SOME RESULTS RELATED TO THE QUANTUM GEOMETRIC LANGLANDS PROGRAM

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ABSTRACT. One of the fundamental results in geometric representation theory is the geometric Satake equivalence, between the category of spherical perverse sheaves on the affine Grassmannian of a reductive group G and the category of representations of its Langlands dual group. The category of spherical perverse sheaves sits naturally in an equivariant derived category, and this larger category was described in terms of the dual group by Bezrukavnikov-Finkelberg. Recently, Finkelberg-Lysenko proved a "twisted" version of the geometric Satake equivalence, which involves perverse sheaves associated to twisted local systems on a line bundle over the affine Grassmannian.

In this thesis we extend the Bezrukavnikov-Finkelberg description of the equivariant derived category to the twisted setting. Our method builds on theirs, but some additional subtleties arise. In particular, we cannot use Ginzburg's results on equivariant cohomology. We get around this by using localization techniques in equivariant cohomology in a more detailed way, allowing as to reduce certain computations to those of Ginzburg and Bezrukavnikov-Finkelberg.

We also use show how our methods can be extended to explain an equivalence between Iwahori-equivariant peverse sheaves and twisted Iwahori-equivariant perverse sheaves on dual affine Grassmannians. This equivalence was observed earlier by Arkhipov-Bezrukavnikov-Ginzburg by combining several deep results, and they posed the problem of finding a more direct explanation. Finally, we explain how our results fit into the (quantum) geometric Langlands program.

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Contents

1. Introduction

In this thesis, we prove some results related to the geometric Langlands program, more specifically its quantum version at a root of unity/rational parameter. The starting point for our investigation is the twisted geometric Satake equivalence of Finkelberg-Lysenko (also proven and generalized by Reich), which extends the geometric Satake equivalence of Lusztig, Ginzburg, Mirkovic-Vilonen, and Beilinson-Drinfeld. Let us summarize the results of this paper:

In the non-twisted setting, the equivariant derived category containing the Satake category as the heart of the perverse t-structure was described (in terms of the dual group) by Ginzburg and Bezrukavnikov-Finkelberg. Their proof makes use of the action of of the equivariant cohomology of the affine Grassmannian on the equivariant cohomology of an IC-sheaf and a description of these cohomologies in terms of the dual group. In the twisted setting the global cohomology of an IC-sheaf is zero, so the naive generalization of this approach does not work. In the non-twisted case, global cohomology is the fiber functor that gives the geometric Satake equivalence, and equivariant cohomology can be seen as an enhancement of it (since in this case we can recover ordinary cohomology by setting the equivariant parameters to zero), which carries a natural action of the equivariant cohomology of the underlying space. Following a suggestion of Bezrukavnikov, we describe a sheaf, which we denote $\mathcal{F}$, such that $\mathrm{Ext}^\bullet(\mathcal{F},\cdot)$ coincides with the fiber functor of Finkelberg-Lysenko. The functor $\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F},\cdot)$ carries a natural action of $\mathrm{Ext}^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F})$, which allows us to argue along the lines of Bezrukavnikov-Finkelberg.

1.1. Background and notation.

1.1. Background and notation. Throughout this paper G and H will denote split reductive groups over a field, typically the complex numbers. Let $B \supset T$ denote a Borel subgroup and maximal torus of G. $\mathcal{O}$ will denote the ring of formal power series, and $\mathcal{K}$ its fraction field, the field of Laurent series. The affine Grassmannian Gr_G is an ind-scheme whose $\mathbb{C}$-points are $G(\mathcal{K})/G(\mathcal{O})$. Under the left $G(\mathcal{O})$-action, its orbits are indexed by $X_*^+(T)$, the set of dominant coweights of G or equivalently the set of dominant weight of its Langlands dual goup $G^\vee$. A subscript of H or a ' will be used throughout this paper to denote corresponding objects for H. Let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ be the abelian category of $G(\mathcal{O})$ equivariant perverse sheaves on Gr_G with finite dimensional support. In [L1], [G], [MV], [BD], following ideas of Lusztig [L1], $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is given the structure of a tensor (symmetric monoidal) category. Consider the diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G(\mathcal{K}) \times \mathrm{Gr}_G \xrightarrow{q} G(\mathcal{K}) \times^G (\mathcal{O})\mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G$$

Here p is projection $G(\mathcal{K}) \to \mathrm{Gr}_G$ times the identity, $G(\mathcal{K}) \times^G (\mathcal{O})\mathrm{Gr}_G$ is the quotient of $G(\mathcal{K}) \times \mathrm{Gr}_G$ under the the action $h \cdot (g, xG(\mathcal{O})) = (gh^{-1}, hxG(\mathcal{O}))$, q is the quotient map, and m is the multiplication map. If $\mathcal{A}, \mathcal{B} \in \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$, $p^*\mathcal{A} \boxtimes \mathcal{B}$ is $G(\mathcal{O})$-equivariant (for the action $h \cdot (g, xG(\mathcal{O})) = (gh^{-1}, hxG(\mathcal{O})))$, so there is a unique sheaf $\mathcal{A}\tilde{\boxtimes}\mathcal{B}$ on $G(\mathcal{K}) \times^G (\mathcal{O})\mathrm{Gr}_G$ such that $q^*(\mathcal{A}\tilde{\boxtimes}\mathcal{B}) = p^*\mathcal{A} \boxtimes \mathcal{B}$. We define the *convolution* $\mathcal{A} * \mathcal{B}$ to be $m_!(\mathcal{A}\tilde{\boxtimes}\mathcal{B})$. This gives the monoidal structure on $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$.

Our starting point is the geometric Satake equivalence (Lusztig, Drinfeld, Ginzburg Mirkovic-Vilonen),

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(G^\vee)$$

an equivalence of tensor categories. A natural question (posed by Drinfeld) in the framework of the geometric Langlands program is to give a Langlands dual description of the larger equivariant derived category $D^b_{G(\mathcal{O})}(\mathrm{Gr}_G)$. Based on results of Ginzburg ([G1], G2]), this question was answered in [BF], where the a dual description of the loop-equivariant derived category was also given.

Finkelberg and Lysenko [FL] gave a geometric analogue of the Satake isomorphism for metaplectic groups, which they called the twisted geometric Satake equivalence. One of our goals in this thesis is to general the equivalence of [BF] to this setting.

1.2. **Statement of the results of Bezrukavnikov-Finkelberg.** Let G be a split reductive group over $\mathbb{C}$, and $G^\vee$ its Langlands dual. In [BF] the authors prove an equivalence of monoidal categories

$$D^b_{G(\mathcal{O})}(Gr_G) \simeq D^{G^\vee}_{perf}(\mathrm{Sym}^\Box(\mathfrak{g}^\vee))$$

Here, the right hand side is the dg-category of perfect complexes of $G^\vee$ equivariant coherent sheaves on $\mathfrak{g}^{\vee *}$ with trivial differential. In particular their results includes a statement about the dg-structure on the left hand side. The proof builds calculations of Ginzburg [G] to relate the Ext groups in the two categories, as well as a purity argument (also due to Ginzburg [G2]) to deduce the dg-formality.

Note that every object of $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is automatically equivariant with respect to the "loop-rotation" action of $\mathbb{G}_m$, namely the $\mathbb{G}_m$ acts by rescaling the formal parameter t. By extending the Ext calculations to the $G(\mathcal{O}) \rtimes \mathbb{G}_m$ equivariant case, the authors are also able to give a description of the loop equivariant category

$$D^b_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_G) \simeq D^{G^\vee}_{perf}(U^\Box_\hbar)$$

where the right hand side is defined as follows: Let $U_\hbar$ be the "graded enveloping algebra" of $\mathfrak{g}^\vee$, the graded $\mathbb{C}[\hbar]$-algebra generated by $\mathfrak{g}^\vee$ with the relations $xy - yx = \hbar[x,y]$. Then $D^{\mathcal{G}^\vee}_{perf}(U^\Box_\hbar)$ is the subcategory of perfect complexes in the derived category of $G^\vee$-equivariant $U_\hbar$-modules, where $U^\Box_\hbar$ means the algebra is considered as a dg-module with zero differential.

We introduce a related definition: Let $\mathcal{HC}_\hbar$ be the category of finitely generated "$\hbar$-Harish-Chandra bimodules", that is $U_\hbar \otimes U_\hbar$-modules with a compatible G-action. The full subcategory of $\mathcal{HC}_\hbar$ of objects where $\hbar$ acts by zero is equivalent to $Coh^{G^\vee}(\mathfrak{g}^{\vee *})$, the category of coherent sheaves on $\mathfrak{g}^{\vee *}$ equivariant under the coadjoint action, so that $\hbar$ corresponds to the extra loop rotation parameter. Finally, let $\mathcal{HC}_\hbar^{fr}$ be the full subcategory generated by objects of the form $U_\hbar \otimes V$, $V \in \mathrm{Rep}(G^\vee)$. Examining the definitions, one sees that an element $M \in \mathcal{HC}_\hbar$ is the same as a $\mathcal{G}^\vee$-equivariant $U_\hbar^\square$-module

To prove the above equivalence of categories, one needs to examine the Ext groups on both sides. The Ext groups on the left hand side are naturally a module over the equivariant cohomology of Gr_G, so the first the is to compute this in terms of the dual group. We have

Theorem 1.1. *(Theorem 1 of [BF])*

$$H^\bullet_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_G) = \otimes_{\pi_1(G)^\wedge}(Coh^{\mathbb{G}_m}(N_{(\mathfrak{t}^*/W)^2}\Delta))$$

where $Coh^{\mathbb{G}_m}(N_{(\mathfrak{t}^*/W)^2}\Delta)$ is the ring of functions on the deformation to the normal cone of the diagonal in $\mathfrak{t}^{\vee *}/W \times \mathfrak{t}^{\vee *}/W$. The deformation to the normal cone is equipped with a morphism to $\mathbb{A}^1$, such that the fiber over $0 \in \mathbb{A}^1$ is $\mathbf{T}(\mathfrak{t}^{\vee *}/W)$, the total space to the tangent bundle of $\mathfrak{t}^{\vee *}/W$.

Let $\phi : U_\hbar \to \mathbb{C}[\hbar]$ be a non-degenerate character. We extend it to $U_\hbar(\mathfrak{n}_-^\vee)^2$ by letting is be trivial on the second factor. Define the Kostant functor $\kappa_\hbar : \mathcal{HC}_\hbar \to \mathrm{Coh}(\mathfrak{t}^{\vee *}/W \times /T/W \times \mathbb{A}^1)$ by

$$\kappa_\hbar(M) = (M \overset{L}{\otimes}_{U_\hbar(\mathfrak{n}_-^\vee)^2} (-\phi))^{N_-^\vee}$$

To see that the image is an $\mathcal{O}(\mathfrak{t}^{\vee *}/W \times /T/W \times \mathbb{A}^1)$-module, note that $\kappa_\hbar(M)$ has an action of $Z(U_\hbar) \otimes_{\mathbb{C}[\hbar]} Z(U_\hbar)$ and use the Harish-Chandra isomorphism. One can see (see [BF] 2.3) that this action extends to an action of $Coh^{\mathbb{G}_m}(N_{(\mathfrak{t}^*/W)^2}\Delta)$, the ring of functions on the deformation to the normal cone of the diagonal inside $\mathfrak{t}^{\vee *}/W$. We then have

Theorem 1.2. *(Theorem 2 of [BF]) The functor* $S : Rep(G^\vee) \to Perv_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_G)$ *(given by the inverse of the geometric Satake equivalence) extends to a full imbedding* $S_\hbar : \mathcal{HC}_\hbar^{fr} \to D_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_G)$ *such that*

$$\kappa_\hbar \simeq H^\bullet_{G(\mathcal{O})\rtimes\mathbb{G}_m} \circ S_\hbar$$

The extension is unique for each such isomorphism of functors.

