Multifield Inflation and Differential Geometry

by

Edward A. Mazenc

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Author ....
Department of Physics
May 16, 2013

Certified by ....
Prof. David I. Kaiser
Germeshausen Professor of the History of Science and Department Head, MIT Program in Science, Technology, and Society, and Senior Lecturer, Department of Physics, MIT
Thesis Supervisor

Accepted by ....
Prof. Nergis Mavalvala
Undergraduate Thesis Coodinator

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Abstract

Cosmic inflation posits that the universe underwent a period of exponential expansion, driven by one or several quantum fields, shortly after the Big Bang. Renormalization requires the fields be non-minimally coupled to gravity. We examine such multi-field models and find a rich geometric structure. After a conformal transformation of spacetime, the target field-space acquires non-trivial curvature. We explore two main consequences. First, we construct a field-space covariant framework to study quantum perturbations, extending prior work beyond the slow-roll approximation by working on the full phase space of the theory. Secondly, we show that a wide class of inflationary models can be understood as a geodesic motion on a suitably related manifold. Our geometric approach provides great insight into the (classical) field dynamics, and we have used them to compute non-gaussianities in the cosmic microwave background radiation spectrum.

Thesis Supervisor: Prof. David I. Kaiser
Title: Germeshausen Professor of the History of Science and Department Head, MIT Program in Science, Technology, and Society, and Senior Lecturer, Department of Physics, MIT

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Chapter 1

Introduction

Since its initial conception by Alan Guth in 1980, inflationary cosmology has received spectacular experimental confirmation [1]. In a nutshell, inflation posits that shortly after the Big Bang, the universe underwent a period of exponential expansion. One might say it explains the "Bang" in Big Bang.

While the general mechanism is well understood, many different models remain compatible with observation. One of the most important outstanding questions is whether the exotic form of matter which drives the period of accelerated expansion is described either by one or by several scalar quantum fields, commonly referred to as the inflaton. The most promising route to answering this question is the study of non-gaussianities in the cosmic microwave background (CMB) [8]. Indeed, the tiny fluctuations detected in the otherwise extremely uniform CMB are thought to originate from quantum fluctuations of the inflaton field, so that statistical analysis of observable CMB radiation data shines light on the quantum field theory underlying inflation.

Non-gaussianities are strongly correlated with diverging classical field trajectories [6]. Hence, understanding the classical dynamics of the theory becomes crucial to successful inflationary model building. In this thesis, we take a geometric approach, borrowing techniques from differential geometry and advanced classical mechanics. Concretely, we find we can reduce the study of both classical and quantum aspects of multifield inflation to understanding geodesic motion on suitably defined manifolds.

In Section 2, we briefly introduce the general inflationary mechanism, focusing on those features most relevant to our subsequent discussion. Section 3 motivates the study of multifield models non-minimally coupled to gravity. Most importantly, it demonstrates i) that non-minimal couplings allow even simple polynomial potential to support prolonged periods of inflation and ii) that non-canonical kinetic terms arise naturally as a result of a conformal transformation between two spacetime frames. These non-canonical kinetic terms induce a curvature in field-space. Field-space diffeomorphism invariance of the action requires the development of a covariant formalism, which we discuss in Section 4.

We study the quantum mechanical aspects of inflation perturbatively, describing the full field behavior as small quantum fluctuations around a classical background. Section 5 outlines a geometric formulation for field-space covariant perturbations using a geodesic construction first introduced in [19, 15], while Section 6 generalizes these results beyond the familiar slow-roll approximations using techniques familiar from Hamiltonian mechanics. Finally, in Section 7, we completely reduce the classical dynamics of $\mathcal{N}$ non-minimally coupled inflaton fields in an FRW spacetime to understanding geodesic motion of a suitably related $\mathcal{N}+1$ dimensional manifold.

Chapter 2

Inflationary Cosmology

In this section, we provide a succinct, and by no means comprehensive, overview of inflationary cosmology [3, 1]. The goal is to introduce many of the salient features of inflationary models, as well as some the terminology we will need in subsequent sections.

