Arithmetic Duality in Algebraic K-theory

by

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Abstract

Let X be a regular arithmetic curve or point (meaning a regular separated scheme of finite type over $\mathbb{Z}$ which is connected and of Krull dimension ≤ 1). We define a compactly-supported variant $K_c(X)$ of the algebraic K-theory spectrum $K(X)$, and establish the basic functoriality of K_c. Briefly, K_c behaves as if it were dual to K.

Then we give this duality some grounding: for every prime ℓ invertible on X, we define a natural ℓ-adic pairing between $K_c(X)$ and $K(X)$. This pairing is of an explicit homotopy-theoretic nature, and reflects a simple relation between spheres, tori, and real vector spaces. Surprisingly, it has the following two properties: first (a consequence of work of Rezk), when one tries to compute it the ℓ-adic logarithm inevitably appears; and second, it can be used to give a new description of the global Artin map, one which makes the Artin reciprocity law manifest.

Thesis Supervisor: Jacob Lurie
Title: Professor, Harvard

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Contents

Chapter 1

Introduction

1.1 Motivation: Euler charactistics and the K-theory of curves

Let X be a smooth complete curve over a field k, and let V be a vector bundle on X. Then the geometrically relevant vector space $H^0(V)$ of global sections of V comes with a seedy companion $H^1(V)$, which can be defined as the cokernel of any number of natural maps whose kernel computes $H^0(V)$. The difference in dimensions of these spaces is an integer

$$\chi(V) := h^0(V) - h^1(V)$$

called the *Euler characteristic* of V. This assignment $\chi : Vect(X) \to \mathbb{Z}$ has two fundamental properties:

(χ.1) Additivity: it is additive in short exact sequences of vector bundles.

(χ.2) Point-modification: for line bundles $\mathcal{L}$, when one performs a modification $\mathcal{L} \mapsto \mathcal{L}(x)$ by allowing sections of $\mathcal{L}$ to have a simple pole at some $x \in X$, the value of χ goes up by $deg(x) := dim_k(k(x))$.

Simple though they are, these properties have interesting consequences for the curve X:

1. Every nonzero rational function f on X with divisor $div(f)$ induces an isomorphism of line bundles $\mathcal{O}(div(f)) \simeq \mathcal{O}$; thus, applying χ and using (χ.2) we deduce that $deg(div(f)) = 0$, i.e. the weighted sum of the zeroes of a nonzero rational function on X is trivial.

2. Every vector bundle V on X is accessible from $\mathcal{O}$ by moves controlled by (χ.1) and (χ.2); from this one sees that (χ.1) and (χ.2) imply the Riemann-Roch formula

$$\chi(V) = \chi(\mathcal{O}) \cdot rk(V) + deg(V).$$

So this function χ and its two basic properties are useful for the study of the curve X. But now we would like to explain that if one reinterprets χ using algebraic K-theory, then not only do these two properties (χ.1) and (χ.2) become transparent, but also the list of their consequences grows substantially.

Recall that algebraic K-theory assigns to every smooth variety Y/k a spectrum denoted $K(Y)$, whose bottom homotopy group $\pi_0 K(Y)$ is the universal abelian group with a map from $Vect(Y)$ satisfying the additivity property as in (χ.1). Furthermore, $K(Y)$ is highly functorial: it pulls back under arbitrary maps, and pushes forward under proper maps.

This proper pushforward functoriality, in particular, already subsumes χ and its two properties (χ.1) and (χ.2). Indeed, the proper pushforward $p_* : K(X) \to K(k)$ along the projection map from X to the point recovers χ on π_0. In light of the universal nature of $\pi_0 K(X)$, this already gives (χ.1); and (χ.2) follows from functoriality of proper pushforward applied to the inclusion $x \to X$ followed by the projection p. Thus the theory of χ follows from the basic functoriality of algebraic K-theory, as developed in Quillen's paper [Q1] or the Thomason-Trobaugh article [TT].

Moreover, promoting the function χ to the map of spectra p_* gives extra information. For example:

1. The above argument for $deg(div(f)) = 0$, if carried out on the level of $\pi_1 p_*$ instead of

$\chi = \pi_0 p_*$, gives a proof of the Weil reciprocity law for X; see [Gi].

2. If we precompose p_* with the tensor product map $K(X) \wedge K(X) \to K(X)$, the result is a "pairing"

$$K(X) \wedge K(X) \to K(k).$$

To illustrate the utility of this construction, assume for simplicity that the field k is separably closed, and choose a prime ℓ invertible in k. Then after completing at ℓ and taking $\pi_1 \wedge \pi_1$, the above pairing produces the Weil pairing

$$T_\ell Jac(X) \otimes T_\ell Jac(X) \to T_\ell \mathbb{G}_m.$$

With some more work one can even see that the Weil pairing is nondegenerate by K-theoretic means.

In conclusion, the function $\chi : Vect(X) \to \mathbb{Z}$ captures interesting properties of the curve X, and its homotopy-theoretic extension $p_* : K(X) \to K(k)$ captures even more.

1.2 This thesis: the arithmetic case

Now let X instead be an *arithmetic* curve (or point), meaning a separated regular scheme of finite type over $\mathbb{Z}$ which is connected and of Krull dimension ≤ 1. For instance X could be *Spec* of the ring of integers of a number field, potentially with finitely many closed points removed. Recall that there is a classical analogy between such arithmetic X and the previously-considered "geometric" curves X/k, giving rise to a long tradition of passing ideas back and forth between the arithmetic and geometric cases. Following in this tradition, our goal is to define and study an arithmetic analog of the map $p_* : K(X) \to K(k)$ discussed above. In doing so, two basic problems arise:

1. First, we have allowed our arithmetic curve X to be non-complete, contrary to our assumption in the geometric case. Completeness seemed crucial in the geometric case,

since it ensured finite-dimensionality of the cohomology groups $H^i(V)$, as is required for the construction χ (and p_*). On the other hand, completeness is unreasonable in the arithmetic case, since the "Archimedean points" will always be missing.

2. Second, there is no base field k in the arithmetic case. So although the source spectrum $K(X)$ exists just fine, the target spectum $K(k)$ needs to be redefined.

To solve the first problem, we will replace K-theory by a "compactly supported" variant, which we define following the usual place-theoretic philosophy:

Definition 1.2.1. *Let X be an arithmetic curve or point, with fraction field denoted $Frac(X)$. Define the "part of X near ∞" as the finite set*

$$X_\infty := \{\textit{equivalence classes of norms } |\cdot| : Frac(X) \to \mathbb{R}_{\geq 0} \textit{ such that } |\mathcal{O}(X)| \textit{ is unbounded}\},$$

and for each $s \in X_\infty$ let F_s be the completion of $Frac(X)$ with respect to the corresponding norm. Then we define a spectrum

$$K_c(X) := Fib(K(X) \to \oplus_{s \in X_\infty} K(F_s)),$$

where Fib stands for homotopy fiber.

One imagines that an $s \in X_\infty$ describes a neighborhood of some missing "point at infinity" of X, the idea being that the corresponding norm $|\cdot|$ measures how much a rational function blows up in this neighborhood. Then, intuitively speaking, F_s is the field of functions defined on this neighborhood. However, the definition also incorporates Archimedean norms, in which case F_s is either $\mathbb{R}$ or $\mathbb{C}$.

