Rigidity and Invariance Properties of Certain Geometric Frameworks

by

Lizhao Zhang

Bachelor of Science, Peking University, June 1995

Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2002

Signature of Author: ....

Department of Mathematics
September 24, 2001

Certified by ....

Daniel J. Kleitman
Professor of Mathematics
Thesis Supervisor

Accepted by ....

Tomasz S. Mrowka
Chair, Departmental Committee on Graduate Students

Rigidity and Invariance Properties of Certain Geometric Frameworks

by

Lizhao Zhang

Submitted to the Department of Mathematics
on September 24, 2001, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

Given a degenerate $(n+1)$-simplex in a n-dimensional Euclidean space R^n, which is embedded in a $(n+1)$-dimensional Euclidean space R^{n+1}. We allow all its vertices to have continuous motion in the space, either in R^{n+1} or restricted in R^n. For a given k, based on certain rules, we separate all its k-faces into 2 groups. During the motion, we give the following restriction: the volume of the k-faces in the 1st group can not increase (these faces are called "k-cables"); the volume of the k-faces in the 2nd group can not decrease ("k-struts"). We will prove that, under more conditions, all the volumes of the k-faces will be preserved for any sufficiently small motion.

We also partially generalize the above result to spherical space S^n and hyperbolic space H^n.

Thesis Supervisor: Daniel J. Kleitman
Title: Professor of Mathematics

Acknowledgments

I am grateful to many people who have made this thesis possible. First, I would like to thank my thesis advisor Professor Daniel J. Kleitman, for all the time and effort that he has devoted to my graduate study. I have benefited very much from his invaluable advice and unique scientific perspective on mathematics.

I would like to thank Professor Richard Stanley, for all the talks he had with me and the suggestions he made on my research. His introduction of other mathematics people to me gave me great help. I thank Professor Alan Edelman to serve on my thesis committee, and invite me to join his group meeting. I also thank Professor Robert Connelly for his pioneering work in rigidity theory and his kindness in helping answer my questions related to rigidity.

I would like to express my deepest gratitude to my former advisor Professor Gian-Carlo Rota, who passed away during my graduate study. His great enthusiasm and philosophical ideas on mathematics are always stimulating.

I give special thanks to Wei Luo, my officemate, who answered me so many questions when I was writing this thesis. His solid background on geometry helped me conquer difficulties when I was studying the rigidity theory on hyperbolic space. I also thank Xun Dong, Behrang Noohi, Lijing Wang, and Huafei Yan for their helpful conversations with me.

True friends are those who share good and bad times with me. I would like to thank Tongwei Liu and Zhonghui Xu, for their friendship over the years. I also thank other friends, who made my life at MIT much more enjoyable.

Finally, I thank my dear parents, their love and emotional support are invaluable to me.

Contents

List of Figures

Chapter 1

Introduction

Rigidity is an area that draws research interest from the old times. One of the first substantial mathematical results concerning rigidity is Cauchy's rigidity theorem [2], which says: "Two convex polyhedra comprised of the same number of equal similarly placed faces are superposable or symmetric." It is natural to ask what if the convexity restriction is removed. Consider a polyhedron in 3-space such that it can change its shape while keeping all its polygonal faces congruent. A longstanding conjecture(mentioned by Euler) has been that the polyhedron is rigid; namely, although adjacent faces of the polyhedron are allowed to rotate along common edges, the polyhedron can only have rigid motion in 3-space. Nevertheless a counterexample was found in Connelly [4]. A question remained as to whether the volume bounded by the surfaces was necessarily constant during the flex. When the polyhedron is homeomorphic to a sphere, the positive answer was given recently in Sabitov [8]. For general polyhedral surface, the positive answer was given in "The bellows conjecture" [5].

The above works share something in common: all the geometric structures they considered have distance restrictions on some pairs of vertices. For a geometric structure, instead of putting distance restrictions on some pairs of its vertices, we can also put volume restrictions on some of its k-faces. This leads to the study of another type of rigidity. There has some detailed discussion of this kind of rigidity in Tay, White and Whiteley([9], [10]). An interesting question is: if all k-faces of a n-simplex have the same volume, then is the n-simplex necessarily regular? Some results were given in McMullen [7].

The main purpose of this thesis is to derive rigidity and invariance properties from geometric structures that have volume restrictions on their k-faces.

In Chapter 2, we will study rigidity properties of certain degenerate $(n+1)$-simplices. The degenerate $(n+1)$-simplices are in a n-dimensional Euclidean space R^n which is embedded in a $(n+1)$-dimensional Euclidean space R^{n+1}. For a degenerate $(n+1)$-simplex, we allow all its vertices to have continuous motion, either in R^{n+1} or restricted in R^n. For each k with $1 \leq k \leq n$, based on certain rules, we separate all the k-faces of the degenerate $(n+1)$-simplex into 2 groups. During the motion, we give the following restriction: the k-volume of the k-faces in the 1st group can not increase; the k-volume of the k-faces in the 2nd group can not decrease. We prove that: if the continuous motion is restricted in R^n and also has some "good" properties, then all k-volumes of its k-faces are preserved for any sufficiently small motion(we say that it is k-unyielding in R^n). When the continuous motion is in R^{n+1} and real analytic, we derive a sequence of constants $c_0, \ldots, c_n$, and prove that the sign of c_{k-1} determines whether it is k-unyielding in R^{n+1}. Our main results in this chapter are Theorem 2.5, Theorem 2.6 and Theorem 2.7.

Chapter 3 is devoted to study the relationship between these k-unyielding properties with different k's by studying the relationship between $c_0, \ldots, c_n$. We define a polynomial

$$f(x) = \sum_{i=0}^{n} (-1)^i c_i x^{n-i},$$

and prove that $f(x)$ only has real roots(Theorem 3.1). We also give the necessary and sufficient condition of when $f(x)$ has n-repeated roots(Theorem 3.2).

Chapter 4 is our attempt to study some rigidity properties in spherical space S^n and hyperbolic space H^n. Most results in Chapter 2 also have analogues in Chapter 4. Some of them are proved while some remain as conjectures. Since R^n, S^n and H^n are spaces with constant sectional curvature 0, 1 and -1 respectively, we hope it can help us find more common rigidity properties among these spaces. Our main results in this chapter are Theorem 4.4, Theorem 4.5 and Theorem 4.6.

Chapter 2

Rigidity and Volume Preserving Deformation in R^n

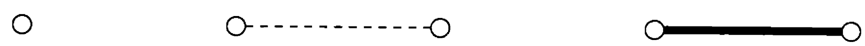
2.1 Definition

In this chapter, we will study some rigidity and volume preserving deformation properties of certain degenerate $(n+1)$-simplices. The degenerate $(n+1)$-simplices are in a n-dimensional Euclidean space R^n embedded in a $(n+1)$-dimensional Euclidean space R^{n+1}. For a degenerate $(n+1)$-simplex, we allow all the vertices to have continuous motion in the space, either in R^{n+1} or restricted in R^n. For each k with $1 \leq k \leq n$, we proved that, under some restrictions on the k-dimensional volumes (or k-volumes) of its k-simplices (or k-faces), all k-volumes of its k-faces are preserved for any sufficiently small motion.

A k-dim simplex is called a *k-cable* if its k-volume can not increase; it is called a *k-strut* if its k-volume can not decrease; it is called a *k-bar* if its k-volume can not change at all. We call a framework to be a *k-tensegrity framework* if some of its k-faces are labeled as either k-cables, k-struts, or k-bars, and the other k-faces are just not labeled as anything. All the k-tensegrity frameworks we consider in this paper are simplices, and *each* their k-face is labeled as either a k-cable, a k-strut, or a k-bar. We can consider k-cable, k-strut, and k-bar as volume restrictions imposed on frameworks. For a framework in R^d, if all k-volumes of its k-faces are preserved for any sufficiently small continuous motion under the volume restriction, then we say that it is *k-unyielding* in R^d.

We will also call 1-cable, 1-strut, and 1-bar as *cable*, *strut*, and *bar* respectively, and call 1-tensegrity framework as *tensegrity framework*. Particularly, for a framework in R^d, we say that it is *rigid* in R^d, if the distance between each pair of vertices can not be changed for any continuous motion under the volume restriction. We notice that, "rigid" is a stronger notion than "k-unyielding", since "rigid" automatically implies "k-unyielding", while not necessarily vice versa. Some definition of other types of rigidity, which is related to tensegrity frameworks, can be found in [3].

When we draw pictures, we use dashed line to represent cable, and use wide solid line to represent strut.



(a) vertex (b) cable (c) strut

Figure 2.1:

Given 2 k-tensegrity frameworks G_1 and G_2 with the same constructions of k-cables, k-struts, and k-bars, we say that G_1 *dominates* G_2, if the k-volume of each k-cable of G_2 is no bigger than the k-volume of the corresponding k-cable of G_1; the k-volume of each k-strut of G_2 is no smaller than the k-volume of the corresponding k-strut of G_1; and the corresponding k-bars of G_1 and G_2 have the same k-volume.

For a framework G_1 in R^d, we say that it is *globally rigid* in R^d, if when G_1 dominates another framework G_2 in R^d, then G_1 and G_2 are congruent. Global rigidity is a very strong notion imposed on a framework, and global rigidity automatically implies that a framework is also rigid in R^d.

Figure 2.2 gives an example of one of the simplest tensegrity frameworks. Suppose A_1, A_2, and A_3 are 3 points in R^1, and A_2 is between A_1 and A_3. Let A_1A_2 and A_2A_3 be struts, and let A_1A_3 be cable(Figure 2.2 (b)). If the continuous motion is restricted in R^1, then it is easy to see that the framework is rigid in R^1; however, if we embed R^1 into R^2 and allow the continuous motion to be in R^2, then the framework is not rigid in R^2. In the above framework, if we switch the role of cable and strut to get a new framework(Figure 2.2 (a)), then it is not hard to see that the new framework is rigid in both R^1 and R^2, and is also globally rigid in R^2.

(a)

(b)

Figure 2.2:

2.2 Construction of k-tensegrity Frameworks

Given $n+2$ points A_1, A_2, $\ldots$, A_{n+2} in R^n in general position, which means that every $n+1$ points are not in a $(n-1)$-dim hyperplane. We can treat these $n+2$ points as the vertices of a degenerate $(n+1)$-simplex. Since A_1, A_2, $\ldots$, A_{n+2} are in R^n in general position, there uniquely exists a sequence of non-zero coefficients α_1, α_2, $\ldots$, α_{n+2} (up to a non-zero factor c), such that $\sum_{i=1}^{n+2} \alpha_i = 0$ and $\sum_{i=1}^{n+2} \alpha_i \overrightarrow{OA_i} = 0$, where O is the origin in R^n. We will use these coefficients α_1, α_2, $\ldots$, α_{n+2} throughout this paper.

Suppose $P_1, P_2, \ldots, P_{k+1}$ are any $k+1$ points in the space, we use $V_k(P_1, \ldots, P_{k+1})$ to denote the k-volume of the k-simplex whose vertices are $P_1, P_2, \ldots, P_{k+1}$.

Proposition 2.1

$$V_n(A_1, \ldots, \hat{A}_i, \ldots, A_{n+2})/|\alpha_i|$$

is independent of i for $1 \leq i \leq n+2$.

Proof. Skipped. □

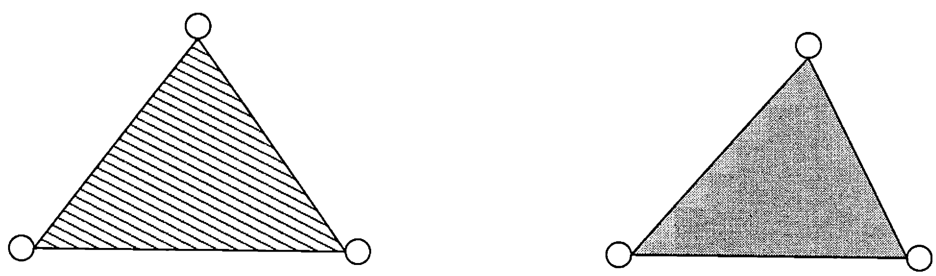
For convenience, we choose α_i to satisfy

$$|\alpha_i| = n! V_n(A_1, \ldots, \hat{A}_i, \ldots, A_{n+2}).$$

Besides, we also choose α_1 to be negative, then all the signs of α_i are determined afterwords. We separate these $n+2$ points into 2 sets X_1 and X_2 by the following rule: A_i is in X_1 if α_i is negative; A_i is in X_2 if α_i is positive. Since α_1 is negative, so A_1 is in set X_1. Indeed, the separation of these points is a following of *Radon's theorem*, which says that every $n+2$ points in R^n can be separated into 2 sets such that the convex hulls of the 2 sets have a non-empty intersection. Given k with $1 \leq k \leq n$, we separate all the k-faces $A_{i_1} \cdots A_{i_{k+1}}$ into 2 sets $Y_{k,1}$ and $Y_{k,2}$ by the following rule: a k-face $A_{i_1} \cdots A_{i_{k+1}}$ is in $Y_{k,1}$ if it has odd number of vertices in X_1; a k-face $A_{i_1} \cdots A_{i_{k+1}}$ is in $Y_{k,2}$ if it has even number of vertices in X_1. Based on the above separation of k-faces, we construct 2 different k-tensegrity frameworks below.

Framework $G_{n,k}$: let all the k-faces in group $Y_{k,1}$ be k-cables, and let all the k-faces in group $Y_{k,2}$ be k-struts.
Framework $F_{n,k}$: let all the k-faces in group $Y_{k,1}$ be k-struts, and let all the k-faces in group $Y_{k,2}$ be k-cables.

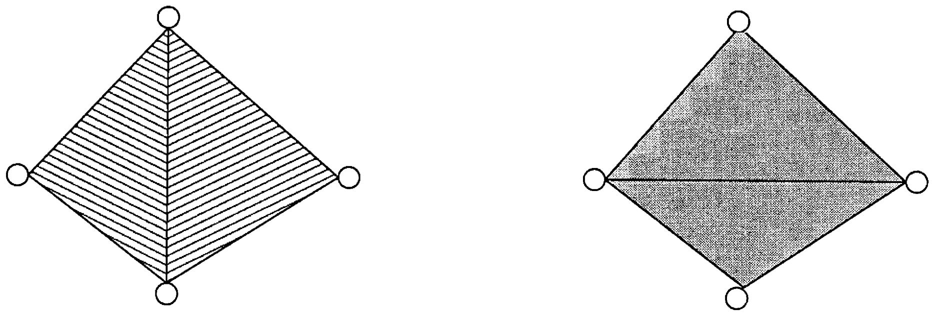
2-cable and 2-strut are shown in Figure 2.3.



(a) 2–cable (b) 2–strut

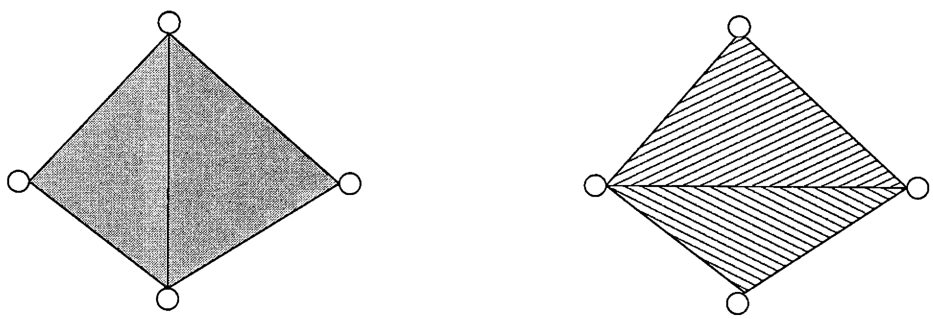
Figure 2.3:

For example, $G_{2,2}$ is shown in Figure 2.4: (a) shows the 2-cables of $G_{2,2}$, and (b) shows the 2-struts of $G_{2,2}$. $G_{2,2}$ is the combination of (a) and (b). $F_{2,2}$ is shown in Figure 2.5: (a) shows the 2-struts of $F_{2,2}$, and (b) shows the 2-cables of $F_{2,2}$. $F_{2,2}$ is the combination of (a) and (b).