2.1 The Friedmann Equations

On the largest scales, our universe appears isotropic and homogeneous. The most general metric ansatz describing this space-time geometry in spherical coordinates is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 sin^2(\theta) d\phi^2 \right) \tag{2.1}$$

We refer to this metric $g_{\mu\nu}$[1] as the Friedmann-Robertson-Walker (often abbreviated as FRW) metric. Since it is always possible to rescale the radial coordinate r, κ can always be chosen to be one of $0, +1$ or -1. The different values of κ determine the global geometry, such that the universe is flat for $\kappa = 0$, closed for $\kappa = +1$ and open for $\kappa = -1$. $a(t)$ is called the scale factor, and parametrizes an overall expansion of the spatial slices of spacetime. As inflation is at heart a theory of the expansion

[1]Note we use the $(-,+,+,+)$ signature for the spacetime metric $g_{\mu\nu}$.

of the universe, $a(t)$ will play a central role in all our subsequent discussions.

In an isotropic and homogeneous spacetime, the stress-energy tensor must take the form of a perfect fluid such that

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \tag{2.2}$$

where ρ is the energy density and p the pressure of the fluid and u^ν is the normalized fluid four-velocity. Recall that the Einstein field equations relate the metric to the stress-energy tensor by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.3}$$

In the above equation, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ where $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann curvature tensor, while $R = g^{\mu\nu}R_{\mu\nu}$ and G is Newton's gravitational constant. Looking at particular components of this tensor equation, we can derive the so-called Friedmann equation:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{2.4}$$

as well as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.5}$$

H is commonly referred to as Hubble's constant, even though it is technically not a constant. Note that κ vanishes when the energy density becomes$\rho = \frac{3H^2}{8\pi G}$. We call this particular value of the density the critical density ρ_{cr}. We also define the dimensionless parameter $\Omega \equiv \frac{\rho}{\rho_{cr}}$; hence, the universe is flat ($\kappa = 0$) for $\Omega = 1$.

2.2 The Flatness Problem

From the Friedmann equations, we find that $\frac{(\Omega-1)}{\Omega} = \left(\frac{3\kappa}{8\pi G}\right)\frac{1}{a^2\rho}$. In this section, we are interested in how cosmological dynamics affect this ratio. Covariant conservation of energy-momentum in the form of $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ is the space-time covariant derivative for an FRW metric, gives us

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{2.6}$$

Note that $\rho(x^\mu) = \rho(t)$ by the isotropy and homogeneity of the FRW spacetime. Assuming an equation of state of the form

$$p = \omega\rho \tag{2.7}$$

we can readily solve for the time-dependence of ρ and find

$$\rho(t) = \rho_0 a(t)^{-3(1+\omega)} \tag{2.8}$$

For relativistic matter (such as a gas of photons), we have $\omega = \frac{1}{3}$, while "ordinary" nonrelativistic matter (such as particle dust) obeys the equation of state with $\omega = 0$. We thus find that

$$\frac{(\Omega(t) - 1)}{\Omega(t)} = \left(\frac{3\kappa}{8\pi G}\right)\frac{a(t)^{(1+3\omega)}}{\rho_0} \propto \begin{matrix} a(t)\, for\, nonrelativistic\, matter \\ a^2(t)\, for\, relativistic\, matter \end{matrix} \tag{2.9}$$

If populated with the usual forms of matter such as photons and electrons, the universe will thus tend to become less and less flat over time. However, current cosmological observations, such as from the Planck satellite, tell us that $\Omega_{obs} = 1.0005 + 0.0065$ [34]. How can this be, unless the universe started with $\Omega_{Big-Bang}$ exponentially close to 1? This is the flatness problem.

2.3 Inflation and Matter Fields

If we could find some type of matter with $\omega < -1/3$, then $\frac{\Omega(t)-1}{\Omega(t)} \propto a(t)^{(1+3\omega)}$ would decrease as $a(t)$ grew over time, i.e. $\Omega(t) \to 1$. The universe would dynamically become flat, no matter how it started out after the big-bang. From the Friedmann equations, one can show that $\omega < -1/3$ is equivalent to $\ddot{a} > 0$, which is nothing but an accelerating expanding universe. Inflation accomplishes this, with the sought after exotic form of matter described by a scalar matter field $\phi(x^\mu)$.

The action for a spacetime with this matter field ϕ is given by

$$\begin{aligned} S &= S_{gravity} + S_{matter} && (2.10) \\ &= \int d^4x\sqrt{-g}\left[\left(\frac{1}{16\pi G}R(x)\right) + \left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)\right] && (2.11) \end{aligned}$$

where $g \equiv det(g_{\mu\nu})$, $R(x)$ is the Ricci scalar and $V(\phi)$ is the potential energy density function for the field ϕ. Varying the action with respect to ϕ gives

$$\frac{\delta S}{\delta \phi} = \nabla_\alpha \nabla^\alpha \phi - \frac{\partial V}{\partial \phi} = 0 \tag{2.12}$$

where again, ∇_α is the spacetime covariant derivative. In an FRW universe, this equation simplifies to

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \tag{2.13}$$

with $H \equiv \frac{\dot{a}}{a}$ the Hubble constant and $V_{,\phi} \equiv \frac{\partial V}{\partial \phi}$.