By construction, there is a natural map $K_c(X) \to K(X)$. It is an equivalence when X_∞ is empty, i.e. when X is proper over a finite field.

The basic result about K_c is that it has opposite variance to K:

Proposition 1.2.2. *The above association $X \mapsto K_c(X)$ is covariantly functorial for all maps and contravariantly functorial for proper maps.*

Let me briefly mention two likely extensions of these ideas, which won't be worked out in this text. First, Definition 2.2.1 works just as well for non-regular (i.e. singular) X. But in that case one should also consider the G-theory $G(X)$ and its compactly supported variant $G_c(X)$, making a total of four theories: K, G, K_c, and G_c. Each would have different functoriality, but they all would be related by various comparisons, dualities, and pairings. Second, and more ambitiously, everything should also extend to higher-dimensional X, by means of the formalism of Berkovich analytic spaces. Such speculative extensions aside, we return to our one-dimensional regular X and continue with the story.

Using $K_c(X)$ instead of $K(X)$ solves the first problem above, that of the potential non-compactness of X. The second problem, that of finding the appropriate target spectrum to replace $K(k)$, is more nuanced. In fact, we will fix an auxiliary prime ℓ and have our spectrum be ℓ-adic in nature. More precisely, we will use $L_{K/\ell}S$, the Bousfield localization of the sphere spectrum at (mod ℓ) complex K-theory (see [Bo]). Assuming ℓ odd for simplicity, the homotopy groups of this spectrum are as follows:

$$\pi_n L_{K/\ell}S = \begin{cases} \mathbb{Z}_\ell & \text{if } n = 0 \\ \mathbb{Z}_\ell/(\ell \cdot (n+1)) & \text{if } n+1 \text{ is a multiple of } 2\cdot(\ell-1) \\ 0 & \text{otherwise} \end{cases}$$

In particular, the group $\mathbb{Z}_\ell$ of ℓ-adic integers will be the target for our Euler characteristics of arithmetic vector bundles.

In these paragraphs, which can be skipped, we will give two reasons why this spectrum $L_{K/\ell}S$ is a plausible arithmetic analog of $K(k)$ in the geometric case. First, recall that the arithmetic-geometric analogy is strongest when the base field k is finite, say $k = \mathbb{F}_q$. Now, on the one hand Quillen's calculation of $K(\mathbb{F}_q)$ ([Q2]) shows that there is an equivalence (for ℓ invertible in $\mathbb{F}_q$)

$$K(\mathbb{F}_q)_{\hat{\ell}} \simeq (K_{\hat{\ell}}^{hq^{\mathbb{Z}}})_{\geq 0},$$

where on the left we have the ℓ-adic completion of $K(\mathbb{F}_q)$ and on the right we have the connective cover of the homotopy fixed points of the cyclic group generated by the q^{th}-power Adams operation acting on ℓ-adic complex

K-theory. But on the other hand ([DH]) there is a natural equivalence

$$L_{K/\ell}S \simeq K_{\hat{\ell}}^{h\mathbb{Z}_\ell^\times}$$

between $L_{K/\ell}S$ and the (continuous) homotopy fixed points of the full profinite group $\mathbb{Z}_\ell^\times$ of ℓ-adic Adams operations acting on $K_{\hat{\ell}}$. So $L_{K/\ell}S$ is a sort of periodized ℓ-adic cousin of all the $K(\mathbb{F}_q)$'s.

For another reason that $L_{K/\ell}S$ is a natural target for our map, note that in the geometric case the map p_* : $K(X) \to K(k)$ is, in a sense, "dual" to the unit map $K(k) \to K(X)$. So we could just as well look for a spectrum to be the source of a natural unit map with target $K(X)$ in the arithmetic case. But in fact there is a ready-made such unit map $S \to K(X)$ with source the sphere spectrum S. Moreover, it follows from work of Voevodsky, Suslin, and Rost on the norm residue isomorphism theorem and its connection with the Quillen-Lichtenbaum conjectures that this unit map, once ℓ-adically completed, factors through a natural map $(L_{K/\ell}S)_{\geq 0} \to K(X)_{\hat{\ell}}$, assuming ℓ invertible on X (CITE mitchell). So, when working ℓ-adically, it seems natural to think of this latter as the "true" unit map, thus again leading to the idea that the spectrum $L_{K/\ell}S$ is analogous to $K(k)$.

Of course, the best reason for choosing $L_{K/\ell}S$ is that it works (and S doesn't, for instance). That will be borne out in the rest of the paper.

Here, then, is our main theorem, the existence of an arithmetic analog of p_*:

Theorem 1.2.3. *Let ℓ be a prime. There is a collection of maps of spectra*

$$j_X : K_c(X) \to L_{K/\ell}S,$$

one for each arithmetic curve or point X on which ℓ is invertible, satisfying the following properties:

(j.1) Functoriality: j_X is covariantly functorial in X.

(j.2) Point-modification: For every prime $p \neq \ell$, the map $j_{\mathbb{F}_p} : K(\mathbb{F}_p) \to L_{K/\ell}S$ sends the unit class $[\mathbb{F}_p] \in K_0(\mathbb{F}_p)$ to the ℓ-adic integer $(1 - \frac{1}{\ell})log(p)$.

In terms of the apparent duality between $K(X)$ and $K_c(X)$, this map j_X should be thought of as dual to the unit map $S \to K(X)$, after K/ℓ-localization. This intuition is made precise by Theorem 1.2.4, at least when ℓ is odd.

The *log* appearing above stands for the ℓ-adic logarithm, i.e. the unique homomorphism $\mathbb{Z}_\ell^\times \to \ell \cdot \mathbb{Z}_\ell$ which is given by the usual power series on $1 + \ell\mathbb{Z}_\ell \subset \mathbb{Z}_\ell^\times$. The appearance of this function *log* above may be surprising; in fact it comes entirely from the remarkable homotopy-theoretic work of Rezk ([R]), and is not really intrinsic to our story. Nonetheless, there is no avoiding it: the functoriality (j.1) together with the point-modification property (j.2) even just for any single prime $p \neq \ell$ uniquely determine the system of maps j_X up to homotopy, as follows from Theorem 1.2.4 (at least for ℓ odd).

The proof of Theorem 1.2.3 is simple: to define j_X in general, by functoriality it suffices to just define $j_{\mathbb{Z}[1/\ell]}$. This in turn is done by an explicit homotopy-theoretic construction relating spheres, tori, and real vector spaces. Then (j.1) holds by definition, and checking (j.2) is a routine matter given the construction of $j_{\mathbb{Z}[1/\ell]}$ and Rezk's work.

Now we state one last theorem, which, unfortunately, we won't have the time to prove here. It justifies the apparent duality between $K_c(X)$ and $K(X)$, at least in the K/ℓ-local setting and for ℓ odd. Consider the composition

$$K(X) \wedge K_c(X) \to K_c(X) \xrightarrow{j_X} L_{K/\ell}S,$$

where the first map is the natural product induced by tensor product of vector bundles. This is the "pairing" between K_c and K referred to in the abstract.