(a) (b)

Figure 2.4:



(a) (b)

Figure 2.5:

We see that $F_{n,k}$ is constructed by switching the role of k-cable and k-strut in $G_{n,k}$. We use the notation $\mathbf{A}(t) = (A_1(t), \ldots, A_{n+2}(t))$ to denote the continuous motion which satisfies the k-volume restrictions with $t \geq 0$ and $A_i(0) = A_i$. The main purpose of this chapter is to see when the above 2 frameworks are k-unyielding in R^n and R^{n+1}.

Before we start the detailed discussion of k-unyielding properties for general k, we first give some results for 2 specific cases $k = 1$ and $k = n$. For convenience, a vector $\overrightarrow{OP}$ will also be written as P when there is no confusion. For example, the inner product $\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}$ and $\overrightarrow{OP}^2$ can also be written as $P_1 \cdot P_2$ and P^2 respectively. The following result was prove in Bezdek and Connelly [1].

Theorem 2.1 *(1) $G_{n,1}$ is globally rigid in R^{n+1}.*

(2) $F_{n,1}$ is rigid in R^n.

Proof. We use the same coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n+2}$ as before which satisfy $\sum \alpha_i = 0$ and $\sum \alpha_i A_i = 0$.

(1) Suppose G' is a tensegrity framework in R^{n+1} which is dominated by $G_{n,1}$, and the vertices of G' are $B_1, B_2, \ldots, B_{n+2}$. We then have

$$0 \le (\alpha_1 B_1 + \cdots + \alpha_{n+2} B_{n+2})^2$$
$$= \sum_i \alpha_i^2 B_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j B_i \cdot B_j$$

since $\sum_{i=1}^{n+2} \alpha_i = 0$

$$= -\sum_i \alpha_i(\alpha_1 + \cdots + \hat{\alpha_i} + \cdots + \alpha_{n+2}) B_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j B_i \cdot B_j$$
$$= -\sum_{i<j} \alpha_i \alpha_j (B_i - B_j)^2.$$

Based on the definition, we have $|B_i - B_j| \le |A_i - A_j|$ if $A_i A_j$ is a cable; and $|B_i - B_j| \ge |A_i - A_j|$ if $A_i A_j$ is a strut. In both cases, we have

$$\alpha_i \alpha_j (B_i - B_j)^2 \ge \alpha_i \alpha_j (A_i - A_j)^2,$$

so

$$-\sum_{i<j} \alpha_i \alpha_j (B_i - B_j)^2 \le -\sum_{i<j} \alpha_i \alpha_j (A_i - A_j)^2$$
$$= -\sum_i \alpha_i(\alpha_1 + \cdots + \hat{\alpha_i} + \cdots + \alpha_{n+2}) A_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j A_i \cdot A_j$$
$$= \sum_i \alpha_i^2 A_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j A_i \cdot A_j$$
$$= (\alpha_1 A_1 + \cdots + \alpha_{n+2} A_{n+2})^2$$
$$= 0.$$

Then $\alpha_i\alpha_j(B_i - B_j)^2 = \alpha_i\alpha_j(A_i - A_j)^2$ must hold. Since each α_i is not 0, so we have $|B_i - B_j| = |A_i - A_j|$, which means that $G_{n,1}$ is globally rigid in R^{n+1}.

(2) Suppose after a small motion, point A_i moves to B_i. When the motion is small enough, $B_1, \ldots, B_{n+2}$ are still in general position in R^n. So we can find a sequence of coefficients $\beta_1, \beta_2, \ldots, \beta_{n+2}$ that satisfy $\sum \beta_i = 0$ and $\sum \beta_i B_i = 0$, and each β_i also has the same sign as α_i. Based on definition, we have $\beta_i\beta_j(B_i - B_j)^2 \leq \beta_i\beta_j(A_i - A_j)^2$. Following from almost the same computation above, we have

$$
\begin{aligned}
0 &= (\beta_1 B_1 + \cdots + \beta_{n+2}B_{n+2})^2 \\
&= -\sum_{i<j} \beta_i\beta_j(B_i - B_j)^2 \\
&\geq -\sum_{i<j} \beta_i\beta_j(A_i - A_j)^2 \\
&= (\beta_1 A_1 + \cdots + \beta_{n+2}A_{n+2})^2 \\
&\geq 0.
\end{aligned}
$$

Then $|B_i - B_j| = |A_i - A_j|$ must hold, which implies that $F_{n,1}$ is rigid in R^n.

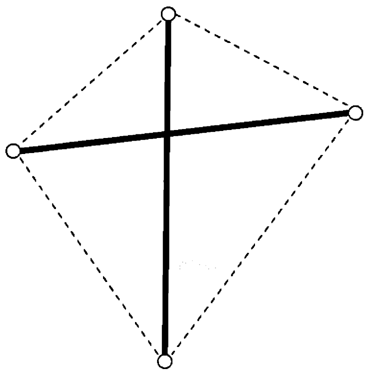
□

Remark. For $k = 1$, besides $G_{n,1}$ and $F_{n,1}$, any other no-bar tensegrity framework constructed on points $A_1, \ldots, A_{n+2}$ is not rigid in R^n; and $G_{n,1}$ is also the only no-bar tensegrity framework constructed on points $A_1, \ldots, A_{n+2}$ to be rigid in R^{n+1}. The proof is left to the reader.

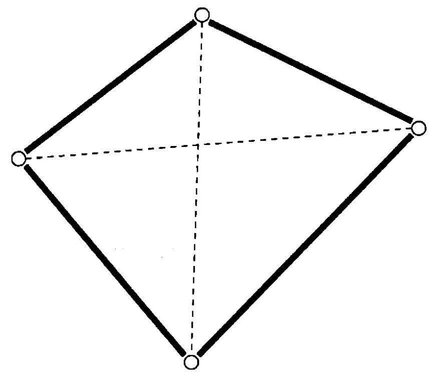
Figure 2.6 (a) is $G_{2,1}$, which is globally rigid in R^3; (b) is $F_{2,1}$, which is rigid in R^2, but not in R^3.

The above rigidity properties of $G_{n,1}$ and $F_{n,1}$ are purely *combinatorial* properties, as they are only determined by how we separate points $A_1, \ldots, A_{n+2}$ into 2 groups. However, when we talk about the k-unyielding properties of $G_{n,k}$ and $F_{n,k}$ later, we will find that they are also determined by other factors.

Theorem 2.2 *Both $G_{n,n}$ and $F_{n,n}$ are n-unyielding in R^n.*



(a)



(b)

Figure 2.6:

Proof. Suppose after a small motion, point A_i moves to B_i. When the motion is small enough, the following equation

$$\sum_{\alpha_i>0} V_n(B_1, \ldots, \hat{B}_i, \ldots, B_{n+2}) = \sum_{\alpha_i<0} V_n(B_1, \ldots, \hat{B}_i, \ldots, B_{n+2})$$

holds, as both sides of the equation are the n-volume of the convex hull of points B_1, $\ldots$, B_{n+2}. We notice that, one side of the above equation is the sum of the n-volume of all the n-cables, and the other side is the sum of the n-volume of all the n-struts, which implies that all these n-volumes are not changed for $G_{n,n}$ and $F_{n,n}$. It means that $G_{n,n}$ and $F_{n,n}$ are n-unyielding in R^n. □

Remark. $G_{n,n}$ and $F_{n,n}$ are not rigid in R^n. For example, there has continuous affine motion to keep all the n-volume of their n-faces fixed while the shape of the n-faces can be changed.

After we proved that $G_{n,n}$ and $F_{n,n}$ are n-unyielding in R^n, a natural question to ask is: are $G_{n,n}$ and $F_{n,n}$ still n-unyielding in R^{n+1}? We will give the relatively more complicated answer after we have more preparation in the following sections.

2.3 Exterior Algebra

Before studying k-unyielding properties for general k, we introduce some mathematical tools to handle the computation of k-volumes. In this section, we are going to prove some k-volume related properties by using exterior algebra.

Let $\Lambda(R^n)$ denote the exterior algebra on R^n. If a and b are two elements of $\Lambda(R^n)$, we will denote the exterior product of a and b by $a \wedge b$ or just ab if it does not cause any confusion. If a is an exterior product of k elements of R^n, we say that a is a *decomposable k-vector.* If a is a linear combination of decomposable k-vectors, then we call a a *k-vector.* We use $\Lambda^k(R^n)$ to denote the vector space that contains all the k-vectors. We have $\Lambda(R^n) = \bigoplus_{k=0}^{n} \Lambda^k(R^n)$ as a vector space.

Suppose $E = \{e_1, \ldots, e_n\}$ is a orthonormal basis of R^n, then

$$E^{(k)} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis of $\Lambda^k(R^n)$.

If P_1, ..., P_{k+1} are $k+1$ points in R^n, we define $\overrightarrow{P_1 \cdots P_{k+1}}$ to be a k-vector in $\Lambda^k(R^n)$ as

$$\overrightarrow{P_1 \cdots P_{k+1}} = \overrightarrow{P_1P_2} \wedge \overrightarrow{P_1P_3} \wedge \cdots \wedge \overrightarrow{P_1P_{k+1}}.$$

From some basic exterior algebra computation, we have

$$\begin{aligned} \overrightarrow{P_1P_2} \wedge \cdots \wedge \overrightarrow{P_1P_{k+1}} &= (P_2 - P_1) \wedge \cdots \wedge (P_{k+1} - P_1) \\ &= \sum_{i=1}^{k+1} (-1)^{i+1} P_1 \wedge \cdots \wedge \hat{P_i} \wedge \cdots \wedge P_{k+1}. \end{aligned}$$

So we also have

$$\overrightarrow{P_1 \cdots P_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i+1} P_1 \wedge \cdots \wedge \hat{P_i} \wedge \cdots \wedge P_{k+1}.$$

Based on the inner product defined on R^n, we can also define inner product on $\Lambda^k(R^n)$. Remember that $e_i \cdot e_j = 1$ if $i = j$; and $e_i \cdot e_j = 0$ is $i \neq j$. We will define the inner product on $\Lambda^k(R^n)$ as

$$(e_{i_1} \wedge \cdots \wedge e_{i_k}) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_k}) = \det(e_{i_l} \cdot e_{j_m})_{1 \leq l,m \leq k}.$$

This definition is equivalent to say that the inner product of every two different elements in $E^{(k)}$ is 0; and the inner product of every element in $E^{(k)}$ to itself is 1. We say that this inner product is naturally well defined, because the definition is independent of the choice of the basis E.

Lemma 2.1 *If $r_1, \ldots, r_k, s_1, \ldots, s_k$ are $2k$ elements in R^n, then we have*

$$(r_1 \wedge \cdots \wedge r_k) \cdot (s_1 \wedge \cdots \wedge s_k) = \det(r_i \cdot s_j)_{1 \le i,j \le k}.$$

Proof. Based on the definition, the above formula is true when each r_i and s_j are elements in the basis E. If we consider each r_i and s_j as variables, then both sides of the formula are linear to each variable. We write each r_i and s_j as linear combinations of $e_1, \ldots, e_n$ and expand both sides of the above formula. We will then notice that they are the same. □

Proposition 2.2 *If $P_1, \ldots, P_{k+1}$ are $k+1$ points in R^n, then we have*

$$\overrightarrow{P_1 \cdots P_{k+1}}^2 = (k!)^2 V_k^2(P_1, \ldots, P_{k+1}).$$

This property tells us that when we want to computer the k-volume of a k-simplex, we can compute the inner product of its associated k-vector instead.

Lemma 2.2 *If we use $l_{i,j}$ to denote $|P_i - P_j|$, then*

$$\overrightarrow{P_1 \cdots P_{k+1}}^2 = \det((l_{1,i}^2 + l_{1,j}^2 - l_{i,j}^2)/2)_{2 \le i,j \le k+1}.$$

Proof. Since we have

$$\begin{aligned} \overrightarrow{P_1 \cdots P_{k+1}}^2 &= (\overrightarrow{P_1P_2} \wedge \cdots \wedge \overrightarrow{P_1P_{k+1}})^2 \\ &= \det(\overrightarrow{P_1P_i} \cdot \overrightarrow{P_1P_j})_{2 \le i,j \le k+1}, \end{aligned}$$

so we only need to prove $\overrightarrow{P_1P_i} \cdot \overrightarrow{P_1P_j} = (l_{1,i}^2 + l_{1,j}^2 - l_{i,j}^2)/2$. This is true because

$$\begin{aligned} 2\overrightarrow{P_1P_i} \cdot \overrightarrow{P_1P_j} &= 2(P_i - P_1) \cdot (P_j - P_1) \\ &= (P_i - P_1)^2 + (P_j - P_1)^2 - (P_i - P_j)^2 \\ &= l_{1,i}^2 + l_{1,j}^2 - l_{i,j}^2. \end{aligned}$$

□

Remark. This result tells us that the square of the k-volume of a k-simplex is not only a polynomial with all its edge lengths $l_{i,j}$ as variables, but also a polynomial with all the $l_{i,j}^2$ as variables.

After we know the relationship between $V_k(P_1, \ldots, P_{k+1})$ and $l_{i,j}$, a interesting question arises: if all $l_{i,j}$ $(1 \leq i, j \leq k+1)$ are first given, when can we find a k-simplex $P_1 \cdots P_{k+1}$ in R^k such that $|P_i - P_j| = l_{i,j}$? A complete answered is given below.

Theorem 2.3 *Given $l_{i,j} \geq 0$ $(1 \leq i, j \leq k+1$, and $l_{i,j} = l_{j,i})$, then we can find a k-simplex $P_1 \cdots P_{k+1}$ in R^k such that $|P_i - P_j| = l_{i,j}$ if and only if: the $k \times k$ matrix $((l_{1,i}^2 + l_{1,j}^2 - l_{i,j}^2)/2)_{2 \leq i,j \leq k+1}$ is positive semi-definite. If we require $P_1 \cdots P_{k+1}$ to be a non-degenerate k-simplex, then the same matrix should be positive definite.*

Proof. Let $A = ((l_{1,i}^2 + l_{1,j}^2 - l_{i,j}^2)/2)_{2 \leq i,j \leq k+1}$. If A is positive semi-definite, then we can find a $k \times k$ matrix C, such that $A = CC^T$. Let $P_1 = O$, and P_{i+1} be the point that with C's i-th row as its coordinate. then the k-simplex $P_1 \cdots P_{k+1}$ satisfies $|P_i - P_j| = l_{i,j}$.

If k-simplex $P_1 \cdots P_{k+1}$ in R^k satisfies $|P_i - P_j| = l_{i,j}$, then let C be a $k \times k$ matrix such that C's i-th row is $\overrightarrow{P_1P_{i+1}}$. Then we have $A = CC^T$, so A is positive semi-definite.

