Equivalent Plastic Strain for the Hill's Yield Criterion under General Three-Dimensional Loading

by

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Abstract

In many industrial applications, an accurate model of the initial yield surface of materials with a significant degree of anisotropy is required. Anisotropy due to preferred orientation can occur in sheet metal parts used in automotive applications due to the rolling processes used to form the sheets. Hill's quadratic yield criterion for anisotropic metals can be used to more accurately model these materials, allowing for improved constitutive models for the prediction of plastic failure and ductile fracture. In this thesis, a derivation of the equivalent plastic strain for plane stress in matrix notation is presented using associated plastic flow and work conjugation. A similar method is attempted for the general three-dimensional case; however, a singularity appears as the six components of the strain increment vector are not independent under plastic incompressibility. To remedy this, a reduced-order system was defined in terms of deviatoric stress, with one normal component eliminated, so that the previous method could be applied; the eliminated component was reintroduced in the final expression. This result was also further expanded to introduce the possibility of defining different plastic potentials and yield criteria under non-associated flow. The result is two expressions for equivalent plastic strain for the Hill's yield criterion in both plane stress and three-dimensional cases that have been partially validated analytically through testing limiting cases such as material isotropy.

Thesis Supervisor: Tomasz Wierzbicki
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Contents

Chapter 1

Introduction

Yield criteria are useful in a variety of structural engineering applications to accurately characterize the initiation of plastic deformation under various forms of loading. This knowledge can be critical for design and production of highly optimized structures. While many popular yield criteria adequately define a yield surface for isotropic materials, in some applications a model of the material's state of anisotropy is required. In this paper, Hill's 1948 quadratic anisotropic yield criterion will be examined, and a model for the equivalent plastic strain under 3D generalized loading will be developed for application in modeling sheet metal materials.

Chapter 1 will discuss the motivation for this work and provide background on the theory of the Hill 48 yield criterion.

Chapter 2 will derive the equivalent plastic strain for a simplified plane stress loading case, and Chapter 3 will present a derivation for the equivalent plastic strain under 3D general loading. Chapter 4 will present conclusions from this work.

1.1 Motivations for Theory of Anisotropic Yield Criteria

Theories to predict the macroscopic yielding of materials are crucial to modeling metallic materials to ensure adequate performance and failure prevention in a variety of applications.

In structural engineering, a yield criterion allows a designer to determine whether a structure under loading will exceed the limits of elastic deformation. An understanding of the structure's yield behavior aids in preventing undesirable material behaviors, including the onset of permanent deformations following yield and possible acceleration of buckling in the plastic regime.

Reasonably accurate anisotropic yield criteria must be incorporated in finite element constitutive models to make predictions of plastic failure and ductile fracture of materials in typical sheet material applications. An example of such a model is developed by Lademo et al, noting that sheet metal parts for automotive applications are often optimized to the verge of material failure, as the parts must be lightweight and crashworthy. [3]

In this model, a correct understanding of the material's yield behavior is required to predict and prevent the dominant plastic failure mode, as well as to identify and model regions of the structure in which other modes such as ductile fracture are dominant in order to prevent overly conservative and heavy designs.

Models of plastic yield are also required in sheet metal forming production processes, which require repeatable permanent deformations of the sheet metal. The yield criterion can be incorporated into the constitutive model to ensure reliable and predictable production in sheet metal forming processes.

The Von Mises yield criterion, a very popular engineering yield criterion for isotropic materials, is, [2]

$$\overline{\sigma} = \sqrt{\frac{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)}{2}}. \tag{1.1}$$

The plastic potential described by this yield criterion has validity for many common metals that deform uniformly in all directions. However, in many circumstances anisotropy can cause significant deviations from the behavior predicted by the Von Mises yield criterion, necessitating the use of an anisotropic yield criterion. The sources and nature of such anisotropic behavior is discussed in the next section.

1.2 Sources and Nature of Anisotropy in Metals

Anisotropy in metals can derive from a variety of sources. The formation of Lüders' bands in annealed mild steels is an early example of yield behavior not modeled by the von Mises yield criterion, cited by Hill in his original paper on anisotropic yielding. [2] Further research has successfully used the Hill 48 yield criterion to analyze these non-uniform deformations. [6]

In most metals, very large strains will result in the formation of crystalline fibers in the direction of greatest loading, along which mechanical properties such as yield stress can vary significantly compared to other directions. This phenomenon is commonly seen in forming processes including cold rolling, drawing, and extrusion; the anisotropy is difficult to eliminate, but can be ameliorated through heat treatment. [2] The anisotropy in such cases due to preferred orientation is modeled well by the use of Hill's yield criterion.

Martensitic phase transformation in some copper alloys and steels can be described as anisotropic as well. [5] Research to develop kinetic laws describing the anisotropic martensite transformation has utilized the Hill's yield criterion to describe the initial yield surface and determine the equivalent plastic strain. [1]

A common way to describe the state of anisotropy of a material is the Lankford ratio,

$$r = \frac{\varepsilon_{22}}{\varepsilon_{33}} = \frac{-\varepsilon_{22}}{(\varepsilon_{11} + \varepsilon_{22})}, \tag{1.2}$$

defining the ratio of strains in the unloaded directions of a sample loaded under uniaxial tension and applying plastic incompressibility. [1]

1.3 The Hill Quadratic Anisotropic Yield Criterion

In his 1948 paper, Hill developed his anisotropic yield criterion building on Von Mises' concept of the plastic potential, defined as

$$f(\sigma_{ij}) = constant, \tag{1.3}$$

where the components of the strain increment tensor can be defined by

$$d\varepsilon_{ij} = \frac{\partial f}{\partial \sigma_{ij}} d\lambda. \tag{1.4}$$

In defining his yield criterion, Hill chooses to define a homogenous quadratic in which no single shear stress can occur linearly. This choice requires that tensile and compressive yield occur at the same yield stress, and satisfies symmetry in shear. Furthermore, in accordance with experimental results, he imposes that hydrostatic pressure will have no effect on yielding. The resulting plastic potential takes the following form:

$$2f \equiv F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2 + 2L\sigma_{23} + 2M\sigma_{31} + 2N\sigma_{12}. \tag{1.5}$$

This form assumes the reference axes are the principle axes of anisotropy, which are orthogonal. [2]