Theorem 1.2.4. *Let X be an arithmetic curve or point on which ℓ is invertible, and suppose that ℓ is odd. Then the above pairing*

$$K(X) \wedge K_c(X) \to L_{K/\ell}S$$

is a perfect pairing in the K/ℓ-local sense. That is, it induces an equivalence of each of $L_{K/\ell}K_c(X)$ and $L_{K/\ell}K(X)$ with the K/ℓ-local functional dual of the other.

This theorem can be thought of as a K-theoretic analog of the Poitou-Tate duality in Galois cohomology. The two statements are related by a (degenerate) Atiyah-Hirzebruch spectral sequence.

We also note that the natural maps $K_c(X)_{\hat{\ell}} \to L_{K/\ell}K_c(X)$ and $K(X)_{\hat{\ell}} \to L_{K/\ell}K(X)$ are isomorphisms on homotopy groups in degrees respectively ≥ 0 and ≥ 1 (this follows from now-standard arguments in the direction of the Quillen-Lichtenbaum conjectures). Thus the K/ℓ-localizations in the above theorem don't lose as much as one might think.

1.3 Number-theoretic applications

All of the applications given above for χ and p_* in the geometric case have an analog for j_X in the arithmetic case. For instance, the analog of the fact that $deg(div(f)) = 0$ is the product formula for valuations, which is essentially what $\pi_0 j_X$ gives, albeit in a distorted form. Furthermore, the analog of Gillet's deduction of the Weil reciprocity law from $\pi_1 p_*$ is a deduction of the quadratic reciprocity law from $\pi_1 j_X$. But all of this is subsumed by the fact that Theorem 1.2.4 almost immediately implies the main theorems of global class field theory completed at ℓ.

However, we won't have time to explain this latter fact; and besides, we aren't even going to prove Theorem 1.2.4. So instead we will content ourselves with an exploration of the consequences of the theorem that we do prove, namely Theorem 1.2.3. More specifically, we would like to show how Theorem 1.2.3 gives rise to a new description of the global Artin map (completed away from 2), one which makes the Artin reciprocity law manifest. This will be done in the final section.

Chapter 2

Work

2.1 Arithmetic curves and points

Here is the class of geometric objects we'll be working with:

Definition 2.1.1. *By an* arithmetic curve *(resp.* point*) we mean a regular separated scheme of finite type over $\mathbb{Z}$ which is connected and of Krull dimension 1 (resp. 0).*

Each arithmetic curve or point X embeds uniquely and functorially as an open subscheme of a maximal arithmetic curve or point X'. When X is a point, we have $X = X' = Spec(\mathbb{F})$ for some finite field $\mathbb{F}$. On the other hand, when X is a curve, X' is either a smooth proper curve over some $\mathbb{F}_p$ ("function field case") or $Spec(\mathcal{O}_F)$ for some number field F ("number field case"). And in either case, X is obtained from X' by deleting finitely many closed points.

2.2 Compactly supported K-theory

Here is our main definition. Unlike in the introduction, we give it in an ad hoc form. This will make for shorter arguments.

Definition 2.2.1. *Let X be an arithmetic curve or point, with maximal extension X' and closed complement $Z = X' - X$. We define the* compactly supported K-theory *of X to be* *the spectrum*

$$K_c(X) := Fib\left(K(X') \to K(X'_{\mathbb{R}} \sqcup X'_{\widehat{Z}})\right).$$

Here Fib stands for homotopy fiber, $X'_{\mathbb{R}}$ stands for the base-change of X' to the real numbers, $X'_{\widehat{Z}}$ stands for $Spec$ of the completed coordinate ring of X' along Z, and $K(-)$ stands for the algebraic K-theory of schemes. Since there are many approaches to defining this K-theory, let us specify: here, by $K(-)$ we mean the Blumberg-Gepner-Tabuada non-connective K-theory of the stable ∞-category $\mathrm{Perf}(-)$ of perfect complexes on our scheme ([BGT]). Then, for instance, the map $K(X') \to K(X'_{\mathbb{R}} \sqcup X'_{\widehat{Z}})$ used in the above definition comes from the pullback of perfect complexes along the natural map $X'_{\mathbb{R}} \sqcup X'_{\widehat{Z}} \to X'$.

Now we can state and prove some basic properties of this $K_c(X)$.

Proposition 2.2.2. *To a map $f : X \to Y$ between arithmetic curves or points can be functorially associated a map of spectra $K_c(X) \to K_c(Y)$.*

Proof. Let $Z = X' - X$ and $W = Y' - Y$, and abusively write $f^{-1}W \subset Z$ for the preimage of W by the map $X' \to Y'$ induced by f. Then the commutative square of schemes

$$\begin{array}{ccc} X' & \longleftarrow & X'_{\mathbb{R}} \sqcup X'_{\widehat{f^{-1}W}} \\ \Big| & & \Big| \\ Y' & \longleftarrow & Y'_{\mathbb{R}} \sqcup Y'_{\widehat{W}} \end{array}$$

is a pullback of the proper left map by the flat bottom map. Thus, by the base change formula for perfect complexes there is an induced commutative square of stable ∞-categories and exact functors

$$\begin{array}{ccc} \mathrm{Perf}(X') & \longrightarrow & \mathrm{Perf}(X'_{\mathbb{R}} \sqcup X'_{\widehat{Z}}) \\ \Big| & & \Big| \\ \mathrm{Perf}(Y') & \longrightarrow & \mathrm{Perf}(Y'_{\mathbb{R}} \sqcup Y'_{\widehat{W}}), \end{array}$$

where the horizontal maps are pullbacks, the left vertical map is pushforward, and the right vertical map is pullback to $X'_{\mathbb{R}} \sqcup X'_{\widehat{f^{-1}W}}$ followed by pushforward. Taking K-theory and passing to horizontal homotopy fibers gives us the desired map $K_c(X) \to K_c(Y)$, and functoriality is evident. $\square$

Another natural structure carried by $K_c(X)$ is that it is a module over $K(X)$, this latter getting its ring structure from the tensor product of vector bundles. Here is the portion of this structure that we will actually use.

Proposition 2.2.3. *Let X be an arithmetic curve or point. Then there is a canonical "action" map*

$$K(X) \wedge K_c(X) \to K_c(X).$$

Furthermore, this action is compatible with the functoriality of K_c, in the sense that if $f : X \to Y$ is a map, then the diagram

$$\begin{tikzcd} K(Y) \wedge K_c(X) \arrow[r] \arrow[d] & K_c(X) \arrow[d] \\ K(Y) \wedge K_c(Y) \arrow[r] & K_c(Y) \end{tikzcd}$$

commutes up to a canonical homotopy. Here the vertical maps come from the functoriality of K_c, the bottom map is the action map for Y, and the top map is the composition of the action map for X with the pullback $f^* : K(Y) \to K(X)$.

Proof. The starting point is that for any scheme S, the tensor product of perfect complexes

$$\mathrm{Perf}(S) \otimes \mathrm{Perf}(S) \to \mathrm{Perf}(S)$$

induces a map of spectra

$$K(S) \wedge K(S) \to K(S),$$

i.e. an action of $K(S)$ on itself. More generally, for any scheme T over S we get an action of $K(S)$ on $K(T)$ by pulling back to T and then taking the tensor product. Furthermore, this is functorial in T/X.