If we instead vary the action with respect to the spacetime metric, we find the Einstein field equations with the stress-energy tensor given by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right) \tag{2.14}$$

Again by isotropy and homogeneity, we must have $\phi(x^\mu) = \phi(t)$, so that $\partial_\mu\phi = \dot{\phi}\delta_{\mu 0}$. Noting that $T_{00} = \rho$ and $T_{ij} = g_{ij}p$ (for $i, j \in \{1,2,3\}$), we find that

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.15}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.16}$$

2.4 Solving the Flatness Problem

One might say that Eqs. (2.15-2.16) make inflation possible. Indeed, the main idea behind inflation is to consider an epoch of the universe where the energy density was dominated by the potential energy of the scalar fields: $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. We say that the fields are *slow-rolling* if this is the case since we neglect the kinetic energy contribution $\frac{1}{2}\dot{\phi}^2$.

Note that for $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, the pressure and density equations, Eq.(2.15) and Eq.(2.16) respectively, become $\rho \approx V(\phi)$ and $p \approx -V(\phi)$, so that the equation of state becomes

$$p \approx (-1)\rho \tag{2.17}$$

We have thus achieved our goal of finding some form of matter for which $\omega < -\frac{1}{3}$. In a universe which we can describe by Eq.(2.11), cosmological dynamics will thus drive Ω towards 1. With inflation, we *expect* our universe to be flat, instead of requiring extreme fine-tuning of initial conditions such that time-evolution does not make Ω to far from its observed value. As we see, the slow-roll approximation is crucial to the inflationary mechanism, allowing it to solve the flatness problem.

Chapter 3

Multifield Inflation & Non-minimal Couplings

3.1 Multifield models & the Jordan Frame

Until July 4th, 2012, the existence of even a single scalar field within the Standard Model of particle physics had never been confirmed. The discovery of the Higgs particle on that day however provided the evidence that scalar fields do naturally occur in nature[35, 36].

Realistic models of high-energy particle physics, from beyond the standard model supersymmetric theories all the way to string theory, predict the existence of many more scalar fields, beyond the now familiar Higgs boson. It is thus well worthwhile to study multifield models in detail.

More concretely, we consider $\mathcal{N}$ scalar fields ϕ^I with $I \in \{1, 2, ..., \mathcal{N}\}$ in a (3+1) dimensional spacetime. As previously noted, we use the metric signature $(-, +, +, +)$. We begin with the following action

$$S_{\text{Jordan}} = \int d^4x \sqrt{-\tilde{g}} \left[f(\phi^I)\tilde{R} - \frac{1}{2}\tilde{\mathcal{G}}_{IJ}\tilde{g}^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J - \tilde{V}(\phi^I) \right], \tag{3.1}$$

We call this the action in the Jordan frame, where space-time is parametrized in terms of the metric $\tilde{g}_{\mu\nu}$ and the Ricci scalar $\tilde{R}$. All quantities in the Jordan frame will be denoted by a tilde. Note how the fields interact directly with gravity through the coupling $\tilde{R}f(\phi^I)$ in the Jordan frame. We say the fields are non-minimally coupled to the Ricci scalar and we refer to $f(\phi^I)$ in Eq.(3.1) as the non-minimal coupling function. Finally, $\tilde{V}(\phi^I)$ is the field potential in the Jordan frame.

The term $f(\phi^I)R$ is in fact required by renormalization of quantum fields in curved space-times. In other words, without this term, we would not be able to make sense of the theory at the high energy scales of inflation[11]. Pioneering work in 1970 [9] showed that $f(\phi^I)$ must take the form

$$f(\phi^I) = \frac{1}{2}\left[M_0^2 + \sum_I \xi_I \left(\phi^I\right)^2\right], \tag{3.2}$$

where M_0 is some mass-scale, and the non-minimal couplings ξ_I are dimensionless constants. We will assume that the fields do not develop any non-zero expectation values, i.e. $<\phi^I>=0$, so that we take $M_0 \approx M_{pl}$, where $M_{pl} \equiv \frac{1}{\sqrt{8\pi G}}$.