The constants F, G, H, L, M, and N describe the material's current condition of anisotropy. These constants can be computed from experimentally determined normal yield stresses,

$$\begin{aligned} F &= \frac{(\sigma^0)^2}{2}\left[\frac{1}{(\sigma^y_{22})^2} + \frac{1}{(\sigma^y_{33})^2} - \frac{1}{(\sigma^y_{11})^2}\right] \\ G &= \frac{(\sigma^0)^2}{2}\left[\frac{1}{(\sigma^y_{33})^2} + \frac{1}{(\sigma^y_{11})^2} - \frac{1}{(\sigma^y_{22})^2}\right] \\ H &= \frac{(\sigma^0)^2}{2}\left[\frac{1}{(\sigma^y_{11})^2} + \frac{1}{(\sigma^y_{22})^2} - \frac{1}{(\sigma^y_{33})^2}\right] \end{aligned} \tag{1.6}$$

and shear yield stresses,

$$\begin{aligned} L &= \frac{3(\tau^0)^2}{2(\tau_{23}^y)^2} \\ M &= \frac{3(\tau^0)^2}{2(\tau_{31}^y)^2} \\ L &= \frac{3(\tau^0)^2}{2(\tau_{12}^y)^2}, \end{aligned} \tag{1.7}$$

where σ^0 and τ^0 are reference stresses. [2]

Thus the equivalent stress for this yield criterion can be found by:

$$\overline{\sigma} = \sqrt{F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2}. \tag{1.8}$$

Hill's paper and much subsequent research demonstrates that the yield criterion above is validated by experiment for many anisotropic metals and loading cases.

Chapter 2

Hill's Equivalent Plastic Strain for 2D Plane Stress

Under fully general three-dimensional loading, the stress state can be expressed via the Cauchy stress tensor,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix}, \tag{2.1}$$

or as a six component vector (due to the symmetry of the tensor),

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}. \tag{2.2}$$

For many loading cases, the stress can be considered to be two-dimensional or plane stress. In this case, the stress state can be further simplified into a two-dimensional tensor,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}, \tag{2.3}$$

or an equivalent vector of three components,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}. \tag{2.4}$$

Using the vector form, the Hill's yield criterion equivalent stress can be expressed in matrix notation, [4]

$$\overline{\sigma} = \sqrt{(\mathbf{P}\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}}. \tag{2.5}$$

In this formulation, $\mathbf{P}$ is a symmetric matrix of the following form, with components selected such that the terms of Hill's quadratic yield criterion can be retrieved from the product above:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix}. \tag{2.6}$$

The expansion of the product in equation 2.5 will be set equal to Hill's equivalent stress with terms corresponding to out-of-plane stress eliminated,

$$\overline{\sigma} = \sqrt{(G+H)\sigma_{11}^2 + (F+H)\sigma_{22}^2 - 2H\sigma_{11}\sigma_{22} + 2N\sigma_{12}^2}, \tag{2.7}$$

creating the following system of equations for the components of $\mathbf{P}$:

$$\begin{bmatrix} P_{11} \\ P_{22} \\ P_{33} \\ 2P_{12} \end{bmatrix} = \begin{bmatrix} G+H \\ F+H \\ 2N \\ -2H \end{bmatrix}. \tag{2.8}$$

Thus matrix $\mathbf{P}$ can be defined in terms of material constants in the form below.

$$\mathbf{P} = \begin{bmatrix} G+H & -H & 0 \\ -H & F+H & 0 \\ 0 & 0 & 2N \end{bmatrix}. \tag{2.9}$$

This matrix can be used to compute the equivalent plastic strain in matrix notation using the derivation below for orthotropic materials.

2.1 Derivation of Equivalent Plastic Strain for Orthotropic Materials

2.1.1 Definition of Plastic Flow Rule

For plastic potentials such as the Hill's yield criterion, a plastic flow rule can be defined based on the chosen plastic potential,

$$\mathbf{d\varepsilon^p} = d\lambda \frac{d\overline{\sigma}}{d\boldsymbol{\sigma}}, \tag{2.10}$$

where $d\lambda$ is a constant multiplier for the given material and $\mathbf{d\varepsilon^p}$ is a vector of the plastic strain increments, defined for plane stress as

$$\mathbf{d\varepsilon^p} = \begin{bmatrix} d\varepsilon^p_{11} \\ d\varepsilon^p_{22} \\ 2d\varepsilon^p_{12} \end{bmatrix}. \tag{2.11}$$

For the equivalent stress defined in equation 2.5, this is equivalent to

$$\mathbf{d\varepsilon^p} = d\lambda \left[\frac{1}{2} \cdot \frac{1}{\overline{\sigma}} \cdot \frac{d((\mathbf{P}\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma})}{d\boldsymbol{\sigma}} \right]. \tag{2.12}$$

For the expression $(\mathbf{P}\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \mathbf{P}\boldsymbol{\sigma}$, the vector identity

$$\frac{d(\boldsymbol{\sigma}^T \mathbf{P}\boldsymbol{\sigma})}{d\boldsymbol{\sigma}} = \mathbf{P}\boldsymbol{\sigma} + \mathbf{P}^T\boldsymbol{\sigma} = 2\mathbf{P}\boldsymbol{\sigma} \tag{2.13}$$

can be used for the symmetric matrix $\mathbf{P}$, resulting in a simplified expression for the plastic strain increment vector.

$$\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = d\lambda\left(\frac{\mathbf{P}\boldsymbol{\sigma}}{\overline{\sigma}}\right). \tag{2.14}$$

2.1.2 Application of Work Conjugation

For plastic deformations (neglecting small elastic deformations), work conjugation can be applied to a plastic potential to determine the equivalent plastic strain for a given equivalent plastic stress.

$$\boldsymbol{\sigma}\cdot\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = \overline{\sigma}\cdot d\overline{\varepsilon}^{p}. \tag{2.15}$$

By rearranging this equation and substituting in the result from equation 2.14, the following expression for the equivalent plastic strain, $d\overline{\varepsilon}^{p}$ can be found:

$$d\overline{\varepsilon}^{p} = \frac{\boldsymbol{\sigma}\cdot\left[d\lambda\cdot\left(\frac{\mathbf{P}\boldsymbol{\sigma}}{\overline{\sigma}}\right)\right]}{\overline{\sigma}} = \frac{\boldsymbol{\sigma}\cdot\mathbf{P}\boldsymbol{\sigma}}{\overline{\sigma}^{2}}d\lambda = d\lambda\left(\frac{\overline{\sigma}^{2}}{\overline{\sigma}^{2}}\right) = d\lambda. \tag{2.16}$$