Letting $S = X'$, and taking $T = X'$ then $T = X'_{\mathbb{R}} \sqcup X'_{\hat{Z}}$, we deduce from this an action of $K(X')$ on $K_c(X) = Fib\left(K(X') \to K(X'_{\mathbb{R}} \sqcup X'_{\hat{Z}})\right)$. Then by virtue of the localization sequence

$$K(Z) \xrightarrow{i_*} K(X') \xrightarrow{j^*} K(X),$$

to make this extend to an action of $K(X)$ we just need to trivialize it on restriction to $K(Z)$. However, from the base-change formula for perfect complexes it follows that for a flat scheme T/X', the action of $K(Z)$ on $K(T)$ via i_* canonically factors through $K(T_Z)$, where by T_Z we mean the fiber product of T and Z over X'. Thus the action of $K(Z)$ on $K_c(X) = Fib\left(K(X') \to K(X'_{\mathbb{R}} \sqcup X'_{\hat{Z}})\right)$ via i_* canonically factors through

$$Fib\left(K((X')_Z) \to K((X'_{\mathbb{R}} \sqcup X'_{\hat{Z}})_Z)\right).$$

But this homotopy fiber vanishes, since both $(X')_Z$ and $(X'_{\mathbb{R}} \sqcup X'_{\hat{Z}})_Z$ compatibly identify with Z. So the action of $K(Z)$ on $K_c(X)$ is trivialized, and hence the action of $K(X')$ on $K_c(X)$ canonically extends to an action of $K(X)$ on $K_c(X)$, as desired.

Similarly, the desired interaction between this action and the functoriality of K_c is obvious on the level of $K(Y')$. Then to make it extend to $K(Y)$, we need only make sure it's compatible with the above trivializations on restriction to $K(W)$, where $W = Y' - Y$. But this follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Perf}(Z) \arrow[d, dash] & \mathrm{Perf}(X') \arrow[l] \arrow[d, dash] \\
\mathrm{Perf}(W) & \mathrm{Perf}(Y'), \arrow[l]
\end{tikzcd}$$

where the horizontal maps are pullbacks, the right vertical map is pushforward, and the left vertical map is restriction to $f'^{-1}(W)$ followed by pushforward. □

2.3 Extracting spheres from manifolds

The purpose of this section is to prove a topological result (Theorem 2.3.1) which will be required for the construction of the j-maps of Theorem 1.2.3. This material follows the well-known philosophy that duality on smooth manifolds is controlled by the one-point compactification operation $V \mapsto S^V$ on real vector spaces. Here we implicitly view this operation as landing in spectra Sp, by letting S^V denote the suspension spectrum of the usual one-point compactification of V.

Some more notation: for a space X we let $C_\cdot(X)$ denote the suspension spectrum of X_+, and we let $C^\cdot(X)$ denote the spectrum of maps from X_+ to the the sphere spectrum S. Note that $C^\cdot(X)$ is a commutative ring spectrum, and $C_\cdot(X)$ is a module over $C^\cdot(X)$. Furthermore, if the space X happens to be pointed, then $C^\cdot(X)$ acquires an augmentation $C^\cdot(X) \to S$. Also, in contrast to the previous section, now we will use the symbol $\otimes$ for the smash product of spectra.

The point of the following theorem will be that for a certain class of pointed manifolds $x \in M$, the sphere S^{T_xM} of the tangent space T_xM can be canonically constructed just from the space M, even when M is taken up to some weak notion of equivalence like stable homotopy equivalence.

Theorem 2.3.1. *Let M be a pointed compact stably parallelizable smooth manifold. Then there is a natural equivalence of spectra*

$$C_\cdot(M) \otimes_{C^\cdot(M)} S \simeq S^{T_xM},$$

where T_xM denotes the tangent space to M at the given point $x \in M$.

In fact, both sides of this equivalence are symmetric monoidal functors

$$(SMan_*, \times) \to (Sp, \otimes),$$

and the above equivalence will respect this structure. Here $SMan_*$ stands for the groupoid of pointed compact stably parallelizable smooth manifolds and isomorphisms between them, and $\times$ stands for cartesian product of such manifolds.

Proof. The first step is to extend the claim, allowing M to have boundary (or even corners), provided that x lies in the interior of M. For this we need only replace the copy of $C_\cdot(M)$ on the left by its relative version $C_\cdot(M, \partial M)$. The gain is that we can now localize: for any neighborhood $N \subset M$ of x the "collapse" $M/\partial M \to N/\partial N$ provides us with a natural restriction map

$$C_\cdot(M, \partial M) \otimes_{C^\cdot(M)} S \to C_\cdot(N, \partial N) \otimes_{C^\cdot(N)} S.$$

Then there are two claims to finish the proof:

1. For every N this map is an equivalence;
2. For a cofinal collection of N's the right-hand side is canonically equivalent to S^{T_xM}.

To prove the first claim, choose a stable parallelization of M and thereby a fundamental class $[M] : S^m \to C_\cdot(M, \partial M)$. According to Atiyah duality ([A]), multiplication by $[M]$ defines an equivalence of $C^\cdot(M)$-modules $C_\cdot(M, \partial M) \simeq \Sigma^m C^\cdot(M)$, and therefore an equivalence of spectra

$$C_\cdot(M, \partial M) \otimes_{C^\cdot(M)} S \simeq S^m.$$

However, since a fundamental class of $M/\partial M$ naturally collapses to a fudamental class of $N/\partial N$, we also get the analogous identification $C_\cdot(N, \partial N) \otimes_{C^\cdot(N)} S \simeq S^m$, and in terms of these identifications the restriction map is the identity $S^m \to S^m$ — thus an equivalence.

To prove the second claim, make a (contractible) choice of Riemannian metric on M, and let N run over all balls around x inside the injectivity radius of the exponential map $exp : T_xM \to M$. Then since N is contractible, the augmentation $C^\cdot(N) \to S$ is an equivalence and so $C_\cdot(N, \partial N) \otimes_{C^\cdot(N)} S \simeq C_\cdot(N, \partial N)$; but on the other hand $S^{T_xM} \simeq C_\cdot(N, \partial N)$ by collapsing along exp. Thus we get the claimed equivalence. $\square$

Note that if we had assumed M only to be orientable instead of stably parallelizable, then using Poincaré duality instead of Atiyah duality we would see that this theorem still holds, except that one gets a homology equivalence rather than a homotopy equivalence. But this is not a big loss, since we could still functorially recover the right-hand side from the left-hand side by $H\mathbb{Z}$-localization.

2.4 Construction of the j-maps

In this section we will prove Theorem 1.2.3. Let ℓ be a fixed prime. Then for every arithmetic curve or point X on which ℓ is invertible, we need to define a map of spectra $j_X : K_c(X) \to L_{K/\ell}S$, in such a way that the functoriality (j.1) and point-modification (j.2) properties hold.

