which need n such lists. The last two chapters solve the same problem for posets on $2n + 1 \geq 7$ elements which need n such lists.

Thesis Supervisor: Curtis Greene
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Introduction

The idea of dimension pervades many branches of mathematics, extending far beyond its geometric origins. In many cases the introduction of an intrinsic dimension has been a fruitful source of new ideas. As a purely combinatorial invariant, however, the dimension of a discrete structure is often extremely hard to calculate or even characterize in any simple way. The case of Kuratowski's theorem and the general problem of testing graphs for planarity is an outstanding exception.

The theory of partially ordered sets (posets) provides a natural but difficult problem of this type: given a poset P, find the smallest integer k such that P can be imbedded in R^k, preserving the natural componentwise ordering of R^k. We define k to be the <u>dimension of P</u>. It then turns out that this invariant has another important combinatorial interpretation: it is the smallest number of lists of the elements of P such that $x \leq y$ in P if and only if x is before y in every list.

Despite the elementary nature of the problem, very few results were known until recently. The first papers - of Szpilrajn [8], Dushnik and Miller [6], etc. - gave simple classes of examples and proved the equivalence of the two definitions given above. The main result was that a product of k chains itself has dimension k. More recently, a paper of Baker, Fishburn and Roberts [1] gives a thorough analysis of the case $k = 2$. The starting point for this paper, however, is the work of Hiraguchi [7] in 1951, the subject of which is primarily concerned with bounds on the dimension of P.

In his paper, Hiraguchi proves a basic inequality: if P is a poset on a set X, then the dimension of P is at most $\frac{1}{2}|X|$. His proof was extremely long and complex, and was shortened somewhat by Bogart [3]. The first main result of this thesis is a strengthening of Hiraguchi's theorem, together with a new proof which is far shorter and more illuminating than either of the previous ones. This proof is contained in chapter 1.

In recent work, Bogart and Trotter were able to characterize those partially ordered sets P for which the dimension is $\frac{1}{2}|X|$ provided that $|X|$ is even, and

$k \geq 3$. With a single exception when $k = 3$, they are obtained by taking the elements of ranks 1 and $k-1$ in a Boolean algebra with k atoms. In chapter 2 we show how the methods of section 1 lead naturally to a short proof of Bogart and Trotter's theorem.

The next two chapters are concerned with solving the same problem when $|X|$ is odd. That is, we characterize those posets with $2k + 1$ elements and dimension k. The final solution is that, aside for 13 exceptions when $k = 3$, every such case is obtained by adding a single point arbitrarily to one of the Bogart-Trotter examples. The proof of this result is much more difficult than the preceding ones, although most of the difficulty lies in handling the small cases. In fact, it is relatively easy to give a short, self-contained proof for the case $k \geq 5$. This is the content of chapter 3.

In chapter 4 we continue the much more difficult case $k = 4$. These arguments in turn are based on a list of exceptional posets for $k = 3$ obtained by exhaustive search of all posets on 7 points.

One of the most difficult problems in this area -

still largely unsolved - is to find good methods for computing the dimension of explicit posets. Even when the number of points is small, the problem can be prohibitively difficult. In the course of proving the main results of this thesis, it was necessary to develop a wide variety of computational tools. In the absence of good algorithms, it is results of this type which may be the most useful for developing a full understanding of the subject.

Chapter 1

Hiraguchi's Theorem

A poset, P, on a set, X, is a partial order relation $P \subseteq X \times X$. A linear extension, L, of P is a linear order $L \subseteq X \times X$ such that $P \subseteq L$. If $\langle x,y \rangle \in P$ we write $x \leq_P y$ or $y \geq_P x$, and we say that x is below y in P or y is above x in P. If the partial order is obvious we simply write $x \leq y$ or $y \geq x$. Two elements x and y of X are said to be incomparable in P if neither $\langle x,y \rangle$ nor $\langle y,x \rangle$ is in P (and we write $x \sim y$). An antichain, A, in P is a subset of X whose elements are mutually incomparable in P. A chain, C, in P is a subset of X linearly ordered by P. For any $x \in X$, $[x] \downarrow = \{y \in X | x \geq y\}$ and is called the <u>closed</u> principal <u>ideal</u> determined by x; dually, $(x) \uparrow = \{y \in X | x < y\}$ and is called the <u>open</u> principal <u>filter</u> determined by x. If $X = \{x_1, x_2, \ldots, x_n\}$ is linearly ordered by L and $x_1 < x_2 < \ldots < x_n$, we will represent this as $L = x_1 x_2 \ldots x_n$. If two disjoint sets X and Y are linearly ordered by K and L, respectively, then concatenation of the two strings, KL will represent $K \cup L \cup (X \times Y)$. Finally, let P be a poset on X and let $Y \subseteq X$. Then $P|Y \equiv P \cap Y \times Y$.

Now, let P be a poset on X. Let $\mathcal{L}$ be the set of all linear extensions of P. Then by a theorem of Szpilrajn [8], we have that $P = \bigcap_{L \in \mathcal{L}} L$. With this we can make the following

<u>Definition</u>: The dimension of P, $d(P)$, is the smallest size of set of linear extensions of P whose intersection is P.

We conclude this preliminary section with one last

<u>Definition</u>: The width of P, $w(P)$, is the greatest size of an antichain of P.

For the rest of the paper we assume that P is a poset on a set X. In this chapter we assume $|X| = n$.

We begin with some basic elementary results concerning the calculation of $d(P)$.

<u>Lemma 1</u>:

(i) If $Y \subseteq X$ then $d[P|Y] \leq d(P)$.

(ii) If P is a poset on $Y \cup Z$ and P is the disjoint union of $P|Y$ and $P|Z$, then

$d(P) = \max \{d(P|Y), d(P|Z), 2\}$.

(iii) If P has a maximal element x, then $d(P) = d[P|X\setminus\{x\}]$.

(iv) If x and y have the same comparabilities ($z < x$ iff $z < y$ and $z > x$ iff $z > y$ for all other $z \in X$), then $d(P) = d(P|X\setminus\{x\})$ if $x \not\sim y$ and $d(P) = \max \{d(P|X\setminus\{x\}), 2\}$ if $x \sim y$.

(v) If X is a chain, then $d(P) = 1$ and if X is an antichain, $d(P) = 2$.

<u>Proof</u>:

(i) If $P = L_1 \cap \ldots \cap L_k$, then $P|Y = L_1|Y \cap \ldots \cap L_k|Y$.

(ii) Let $P|Y = L_1 \cap \ldots \cap L_k$ and let $P|Z = M_1 \cap \ldots \cap M_j$ where $j \leq k$. Then $P = L_1M_1 \cap M_2L_2 \cap \ldots \cap M_jL_j \cap \ldots \cap M_jL_k$

(iii) Let $P|X\setminus\{x\} = L_1 \cap \ldots \cap L_k$. Then $P = L_1x \cap \ldots \cap L_kx$.

(iv) Let $P|X\setminus\{x\} = L_1 \cap \ldots \cap L_k$. Then $L_i = M_iyN_i$ for some M_i and N_i for $1 \leq i \leq k$. If $y < x$, then

$P = M_1yxN_1 \cap \ldots \cap M_kyxN_k$. If $y \sim x$, then

$P = M_1yxN_1 \cap M_2xyN_2 \cap \ldots \cap M_kxyN_k$.

(v) If X is a chain then $|\mathcal{L}| = 1$. If X is an antichain, then $P = (x_1x_2 \ldots x_n) \cap (x_n \ldots x_2x_1)$.

Lemma 2: If $x \in X$, then $d(P) \leq 1 + d[P|X\backslash\{x\}]$.

Proof: Let $P|X\backslash\{x\} = L_1 \cap \ldots \cap L_k$. For $1 \leq i < k$ let L_i^* be a linear extension of P such that $L_i = L_i^*|X\backslash\{x\}$. Let $L_k^* = [L_k|(x)\downarrow]x[L_k|X\backslash[x]\downarrow]$ and let $L_{k+1}^* = [L_k|X\backslash[x]\uparrow]x[L_k|(x)\uparrow]$. Then $P = L_1^* \cap \ldots \cap L_{k+1}^*$.

Corollary 3: If $Y \subseteq X$ then $d(P) \leq |X\backslash Y| + d(P|Y)$.

Lemma 4: Let C be a chain in P. Then there exists a linear extension of P in which every element of C is above everything with which it is incomparable.

Proof: Let $C = \{x_1, \ldots, x_m\}$ with $x_1 < \ldots < x_m$. Let $X_0 = X\backslash[x_1]\uparrow$, let $X_i = [x_i]\uparrow\backslash[x_{i+1}]\uparrow$ for

$1 \leq i < m$ and let $X_m = [x_m]\uparrow$. For $0 \leq i \leq m$, let L_i be a linear extension of $P|X_i$. Then $L = L_0 L_1 \ldots L_m$ is the required linear extension.

We are now in a position to prove two basic inequalities for $d(P)$. The first was known to Hiraguchi [7]. The second is new.