In the limit as $\hbar \to 0$, $\kappa_\hbar$ becomes the functor $\kappa : \mathrm{Coh}^{G^\vee\times\mathbb{G}_m}(\mathfrak{g}^{\vee *}\mathrm{Coh}^{\mathbb{G}_m}\mathbf{T}^{\mathbb{G}_m}(\mathfrak{t}^{\vee *}/W))$. Here $\mathbf{T}^{\mathbb{G}_m}(\mathfrak{t}^{\vee *}/W)$ is the total space of the tangent bundle of $\mathfrak{t}^{\vee *}/W$. We then have

Theorem 1.3. *(Theorem 4 of [BF]) The functor* $\widetilde{S} : Rep(G^\vee) \to Perv_{G(\mathcal{O})}(Gr_G)$ *given by the inverse of the geometric Satake equivalence extends to a full imbedding* $\widetilde{S}_{qc}$: $Coh_{fr}^{G^\vee\times\mathbb{G}_m}(\mathfrak{g}^{\vee *}) \to D_{G(\mathcal{O})}(Gr_G)$, *such that*

$$\kappa \simeq H^\bullet_{G(\mathcal{O})} \circ \widetilde{S}_{qc}$$

The extension is unique for each such isomorphism of functors.

1.3. **Statement of the results of Finkelberg-Lysenko.** Let G be a split almost simple group, and let Gr_G be the associated affine Grassmannian. Its Picard group is isomorphic to $\mathbb{Z}$, and it carries a natural $G(\mathcal{O})$-equivariant line bundle *det*, the determinant line bundle associated to the adjoint representation of G [Ku]. It is the $2\check{h}^{th}$ power of an ample generator of $\mathrm{Pic}(\mathrm{Gr}_G)$, where $\check{h}$ is the dual Coxeter number of G. Let Gra_G be the punctured total space of *det*. Fix an integer N, and let ζ be a fixed $2N\check{h}/d^{th}$ order character, where d is the divisor of $\check{h}$ defined below. Let $\mathcal{L}^\zeta$ be the rank one local system on $\mathbb{G}_m$ corresponding to ζ. Finkelberg and Lysenko consider $\mathbb{P}\mathrm{erv}_N$, the category of $G(\mathcal{O})$-equivariant perverse sheaves on Gra_G shifted one degree to the left, which are $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant, and prove

$$\mathbb{P}\mathrm{erv}_N \simeq \mathrm{Rep}(G_N^\vee)$$

Here, $G_N^\vee$ is a split almost simple group which *depends on* N. For example, if $G = SL(2)$, $G_N^\vee = PGL(2)$ if N is odd and $= SL(2)$ if N is even. A comprehensive list of examples if given in [FL]. The proof follows the arguments of [MV] with some modifications. In particular, one can't use $H^\bullet(\cdot)$ as the fiber functor, but one still can do the analogue of integrating along the "semi-infinite" $N(\mathcal{K})$-orbits. The description of the simple objects of $\mathbb{P}\mathrm{erv}_N$ is as follows: Let $\iota : X_*(T) \to X^*(T) \otimes \mathbb{Q}$ be the map induced by the pairing $(\cdot,\cdot) : X_*(T^{sc}) \times X_*(T^{sc}) \to \mathbb{Z}$ such that $(\alpha, \alpha) = 2$ for a short coroot α. Let d be the smallest positive integer such that $d\iota(X_*(T)) \subset X^*(T)$. Let $X_*(T)$ denote the weight lattice of G, and let

$$X^{*+}(T_N^\vee) = \{\lambda \in X_*^+(T) | d\iota(\lambda) \in NX^*(T)\}$$

Then according to [FL] Lemma 2, the orbits Gra_G^λ admits a $G(\mathcal{O})$-equivariant, $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant local system if and only if $\lambda \in X^{*+}(T_N^\vee)$. The is chosen because of $X^*(T_N^\vee)$ turns out to be the weight lattice of $T_N^\vee$, the maximal torus of $G_N^\vee$.

The authors also note the following useful 'adjunction' formula between Ext groups (always in the derived caegory)

$$\mathrm{Ext}^\bullet(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{A}_3) \simeq \mathrm{Ext}^\bullet(\mathcal{A}_1, (\mathcal{A}_2)^\vee * \mathcal{A}_3)$$

Here $(\cdot)^\vee$ is the involution given by the composition of the pullback along inversion map of $G(\mathcal{K})$ and Verdier duality. For example, $(\mathcal{A}_\lambda)^\vee = \mathcal{A}_{-w_0(\lambda)}$.

1.4. **Statement of the results of Arkhipov-Bezrukavnikov-Ginzburg.** Let I be the Iwahori subgroup of $G(\mathcal{K})$, that is the preimage of B under the map $G(\mathcal{O}) \to G$. In

[ABG] the authors consider the category of Gr_G constructible along the stratification by I-orbits, and prove the following chain of equivalences of triangulated categories. Note that in [ABG] the roles of G and $G^\vee$ are reversed

$$D_{Schubert}(\mathrm{Gr}_G) \simeq D^{G^\vee}_{coh}(\widetilde{\mathcal{N}}) \simeq D_{quantum}(\mathfrak{g}^\vee)$$

Here, the middle term is related to the $G^\vee$-equivariant bounded derived category of *coherent* sheaves on the Springer resolution associated to the Langlands dual group $G^\vee$, and the right hand term is the derived category of the principal block of finite dimensional representation of the quantized enveloping algebra of $\mathfrak{g}^\vee$ at an (any) odd root of unity. In addition, they prove that the combined equivalence is compatible with the t-structures on both sides, hence induces an equivalence

$$\mathrm{Perv}_{Schubert}(\mathrm{Gr}_G) \simeq \mathrm{block}(\mathfrak{g}^\vee)$$

of abelian categories. They also note that combined with (deep) results of Kazhdan-Lusztig and Kashiwara-Tanisaki, one gets an equivalence

$$\mathrm{Perv}_{Schubert}(\mathrm{Gr}_G) \simeq \mathrm{Perv}^{\zeta}_{Schubert}(\mathrm{Gra}_{G^\vee})$$

where the right hand side is a definied analogously using $\mathcal{L}^\zeta$-equivariant sheaves. One of our goals is to explain this equivalence only using the methods of [ABG].

1.5. **Main results and strategy of the proof.** Our goal is to extend the result of [BF] to the twisted setting. The method of [BF] relies on looking at the equivariant hypercohomology of the irreducible perverse sheaves as modules over $H^\bullet_{G(\mathcal{O})}(\mathrm{Gr}_G)$ (resp. $H^\bullet_{G(\mathcal{O})\rtimes\mathbb{G}_m}(\mathrm{Gr}_G)$) and computing this module structure in terms of $G^\vee$. However the (equivariant) hypercohomology of an $\mathcal{L}^\zeta$-equivariant sheaf will be zero for trivial reasons. Therefore we have to modify their approach. Notice that the functor $H^\bullet(\cdot)$ is the same as the functor $\mathrm{Ext}^\bullet(\mathbb{C},\cdot)$. The idea, suggested by Bezrukavnikov, is to replace

$\mathbb{C}$ with a suitable $\mathcal{L}^\zeta$ equivariant sheaf defined below, which we denote by $\mathcal{F}$. Then we can analyze the action of $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ on $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{A}_\lambda)$ (and its equivariant versions), but some additional technicalities arise. This is somewhat analogous to integrating a genuine representation against an anti-genuine function in the setting of metaplectic groups.

Recall that $G_N^\vee$ is the twisted dual group of [FL]. Let H be its Langlands dual. Since the $\mathcal{L}^\zeta$-equivariant geometry of Gra_G is related to the representation theory of H, we expect it to behave like the geometry of Gr_H. In fact, we will take the approach of directly comparing Gra_G and GrH, and then deduce our results from the results of [BF]. We will use this frequently without comment to identify weight lattices, equivariant parameters, etc.

Our first main result is

Theorem 1.4. *There are equivalences of DG-categories (here $U_\hbar$ is the graded enveloping algebra for $\mathfrak{g}_N^\vee$)*

$$D^{b,\zeta}_{G(\mathcal{O})}(Gra_G) \simeq D^{G_N^\vee}_{perf}(Sym^\square(\mathfrak{g}_N^\vee))$$

$$D^{b,\zeta}_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gra_G) \simeq D^{G_N^\vee}_{perf}(U_\hbar^\square)$$

Our second main result is a proof of

Theorem 1.5.

$$Perv_{Schubert}(Gr_G) \simeq Perv^\zeta_{Schubert}(Gra_{G^\vee})$$

While this result is already already known, our contribution is to give a proof completely within the framework of [ABG]. In particular we don't have to appeal to the deep results of Kahiwara-Tanisaki and Kazhdan-Lusztig.

Note that $\mathcal{L}^\zeta$-equivariant sheaves are (by definition) not equivariant with respect to the natural $\mathbb{G}_m$ action on the fibers of Gra_G, but they are equivariant with respect to the N^{th} power of this action. All Ext groups between such sheaves are assumed to be equivariant with respect to this latter action, which will be suppressed from our notation (we do this to avoid confusion with the $\mathbb{G}_m$ loop equivariance). All other equivariances will be made explicit.

1.6. **Localization in equivariant cohomology.** We will make frequent use of the localization techniques in equivariant cohomology, based on the method of Chang and Skjelbred . Our reference for this is [GKM1], though many of the ideas go back much further (see [GKM1] for the history and references). Let X be a space with the action of a torus T. Let $\mathcal{A} \in D_T(X)$. Then $H_T^\bullet(X;\mathcal{A})$ is naturally a module over S, the equivariant cohomology ring of a point. If $H_T^\bullet(X;\mathcal{A})$ is a free S module - in this case we say $\mathcal{A}$ is "equivariantly formal" - the localization theorem takes a particularly nice form. Let X_0 be the fixed point set of T acting on X, and let X_1 be the union of the fixed points and 1-dimensional T-orbits.

Theorem 1.6. *([GKM1] Theorem 6.3) Suppose $\mathcal{A}$ is equivariant formal. Then the following sequence of S-modules*

$$0 \to H_T^\bullet(X;A) \to H_T^\bullet(X_0,\mathcal{A}) \to H_T^\bullet(X_1,X_0;\mathcal{A})$$

is exact.

Note that for general $\mathcal{A}$, if we tensor with the fraction field of S, the last term disappears and we have an isomorphism

$$H_T^\bullet(X;A) \otimes_S \mathrm{Frac}(S) \to H_T^\bullet(X_0,\mathcal{A}) \otimes_S \mathrm{Frac}(S)$$

however inverted the equivariant parameters a lot of information. For reason reason equivariantly formal sheaves are especially nice to work with. Equivariant formality holds for a $\mathcal{A}$ pure, and for $\mathcal{A}$ such that $H^\bullet(X;A)$ vanishes in odd degrees. The latter condition holds for example when $\mathcal{A}$ is constructible along a stratification by affine linear spaces, and the cohomology sheaves of $\mathcal{A}$ restricted to any stratum vanish in a given parity. For a an exhaustive list of sufficient condition for equivariant formality, see [GKM1] Section 14.