2.4.1 A more refined construction involving ℓ-adic spheres

Actually, we will even produce maps with a more refined target:

$$J_X : K_c(X) \to \Sigma^{-1} Sp_{\hat{\ell}}^{\times}.$$

Here $Sp_{\hat{\ell}}^{\times}$ is a spectrum defined as follows. Consider the ∞-category $Sp_{\hat{\ell}}$ of ℓ-adic spectra, by which we mean the Bousfield localization of spectra with respect to a (mod ℓ) Moore spectrum ([Bo]). (We will denote the resulting Bousfield localization functor by $(-)_{\hat{\ell}} : Sp \to Sp_{\hat{\ell}}$, and call it ℓ-adic completion.) This ∞-category $Sp_{\hat{\ell}}$ carries a canonical symmetric monoidal structure, namely the ℓ-adic completion of the smash product on spectra Sp. Therefore, if we restrict to invertible objects in $Sp_{\hat{\ell}}$ and invertible morphisms between these objects, what we get is a group-like symmetric monoidal ∞-groupoid, or a group-like E_∞-space. Such an object canonically deloops to a connective spectrum ([S]), and it is this spectrum which we denote by $Sp_{\hat{\ell}}^{\times}$.

Thus, informally speaking, $Sp_{\hat{\ell}}^{\times}$ is the spectrum which classifies smash-invertible ℓ-adic spectra. In particular, we have

$$\pi_0 Sp_{\hat{\ell}} = \mathbb{Z},$$

since the smash-invertible ℓ-adic spectra are exactly the ℓ-adic spheres $(S^n)_{\hat{\ell}}$ for $n \in \mathbb{Z}$; and

$$\pi_1 Sp_{\hat{\ell}} = \mathbb{Z}_\ell^\times,$$

since group of automorphisms of $S_{\hat{\ell}}$ up to homotopy is this group of ℓ-adic units. More generally, the loop space $\Omega\Omega^\infty Sp_{\hat{\ell}}^\times$ identifies with $Aut(S_{\hat{\ell}})$, the space of automorphisms of $S_{\hat{\ell}}$. This space is just a union of components of $Map(S_{\hat{\ell}}, S_{\hat{\ell}}) \simeq \Omega^\infty S_{\hat{\ell}}$; it follows that the higher homotopy groups of $Sp_{\hat{\ell}}$ are a shifted copy of the ℓ-stable stem $(\pi_*^s)_{(\ell)}$. Thus the new target $\Sigma^{-1} Sp_{\hat{\ell}}^\times$ is more topologically intuitive, but less computationally tractable, than the old target $L_{K/\ell}S$.

The connection between these two targets is as follows. Since $\Omega^\infty\Sigma^{-1}Sp_{\hat{\ell}}$ and $\Omega^\infty S_{\hat{\ell}}$ have canonically equivalent connected components (by translating the identity map $S_{\hat{\ell}} \to S_{\hat{\ell}}$ to the zero map $S_{\hat{\ell}} \to S_{\hat{\ell}}$ using subtraction), it follows from the existence of the Bousfield-Kuhn functor (see [Bo2] in this case) that there is a canonical equivalence of spectra

$$L_{K/\ell}(\Sigma^{-1}Sp_{\hat{\ell}}) \simeq L_{K/\ell}(S_{\hat{\ell}}) = L_{K/\ell}S,$$

and thus a canonical map

$$log : \Sigma^{-1}Sp_{\hat{\ell}} \to L_{K/\ell}S.$$

This map has been comprehensively studied by Rezk in [R]; in particular, a consequence of the formula given in Theorem 1.9 of that paper is that the homomorphism $\pi_0 log$ identifies with

$$(1-\frac{1}{\ell})log(-) : \mathbb{Z}_\ell^\times \to \mathbb{Z}_\ell.$$

(Note that when $\ell = 2$, the group $\pi_0 L_{K/\ell}S$ is larger than $\mathbb{Z}_2$: in fact it is $\mathbb{Z}_2 \oplus \mathbb{Z}/2$. In this case we only mean to claim that the projection of $\pi_0 log$ onto the first factor is given by $(1-\frac{1}{\ell})log(-)$. As it turns out the projection onto the second factor vanishes, but this doesn't follow directly from Rezk's work.)

In summary, instead of producing the maps $j_X : K_c(X) \to L_{K/\ell}S$ satisfying properties (j.1) and (j.2), we can instead produce maps $J_X : K_c(X) \to \Sigma^{-1}Sp_{\hat{\ell}}$ satisfying properties

(J.1) J_X is covariantly functorial in X;

(J.2) For all primes $p \neq \ell$, the map $J_{\mathbb{F}_p} : K(\mathbb{F}_p) \to \Sigma^{-1}Sp^{\times}_{\hat{\ell}}$ sends the unit class $[\mathbb{F}_p] \in \pi_0 K(\mathbb{F}_p)$ to the class $p \in \mathbb{Z}^{\times}_{\ell} = \pi_0\Sigma^{-1}Sp^{\times}_{\hat{\ell}}$.

Then by setting $j_X = log \circ J_X$ we will fulfill the needs (j.1) and (j.2) of Theorem 1.2.3.

A side remark: it's reasonable to ask for a more refined verstion of (J.2), one which identifies the full map of spectra $J_{\mathbb{F}_p}$ in topological terms. (Such a refinement would be unnecessary in the case of $j_{\mathbb{F}_p}$: as we will see later, $j_{\mathbb{F}_p}$ is completely determined by what it does on π_0). In fact such a refinement does exist, and was given in [C].

2.4.2 The base case: $\mathbb{Z}[1/\ell]$

We start with the case $X = Spec(\mathbb{Z}[1/\ell])$. Thus in this subsection we will produce the map

$$J_{\mathbb{Z}[1/\ell]} : K_c(\mathbb{Z}[1/\ell]) \to \Sigma^{-1}Sp^{\times}_{\hat{\ell}}.$$

By definition, $K_c(\mathbb{Z}[1/\ell]) = Fib(K(\mathbb{Z}) \to K(\mathbb{Z}_\ell) \oplus K(\mathbb{R}))$. Equivalently, we can say that $\Sigma K_c(\mathbb{Z}[1/\ell])$ fits into a canonical pushout square

$$\begin{array}{ccc} K(\mathbb{Z}) & \longrightarrow & K(\mathbb{Z}_\ell) \\ \big| & & \big| \\ K(\mathbb{R}) & \longrightarrow & \Sigma K_c(\mathbb{Z}[1/\ell]). \end{array}$$

Thus, giving a map $K_c(\mathbb{Z}[1/\ell]) \to \Sigma^{-1}Sp_{\hat{\ell}}$ is equivalent to giving the following data:

1. A map $K(\mathbb{Z}) \to Sp_{\hat{\ell}}$;

2. A factoring of this map through $K(\mathbb{R})$;

3. A separate factoring of this map through $K(\mathbb{Z}_\ell)$.

Now, for the purposes of producing this data it is no longer convenient to have our model of K-theory be based on perfect complexes. Thus, for a commutative ring R, let $K^\oplus(R)$ denote the connective spectrum defined, as in [S], by taking the group completion of the symmetric monoidal groupoid of finitely-generated projective R-modules under direct sum. There is a natural map

$$K^\oplus(R) \to K(R) = K(Spec(R)),$$

and this map is an equivalence ([Gr]). Thus we can use $K^\oplus(-)$ instead of $K(-)$ above. Then we produce the required data as follows (compare with [K] and [B] applied to tori):

1. The map $K^\oplus(\mathbb{Z}) \to Sp^\times_{\hat{\ell}}$ is defined by sending a finite free $\mathbb{Z}$-module A to the ℓ-adic completion of the sphere

$$Sph(\mathbb{T}_A) := C.(\mathbb{T}_A) \otimes_{C^\cdot(\mathbb{T}_A)} S$$

of Theorem 2.3.1, where $\mathbb{T}_A$ denotes the torus $A \otimes (\mathbb{R}/\mathbb{Z})$ pointed by 0. This association $A \mapsto Sph(\mathbb{T}_A)_{\hat{\ell}}$ takes direct sum of $\mathbb{Z}$-modules to ℓ-adic smash product of ℓ-adic spheres, thus defining the required map of spectra $K^\oplus(\mathbb{Z}) \to Sp^\times_{\hat{\ell}}$.