Theorem 5:

(i) $d(P) \leq w(P)$

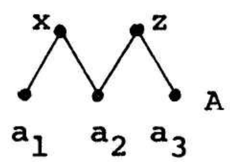
(ii) $d(P) \leq \max\{2, n-w(P)\}$

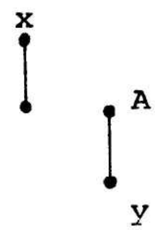
Proof:

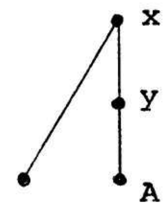
(i) By Dilworth's Theorem [5] there is a set of $w(P)$ chains in P whose union is X. For each chain of such a set construct a linear extension of P as in Lemma 4. Then P is the intersection of these linear extensions.

(ii) Let $A \subseteq X$ be an antichain of P of size $w(P)$ and assume that $|X \backslash A| \geq 2$. Let x and y be any two elements of $X \backslash A$. Then by Corollary 3, $d(P) \leq d(P|A \cup \{x,y\}) + n-w(P) - 2$. But by the reduction of Lemma 1, it is clear that $P|A \cup \{x,y\}$

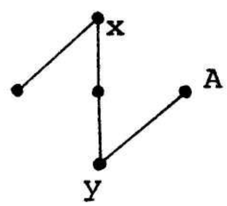
has the same dimension as one of the four following posets:







or



It is easily verified that all of these posets have dimension 2; for instance, the first is $a_1a_2xa_3y \cap a_3a_2ya_1x$. Thus, $d(P) \leq 2 + n - w(P) - 2 = n - w(P)$ provided that $n - w(P) \geq 2$. Otherwise $d(P) \leq 2$.

We obtain as an immediate corollary the following

<u>Corollary 6</u>: (Hiraguchi) Let P be a poset on a set X containing $n \geq 4$ elements. Then $d(P) \leq [\frac{n}{2}]$.

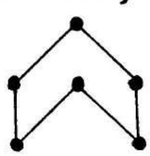
This was the main result of Hiraguchi's 1951 paper [7]. His proof was extremely long and Bogart recently gave a shortened version [3]. In a sense, Theorem 5 can be regarded as a refinement of Hiraguchi's theorem. The proof is much simpler and more direct than those of Bogart and Hiraguchi and it contains more information.

In fact the technique used here is the basis for our attack on the next two problems of this paper.

Chapter 2

Characterization of 2n element Posets with Dimension n.

In 1941 Dushnik and Miller [6] gave an example of a poset, P, on a set, X, where $|X| = 2n$ and $d(P) = n$; namely, let X be the set of n-1 element subsets and 1 element subsets of the set $\{1,\ldots,n\}$ and let P be set inclusion. We call this poset S_{2n}. Furthermore, there is a six element poset,

$C_6 \cong$ , of dimension 3. Bogart and Trotter [4] showed that for $n \geq 3$ these are the only examples. In this chapter we show how the methods of Chapter 1 can be used to give a relatively short proof of this result.

The main idea is to find conditions under which the basic inequality $d(P) \leq n - w(P)$ can be made strict. In Lemmas 9, 12 and 13 we give three important conditions of this type. By Corollary 3 it is sufficient to give conditions under which $|X\setminus A| = 3$ and $d(P) \leq 2$ since adding more elements preserves the relation $d(P) < |X|-|A|$.

But first we need some ground work. In this chapter, $A \subseteq X$ will be a maximal antichain in P. Given A, we define A^* to be the elements of $X \backslash A$ above some element of A and we define A_* dually. Thus, X is the disjoint union of A, A^* and A_*.

In Lemma 1 we gave some conditions whereby we could remove an element from a poset without lowering its dimension. In such a case we say, after Hiraguchi [7], that x is d-removable. We now give another such condition

<u>Definition</u>: $x \prec y$ (y covers x) iff $x < y$ and there is no $z \in X$ such that $x < z < y$. Furthermore, we refer to such a pair, $x \succ y$ as a cover.

<u>Lemma 7</u>: Let y be a maximal element of P covering precisely one element x. Suppose that the element of X incomparable with x have a minimum or a maximum element. Then unless $X \backslash \{y\}$ is a chain, y is d-removable.

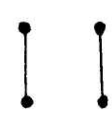
<u>Proof</u>: Let $P|X\backslash\{y\} = L_1 \cap \ldots \cap L_k$, $k \geq 2$. Suppose that the elements incomparable with x have a minimum element, z. We may assume that in L_1, x is below z.

Let L_1' be L_1 with y placed immediately above x. Then $P = L_1' \cap L_2y \cap \ldots \cap L_ky$. Suppose then that the elements incomparable with x have a maximum element z. We may assume that in L_1 x is above z. Then for $2 \leq i \leq k$, let L_i' be L_i with y placed immediately above x. Then $P = L_1y \cap L_2' \cap \ldots \cap L_k'$.

<u>Corollary 8</u>: Let y and x be as in Lemma 7. Suppose that the elements of X incomparable with x have k minimal elements. Then if $d(P|X\backslash\{y\}) > k$, y is d-removable.

<u>Proof</u>: Let $P|X\backslash\{y\} = L_1 \cap \ldots \cap L_m$, $m > k$. We may assume that x is below each of the k minimally incomparable elements at least once in the first k linear extensions. For $1 \leq i \leq k$, let L_i' be L_i with y placed immediately above x. Then
$P = L_1' \cap \ldots \cap L_k' \cap \ldots \cap L_{k+1}y \cap \ldots \cap L_my$.

<u>Lemma 9</u>: Suppose A_* is empty, $w(P|A^*) \leq 2$ and $P|A^*$ does not contain a subposet isomorphic to



Then $d(P) \leq 2$.

<u>Proof</u>: By induction on $|A^*|$. By Theorem 5 the assertion is true for $|A^*| = 1, 2$. If $|A^*| = n \geq 3$ and $P|A^*$ has a maximum element, then by application of Lemma 1 we can use the inductive step. Suppose then that $w(P|A^*) = 2$. Then by Dilworth's Theorem, there are two maximal chains, C_1 and C_2, in $P|A^*$ such that $C_1 \cup C_2 = A^*$. Suppose C_1 has one element, x, and C_2 has n-1 elements, $y_1 > \ldots > y_{n-1}$. By Lemma 1 we can assume that everything in A is below either x or y_1. Let $X_i = \{a \in A | a \leq x$ and $a \leq y_i$ and $a \nleq y_{i+1}\}$ for $1 \leq i < n$. Let $Y_i = \{a \in A | a \nleq x$ and $a \leq y_i$ and $a \nleq y_{i+1}\}$ for $1 \leq i < n$. By Lemma 1, we can assume that $X_i = \{a_i\}$ and $Y_i = \{b_i\}$ for $1 \leq i < n$. Then $P = b_1 \ldots b_{n-1}a_{n-1}y_{n-1} \ldots a_1y_1x \cap a_1 \ldots a_{n-1}xb_{n-1}y_{n-1} \ldots b_1y_1$. We can therefore assume that C_1 and C_2 both contain at least two elements. Let $w > x$ be the top two of C_1 and $y > z$ be the top two of C_2. We can assume $w \neq y$ by Lemma 1 and we can assume that $x \neq z$ by Lemmas 1 and 7. Now, by hypothesis we must have $w > z$ or $y > x$. If both, we can again use Lemmas 1 and 7 to reduce to the

inductive step. Thus, we can assume $w > z$ and $y \sim x$. By Lemmas 1 and 7 we can assume everything in A is below both w and y. By the inductive step $P|X\backslash\{w\} = L_1 \cap L_2$. Since $P|X\backslash\{w\}$ contains two maximal elements, x and y, we have $L_1 = M_1x$ and $L_2 = M_2y$ for suitable M_1 and M_2. Then $P = M_1xw \cap M_2wy$.

<u>Lemma 10</u>: Suppose x is maximal in P and y is minimal in P and $x \sim y$. Then $d(P) \leq 1 + d(P|X\backslash\{x,y\})$.

<u>Proof</u>: Suppose $P|X\backslash\{x,y\} = L_1 \cap \ldots \cap L_k$. Then

$$P = yL_1x \cap \ldots \cap yL_kx \cap (L_k|(x)\downarrow)x(L_k|X\backslash[x]\downarrow\backslash[y]\downarrow)y(L_k|(y)\uparrow).$$

<u>Definition</u>: A cover, $x \prec y$, is said to be a cover of rank k iff there are precisely k pairs, $\langle a,b\rangle$, such that $x < a$, $b < y$ and $a \sim b$.

The following very useful lemma is due to Hiraguchi [7].

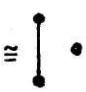
<u>Lemma 11</u>: If $x \prec y$ is a cover of rank 0 or 1, then $d(P) \leq 1 + d(P|X\backslash\{x,y\})$.

<u>Proof</u>: Let $P|X\backslash\{x,y\} = L_1 \cap \ldots \cap L_k$. If $x \prec y$ is a cover of rank 1, let $\langle a,b\rangle$ be the pair of incomparable

elements such that $x < a$ and $b < y$ and assume L_k is a linear extension of $P|X\backslash\{x,y\}$ in which $b < a$. Now, let $L_k' = (L_k|(x)\downarrow)x(L_k|(y)\downarrow\backslash[x]\downarrow)y(L_k|X\backslash[y]\downarrow)$ and let

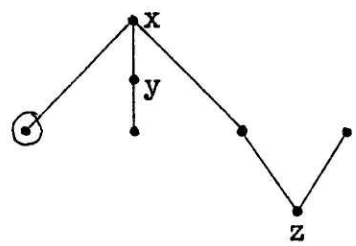
$L_{k+1}' = (L_k|X\backslash[x]\uparrow)x(L_k|(x)\uparrow\backslash[y]\uparrow)y(L_k|(y)\uparrow)$. For $1 \leq i < k$ let L_i' be a linear extension of P such that $L_i = L_i'|X\backslash\{x,y\}$. Then $P = L_1' \cap \ldots \cap L_{k+1}'$.