We will need to apply this idea below to groups of the form $\mathrm{Ext}^\bullet_T(\mathcal{A}, \mathcal{B})$ between two T equivariant sheaves. Suppose that $\mathrm{Ext}^\bullet_T(\mathcal{A}, B)$ is a free S-module. For example this holds if $\mathrm{Ext}^\bullet(\mathcal{A}, \mathcal{B}) = H^\bullet \mathcal{E}\mathrm{xt}(\mathcal{A}, \mathcal{B})$ vanishes in odd degrees, which will always hold in the cases we consider. Then the proof of [GKM1] Theorem 6.3 can be adapted to show

$$0 \to \mathrm{Ext}^\bullet_T(\mathcal{A}, B) \to \mathrm{Ext}^\bullet_T(X_0; \mathcal{A}, B) \to \mathrm{Ext}^\bullet_T(X_1, X_0; \mathcal{A}, B)$$

is exact. For a sheaf-theotretic definition of relative cohomology and Ext groups, see [GKM1] Section 5.7. If G is a reductive group with maximal torus T and Weyl group W, we can recover G-equivariant cohomoology from T-equivariant cohomology by taking W invariants. Conversely we can recover T equivariant cohomology form G-equivariant cohomology by tensoring with S over S^W.

2. Description of the twisted derived Satake category

2.1. The sheaf $\mathcal{F}$. Consider the moduli stack of G-bundles on $\mathbb{P}^1$, $\mathrm{Bun}_G(\mathbb{P}^1)$. It is well known that it has a presentation as a double quotient

$$\mathrm{Bun}_G(\mathbb{P}^1) = G(\mathbb{C}[t^{-1}]) \backslash G(\mathcal{K}) / G(\mathcal{O})$$

The $G(\mathbb{C}[t^{-1}])$-orbits on Gr_G are parametrized by dominant coweights of G, so that

$$\mathrm{Gr}_G = \coprod_\lambda \mathrm{Gr}_G^\lambda$$

The orbits satisfy the closure relation

$$\overline{\mathrm{Gr}_G^\lambda} = \coprod_{\mu \geq \lambda} \mathrm{Gr}_G^\mu$$

Thus as a set $\mathrm{Bun}_G(\mathbb{P}^1)$ is a union of points indexed by $X_*^+(T)$. Similar descriptions hold for $\widetilde{\mathrm{Bun}}_G(\mathbb{P}^1)$, the punctured total space of the determinant line bundle on $\mathrm{Bun}_G(\mathbb{P}^1)$ (see [Ku]), for and Gra_G. One shows analogously to [FL] Lemma 2 that Gra_G^λ admits a $G(\mathbb{C}[t^{-1}])$-equivariant local system with $\mathbb{G}_m$-monodromy ζ_a if and only if $\lambda \in X^{*+}(T_N^\vee) = X_*^+(H)$. Let $I \subset \lambda \in X^{*+}(T_N^\vee) = X_*^+(H)$ be the set of minimal elements in the dominance order, namely the basepoints of the connected components of Gr_H. Define $\mathcal{F}$ to be the Goresky-MacPherson extension to $\widetilde{\mathrm{Bun}}_G(\mathbb{P}^1)$ of the direct sum over $\lambda \in I$ of the local system $\mathcal{L}^\zeta$ on each Gr_G^λ.

The sheaf $\mathcal{F}$ can also be thought of a pro-object of the $G \times \mathbb{G}_m$-equivariant derived category of Gra_G, so that we can take $\mathrm{Ext}^\bullet_{G \times \mathbb{G}_m}(\mathcal{F}, \cdot)$ of $G(\mathcal{O})$-equivariant sheaves. Our motivation for introducing it is that the functor $\mathrm{Ext}^\bullet(\mathcal{F}, \cdot)$ is non-zero on $\mathbb{P}\mathrm{erv}_N$. In this way, $\mathcal{F}$ plays the same role that the constant sheaf does in the non-twisted setting. As in [FL] Lemma 2, the $G(\mathbb{C}[t^{-1}])$ orbits that support a non-zero $\mathcal{L}^\zeta$-equivariant local system correspond exactly to $\lambda \in X_+^*(T_N^\vee)$. Therefore, $\mathcal{F}$ can only have non-zero stalks on these orbits. Suppose we were working in the non-twisted case on $\mathrm{Bun}_H(\mathbb{P}^1)$. Then since $\mathrm{Bun}_H(\mathbb{P}^1)$ is smooth, $\mathcal{F}$ would just be the constant sheaf, and these stalks would be 1-dimensional. The stalks of $\mathcal{F}$ can also be described by inverse Kazhdan-Lusztig polynomials for the affine Weyl group (see [KT] Section 5). These polynomials will be the same in the twisted case for G and non-twisted case for H, hence all non-zero stalks of $\mathcal{F}$ will be 1-dimensional in the twisted case as well.

Theorem 2.1.

$$Ext^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F}) \cong \oplus_{Z(G_N^\vee)} \mathcal{O}(N_{\mathfrak{t}_N^{\vee *}/W \times \mathfrak{t}_N^{\vee *}/W} \Delta) \cong H^\bullet_{H(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_H)$$

$$Ext_T(\mathcal{F},\mathcal{F}) \simeq H^\bullet_{T_H\times\mathbb{G}_m}(Gr_H)$$

Proof. Let I^- be the subgroup of $G(\mathbb{C}[t^{-1}])$ of elements that are in B^- when $t\to\infty$. Consider the spectral sequence associated to the filtration of Gra_G by I^--orbits. By Theorem 5.3.5 of [KT], the stalks of $\mathcal{F}$ satisfy parity-vanishing, so this spectral sequence degenerates. Via $X^*(T_N^\vee)\simeq X_*(T_H)$, the I^--orbits where $\mathcal{F}$ is non-zero correspond exactly to the corresponding orbits in Gr_H, and by [KT] (5.3.10), so do the degrees of the stalks/costalks. Therefore $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ is computed by the same (degenerate) spectral sequence as $H^\bullet(\mathrm{Gr}_H)$, and in particular is isomorphic to $H^\bullet(\mathrm{Gr}_H)$ as a graded vector space. Note that according to [G1], the latter is a polynomial algebra in variables whose degrees are twice the exponents of $\mathfrak{g}_N^\vee$.

We now argue exactly as in [BF] 3.1. We have two morphisms $pr_1^*, pr_2^* : \sum \mathcal{O}(\mathfrak{t}_N^{\vee *}/W)\to \mathrm{Ext}^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F})$, and a morphism $pr^* : \mathbb{C}[\hbar]\to \mathrm{Ext}^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F})$ We claim that $pr_1^*|_{\hbar=0} = pr_2^*|_{\hbar=0}$. To see this, note that the parity vanishing of $\mathcal{F}$ implies parity vanishing of $\mathcal{E}\mathrm{xt}(\mathcal{F},\mathcal{F})$, so the latter is equivariantly formal, so $Ext_T(\mathcal{F},\mathcal{F})\hookrightarrow \oplus_\lambda \mathrm{Ext}^\bullet(X_0;\mathcal{F}\mathcal{F}) = \oplus_\lambda H_{T\times\mathbb{G}_m}(\lambda)$. By [BF] 3.2 (which is doesn't depend on their 3.1), the left $\mathcal{O}(\mathfrak{t}_N^{\vee *})$-action and the right $\mathcal{O}(\mathfrak{t}_N^{\vee *}/W)$-action on $H_{T\times\mathbb{G}_m}(\lambda)$ commute when $\hbar=0$, which proves the claim.

The claim implies that the morphism (pr_1^*, pr_2^*, pr^*) factors through a morphism $\alpha : \oplus_{Z(G_N^\vee)}\mathcal{O}(N_{\mathfrak{t}_N^{\vee *}/W\times\mathfrak{t}_N^{\vee *}/W}\Delta)\to \mathrm{Ext}^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F})$. Since the localization

$$\alpha_{loc} : \oplus_{Z(G_N^\vee)}\mathcal{O}(N_{\mathfrak{t}_N^{\vee *}/W\times\mathfrak{t}_N^{\vee *}/W}\Delta)\otimes_{\mathcal{O}(\mathfrak{t}_N^{\vee *}/W\times\mathbb{A}^1)}\mathrm{Frac}(\mathcal{O}(\mathfrak{t}_N^{\vee *}\times\mathbb{A}^1))$$
$$\to \mathrm{Ext}^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F})\otimes_{\mathcal{O}(\mathfrak{t}_N^{\vee *}/W\times\mathbb{A}^1)}\mathrm{Frac}(\mathcal{O}(\mathfrak{t}_N^{\vee *}\times\mathbb{A}^1))$$

is injective, so is α. Since $\mathrm{Ext}^\bullet_{G(\mathbb{C}[t^{-1}])\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{F})$ has the same graded dimension as $H^\bullet_{H(\mathcal{O})\rtimes\mathbb{G}_m}(\mathrm{Gr}_H)$, which by [BF] 3.1 has the same graded dimension as $\oplus_{Z(G_N^\vee)}\mathcal{O}(N_{\mathfrak{t}_N^{\vee *}/W\times\mathfrak{t}_N^{\vee *}/W}\Delta)$,

α is an isomorphism. The second statement of the theorem follows by tensoring both sides with $\mathcal{O}(\mathfrak{t}_N^{\vee *})$ over $\mathcal{O}(\mathfrak{t}_N^{\vee *}/W)$. □

Remark 2.2. *Notice that if $G_N^\vee$ is not adjoint, $Ext^\bullet(\mathcal{F},\mathcal{F})$ looks like the cohomology of a disconnected space, even though Gra_G may be connected (e.g. for $G = SL_2$, $N = 2$). One we to see this as is follows: by parity vanishing, the sheaf $\mathcal{E}xt(\mathcal{F},\mathcal{F})$ is equivariantly formal in the sense of* [GKM1], *hence its cohomology embeds into the its cohomology restricted to the fixed points, and further $Ext^\bullet(\mathcal{F},\mathcal{F})$ is defined as the kernel of the boundary map to X_1, the union of 0- and 1-dimensional orbits. The components of X_1 are labeled by $Z(G_N^\vee)$. This is why we have to take a direct sum in the definition of $\mathcal{F}$.*

2.2. **Fiber functor and canonical filtration.** We first recall the fiber functor of Finkelberg-Lysenko. The $N_-(\mathcal{K})$-orbits on Gra_G are indexed by $\lambda \in X_*(T)$. We denote the orbit through λ by $\mathcal{I}_\lambda$ We will only need to consider $\lambda \in X^*(T_N^\vee) \subset X_*(T)$. Write i_λ for the locally closed embedding of $\mathcal{I}_\lambda$ into Gra_G, and $p_\lambda : \mathcal{I}_\lambda \to \rho^{-1}(\lambda)$. For $\mathcal{A} \in \mathbb{P}\mathrm{erv}_N$, let $F_\lambda(\mathcal{A})$ be the stalk of $p_{\lambda !} i_\lambda^*(\mathcal{A})$ at λ. If $\jmath_\lambda$ is the inclusion of λ into $\mathcal{I}_\lambda$, then by the hyperbolic localization theorem of Braden [Br], $F_\lambda(\mathcal{A}) = j_\lambda^* i_\lambda^!(\mathcal{A})$. The Finkelberg-Lysenko fiber functor is defined by $F = \oplus_\lambda F_\lambda$.

Each $N_-(\mathcal{K})$-orbit is contained in a $G(\mathbb{C}[t^{-1}])$-orbits, so $\mathcal{F}$ is constant along these orbits with 1-dimensional stalks. Using the filtration by $N_-(\mathcal{K})$-orbits, one sees that $F(\mathcal{A}) = \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{A}_\lambda)$. This proves

Theorem 2.3. *(Forgetting the grading), the functor $Ext^\bullet(\mathcal{F},\cdot)$ coincides with the fiber functor F.*

By definition, there is a natural action of $\mathcal{O}(\mathfrak{t}_N^{\vee *} \times \mathfrak{t}_N^{\vee *}/W \times \mathbb{A}^1)$ on $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A})$, and for $\mathcal{A} = \mathcal{A}_\lambda$ this action factors through the diagonal morphism

$$\mathcal{O}(\mathfrak{t}_N^{\vee *}/W \times \mathfrak{t}_N^{\vee *}/W \times \mathbb{A}^1) \to \oplus_{Z(G_N^\vee)} \mathcal{O}(N_{\mathfrak{t}_N^{\vee *}/W \times \mathfrak{t}_N^{\vee *}/W}\Delta) \simeq Ext_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{F})$$

Let $\pi : \mathfrak{t}_N^{\vee *} \to \mathfrak{t}_N^{\vee *}/W$ denote the projection. Thus (π^*, Id, Id) is the natural map from G- to T-equivariant cohomology. Let $\Gamma_\lambda \subset \mathfrak{t}_N^{\vee *} \times \mathfrak{t}_N^{\vee *} \times \mathbb{A}^1$ be the subscheme defined by the equation $x_2 = x_1 + a\lambda$.