By substituting this result for $d\lambda$ into equation 2.14, an expression for $\boldsymbol{\sigma}$ can be obtained.

$$\begin{aligned}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} &= d\overline{\varepsilon}^{p}\left(\frac{\mathbf{P}\boldsymbol{\sigma}}{\overline{\sigma}}\right)\\ \boldsymbol{\sigma} &= \frac{\overline{\sigma}}{d\overline{\varepsilon}^{p}}[\mathbf{P}^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}].\end{aligned} \tag{2.17}$$

This expression can be used with the expression for work conjugation above to determine a formula for the equivalent plastic strain in terms of the plastic strain increment vector:

$$\begin{aligned}\frac{\overline{\sigma}}{d\overline{\varepsilon}^{p}}[(\mathbf{P}^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}})\cdot\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}] &= \overline{\sigma}(d\overline{\varepsilon}^{p})\\ d\overline{\varepsilon}^{p} &= \sqrt{(\mathbf{P}^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}})\cdot\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}}.\end{aligned} \tag{2.18}$$

For an orthrotropic material, the matrix $\mathbf{P}^{-1}$ has the following form (derived in Appendix A):

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{F+H}{FH+FG+GH} & \frac{H}{FH+FG+GH} & 0 \\ \frac{H}{FH+FG+GH} & \frac{G+H}{FH+FG+GH} & 0 \\ 0 & 0 & \frac{1}{2N} \end{bmatrix}. \tag{2.19}$$

Thus the expanded form of the equivalent plastic strain for plane stress in an orthrotropic material can be found by

$$\overline{d\varepsilon^p} = \sqrt{\frac{1}{FH+FG+GH}[(F+H)(d\varepsilon^p_{11})^2 + 2Hd\varepsilon^p_{11}d\varepsilon^p_{22} + (G+H)(d\varepsilon^p_{22})^2] + \frac{2(d\varepsilon^p_{12})^2}{N}}. \tag{2.20}$$

2.2 Derivation of Equivalent Plastic Strain for Isotropic Materials

Equation 2.20 can be simplified further in the case of isotropic materials. In this case, the normal material constants defined in equation 1.7 can all be reduced to $\frac{1}{2}$, and the shear material constants defined in equation 1.8 can be reduced to $\frac{3}{2}$ to reflect an equal yield stress in all directions. Substituting this into Hill's yield criterion will recover Von Mises' yield criterion for isotropic materials. In this case, the equivalent plastic strain of equation 2.20 can be simplified to

$$d\overline{\varepsilon}^p = \sqrt{\frac{4}{3}[(d\varepsilon^p_{11})^2 + d\varepsilon^p_{11}d\varepsilon^p_{22} + (d\varepsilon^p_{22})^2 + (d\varepsilon^p_{12})^2]}. \tag{2.21}$$

The equivalent plastic strain for the Von Mises' yield criterion can be found by

$$d\overline{\varepsilon}^p = \sqrt{\frac{2}{3}d\varepsilon^p_{ij}d\varepsilon^p_{ij}} = \sqrt{\frac{2}{3}[(d\varepsilon^p_{11})^2 + (d\varepsilon^p_{22})^2 + (d\varepsilon^p_{33})^2] + \frac{4}{3}[(d\varepsilon^p_{23})^2 + (d\varepsilon^p_{31})^2 + (d\varepsilon^p_{12})^2]}. \tag{2.22}$$

For the plane stress case, out-of-plane shear terms can be eliminated, and $d\varepsilon_{33}^{P}$ can be eliminated from the expression using plastic incompressibility,

$$d\varepsilon_{33}^{p} = -(d\varepsilon_{11}^{p} + d\varepsilon_{22}^{p}). \tag{2.23}$$

The resulting expression is

$$d\bar{\varepsilon}^{p} = \sqrt{\frac{2}{3} d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}} = \sqrt{\frac{2}{3}[(d\varepsilon_{11}^{p})^{2} + (d\varepsilon_{22}^{p})^{2} + (d\varepsilon_{11}^{p} + d\varepsilon_{22}^{p})^{2}] + \frac{4}{3}(d\varepsilon_{12}^{p})^{2}}. \tag{2.24}$$

The expansion of the square of sums yields the identical result in equation 2.21. This result validates that the expression derived for the equivalent plastic strain of orthotropic materials can correctly reproduce the expected model for the limiting case of isotropic materials.

Chapter 3

Hill's Equivalent Plastic Strain for 3D General Loading

In this chapter, the derivations of the previous chapter will be expanded to the general three-dimensional case, with a stress state described by the Cauchy stress tensor,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix}, \tag{3.1}$$

or as a six component vector (due to the symmetry of the tensor),

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}. \tag{3.2}$$

Using the expanded stress vector in equation 3.2, the Hill's yield criterion could again be expressed using matrix notation, [4]

$$\overline{\sigma} = \sqrt{(\mathbf{P}\boldsymbol{\sigma}) \cdot \boldsymbol{\sigma}}, \tag{3.3}$$

requiring a new expanded matrix, **P**:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & P_{31} & 0 & 0 & 0 \\ P_{12} & P_{22} & P_{23} & 0 & 0 & 0 \\ P_{31} & P_{23} & P_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & P_{66} \end{bmatrix}. \tag{3.4}$$

The expansion of the product in equation 3.3 will be set equal to Hill's equivalent stress,

$$\begin{aligned} \overline{\sigma}^2 = (G+H)\sigma_{11}^2 + (F+H)\sigma_{22}^2 + (F+G)\sigma_{33}^2 - 2H\sigma_{11}\sigma_{22} \\ - 2F\sigma_{22}\sigma_{33} - 2G\sigma_{11}\sigma_{33} + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2, \end{aligned} \tag{3.5}$$

creating the following system of equations for the components of **P**:

$$\begin{bmatrix} P_{11} \\ P_{22} \\ P_{33} \\ 2P_{12} \\ 2P_{23} \\ 2P_{31} \\ P_{44} \\ P_{55} \\ P_{66} \end{bmatrix} = \begin{bmatrix} G+H \\ F+H \\ F+G \\ -2H \\ -2F \\ -2G \\ 2L \\ 2M \\ 2N \end{bmatrix}. \tag{3.6}$$