<u>Example</u>: Let $x \prec y$ be a cover in which x is covered only by y. Then $x \prec y$ is a cover of rank 0.

<u>Lemma 12</u>: Suppose $P|(A^* \cup A_*) \cong$  or $P|A^* \cong$  and A_* has one element. Then $d(P) \leq 2$.

<u>Proof</u>: If either A^* or A_* is empty, then the assertion follows from Lemma 9. Assume then that $A^* = \{x,y\}$ and $A_* = \{z\}$. Then the following cases are possible: $x > y > z$; $x > y$ and $x,y \sim z$; $x > z$ and $x,z \sim y$; or $x > y,z$ and $y \sim z$. In the first case we can assume by Lemmas 1 and 7 that everything is below x. Then by Lemma 1 and Theorem 5, the result holds. The second case follows immediately from Lemma 1 and Theorem 5. In the third case we can assume, using Lemma 7, that everything in A is either

between x and z or below both x and y. By Lemma 1 we can assume that A has two elements. The resulting poset has 5 elements, and hence its dimension is 2. In the last case P has the same dimension as



We may remove the circled element by Lemma 7 and we may remove y by Lemma 1. Thus we have $d(P) \leq 2$.

Lemma 13: Suppose $|A^* \cup A_*| = 4$ and neither A^* nor A_* is empty. Then $d(P) \leq 3$.

Proof: By use of the elementary reductions from the preceding lemmas (incomparable minimal and maximal elements or covers of rank 0) we reduce the cases to the following two: $A^* = \{a,b,c\}$ and $A_* = \{d\}$ with A^* an antichain and $a,b,c > d$ or $A^* = \{a,b\}$ and $A_* = \{c,d\}$ with both being antichains and $a,b > c,d$. Using Lemma 1 we can assume that A has at most 15 elements - a_1, comparable only to a; a_2, comparable only to b; a_3 comparable only to c; a_4 comparable only to d; a_5 comparable only to a and b; a_6, comparable only to a and c; a_7, comparable only to a and d; a_8, comparable only to b and c; a_9; comparable only to b and d; a_{10}, comparable only to c and d; a_{11}, comparable to all but d; a_{12}, comparable to all but c; a_{13}, comparable to all but b; a_{14}, comparable to all but a; and a_{15}, comparable to all four. Then in the first case, P =

$a_1 a_3 a_6 a_8 a_{11} d a_{15} a_{13} a_{14} a_{10} c a_{12} a_9 a_7 a_5 a a_2 b a_4 \cap \ldots$

$\cap\ a_2 a_5 d a_7 a_6 a_{13} a_{11} a_{12} a_{15} a_1 a a_9 a_8 a_{14} b a_{10} a_3 c a_4 \cap \ldots$

$\cap\ d a_4 a_{10} a_{14} a_8 a_9 a_{15} a_{12} a_{11} a_5 a_2 b a_{13} a_6 a_7 a_3 c a_1 a$

In the second case, P =

$d a_4 a_1 a_7 a_{12} c a_{10} a_{13} a_6 a_{15} a_{11} a_5 a a_{14} a_8 a_9 a_2 b a_3 \cap \ldots$

$\cap\ a_2 d a_9 c a_3 a_{10} a_8 a_{14} a_5 a_{11} a_{15} a_{12} b a_6 a_{13} a_7 a_1 a a_4 \cap \ldots$

$\cap\ a_1 a_5 c a_3 a_2 a_{11} a_6 a_8 d a_{14} a_{12} a_{13} a_{15} a_9 a_7 b a a_{10} a_4$

The above proof is an example of a proof by "brute force". There appears to be no elegant reduction in view of the fact that any of the elements of $A^* \cup A_*$ may be removed and still leave a poset of dimension 3. Furthermore, there are plenty of examples of posets where removing two maximal or two comparable extremal elements lowers the dimension by 2. The only recourse then was to find three linear extensions of P whose intersection yielded P. These were found by the author in a relatively short time, but they were done mainly by trial and error. Thus, if the reader attempts to find the motivation for these lists, he may encounter some difficulty.

We can summarize the results of Lemmas 9, 12 and 13 in the following form.

<u>Corollary 14</u> Suppose A is a maximal antichain and $|X \backslash A| \geq 3$. Then $d(P) < |X \backslash A|$ except possibly in the following cases:

(i) $X \backslash A$ is an antichain lying either entirely above or entirely below A

(ii) $|X \backslash A| = 3$ and $P|X \backslash A \cong$  or 

with A lying between the minimal and maximal elements of $X \backslash A$.

Proof: If $|X\backslash A| \geq 4$ and $A^* \neq \phi \neq A_*$, then the corollary follows from Lemma 13. If $A_* = \phi$ and $X\backslash A$ is not an antichain, then the corollary follows from Lemma 9. Hence (i) must hold. If $|X\backslash A| = 3$, then either (i) or (ii) holds by Lemmas 9 and 12.

The main theorem now follows easily:

Theorem 15: (Bogart and Trotter [4]). Suppose $d(P) = n$ $|X| = 2n$ and $n \geq 3$. Then $P \cong S_{2n}$ or C_6 (or the dual of C_6).

Proof: If $n = 3$, then there are only a few six element posets satisfying the restrictions of Theorem 5 and Corollary 14. Examination of these yields the conclusion. If $n \geq 4$ then we may assume from Theorem 5 and Corollary 14 that $X = A \cup A^*$ for two n element antichains A and A^*. Now, let $x_1 \in A$ and $y_1 \in A^*$ be incomparable. If no such pair exists, then $d(P) = 2 < n$. Thus, by Lemma 10, $d(P|X\backslash\{x_1,y_1\}) = n - 1$ and by induction $P|X\backslash\{x_1,y_1\} \cong S_{2n-2}$. Thus, $A = \{x_1,\ldots,x_n\}$ and $A^* = \{y_1,\ldots,y_n\}$ where $x_i \sim y_i$ for $1 \leq i \leq n$. Removing x_1 and y_1 allows us to conclude that $x_i < y_j$ for $2 \leq i, j \leq n$

and $i \neq j$. Removing x_2 and y_2 allows us to conclude that $x_i < y_j$ for $1 \leq i, j \leq n$ and $2 \neq i \neq j \neq 2$. Finally, removing x_3 and y_3 allows us to conclude $x_1 < y_2$ and $x_2 < y_1$. Thus $P \cong S_{2n}$.

Chapter 3

Characterization of 2n+1 element Posets with Dimension n, $n \geq 5$

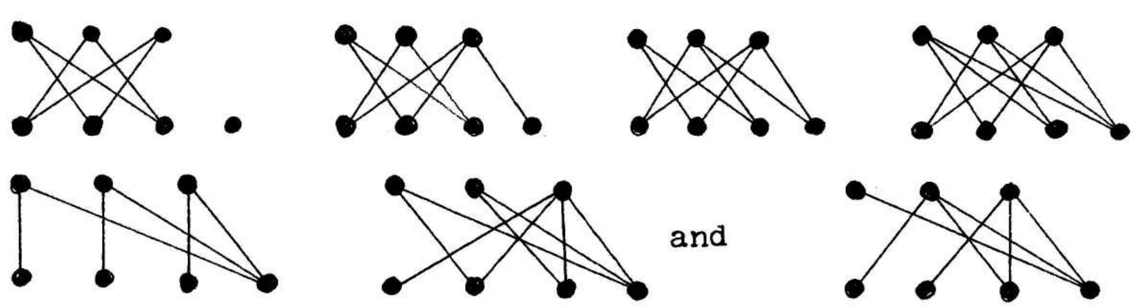
After Bogart and Trotter proved the preceding theorem, they posed the question of characterizing maximal dimensional posets with an odd number of elements. The techniques they had used to prove Theorem 15 were too cumbersome to attack this problem. They had not discovered Theorem 5 (ii), which is extraordinarily useful. Using this fact, we conclude that a 2n+1 element poset having dimension n must have width n or n+1. Armed with this knowledge, we proceed in the following manner; first, if the poset has width n+1, we show that it consists of two antichains, one of size n and the other of size n+1. If the width is n, we show that the poset has two disjoint n element antichains. If $n \geq 5$ we then show that the poset in fact has a subposet which is isomorphic to S_{2n}. This says that there are no really new examples of posets of dimension n, if we are allowed 2n+1 elements in our poset instead of only 2n. To make this more precise, we first make the following

<u>Definition</u>: A poset P on X is said to be irreducible if for every $x \in X$, $d(P|X\setminus\{x\}) = d(P) - 1$. In other words, if we remove any element of X, we lower the dimension.

In this chapter, we will prove the following

<u>Proposition</u>: For $n \geq 5$ there are no irreducible $2n+1$ element posets of dimension n.