Following [BF] 3.2 we define the canonical filtration on $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A})$ for any $\mathcal{A} \in \mathbb{P}\mathrm{erv}_N$. Let $\mathcal{I}_\lambda$ be inverse image under ρ of the semi-infinite $N_-(\mathcal{K})$-orbit through λ. We filter $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A})$ by the closures of $\mathcal{I}_\lambda$. The associated graded of this filtration is

$$\bigoplus_{\lambda\in X^*(T_N^\vee)} \mathrm{Ext}^\bullet_{\mathcal{I}_\lambda, T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A})$$

Let i_λ be the locally closed embedding of $\mathcal{I}_\lambda$ in Gra_G, and j_λ the embedding of $\rho^{-1}\lambda$ in $\mathcal{I}_\lambda$. Then

$$\mathrm{Ext}^\bullet_{\mathcal{I}_\lambda, T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}) = \mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(j_\lambda^* i_\lambda^* \mathcal{F}, j_\lambda^* i_\lambda^! \mathcal{A}) \otimes F_\lambda(\mathcal{A})$$

On $\rho^{-1}\lambda$ both $j_\lambda^* i_\lambda^* \mathcal{F}$ and $j_\lambda^* i_\lambda^! \mathcal{A}$ reduce to L^ζ, and $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(j_\lambda^* i_\lambda^* \mathcal{F}, j_\lambda^* i_\lambda^! \mathcal{A}) \simeq H^\bullet_{T\times\mathbb{G}_m}(\lambda) \simeq (Id, \pi, Id)_* \mathcal{O}(\Gamma_\lambda)$ by [BF] 3.2. If $\mathcal{A} = S(V)$, where V is a representation of $G_N^\vee$, then $F_\lambda(\mathcal{A})$ is the λ weight space of V, which we write V^λ. This proves

Lemma 2.4. *For $V \in Rep(G_N^\vee)$, the $\mathcal{O}(\mathfrak{t}_N^{\vee *} \times (\mathfrak{t}_N^{\vee *}/W) \times \mathbb{A}^1)$-module $Ext^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\lambda)$ has a canonical filtration with associated graded $\oplus_\lambda (Id, \pi, Id)_* \mathcal{O}(\Gamma_\lambda) \otimes V^\lambda$, in particular $Ext^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\lambda)$ is flat as an $\mathcal{O}(T_N^\vee \times \mathbb{A}^1)$-module.*

This canonical filtration is compatible with the restriction to a Levi subgroup. Let M be a Levi subgroup of G, and $M_N^\vee$ corresponding subgroup of $G_N^\vee$ by [FL]. Let P_M^- be the parabolic subgroup of G generated by M and B^-, and let $\pi_M : \mathfrak{t}_N^{\vee *}/W_M \to \mathfrak{t}_N^{\vee *}/W$ be the projection. Let $X_*^{+M}(T) \subset X_*(T)$ be the coweights of T that are dominant for M. This set indexes the P_M^--orbits on Gra_G, and the oribts which admit an $M(\mathcal{O})$-equivariant local system with monodromy ζ are indexed by

$$X^{*+}_M(T^\vee_N) := \{\lambda \in X^{+M}_*(T) | d\iota(\lambda) \in NX^*(T)\}$$

Let $\mathcal{I}_{M,\lambda}$ denote the P^-_M-orbit through λ, and $p^\lambda : \mathcal{I}_{M,\lambda} \to \mathrm{Gra}_M$ the natural projection. Now $(\pi_M, Id, Id)^*\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}) = \mathrm{Ext}^\bullet_{L\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\lambda)$ is filtered by the $\mathcal{I}_{M,\lambda}$, with associated graded

$$\bigoplus_{\lambda\in X^*_M(T^\vee_N)} \mathrm{Ext}^\bullet_{\mathcal{I}_{M,\lambda}, M\times\mathbb{G}_m}(\mathcal{F}, S(V)) =$$
$$\bigoplus_{\lambda\in X^*_M(T^\vee_N)} \mathrm{Ext}^\bullet_{M\times\mathbb{G}_m}(\mathrm{Gra}_M; p^\lambda_* i^!_\lambda \mathcal{F}, p^\lambda_* i^!_\lambda S(V)) =$$
$$(Id, \pi_M, Id)_* \mathrm{Ext}^\bullet_{M\times\mathbb{G}_m}(\mathrm{Gra}_M, S_M(V_{M^\vee_N}))$$

The following are proved exactly as in [BF] 3.4 and 3.5.

Lemma 2.5. *The canonical filtration is compatible with restriction to a Levi subgroup.*

Lemma 2.6. *The canonical filtration is compatible with the tensor structure on* $Ext^\bullet_{G\times\mathbb{G}_m}(\mathcal{F}, \cdot)$ *given by convolution.*

2.3. **The rank one case.** We begin by recalling some results of Goresky-Kottwitz-Macpherson. In [GKM2], which was the inspirationfor our argument, they use the localization techniques of [GKM1] to study affine Springer fibers. Their proofs involve a detailed study of the rank one case.

Lemma 2.7. *(5.12 of [GKM2]) The union of 0- and 1-dimensional orbits of T on Gr_G is*

$$X_1 = \bigcup_{\alpha\in\Phi^+} Gr_{G_\alpha}$$

If $\alpha, \beta \in \Phi^+$ and $\alpha \neq \beta$, then $Gr_{G_\alpha} \cup Gr_{G_\beta} = X_*(T)$.

Lemma 2.8. *(6.4 of [GKM2]) Let $G = SL(2)$. For any two $\lambda, \mu \in X_*(T)$, there exists a unique 1-dimensional $T \times \mathbb{G}_m$-orbit $\mathcal{O}_{\lambda,\mu}$ which connects them. These are all of the $T \times \mathbb{G}_m$ in Gr_G.*

We apply Lemma 2.8 to the rank one case, but first we explain another way to compute $Ext_T(\mathcal{F},\mathcal{F})$. Let $i : X_0 \to \mathrm{Gra}_G$ denotes the union of the fibers over the T-fixed points and $j : X_1 \to \mathrm{Gra}_G$ denote the union of the fibers over the 0- and 1-dimensional orbits. By a parity vanishing argument, $\mathcal{E}\mathrm{xt}(\mathcal{F},\mathcal{F})$ is equivariantly formal in the sense of [GKM1]. Therefore we have an exact sequence

$$0 \to Ext_T(\mathcal{F},\mathcal{F}) \to \mathrm{Ext}^\bullet_{X_0,T\times\mathbb{G}_m}(i^*\mathcal{F}, i^!\mathcal{F}) \to \mathrm{Ext}^\bullet_{X_1,T\times\mathbb{G}_m}(j^*\mathcal{F}, j^!\mathcal{F})$$

For a given $\lambda \in X^*(T_N^\vee) = X_*(T_H)$, $i^!_\lambda\mathcal{F}$ is a local system of rank one. By Lemmas 2.7, 2.8 there is a unique one-dimensional orbit connecting each pair of fixed points λ and μ on Gr_G. Let $\mathfrak{t}^{\vee *}_{N,\lambda,\mu}$ be the lie algebra of the stabilizer of such a connecting orbit. The above exact sequence says that $Ext_T(\mathcal{F},\mathcal{F})$ is the subalgebra of functions in $(f_\lambda) \in \oplus_\lambda \mathbb{C}[\mathfrak{t}^{\vee *}, \hbar] \otimes \mathbb{C}_\lambda$ such that $f_\lambda|_{\mathbb{C}[\mathfrak{t}^{\vee *}_{N,\lambda,\mu}]} = f_\mu|_{\mathbb{C}[\mathfrak{t}^{\vee *}_{N,\lambda,\mu}]}$ for all pairs λ, μ. One can see that this argument applied to Gr_H gives the exact same answer.

We now assume that G has rank one, so $G_N^\vee$ does as well.

Lemma 2.9. *$Ext^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_n) \simeq H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_n)$ as $\mathcal{O}(\mathfrak{t}_N^{\vee *} \times \mathfrak{t}_N^{\vee *} \times \mathbb{A}^1)$-modules in a way that is compatible with the canonical filtrations on each. This isomorphism intertwines the actions of $Ext_T(\mathcal{F},\mathcal{F})$ and $H_{T_H\times\mathbb{G}_m}(Gr_H)$.*

Proof. Using a filtration and parity the vanishing of stalks, one sees that $\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{A}_n)$ vanishes in odd degrees, hence the sheaf $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{A}_n)$ is equivariantly formal. Let $i : X_0 \to \mathrm{Gra}_G$ denotes the union of the T-fixed points and $j : X_1 \to \mathrm{Gra}_G$ denote the union of the 0- and 1-dimensional orbits. Applying (6.3) of [GKM1] gives

$$0 \to \mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_n) \to \mathrm{Ext}^\bullet_{X_0,T\times\mathbb{G}_m}(i^*\mathcal{F}, i^!\mathcal{A}_n) \to \mathrm{Ext}^\bullet_{X_1,T\times\mathbb{G}_m}(j^*\mathcal{F}, j^!\mathcal{A}_n)$$

Here the middle term is equal to $\oplus_{i\le n}\mathcal{O}[T_N^\vee]\otimes V^i$, and the last term by $\sum_{j<i\le n}\mathcal{O}(\mathfrak{t}_{N,ij}^{\vee *})\otimes \mathbb{C}_i$, where $\mathfrak{t}_{N,ij}^{\vee *}$ is the Lie algebra of the stabilizer of a 1-dimensional orbits between points over i and j. Here i runs over the weights of the representation V_n of $G_N^\vee$, and V^i is the corresponding one-dimensional weight space. We can write down the corresponding exact sequence for H

$$0 \to H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_n) \to H_{X_{0,H},T\times\mathbb{G}_m}(i^!\mathcal{A}'_n) \to H_{X_{1,H},T\times\mathbb{G}_m}(j^!\mathcal{A}'_n)$$

Here the middle sequence is also given by $\oplus_{i\le n}\mathcal{O}[\mathfrak{t}_H^{\vee *}]\otimes V^i$, and the last term by $\oplus_{j<i\le n}\mathcal{O}(\mathfrak{t}_{N,ij}^{\vee *})\otimes V^i$, and we can canonically identify the middle and last terms of these two exact sequences as $\mathcal{O}(\mathfrak{t}_N^{\vee *}\times\mathbb{A}^1)$-modules, as well as the boundary between them. Equating the kernels of the boundary maps, this allows us to identify $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_n)$ and $H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_n)$ as $\mathcal{O}(\mathfrak{t}_N^{\vee *}\times\mathbb{A}^1)$-modules, where the action is via $(\pi^*(pr_1^*), pr^*)$. A similar argument, considering Gra_G as a left quotient by $G(\mathcal{O})$ shows the same where the action is via $(\pi^*(pr_2^*), pr^*)$. Thus we can identify $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_n)$ and $H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_n)$ as filtered $\mathcal{O}(\mathfrak{t}_N^{\vee *}\times\mathfrak{t}_N^{\vee *}\times\mathbb{A}^1)$-modules without passing to the associated graded. In particular, it implies that if s is the non-trivial element of W, the action s on the associated graded of the canonical filtration coming from Gra_G is the same as the action coming from Gr_H. Since the actions of $Ext_T(\mathcal{F},\mathcal{F})$ and $H_{T_H\times\mathbb{G}_m}(\mathrm{Gr}_H)$ are determined by the action of $\mathcal{O}(\mathfrak{t}_N^{\vee *}\times\mathfrak{t}_N^{\vee *}\times\mathbb{A}^1)$, these actions also coincide. □

Remark 2.10. *The proof of the above lemma relies on the fact that the stalks of $\mathcal{A}_n$, $\mathcal{A}'_n$ are one-dimensional. In general we know that the stalks of both $\mathcal{A}_\lambda$ and $\mathcal{A}'_\lambda$ at μ have the same dimension as the μ weight space of V_λ, but we don't have a natural way to identify the stalks themselves. If we can define a Hopf algebra structure on $Ext^\bullet(\mathcal{F},\mathcal{F})$ and show that this Hopf algebra is isomorphic to the enveloping algebra of the centralizer of a principal nilpotent in $\mathfrak{g}_N^\vee$, one can use the Brylinski filtration to canonically identify the stalks of $\mathcal{A}_\lambda$ and $\mathcal{A}'_\lambda$. It should be possible to do this using the Beilinson-Drinfeld*

Grassmannian. Another possible approach to the general case might be to use Lusztig's canonical basis of the representation V_λ, combined with it geometric realization in [BG].