□

We already know that $\overrightarrow{P_1 \cdots P_{k+1}}^2$ is a polynomial with all $l_{i,j}^2$ as its variables where $l_{i,j} = |P_i - P_j|$. The partial derivative of $\overrightarrow{P_1 \cdots P_{k+1}}^2$ over $l_{i,j}^2$ is also a polynomial, and we will write it as $\partial_{l_{i,j}^2} \overrightarrow{P_1 \cdots P_{k+1}}^2$. We found that $\partial_{l_{i,j}^2} \overrightarrow{P_1 \cdots P_{k+1}}^2$ can be represented as the inner product of two $(k-1)$-vectors, which is more convenient to be used than its polynomial form in terms of computation. The two $(k-1)$-vectors are given below.

Lemma 2.3

$$\partial_{l_{i,j}^2} \overrightarrow{P_1 \cdots P_{k+1}}^2 = \overrightarrow{P_i P_1 \cdots \hat{P}_i \cdots \hat{P}_j \cdots P_{k+1}} \cdot \overrightarrow{P_j P_1 \cdots \hat{P}_i \cdots \hat{P}_j \cdots P_{k+1}}$$

Proof. Let $A = ((l_{1,i}^2 + l_{1,j}^2 - l_{i,j}^2)/2)_{2 \le i,j \le k+1}$. From Lemma 2.2, we have $\overrightarrow{P_1 \cdots P_{k+1}}^2 = \det(A)$.

If $1 < i$ and $1 < j$, then $l_{i,j}^2$ only appears inside the $(i-1, j-1)$-th and $(j-1, i-1)$-th item of matrix A. Since A is symmetric, so $\partial_{l_{i,j}^2} \det(A)$ is $(-1)^{i+j+1}$ times the determinant of a $(k-1) \times (k-1)$ matrix B, which is the rest of matrix A without the $(i-1)$-th row and $(j-1)$-th column. Since

$$\begin{aligned} \det(B) &= (\overrightarrow{P_1P_2} \wedge \cdots \wedge \widehat{\overrightarrow{P_1P_i}} \wedge \cdots \wedge \overrightarrow{P_1P_{k+1}}) \cdot (\overrightarrow{P_1P_2} \wedge \cdots \wedge \widehat{\overrightarrow{P_1P_j}} \wedge \cdots \wedge \overrightarrow{P_1P_{k+1}}) \\ &= \overrightarrow{P_1 \cdots \hat{P}_i \cdots P_{k+1}} \cdot \overrightarrow{P_1 \cdots \hat{P}_j \cdots P_{k+1}}, \end{aligned}$$

so

$$\partial_{l_{i,j}^2} \det(A) = \overrightarrow{P_i P_1 \cdots \hat{P}_i \cdots \hat{P}_j \cdots P_{k+1}} \cdot \overrightarrow{P_j P_1 \cdots \hat{P}_i \cdots \hat{P}_j \cdots P_{k+1}}.$$

If $i = 1$ or $j = 1$, the formula still holds because of symmetry. □

Now we will start considering the cases when continuous motion $P_i(t)$ ($t \ge 0$ and $P_i(0) = P_i$) is involved.

For convenience, also for trying to avoid writing a formula that is to long in one line, we will sometime use p_i to replace $P_i(t)$ if it does not cause any confusion.

Corollary 2.1 *Suppose $P_1(t)$, ..., $P_{k+1}(t)$ ($t \ge 0$ and $P_i(0) = P_i$) are continuous motion, and each their coordinate is a C^1 function over t, then*

$$(\overrightarrow{p_1 \cdots p_{k+1}}^2)' = \sum_{i<j} (\overrightarrow{p_i p_1 \cdots \hat{p}_i \cdots \hat{p}_j \cdots p_{k+1}} \cdot \overrightarrow{p_j p_1 \cdots \hat{p}_i \cdots \hat{p}_j \cdots p_{k+1}})(l_{i,j}^2(t))',$$

where $p_i = P_i(t)$ and $l_{i,j}(t) = |P_i(t) - P_j(t)|$.

Proof. Use chain rule and Lemma 2.3. □

Lemma 2.4 *Embed R^n into a bigger space R^d. For a given l, let ω_1 and ω_2 be any two l-vectors in R^d, then we have*

(1)

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}(\omega_1\wedge\overrightarrow{A_{i_1}\cdots A_{i_{k+1}}})\cdot(\omega_2\wedge\overrightarrow{A_{i_1}\cdots A_{i_{k+1}}})=0,$$

(2)

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}\overrightarrow{A_{i_1}\cdots A_{i_{k+1}}}^2=0.$$

Proof. Since (2) is a special case of (1), we only need to prove (1). In order to make a formula not too wide in a line, we will omit "$\wedge$" if there is no confusion in the following. For example, we will write $A_i\wedge A_j$ as A_iA_j. Then we have

$$\begin{aligned}
&\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}(\omega_1\wedge\overrightarrow{A_{i_1}\cdots A_{i_{k+1}}})\cdot(\omega_2\wedge\overrightarrow{A_{i_1}\cdots A_{i_{k+1}}})\\
=&\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}(\omega_1\sum_{j=1}^{k+1}(-1)^{j+1}A_{i_1}\cdots\hat{A_{i_j}}\cdots A_{i_{k+1}})\\
&\cdot(\omega_2\sum_{j=1}^{k+1}(-1)^{j+1}A_{i_1}\cdots\hat{A_{i_j}}\cdots A_{i_{k+1}})\\
=&\sum_{i_1<\cdots<i_k} \alpha_{i_1}\cdots\alpha_{i_k}(\sum_{i=1}^{n+2}\alpha_i-\sum_{j=1}^{k}\alpha_{i_j})(\omega_1A_{i_1}\cdots A_{i_k})\cdot(\omega_2A_{i_1}\cdots A_{i_k})\\
&-\sum_{i_1<\cdots<i_{k-1}}\sum_{i_k\neq i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}(\omega_1A_{i_1}\cdots A_{i_{k-1}}A_{i_k})\cdot(\omega_1A_{i_1}\cdots A_{i_{k-1}}A_{i_{k+1}})
\end{aligned}$$

since $\sum_{i=1}^{n+2}\alpha_i=0$

$$\begin{aligned}
=&-\sum_{i_1<\cdots<i_k} \alpha_{i_1}\cdots\alpha_{i_k}(\sum_{j=1}^{k}\alpha_{i_j})(\omega_1A_{i_1}\cdots A_{i_k})\cdot(\omega_2A_{i_1}\cdots A_{i_k})\\
&-\sum_{i_1<\cdots<i_{k-1}}\sum_{i_k\neq i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}(\omega_1A_{i_1}\cdots A_{i_{k-1}}A_{i_k})\cdot(\omega_1A_{i_1}\cdots A_{i_{k-1}}A_{i_{k+1}})\\
=&-\sum_{i_1<\cdots<i_{k-1}}\sum_{i_k}\sum_{i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}}(\omega_1A_{i_1}\cdots A_{i_{k-1}}A_{i_k})\cdot(\omega_1A_{i_1}\cdots A_{i_{k-1}}A_{i_{k+1}})\\
=&-\sum_{i_1<\cdots<i_{k-1}} \alpha_{i_1}\cdots\alpha_{i_{k-1}}(\omega_1A_{i_1}\cdots A_{i_{k-1}}(\sum_{i_k}\alpha_{i_k}A_{i_k}))\\
&\cdot(\omega_1A_{i_1}\cdots A_{i_{k-1}}(\sum_{i_{k+1}}\alpha_{i_{k+1}}A_{i_{k+1}}))
\end{aligned}$$

since $\sum_{i=1}^{n+2} \alpha_i A_i = 0$

$= 0.$

□

Lemma 2.5 *Embed R^n into a bigger space R^d and P is a point inside R^d. For a given l, let ω_1 be a l-vector and ω_2 be a $(l+1)$-vector in R^d. Then*

$$\sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k} (\omega_1 \wedge \overrightarrow{PA_{i_1} \cdots A_{i_k}}) \cdot (\omega_2 \wedge \overrightarrow{A_{i_1} \cdots A_{i_k}}) = 0.$$

Proof. We use $b_k(\omega_1, \omega_2; P)$ to denote the right side of the formula. In order to make a formula not too wide in a line, we will omit "$\wedge$" if there is no confusion in the following. By symmetry, we only need to prove that $b_k(\omega_1, \omega_2; O) = 0$. We have

$$\begin{aligned}
& b_k(\omega_1, \omega_2; O) \\
&= \sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k} (\omega_1 A_{i_1} \cdots A_{i_k}) \cdot (\omega_2 \sum_{j=1}^{k} (-1)^{j+1} A_{i_1} \cdots \hat{A_{i_j}} \cdots A_{i_k}) \\
&= \sum_{i_1<\cdots<i_{k-1}} \sum_{i_k} \alpha_{i_1} \cdots \alpha_{i_k} (\omega_1 A_{i_k} A_{i_1} \cdots A_{i_{k-1}}) \cdot (\omega_2 A_{i_1} \cdots A_{i_{k-1}}) \\
&= \sum_{i_1<\cdots<i_{k-1}} \alpha_{i_1} \cdots \alpha_{i_{k-1}} (\omega_1 (\sum_{i_k} \alpha_{i_k} A_{i_k}) A_{i_1} \cdots A_{i_{k-1}}) \cdot (\omega_2 A_{i_1} \cdots A_{i_{k-1}})
\end{aligned}$$

since $\sum_{i=1}^{n+2} \alpha_i A_i = 0$

$= 0.$

□

Lemma 2.6 *Suppose P and Q are any two points in a higher dimensional space R^d. For a given l, let ω_1 and ω_2 be any two l-vectors in R^d. Then*

$$c_k(\omega_1, \omega_2; P, Q) := \sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k} (\omega_1 \wedge \overrightarrow{PA_{i_1} \cdots A_{i_k}}) \cdot (\omega_2 \wedge \overrightarrow{QA_{i_1} \cdots A_{i_k}})$$

is independent of the choice of points P and Q. Then we can define a sequence of constants c_0, ..., c_n such that $c_0 = 1$, and

$$c_k = \sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k} \overrightarrow{PA_{i_1} \cdots A_{i_k}} \cdot \overrightarrow{QA_{i_1} \cdots A_{i_k}} \tag{2.1}$$

for $1 \leq k \leq n$, which is independent of the choice of points P and Q.

Proof.

$$\begin{aligned} & c_k(\omega_1, \omega_2; P, Q) - c_k(\omega_1, \omega_2; P, O) \\ = & - \sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k} (\omega_1 \wedge \overrightarrow{PA_{i_1} \cdots A_{i_k}}) \cdot (\omega_2 \wedge Q \wedge \overrightarrow{A_{i_1} \cdots A_{i_k}}) \\ = & \ 0. \end{aligned}$$

In above, the last "=' comes from Lemma 2.5.

By symmetry, we have

$$c_k(\omega_1, \omega_2; P, Q) = c_k(\omega_1, \omega_2; P, O) = c_k(\omega_1, \omega_2; O, O),$$

which is independent of the choice of points P and Q. □

A geometric optimization problem can be easily solved by using the above results. Suppose P_1, P_2, ..., P_{n+1} are $n+1$ points in R^n, then the question is: at which point Q does the expression

$$\sum_{1\leq i_1<\cdots<i_k} V_k^2(Q, P_{i_1}, \ldots, P_{i_k})$$

get minimized? When $k = 1$, a well-known result tells us that the expression is minimized when Q is the *centroid* (or called center of gravity) of points P_1, P_2, ..., P_{n+1}. The following theorem gives the solution for general k.

Theorem 2.4 *For each k, the expression*

$$\sum_{1\leq i_1<\cdots<i_k} \overrightarrow{QP_{i_1} \cdots P_{i_k}}^2$$

is always minimized when Q is the centroid of points P_1, P_2, ..., P_{n+1}.

Proof. Suppose P_0 is the centroid of points $P_1, \ldots, P_{n+1}$. Let $\gamma_0 = -n-1$ and $\gamma_i = 1$ for $1 \le i \le n+1$, then $\sum_{i=0}^{n+1} \gamma_i = 0$ and $\sum_{i=0}^{n+1} \gamma_i P_i = 0$. Using Lemma 2.6, we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_k} \overrightarrow{P_0P_{i_1}\cdots P_{i_k}}^2 &= \sum_{0\le i_1<\cdots<i_k} \gamma_{i_1}\cdots\gamma_{i_k}\overrightarrow{QP_{i_1}\cdots P_{i_k}}^2 \\ &= \sum_{1\le i_1<\cdots<i_k} \overrightarrow{QP_{i_1}\cdots P_{i_k}}^2 - (n+1)\sum_{1\le i_1<\cdots<i_{k-1}} \overrightarrow{QP_0P_{i_1}\cdots P_{i_{-1}}}^2 \\ &\le \sum_{1\le i_1<\cdots<i_k} \overrightarrow{QP_{i_1}\cdots P_{i_k}}^2,\end{aligned}$$

and "=" holds if and only if Q is the centroid P_0. □

2.4 Main Theorems

When we consider the continuous motion of points $A_1, \ldots, A_{n+2}$, we use the notation $\mathbf{A}(t) = (A_1(t), \ldots, A_{n+2}(t))$ to denote the continuous motion with $t \ge 0$ and $A_i(0) = A_i$. When the continuous motion $\mathbf{A}(t)$ is restricted in R^n, similar to the way that we get α_i which satisfies $\sum_{i=1}^{n+2} \alpha_i = 0$ and $\sum_{i=1}^{n+2} \alpha_i A_i = 0$, we can get coefficients $\alpha_1(t)$, $\ldots$, $\alpha_{n+2}(t)$ such that they satisfy $\sum_{i=1}^{n+2} \alpha_i(t) = 0$ and $\sum_{i=1}^{n+2} \alpha_i(t)A_i(t) = 0$. Besides, we can also require $\alpha_i(t)$ to satisfy

$$\begin{aligned}|\alpha_i(t)| &= n!V_n(A_1(t), \ldots, \hat{A_i(t)}, \ldots, A_{n+2}(t)) \\ &= |\overrightarrow{A_1(t)\cdots\hat{A_i(t)}\cdots A_{n+2}(t)}|.\end{aligned}$$

When t is small enough, $\alpha_i(t)$ has the same sign as $\alpha_i(0) = \alpha_i$.

For convenience, also for trying to avoid writing a formula that is too long in a line, we will sometimes use a_i to replace $A_i(t)$ when there is no confusion. Similar to the definition of c_k, we can define $c_0(t) = 1$ and

$$c_k(t) = \sum_{i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)\overrightarrow{Pa_{i_1}\cdots a_{i_k}} \cdot \overrightarrow{Qa_{i_1}\cdots a_{i_k}},$$

which is independent of the choice of points P and Q.

Lemma 2.7 *When each coordinate of $A_i(t)$ is a C^1 function over t, we have*

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' = 0,$$

where $a_i = A_i(t)$.

Proof. When $k = 1$, we have

$$\sum_{i_1<i_2} \alpha_{i_1}(t)\alpha_{i_2}(t)(\overrightarrow{a_{i_1}a_{i_2}}^2)'$$
$$= -\sum_{i_1<i_2} (\overrightarrow{a_{i_1}a_1\cdots\hat{a_{i_1}}\cdots\hat{a_{i_2}}\cdots a_{n+2}} \cdot \overrightarrow{a_{i_2}a_1\cdots\hat{a_{i_1}}\cdots\hat{a_{i_2}}\cdots a_{n+2}})(\overrightarrow{a_{i_1}a_{i_2}}^2)'$$
$$= -(\overrightarrow{a_1\cdots a_{n+2}}^2)'$$
$$= -0'$$
$$= 0.$$

In above, the second "=" comes from Corollary 2.1.