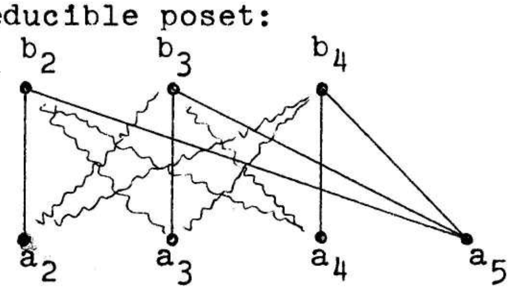
Suppose then that $|X| = 2n+1$ and $d(P) = n \geq 3$. We know that $w(P) = n$ or $n+1$. Suppose for now that $w(P) = n + 1$. Let $n = 3$ and let $A \subseteq X$ be a four element antichain. Assume that A_* is empty. By Lemma 9, A^* must be a three element antichain. Examination of the seven element posets satisfying the above oonditions yields the following seven examples:



Note that first four contain a subposet isomorphic to S_6, and the last three are irreducible. We are now in a position to state

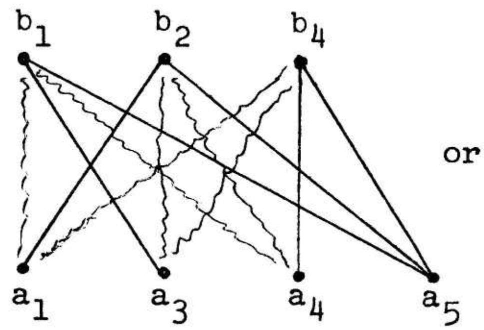
<u>Lemma 16</u>: Suppose $|X| = 2n + 1$, $d(P) = n$ and $w(P) = n + 1$ for $n \geq 4$. Then P contains a d-removable element.

<u>Proof</u>: Let A be an antichain of size $n + 1$. By Corollary 14 we may assume that A^* is also an antichain and A_* is empty. Then there is at least one pair of incomparable minimal and maximal elements. Suppose $n = 4$, $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $A^* = \{b_1, b_2, b_3, b_4\}$ with $a_1 \sim b_1$. By removing them we must be left with one of the seven posets of dimension 3 listed above. Suppose we are left with the first irreducible poset:

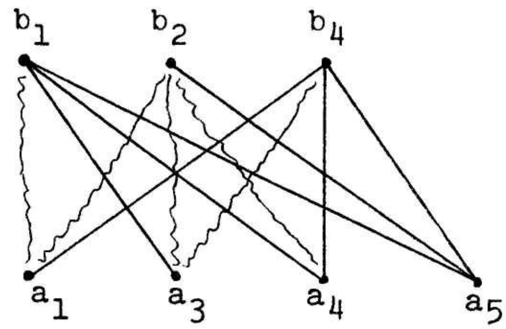


If $a_5 \sim b_1$, then removing them allows us to conclude that $a_1 < b_2, b_3, b_4$. But then either a_1 or a_5 would be d-removable. So we may assume $a_5 < b_1$.

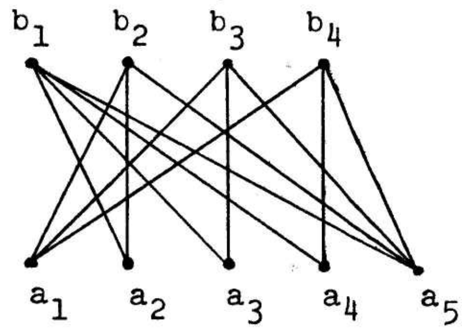
Now removing a_2, b_3 must leave either



or

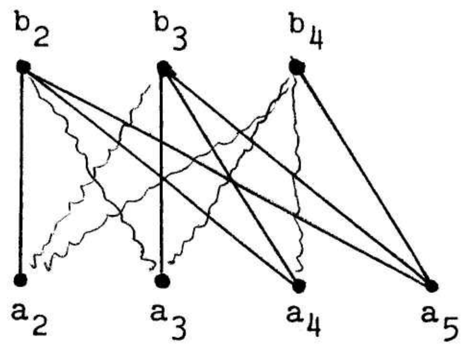


In both instances we conclude that $b_1 > a_3$. By symmetry we are also led to conclude $b_1 > a_2, a_4$. Now if $a_1 \sim b_2$, then removing b_3 and a_4 leaves a poset of dimension 2. Again by symmetry we may thus conclude that $a_1 < b_2, b_3, b_4$. Thus

P = 

But then removing b_2, a_3 leaves a poset of dimension 2, contradicting the hypothesis that $d(P) = 4$.

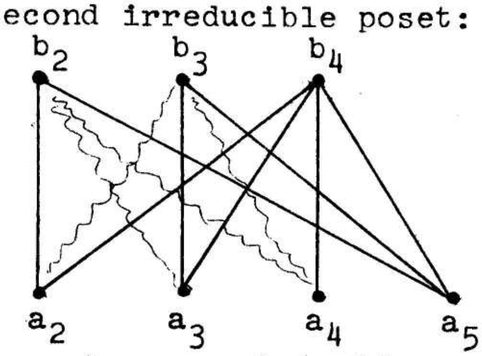
Suppose that removing a_1, b_1 leaves the last irreducible poset:



As before, we may assume that $a_5 < b_1$.

Now removing a_4, b_4 allows us to conclude that $a_1 < b_2, b_3$. But then removing a_3, b_4 leaves a poset of dimension 2.

Finally, suppose that removing a_1, b_1 leaves the second irreducible poset:



Removing a_2, b_3 allows us to deduce that $a_1 < b_2, b_4$ and $a_5 < b_1$. Removing a_5, b_2 allows us to deduce that $a_1 < b_3$. Removing a_4, b_3 then leaves a poset of dimension 2.

Thus, by induction removing a pair of incomparable minimal and maximal elements leaves a poset containing S_{2n-2}. Proceeding in a manner similar to above allows us to conclude that P contains a subposet isomorphic to S_{2n}, thus establishing the Lemma.

We now proceed to develop the tools necessary for handling the case $w(P) = n$. The following lemma gives important conditions under which $d(P) < n - w(P) - 1$. The proof is very much like the proof to Lemma 12 and the lists used here are based on those of Lemma 12. Due to the length of the proof and the fact that it has little expository value, it is given in the Appendix.

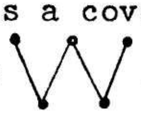
Lemma 17. Let A be a maximal antichain of P such that neither A^* nor A_* is empty and at least one is not an antichain. If $|A^* \cup A_*| \leq 5$, then $d(P) \leq 3$.

Proof: Given in the Appendix.

Lemma 18: Let A be the set of all minimal elements of a poset, P. Then $d(P) \leq 1 + w(P|A^*)$.

Proof: Let $P|A^* = C_1 \cup \ldots \cup C_k$ where each C_i is a chain. Let L_i, $1 \le i \le k$ be a linear extension of P in which every element of C_i is below everything incomparable to it. Let $L_{k+1} = (M|A)(L_1|A^*)$ where M reverses the order of L_1 on A. Then $P = L_1 \cap \ldots \cap L_{k+1}$.

<u>Lemma 19</u>: Let A be the set of minimal elements of P. If $|A^*| = 5$ and $w(P|A^*) \le 3$, then $d(P) \le 3$.

<u>Proof</u>: If $w(P|A^*) \le 2$, then $d(P) \le 3$ by the previous lemma. Hence, we may assume that $w(P|A^*) = 3$. Then $A^* = C_1 \cup C_2 \cup C_3$ where these are disjoint chains. By the now legendary pigeon-hole principle we may conclude that C_1 has only one element. Then $d(P) \le 1 + d(P|A \cup C_2 \cup C_3) \le 3$ by Lemma 9 unless $P|C_2 \cup C_3 \cong$ . We can remove either C_1 or C_2 and use Lemma 9 unless neither is a cover of rank 0 which can happen only if $P|A^* \cong$ . But then removing the top element of C_2 allows us to use Lemma 9.

We can summarize the results of Lemmas 17 and 19 in

Corollary 20: Suppose A is a maximal antichain and $|X\backslash A| \geq 5$. Then $d(P) < |X\backslash A| - 1$ except possibly in the following cases:

(i) $X\backslash A$ lies either entirely above or below A and $w(P|X\backslash A) \geq |X\backslash A| - 1$

(ii) $X\backslash A = A^* \cup A_*$ and A^* and A_* are both nonempty antichains.

Proof:

(i) follows from Lemma 19 and

(ii) follows from Lemma 17.

We are now ready to start on the case where $w(P) = n$.

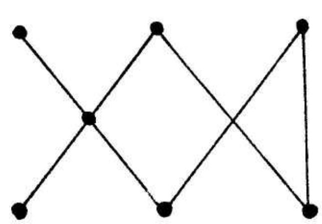
Lemma 21: Suppose $|X| = 2n+1$, $d(P) = w(P) = n \geq 5$. Furthermore, suppose that for some n-element antichain, A, in P, A^* is an n element set. Then P contains a subposet isomorphic to S_{2n}.