2.4. **Comaprison of Ext groups.** Following the method of [BF] section 6, we deduce the general case from the rank one case.

Theorem 2.11. *We have a natural isomorphism of the $\mathcal{O}(\mathfrak{t}_N^{\vee *}/W \times /\mathfrak{t}_N^{\vee *}/W \times \mathbb{A}^1)$-modules $Ext^\bullet_{G\rtimes\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\lambda)$ and $H^\bullet_{H(\mathcal{O})\rtimes\mathbb{G}_m}(\mathcal{A}'_\lambda)$ that preserves the canonical filtrations and induces the identity on the associated graded pieces.*

Proof. We follow [BF] Theorem 6. We have

$$(\pi, Id, Id)^*\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1) =$$

$$gr(\pi, Id, Id)^*\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1) =$$

$$\bigoplus_\lambda (Id, \pi, Id)_*\mathcal{O}(\Gamma_\lambda) \otimes_\lambda V \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1) =$$

$$gr(\pi, Id, Id)^* H_{T_H\times\mathbb{G}_m}(\mathcal{A}') \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1) =$$

$$(\pi, Id, Id)^* H_{T_H\times\mathbb{G}_m}(\mathcal{A}') \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$$

By Lemma 2.9, and compatibility with Levi subgroups (Lemma 2.5), the action of simple reflection $s \in W$ on

$$\bigoplus_\lambda (Id, \pi, Id)_*\mathcal{O}(\Gamma_\lambda) \otimes_\lambda V \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$$

coming from

$$(\pi, Id, Id)^*\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}) \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$$

is the same as the action coming from

$$(\pi, Id, Id)^* H_{T_H\times\mathbb{G}_m}(\mathcal{A}') \otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)} \mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$$

hence we have a naturally defined action of W.

Recall the submodule $\cap_w^\alpha w(M_\alpha) \subset \oplus_\lambda (Id,\pi,Id)_*\mathcal{O}(\Gamma_\lambda)\otimes({}_\lambda V)\otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)}\mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$. By [BF] 5.2 (7) $\cap_w^\alpha w(M_\alpha)$ contains $(\pi,Id,Id)^* H_{T_H\times\mathbb{G}_m}(\mathcal{A}')\otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)}\mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$, so by Lemma 2.9 it also contains $(\pi,Id,Id)^*\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F},\mathcal{A})\otimes_{\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)}\mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$. By [BF], any section of $\cap_w^\alpha w(M_\alpha)$ is regular away from a codimension 2 subvariety. By Lemma 2.4 and [BF] 3.3, both $(\pi,Id,Id)^*\mathrm{Ext}^\bullet_{G\times\mathbb{G}_m}(\mathcal{F},\mathcal{A})$ and $(\pi,Id,Id)^* H_{T_H\times\mathbb{G}_m}(\mathcal{A}')$ are flat $\mathcal{O}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$-modules that equal $\cap_w^\alpha w(M_\alpha)$ after tensoring with $\mathbf{k}(\mathfrak{t}^{\vee *}\times\mathbb{A}^1)$, hence both are in fact equal to $\cap_w^\alpha w(M_\alpha)$.

□

2.5. **Purity and the derived category.** Using a $\mathbb{G}_m$-action, one may show that the $\mathcal{A}_\lambda$ are pointwise pure (see e.g. [KT], remark preceding Theorem 5.3.5). Recall the following "adjunction" formula from [FL]

$$\mathrm{Ext}^\bullet(\mathcal{A}_1 * \mathcal{A}_2, \mathcal{A}_3) \simeq \mathrm{Ext}^\bullet(\mathcal{A}_1, \mathcal{A}_3 * \mathcal{A}_2^\vee)$$

Taking $\mathcal{A}_1 = \mathcal{A}_0$ (corresponding to the trivial representation of $G_N^\vee$), $\mathcal{A}_2 = \mathcal{A}_\lambda$, and $\mathcal{A}_3 = \mathcal{A}_\mu$ we have

$$\mathrm{Ext}^\bullet(\mathcal{A}_\lambda,\mathcal{A}_\mu) \simeq \mathrm{Ext}^\bullet(\mathcal{A}_0 * \mathcal{A}_\lambda,\mathcal{A}_\mu) \simeq \mathrm{Ext}^\bullet(\mathcal{A}_0,\mathcal{A}_\mu * \mathcal{A}_\lambda^\vee) \simeq \bigoplus_\nu \mathrm{Ext}^\bullet(\mathcal{A}_0,\mathcal{A}_\nu)\otimes V(\nu)$$

for some multiplicity spaces $V(\nu)$. The pointwise purity of the $\mathcal{A}_\nu$ at $\tilde{0}$ then implies that $\mathrm{Ext}^\bullet(\mathcal{A}_\lambda,\mathcal{A}_\mu)$ is pure. To extend this to $T\times\mathbb{G}_m$-equivariant Ext's, let P_i be finite dimensional approximations to the classifying space of $T\times\mathbb{G}_m$, so that we have ind-varieties $P_i\mathrm{Gra}_G$ fibred over P_i with fiber Gra_G such that $P_i\mathcal{A}$ be the sheaf on $P_i\mathrm{Gra}_G$ giving the equivariant structure on $\mathcal{A}$. By defining a suitable $\mathbb{G}_m$-action, one can show the sheaf $P_i\mathcal{A}$ is also pointwise pure. By definition,

$$\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{A}_\lambda,\mathcal{A}_\mu)=\varinjlim \mathrm{Ext}^\bullet_{P_i\mathrm{Gra}_G}(P_i\mathcal{A}_\lambda,P_i\mathcal{A}_\mu)$$

We have a corresponding 'adjunction' formula on each $P_i\mathrm{Gra}_G$, due to the degeneracy of the Leray-Hirsch spectral sequence, which allows us to prove purity by the same argument in the equivariant case.

We now turn to the computation of $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{A}_\lambda,\mathcal{A}_\mu)$. We have a canonical map

$$\alpha:\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{A}_\lambda,\mathcal{A}_\mu)\to \mathrm{Hom}^\bullet_{Ext_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{F})}(\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\lambda),\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\mu))$$

Theorem 2.12. *α is an isomorphism of graded vector spaces.*

Proof. Let β be α composed with the natural inclusion

$$\mathrm{Hom}^\bullet_{Ext_T(\mathcal{F},\mathcal{F})}(\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\lambda),\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\mu))\hookrightarrow \mathrm{Hom}^\bullet_{\mathbb{C}}(\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\lambda),\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F},\mathcal{A}_\mu))$$

Consider the filtration given by I-orbits indexed by p, along which $\mathcal{A}_\lambda$ is locally constant. By parity considerations this spectral sequence degenerates. Suppose we have $f\in\mathrm{Ext}^\bullet_{T\rtimes\mathbb{G}_m}(\mathcal{A}_\lambda,\mathcal{A}_\mu)$ such that $\beta(f)=0$. Since the spectral sequence degenerates, for each p we have

$$0=\beta(i_{p*}i_p^!f)\in\mathrm{Hom}^\bullet_{\mathbb{C}}(\mathrm{Ext}^\bullet_{T\rtimes\mathbb{G}_m}(\mathcal{F},i_{p*}i_p^!\mathcal{A}_\lambda),\mathrm{Ext}^\bullet_{T\rtimes\mathbb{G}_m}(\mathcal{F},i_{p*}i_p^!\mathcal{A}_\mu))$$

Since each orbit p contains at most a single fixed point, this implies that $0=i_p^!f\in\mathrm{Ext}^\bullet_{T\rtimes\mathbb{G}_m}(i_{p*}i_p^!\mathcal{A}_\lambda,i_{p*}i_p^!\mathcal{A}_\mu)$. By induction on the filtration as in [BGS] 3.4.2, we conclude that $f=0$, so β and hence α is injective. To see that α is surjective, it suffices to note that both sides have the same graded dimensions:

By [L2], $\mathrm{Ext}^\bullet(\mathcal{A}_\lambda, \mathcal{A}_\mu) = \mathrm{Ext}^\bullet(\mathcal{A}_0, \mathcal{A}_\lambda^\vee)$ and $\mathrm{Ext}^\bullet(\mathcal{A}'_\lambda, \mathcal{A}'_\mu) = \mathrm{Ext}^\bullet(\mathcal{A}'_0, \mathcal{A}'^\vee_\lambda)$ have the same graded dimensions, so by equivariant formality $\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{A}_\lambda, \mathcal{A}_\mu)$ and $\mathrm{Ext}^\bullet_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda, \mathcal{A}'_\mu)$ do as well.

By Theorem 2.11

$$\mathrm{Hom}^\bullet_{Ext_T(\mathcal{F},\mathcal{F})}(\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\lambda), \mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\mu)) \simeq \mathrm{Hom}^\bullet_{H_{T_H\times\mathbb{G}_m}}(H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda), H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\mu))$$

and by [BF] we have

$$\mathrm{Ext}^\bullet_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda, \mathcal{A}'_\mu) \simeq \mathrm{Hom}^\bullet_{H_{T_H\times\mathbb{G}_m}(\mathrm{Gr}_H)}(H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda), H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\mu))$$

Putting these together gives the desired equality of graded dimensions.

Corollary 2.13. *$Ext^\bullet_{T\times\mathbb{G}_m}(\mathcal{A}_\lambda, \mathcal{A}_\mu)$ and $Ext^\bullet_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda, \mathcal{A}'_\mu)$ are canonically isomorphic, and respect the composition of Ext's.*

Proof. The isomorphisms

$$\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{A}_\lambda, \mathcal{A}_\mu) \simeq \mathrm{Hom}^\bullet_{Ext_T(\mathcal{F},\mathcal{F})}(\mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\lambda), \mathrm{Ext}^\bullet_{T\times\mathbb{G}_m}(\mathcal{F}, \mathcal{A}_\mu))$$
$$\simeq \mathrm{Hom}^\bullet_{H_{T_H\times\mathbb{G}_m}}(H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda), H_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\mu)) \simeq \mathrm{Ext}^\bullet_{T_H\times\mathbb{G}_m}(\mathcal{A}'_\lambda, \mathcal{A}'_\mu)$$

are all canonical. The first and last isomorphisms respect the composition of Ext's by functoriality. The middle isomorphism respects the composition of Hom's by Theorem 2.11. □

□

We can now prove Theorem 1.4.