2. The factoring of this map through $K^\oplus(\mathbb{R})$ comes from Theorem 2.3.1, which identifies

$$Sph(\mathbb{T}_A) \simeq S^{A \otimes \mathbb{R}},$$

since the tangent space to $\mathbb{T}_A$ at 0 canonically identifies with $A \otimes \mathbb{R}$.

3. The factoring of this map through $K^\oplus(\mathbb{Z}_\ell)$ comes from the fact that the natural map $\mathbb{T}_A \simeq BA \to B(A \otimes \mathbb{Z}_\ell)$ is a (mod ℓ) homology equivalence, and so the ℓ-adic completion of $Sph(\mathbb{T}_A)$ can be accessed as an analogous functor $Sph_{\hat{\ell}}(B(A \otimes \mathbb{Z}_\ell))$ of $A \otimes \mathbb{Z}_\ell$.

This finishes the construction of $J_{\mathbb{Z}[1/\ell]} : K_c(\mathbb{Z}[1/\ell]) \to \Sigma^{-1} Sp^\times_{\hat{\ell}}$. To recap, we can describe it loosely as follows. A point in the source $K_c(\mathbb{Z}[1/\ell])$ can be given by a point $[A]$ in $K(\mathbb{Z})$

together with both an ℓ-adic and a real trivialization. Then consider the chain of equivalences

$$S_{\hat{\ell}} \simeq (S^{A\otimes\mathbb{R}})_{\hat{\ell}} \simeq Sph(\mathbb{T}_A)_{\hat{\ell}} \simeq Sph_{\hat{\ell}}(B(A\otimes\mathbb{Z}_\ell)) \simeq S_{\hat{\ell}}.$$

Here the outer equivalences come from the real and ℓ-adic trivializations of $[A]$, and the middle equivalences come from items 2 and 3 above. Then the automorphism of $S_{\hat{\ell}}$ given by composing this chain of equivalences defines the required point of the target $\Sigma^{-1}Sp_{\hat{\ell}}^{\times}$.

2.4.3 The case of general X

Let X now be an arbitrary arithmetic curve or point on which ℓ is invertible. Then there is a unique map $X \to Spec(\mathbb{Z}[1/\ell])$, and hence, by functoriality of $K_c(-)$, a canonical map $K_c(X) \to K_c(\mathbb{Z}[1/\ell])$. We define J_X to be the composition of $J_{\mathbb{Z}[1/\ell]}$ with this map. Thus, the functoriality property (J.1) holds by definition.

2.4.4 The calculation on points

To finish the proof of Theorem 1.2.3 we need to prove the property (J.2), which says that for a prime $p \neq \ell$, the map

$$J_{\mathbb{F}_p} : K(\mathbb{F}_p) \to \Sigma^{-1}Sp_{\hat{\ell}}^{\times}$$

sends the unit class $[\mathbb{F}_p] \in K_0(\mathbb{F}_p)$ to the number $p \in \mathbb{Z}_\ell^\times = \pi_0\Sigma^{-1}Sp_{\hat{\ell}}$.

Now, $J_{\mathbb{F}_p}$ is by definition the composition

$$K(\mathbb{F}_p) \xrightarrow{K_c(-)} K_c(\mathbb{Z}[1/\ell]) \xrightarrow{J_{\mathbb{Z}[1/\ell]}} \Sigma^{-1}Sp_{\hat{\ell}}^{\times},$$

so we will just trace through where $[\mathbb{F}_p]$ goes under π_0 of each of these maps.

To trace $[\mathbb{F}_p]$ under $K_c(-)$ it is convenient to redescribe $\pi_0K_c(\mathbb{Z}[1/\ell])$ as π_0 of the fiber of the map on zeroth spaces of

$$K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}_\ell) \oplus K(\mathbb{R})$$

over the *unit* element $([\mathbb{Z}_\ell], [\mathbb{R}])$ instead of over the 0 element. In these terms, the image of $[\mathbb{F}_p]$ in $\pi_0 K_c(\mathbb{Z}[1/\ell])$ identifies with the class of the finite free $\mathbb{Z}$-module $\mathbb{Z}$, with trivializations $\mathbb{Z} \otimes \mathbb{Z}_\ell \simeq \mathbb{Z}_\ell$ and $\mathbb{Z} \otimes \mathbb{R} \simeq \mathbb{R}$ both given by multiplication by p.

Following this class in $\pi_0 K_c(\mathbb{Z}[1/\ell])$ under the next map $J_{\mathbb{Z}[1/\ell]}$, we arrive at the element in $\pi_0 \Sigma^{-1} Sp^\times_{\hat{\ell}}$ corresponding to the automorphism of $S^1_{\hat{\ell}}$ obtained by composing the map $S^{\mathbb{R}} \to S^{\mathbb{R}}$ induced by multiplication by p^{-1} on $\mathbb{R}$ with the map $Sph_{\hat{\ell}}(B\mathbb{Z}_\ell) \to Sph_{\hat{\ell}}(B\mathbb{Z}_\ell)$ induced by multiplication by p on $\mathbb{Z}_\ell$. The first map is the identity, since p^{-1} connects to 1 by a path in $\mathbb{R}^\times$; however the second map is of degree p, since it identifies with the multiplication by p map on S^1. Thus the image of $[\mathbb{F}_p]$ in $\pi_1 Sp^\times_{\hat{\ell}} \simeq \pi_0 S^\times_{\hat{\ell}} \simeq \mathbb{Z}^\times_\ell$ is just p, as desired.

This finishes the proof of properties (J.1) and (J.2) of the maps J_X, and therefore the proof of our main theorem (Theorem 1.2.3).

2.5 A homotopy-theoretic description of the global Artin map

In this section we will see that the existence of the maps j_X as in Theorem 1.2.3 more-or-less implies the Artin reciprocity law.

2.5.1 The Artin reciprocity law

We start by giving a statement of the Artin reciprocity law. Let X be an arithmetic curve, assumed for simplicity to be flat over $\mathbb{Z}$ (number field case). To X we can associate two abelian groups which, though of a different nature, share remarkable features:

1. (Galois side) Let $H_1(X_{et})$ denote the profinite abelian group which classifies finite abelian covers of X. To every closed point $x \in X$ we can associate a canonical element $Frob_x$ of $H_1(X_{et})$, namely the image of the Frobenius under the map

$$Gal(\overline{\mathbb{F}_x}/\mathbb{F}_x) \simeq H_1(x_{et}) \to H_1(X_{et})$$

induced by the residue field inclusion $x \to X$. Note that $Frob_x$ acts trivially on a given cover if and only x splits completely in that cover. Since only the trivial cover is split completely everywhere, it follows that the $Frob_x$ generate $H_1(X_{et})$ as a profinite group.