Thus matrix **P** can be defined in terms of material constants in the form below.

$$\mathbf{P} = \begin{bmatrix} G+H & -H & -G & 0 & 0 & 0 \\ -H & F+H & -F & 0 & 0 & 0 \\ -G & -F & F+G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N \end{bmatrix}. \tag{3.7}$$

However, problems arise in attempting to implement this method as described in Chapter 2. The derivation finds the following form for the equivalent plastic strain,

$$d\bar{\varepsilon}^p = \sqrt{(\mathbf{P}^{-1}\mathbf{d}\boldsymbol{\varepsilon}^\mathbf{p}) \cdot \mathbf{d}\boldsymbol{\varepsilon}^\mathbf{p}}. \tag{3.8}$$

However, it can be shown that for any choice of the material constants F, G, H, L, M, N, the matrix $\mathbf{P}$ is singular (derived in Appendix A):

$$\det \mathbf{P} = 0. \tag{3.9}$$

This result occurs because in defining six independent strain increments,the system is overconstrained, due to additional physical constraints related to plastic incompressibility. Experimental evidence shows that in the plastic regime, hydrostatic pressure results in no change in volume, thus applying an additional constraint,

$$d\varepsilon^p_{11} + d\varepsilon^p_{22} + d\varepsilon^p_{33} = 0. \tag{3.10}$$

Thus the solution for the equivalent plastic strain requires a different method, involving a reduction in the order of the matrices used to define the system.

3.1 Derivation of Equivalent Plastic Strain for Orthotropic Materials

3.1.1 Formulation of Equivalent Stress via Deviatoric Stress

The stress state can be equivalently defined by the deviatoric stress, $\mathbf{S}$, defined as

$$S_{ij} = \sigma_{ij} - \delta_{ij}\frac{1}{3}\sigma_{kk}. \tag{3.11}$$

While this deviatoric stress can be described in a vector as

$$\mathbf{S} = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{31} \\ S_{12} \end{bmatrix}, \tag{3.12}$$

the definition of the deviatoric stress implies that the six components are not all independent, as the definition requires that

$$S_{11} + S_{22} + S_{33} = 0. \tag{3.13}$$

Therefore, one of the normal components (chosen arbitrarily) can be removed to express the stress state instead in terms of five components:

$$\mathbf{S}_{reduced} = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{23} \\ S_{31} \\ S_{12} \end{bmatrix}. \tag{3.14}$$

Using the new vector $\mathbf{S}_{red}$, the Hill's yield criterion can again be expressed in matrix notation,

$$\overline{\sigma} = \sqrt{(\mathbf{Q}\mathbf{S}_{red}) \cdot \mathbf{S}_{red}}, \tag{3.15}$$

where $\mathbf{Q}$ is a matrix of the form

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix}. \tag{3.16}$$

By expanding this product and substituting the definition of deviatoric stress to obtain an expression in terms of stress, and equating it to the equivalent stress found in equation 3.5, the system of equations below was generated.

$$\begin{gathered} \frac{4}{9}Q_{11} - \frac{4}{9}Q_{12} + \frac{Q_{22}}{9} = G + H \\ \frac{Q_{11}}{9} - \frac{4}{9}Q_{12} + \frac{4}{9}Q_{22} = F + H \\ \frac{Q_{11}}{9} + \frac{Q_{22}}{9} + \frac{2}{9}Q_{12} = F + G \\ -\frac{4}{9}Q_{11} + \frac{10}{9}Q_{12} - \frac{4}{9}Q_{22} = -2H \\ -\frac{4}{9}Q_{11} - \frac{2}{9}Q_{12} + \frac{2}{9}Q_{22} = -2G \\ \frac{2}{9}Q_{11} - \frac{2}{9}Q_{12} - \frac{4}{9}Q_{22} = -2F \\ Q_{33} = 2L \\ Q_{44} = 2M \\ Q_{55} = 2N \end{gathered} \tag{3.17}$$

This system can be solved to find the components of matrix $\mathbf{Q}$, resulting in the matrix below:

$$\mathbf{Q} = \begin{bmatrix} F+4G+H & 2F+2G-H & 0 & 0 & 0 \\ 2F+2G-H & 4F+G+H & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2N \end{bmatrix}. \tag{3.18}$$

3.1.2 Application of the Plastic Flow Rule

The same plastic flow rule will be used as in the two-dimensional plane stress case. In the plastic regime, the strain increment and deviatoric strain increment vectors are equivalent due to incompressibility,

$$\mathbf{d\varepsilon^p} = \mathbf{de^p} = \begin{bmatrix} d\varepsilon^p_{11} \\ d\varepsilon^p_{22} \\ d\varepsilon^p_{33} \\ 2d\varepsilon^p_{23} \\ 2d\varepsilon^p_{31} \\ 2d\varepsilon_{12} \end{bmatrix}. \tag{3.19}$$

It is also possible to reduce this vector as with the deviatoric stress vector,

$$\mathbf{d\varepsilon^p_{red}} = \mathbf{de^p_{red}} = \begin{bmatrix} d\varepsilon^p_{11} \\ d\varepsilon^p_{22} \\ 2d\varepsilon^p_{23} \\ 2d\varepsilon^p_{31} \\ 2d\varepsilon_{12} \end{bmatrix}. \tag{3.20}$$

The components of these vectors can be calculated using the flow rule,

$$\mathbf{d\varepsilon^p} = d\lambda \frac{d\overline{\sigma}}{d\boldsymbol{\sigma}}. \tag{3.21}$$

The derivative in the flow rule can be expanded using the chain rule,

$$\frac{d\overline{\sigma}}{d\boldsymbol{\sigma}} = \frac{d\mathbf{S}_{red}}{d\boldsymbol{\sigma}} \frac{d\overline{\sigma}}{d\mathbf{S}_{red}}, \tag{3.22}$$

where the vector derivative $\frac{d\mathbf{S}_{red}}{d\boldsymbol{\sigma}}$ can be found using the definition of the deviatoric stress, and denoted by the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.23}$$