There are no clear restrictions on the range of values for the ξ_I. In this thesis, we will only consider positive non-minimal couplings: $\xi_I > 0$. The renormalization group flow analysis of how the value of ξ_I depends on energy scale is sensitive to the matter sector of the Lagrangian under study. Assuming matter content akin to the Standard model, computation of the beta function for ξ_I shows that the ξ_I should grow logarithmically with the energy scale. Furthermore, $\beta(\xi_I)$ does not have any fixed point; hence, the ξ_I grow without bound[12]. Previous studies of Higgs inflation showed in fact that [22]

$$\frac{\xi_I(Infl.scale)}{\xi_I(ElectroW.scale)} \approx \mathcal{O}(10^1 - 10^2) \tag{3.3}$$

. If we expect non-minimal couplings of $\mathcal{O}(1)$ at low energies, it is thus completely reasonable to consider inflationary models with ξ_I of $\mathcal{O}(10^2)$.

Simple mass-dimension power-counting shows that renormalization equally restricts the form of the Jordan frame potential $\tilde{V}(\phi)$. Assuming a polynomial form[1],

[1]Non-polynomial potentials remain an open problem in field theory[22].

only terms up to fourth powers in the fields may be included. We have thus motivated the models we will be studying, and have shown how consistency considerations clearly delimit what terms we may sensibly include in our theory.

3.2 Transforming to the Einstein Frame

Recall the action for a multifield model in the Jordan frame, assuming canonical kinetic terms such that $\tilde{\mathcal{G}}_{IJ} = \tilde{\delta}_{IJ}$

$$S_{\text{Jordan}} = \int d^4x \sqrt{-\tilde{g}} \left[f(\phi^I)\tilde{R} - \frac{1}{2}\tilde{\delta}_{IJ}\tilde{g}^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J - \tilde{V}(\phi^I) \right], \tag{3.4}$$

While well motivated, this model is at first somewhat cumbersome to work with in this form, because the gravitational sector differs from regular Einstein gravity. We can side-step this difficulty by performing a conformal transformation of the metric $\tilde{g}_{\mu\nu}$. Note that such a transformation is essentially a locally defined stretching of the metric $\tilde{g}_{\mu\nu}$ and is not a coordinate transformation.

We thus define the metric in this new frame, $g_{\mu\nu}$, as

$$g_{\mu\nu}(x) = \Omega^2(x)\,\tilde{g}_{\mu\nu}(x), \tag{3.5}$$

where

$$\Omega^2(x) = \frac{2}{M_{\text{pl}}^2} f(\phi^I(x)). \tag{3.6}$$

We call this new frame the Einstein frame, since the gravity sector now becomes the standard Einstein-Hilbert action:

$$S_{\text{Einstein}} = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2}{2} R - \frac{1}{2}\mathcal{G}_{IJ} g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J - V(\phi^I) \right]. \tag{3.7}$$

However, note the kinetic term for the fields has acquired a non-canonical form, akin to a non-linear sigma model familiar in particle physics. Under this transformation, we have $\tilde{\delta}_{IJ} \to \mathcal{G}_{IJ}$, where

$$\mathcal{G}_{IJ}(\phi^K) = \frac{M_{\rm pl}^2}{2f(\phi^I)}\left[\tilde{\delta}_{IJ} + \frac{3}{f(\phi^I)} f_{,I} f_{,J}\right] \tag{3.8}$$

with $f_{,I} \equiv \frac{\partial f}{\partial \phi^I}$. We refer to $\mathcal{G}_{IJ}$ as the field-space metric. The potential in the Einstein frame differs from the Jordan frame potential, and can be written as

$$V(\phi^I) = \frac{1}{\Omega^4(x)}\tilde{V}(\phi^I) = \frac{M_{\rm pl}^4}{4f^2(\phi^I)}\tilde{V}(\phi^I). \tag{3.9}$$

These two features deserve close attention, as they result in phenomenologically important characteristics of our model.

First, we may view the $\mathcal{N}$ fields $\phi^I(x^\mu)$ as maps from the spacetime manifold $\mathcal{M}$ (which we will assume to be described by an FRW metric) to a target field-manifold $\mathcal{F}$, i.e. $\phi^I(x^\mu) : \mathcal{M} \to \mathcal{F}$. More precisely, the fields are a composition of two maps: $\phi^I(x^\mu) : \mathbb{R}^{3,1} \to \mathcal{M} \to \mathcal{F}$, so that the ϕ^I are in fact coordinates on the $\mathcal{N}$ dimensional field-manifold $\mathcal{M}$. In this light, $\mathcal{G}_{IJ}(\phi^K)$ is a (local) pseudo-Riemannian metric on $\mathcal{M}$. Computing curvature invariants such as the Ricci scalar show that $\mathcal{M}$ has non-vanishing intrinsic curvature. We say that the conformal transformation between the Jordan and Einstein frames has induced a curved field-manifold[13]. As we shall see in subsequent sections, this curvature can substantially alter the inflationary dynamics.