Proof: Let $A_* = \{c\}$, $A = \{b_1,\ldots,b_n\}$ and $A^* = \{a_1,\ldots,a_n\}$. Then Corollary 20 says that A^* is an antichain. We may assume that $c < b_n$. Removing $\{c,b_n\}$ by Lemma 11 allows us to conclude that

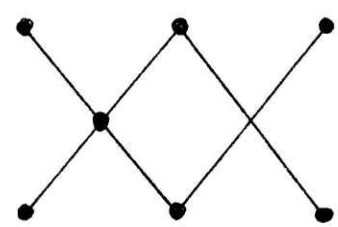
$P|A \cup A^*\setminus\{a_n,b_n\} \cong S_{2n-2}$, with $a_i \sim b_i$ for $1 \leq i \leq n-1$. Now for $1 \leq i \leq n-1$, let $L_i \supseteq P$ be constructed as as follows. Place b_i above a_i. Now c is either incomparable to or below a_i. If it is incomparable to a_i place it above a_i; otherwise, it must be put below a_i. Do the same for b_n, keeping in mind that $b_n > c$. Now, if a_n is incomparable to b_i, place it below b_i but above a_i and b_i. Otherwise it must be place above b_i. Now place the rest of A^* above b_i and a_n arbitrarily, place the elements of A above c between c and b_n and place the remaining elements below c and a_i. Note that if $c < b_j$ for some $j \neq i$, then $c < a_i$ by hypothesis. Now, if $a_n > b_n$, then $P = L_1 \cap \ldots \cap L_{n-1}$. So we may suppose $a_n \sim b_n$. If $a_n > c$, then we use the same linear orders as above with the following modifications. We find some i between 1 and $n-1$ such that $b_n \sim a_i$ or $a_n \sim b_i$. If none exists, then $P|A \cup A^* \cong S_{2n}$. If $b_n \sim a_i$ and $a_n \sim b_i$, then b_n is immediately below a_n in L_{ij} reverse them. Then $P = L_1 \cap \ldots \cap L_{n-1}$. If $b_n \sim a_i$ and $a_n > b_i$, place b_n immediately above a_n and for some $j \neq i$, for which $b_n < a_j$, place

b_i immediately above b_n in L_j - note that such a_j must exist or else $w(P) = n + 1$. Then $P = L_1 \cap \ldots \cap L_{n-1}$. If $b_n < a_i$ and $a_n \sim b_i$, then we place a_n immediately below b_n and for some $j \neq i$, for which $a_n > b_j$, place a_i immediately below a_n in L_j - again such a_j must exist. Then $P = L_1 \cap \ldots \cap L_{n-1}$. If $a_n \sim c$, then follow the above instructions replacing b_n by c. Again, if this is not possible, then $P|X\setminus\{b_n\} \cong X_{2n}$.

Now suppose we have a seven element poset with three minimal elements and three maximal elements. Furthermore, suppose that the remaining element is below at least two of the maximal elements and above at least two of the minimal elements. The only two such posets which don't have a d-removable element by Lemma 1 are



or



The ones which do have a d-removable element all have dimension 2 and so do these two. Thus, all such posets have dimension 2. We are now ready for

Lemma 22: Suppose $|X| = 9$, $d(P) = w(P) = 4$; furthermore, suppose P has four maximal elements, four minimal elements and a ninth element below at least two of the maximal elements and above at least two of the minimal elements. Then P restricted to the maximal and minimal elements is isomorphic to S_8.

Proof: Let a_1, a_2, a_3, a_4 be the minimal elements, let b_1, b_2, b_3, b_4 be the maximal elements and let $a_1, a_2 < c < b_3, b_4$. If either a_3 or a_4 is incomparable to either b_1 or b_2, then removing them leaves a poset of dimension 2. Now a_1 and a_2 must have different comparabilities and so must a_3 and a_4, b_1 and b_2, and b_3 and b_4. We may therefore assume $b_1 > a_1 \sim b_2$ and $b_4 > a_4 \sim b_3$. Now, if $a_2 \sim b_2$, then b_2 is d-removable by Corollary 8. Likewise for a_3 if $a_3 \sim b_3$. If $a_2 < b_1$, then $c \prec b_2$ is a cover of rank 0, and removing it leaves a poset of dimension 2. Thus, $a_2 \sim b_1$. Similarly, $a_3 \sim b_4$. But then $P|X \setminus \{c\} \cong S_8$. If a_3 is also below c, then we must have a_1, a_2, a_3 all below

b_1 and b_2 or else removing two leaves a poset of dimension 2. But then a_1, a_2 or a_3 is d-removable. Thus, we have the lemma.

<u>Lemma 23</u>: Suppose $|X| = 2n + 1$, $d(P) = w(P) = n \geq 5$ and P has n minimal elements. Then P contains a subposet isomorphic to S_{2n}.

<u>Proof</u>: Let $A = \{a_1,\ldots,a_n\}$ be the set of minimal elements. Then by Corollary 20, $w(P|A^*) = n$. Let $B = \{b_1,\ldots,b_n\} \subseteq A^*$ be an antichain and let c be the element of $A^*\setminus B$. Now if c is not above at least two elements of A and below at least two elements of B, then the lemma is established by Lemma 21. Suppose then that $a_1, a_2 < c < b_{n-1}, b_n$. Now suppose that $a_3,\ldots,a_n < b_1,\ldots,b_{n-2}$. Then $P|X\setminus\{a_1,b_n\}$ has dimension 2 which contradicts the hypothesis that $d(P) = n \geq 5$. Assume then that $a_3 \sim b_3$. Removing them gives a poset satisfying the hypothesis of Lemma 22. Thus we conclude that $a_i \sim b_i$ for $1 \leq i \leq n$ and $a_i < b_j$ for $2 \neq i \neq j \neq 2$. Now, if $a_3 \sim b_i$, $i \leq i \leq 2$ then removing them implies that b_3 has the same comparabilities as b_i and is therefore

d-removable. Thus $a_3 < b_1, b_2$. Similarly, $b_3 > a_{n-1}, a_n$. Now $a_1 < c$ is a cover of rank 1. Removing it implies that $a_3 < b_i$, $1 \leq i \leq n$ and $i \neq 3$, and removing c, b_n implies that $b_3 > a_i$, $1 \leq i \leq n$ and $i \neq 3$. Thus $P|A \cup B \cong S_{2n}$.

We wrap us the remaining difficulties with

Lemma 24: Suppose A is a maximal antichain in P such that $|A^*|, |A_*| \geq 2$ and $|A^* \cup A_*| = 6$. Then $d(P) \leq 4$.

Proof: Given in the Appendix.

This immediately strengthens Corollary 20.

Corollary 25: Suppose A is a maximal antichain and $|X\backslash A| \geq 5$. Then $d(P) < |X\backslash A| - 1$ except possibly in the following cases:

(i) either A^* or A_* contains an antichain of size $|X\backslash A| - 1$ or

(ii) $|X\backslash A| = 5$, $|A^*|, |A_*| \geq 2$ and both are antichains.

Lemmas 16, 21 and 23, and Corollary 25 are summarized in

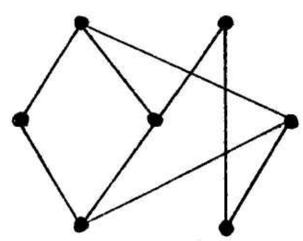
<u>Theorem 26</u>: For $n \geq 5$, there are no irreducible $2n+1$ element posets of dimension n.

Chapter 4

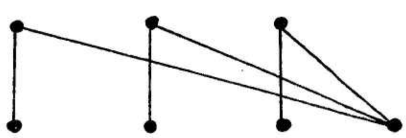
Characterization of 2n+1 element Posets with Dimension n, $n \leq 4$

In Chapter 3 we were able to show that there were no irreducible 2n+1 element posets of dimension n for $n \geq 5$. In this chapter we will show that this same statement holds for $n = 4$, but not for $n = 3$.

The existence of seven element irreducible posets of dimension 3 has been known for some time. Baker, Fishburn and Roberts [1] give an example isomorphic to

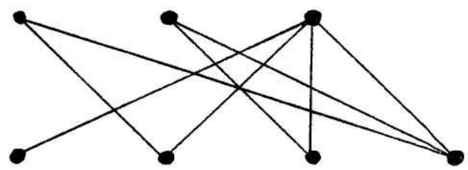
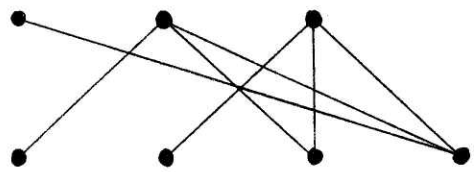


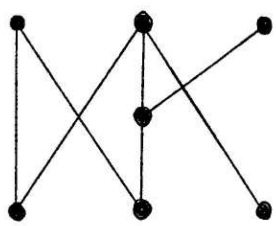
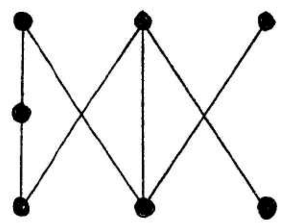
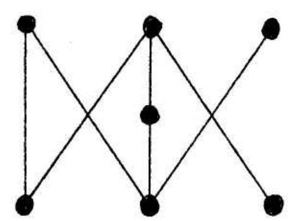
and Bogart and Trotter give the following example:

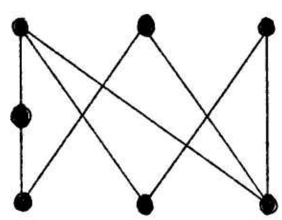
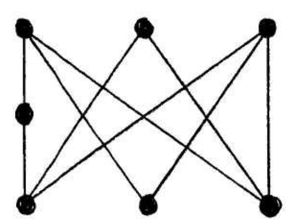
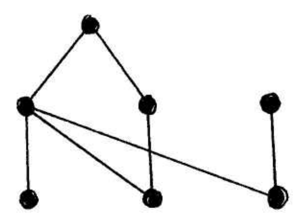


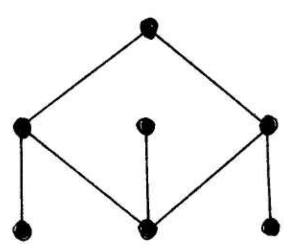
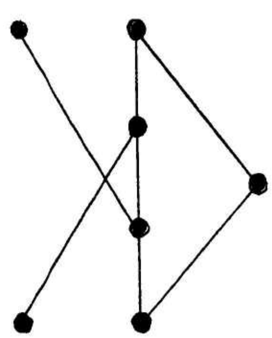
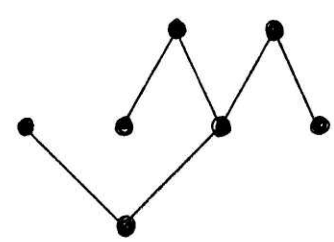
Subsequently Trotter and this author have independently examined all seven element posets of width 3 and 4 to discover all those irreducible of dimension 3.

Including the first two mentioned, here - along with duals of course - is a complete list:

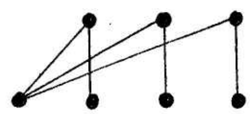




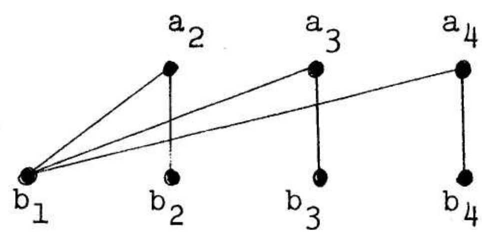




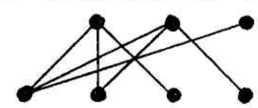
With this knowledge we can ask the question, "are there any irreducible nine element posets of dimension 4?" If the width is 5, the answer is no by Lemma 16. If the width is four, then by the previous lemmas we can conclude that such a poset has two disjoint antichains of size 4 or it has a four element antichain, A, such that A^* and A_* are antichains of sizes 2 and 3. In the first case we can assume by Lemma 22 that we have a four element antichain A, such that A^* has one element and A_* is also an antichain.

Lemma 27: Suppose $|X| = 9$ and $d(P) = w(P) = 4$. Furthermore, suppose that P has two disjoint antichains of size 4. Then P contains a subposet isomorphic to S_8.

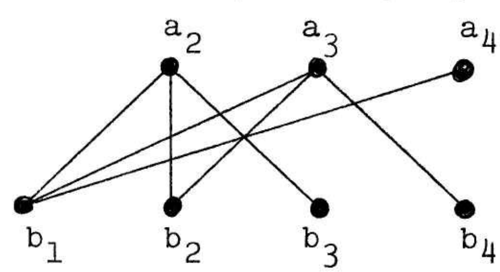
Proof: The only case not covered by Lemmas 20 and 21 is the case in which the remaining element is above both antichains, and removing it along with an element it covers, leaves an irreducible poset of dimension 3 in seven elements. Let A be a four element antichain. Then we can assume $A = \{a_1, a_2, a_3, a_4\}$, $A_* = \{b_1, b_2, b_3, b_4\}$ and $A^* = \{c\}$. Clearly, if c is above a_1, a_2, a_3 and a_4, then $P|A \cup A_* \cong S_8$.

Assume that $a_4 \sim c \geq a_1, a_2, a_3$. Suppose the removal of a_1, c leaves  . Then we can assume without loss of generality that the poset is labelled as follows:

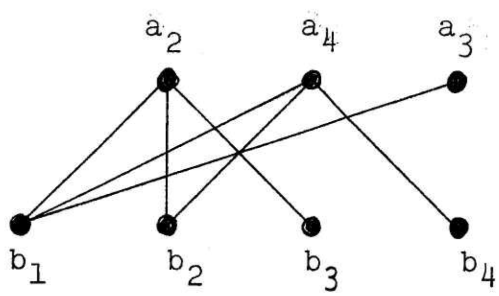


Now, by removing c, a_2, we can conclude that $b_1, b_2 < a_1$. Removing c, a_3 we can also conclude that $b_3, b_4 < a_1$. But then removal of a_4, b_3 leaves a poset of dimension 2. Suppose that the removal of a_1, c leaves 

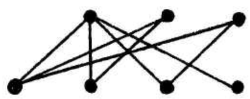
Then it can be labelled either



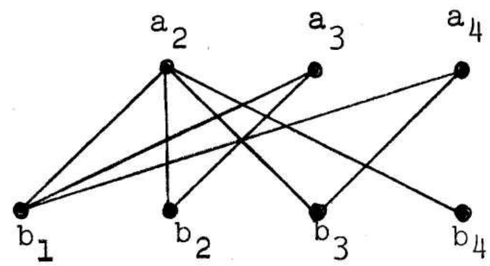
or



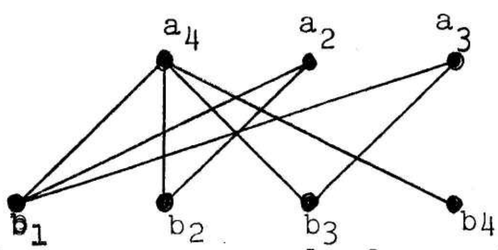
In either case, removal of c, a_2 allows us to conclude that $b_4 \sim a_1 > b_1, b_2, b_3$. Then a_1 or a_2 can be removed. Finally, suppose removal of a_1, c leaves



Then it can be labelled either

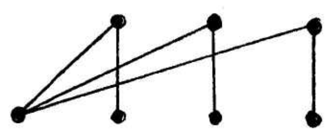


or

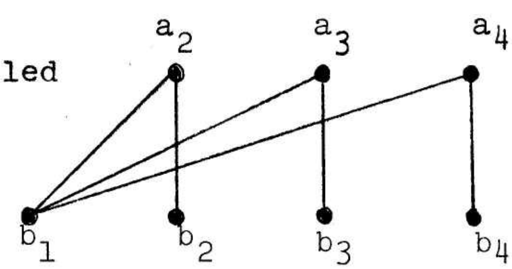


In either case, removal of a_4, b_4 leaves a poset of dimension 2.

Assume that $a_1, a_2 < c \sim a_3, a_4$. Suppose removal of c, a_1 leaves

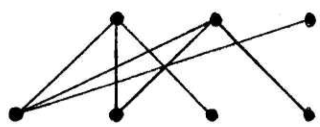


Again we can assume it is labelled

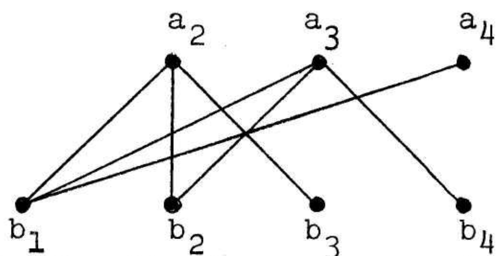


Removing c, a_2 we can conclude $b_1, b_2 < a_1 \sim b_3, b_4$ or $a_1 > b_1, b_2, b_3, b_4$. In the first case either a_1 or a_2 can be removed. In the second case $a_4 > b_4$ is a cover of rank 0 and removing it leaves a poset of dimension 2.

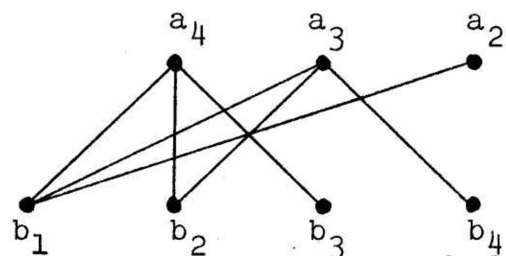
Suppose that removal of c, a_1 leaves



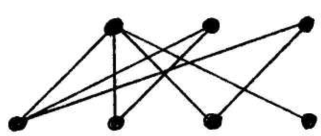
Then it can be labelled either



or

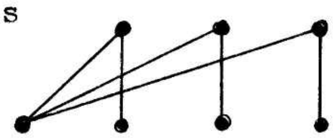


In the first case, removing c, a_2 allows us to conclude that $b_1, b_3, b_4 < a_1 \sim b_2$. But then removing a_4, b_1 leaves a poset of dimension 2. In the second case, removing c, a_2 allows us to conclude that $b_1 < a_1 \sim b_2, b_3, b_4$. But then either a_1 or a_2 can be removed. Suppose then that removing c, a_1 leaves

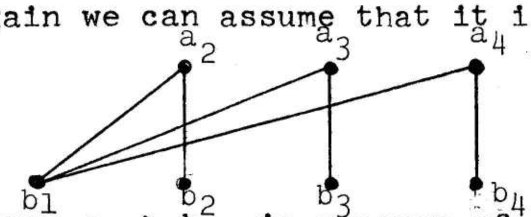


For any way that this is labelled, removal of c, a_2 gives a_1 the same comparabilities as a_2, and hence, either can be removed.