Using the identities in the previous chapter,

$$\frac{d\overline{\sigma}}{d\mathbf{S}_{red}} = \frac{\mathbf{Q}\mathbf{S}_{red}}{\overline{\sigma}}, \tag{3.24}$$

so that the strain increment vector can be found by

$$\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = \frac{d\lambda}{\overline{\sigma}} \mathbf{M}\mathbf{Q}\mathbf{S}_{red}. \tag{3.25}$$

3.1.3 Application of Work Conjugation

Work conjugation can be applied using the previously defined expression,

$$\boldsymbol{\sigma} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = \overline{\sigma}(d\overline{\varepsilon}^{p}), \tag{3.26}$$

or equivalently in terms of deviatoric stress and strain increment,

$$\mathbf{S} \cdot \mathbf{d}\mathbf{e}^{\mathbf{p}} = \overline{\sigma}(d\overline{\varepsilon}^{p}). \tag{3.27}$$

It can be shown that both expressions result in the same equivalent plastic strain increments. As the strain increment and deviatoric strain increment vectors are equivalent (as are the equivalent stress and equivalent deviatoric stress), the products $\boldsymbol{\sigma} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}$ and $\mathbf{S} \cdot \mathbf{de}^{\mathbf{p}}$ can be computed using the result in equation 3.25:

$$\boldsymbol{\sigma} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = \mathbf{S} \cdot \mathbf{de}^{\mathbf{p}} = \overline{\sigma} d\lambda. \tag{3.28}$$

The result is physically intuitive; due to plastic incompressibility, hydrostatic pressure can result in no volume change in the material, and thus cannot perform work. From this result, it can be shown, as before, that

$$d\lambda = d\overline{\varepsilon}^{p} = d\overline{e}^{p}. \tag{3.29}$$

Following this result, a reduced system can be solved to define $\mathbf{S}_{red}$ in terms of $\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}$. Removing the equation corresponding to $d\varepsilon_{33}^{p}$ from the system defined in equation 3.25, the reduced system can be described by

$$\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} = \mathbf{M}_{red}\mathbf{Q}\mathbf{S}_{red}\frac{d\overline{\varepsilon}^{p}}{\overline{\sigma}}, \tag{3.30}$$

where $\mathbf{M}_{red}$ is defined as

$$\mathbf{M}_{red} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.31}$$

The resulting solution to the system is found by

$$\mathbf{S}_{red} = \frac{\overline{\sigma}}{d\overline{\varepsilon}^{p}}(\mathbf{M}_{red}\mathbf{Q})^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}, \tag{3.32}$$

where the inverse $(\mathbf{M}_{red}\mathbf{Q})^{-1}$ is

$$(\mathbf{M}_{red}\mathbf{Q})^{-1} = \begin{bmatrix} \frac{2F+H}{3(FG+FH+GH)} & \frac{-G+H}{3(FG+FH+GH)} & 0 & 0 & 0 \\ \frac{-F+H}{3(FG+FH+GH)} & \frac{2G+H}{3(FG+FH+GH)} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2L} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2M} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2N} \end{bmatrix}. \tag{3.33}$$

Based on this reduction, work conjugation can then be broken down into a sum of two components,

$$\mathbf{S}_{red} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} + S_{33} \cdot d\varepsilon^{p}_{33} = \overline{\sigma}(d\overline{\varepsilon}^{p}). \tag{3.34}$$

The first component can be calculated from equation 3.32 above. The second can be calculated based on the relations previously derived:

$$S_{33} \cdot d\varepsilon^{p}_{33} = -(S_{11} + S_{22}) \cdot d\varepsilon^{p}_{33}. \tag{3.35}$$

In this expression, S_{11} and S_{22} can be calculated from equation 3.32 above.

The resulting equation is

$$(\mathbf{M}_{red}\mathbf{Q})^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} - \left[\frac{(F+2H)d\varepsilon^{p}_{11} + (2H+G)d\varepsilon^{p}_{22}}{3(FG+FH+GH)} \cdot d\varepsilon^{p}_{33}\right] = (d\overline{\varepsilon}^{p})^2. \tag{3.36}$$

This can be solved to calculate the equivalent plastic strain increment using the expression below.

$$\begin{cases} A = 2F + H \\ B = F + 2H \\ C = 2G + H \\ D = 2H + G \\ E = 2H - G - F \end{cases}$$

$$d\bar{\varepsilon}^p = \sqrt{\frac{A(d\varepsilon^p_{11})^2 + E(d\varepsilon^p_{11}d\varepsilon^p_{22} + C(d\varepsilon^p_{22})^2 - Bd\varepsilon^p_{11}d\varepsilon^p_{33} - Dd\varepsilon^p_{22}d\varepsilon^p_{33}}{3(FG+FH+GH)} + \frac{2(d\varepsilon^p_{23})^2}{L} + \frac{2(d\varepsilon^p_{31})^2}{M} + \frac{2(d\varepsilon^p_{12})^2}{N}} \tag{3.37}$$

This expression can be simplified to:

$$d\bar{\varepsilon}^p = \sqrt{\frac{F(d\varepsilon^p_{11})^2 + G(d\varepsilon^p_{22})^2 + H(d\varepsilon^p_{33})^2}{FG+FH+GH} + \frac{2(d\varepsilon^p_{23})^2}{L} + \frac{2(d\varepsilon^p_{31})^2}{M} + \frac{2(d\varepsilon^p_{12})^2}{N}}. \tag{3.38}$$

3.2 Equivalent Plastic Strain for Isotropic Materials

As in the two-dimensional case, in a case where the material can be assumed isotropic, this expression can be simplified using $\frac{1}{2}$ for the normal Hill's coefficients and $\frac{3}{2}$ for all the shear coefficients. The results of this simplification are below.

$$d\bar{\varepsilon}^p = \sqrt{\frac{2[(d\varepsilon^p_{11})^2 + (d\varepsilon^p_{22})^2 + (d\varepsilon^p_{33})^2]}{3} + \frac{4(d\varepsilon^p_{23})^2}{3} + \frac{4(d\varepsilon^p_{31})^2}{3} + \frac{4(d\varepsilon^p_{12})^2}{3}}. \tag{3.39}$$

For comparison, the Von Mises' equivalent plastic strain can be found by

$$d\bar{\varepsilon}^p = \sqrt{\frac{2}{3}d\varepsilon^p_{ij}d\varepsilon^p_{ij}} = \sqrt{\frac{2}{3}[(d\varepsilon^p_{11})^2 + (d\varepsilon^p_{22})^2 + (d\varepsilon^p_{33})^2] + \frac{4}{3}[(d\varepsilon^p_{23})^2 + (d\varepsilon^p_{31})^2 + (d\varepsilon^p_{12})^2]}. \tag{3.40}$$

These two expressions will be identical in all cases, thus showing that the derived formula for equivalent plastic strain for general loading can correctly reproduce the Von Mises' equivalent strain with appropriate simplifications for isotropy.