When $k \geq 2$, we have

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)'$$
$$= \sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\sum_{j<l}(\overrightarrow{a_{i_j}a_{i_1}\cdots\hat{a_{i_j}}\cdots\hat{a_{i_l}}\cdots\ a_{i_{k+1}}}$$
$$\cdot\ \overrightarrow{a_{i_l}a_{i_1}\cdots\hat{a_{i_j}}\cdots\hat{a_{i_l}}\cdots\ a_{i_{k+1}}})(\overrightarrow{a_{i_j}a_{i_l}}^2)')$$
$$= \sum_{j<l}\alpha_j(t)\alpha_l(t)(\overrightarrow{a_ja_l}^2)' \sum_{i_1<\cdots<i_{k-1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k-1}}(t)\overrightarrow{a_ja_{i_1}\cdots a_{i_{k-1}}} \cdot \overrightarrow{a_la_{i_1}\cdots a_{i_{k-1}}}$$
$$= c_{k-1}(t)\sum_{j<l}\alpha_j(t)\alpha_l(t)(\overrightarrow{a_ja_l}^2)'$$
$$= 0$$

In above, the first "=" comes from Corollary 2.1; the third "=" comes from Lemma 2.6; and the last "=" comes from the special case $k = 1$ which we just proved above. □

This property leads to our first main theorem.

Theorem 2.5 *If* $\mathbf{A}(t)$ *is a continuous motion that satisfies the* k*-volume restrictions, and* $\mathbf{A}(t)$ *is also* C^1 *over* t*, then both* $G_{n,k}$ *and* $F_{n,k}$ *are* k*-unyielding in* R^n *for each* k*.*

Proof. If $A_{i_1}\cdots A_{i_{k+1}}$ is a k-strut, then we have

$$(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' \geq 0;$$

if $A_{i_1}\cdots A_{i_{k+1}}$ is a k-cable, then

$$(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' \leq 0.$$

So for $G_{n,k}$(or $F_{n,k}$), we have

$$\alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' \geq (\text{or} \leq)0.$$

By using Lemma 2.7, we find that

$$(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' = 0$$

must hold for Small $t \geq 0$. So both $G_{n,k}$ and $F_{n,k}$ are k-unyielding in R^n. □

Remark. For the above theorem, we proved two special cases $k = 1$ and $k = n$ before without using the restriction "$\mathbf{A}(t)$ is also C^1 over t". We believe that, $G_{n,k}$ and $F_{n,k}$ can also be proved to be k-unyielding in R^n without the restriction "$\mathbf{A}(t)$ is also C^1 over t".

We are now starting to discuss that whether $G_{n,k}$ and $F_{n,k}$ are also k-unyielding in R^{n+1}. Remind that we derived a sequence of constants $c_0, c_1, \ldots, c_n$ in Lemma 2.6, where $c_0 = 1$ and

$$c_k = \sum_{i_i<\cdots<i_k} \alpha_{i_1}\cdots\alpha_{i_k}\overrightarrow{PA_{i_1}\cdots A_{i_k}}\cdot\overrightarrow{QA_{i_1}\cdots A_{i_k}},$$

which is independent of the choice of points P and Q. These constants play the main roles in deciding whether $G_{n,k}$ and $F_{n,k}$ are k-unyielding in R^{n+1}.

Suppose $\mathbf{A}(t)$ is a continuous motion in R^{n+1}. When t is small, let $B_1(t)$ be $A_1(t)$'s orthogonal projection onto the hyperplane which contains points $A_2(t), \ldots, A_{n+2}(t)$. Easy to see that, if $\mathbf{A}(t)$ is real analytic over t, $B_1(t)$ is also real analytic over t, and $B_1(0) = A_1$. For small t, we can find a sequence of coefficients $\alpha_1(t), \ldots, \alpha_{n+2}(t)$ such that $\sum\alpha_i(t) = 0$,

$$\alpha_1(t)B_1(t) + \sum_{i\geq 2}\alpha_i(t)A_i(t) = 0.$$

and

$$|\alpha_i(t)| = \begin{cases} n!\, V_n(A_2(t), \ldots, A_{n+2}(t)) & \text{if } i = 1, \\ n!\, V_n(B_1(t), A_2(t), \ldots, \hat{A_i(t)}, \ldots, A_{n+2}(t)) & \text{if } i \geq 2. \end{cases}$$

When t is small, $\alpha_i(t)$ and $\alpha_i(0) = \alpha_i$ has the same sign.

For convenience, we define $B_i(t) = A_i(t)$ for $i \geq 2$. In order to avoid writing a formula which is too long in one line, we sometimes use a_i and b_i to replace $A_i(t)$ and $B_i(t)$ respectively. Similar to the definition of c_k, we can define $c_0(t) = 1$ and

$$c_k(t) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1}(t) \cdots \alpha_{i_k}(t) \overrightarrow{Pb_{i_1} \cdots b_{i_k}} \cdot \overrightarrow{Qb_{i_1} \cdots b_{i_k}},$$

which is independent of the choice of points P and Q, and has $c_k(0) = c_k$.

Lemma 2.8 *Suppose* $\mathbf{A}(t)$ *is real analytic over* t *for small* $t \geq 0$. *Use* a_i *to replace* $A_i(t)$.

(1) If $c_{k-1} > 0$, *then for small* $t \geq 0$,

$$\sum_{i_1 < \cdots < i_{k+1}} \alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t) (\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' \leq 0,$$

and the equality holds if and only if $B_1(t) = A_1(t)$.

(2) If $c_{k-1} < 0$, *then for small* $t \geq 0$,

$$\sum_{i_1 < \cdots < i_{k+1}} \alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t) (\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' \geq 0,$$

and the equality holds if and only if $B_1(t) = A_1(t)$.

Proof. We will prove (1) and (2) at the same time.

$$\begin{aligned}
&\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' \\
&= \sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1}\cdots a_{i_{k+1}}}^2)' - \sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\overrightarrow{b_{i_1}\cdots b_{i_{k+1}}}^2)' \\
&= \alpha_1(t) \sum_{2\le i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)(\overrightarrow{a_1 b_{i_1}\cdots b_{i_k}}^2 - \overrightarrow{b_1 b_{i_1}\cdots b_{i_k}}^2)' \\
&= \alpha_1(t) \sum_{2\le i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)((\overrightarrow{b_1 b_{i_1}\cdots b_{i_k}} - \overrightarrow{b_1 a_1}\wedge \overrightarrow{b_{i_1}\cdots b_{i_k}})^2 - \overrightarrow{b_1 b_{i_1}\cdots b_{i_k}}^2)' \\
&= \alpha_1(t) \sum_{2\le i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)((\overrightarrow{b_1 a_1}^2)(\overrightarrow{b_{i_1}\cdots b_{i_k}})^2)'.
\end{aligned}$$

In above, the first "=" comes from Lemma 2.7; the last "=" is because $\overrightarrow{b_1 a_1}\cdot\overrightarrow{b_i b_j}=0$ for all i and j.

If $A_1(t)=B_1(t)$, then the above formula $=0$.

In the following, we suppose $A_1(t)\neq B_1(t)$. In terms of the formal power series at $t=0$, if two functions have the same leading term lt^m, then we will use the symbol "$\approx$".

$(\overrightarrow{b_1 a_1}^2)'$ dominates $\overrightarrow{b_1 a_1}^2$ in terms of formal power series at $t=0$. When $c_{k-1}\neq 0$, we have

$$\begin{aligned}
&\alpha_1(t) \sum_{2\le i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)((\overrightarrow{b_1 a_1}^2)(\overrightarrow{b_{i_1}\cdots b_{i_k}})^2)' \\
&\approx \alpha_1(t) \sum_{2\le i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)\, \overrightarrow{b_{i_1}\cdots b_{i_k}}^2(\overrightarrow{b_1 a_1}^2)' \\
&= \alpha_1(t) \sum_{i_1<\cdots<i_k} \alpha_{i_1}(t)\cdots\alpha_{i_k}(t)\, \overrightarrow{b_{i_1}\cdots b_{i_k}}^2(\overrightarrow{b_1 a_1}^2)' \\
&\quad - \alpha_1^2(t) \sum_{2\le i_2<\cdots<i_k} \alpha_{i_2}(t)\cdots\alpha_{i_k}(t)\, \overrightarrow{b_1 b_{i_2}\cdots b_{i_k}}^2(\overrightarrow{b_1 a_1}^2)'
\end{aligned}$$

by Lemma 2.4

$$\begin{aligned}
&= -\alpha_1^2(t) \sum_{i_2<\cdots<i_k} \alpha_{i_2}(t)\cdots\alpha_{i_k}(t)\, \overrightarrow{b_1 b_{i_2}\cdots b_{i_k}}^2(\overrightarrow{b_1 a_1}^2)' \\
&= -\alpha_1^2(t)c_{k-1}(t)(\overrightarrow{b_1 a_1}^2)'.
\end{aligned}$$

Since $\mathbf{A}(t)$ is real analytic over t, so $(\overrightarrow{b_1 a_1}^2)' > 0$ for small $t > 0$. So when $c_{k-1} \neq 0$, c_{k-1} and

$$\sum_{i_1 < \cdots < i_{k+1}} \alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t) (\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)'$$

have the opposite signs for small $t > 0$. □

Theorem 2.6 *Suppose $\mathbf{A}(t)$ is a continuous motion in R^{n+1} that satisfies the k-volume restriction. Also suppose $\mathbf{A}(t)$ is real analytic over t.*

(1) If $c_{k-1} > 0$, then for $G_{n,k}$, points $A_1(t), \ldots, A_{n+2}(t)$ will keep staying in a common n-dim hyperplane in R^{n+1} for small $t \geq 0$.

(2) If $c_{k-1} < 0$, then for $F_{n,k}$, points $A_1(t), \ldots, A_{n+2}(t)$ will keep staying in a common n-dim hyperplane in R^{n+1} for small $t \geq 0$.

Proof. Still let $B_1(t)$ be $A_1(t)$'s orthogonal projection onto the hyperplane that contains points $A_2(t), \ldots, A_{n+2}(t)$.

(1) Suppose $c_{k-1} > 0$. If points $A_1(t), \ldots, A_{n+2}(t)$ do not keep staying in a common n-dim hyperplane in R^{n+1} for small $t \leq 0$, then $A_1(t) \neq B_1(t)$. By Lemma 2.8, we have

$$\sum_{i_1 < \cdots < i_{k+1}} \alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t) (\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' < 0.$$

However, for $G_{n,k}$, we have

$$\alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t) (\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' \geq 0$$

for all the k-faces $A_{i_1} \cdots A_{i_{k+1}}$, which is a contradiction.

(2) The same as (1).

□

Remark. We believe that the above theorem can be proved without the restriction "$\mathbf{A}(t)$ is real analytic over t".

Theorem 2.7 *Suppose $\mathbf{A}(t)$ is a continuous motion in R^{n+1} that satisfies the k-volume restrictions. Also suppose $\mathbf{A}(t)$ is real analytic over t.*

(1) If $c_{k-1} > 0$, then $G_{n,k}$ is k-unyielding in R^{n+1}.

(2) If $c_{k-1} < 0$, then $F_{n,k}$ is k-unyielding in R^{n+1}.

Proof.

(1) Suppose $c_{k-1} > 0$. For $G_{n,k}$, we have

$$\alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' \geq 0$$

for all the k-faces $A_{i_1} \cdots A_{i_{k+1}}$. By Lemma 2.8, we have

$$\sum_{i_1 < \cdots < i_{k+1}} \alpha_{i_1}(t) \cdots \alpha_{i_{k+1}}(t)(\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' \leq 0,$$

which implies that

$$(\overrightarrow{a_{i_1} \cdots a_{i_{k+1}}}^2)' = 0.$$

So $G_{n,k}$ is k-unyielding in R^{n+1}.

(2) The same as (1).

□

Remark. We proved "$G_{n,1}$ is rigid in R^{n+1}" before without the restriction " $\mathbf{A}(t)$ is real analytic over t". We believe that the above theorem can also be proved without the restriction "$\mathbf{A}(t)$ is real analytic over t".

As $c_0 = 1$ is bigger than 0, the above theorem gives another explanation that why $G_{n,1}$ is always rigid in R^{n+1}. If $c_{k-1} \neq 0$, based on the sign of c_{k-1}, we can tell which one of $G_{n,k}$ and $F_{n,k}$ is k-unyielding in R^{n+1}. When $k \geq 2$, it is possible that $c_{k-1} = 0$. If $c_{k-1} = 0$, then we are interested in knowing what the framework looks like. We will give detailed discussion for the case $k = 2$ in the next section.

2.5 2-tensegrity Frameworks $G_{n,2}$ and $F_{n,2}$

In the previous section, we proved that the sign of c_1 determines which one of $G_{n,2}$ and $F_{n,2}$ is 2-unyielding. Then a natural question to ask is: when does $c_1 = 0$ happen? The answer is amazingly simple: $c_1 = 0$ if and only if points $A_1, \ldots, A_{n+2}$ lie on a common sphere in R^n. Consequently, if we use S_i^{n-1} to denote the sphere in R^n that contains points $A_1, \ldots, \hat{A}_i, \ldots, A_{n+2}$, then $c_1 > 0$ if and only if A_1 is inside the sphere S_1^{n-1}; and $c_1 < 0$ if and only if A_1 is outside the sphere S_1^{n-1}.

Proposition 2.3 *$c_1 = 0$ if and only if points $A_1, \ldots, A_{n+2}$ lie on a common sphere in R^n.*

Proof. We define a transformation f in R^n, such that for any point $P \neq A_1$, $f(P)$ is a point that satisfies

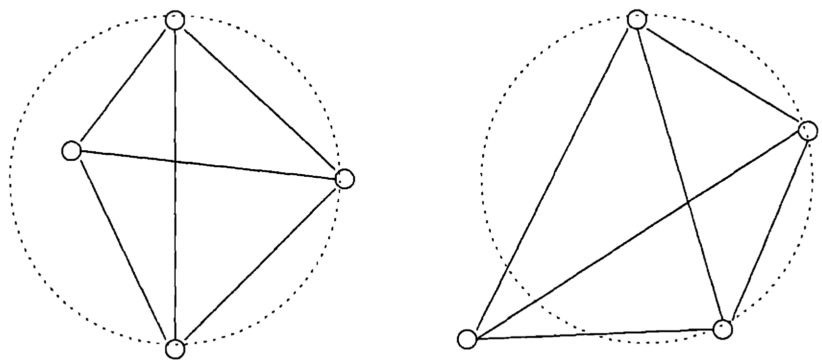
$$\overrightarrow{A_1 f(P)} = \overrightarrow{A_1 P}/\overrightarrow{A_1 P}^2.$$

Since $\sum_{i=2}^{n+2} \alpha_i \overrightarrow{A_1 A_i} = 0$, so we have

$$\sum_{i=2}^{n+2} (\alpha_i \overrightarrow{A_1 A_i}^2) \overrightarrow{A_1 f(A_i)} = 0.$$

A basic property in *inversion geometry* tells us that(proof skipped): A_1 lies on S_1^{n-1} if and only if $f(S_1^{n-1})$ is a $(n-1)$-dim hyperplane. Since $f(S_1^{n-1})$ is a $(n-1)$-dim hyperplane if and only if $\sum_{i=2}^{n+2} \alpha_i \overrightarrow{A_1 A_i}^2 = 0$(which is also equivalent to $c_1 = 0$), so A_1 lies on S_1^{n-1} if and only if $c_1 = 0$. □

To show some geometric feature of $G_{n,2}$ and $F_{n,2}$, we use $n = 2$ as example. Suppose that A_1, A_2, A_3 and A_4 are in such a position that the segments A_1A_3 and A_2A_4 share and inner point. Combinatorially, it is very hard for us to tell the difference between $G_{2,2}$ and $F_{2,2}$, so how can we determine that one of them is 2-unyielding in R^3 while the other is not? In Figure 2.7 (a) and (b), A_1 is the point that is not on the circle. If $\angle A_1 + \angle A_3 > \pi$ (Figure 2.7 (a)) and $\mathbf{A}(t)$ is a real analytic motion over t, then Theorem 2.7 tells us that $c_1 > 0$ and $G_{2,2}$ is 2-unyielding in R^3; and Theorem 2.6 tells us that $A_1(t), \ldots, A_4(t)$ will keep staying in a common 2-dim



(a) (b)

Figure 2.7:

plane in R^3. If $\angle A_1 + \angle A_3 < \pi$ (Figure 2.7 (b)), then $c_1 < 0$ and $F_{2,2}$ is 2-unyielding in R^3.