Proof. (of 1.4)

Let $\mathcal{E}_\lambda$ be the endomorphism algebra of the sum of the simple objects of $D^{b,\zeta}_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gra_G)$ over $\mu \leq \lambda$. Let $\mathcal{E}'_\lambda$ be the corresponding algebra for $D^b_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_H)$. The above that

$\mathcal{E}_\lambda \simeq \mathcal{E}'_\lambda$. Let $D^{b,\zeta}_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gra_G)^{\leq\lambda}$ be the sub triangulated category of objects supported on orbits $\leq \lambda$. By the purity of the equivariant Ext groups, and Proposition 6 of [BF], we have a natural functor $\Phi_\lambda : D_{perf}(\mathcal{E}_\lambda) \to D^{b,\zeta}_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gra_G)^{\leq\lambda}$ that sends the free module to the $\oplus_{\mu\leq\lambda}\mathcal{A}_\mu$, and induces and isomorphism on Ext groups by Theorem 4.10, 4.11, so Φ_λ is an isomorphism. An identical argument for $\mathcal{E}'_\lambda$, and taking the direct limit over λ gives

$$D^{b,\zeta}_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gra_G) \simeq \lim_{\lambda} D_{perf}(\mathcal{E}_\lambda) \simeq \lim_{\lambda} D_{perf}(\mathcal{E}'_\lambda) \simeq D^{b}_{G(\mathcal{O})\rtimes\mathbb{G}_m}(Gr_H) \simeq D^{G_N^\vee}_{perf}(U_\hbar^\square)$$

It remains to show the compatibility of the monoidal structures. By [BF] Proposition 7, Remark 2, this follows from the fact that functors from $\mathcal{E}^2$ (resp. $\mathcal{E}^3$) to the convolution (resp. triple convolution) space given by [BF] Proposition 6 are given by the twisted external product of Φ, which in turn follows from [BF] Proposition 6, Remark 3.

Similar arguments give the result for $D^{b,\zeta}_{G(\mathcal{O})}(Gra_G)$.

□

2.6. **Quantum geometric Langlands duality for $\mathbb{P}^1$.** Let $G^\vee$ be the dual group of G. Let M be an integer relatively prime to N. Let ζ' be an $(2Mh/d)^{th}$ order character, and let $\mathbb{P}\mathrm{erv}_M$ be defined exactly as $\mathbb{P}\mathrm{erv}_N$, but with $G^\vee$ and M replacing G and N. Here, h is the dual Coxeter number of $G^\vee$, which is just the Coxeter number of G. Let $(G^\vee)_M$ be the twisted dual group of $G^\vee$. Suppose that $(G^\vee)_M = G^\vee_N$. Then by [FL] and Theorem 3.4 respectively

$$\mathbb{P}\mathrm{erv}_N \simeq \mathbb{P}\mathrm{erv}_M$$

$$D^{b,\zeta}_{G(\mathcal{O})}(Gra_G) \simeq D^{\zeta'}_{G^\vee(\mathcal{O})}(\mathrm{Gra}_{G^\vee})$$

Furthermore by the argument of [La], the latter can be used to show that

$$D^{\zeta}(\widetilde{\mathrm{Bun}}_G(\mathbb{P}^1)) \simeq D^{\zeta'}(\widetilde{\mathrm{Bun}}_{G^\vee}(\mathbb{P}^1))$$

According to the quantum geometric Langlands conjecture (see [Sto]), not only should this equivalence be true, but it should be be realized by an explicit kernel on $\mathrm{Gra}_G \times \mathrm{Gra}_{G^\vee}$. It would be a very interesting problem to find this kernel (the author would like to Sergey Lysenko for pointing this out).

2.7. **Factorizable structure of $\mathcal{F}$.** The purpose of this section is to indicate a possible for generalizing the results of [G1] the the twisted setting, which would allow one to imitate the proofs of [BF] and [ABG] directly. We will not give detailed proofs in this section, nor will we use it later.

Let $\mathcal{F}$ be the sheaf defined above. We would like to put a Hopf algebra structure on $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$. To do this we need a factorizable version of the 'thick' Grassmannian (see [Zh]). We note that the determinant line bundle is factorizable [FL].

Let $\mathrm{Gr}_{G,\mathbb{A}^2} \to \mathbb{A}^2$ be the Beilinson-Drinfeld (factorizable) Grassmannian whose fiber is given by Gr_G over the diagonal Δ and by $\mathrm{Gr}_G \times \mathrm{Gr}_G$ over $\mathbb{A}^2 \backslash \Delta$, and $\pi : \mathrm{Gra}_{G,\mathbb{A}^2} \to \mathbb{A}^2$ its base change to Gra_G.

Let $G_{\mathbb{A}^2}$ be the group scheme over $\mathbb{A}^2$ classifying $\{x_1, x_2) \in \mathbb{A}^2, \alpha\}$, where α is an automorphism of the trivial principal G-bundle restricted to the complement of $\{x_1, x_2\}$ in $\mathbb{A}^1$. Using the action of $\mathcal{G}^2_{\mathbb{A}}$ on $\mathrm{Gra}_{G,\mathbb{A}^2}$ one can define $\mathcal{F}^2_{\mathbb{A}}$, a factorizable analogue of the sheaf $\mathcal{F}$. Let $\mathcal{G}$ be the sheaf $\mathcal{E}\mathrm{xt}(\mathcal{F}^2_{\mathbb{A}}, \mathcal{F}^2_{\mathbb{A}})$. Then $\pi_*\mathcal{G}$ has stalks $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ at points of Δ, and $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F}) \times \mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ at points of $\mathbb{A}^2 \setminus \Delta$. The cospecialization between these stalks gives a comultiplication map, and this can be shown to give $\mathrm{Ext}^\bullet(\mathcal{F},\mathcal{F})$ a Hopf algebra structure.

3. Description of the twisted Iwahori-equivariant derived category

3.1. **Recollection of [ABG].** In this section we recall the equivalence $D^b_{Schubert}(\mathrm{Gr}_G) \simeq D^{G^\vee}_{coh}(\widetilde{\mathcal{N}})$ in some more detail. Recall that the left hand is the bounded derived category of perverse sheaves on Gr_G that are smooth along the stratification by I orbits. The right hand side is a certain DG-enhancement of of the G-equivariant bounded derived category of coherent sheaves on $\widetilde{\mathcal{N}}$ (see [ABG] 3.2). Note that in [ABG] the roles of G and $G^\vee$ are reversed.

For a DG-algebra A with the action of $G^\vee$, we let $DGM_f^{G^\vee}(A)$ be the category of differential graded A-modules M such that the $G^\vee$-action preserves the grading and commutes with the differential of M, and such that $H^\bullet(M)$ is a finitely generated $H^\bullet(A)$ module. We let $D_f^{G^\vee}(A)$ be the triangulated category obtained by localizing $DGM_f^{G^\vee}(A)$ at quasi-isomorphisms.

As a corollary of the proof in [ABG], there are equivalences ([ABG] (9.7.6), Theorem 9.1.4)

$$D^b_{Schubert}(\mathrm{Gr}_G) \simeq D_f^{G^\vee}(\mathbf{E}(\mathbf{1},\mathbf{1})) \simeq D^{G^\vee}_{coh}(\widetilde{\mathcal{N}})$$

where $\mathbf{E}(\mathbf{1},\mathbf{1})$ is a certain DG-algebra built from the Ext groups between perverse sheaves on Gr_G. Our strategy will be to definite an analogue of the functor from the twsited version of $D^b_{Schubert}(\mathrm{Gr}_G)$ to $D_f^{G^\vee}(\mathbf{E}(\mathbf{1},\mathbf{1}))$. The fact that we land in DG-modules over the same algebra will follow from the comparison of Ext groups in the first half. We will not need to work with the dual nilpotent cone directly, but we will sometimes mention it for motivation. We will switch notations below and use the groups G and H on the perverse side, and $G_N^\vee$ on the coherent side.

3.2. **Definitions and Notations.** Recall that Gra_G is the punctured total space of the line bundle *det* over Gr_G. Recall the Iwahori subgroup $I \subset G(\mathcal{O})$, defined as the preimage of B under

$$t \mapsto 0 : G(\mathcal{O}) \to G$$

The affine flag variety of G, Fl_G is the ind-scheme whose $\mathbb{C}$-points are $G(\mathcal{K})/I$. Let W_{aff} be the extended affine Weyl group of G, and let W be the finite Weyl group. Then under the left action of I, Fl_G decomposes into orbits indexed by $w \in W_{aff}$. The orbit corresponding to w is an affine space of dimension $\ell(w)$. There is a natural projection $\rho : \mathrm{Fl}_G \to \mathrm{Gr}_G$ with fibers isomorphic to the finite flag variety G/B. We can pull back *det* under ϱ to get a line bundle on Fl_G. Let Fla_G be the punctured total space of this line bundle. By abuse of notation we will also use ϱ to denote the projection from Fla_G to Gra_G.

The stratification of Gra_G by I-orbits, the Schubert stratification, gives a refinement of the stratification by $G(\mathcal{O})$-orbits. The I orbits are affine space labelled by elements $\lambda \in X_*(T)$. The $G(\mathcal{O})$-orbit containing the I-orbit corresponding to λ corresponds to the unique dominant coweight in the W^0-orbit of λ.

We now recall some results of Lusztig [L2] about perverse sheaves constructible along the Schubert stratification of Gra_G, which are $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant along the fibers of *det*. Lusztig works with the affine flag variety, but his results apply to our situation. The orbit $\mathcal{O}_\lambda$ admits a unique (up to isomorphism) $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant local system of rank one $\mathcal{L}^\zeta_\lambda$. Let $\mathcal{B}_\lambda$ be the Goresky-Macpherson extension of of $\mathcal{L}^\zeta_\lambda$. For another coweight μ, Proposition 5.4 of [L2] implies that the stalk of $\mathcal{B}_\lambda$ at μ vanishing if $\lambda + X^*(T_N^\vee) \neq \mu + X^*(T_N^\vee)$.

Consider the category of $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant perverse sheaves on Gra_G constructible along the Schubert stratification. Let $\mathrm{Perv}^\zeta(\mathrm{Gra}_G)$ be the subcategory of sheaves $\mathcal{B}$, such that simple subquotients of $\mathcal{B}$ are of the form $\mathcal{B}_\lambda$, $\lambda \in X^*(T_N^\vee)$. By the above vanishing, it splits off as a direct summand.

3.3. **The Regular sheaf.** Let (e, h, f) be a principal $\mathfrak{sl}_2$-triple in $\mathfrak{g}_N^\vee$. By Theorem 2.1 and [ABG] Corollary 6.5.4 we have that

$$\mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{F}) \simeq H^\bullet(\mathrm{Gr}_H) \simeq U((\mathfrak{g}_N^\vee)^e)$$

are isomorphisms of graded algebras, and that the grading on $V = \mathrm{Ext}^\bullet(\mathcal{F}, \mathcal{A}_\lambda)$ corresponds to the eigenvalues of the action of h on V.

Consider the Ind-object of $\mathbb{P}\mathrm{erv}_N$ corresponding to $\mathbb{C}[G_N^\vee] = \oplus_\lambda V_\lambda \otimes V_\lambda^*$ under the twisted Satake equivalence, namely

$$\mathcal{R} := \oplus_{\lambda \in X_*^+(T_N^\vee)} \mathcal{A}_\lambda \otimes V_\lambda^*$$

Let $\mathbf{1} = \mathcal{A}_0$. Right translation by an element of $G_N^\vee$ gives an action of $G_N^\vee$ on $\mathbb{C}[G_N^\vee]$, and hence by the twisted Satake equivalence an action of $\mathcal{R}$, and hence for any $\mathcal{A}, \mathcal{B}$ an action on $\mathrm{Ext}^\bullet(\mathcal{A}, \mathcal{B} * \mathcal{R})$.