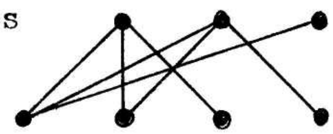
Finally, assume that $a_1 < c \sim a_2, a_3, a_4$. Suppose that removal of c, a_1 leaves



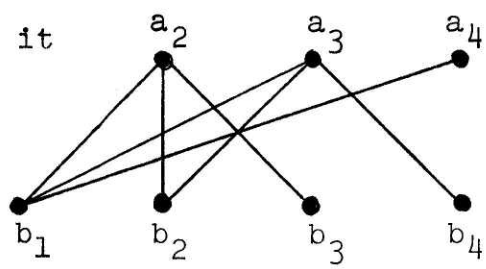
Again we can assume that it is labelled



Then $a_4 > b_4$ is a cover of rank at most 1 unless $b_4 < a_1$ and $c \sim b_1$. In the first case, removing a_4, b_4 leaves a poset of dimension 2. In the second case removing a_4, b_3 leaves a poset of dimension 2. Suppose that removal of c, a_1 leaves

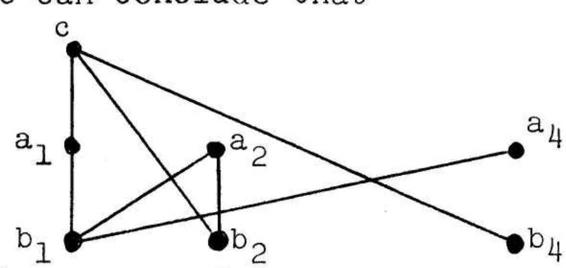


Then we can label it



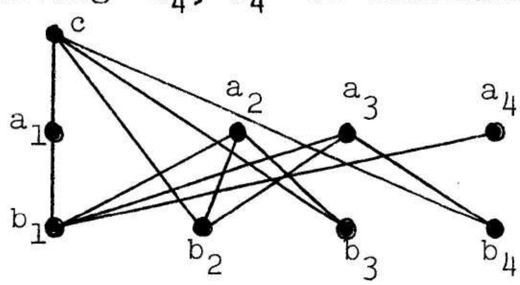
Removing a_3, b_3 we can conclude that

$P|X\backslash\{a_3, b_3\} =$



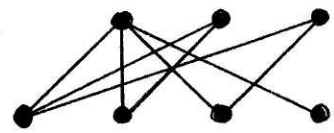
Then removing a_4, b_4 we conclude

$P =$

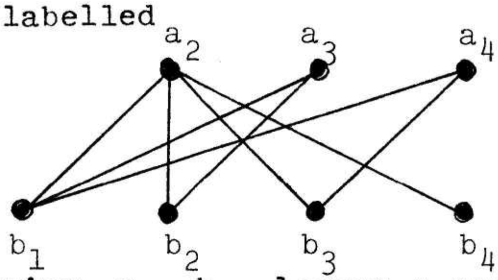


But then removing a_1, b_1 leaves a poset of dimension 2.

So we may assume that removing c, a_1 leaves



which can be labelled



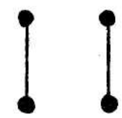
But then removing a_3, b_3 leaves a poset of dimension 2.

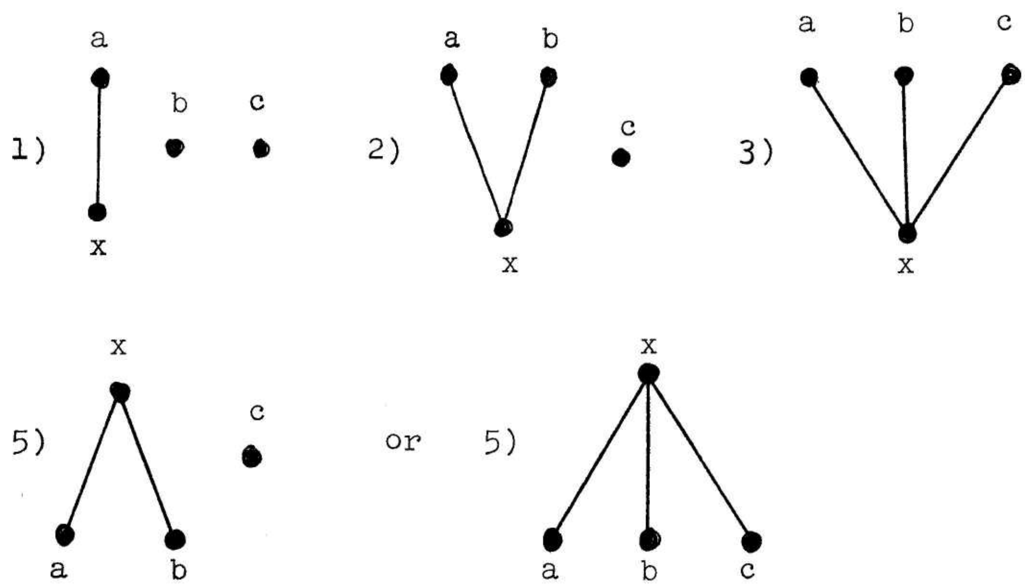
The proof of this last statement illustrates the straightforward technique used to show that certain types of nine element partially ordered sets have dimension three or less. The same technique can be used to show that for the type of nine element poset mentioned immediately before Lemma 27, its dimension is at most 3. Since the proof is tedious and unenlightening, consisting essentially as an exhaustion of such posets, only the statement of this result is given here. It has also been verified by Trotter.

In view of Lemma 27, the statement of Theorem 26 holds for $n = 4$. Thus, summarizing the results of this chapter gives us

Theorem 28: There are 13 seven element irreducible posets of dimension 3 up to isomorphism and dual isomorphism classes. For $n \geq 4$, there are no $2n+1$ element irreducible posets of dimension n.

Appendix

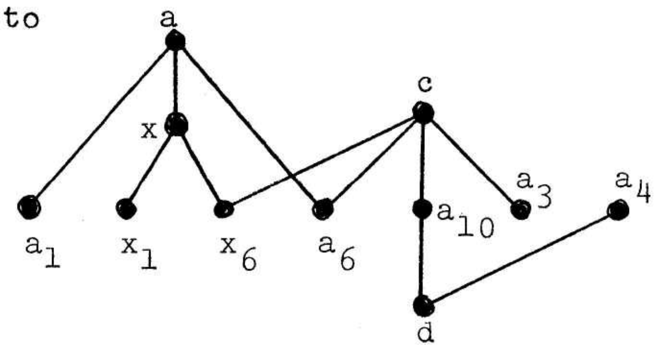
Proof to Lemma 17: We have two possibilities: $|A^*| = 3$ or 4. Suppose $|A^*| = 4$. If $w(P|A^*) \leq 2$, then $d(P|A \cup A^*) \leq 2$ unless $P|A^* \cong$  by Lemma 9. In that case, removal of one of the two top chains by Lemma 11 and Corollary 14 gives the conclusion. Thus we may assume that $w(P|A^*) = 3$. This gives us five cases:



Let d be the element of A_*. For the rest of this lemma, let a_i, $1 \leq i \leq 15$ have the same interpretation as in Lemma 13; also, let x_i, $1 \leq i \leq 15$, be comparable only to x, and the same things to which a_i is comparable.

Let x_0 be comparable only to x. As in Lemma 13, we can assume that x has only the elements listed.

Case 1: By removing a, x we can assume that $b, c > d$. If $a \sim d$, then removing b leaves a poset equivalent to



Using Lemma 7, we can replace a_4 by an element covering a and a_3. But this poset has dimension 2 by Lemma 9. Thus, we may assume that $a > d$.

When $x > d$, P =

$$x_1a_1a_3x_6a_6a_8x_{11}a_{11}dx_{15}a_{15}x_{13}a_{13}a_{14}a_{10}cx_{12}a_{12}a_9x_7a_7x_5xa_5aa_2ba_4 \cap \ldots$$

$$a_2x_5a_5dx_7x_6x_{13}x_{11}x_{12}x_{15}x_1xa_7a_6a_{13}a_{11}a_{12}a_{15}a_1aa_9a_8a_{14}ba_{10}a_3ca_4 \cap \ldots$$

$$\cap\, da_4a_{10}a_{14}a_8a_9a_{15}x_{15}a_{12}x_{12}a_{11}x_{11}a_5x_5a_2ba_{13}x_{13}a_6x_6a_7x_7a_3ca_1x_1xa$$

When $x \sim d$, we use the same linear orders as above with the following modifications: we drop x_7, x_{12}, x_{13} and x_{15}; and in the second linear order we put d immediately above x.

Case 2: As above we can assume that $a, b, c > d$. We can also use the same linear orders with the following modifications: we drop x_6, x_7, x_{14} and x_{15}; and in the third linear order we put x immediately below b.

Case 3: As in case 1 we can assume $a, b, c > d$. We can also use the same linear orders with the following modifications: we drop x_{11} and x_{12}; and in the first linear order we put x immediately below c.