2. (Bundle side) Let $Pic_c(X)$ denote the abelian group of isomorphism classes of line bundles on X' together with a trivialization on $X'_{\mathbb{R}} \sqcup X'_{\widehat{\mathbb{Z}}}$ (c.f. Definition 2.2.1). To every closed point $x \in X$ we can associate a canonical element $\mathcal{O}(x)$ of $Pic_c(X)$, namely the line bundle of rational functions with at worst a simple pole at x, the required trivialization being given by the inclusion of the structure sheaf. It follows from the "weak approximation theorem" that these $\mathcal{O}(x)$ generate the profinite completion of $Pic_c(X)$ as a profinite group.

The Artin reciprocity law performs the remarkable task of connecting these two sides:

Theorem 2.5.1. *Let X be an arithmetic curve, flat over $\mathbb{Z}$. Then there exists a homomorphism $Art : Pic_c(X) \to H_1(X_{et})$ such that $Art(\mathcal{O}(x)) = Frob_x$ for every closed point $x \in X$.*

Such a homomorphism is necessarily unique, since the $\mathcal{O}(x)$ generate $Pic_c(X)$ in the profinite sense. So the content in Theorem 2.5.1 is that relations among the $\mathcal{O}(x)$ imply relations among the $Frob_x$. For example, it is a consequence of Theorem 2.5.1 that, given any finite abelian cover $\widetilde{X} \to X$, the question of whether a closed point $x \in X$ splits in $\widetilde{X}$ can be answered by a congruence condition on x.

This is not a standard formulation of the Artin reciprocity law. However, open quotients of $H_1(X_{et})$ correspond to finite abelian extensions of the fraction field of X which are unramified at all closed points of X, and from this it is easy to see that Theorem 2.5.1 implies the "existence of a conductor" for the Artin map, this being a standard formulation of the Artin reciprocity law. See Lang's book [L] for this standard formulation, and for an explication of what is meant by "congruence condition" above.

2.5.2 An ℓ-adic analog

Now suppose that ℓ is a prime, assumed to be odd for simplicity. Our goal for the rest of the paper will be to deduce from our main theorem (1.2.3) the following ℓ-adic analog of Theorem 2.5.1:

Theorem 2.5.2. *Let X be an arithmetic curve, flat over $\mathbb{Z}[1/\ell]$. Then there exists a homomorphism $Art : Pic_c(X) \to H_1(X_{et};\mathbb{Z}_\ell)$ such that $Art(\mathcal{O}(x)) = Frob_x$ for every closed point $x \in X$.*

Thus Theorem 2.5.2 differs from Theorem 2.5.1 in that we require ℓ to be invertible on X, and we replace $H_1(X_{et})$ by its ℓ-adic completion. Actually, somewhat surprisingly, this weaker Theorem 2.5.2 implies the full Theorem 2.5.1 (away from 2), the reason being that the conclusion of Theorem 2.5.1 holds for an X whenever it holds for a nonempty open subset of X. But let's not concern ourselves with that here, and only focus on deducing Theorem 2.5.2 from Theorem 1.2.3.

The crucial idea is to use the action map $K(X) \wedge K_c(X) \to K_c(X)$ of Proposition 2.2.3 to promote the map $j_X : K_c(X) \to L_{K/\ell}S$ to a "duality" map

$$\mathbb{D}_X : K_c(X) \to Map(K(X), L_{K/\ell}S),$$

where on the right we mean the spectrum of maps from $K(X)$ to $L_{K/\ell}S$. Now the source spectrum $K_c(X)$ evidently lives on the bundle side — for example, sending $\mathcal{L}$ to $\mathcal{L}-\mathcal{O}$ defines an isomorphism $Pic_c(X) \simeq \pi_0 K_c(X)$ — so the game will be to see that the target spectrum $Map(K(X), L_{K/\ell}S)$ has a secret life on the Galois side. (In the end, this will follow from the work [T] of Thomason.) More particularly, we will show the following theorem, whose third point produces from $\mathbb{D}_X$ a homomorphism $Pic_c(X) \to H_1(X_{et};\mathbb{Z}_\ell)$ verifying Theorem 2.5.2.

Theorem 2.5.3. *Let ℓ be an odd prime, and let X be an arithmetic curve or point on which*

ℓ is invertible. Then there is a natural isomorphism

$$H_1(X_{et};\mathbb{Z}_\ell) \xrightarrow{\sim} [K(X), L_{K/\ell}S].$$

Furthermore, if X is a point, then the resulting composition

$$\mathbb{Z} = \pi_0 K_c(X) \xrightarrow{\pi_0\mathbb{D}_X} [K(X), L_{K/\ell}S] \xleftarrow{\sim} H_1(X_{et};\mathbb{Z}_\ell)$$

sends 1 to the Frobenius generator of $H_1(X_{et};\mathbb{Z}_\ell)$; and if X is a curve, then the composition

$$Pic_c(X) \to \pi_0 K_c(X) \xrightarrow{\pi_0\mathbb{D}_X} [K(X), L_{K/\ell}S] \xleftarrow{\sim} H_1(X_{et};\mathbb{Z}_\ell)$$

sends $\mathcal{O}(x)$ to $Frob_x$.

We note right away that the third statement (about curves X) follows immediately from the second statement (about points X), by the functoriality of K_c established in Proposition 2.2.3. So we will only worry about the first two statements.

The deduction of Theorem 2.5.3 from Theorem 1.2.3, which will occupy the rest of this thesis, surprisingly makes very little use of number theory. One ingredient (Proposition 2.5.4) is pure homotopy theory, and another (Thomason's theorem [T]) is pure K-theory of schemes. The only fact we require from number theory is that the (mod ℓ) etale cohomological dimension of X is ≤ 2 (see [Se]). Admittedly, this fact is usually tied up with the development of class field theory.

We start the proof with a more general discussion of homotopy classes of maps $[Z, L_{K/\ell}S]$, where Z is an arbitrary spectrum. The first thing to mention is that Z can always be assumed to be K/ℓ-local. And in that case, we have the following result.

Proposition 2.5.4. *Let ℓ be an odd prime, and Z a K/ℓ-local spectrum. Then the abelian group $[Z, L_{K/\ell}S]$ carries a natural ℓ-profinite topology, and moreover there is a functorial isomorphism*

$$[Z, L_{K/\ell}S] \simeq (\mathrm{colim}_n\, \pi_{-1} Z/\ell^n)^*.$$

Here $(-)^*$ means Pontryagin dual, and the colimit is taken along the maps $Z/\ell \to Z/\ell^2 \to \ldots$ given by multiplication by ℓ.

This is an expression of the calculation of the Brown-Comenetz dual of $L_{K/\ell}S$; compare [HM].