The multiplication map $m : \mathbb{C}[G_N^\vee] \to \mathbb{C}[G_N^\vee]$ induces a map, which by abuse we write $m : \mathcal{R} \to \mathcal{R}$. We define a graded algebra structure on $\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{R})$ exactly as in [ABG] 7.2:

Let x and y be elements of $\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{R})$ considered as "derived morphisms" in $D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G)$ of degrees i and j. Then $y \cdot x$ is defined as the composition

$$\mathbf{1} \xrightarrow{y} \mathcal{R}[j] = (\mathcal{R} * \mathbf{1})[j] \xrightarrow{Id * x} \mathcal{R} * \mathcal{R}[i+j] \xrightarrow{m} \mathcal{R}[i+j]$$

Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}_N^\vee$ and $\widetilde{\mathcal{N}}$ the Springer resolution.. Recall that H denotes the Langlands dual group of $G_N^\vee$. Then Theorem 4.10 implies that

$$\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{R}) \simeq \mathrm{Ext}^\bullet(\mathbf{1}_\mathbf{H}, \mathcal{R}_H) \simeq \mathbb{C}[\mathcal{N}]$$

by [ABG] 7.3.1. Here we have used the indentification of Yoneda Ext's from Section 4, as well as the identification of m with m_H (since both come from the multiplication map on $\mathbb{C}[G_N^\vee]$.

3.4. **Wakimoto sheaves and an Ext algebra.** This section restates several definitions and results of [ABG] 8 in the twisted setting. We will omit many proofs in the cases that they are completely identical to those [ABG].

Let Fla_G be the pull back of *det* to the affine flag variety of G. Let I be the Iwahori subgroup of $G(\mathcal{O})$. The I orbits on Fla_G are paramterized by the extended affine Weyl group $= X_*(T) \rtimes W$. Let j_w be the includsion of the corresponding orbit, which is isomorphic to a $\ell(w)$-dimensional vector space.

Let $\mathcal{L}_w^\zeta$ be a $(\mathbb{G}_m, \mathcal{L}^\zeta)$-equivariant rank-one local system on the I-orbit corresponding to w. For $w \in W_N$ define $\mathcal{M}_w = j_{w!}\mathcal{L}_w^\zeta[\ell(w)+1]$ and $\mathcal{M}_w^\vee = j_{w*}\mathcal{L}_w^\zeta[\ell(w)+1]$. Since j_w is affine, $\mathcal{M}_w$ and $\mathcal{M}_w^\vee$ are perverse. For $\lambda \in X^*(T_N^\vee)$, write $\lambda = \mu - \nu$ with $\mu, \nu \in X_+^*(T_N^\vee)$ and define

$$\mathcal{W}_\lambda = \mathcal{M}_\mu^\vee * \mathcal{M}_\nu$$

This definition does not depend on the choice of μ and ν.

Theorem 3.1. *(I. Mirkovic)*

$$\mathcal{W}_\lambda * \mathcal{W}_\mu = \mathcal{W}_{\lambda+\mu}$$

For a proof see [ABG] and [AB]. While the work in the non-twisted setting, the proof carries over immediately.

Notice that for any $\mathcal{A} \in D_I(\mathrm{Gr}_G)$, $\mathcal{W}_\lambda * \mathcal{A} \in D_I(\mathrm{Gr}_G)$. For any $\mathcal{A}_1, \mathcal{A}_2 \in D_I(\mathrm{Gr}_G)$, we define

$$\mathbf{E}(\mathcal{A}_1, \mathcal{A}_2) = \oplus_{\lambda \in X_+^*(T_N^\vee)} \mathrm{Ext}^\bullet(\mathcal{A}_1, \mathcal{W}_\lambda * \mathcal{A}_2 * \mathcal{R})$$

We will be particularly interested in

$$\mathbf{E}(\mathbf{1}, \mathbf{1}) = \oplus_{\lambda \in X_+^*(T_N^\vee)} \mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{W}_\lambda * \mathcal{R})$$

Using an analogous construction to the previous section, and Theorem 3.1, one can give $\mathbf{E}(\mathbf{1}, \mathbf{1})$ a graded algebra structure. For $\lambda \in X_+^*(T_N^\vee)$

$$\mathcal{W}_{-\lambda} * \mathbf{1} = \varrho_* \mathcal{W}_{-\lambda} = \varrho_* \mathcal{M}_{-\lambda} = \Delta_{-\lambda}[-l(w)]$$

Since $(\mathcal{W}_{-\lambda} * \mathcal{W}_\lambda)$ is the identity functor, $(\mathcal{W}_{-\lambda} *)$ is an equivalence of categories hence preserves the Ext groups, and we have

$$\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{W}_\lambda * \mathcal{R}) = \mathrm{Ext}^\bullet(\mathcal{W}_{-\lambda} * \mathbf{1}, \mathcal{W}_{-\lambda} * \mathcal{W}_\lambda * \mathcal{R})$$

$$= \mathrm{Ext}^\bullet(\mathcal{W}_{-\lambda} * \mathbf{1}, \mathcal{R}) = \mathrm{Ext}^\bullet(\Delta_{-\lambda}[-l(w)], \mathcal{R})$$

Now for any $\mathcal{A}$

$$\mathrm{Ext}^\bullet(\Delta_{-\lambda}, \mathcal{A}) = \mathrm{Ext}^\bullet(\rho^{-1}(-\lambda); \mathcal{L}^\zeta, i^!_{-\lambda}\mathcal{A}[l(w) + ht(\lambda)])$$

So

$$\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{W}_\lambda * \mathcal{R}) = \mathrm{Ext}^{\bullet + ht(\lambda)}(\rho^{-1}(\lambda); \mathcal{L}^\zeta, i^!_{-\lambda}\mathcal{A}[ht(\lambda)])$$

Here, $ht(\lambda) = \mathrm{height}(\lambda)$. By the results of the first part, this last group is isomorphic to

$$H^{\bullet + ht(\lambda)}(\mathrm{Gr}_H; i^!_{H,-\lambda}\mathcal{R}_H)$$

It remains to check that the resulting isomorphism on $\mathbf{E}$ is compatible with the product map: From Lemma 2.11-2.13, it follows that the isomorphism

$$\mathrm{Ext}^{\bullet + ht(\lambda)}(\rho^{-1}(\lambda); \mathcal{L}^\zeta, i^!_{-\lambda}\mathcal{A}[ht(\lambda)]) \simeq H^{\bullet + ht(\lambda)}(\mathrm{Gr}_H; i^!_{H,-\lambda}\mathcal{A}')$$

where $\mathcal{A} \in \mathbb{P}\mathrm{erv}_N$ and $\mathcal{A}' \in Perv_{H(\mathcal{O})}(\mathrm{Gr}_H)$ correspond to the same representation of $G_N^\vee$, is compatible with the product map. Then we can conclude as on [ABG] pg. 55.

Recall Theorem 2.11. By taking a direct limit we an isomorphism

$$\mathrm{Ext}^\bullet_T(\mathcal{F}, \mathcal{A}) \simeq H^\bullet_{T_H}(\mathcal{A}')$$

that is compatible with the canonical filtration. Specializing $\mathbb{C}[T_N^\vee]$ to a regular element t, gives, by the localization theorem

$$\oplus_\lambda \mathrm{Ext}_t^\bullet(\rho^{-1}(\lambda); \mathcal{L}^\zeta, i_{-\lambda}^! \mathcal{A}) \simeq \mathrm{Ext}_t^\bullet(\mathcal{F}, \mathcal{A}) \simeq H_t^\bullet(\mathcal{A}') \simeq \oplus_\lambda H_t^\bullet(i_\lambda^! \mathcal{A}')$$

which identifies the λ-graded pieces, which are now filtered (but not graded) by degree). Taking the associated graded of this filtration on each λ-graded part gives the desired identification, which by Lemma 2.6 is compatible with the product structure. This combined with [ABG] 8.7 proves

Theorem 3.2. *Let* $\mathbf{E}(\mathbf{1}_H, \mathbf{1}_H) = \oplus_{\lambda \in X_\bullet^+(T_H)} Ext^\bullet(\mathbf{1}_H, \mathcal{W}'_\lambda \otimes \mathcal{R}_H)$. *Then we have an isomorphism*

$$\mathbf{E}(\mathbf{1}, \mathbf{1}) \simeq \mathbf{E}(\mathbf{1}_H, \mathbf{1}_H)$$

3.5. **The comparison.** In this section we recall some constructions from [ABG] section 9.

Let $Perv^\zeta_{Schubert}(\mathrm{Gra}_G)$ be the category of perverse sheaves that are constructible along the pullback to Gra_G of the Schubert stratification, which have finite dimensional support, and which have $\mathbb{G}_m$-monodromy $\mathcal{L}^\zeta$. Recall the full subcategory $\mathrm{Perv}^\zeta(\mathrm{Gra}_G)$. The goal of this section is to prove Theorem 1.5., which states

$4Perv^\zeta(\mathrm{Gra}_G) \simeq Perv(\mathrm{Gr}_H)$

The idea is to realize both categories (more precisely their derived versions) using dg-categories of modules over certain algebras. These algebras will turn out to be $\mathbf{E}(\mathbf{1}, \mathbf{1})$ and $\mathbf{E}(\mathbf{1_H}, \mathbf{1_H})$ respectively. By the calculation in the previous section, these algebras are isomorphic. This was raelization was given for $Perv(\mathrm{Gr}_H)$ in [ABG] 9.7.6 and Theorem 9.1.4. Below we will repeat their construction (which is more or less identical) for $Perv^\zeta(\mathrm{Gra}_G)$. Since many of the proofs translate verbatim, we will omit many details, and refer the reader the the corresponding sections of [ABG]. A priori, we don't know that the map from $D^b Perv^\zeta(\mathrm{Gra}_G)$ is an equivalence, but we will show that it is fully faithful, and that the images of of a set of generating objects for both sides coincide, which is enough to conclude Theorem 1.5.

The proof of [BGS] Corollary 3.3.2 adapts to give

$$\mathrm{Ext}^\bullet_{Perv^\zeta(\mathrm{Gra}_G)}(\mathcal{A}_1, \mathcal{A}_2) \simeq \mathrm{Ext}^\bullet_{D^{b,\zeta}(\mathrm{Gra}_G)}(\mathcal{A}_1, \mathcal{A}_2)$$

We also have

Lemma 3.3. *([ABG] Lemma 9.3.4) Any $\mathcal{A} \in Perv^\zeta(Gra_G)$ is a quotient of a projective pro-object P*

Let $D^\zeta_{proj}(\mathrm{Gra}_G)$ be the full subcategory of the homotopy category of complexes $C^\bullet = (\cdots \to C^i \to C^{i+1} \to \dots)$ in $\lim_{proj} Perv^\zeta(\mathrm{Gra}_G)$ such that $C^i = 0$ for $i \gg 0$ and $H^i(C^\bullet) = 0$ for $i \ll 0$.

Theorem 3.4. *([ABG] Corollary 9.3.8) The natural functor $\Theta : D^\zeta_{proj}(Gra_G) \to D^b Perv^\zeta(Gra_G)$, induced by the natural functor from the homotopy category to the derived category, is an equivalence.*

By Lemma 3.3 we have a projective resolution

$$\dots P^{-1} \to P^0 \twoheadrightarrow 1$$

Define $P = \oplus_{i \le 0} P^i$. P is a projective pro-object in $Perv^\zeta(\mathrm{Gra}_G$. Using the differential in the resolution, we can consider P as a dg-object quasi-isomorphic to $\mathbf{1}$.