3.3 Equivalent Plastic Strain for Non-Associated Flow

This method can be extended by a similar process for non-associated flow, where the yield surface and plastic potential are defined separately. In this case, the previously defined function

$$q(\mathbf{S}_{red}) = \sqrt{(\mathbf{QS}_{red}) \cdot \mathbf{S}_{red}}. \tag{3.41}$$

will be used to define the plastic potential of the material. An additional function will be defined to characterize the yield criterion:

$$\overline{\sigma} = r(\mathbf{S}_{red}) = \sqrt{(\mathbf{RS}_{red}) \cdot \mathbf{S}_{red}}, [4] \tag{3.42}$$

where $\mathbf{R}$ is a matrix used to define a yield criterion, such as Von Mises' or a different Hill's yield criterion, in the equation above:

$$\mathbf{R} = \begin{bmatrix} P_{11} & P_{12} & 0 & 0 & 0 \\ P_{12} & P_{22} & 0 & 0 & 0 \\ 0 & 0 & P_{33} & 0 & 0 \\ 0 & 0 & 0 & P_{44} & 0 \\ 0 & 0 & 0 & 0 & P_{55} \end{bmatrix}. \tag{3.43}$$

The plastic strain increment vector computed from the plastic flow rule can be rewritten using the function definitions above.

$$\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = d\lambda \frac{dq}{d\boldsymbol{\sigma}} = \frac{d\lambda \mathbf{MQS}_{red}}{q}. \tag{3.44}$$

Work conjugation can be used as before with this equivalent expression for the plastic flow rule,

$$\boldsymbol{\sigma} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = \overline{\sigma}(d\overline{\varepsilon}^{p})$$

$$\boldsymbol{\sigma} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}} = \boldsymbol{\sigma} \cdot \frac{d\lambda \mathbf{MQS}_{red}}{q} = q d\lambda = r(d\overline{\varepsilon}^{p}). \tag{3.45}$$

yielding the following expression for $d\lambda$,

$$d\lambda = d\overline{\varepsilon}^{p} \frac{r}{q}. \tag{3.46}$$

This can be substituted back into the expression for the plastic flow rule as in the previous derivation, and a reduced system can be formed to solve for an expression for $\mathbf{S}_{red}$:

$$\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} = \mathbf{M}_{red}\mathbf{Q}\mathbf{S}_{red}\frac{r}{q^2}d\overline{\varepsilon}^{p}$$

$$\mathbf{S}_{red} = \left(\frac{q^2}{r d\overline{\varepsilon}^{p}}\right)(\mathbf{M}_{red}\mathbf{Q})^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}}. \tag{3.47}$$

It can be seen that the same inverse appears in this expression as for the previous derivation. This expression can then be substituted into work conjugation as before in order to obtain an expression for equivalent plastic strain:

$$\mathbf{S}_{red} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} + S_{33} \cdot d\varepsilon^{p}_{33} = \overline{\sigma}(d\overline{\varepsilon}^{p}).$$

$$S_{33} \cdot d\varepsilon^{p}_{33} = -(S_{11} + S_{22}) \cdot d\varepsilon_{33}. \tag{3.48}$$

Substitution for the $(S_{11} + S_{22})$ term from the previously solved system results in the following equation:

$$\frac{q^2}{r^2}\left\{(\mathbf{M}_{red}\mathbf{Q})^{-1}\mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} \cdot \mathbf{d}\boldsymbol{\varepsilon}^{\mathbf{p}}_{\mathbf{red}} - \left[\frac{(F+2H)d\varepsilon^{p}_{11} + (2H+G)d\varepsilon^{p}_{22}}{3(FG+FH+GH)} \cdot d\varepsilon^{p}_{33}\right]\right\} = (d\overline{\varepsilon}^{p})^2. \tag{3.49}$$

The resulting expression for equivalent plastic strain is quite similar to the expression derived previously for associated flow, and the expression can be simplified to yield the expression in equation B.4 above.

$$d\overline{\varepsilon}^{p} = \frac{q}{r}\sqrt{\frac{F(d\varepsilon^{p}_{11})^2 + G(d\varepsilon^{p}_{22})^2 + H(d\varepsilon^{p}_{33})^2}{FG+FH+GH} + \frac{2(d\varepsilon^{p}_{23})^2}{L} + \frac{2(d\varepsilon^{p}_{31})^2}{M} + \frac{2(d\varepsilon^{p}_{12})^2}{N}}. \tag{3.50}$$

Chapter 4

Conclusion

In this thesis, various methods have been used to derive formulas for the equivalent plastic strain for the Hill's yield criterion. For the two-dimensional case, it has been shown that the stress and strain increments can be expressed as vectors, and the Hill's coefficients can be expressed in a 3x3 matrix, in order to define the formula for the Hill's equivalent stress in matrix notation. This approach was further developed via the use of an associated plastic flow rule and work conjugation in order to arrive at a formula in matrix notation for the equivalent plastic strain. An expansion of this method was attempted for the three-dimensional case; however, a singularity arose as the 6x6 matrix of Hill's coefficients corresponding to the fully general case was not invertible. The source of the singularity was thought to be a lack of six independent variables in the strain space; as the normal strain increments are related through plastic incompressibility, specifying six independent strain increments overconstrained the system.

In order to solve a reduced-order system and eliminate this problem, the yield criterion was reformulated in terms of five components of deviatoric stress (the last normal component can be eliminated due to the definition of deviatoric stress). The resulting reduced-order system can be solved using the associated plastic flow rule and work conjugation as in the two-dimensional case, with the addition of an extra term at the end to include the contribution of the third normal deviatoric stress, resulting in a formula for equivalent plastic strain for the general three-dimensional case (which can be shown to reduce to the two-dimensional result previously found). This method was further extended to consider non-associated plastic flow, and an additional formulation for equivalent plastic strain was derived for this case.