It will be interesting to know the geometric meaning of $c_k = 0$ when $k \geq 2$.

Chapter 3

Characteristic Polynomial of $n+2$ points in R^n

3.1 Definition of Characteristic Polynomial

In the previous chapter, we have studied some k-unyielding properties of $G_{n,k}$ and $F_{n,k}$ for $1 \leq k \leq n$. In this chapter, we are going to show that there is some relationship between these k-unyielding properties with different k's. Remember that we derived a sequence of constants c_0, c_1, $\dots$, c_n in Lemma 2.6, where $c_0 = 1$ and

$$c_k = \sum_{i_i < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k} \overrightarrow{PA_{i_1} \cdots A_{i_k}} \cdot \overrightarrow{QA_{i_1} \cdots A_{i_k}},$$

which is independent of the choice of points P and Q.

We define a polynomial as

$$f(x) = c_0 x^n - c_1 x^{n-1} + \cdots + (-1)^i c_i x^{n-i} + \cdots + (-1)^n c_n.$$

We call it the *characteristic polynomial* of the points configuration A_1, $\dots$, A_{n+2}. Suppose $(\lambda_1, \dots, \lambda_n)$ are the roots of $f(x)$. Remember that the sign of c_{k-1} plays the main role in deciding whether $G_{n,k}$ and $F_{n,k}$ are k-unyielding in R^{n+1}. We will show that these k-unyielding properties with different k's do have some relationship by showing that $(\lambda_1, \dots, \lambda_n)$ are all real roots.

3.2 Properties of Characteristic Polynomial

When we define the constant c_k in formula (2.1), if we use $(a\alpha_1, \ldots, a\alpha_{n+2})$ to substitute $(\alpha_1, \ldots, \alpha_{n+2})$, then the characteristic polynomial $f(x)$ will become $a^n f(x/a)$ instead. The roots of $a^n f(x/a)$ are then $(a\lambda_1, \ldots, a\lambda_n)$. In this paper, we are only interested in the roots of $f(x)$ up to a non-zero factor.

Theorem 3.1 *(1) All λ_i are non-zero real numbers.*

(2) If $\alpha_1, \ldots, \alpha_{n+2}$ have s negative numbers and $n+2-s$ positive numbers, then all λ_i have $s-1$ negative numbers and $n+1-s$ positive numbers.

We will give the proof after we prove Lemma 3.2.

This is again a combinatorial property, which is only determined by how we separates points $A_1, \ldots, A_{n+2}$ into two groups.

The idea to prove Theorem 3.1 is to prove that $f(x)$ is the characteristic polynomial of a $n \times n$ symmetric matrix $A^T D A$ where D is diagonal. In the following we will show how to construct A and D.

Suppose $\alpha_1, \ldots, \alpha_s < 0$, and $\alpha_{s+1}, \ldots, \alpha_{n+2} > 0$. Since $\sum_{i=1}^{n+2} \alpha_i = 0$, so when $i \geq 2$,

$$\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n+2} > 0.$$

If we let $B_{n+1} = A_{n+2}$, then this property guarantees that line $A_{n+1}B_{n+1}$ intersects with the $(n-1)$-dim plane containing points $A_1, \ldots, A_n$ at a point B_n; $\ldots$; line $A_{k+1}B_{k+1}$ intersects with the $(k-1)$-dim plane containing points $A_1, \ldots, A_k$ at a point B_k; $\ldots$. Finally, we also let $B_1 = A_1$.

Let $\beta_1 = -\alpha_1, \ldots, \beta_k = -\sum_{i=1}^{k} \alpha_i, \ldots, \beta_{n+1} = -\sum_{i=1}^{n+1} \alpha_i = \alpha_{n+2}$.

Proposition 3.1 *(1) $\sum_{i=1}^{k} \alpha_i + \beta_k = 0$, and $\sum_{i=1}^{k} \alpha_i A_i + \beta_k B_k = 0$.*

(2) $-\beta_k + \beta_{k+1} + \alpha_{k+1} = 0$, and $-\beta_k B_k + \beta_{k+1} B_{k+1} + \alpha_{k+1} A_{k+1} = 0$.

Proof.

(1) B_k is the intersection point of $(k-1)$-dim plane $A_1 \cdots A_k$ and $(n+1-k)$-dim plane $A_{k+1} \cdots A_{n+2}$. Since $\sum_{i=1}^{k} \alpha_i A_i = -\sum_{i=k+1}^{n+2} \alpha_i A_i$ and $\beta_k = -\sum_{i=1}^{k} \alpha_i$, so we have

$$\sum_{i=1}^{k} \alpha_i A_i + \beta_k B_k = 0.$$

(2)

$$-\beta_k + \beta_{k+1} + \alpha_{k+1} = \sum_{i=1}^{k} \alpha_i - \sum_{i=1}^{k+1} \alpha_i + \alpha_{k+1} = 0.$$

By (1), we have

$$\begin{aligned} 0 &= (\sum_{i=1}^{k+1} \alpha_i A_i + \beta_{k+1} B_{k+1}) - (\sum_{i=1}^{k} \alpha_i A_i + \beta_k B_k) \\ &= -\beta_k B_k + \beta_{k+1} B_{k+1} + \alpha_{k+1} A_{k+1}. \end{aligned}$$

□

For $1 \le i \le n$, let vector v_i point the same direction as $\overrightarrow{B_i B_{i+1}}$, and the length of v_i be

$$|\alpha_{i+1} \overrightarrow{B_i A_{i+1}}^2 + \beta_{i+1} \overrightarrow{B_i B_{i+1}}^2|^{1/2}.$$

Let $d(i)$ be the sign function of

$$\alpha_{i+1} \overrightarrow{B_i A_{i+1}}^2 + \beta_{i+1} \overrightarrow{B_i B_{i+1}}^2,$$

so

$$v_i^2 d(i) = \alpha_{i+1} \overrightarrow{B_i A_{i+1}}^2 + \beta_{i+1} \overrightarrow{B_i B_{i+1}}^2. \tag{3.1}$$

Definition 3.1 *If we consider v_i as a row vector or a $1 \times n$ matrix, then we define A to be a $n \times n$ matrix with v_i as its i-th row. We define D to be a $n \times n$ diagonal matrix with $d(i)$ as its i-th diagonal element.*

$$A = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad D = \begin{bmatrix} d(1) & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d(n) \end{bmatrix}$$

Let

$$g(x) := \det(xI - AA^T D)$$

be the characteristic polynomial of matrix $AA^T D$.

Lemma 3.1 *(1) The coefficient of x^{n-k} in $g(x)$ is*

$$(-1)^k \sum_{i_1<\cdots<i_k} (v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1) \cdots d(i_k). \tag{3.2}$$

(2) $g(x)$ is the characteristic polynomial of matrix $A^T DA$.

(3) $d(i) = -1$ if $1 \le i \le s-1$; $d(i) = 1$ if $s \le i \le n$.

Proof.

(1) The coefficient of x^{n-k} in $g(x)$ is $(-1)^k$ times the sum of the determinants of all $k \times k$ main diagonal submatrices of $AA^T D$. It is exactly formula (3.2).

(2) $A^T DA = A^{-1}(AA^T D)A$, which is similar to $AA^T D$. So $A^T DA$ and $AA^T D$ have the same characteristic polynomial.

(3) We have $\alpha_1, \ldots, \alpha_s < 0$; $\alpha_{s+1}, \ldots, \alpha_{n+2} > 0$; and $\beta_k = -\sum_{i=1}^k \alpha_i > 0$. When $s \le i \le n$, we have

$$\alpha_{i+1} \overrightarrow{B_i A_{i+1}}^2 + \beta_{i+1} \overrightarrow{B_i B_{i+1}}^2 > 0.$$

So $d(i) = 1$ when $s \le i \le n$. If we use Proposition 3.1 (2), similarly to the way we proved Lemma 2.6, we can get

$$\begin{aligned} \alpha_{i+1} \overrightarrow{B_i A_{i+1}}^2 + \beta_{i+1} \overrightarrow{B_i B_{i+1}}^2 &= \alpha_{i+1} \overrightarrow{PA_{i+1}}^2 + \beta_{i+1} \overrightarrow{PB_{i+1}}^2 - \beta_i \overrightarrow{PB_i}^2 \\ &= \alpha_{i+1} \overrightarrow{B_{i+1} A_{i+1}}^2 - \beta_i \overrightarrow{B_{i+1} B_i}^2. \end{aligned}$$

When $1 \le i \le s-1$, since $\alpha_{i+1} < 0$, we have

$$\alpha_{i+1} \overrightarrow{B_{i+1} A_{i+1}}^2 - \beta_i \overrightarrow{B_{i+1} B_i}^2 < 0.$$

So $d(i) = -1$ when $1 \le i \le s-1$.

□

Lemma 3.2 $f(x) = g(x)$

Proof. By lemma 3.1, we only need to prove

$$c_k = \sum_{i_1 < \cdots < i_k} (v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1) \cdots d(i_k).$$

Embed R^n into a bigger space R^d. For a given l, let ω be a l-vector in R^d. By lemma 2.6,

$$c_k(\omega) := \sum_{i_1 < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k} (\omega \wedge \overrightarrow{PA_{i_1} \cdots A_{i_k}})^2$$

is independent of the choice of point P. Let us prove a stronger result:

$$c_k(\omega) = \sum_{i_1 < \cdots < i_k} (\omega \wedge v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1) \cdots d(i_k). \tag{3.3}$$

We will prove by induction on n.

When $n = 1$, we have

$$c_1(\omega) = \sum_{i=1}^{3} \alpha_i (\omega \wedge \overrightarrow{A_1 A_i})^2 = \alpha_2 (\omega \wedge \overrightarrow{A_1 A_2})^2 + \beta_2 (\omega \wedge \overrightarrow{A_1 B_2})^2 = (\omega \wedge v_1)^2 d(1).$$

Suppose when $n \leq m$, formula (3.3) is true. Then when $n = m + 1$, we have

$$\begin{aligned} c_k(\omega) = & \sum_{i_1 < \cdots < i_k \leq m+3} \alpha_{i_1} \cdots \alpha_{i_k} (\omega \wedge \overrightarrow{B_{m+1} A_{i_1} \cdots A_{i_k}})^2 \\ = & \sum_{i_1 < \cdots < i_k \leq m+1} \alpha_{i_1} \cdots \alpha_{i_k} (\omega \wedge \overrightarrow{B_{m+1} A_{i_1} \cdots A_{i_k}})^2 \\ & + \sum_{i_1 < \cdots < i_{k-1} \leq m+1} \alpha_{i_1} \cdots \alpha_{i_{k-1}} \alpha_{m+2} (\omega \wedge \overrightarrow{B_{m+1} A_{i_1} \cdots A_{i_{k-1}} A_{m+2}})^2 \\ & + \sum_{i_1 < \cdots < i_{k-1} \leq m+1} \alpha_{i_1} \cdots \alpha_{i_{k-1}} \alpha_{m+3} (\omega \wedge \overrightarrow{B_{m+1} A_{i_1} \cdots A_{i_{k-1}} A_{m+3}})^2 \end{aligned}$$

by induction

$$\begin{aligned} = & \sum_{i_1 < \cdots < i_k \leq m} (\omega \wedge v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1) \cdots d(i_k) \\ & + \sum_{i_1 < \cdots < i_{k-1} \leq m+1} \alpha_{i_1} \cdots \alpha_{i_{k-1}} \alpha_{m+2} (\omega \wedge \overrightarrow{B_{m+1} A_{m+2}} \wedge \overrightarrow{B_{m+1} A_{i_1} \cdots A_{i_{k-1}}})^2 \\ & + \sum_{i_1 < \cdots < i_{k-1} \leq m+1} \alpha_{i_1} \cdots \alpha_{i_{k-1}} \alpha_{m+3} (\omega \wedge \overrightarrow{B_{m+1} A_{m+3}} \wedge \overrightarrow{B_{m+1} A_{i_1} \cdots A_{i_{k-1}}})^2 \end{aligned}$$

by (3.1)

$$= \sum_{i_1<\cdots<i_k\le m} (\omega \wedge v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1)\cdots d(i_k)$$
$$+ \sum_{i_1<\cdots<i_{k-1}\le m+1} \alpha_{i_1}\cdots\alpha_{i_{k-1}} (\omega \wedge v_{m+1} \wedge \overrightarrow{B_{m+1}A_{i_1}\cdots A_{i_{k-1}}})^2 d(m+1)$$

by induction

$$= \sum_{i_1<\cdots<i_k\le m} (\omega \wedge v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1)\cdots d(i_k)$$
$$+ \sum_{i_1<\cdots<i_{k-1}\le m} (\omega \wedge v_{m+1} \wedge v_{i_1} \wedge \cdots \wedge v_{i_{k-1}})^2 d(i_1)\cdots d(i_{k-1}) d(m+1)$$
$$= \sum_{i_1<\cdots<i_k\le m+1} (\omega \wedge v_{i_1} \wedge \cdots \wedge v_{i_k})^2 d(i_1)\cdots d(i_k).$$

So formula (3.3) is also true for $n = m + 1$.

□

Proof of Theorem 3.1. By Lemma 3.1 and Lemma 3.2, $f(x)$ is the characteristic polynomial of matrix $A^T DA$, so all λ_i are real. Easy to see that A is non-singular. Since D has $s-1$ negative and $n+1-s$ positive items on its diagonal(by Lemma 3.1), so $f(x)$ has $s-1$ negative roots and $n+1-s$ positive roots. □

Theorem 3.2 *The necessary and sufficient condition for the characteristic polynomial $f(x)$ to have n-repeated roots is: $\overrightarrow{A_{i_1}A_{i_2}} \cdot \overrightarrow{A_{i_3}A_{i_4}} = 0$ for all distinct numbers i_1, i_2, i_3, i_4. It is also equivalent to say that each A_i is the* orthocenter *of the n-simplex with the other $n+1$ points as vertices.*

Proof.

(1) If $\lambda_1 = \cdots = \lambda_n$, then $d(1), \ldots, d(n)$ have the same sign, so $D = I$ or $-I$. Since $f(x) = g(x)$ (Lemma 3.2) is the characteristic polynomial of $AA^T D$ (Definition 3.1), so AA^T is a multiple of I. So $v_i \cdot v_j = 0$ for $i \neq j$. Specifically we have $v_1 \cdot v_i = 0$ for $i \neq 1$. This implies that line A_1A_2 is perpendicular to $(n-1)$-dim plane $A_3 \cdots A_{n+2}$. So $\overrightarrow{A_1A_2} \cdot \overrightarrow{A_iA_j} = 0$ for $3 \le i < j$. By symmetry, we have $\overrightarrow{A_{i_1}A_{i_2}} \cdot \overrightarrow{A_{i_3}A_{i_4}} = 0$ for all distinct numbers i_1, i_2, i_3, i_4.