Proof. The natural topology on $[Z, L_{K/\ell}S]$ comes from writing Z is a filtered colimit of finite spectra, and $L_{K/\ell}S$ as the inverse limit of the $L_{K/\ell}S/\ell^n$. It is ℓ-profinite since the homotopy groups of this latter spectrum are all finite ℓ-groups. Given this topology, there is an evident natural continuous pairing

$$[Z, L_{K/\ell}S] \otimes \mathrm{colim}_n\, \pi_{-1}Z/\ell^n \to \mathrm{colim}_n\, \pi_{-1}L_{K/\ell}S/\ell^n.$$

Thus it suffices to see that $\mathrm{colim}_n\, \pi_{-1}L_{K/\ell}S/\ell^n$ identifies with $\mathbb{Z}/\ell^\infty$, and that this is a perfect pairing. The first claim is an easy calculation from the standard fiber sequence $L_{K/\ell}S \to K_{\hat{\ell}} \xrightarrow{\psi^g-1} K_{\hat{\ell}}$ (see [Bo]), where g is a chosen generator of $\mathbb{Z}_\ell^\times$. As for the second claim, note that the adjoint map

$$\mathrm{colim}_n\, \pi_{-1}Z/\ell^n \to Hom_{cont}([Z, L_{K/\ell}S], \mathbb{Z}/\ell^\infty).$$

defines a natural transformation of K/ℓ-local homology theories (for the right-hand side, this is clear; for the left-hand side one needs to remember that, once taken (mod ℓ^n), filtered colimits in $L_{K/\ell}Sp$ are the same as in Sp). Thus it suffices to check that this natural transformation is an isomorphism when Z is a shift of $L_{K/\ell}S$. And again, this is a simple calculation from the fiber sequence $L_{K/\ell}S \to K_{\hat{\ell}} \xrightarrow{\psi^g-1} K_{\hat{\ell}}$. □

Note that the proof made use of a choice of generator $g \in \mathbb{Z}_\ell^\times$, and so was non-canonical. In fact, a functorial isomorphism as in Proposition 2.5.4 is unique only up to ℓ-adic units. Let us therefore pin down a particular choice as follows. Taking $Z = \Sigma^{-1}K$, there is a canonical generator

$$j \in [\Sigma^{-1}K, L_{K/\ell}S]$$

given by K/ℓ-localizing the complex J-homomorphism $J_{\mathbb{C}} : ku \to Sp^{\times}$ (defined by $V \mapsto S^V$) and composing with the equivalence $log : L_{K/\ell}Sp^{\times} \simeq \Sigma L_{K/\ell}S$ coming from the Bousfield-Kuhn functor (c.f. Section 2.4.1). On the other hand, there is also a canonical generator of

$$(\mathrm{colim}_n\, \pi_0 K/\ell^n)^* \simeq (\mathrm{colim}_n\, \mathbb{Z}/\ell^n)^* \simeq \mathbb{Z}_\ell,$$

namely 1. We normalize the isomorphism of Proposition 2.5.4 so that these classes match up.

Now, taking $Z = L_{K/\ell}K(X)$ in Proposition 2.5.4, we see that the group $[K(X), L_{K/\ell}S]$ canonically identifies with the Pontryagin dual to

$$\mathrm{colim}_n\, \pi_{-1}\left(L_{K/\ell}K(X)/\ell^n\right).$$

However, Thomason ([T]) showed that $L_{K/\ell}K(X)/\ell^n$ satisfies etale descent in X, and combined this with the Gabber-Suslin calculation $L_{K/\ell}K(R)/\ell^n \simeq K/\ell^n$ for R strictly henselian to give a natural spectral sequence of Atiyah-Hirzebruch style

$$H^p(X_{et}; \mathbb{Z}/\ell^n(q)) \Longrightarrow \pi_{2q-p}\left(L_{K/\ell}K(X)/\ell^n\right),$$

valid for very general X (and now known in even greater generality — and with a cleaner proof — by work of Suslin, Voevodsky, and Rost in motivic cohomology).

Furthermore, Thomason's spectral sequence is functorial in n with respect to the multiplication by ℓ maps. Therefore, we can take colimits and pass to Pontryagin duals to deduce a spectral sequence of ℓ-profinite abelian groups

$$H_p(X_{et}; \mathbb{Z}_\ell(q)) \Longrightarrow \pi_{p+2q-1}Map(K(X), L_{K/\ell}S).$$

(Compare with Mitchell's approach in [M].) Since in our case X has (mod ℓ) etale cohomological dimension ≤ 2, this spectral sequence degenerates, and in particular the natural "edge" map

$$H_1(X_{et};\mathbb{Z}_\ell) \to [K(X), L_{K/\ell}S]$$

is an isomorphism. This proves the first claim of Theorem 2.5.3.

Furthermore, our normalization of the isomorphisms of Proposition 2.5.4 implies the following "concrete" description of this edge map: Let F be the fraction field of X. Then the composition

$$Gal(\overline{F}/F) \to H_1(X_{et};\mathbb{Z}_\ell) \to [K(X), L_{K/\ell}S]$$

identifies with the map which sends a $\sigma \in Gal(\overline{F}/F)$ to the composition

$$K(X) \to K(F) \xrightarrow{J^{et}(\sigma)} \Sigma^{-1}Sp^{\times}_{\hat{\ell}} \xrightarrow{log} L_{K/\ell}S,$$

where the first map is pullback to the generic point, and the second map $J^{et}(\sigma)$ sends a vector space V/F to the automorphism of

$$S_{\hat{\ell}} \simeq S^{\overline{V}} \wedge (S^{\overline{V}})^{-1}$$

given by the action of σ on the first factor and the identity on the second factor. Here $\overline{V}$ stands for the base-change of V to the separable closure $\overline{F}/F$, and $S^{\overline{V}}$ stands for the ℓ-adic homotopy type of the cofiber of varieties

$$\overline{V}/(\overline{V}-0).$$

Thus, to finish the proof of Theorem 2.5.3, we need only see that, when we take $X = Spec(\mathbb{F}_q)$ and $\sigma = Frob$, this composition

$$K(\mathbb{F}_q) \xrightarrow{J^{et}(Frob)} \Sigma^{-1}Sp^{\times}_{\hat{\ell}} \xrightarrow{log} L_{K/\ell}S$$

is homotopic to $j_{\mathbb{F}_q}$. However, we know already that the group of homotopy classes of maps $[K(\mathbb{F}_q), L_{K/\ell}S]$ identifies with $H_1(X_{et};\mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell$; it follows that we need only check that the two maps in question have the same effect on π_0. And in fact, this is true even before composing with log: both $J_{\mathbb{F}_q}$ and $J^{et}(Frob)$ send the unit class to the class $q \in \pi_0\Sigma^{-1}Sp_{\hat{\ell}}^{\times}$. For $J_{\mathbb{F}_q}$, this was verified in Section 2.4.4 when q is prime, and the general case follows by functoriality; and for $J^{et}(Frob)$, it follows from the fact that $S^{\overline{\mathbb{F}_q}} \simeq \Sigma\mathbb{G}_m$, and that the action of Frobenius on the ℓ-adic Tate module of $\mathbb{G}_m$ identifies with multiplication by q.

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