Further work is necessary to apply and empirically validate the results of this derivation. Tests of anisotropic materials with multiaxial strain measurements will be necessary to validate the model; different loading cases should be examined to test the behavior of this formulation. Preliminary analytical work to assess the results has successfully demonstrated consistency between the two-dimensional and three-dimensional results. Both formulas were also shown to reduce appropriately to the Von Mises' equivalent strain when assumptions of isotropy were applied.

In conclusion, this paper successfully derives equivalent plastic strain for anisotropic materials using Hill's yield criterion for both two-dimensional and three-dimensional general loading. The relations derived here could find wide application in modeling plastic failure in sheet metals and other optimized structures, and the methods could easily be extended to other yield criteria.

Appendix A

Matrix Determinants and Inversion

A.1 Matrix Inversion for P for Plane Stress

For the 2D plane stress case, inversion of the matrix **P** below is required.

$$\mathbf{P} = \begin{bmatrix} G+H & -H & 0 \\ -H & F+H & 0 \\ 0 & 0 & 2N \end{bmatrix}. \tag{A.1}$$

The derivation of the matrix inverse, $\mathbf{P}^{-1}$, requires calculating the determinant of the matrix:

$$\det \mathbf{P} = (G+H) \cdot \begin{vmatrix} F+H & 0 \\ 0 & 2N \end{vmatrix} - (-H) \cdot \begin{vmatrix} -H & 0 \\ 0 & 2N \end{vmatrix} + 0 \cdot \begin{vmatrix} -H & F+H \\ 0 & 0 \end{vmatrix}. \tag{A.2}$$

$$\det \mathbf{P} = (G+H)(F+H)(2N) - H^2(2N) = 2N(FG+FH+GH). \tag{A.3}$$

Next the transpose of the matrix should be calculated; since **P** is symmetric, the transpose will be identical. Then the matrix of cofactors is calculated based on the determinants of the 2x2 minor matrices, which gives the following matrix:

$$\begin{bmatrix} 2N(F+H) & 2NH & 0 \\ 2NH & 2N(G+H) & 0 \\ 0 & 0 & FG+FH+GH \end{bmatrix}. \tag{A.4}$$

This matrix divided by the determinant of **P** gives the inverse:

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{F+H}{FH+FG+GH} & \frac{H}{FH+FG+GH} & 0 \\ \frac{H}{FH+FG+GH} & \frac{G+H}{FH+FG+GH} & 0 \\ 0 & 0 & \frac{1}{2N} \end{bmatrix}. \tag{A.5}$$

A.2 Matrix Determinant for P for 3D Loading

While a similar inversion was attempted for the three dimensional case, the matrix could not be inverted, as it was found that the determinant was zero for all values of the constants F, G, H, L, M, N. The derivation of this result is expanded below.

$$\det \mathbf{P} = (G+H) \begin{vmatrix} F+H & -F & 0 & 0 & 0 \\ -F & F+G & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2N \end{vmatrix} - (-H) \begin{vmatrix} -H & -F & 0 & 0 & 0 \\ -G & F+G & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2M \end{vmatrix}$$
$$+ (-G) \begin{vmatrix} -H & F+H & 0 & 0 & 0 \\ -G & -F & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2N \end{vmatrix}. \tag{A.6}$$

$$
\begin{aligned}
\det \mathbf{P} = (G+H) & \left\{ (F+H) \begin{vmatrix} F+G & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 2M & 0 \end{vmatrix} - (-F) \begin{vmatrix} -F & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} \right\} \\
- (-H) & \left\{ (-H) \begin{vmatrix} F+G & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} - (-F) \begin{vmatrix} -G & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} \right\} \\
+ (-G) & \left\{ (-H) \begin{vmatrix} -F & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} - (F+H) \begin{vmatrix} -G & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} \right\}
\end{aligned}
\tag{A.7}
$$

$$
\det \mathbf{P} = 8LMN[(G+H)(F+H)(F+G) - (G+H)F^2 - H(F+G) - FGH - FGH - G^2(F+H)] \tag{A.8}
$$

$$
\begin{aligned}
\det \mathbf{P} = 8LMN(&F^2G + FG^2 + FGH + G^2H + F^2H + FGH + FH^2 + GH^2 \\
&- F^2G - F^2H - H^2F - H^2G - FGH - FGH - G^2F - G^2H)
\end{aligned}
\tag{A.9}
$$

As all terms in the final equation cancel out, it has been shown that the matrix's determinant is identically zero for all values of the coefficients. This result provides the motivation for the alternative method applied in Chapter 3.

$$
\det \mathbf{P} = 0. \tag{A.10}
$$

A.3 Matrix Inversion for Reduced System

In Chapter 3, following the failure to invert the matrix $\mathbf{P}$ as described above, a reduced system is developed, forming the matrix $\mathbf{M}_{red}\mathbf{Q}$ to be inverted. A derivation of this inversion is presented below.

The matrices are defined as follows:

$$\mathbf{M}_{red} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A.11}$$

$$\mathbf{Q} = \begin{bmatrix} F+4G+H & 2F+2G-H & 0 & 0 & 0 \\ 2F+2G-H & 4F+G+H & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2N \end{bmatrix}. \tag{A.12}$$

Their product can be found by:

$$\mathbf{M}_{red}\mathbf{Q} = \begin{bmatrix} 2G+H & G-H & 0 & 0 & 0 \\ F-H & 2F+H & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2N \end{bmatrix}. \tag{A.13}$$

The determinant of matrix $\mathbf{M}_{red}\mathbf{Q}$ is:

$$\det \mathbf{M}_{red}\mathbf{Q} = (2G{+}H)\begin{vmatrix} 2F+H & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} - (G{-}H)\begin{vmatrix} F-H & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix}. \tag{A.14}$$

$$\det \mathbf{M}_{red}\mathbf{Q} = 8LMN[(2G{+}H)(2F{+}H){-}(G{-}H)(F{-}H)] = 24LMN(FG{+}FH{+}GH). \tag{A.15}$$

The matrix is invertible, as the determinant is nonzero.