(2) Suppose $\overrightarrow{A_{i_1}A_{i_2}} \cdot \overrightarrow{A_{i_3}A_{i_4}} = 0$ for all distinct numbers i_1, i_2, i_3, i_4. It is easy to see that $v_i \cdot v_j = 0$ for $i \neq j$. We will prove $v_1^2 = \cdots = v_n^2$, and $d(1) = \cdots = d(n)$.

Consider these 5 points B_k, B_{k+1}, B_{k+2}, A_{k+1}, A_{k+2}, where B_{k+1} is the intersection of line B_kA_{k+1} and line $B_{k+2}A_{k+2}$. Easy to prove that B_{k+2} is the orthocenter of triangle $B_kA_{k+1}A_{k+2}$. Then we have $|B_{k+1}B_k||B_{k+1}A_{k+1}| = |B_{k+1}B_{k+2}||B_{k+1}A_{k+2}|$. Apply Proposition 3.1 and formula 3.1, we can get $v_k^2 d(k) = v_{k+1}^2 d(k+1)$. So $v_1^2 = \cdots = v_n^2$ and $d(1) = \cdots = d(n)$. Then AA^TD is a multiple of I, which means that $\lambda_1 = \cdots = \lambda_n$.

□

Chapter 4

Rigidity and Volume Preserving Deformation in S^n and H^n

4.1 Elementary Geometry in S^n and H^n

In this chapter, we study other kind of k-tensegrity frameworks, which are in a n-dimensional spherical space S^n or a hyperbolic space H^n. For some basic properties related to S^n and H^n, we will just list them as facts without giving proofs. Some basic notions can be found in Fenchel's book [6].

S^n is defined as a sphere with radius 1 in an Euclidean space R^{n+1}. If we use $(x_1, \ldots, x_{n+1})$ as the coordinates of R^{n+1}, then the equation that S^n satisfies is

$$x_1^2 + \cdots + x_{n+1}^2 = 1.$$

We use S_+^n to denote the semisphere with $x_1 > 0$. We will use the standard *Riemannian metric* on S^n throughout this paper, and we know that S^n has *constant sectional curvature* 1. Besides, S_+^n is *geodesic convex*, which means that for any 2 points P and Q in S_+^n, their has a unique *geodesic* connecting P and Q. We use $\widetilde{PQ}$ to denote the *geodesic distance* between P and Q.

The definition of hyperbolic space H^n is related to non-Euclidean geometry. We define $R^{n,1}$(not R^{n+1}) as a $(n+1)$-dimensional linear space. If we use $(x_1, \ldots, x_{n+1})$

as the coordinates of $R^{n,1}$, then the equation that H^n satisfies is

$$-x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -1, \quad \text{and} \quad x_1 > 0.$$

The restriction $x_1 > 0$ makes H^n to be simply connected.

Similar to the way that we defined inner product "·" in R^n, we define a bilinear product "·" in $R^{n,1}$ as

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = -x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1},$$

where A and B are 2 points in $R^{n,1}$ with coordinates $(x_1, \ldots, x_{n+1})$ and $(y_1, \ldots, y_{n+1})$ separately. Notice that the only difference is the "−" sign at the 1st coordinate part. If A is a point in H^n, then $\overrightarrow{OA}^2 = \overrightarrow{OA} \cdot \overrightarrow{OA} = -1$.

Remark. When $\overrightarrow{OB}^2 = 0$, $\overrightarrow{OA} \cdot \overrightarrow{OB}$ does not have to be 0. This is very different from the R^n case.

This operation "·" gives a quadratic form in $R^{n,1}$, which is not positive definite. However, the restriction of this quadratic form on the tangent space at any point in H^n is positive definite, then it induces a Riemannian metric on H^n. Besides, H^n has constant sectional curvature -1 and is geodesic convex. We will use $\widetilde{PQ}$ to denote the geodesic distance between any 2 points P and Q in H^n.

R^n is of constant sectional curvature 0. S^n_+ and H^n are also constant sectional curvature spaces, and we can apply the notion "simplex" from R^n to S^n_+ and H^n as well. We first define the notion "simplex" in H^n, and it will follow similarly in S^n_+. Suppose $P_1, \ldots, P_{k+1}$ are $k+1$ points in H^n, then all the linear combination $\sum_{i=1}^{k+1} \gamma_i P_i$ with $\gamma_i \geq 0$ span a *cone* in $R^{n,1}$. The intersection of this cone with H^n is called a *hyperbolic k-simplex*, or *H^k-simplex* for convenience. The vertices of this hyperbolic k-simplex are $P_1, \ldots, P_{k+1}$. One important fact is: when $k = 1$, the hyperbolic 1-simplex is exactly the geodesic that connects points P_1 and P_2. Another fact is: the hyperbolic k-simplex stays in the k-dimensional *totally geodesic submanifold* in H^n that contains points $P_1, \ldots, P_{k+1}$. In the S^n_+ case, it is called *spherical k-simplex*, or *S^k_+-simplex* for convenience. We use the notation

$$\tilde{V}_k(P_1, \ldots, P_{k+1})$$

to denote the k-dim volume of the hyperbolic(or spherical) k-simplex under the Riemannian metric on H^n(or S^n_+). Notice that $\tilde{V}_1(P_1, P_2) = \widetilde{P_1P_2}$.

Similar to the way that we defined exterior algebra in Euclidean space, we now give a very brief introduction about exterior algebra in $R^{n,1}$. Remember in the R^n case, based on the inner product "$\cdot$" defined in R^n, the definition of the inner product "$\cdot$" in $\Lambda^k(R^n)$ is induced and well defined. In the $R^{n,1}$ case, a bilinear product "$\cdot$" in $\Lambda^k(R^{n,1})$ is also induced from the definition of the bilinear product "$\cdot$" in $R^{n,1}$, and has the following property.

Lemma 4.1 *If $r_1, \ldots, r_k, s_1, \ldots, s_k$ are $2k$ elements in $R^{n,1}$, then we have*

$$(r_1 \wedge \cdots \wedge r_k) \cdot (s_1 \wedge \cdots \wedge s_k) = \det(r_i \cdot s_j)_{1 \le i,j \le k},$$

where the "$\cdot$" in right side is the bilinear product defined in $R^{n,1}$.

Proof. The same as the R^n case. □

If $P_1, \ldots, P_{k+1}$ are $k+1$ points in $R^{n,1}$, we define $\overrightarrow{P_1 \cdots P_{k+1}}$ in $\Lambda^k(R^{n,1})$ to be

$$\overrightarrow{P_1 \cdots P_{k+1}} = \overrightarrow{P_1P_2} \wedge \overrightarrow{P_1P_3} \wedge \cdots \wedge \overrightarrow{P_1P_{k+1}}.$$

Lemma 4.2 *If $P_1, \ldots, P_{k+1}$ are $k+1$ points in H^n, then*

$$\overrightarrow{OP_1 \cdots P_{k+1}}^2 \le 0,$$

and "$= 0$" happens if and only if $\overrightarrow{OP_1}, \ldots, \overrightarrow{OP_{k+1}}$ are linearly dependent.

Proof. First suppose $k = n$. We define A to be a $(n+1) \times (n+1)$ matrix with P_i's coordinates as its i-th row. We also define D to be a $(n+1) \times (n+1)$ diagonal matrix with -1 as its 1st diagonal element, and 1 as its rest diagonal elements.

$$A = \begin{bmatrix} \overrightarrow{OP_1} \\ \vdots \\ \overrightarrow{OP_{n+1}} \end{bmatrix}, \quad D = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Then we have

$$\begin{aligned}\overrightarrow{OP_1\cdots P_{n+1}}^2 &= (\overrightarrow{OP_1}\wedge\cdots\wedge\overrightarrow{OP_{n+1}})^2\\ &= \det(ADA^T)\\ &= -(\det(A))^2\\ &\leq 0,\end{aligned}$$

and $\det(A)=0$ if and only if the coordinates of $P_1, \ldots, P_{n+1}$ are linearly dependent.

If $k<n$, $P_1, \ldots, P_{k+1}$ stay in a smaller hyperbolic space H^k which is embedded in H^n. The problem is then reduced to H^k, so we have

$$\overrightarrow{OP_1\cdots P_{k+1}}^2 \leq 0,$$

and "$=0$" happens if and only if $\overrightarrow{OP_1}, \ldots, \overrightarrow{OP_{k+1}}$ are linearly dependent. □

We define

$$|\overrightarrow{OP_1\cdots P_{k+1}}| = |\overrightarrow{OP_1\cdots P_{k+1}}^2|^{1/2}.$$

Proposition 4.1 *If A and B are 2 different points in H^n, then $\overrightarrow{AB}^2>0$, and*

$$\overrightarrow{AB}^2 = 2\cosh\widetilde{AB}-2, \quad A\cdot B = -\cosh\widetilde{AB};$$

if A and B are in S^n_+, then

$$\overrightarrow{AB}^2 = 2-2\cos\widetilde{AB}, \quad A\cdot B = \cos\widetilde{AB}.$$

Proof. Skipped. □

Proposition 4.2 *If A and B are 2 points in S^n_+ or H^n, then we have*

$$\frac{d(\overrightarrow{AB}^2)}{d\widetilde{AB}} = 2|\overrightarrow{OAB}|$$

in both cases.

Proof. For the S^n_+ case, by using Proposition 4.1, we have

$$\frac{d(\overrightarrow{AB}^2)}{d\widetilde{AB}} = 2\sin\widetilde{AB} = 2|\overrightarrow{OAB}|.$$

For the H^n case, by using Proposition 4.1 again, we have

$$\frac{d(\overrightarrow{AB}^2)}{d\widetilde{AB}} = 2\sinh\widetilde{AB}$$

and

$$\begin{aligned}\overrightarrow{OAB}^2 &= (A\wedge B)^2\\ &= \det(\begin{bmatrix} -1 & -\cosh\widetilde{AB} \\ -\cosh\widetilde{AB} & -1\end{bmatrix})\\ &= -\sinh^2\widetilde{AB},\end{aligned}$$

so

$$\frac{d(\overrightarrow{AB}^2)}{d\widetilde{AB}} = 2|\overrightarrow{OAB}|.$$

□

The above proposition tells us that, when $\widetilde{AB}$ increases(or decreases), $\overrightarrow{AB}^2$ increases(or decreases).

Proposition 4.3 *Suppose $P_1, \ldots, P_{k+1}$ are $k+1$ points in S^n_+(or H^n), then both $\tilde{V}_k(P_1, \ldots, P_{k+1})$ and $\overrightarrow{OP_1\cdots P_{k+1}}^2$ are functions with all $\widetilde{P_iP_j}$ as the variables; they are also functions with all $\overrightarrow{P_iP_j}^2$ as the variables.*

Proof. Skipped. □

We will write the partial derivative of $\tilde{V}_k(P_1, \ldots, P_{k+1})$ over $\widetilde{P_iP_j}$ and $\overrightarrow{P_iP_j}^2$ as $\partial_{\widetilde{P_iP_j}}\tilde{V}_k(P_1, \ldots, P_{k+1})$ and $\partial_{\overrightarrow{P_iP_j}^2}\tilde{V}_k(P_1, \ldots, P_{k+1})$ separately.

4.2 Definition of k-tensegrity Frameworks in S^n_+ and H^n

We already defined spherical k-simplex and hyperbolic k-simplex, then some notions like "k-tensegrity framework" can also be moved over from R^n to S^n_+ and H^n.

A S_+^k(or H^k)-simplex is called a k-cable if its k-volume can not increase; it is called a k-strut if its k-volume can not decrease; it is called a k-bar if its k-volume can not change at all. We call a framework in S_+^n or H^n to be a *k-tensegrity framework* if some of its k-faces(S_+^k or H^k-simplices) are labeled as either k-cables, k-struts, or k-bars, and the other k-faces are just not labeled as anything. We can consider k-cable, k-strut, and k-bar as volume restrictions imposed on frameworks. For a framework in S_+^d(or H^d), if all k-volumes of its k-faces are preserved for any sufficiently small continuous motion under the volume restriction, then we say that it is k-unyielding in S_+^d(or H^d).

We will also call 1-cable, 1-strut, and 1-bar as cable, strut, and bar respectively, and call 1-tensegrity framework as tensegrity framework. Particularly, for a framework in S_+^d(or H^d), we say that it is rigid in S_+^d(or H^d), if the geodesic distance between each pair of vertices can not be changed for any continuous motion under the volume restriction.

Given 2 k-tensegrity frameworks G_1 and G_2 in S_+^d(or H^d) with the same constructions of k-cables, k-struts, and k-bars, we say that G_1 dominates G_2, if the k-volume of each k-cable of G_2 is no bigger than the k-volume of the corresponding k-cable of G_1; the k-volume of each k-strut of G_2 is no smaller than the k-volume of the corresponding k-strut of G_1; and the corresponding k-bars of G_1 and G_2 have the same k-volume.

We say 2 frameworks G_1 and G_2 in S_+^d(or H^d) are congruent, if the geodesic distance between any pair of points in G_1 is the same as the geodesic distance between the corresponding pair of points in G_2. For a framework G_1 in S_+^d(or H^d), we say that it is globally rigid in S_+^d(or H^d), if when G_1 dominates another framework G_2 in S_+^d(or H^d), then G_1 and G_2 are congruent. Global rigidity is a very strong notion imposed on a framework, and global rigidity automatically implies that a framework is also rigid in R^d.

4.3 Construction of k-tensegrity Frameworks in S^n_+ and H^n

Given $n+2$ points A_1, A_2, $\ldots$, A_{n+2} in S^n_+(or H^n) in general position, which means that every $n+1$ points are not in a lower dimensional space S^{n-1}_+(or H^{n-1}). We can treat these $n+2$ points as the vertices of a degenerate S^{n+1}_+(or H^{n+1})-simplex. Since A_1, A_2, $\ldots$, A_{n+2} are in general position, there uniquely exists a sequence of non-zero coefficients α_1, $\ldots$, α_{n+2} (up to a non-zero factor c), such that $\sum_{i=1}^{n+2} \alpha_i \overrightarrow{OA_i} = 0$.

Proposition 4.4

$$|\overrightarrow{OA_1 \cdots \hat{A}_i \cdots A_{n+2}}|/|\alpha_i|$$

is independent of i for $1 \leq i \leq n+2$.

Proof. Skipped. □

For convenience, we choose α_i to satisfy

$$|\alpha_i| = |\overrightarrow{OA_1 \cdots \hat{A}_i \cdots A_{n+2}}|.$$

Besides, we also choose α_1 to be negative, then all the signs of α_i are determined afterwords. We separate these $n+2$ points into 2 sets X_1 and X_2 by the following rule: A_i is in X_1 if α_i is negative; A_i is in X_2 if α_i is positive. Since α_1 is negative, so A_1 is in set X_1. Given k with $1 \leq k \leq n$, we separate all the S^k_+(or H^k)-simplices $A_{i_1} \cdots A_{i_{k+1}}$ into 2 sets $Y_{k,1}$ and $Y_{k,2}$ by the following rule: a S^k_+(or H^k)-simplex $A_{i_1} \cdots A_{i_{k+1}}$ is in $Y_{k,1}$ if it has odd number of vertices in X_1; it is in $Y_{k,2}$ if it has even number of vertices in X_1. Based on the above separation of k-faces, we construct 2 different k-tensegrity frameworks below.