Case 4: Again we can assume that $a, b, c > d$. Then $P =$

$$x_1 a_3 x_3 x_6 x_8 x_{11} d x_{15} x_{13} x_{14} a_{10} x_{10} c x_{12} x_9 x_7 x_5 a x_2 b a_4 x_4 x_0 x \cap \ldots$$

$$\cap\ x_0 x_2 x_5 d x_7 x_6 x_{13} x_{11} x_{12} x_{15} x_1 a x_9 x_8 x_{14} b x_{10} x_3 x_4 x a_{10} a_3 c a_4 \cap \ldots$$

$$\cap\ x_0 d x_4 a_4 x_{10} a_{10} x_{14} x_8 x_9 x_{15} x_{12} x_{11} x_5 x_2 b x_{13} x_6 x_7 x_3 a_3 c s_1 a x$$

Case 5: Again we can assume that $a, b, c > d$. Then we can use the above linear orders with the following modifications: we drop a_3 and a_{10}; and in the second linear order we put x immediately above c.

Suppose $|A^*| = 3$. If A_* is a chain, then removal of

A_* by Lemma 11 gives the result unless A^* is an antichain. Let $A_* = \{x > d\}$ and $A^* = \{a, b, c\}$. Then we can assume that $a, b, c > d$, and we have the following cases

Case 1: $x < a, b, c$. Then $P =$

$$a_1 a_3 a_6 a_8 a_{11} d x x_{15} a_{15} x_{13} a_{13} x_{14} a_{14} x_{10} a_{10} c x_{12} a_{12} x_9 a_9 x_7 a_7 a_5 a a_2 b x_4 a_4 \cap \ldots$$

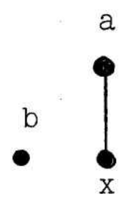
$$\cap\ a_2 a_5 d a_7 a_6 a_{13} x x_7 x_{13} a_{11} a_{12} x_{12} a_{15} x_{15} a_1 a a_9 x_9 a_8 x_{14} a_{14} b x_{10} a_{10} a_3 c x_4 a_4 \cap \ldots$$

$$\cap\ d a_4 a_{10} a_{14} a_9 a_{15} a_{12} x x_4 x_{10} x_{14} a_8 x_9 x_{15} x_{12} a_{11} a_5 a_2 b a_{13} x_{13} x_7 a_6 a_7 a_3 c a_1 a$$

Case 2: $c \sim x < a, b$. Then we can use the above linear orders with the following modifications: we drop x_{10}, x_{13}, x_{14} and x_{15}; and in the first linear order we place x immediately above c.

Case 3: $a, c \sim x < b$. Then we use the linear orders of case 2 with the following modifications: we drop x_7 and x_{12}; and in the second linear order we place x immediately above a.

Now, assume $A_* = \{c,d\}$ is an antichain.

Then $P|A^{*} \cong$ 1) 

2) 

, 3) 

or

4) 

.

Case 1: Using elementary facts we can assume that $a, b > c, d$. If $x > c, d$, then $P =$

$$d a_4 x_1 a_1 a_7 x_7 a_{12} x_{12} c a_{10} x_{13} x_6 x_{15} x_{11} x_5 x a_{13} a_6 a_{15} a_{11} a_5 a a_{14} a_8 a_9 a_2 b a_3 \cap \ldots$$

$$\cap\ a_2 d a_9 c a_3 a_{10} a_8 a_{14} x_5 a_5 x_{11} a_{11} x_{15} a_{15} x_{12} a_{12} b a_6 x_6 x_{13} x_7 x_1 x a_{13} a_7 a_1 a a_4 \cap \ldots$$

$$\cap\ x_1 a_1 a_5 x_5 c a_3 a_2 a_{11} x_{11} a_6 x_6 a_8 d a_{14} a_{13} x_{13} a_{15} x_{15} a_9 a_7 x_7 x_{12} x a_{12} b a a_{10} a_4$$

If $d \sim x > c$, then we can use the same linear orders as above with the following modifications: we drop x_7, x_{12}, x_{13} and x_{15}; and we put $x_6 x_{11} x_5 x$ immediately below c in the first linear order.

If $x \sim c, d$, then we can use the preceding linear orders with the following modifications: we drop x_6 and x_{11}; we put $x_5 x$ immediately below d in the first linear order and we place a_4 immediately below x in the third linear order.

Case 2: By removing x, a and x, b we can assume that $a, b > c, d$. Then $P =$

$$d a_4 x_4 x_1 x_7 x_{12} c a_{10} x_{10} x_{13} x_6 x_{15} x_{11} x_5 a x_{14} x_8 x_9 x_2 b x_3 x_0 x a_3 \cap \ldots$$

$$\cap\, x_2 d x_9 c a_3 x_3 x_{10} a_{10} x_8 x_{14} x_5 x_{11} x_{15} x_{12} b x_6 x_{13} x_7 x_1 a x_4 x_0 x a_4 \cap \ldots$$

$$\cap\, x_0 x_1 x_5 c a_3 x_3 x_2 x_{11} x_6 x_8 d x_{14} x_{12} x_{13} x_{15} x_9 x_7 b a x_{10} x_4 x a_{10} a_4$$

Case 3: By removing a and then b, we can assume that $a, b > c, d$. Then we can use the same linear orders as in case 1 with the following modifications: we drop x_1, x_6, x_7 and x_{13}; and we place x immediately below b in the second linear order.

Case 4: By removing c we are left with a poset of dimension 2.

Proof to Lemma 24: Let $A^* \cup A_* = \{a, b, c, d, x, y\}$. We can assume that $A = \{a_{ij} \mid 0 \le i, j \le 7\}$, as follows: of the three elements $\{a, b, x\}$ a_{0j} is comparable to none, a_{1j} is comparable only to a, a_{2j} is comparable only to b, a_{3j} is comparable only to x,

a_{4j} is comparable only to a and b, a_{5j} is comparable only to a and x, a_{6j} is comparable only to b and x, and a_{7j} is comparable to all three for $0 \leq j \leq 7$; of the three elements $\{c, d, y\}$ a_{i0} is comparable to none, a_{i1} is comparable only to c, a_{i2} is comparable only to d, a_{i3} is comparable only to y, a_{i4} is comparable only to c and d, a_{i5} is comparable only to c and y, a_{i6} is comparable only to d and y, and a_{i7} is comparable to all three for $0 \leq i \leq 7$. Let M_i be a linear ordering of $\{a_{ij} | 0 \leq j \leq 7\}$ for $0 \leq i \leq 7$. Let N_j be a linear ordering of $\{a_{ij} | 0 \leq i \leq 7\}$ for $0 \leq j \leq 7$. Suppose $A^* = \{a, b, y\}$ and $A_* = \{c, d, x\}$. Then by Corollary 20 and Lemma 13 we can assume that both are antichains and $a, b, y > c, d, x$. Let

$L_1 = dxM_0M_1M_4xM_5M_7aM_6M_2bM_3y$,

$L_2 = dcM_2xM_3M_6M_7M_4bM_5M_1aM_0z$,

$L_3 = xdN_1N_5cN_4N_7N_6N_3yN_2N_0ba$ and

$L_4 = xN_0N_3cN_2N_6dN_7N_5yN_4N_1ba$. Let K_1 be the same as L_1 with the following changes: put a_{34} immediately above y and put a_{45} just above a_{54}. Let K_2 be the same as L_2 with the following modifications: put a_{52} immediately below a_{43} and a_{23} at the bottom. Let K_3 be the same as L_3 with the following modifications: put a_{52} right above b and put a_{45} right below d. Let K_4 be the same as L_4 with the following changes: put a_{34} just below a_{25} and put a_{23} immediately above a_{32}. It is also necessary for $a_{43}a_{40}$ to be the last two elements of M_4, a_{50} to be the first element of M_5, a_{32} to be the bottom element of N_2 and $a_{25}a_{45}$ to be the last two elements of N_5. Then $P = K_1 \cap K_2 \cap K_3 \cap K_4$.

Now suppose $A^* = \{a, b, c, d\}$ and $A_* = \{x, y\}$. Again we can assume both are antichains and $a, b, c, d > x, y$. Let

$L_1 = yM_0M_1M_4xM_5M_7aM_6M_2bM_3dc$,

$L_2 = yM_2xM_3M_6M_7M_4bM_5M_1aM_0cd$,

$L_3 = xN_0N_1N_4yN_5N_7cN_6N_2dN_3ba$ and

$L_4 = xN_2yN_3N_6N_7N_4dN_5N_1cN_0$. Let K_1 be the same as L_1 with the following changes: put $a_{44}a_{42}$ immediately above a_{55} and $a_{33}a_{35}$ between c and c. Let K_2 be the same as L_2. Let K_3 be the same as L_3

with the following modifications: put a_{33} immediately below a_{22} and put a_{44} at the bottom. Let K_4 be the same as L_4 with the following changes: put a_{35} immediately below a_{24} and a_{42} at the bottom. It is necessary for a_{51} to be below a_{55} in M_5. Then $P = K_1 \cap K_2 \cap K_3 \cap K_4$.

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Biographical Sketch

The author was born in Allentown, Pennsylvania on 9 July 1948. He graduated from William Allen High School there in June 1966. At the end of that month he entered the United States Naval Academy from which he graduated four years later with majors in Physics and Mathematics. Since that time he has been a Lieutenant in the U.S. Marine Corps studying at M.I.T.