To find the inverse of the matrix, first, the transpose of the matrix is found:

$$\mathbf{M}_{red}\mathbf{Q}^T = \begin{bmatrix} 2G+H & F-H & 0 & 0 & 0 \\ G-H & 2F+H & 0 & 0 & 0 \\ 0 & 0 & 2L & 0 & 0 \\ 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 2N \end{bmatrix}. \tag{A.16}$$

Next, the matrix of minors is found. An example of the formula for one minor is:

$$M_{11} = \begin{vmatrix} 2F+H & 0 & 0 & 0 \\ 0 & 2L & 0 & 0 \\ 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 2N \end{vmatrix} = 8LMN(2F+H). \tag{A.17}$$

The full matrix of minors is:

$$\alpha = FG + FH + GH$$

$$\begin{bmatrix} 8LMN(2F+H) & 8LMN(G-H) & 0 & 0 & 0 \\ 8LMN(F-H) & 8LMN(2F+H) & 0 & 0 & 0 \\ 0 & 0 & 12MN\alpha & 0 & 0 \\ 0 & 0 & 0 & 12LN\alpha & 0 \\ 0 & 0 & 0 & 0 & 12LM\alpha \end{bmatrix}. \quad (A.18)$$

This can then be converted to the matrix of cofactors by multiplying each element whose indices sum to an odd number by -1:

$$\begin{bmatrix} 8LMN(2F+H) & 8LMN(-G+H) & 0 & 0 & 0 \\ 8LMN(-F+H) & 8LMN(2F+H) & 0 & 0 & 0 \\ 0 & 0 & 12MN\alpha & 0 & 0 \\ 0 & 0 & 0 & 12LN\alpha & 0 \\ 0 & 0 & 0 & 0 & 12LM\alpha \end{bmatrix}. \quad (A.19)$$

The inverse is then given by this matrix of cofactors divided by the determinant:

$$(\mathbf{M}_{red}\mathbf{Q})^{-1} = \begin{bmatrix} \frac{2F+H}{3(FG+FH+GH)} & \frac{-G+H}{3(FG+FH+GH)} & 0 & 0 & 0 \\ \frac{-F+H}{3(FG+FH+GH)} & \frac{2G+H}{3(FG+FH+GH)} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2L} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2M} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2N} \end{bmatrix}. \quad (A.20)$$

This result was used in the derivation of equivalent plastic strain in three-dimensional cases in Chapter 3.

Appendix B

Reduction of 3D Equivalent Plastic Strain under Plane Stress

B.1 Simplification of General Formula

In order to establish consistency between the formula derived for three dimensions and that derived for plane stress, the more general formula can be simplified using the assumptions of plane stress. From Chapter 3, the general formula is:

$$d\bar{\varepsilon}^p = \sqrt{\frac{F(d\varepsilon_{11}^p)^2 + G(d\varepsilon_{22}^p)^2 + H(d\varepsilon_{33}^p)^2}{FG + FH + GH} + \frac{2(d\varepsilon_{23}^p)^2}{L} + \frac{2(d\varepsilon_{31}^p)^2}{M} + \frac{2(d\varepsilon_{12}^p)^2}{N}}. \quad \text{(B.1)}$$

This formula can be shown to be equivalent to the expression below under plastic incompressibility.

$$d\bar{\varepsilon}^p = \sqrt{\frac{F(d\varepsilon_{22}^p + d\varepsilon_{33}^p)^2 + G(d\varepsilon_{11}^p + d\varepsilon_{33}^p)^2 + H(d\varepsilon_{11}^p + d\varepsilon_{22}^p)^2}{FG + FH + GH} + \frac{2(d\varepsilon_{23}^p)^2}{L} + \frac{2(d\varepsilon_{31}^p)^2}{M} + \frac{2(d\varepsilon_{12}^p)^2}{N}}. \quad \text{(B.2)}$$

Based on the assumptions of the two-dimensional case, the out-of-plane shears will be eliminated,

$$d\varepsilon^p_{31} = d\varepsilon^p_{23} = 0, \tag{B.3}$$

producing the following expression:

$$d\bar{\varepsilon}^p = \sqrt{\frac{F(d\varepsilon^p_{22} + d\varepsilon^p_{33})^2 + G(d\varepsilon^p_{11} + d\varepsilon^p_{33})^2 + H(d\varepsilon^p_{11} + d\varepsilon^p_{22})^2}{FG + FH + GH} + \frac{2(d\varepsilon^p_{12})^2}{N}}. \tag{B.4}$$

The two-dimensional formula is:

$$d\bar{\varepsilon}^p = \sqrt{\frac{1}{FH + FG + GH}[(F + H)(d\varepsilon^p_{11})^2 + 2H d\varepsilon^p_{11} d\varepsilon^p_{22} + (G + H)(d\varepsilon^p_{22})^2] + \frac{2(d\varepsilon^p_{12})^2}{N}}. \tag{B.5}$$

Further manipulation is required to determine the equivalence of the two forms.

B.2 Application of Plastic Incompressibility

Plastic incompressibility can be applied to this problem.

$$d\varepsilon^p_{11} + d\varepsilon^p_{22} + d\varepsilon^p_{33} = 0. \tag{B.6}$$

Thus $d\varepsilon^p_{33}$ can be eliminated from both expressions with the following substitution:

$$d\varepsilon^p_{33} = -(d\varepsilon^p_{11} + d\varepsilon^p_{22}). \tag{B.7}$$

Substituting this into the three-dimensional equation yields:

$$d\bar{\varepsilon}^p = \sqrt{\frac{F(-d\varepsilon^p_{11})^2 + G(-d\varepsilon^p_{22})^2 + H(d\varepsilon^p_{11} + d\varepsilon^p_{22})^2}{FG + FH + GH} + \frac{2(d\varepsilon^p_{12})^2}{N}}. \tag{B.8}$$

Equation B.9 can be expanded and rearranged to the form below.

$$d\bar{\varepsilon}^p = \sqrt{\frac{(F + H)(d\varepsilon^p_{11})^2 + (G + H)(d\varepsilon^p_{22})^2 + 2H d\varepsilon^p_{11} d\varepsilon^p_{22}}{FG + FH + GH} + \frac{2(d\varepsilon^p_{12})^2}{N}}. \tag{B.9}$$

This shows that the three-dimensional formula for equivalent plastic strain derived here can be identically reduced to match the two-dimensional formula for all values of the coefficients.

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