Framework $G_{n,k}$: let all the k-faces in group $Y_{k,1}$ be k-cables, and let all the k-faces in group $Y_{k,2}$ be k-struts.
Framework $F_{n,k}$: let all the k-faces in group $Y_{k,1}$ be k-struts, and let all the k-faces in group $Y_{k,2}$ be k-cables.

We see that $F_{n,k}$ is constructed by switching the role of k-cable and k-strut in $G_{n,k}$. We use the notation $\mathbf{A}(t) = (A_1(t), \ldots, A_{n+2}(t))$ to denote the continuous motion with $t \geq 0$ and $A_i(0) = A_i$. The main purpose of this chapter is to see when the above 2 frameworks are k-unyielding in S^n_+(or H^n) and S^{n+1}_+(or H^{n+1}).

Before we start the detailed discussion of k-unyielding properties for general k, we first give a partial result for the specific case $k = n$.

Theorem 4.1 *$G_{n,n}$ and $F_{n,n}$ are n-unyielding in S^n_+(or H^n).*

Proof. For $G_{n,n}$(or $F_{n,n}$), the sum of the n-volume of its n-cables equals the sum of the n-volume of its n-struts. It is still true during the continuous motion when t is small, so non of the n-volume changes. It means that $G_{n,n}$ and $F_{n,n}$ are n-unyielding in S^n_+(or H^n). □

4.4 Main Theorems and Conjectures

Most results in this section have analogues in the R^n case.

Theorem 4.2 *(1) $G_{n,1}$ is globally rigid in S^{n+1}.*

(2) $F_{n,1}$ is rigid in S^n_+.

Proof. It is easy to see that $\widetilde{B_iB_j} \leq \widetilde{A_iA_j}$ is equivalent to $(B_i - B_j)^2 \leq (A_i - A_j)^2$.

(1) Suppose $B_1, \ldots, B_{n+2}$ are $n+2$ points in S^{n+1} that satisfy the cable-strut restriction set by $G_{n,1}$. Then we have

$$\begin{aligned}
0 &\leq (\alpha_1 B_1 + \cdots + \alpha_{n+2} B_{n+2})^2 \\
&= \sum_i \alpha_i^2 B_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j B_i \cdot B_j \\
&= \sum_j \alpha_j \sum_i \alpha_i B_i^2 - \sum_{i<j} \alpha_i \alpha_j (B_i - B_j)^2
\end{aligned}$$

since $B_i^2 = A_i^2 = 1$ and $\alpha_i\alpha_j(B_i - B_j)^2 \geq \alpha_i\alpha_j(A_i - A_j)^2$

$$\begin{aligned} &\leq \sum_j \alpha_j \sum_i \alpha_i A_i^2 - \sum_{i<j} \alpha_i\alpha_j(A_i - A_j)^2 \\ &= (\alpha_1 A_1 + \cdots + \alpha_{n+2}A_{n+2})^2 \\ &= 0. \end{aligned}$$

Then $\alpha_i\alpha_j(B_i - B_j)^2 = \alpha_i\alpha_j(A_i - A_j)^2$ must hold, which means that $G_{n,1}$ is globally rigid in S^{n+1}.

(2) Suppose $B_1, \ldots, B_{n+2}$ are $n+2$ points in S_+^n that satisfy the cable-strut restriction set by $F_{n,1}$. We can find a sequence of coefficients $\beta_1, \ldots, \beta_{n+2}$ that satisfy $\sum \beta_i B_i = 0$. When the motion is small enough, β_i also has the same sign as α_i. Following from almost the same computation in (1), we have

$$\begin{aligned} 0 &= (\beta_1 B_1 + \cdots + \beta_{n+2}B_{n+2})^2 \\ &= \sum_j \beta_j \sum_i \beta_i B_i^2 - \sum_{i<j} \beta_i\beta_j(B_i - B_j)^2 \\ &\geq \sum_j \beta_j \sum_i \beta_i A_i^2 - \sum_{i<j} \beta_i\beta_j(A_i - A_j)^2 \\ &= (\beta_1 A_1 + \cdots + \beta_{n+2}A_{n+2})^2 \\ &\geq 0. \end{aligned}$$

Then $\beta_i\beta_j(B_i - B_j)^2 = \beta_i\beta_j(A_i - A_j)^2$ must hold, which means that $F_{n,1}$ is rigid in S_+^n.

□

Figure 4.1 shows a example of $G_{2,1}$ in S_+^2. Surprisingly, the above proof can not be applied to prove the H^n case. The main reason is: for a point B in $R^{n,1}$, it is possible that $\overrightarrow{OB}^2 < 0$. We use a different method the prove the H^n case in the following.

Theorem 4.3 *Suppose* $\mathbf{A}(t)$ *is a continuous motion that satisfies the cable-strut restriction.*

(1) If $\mathbf{A}(t)$ *is also real analytic over* t*, then* $G_{n,1}$ *is rigid in* H^{n+1}.

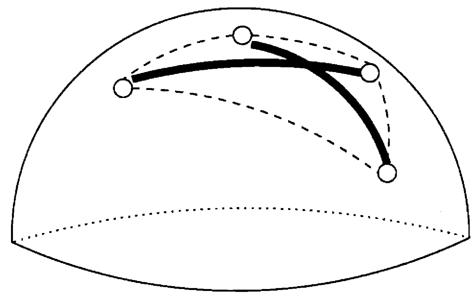


Figure 4.1:

(2) If $\mathbf{A}(t)$ *is also* C^1 *over* t*, then* $F_{n,1}$ *is rigid in* H^n*.*

Proof. In order to make a formula not too wide in a line, we will use a_i to replace $A_i(t)$.

(1) For small $t \geq 0$, we have

$$0 \geq (\overrightarrow{Oa_1 \cdots a_{n+2}}^2)'$$

by Corollary 2.1

$$= \sum_{i<j} \overrightarrow{Oa_ia_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_{n+2}} \cdot \overrightarrow{Oa_ja_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_{n+2}} (\overrightarrow{a_ia_j}^2)'$$

$$= \sum_{i<j} \alpha_i(t)\alpha_j(t)(\overrightarrow{a_ia_j}^2)'$$

$$\geq 0.$$

In above, the 1st "$\geq$" comes from Lemma 4.2. So $(\overrightarrow{a_ia_j}^2)' = 0$ must hold for small $t \geq 0$, which means that $G_{n,1}$ is rigid in H^{n+1}.

(2) The computation is very similar to (1). We just need to change the 1st "$\geq$" to "$=$", and change the last "$\geq$" to "$\leq$". So $(\overrightarrow{a_ia_j}^2)' = 0$ holds for small $t \geq 0$, which means that $F_{n,1}$ is rigid in H^n.

□

Remark. We believe that the above theorem can be proved without the restriction "$\mathbf{A}(t)$ is also real analytic(or C^1) over t".

Since $G_{n,1}$ is globally rigid in R^{n+1} and S_+^{n+1}, we have the following conjecture.

Conjecture 4.1 *$G_{n,1}$ is globally rigid in H^{n+1}.*

In order to see that whether $G_{n,k}$ and $F_{n,k}$ are k-unyielding in S^n_+(or H^n) or S^{n+1}_+(or H^{n+1}), we give the following main conjecture.

Conjecture 4.2 *Embed S^n_+(or H^n) into a bigger space S^d_+(or H^d) with $d > n$, and choose any 2 points P and Q in S^d_+(or H^d). Define*

$$d(P, Q; A_{i_1}, \ldots, A_{i_k}) = 2(k+1)!\,|\overrightarrow{OPQA_{i_1}\cdots A_{i_k}}|\,\partial_{\overrightarrow{PQ}^2}\tilde{V}_{k+1}(P, Q, A_{i_1}, \ldots, A_{i_k}).$$

We also define $c_0 = d(P, Q)$, and

$$c_k = \sum_{i_1<\cdots<i_k} \alpha_{i_1}\cdots\alpha_{i_k} d(P, Q; A_{i_1}, \ldots, A_{i_k})$$

when $1 \leq k \leq n$, then c_k is independent of the choice of points P and Q.

Remark. This is the analogue of Lemma 2.1. Conjecture 4.2 is true for $k = 0$ and $c_0 = 1$, which is proved directly by using Proposition 4.2. Conjecture 4.2 is also true for $k = 1$, and $c_1 = 2\sum_{i=1}^{n+2}\alpha_i$ for the S^n_+ case; $c_1 = -2\sum_{i=1}^{n+2}\alpha_i$ for the H^n case. It will be proved in the next section.

We use the notation $\mathbf{A}(t) = (A_1(t), \ldots, A_{n+2}(t))$ to denote the continuous motion with $t \geq 0$ and $A_i(0) = A_i$. When the continuous motion $\mathbf{A}(t)$ is restricted in S^n_+(or H^n), similar to the way that we get α_i, we can get coefficients $\alpha_1(t),\ldots,\alpha_{n+2}(t)$ such that they satisfy $\sum_{i=1}^{n+2}\alpha_i(t)A_i(t) = 0$. Besides, we can also require $\alpha_i(t)$ to satisfy

$$|\alpha_i(t)| = |\overrightarrow{OA_1(t)\cdots \hat{A_i}(t)\cdots A_{n+2}(t)}|.$$

When t is small enough, $\alpha_i(t)$ has the same sign as $\alpha_i(0) = \alpha_i$.

Lemma 4.3 *Suppose Conjecture 4.2 is true. When $\mathbf{A}(t)$ is restricted in S^n_+(or H^n) and is C^1 over t, then we have*

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)\,|\overrightarrow{OA_{i_1}(t)\cdots A_{i_{k+1}}(t)}|\,(\tilde{V}_k(A_{i_1}(t), \ldots, A_{i_{k+1}}(t)))' = 0.$$

Proof. In order to make a formula not too wide in a line, we use α_i to replace $A_i(t)$ in the following.

When $k = 1$, we have

$$
\begin{aligned}
& 2\sum_{i_1<i_2} \alpha_{i_1}(t)\alpha_{i_2}(t)\,\overrightarrow{Oa_{i_1}a_{i_2}}\,(\tilde{V}_1(a_{i_1}, a_{i_2}))' \\
&= \sum_{i_1<i_2} \alpha_{i_1}(t)\alpha_{i_2}(t)\,(\overrightarrow{a_{i_1}a_{i_2}}^2)' \\
&= \mp \sum_{i_1<i_2} (\overrightarrow{Oa_{i_1}a_1\cdots\hat{a_{i_1}}\cdots\hat{a_{i_2}}\cdots a_{n+2}} \cdot \overrightarrow{Oa_{i_2}a_1\cdots\hat{a_{i_1}}\cdots\hat{a_{i_2}}\cdots a_{n+2}})(\overrightarrow{a_{i_1}a_{i_2}}^2)' \\
&= \mp(\overrightarrow{Oa_1\cdots a_{n+2}}^2)' \\
&= \mp 0' \\
&= 0.
\end{aligned}
$$

In above, the first "=" comes from Proposition 4.2; in the second "=", "−" is for S^n_+ case and "+" is for H^n case; the third "=" comes from Corollary 2.1.

When $k \geq 2$, we have

$$
\begin{aligned}
& 2k! \sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)\,|\overrightarrow{Oa_{i_1}\cdots a_{i_{k+1}}}|\,(\tilde{V}_k(a_{i_1},\ldots,a_{i_{k+1}}))' \\
&= 2k! \sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)|\overrightarrow{Oa_{i_1}\cdots a_{i_{k+1}}}|(\sum_{j<l} \partial_{\overrightarrow{a_{i_j}a_{i_l}}^2}\tilde{V}_k(a_{i_1},\ldots,a_{i_{k+1}})(\overrightarrow{a_{i_j}a_{i_l}}^2)') \\
&= 2k! \sum_{j<l} \alpha_j(t)\alpha_l(t)(\overrightarrow{a_ja_l}^2)'(\sum_{i_1<\cdots<i_{k-1}} \alpha_{i_1}(t)\cdots\alpha_{i_{k-1}}(t)\,|\overrightarrow{Oa_ja_la_{i_1}\cdots a_{i_{k-1}}}| \\
&\quad \partial_{\overrightarrow{a_ja_l}^2}\tilde{V}_k(a_j, a_l, a_{i_1},\ldots,a_{i_{k-1}})) \\
&= c_{k-1}(t)\sum_{j<l} \alpha_j(t)\alpha_l(t)(\overrightarrow{a_ja_l}^2)' \\
&= 0.
\end{aligned}
$$

In above, the third "=" comes from Conjecture 4.2; the last "=" comes from the special case $k = 1$ which we just proved above. □

This property leads to the following main theorem.

Theorem 4.4 *Suppose Conjecture 4.2 is true. If* $\mathbf{A}(t)$ *is a continuous motion that* *satisfies the k-volume restrictions, and $\mathbf{A}(t)$ is also C^1 over t, then both $G_{n,k}$ and $F_{n,k}$ are k-unyielding in S^n_+(or H^n) for each k.*

Proof. For $G_{n,k}$, we have

$$\alpha_{i_1}(t)\cdots\alpha_{i_{k+1}}(t)(\tilde{V}_k(A_{i_1},\ldots,A_{i_{k+1}}(t))'\geq 0.$$

By using Lemma 4.3, we find that

$$(\tilde{V}_k(A_{i_1},\ldots,A_{i_{k+1}}(t))'=0$$

must hold for small t. So $G_{n,k}$ is k-unyielding in S^n_+(or H^n). The same proof works for $F_{n,k}$. □

Remark. We believe that the above theorem can be proved without the restriction "$\mathbf{A}(t)$ is also C^1 over t".

We are now starting to discuss that whether $G_{n,k}$ and $F_{n,k}$ are also k-unyielding in S^{n+1}_+(or H^{n+1}). Suppose $\mathbf{A}(t)$ is a continuous motion in S^{n+1}_+(or H^{n+1}). When t is small, there has a unique S^n(or H^n) that contains points $A_2(t)$, ..., $A_{n+2}(t)$. Let $B_1(t)$ be the point on this S^n(or H^n) such that $\overrightarrow{B_1(t)A_1(t)}^2$ reaches the minimum. Easy to see that, if $\mathbf{A}(t)$ is real analytic over t, then $B_1(t)$ is also real analytic over t, and $B_1(0)=A_1$. For convenience, we define $B_i(t)=A_i(t)$ for $i\geq 2$. For small t, we can find a sequence of coefficients $\alpha_1(t)$, ..., $\alpha_{n+2}(t)$ such that $\sum_{i=1}^{n+2}\alpha_i B_i(t)=0$ and

$$|\alpha_i(t)|=|\overrightarrow{OB_1(t)\cdots\hat{B_i}(t)\cdots B_{n+2}(t)}|.$$

When t is small, $\alpha_i(t)$ and $\alpha_i(0)=\alpha_i$ has the same sign.

Similar to the definition of c_k, we can define $c_0(t)=1$ and

$$c_k(t)=\sum_{i_1<\cdots<i_k}\alpha_{i_1}(t)\cdots\alpha_{i_k}(t)\,d(P,Q;B_{i_1}(t),\ldots,B_{i_k}(t)),$$

which is independent of the choice of points P and Q if Conjecture 4.2 is true. Besides, $c_k(0)=c_k$.

Conjecture 4.3 *Suppose Conjecture 4.2 is true and $\mathbf{A}(t)$ is real analytic over t for small $t\geq 0$ in S^{n+1}_+(or H^{n+1}).*

(1) If $c_{k-1} > 0$, then

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}} |\overrightarrow{OA_{i_1}(t)\cdots A_{i_{k+1}}(t)}| (\tilde{V}_k(A_{i_1}(t),\ldots,A_{i_{k+1}}(t))' \leq 0$$

for small $t \geq 0$, and the equality holds if and only if $B_1(t) = A_1(t)$.