Theorem 3.5. *(Lusztig [L3]) For any Schubert-constructible perverse sheaf $\mathcal{A}$ on Gra_G with $\mathbb{G}_m$-monodromy $\mathcal{L}^\zeta$, and any $\mathcal{A}_0 \in \mathbb{P}erv_N$, $\mathcal{A} * \mathcal{A}_0$ is perverse*

Proof. One can check that the proof of [L3] carries over to the twisted case. □

Lemma 3.6. *([ABG] Lemma 9.3.6) For $P_1, P_2 \in \lim_{proj} Perv^{\zeta}(Gra_G)$, $\lambda, \mu \in X^*_+(T^\vee_N)$*

$$Ext^n(P_1, \mathcal{W}_\lambda * P_2 * \mathcal{A}_\mu) = 0$$

for $n \neq 0$.

By Lemma 3.6

$$\mathbf{E}(P,P) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\{(i,j)\in\mathbb{Z}^2 | i-j=n, \lambda\in X^*_+(T^\vee_N)\}} \mathrm{Ext}^0(P^i, \mathcal{W}_\lambda * P^j * \mathcal{R})$$

Let

$$\mathbf{E}^n(P,P) = \bigoplus_{\{(i,j)\in\mathbb{Z}^2 | i-j=n, \lambda\in X^*_+(T^\vee_N)\}} \mathrm{Ext}^0(P^i, \mathcal{W}_\lambda * P^j * \mathcal{R})$$

.

The map $d : \mathbf{E}^n(P,P) \to \mathbf{E}^{n+1}(P,P)$, defined by taking the commutator of an element of $\mathbf{E}^n(P,P)$ with the differential of $P^\bullet$, gives a differential making $\mathbf{E}(P,P)$ into a dg-algebra

Theorem 3.7. *([ABG] Proposition 9.5.2) The dg-algebra is formal: it is quasi-isomorphic to $\mathbf{E}(\mathbf{1},\mathbf{1})$ with zero differential.*

The proof is that of [ABG] verbatim, but we should remark that we are surpressing a lot of detail: one has to introduce a mixed version of of the category $\mathrm{Perv}^{\zeta}(\mathrm{Gra}_G)$, and use the additional grading on the Ext groups combined with a purity argument to deduce formality. Since this would force us to introduce a lot of extra notation without any original arguments, we refer the reader to [ABG] Section 9 for the details.

Let $\mathbf{Mod}^{G^\vee_N}(\mathbf{E}(P,P))$ be the category of $G^\vee_N$-equivariant (via the $G^\vee_N$ action on $\mathcal{R}$ defined above) algebraic $\mathbf{E}(P,P)$ modules, and let $\mathbf{Comp}^{G^\vee_N}(\mathbf{E}(P,P))$ be the homotopy category of finitely generated $G^\vee_N$-equivariant algebraic differential $\mathbf{E}(P,P)$ modules. We make the same definitions for $\mathbf{E}(\mathbf{1},\mathbf{1})$.

Consider the map

$$C \mapsto \mathbf{E}(P, C) : D_{proj}(\mathrm{Gra}_G) \to \mathbf{Comp}^{G_N^\vee}(\mathbf{E}(P, P))$$

This defines a functor $D_{proj}(\mathrm{Gra}_G) \to \mathbf{Comp}^{G_N^\vee}(\mathbf{E}(P, P))$.

By Theorem 5.7

$$\mathbf{Comp}^{G_N^\vee}(\mathbf{E}(P, P)) \simeq \mathbf{Comp}^{G_N^\vee}(\mathbf{E}(\mathbf{1}, \mathbf{1}))$$

We have a natural functor

$$\mathbf{Comp}^{G_N^\vee}(\mathbf{E}(\mathbf{1}, \mathbf{1})) \to D_f^{G_N^\vee}(\mathbf{E}(\mathbf{1}, \mathbf{1}))$$

Now define χ as the composition

$$\chi : D^{\zeta}_{proj}(\mathrm{Gra}_G) \simeq D_{proj}(\mathrm{Gra}_G) \to \mathbf{Comp}^{G_N^\vee}(\mathbf{E}(P, P))$$
$$\simeq \mathbf{Comp}^{G_N^\vee}(\mathbf{E}(\mathbf{1}, \mathbf{1})) \to D_f^{G_N^\vee}(\mathbf{E}(\mathbf{1}, \mathbf{1}))$$

The corresponding functor

$$\chi_H : D^b Perv(\mathrm{Gr}_H) \to D^b\mathbf{Mod}^{G_N^\vee}(\mathbf{E}(\mathbf{1_H}, \mathbf{1_H})) \simeq D^b\mathbf{Mod}^{G_N^\vee}(\mathbf{E}(\mathbf{1}, \mathbf{1}))$$

was defined in [ABG] Section 9.

From [ABG] 9.7.5, we know that χ_H is an equivalence. A priori, we don't know that χ is an equivalence, but it is enough for out purposes to show that χ is fully faithful and that the images of χ and χ_H coincide.

Let X_H be the set

$$X_H := \{\varrho_{H*}\mathcal{W}'_\lambda, \forall\lambda \ngeq 0; \mathcal{W}'_\mu * \mathcal{A}', \forall\mu \in X^*_+(T_N^\vee), \mathcal{A} \in Perv_{H(\mathcal{O})}(\mathrm{Gr}_H)\}$$

According to [ABG] Lemma 9.8.4, the smallest triangulated subcategory in $D^bPerv(\mathrm{Gr}_H)$ that contains X_H is the whole $D^bPerv(\mathrm{Gr}_H)$. If we define

$$X := \{\varrho_*\mathcal{W}_\lambda, \forall\lambda \not\geq 0; \mathcal{W}_\mu * \mathcal{A}, \forall\mu \in X^*_+(T^\vee_N), \mathcal{A} \in \mathbb{P}\mathrm{erv}_N\}$$

then the same holds for $D^bPerv^\zeta(\mathrm{Gra}_G)$.

Lemma 3.8. *([ABG] Lemma 9.8.8 (i))* $Ext^\bullet(\mathbf{1}, \varrho_*\mathcal{W}_\lambda) = 0$ *for* $\lambda \not\geq 0$.

Proof. Recall that $\mathcal{W}_\mu$ is an equivalence for any $\mu \in X^*(T^\vee_N)$. Take μ such that both μ and and λ are anti-dominant, so that $\varrho_*\mathcal{W}_\mu = j_{\mu,!}\mathcal{L}^\zeta_\mu$ and $\varrho_*\mathcal{W}_{\mu+\lambda} = j_{\mu+\lambda,!}\mathcal{L}^\zeta_{\mu+\lambda}$. For μ sufficiently anti-dominant, $\mathrm{Gra}^\mu_G \not\subseteq \mathrm{Gra}^{\mu+\lambda}_G$, so that $j^!_\mu j_{\mu+\lambda,!}\mathcal{L}^\zeta_{\mu+\lambda} = 0$. Then

$$\mathrm{Ext}^\bullet(\mathbf{1}, \varrho_*\mathcal{W}_\lambda) = \mathrm{Ext}^\bullet(\varrho_*\mathcal{W}_\mu, \varrho_*\mathcal{W}_{\mu+\lambda})$$

$$= \mathrm{Ext}^\bullet(j_{\mu,!}\mathcal{L}^\zeta_\mu, j_{\mu+\lambda,!}\mathcal{L}^\zeta_{\mu+\lambda}) = j^!_\mu j_{\mu+\lambda,!}\mathcal{L}^\zeta_{\mu+\lambda} = 0$$

□

The analogous result for $\mathcal{W}'_\lambda$ is [ABG] Lemma 9.8.8 (i), so these Ext groups agree. The equality of Ext's for $\mathcal{W}_\mu * \mathcal{A}$ ($\mu \in X^*_+(T^\vee_N)$, $\mathcal{A} \in \mathbb{P}\mathrm{erv}_N$) follows from the argument of 3.4:

$$\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{W}_\lambda * \mathcal{R}) = \mathrm{Ext}^{\bullet+ht(\lambda)}(\rho^{-1}(\lambda); \mathcal{L}^\zeta, i^!_{-\lambda}\mathcal{A}[ht(\lambda)])$$

$$\simeq H^{\bullet+ht(\lambda)}(\mathrm{Gr}_H; i^!_{H,-\lambda}\mathcal{A}') = \mathrm{Ext}^\bullet(\mathbf{1_H}, \mathcal{W}'_\lambda * \mathcal{R}_H)$$

Now consider the map

$$\alpha := \chi_H \circ \chi : D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G) \to D^bPerv(\mathrm{Gr}_H)$$

The proof of [ABG] Proposition 9.8.1 shows that for any $\mathcal{B} \in D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G)$ and $V \in \mathrm{Rep}(G^\vee_N)$

$$\alpha(\mathcal{W}_\mu * \mathcal{A} * S(V)) = \mathcal{W}'_\mu * \alpha(\mathcal{A}) * S_H(V)$$

and $\alpha(\mathbf{1}) = \mathbf{1}_H$. Combined with the above, this shows that α sends a set of generating objects X to a set of generating objects X_H, and for any $\mathcal{A} \in X$, $\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{A}) \simeq \mathrm{Ext}^\bullet(\alpha(\mathbf{1}), \alpha(\mathcal{A}))$. Now let $\mathcal{B} \in D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G)$ be arbitrary, and let $\mu \in X^*(T_N^\vee)$. Since X generates $D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G)$, $\mathrm{Ext}^\bullet(\mathbf{1}, \mathcal{W}_\mu * \mathcal{B}) \simeq \mathrm{Ext}^\bullet(\alpha(\mathbf{1}), \mathcal{W}'_\mu \alpha(\mathcal{B}))$.

Now since α takes $\mathcal{W}_\mu *$ to $\mathcal{W}'_\mu *$, and both are equivalences of categories, we get

$$\mathrm{Ext}^\bullet(\mathcal{W}_\mu, \mathcal{B}) \simeq \mathrm{Ext}^\bullet(\mathcal{W}'_\mu, \alpha(\mathcal{B}))$$

for arbitrary μ and $\mathcal{B}$. Finally, since $\mathcal{W}_\mu$ and $\mathcal{W}'_\mu$ generate their respective categories as μ runs though $X^*(T_N^\vee)$, we get that

$$\mathrm{Ext}^\bullet(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Ext}^\bullet(\alpha(\mathcal{A}), \alpha(\mathcal{B}))$$

for arbitrary $\mathcal{A}, \mathcal{B}$. This proves that α is fully faithful. Since we already showed that α sends a generating set to a generating set, α is an equivalence of triangulated categories. This proves

$$D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G) \simeq D^b\mathrm{Perv}(\mathrm{Gr}_H)$$

Proof. (of Theorem 1.5) It remains to show that the equivalence of trangulated categories is compatible with the perverse t-structures. For $\lambda \in X^*_+(T_N^\vee)$, let $D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G)_{\leq\lambda}$ and $D^b\mathrm{Perv}(\mathrm{Gr}_H)_{\leq\lambda}$ respectively be the full triangulated subcategories of objects supported on $\overline{\mathrm{Gra}_{G,\lambda}}$ and $\overline{\mathrm{Gr}_{H,\lambda}}$ respectively. Then we have an equivalence

$$D^b\mathrm{Perv}^\zeta(\mathrm{Gra}_G)_{\leq\lambda} \simeq D^b\mathrm{Perv}(\mathrm{Gr}_H)_{\leq\lambda}$$

We also know that α sends $\mathcal{A}_\lambda$ to $\mathcal{A}'_\lambda$. This shows that the conditions of [ABG] Lemma 9.10.5 are satisfied, so the compatibility of t-structures follows from [ABG] Lemma 9.10.6.

□

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