(2) If $c_{k-1} < 0$, then

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}} |\overrightarrow{OA_{i_1}(t)\cdots A_{i_{k+1}}(t)}| (\tilde{V}_k(A_{i_1}(t),\ldots,A_{i_{k+1}}(t))' \geq 0$$

for small $t \geq 0$, and the equality holds if and only if $B_1(t) = A_1(t)$.

Theorem 4.5 *Suppose Conjecture 4.2 and Conjecture 4.3 are true, and $\mathbf{A}(t)$ is a continuous motion in S^{n+1}_+(or H^{n+1}) that satisfies the k-volume restriction. Also suppose $\mathbf{A}(t)$ is real analytic over t for small $t \geq 0$.*

(1) If $c_{k-1} > 0$, then for $G_{n,k}$, points $A_1(t),\ldots,A_{n+2}(t)$ will keep staying in a common S^n_+(or H^n) which is embedded in S^{n+1}_+(or H^{n+1}) for small $t \geq 0$.

(2) If $c_{k-1} < 0$, then for $F_{n,k}$, points $A_1(t),\ldots,A_{n+2}(t)$ will keep staying in a common S^n_+(or H^n) which is embedded in S^{n+1}_+(or H^{n+1}) for small $t \geq 0$.

Proof.

(1) Suppose $c_{k-1} > 0$. If points $A_1(t),\ldots,A_{n+2}(t)$ do not keep staying in a common S^n_+(or H^n) which is embedded in S^{n+1}_+(or H^{n+1}) for small $t \geq 0$, then $A_1(t) \neq B_1(t)$. By Conjecture 4.3, we have

$$\sum_{i_1<\cdots<i_{k+1}} \alpha_{i_1}\cdots\alpha_{i_{k+1}} |\overrightarrow{OA_{i_1}(t)\cdots A_{i_{k+1}}(t)}| (\tilde{V}_k(A_{i_1}(t),\ldots,A_{i_{k+1}}(t))' < 0.$$

However, for $G_{n,k}$, we have

$$\alpha_{i_1}\cdots\alpha_{i_{k+1}}(\tilde{V}_k(A_{i_1}(t),\ldots,A_{i_{k+1}}(t))' \geq 0,$$

which is a contradiction.

(2) The same as (1).

□

Remark. We believe that the above theorem can be proved without the restriction "$\mathbf{A}(t)$ is real analytic over t".

Finally, we have the following main theorem.

Theorem 4.6 *Suppose Conjecture 4.2 and Conjecture 4.3 are true, and* $\mathbf{A}(t)$ *is a continuous motion in* S_+^{n+1}*(or* H^{n+1}*) that satisfies the* k*-volume restrictions. Also suppose* $\mathbf{A}(t)$ *is real analytic over* t *for small* $t \geq 0$*.*

(1) If $c_{k-1} > 0$*, then* $G_{n,k}$ *is* k*-unyielding in* S_+^{n+1}*(or* H^{n+1}*).*

(2) If $c_{k-1} < 0$*, then* $F_{n,k}$ *is* k*-unyielding in* S_+^{n+1}*(or* H^{n+1}*).*

Proof.

(1) Suppose $c_{k-1} > 0$. For $G_{n,k}$, we have

$$\alpha_{i_1} \cdots \alpha_{i_{k+1}} (\tilde{V}_k(A_{i_1}(t), \ldots, A_{i_{k+1}}(t))' \geq 0.$$

By Conjecture 4.3, we have

$$\sum_{i_1 < \cdots < i_{k+1}} \alpha_{i_1} \cdots \alpha_{i_{k+1}} \, |\overrightarrow{OA_{i_1}(t) \cdots A_{i_{k+1}}(t)}| \, (\tilde{V}_k(A_{i_1}(t), \ldots, A_{i_{k+1}}(t))' \leq 0,$$

which implies that

$$(\tilde{V}_k(A_{i_1}(t), \ldots, A_{i_{k+1}}(t))' = 0.$$

So $G_{n,k}$ is k-unyielding in S_+^{n+1}(or H^{n+1}.

(2) The same as (1).

□

Remark. We believe that the above theorem can be proved without the restriction "$\mathbf{A}(t)$ is real analytic over t".

4.5 2-tensegrity Frameworks $G_{n,2}$ and $F_{n,2}$ in S^n_+ and H^n

In this section, we are going to show that Conjecture 4.2 is true for $k = 1$. For some basic properties related to S^n and H^n, we will just list them as facts without giving proofs.

Suppose points A, B, C are in S^n_+. We use a, b, c to replace $\widetilde{BC}$, $\widetilde{CA}$, $\widetilde{AB}$ respectively. We denote α to be the angle of the spherical triangle ABC at point A, and the analogues for β and γ. For the H^n case, a, b, c, α, β, γ are defined similarly.

Proposition 4.5 *For the S^n_+ case, we have $\tilde{V}_2(A, B, C) = \alpha + \beta + \gamma - \pi$,*

$$\cos\alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

and the analogues for $\cos\beta$ and $\cos\gamma$.

Proposition 4.6 *For the H^n case, we have $\tilde{V}_2(A, B, C) = \pi - \alpha - \beta - \gamma$,*

$$\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c},$$

and the analogues for $\cos\beta$ and $\cos\gamma$.

Proposition 4.7 *For the S^n_+ case, we have*

$$d(B, C; A) = \frac{2}{1 + \cos a} \overrightarrow{BA} \cdot \overrightarrow{CA}$$

Proof. Using Proposition 4.5. Detail left to the reader. □

Proposition 4.8 *For the H^n case, we have*

$$d(B, C; A) = \frac{2}{1 + \cosh a} \overrightarrow{BA} \cdot \overrightarrow{CA}$$

Proof. Using Proposition 4.6. Detail left to the reader. □

We can solve a geometric optimization problem by using the above 2 propositions.

Corollary 4.1 *A, B and C are 3 points in S^2(or H^2). Suppose b and c are fixed, while a can change.*

(1) For the S^2 case, if $b + c < \pi$, then $\tilde{V}_2(A, B, C)$ reaches its maximum when $\overrightarrow{BA} \cdot \overrightarrow{CA} = 0$.

(2) For the H^2 case, $\tilde{V}_2(A, B, C)$ reaches its maximum when $\overrightarrow{BA} \cdot \overrightarrow{CA} = 0$.

Proof. (1) is proved by using Proposition 4.7, and (2) is proved by using Proposition 4.8. See Figure 4.2. □

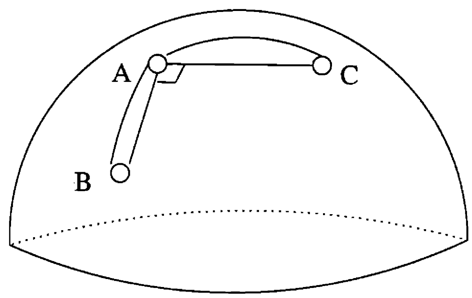


Figure 4.2:

Theorem 4.7 *Conjecture 4.2 is true for $k = 1$, and $c_1 = 2\sum_{i=1}^{n+2} \alpha_i$ for the S^n_+ case; $c_1 = -2\sum_{i=1}^{n+2} \alpha_i$ for the H^n case.*

Proof. For convenience, we let $A_0 = O$ and $\alpha_0 = -\sum_{i=1}^{n+2} \alpha_i$. For the S^n_+ case, by using Proposition 4.7, we have

$$\begin{aligned}
\sum_{i=1}^{n+2} \alpha_i d(P, Q; A_i) &= \frac{2}{1 + \cos \widetilde{PQ}} \sum_{i=1}^{n+2} \alpha_i \overrightarrow{PA_i} \cdot \overrightarrow{QA_i} \\
&= \frac{2}{1 + \cos \widetilde{PQ}} (\sum_{i=0}^{n+2} \alpha_i \overrightarrow{PA_i} \cdot \overrightarrow{QA_i} - \alpha_0 \overrightarrow{PO} \cdot \overrightarrow{QO}) \\
&= \frac{2}{1 + \cos \widetilde{PQ}} (\sum_{i=0}^{n+2} \alpha_i \overrightarrow{OA_i} \cdot \overrightarrow{OA_i} - \alpha_0 \overrightarrow{PO} \cdot \overrightarrow{QO}) \\
&= \frac{2}{1 + \cos \widetilde{PQ}} (\sum_{i=0}^{n+2} \alpha_i + \sum_{i=0}^{n+2} \alpha_i \cos \widetilde{PQ}) \\
&= 2\sum_{i=1}^{n+2} \alpha_i.
\end{aligned}$$

In above, the 3rd "=" comes from Lemma 2.6; the 4th "=" comes from Proposition 4.1.

Similarly, for the H^n case, by using Proposition 4.8, we have

$$\sum_{i=1}^{n+2} \alpha_i d(P,Q;A_i) = \frac{2}{1+\cosh \widetilde{PQ}} \sum_{i=1}^{n+2} \alpha_i \overrightarrow{PA_i} \cdot \overrightarrow{QA_i}$$

$$= \frac{2}{1+\cosh \widetilde{PQ}} (\sum_{i=0}^{n+2} \alpha_i \overrightarrow{OA_i} \cdot \overrightarrow{OA_i} - \alpha_0 \overrightarrow{PO} \cdot \overrightarrow{QO})$$

$$= \frac{2}{1+\cosh \widetilde{PQ}} (-\sum_{i=0}^{n+2} \alpha_i - \sum_{i=0}^{n+2} \alpha_i \cosh \widetilde{PQ})$$

$$= -2 \sum_{i=1}^{n+2} \alpha_i.$$

In above, the 3rd "=" comes from Proposition 4.1. □

The geometric meaning of $c_1 = 0$ is very simple, which is listed in the following proposition.

Proposition 4.9 *For $G_{n,2}$ and $F_{n,2}$ in S^n_+(or H^n), $c_1 = 0$ if and only if points A_1, $\ldots$, A_{n+2} are in a n-dim hyperplane in R^{n+1}(or $R^{n,1}$).*

Proof. We have $c_1 = 2\sum_{i=1}^{n+2} \alpha_i$ for the S^n_+ case; and $c_1 = -2\sum_{i=1}^{n+2} \alpha_i$ for the H^n case. Since $\sum_{i=1}^{n+2} \alpha_i A_i = 0$, so $c_1 = 0$ if and only if points A_1, $\ldots$, A_{n+2} are in a n-dim hyperplane. □

In Figure 4.3, on S^2_+, A_1 is the point that outside the dotted circle, and $c_1 < 0$. In Figure 4.4, on H^2, A_1 is the point that outside the dotted loop, and $c_1 < 0$.

It will be interesting to know the geometric meaning of $c_k = 0$ when $k \geq 2$.

4.6 A Special Version of k-tensegrity Frameworks $G_{n,k}$ and $F_{n,k}$ in S^n

In the previous sections, we discussed the situation when points A_1, $\ldots$, A_{n+2} are in a semisphere S^n_+. We are now going to discuss a special case in this section. Suppose $A_1, \ldots, A_{n+2}$ are in general position in S^n, and the origin O is inside the $n+1$-simplex

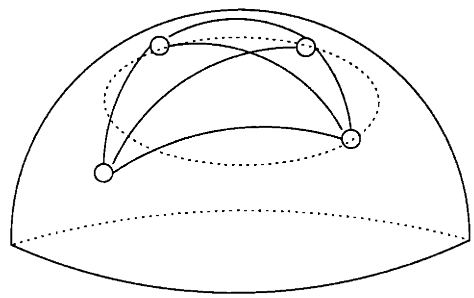


Figure 4.3:

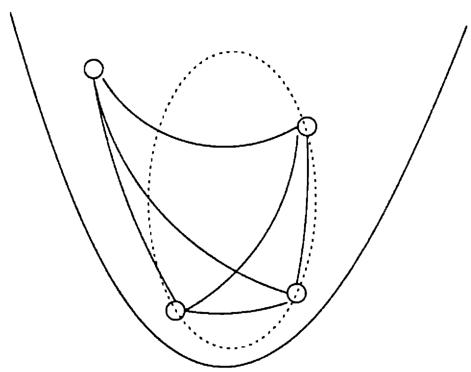


Figure 4.4:

$A_1 \cdots A_{n+2}$. We can still find a sequence of coefficients $\alpha_1, \ldots, \alpha_{n+2}$ as before, such that $\sum_{i=1}^{n+2} \alpha_i A_i = 0$. In this case, all α_i's have the same sign. Given $1 \leq k \leq n$, we construct 2 different k-tensegrity frameworks below.

Framework $G_{n,k}$: let all the S_+^k-simplices $A_{i_1} \cdots A_{i_{k+1}}$ be k-struts.
Framework $F_{n,k}$: let all the S_+^k-simplices $A_{i_1} \cdots A_{i_{k+1}}$ be k-cables.

Conjecture 4.4 *(1) $G_{n,k}$ is k-unyielding in S^{n+1}.*

(2) $F_{n,k}$ is k-unyielding in S^n.

The proof of the following theorem is similar as before, so we will just skip the proof.

Theorem 4.8 *(1) Conjecture 4.4 is true for* $k = 1$.

(2) If $\mathbf{A}(t)$ *is real analytic over* t *for small* $t \geq 0$*, then Conjecture 4.4 is true.*

Remark. We believe that the above theorem can be proved without the restriction "$\mathbf{A}(t)$ is real analytic over t".

Theorem 4.9 *Both* $G_{n,n}$ *and* $F_{n,n}$ *are* n*-unyielding in* S^n.

Proof. Suppose $\mathbf{A}(t)$ is in S^n. For small $t \geq 0$,

$$\sum_{i=1}^{n+2} \tilde{V}_n(A_1(t), \ldots, \hat{A_i(t)}, \ldots, A_{n+2}(t))$$

is a constant, which is the n-volume of sphere S^n. Then both $G_{n,n}$ and $F_{n,n}$ are n-unyielding in S^n. □

Bibliography

[1] K. Bezdek and R. Connelly, *Two-distance preserving functions from Euclidean space*, Discrete geometry and rigidity(Budapest, 1999), Period. Math. Hungar., **39** (1999), 185-200.

[2] A.L. Cauchy, *Sur les polygones et les polyèdres*, J. École Polytechnique, XVIe Cahier, Tome IX, 87-98.

[3] R. Connelly, *Rigidity*, in: Handbook of convex geometry, Vol. A, Amesterdam; New York: North-Holland, 1993, pp. 223–271.

[4] R. Connelly, *A counterexample to the rigidity conjecture for polyhedra*, Inst. Hautes Études Sci. Publ. Math., **47**, (1978), 333-338.

[5] R. Connelly, I. Sabitov and A. Walz, *The bellows conjecture*, Contributions to Algebra and Geometry, **38** (1997), no.1, 1-10.

[6] W. Fenchel, Elementary Geometry in Hyperbolic Space, Berlin; New York: W. de Gruyter, 1989.

[7] P. McMullen, *Simplices with equiareal faces*, Discrete Comput. Geometry, **24** (2000), 397-411.

[8] I. Sabitov, *On the problem of invariance of the volume of a flexible polyhedron*, Russian Mathematical Surveys, **50** (1995), no.2, 451-452.

[9] T.-S. Tay, N. White and W. Whiteley, *Skeletal rigidity of simplicial complexes, I*, European J. Combinatorics, **16** (1995), 381-403.

[10] T.-S. Tay, N. White and W. Whiteley, *Skeletal rigidity of simplicial complexes, II*, European J. Combinatorics, **16** (1995), 503-523.