One might ask whether it may be possible to define some sort of additional conformal transformation, this time on $\mathcal{G}_{IJ}$, such that $\mathcal{G}_{IJ}(\phi^K) \stackrel{?}{=} \lambda^2(\phi^K)\delta_{IJ}$. In a certain sense, this would make things easier, just as we recovered Einstein gravity when passing from the Jordan to the Einstein frame. However, it is shown in [13] that such a transformation cannot exist with $\lambda(\phi^K)$ defined globally on $\mathcal{F}$. In other words, we must content ourselves with the action given by Eq.(3.7). Note this implies that fields in the Jordan frame will necessarily have non-canonical kinetic terms in the Einstein frame, even if, as we assumed, $\tilde{\mathcal{G}}_{IJ} = \tilde{\delta}_{IJ}$ in the Jordan frame.

Secondly, recall that renormalizability of the field theory in the Jordan frame stipulates that the potential $\tilde{V}(\phi)$ contains terms at most quartic in the fields. For sake of concreteness, we will assume the following general form for $\tilde{V}(\phi)$:

$$\tilde{V}(\phi^I) = \frac{1}{2}\sum_I m_I^2 \left(\phi^I\right)^2 + \frac{1}{2}\sum_{I<J} g_{IJ} \left(\phi^I\right)^2 \left(\phi^J\right)^2 + \frac{1}{4}\sum_I \lambda_I \left(\phi^I\right)^4 \tag{3.10}$$

In the Einstein frame, the potential acquires the remarkable feature that it is asymptotically flat in all field-space directions. This is crucial for inflationary models, so that there will exist a non-negligible window where the slow-roll approximation holds and $\rho \approx -p$ in order to solve the flatness problem. Concretely, in the limit as the J^{th} component of the field grows arbitrarily large, we have:

$$V(\phi^I) = \frac{M_{\rm pl}^4}{4}\frac{\tilde{V}(\phi^I)}{f^2(\phi^I)} \to \frac{M_{\rm pl}^4}{4}\frac{\lambda_J}{\xi_J^2} \tag{3.11}$$

(no sum on J). The flatness of the potential in the Einstein frame thus arises naturally, instead of being an additional requirement on $V(\phi)$. It is equally noteworthy that all of inflation can occur with $|\phi^J| < M_{pl}$, since renormalization group flow analysis show that $\xi_J \gg 1$ at the energy scales of inflation. This is unlike most models of ordinary chaotic inflation with minimal couplings where our current understanding of particle physics is relied upon at energy scales above M_{pl}, at which we no longer expect results to remain valid.

3.3 Dynamics in the Einstein Frame

To find the relevant equations of motion for our system in the Einstein frame, we first vary the action in Eq.(3.7) with respect to the metric and subsequently with respect to the scalar fields ϕ^I.

Variation of Eq.(3.7) with respect to $g^{\mu\nu}(x)$ gives the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{\rm pl}^2}T_{\mu\nu}, \tag{3.12}$$

with the stress-energy tensor $T_{\mu\nu}$ given by

$$T_{\mu\nu} = \mathcal{G}_{IJ}\partial_\mu\phi^I\partial_\nu\phi^J - g_{\mu\nu}\left[\frac{1}{2}\mathcal{G}_{IJ}g^{\alpha\beta}\partial_\alpha\phi^I\partial_\beta\phi^J + V(\phi^I)\right]. \tag{3.13}$$

The equation of motion for the $\mathcal{N}$ fields in turn is given by

$$\Box\phi^I + g^{\mu\nu}\Gamma^I_{JK}\partial_\mu\phi^J\partial_\nu\phi^K - \mathcal{G}^{IK}V_{,K} = 0, \tag{3.14}$$

where the covariant D'Alembertian $\Box$ is defined in terms of the spacetime covariant derivative ∇_μthrough $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. $\Gamma^I_{JK}(\phi)$ is the Levi-Civita connection compatible with the field-space metric $\mathcal{G}_{IJ}$.

We can combine the 00 and i, j components of Eq.(3.12) with the equations of motion for ϕ to derive the Friedmann equations for our model. We find

$$\begin{aligned} H^2 &= \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\mathcal{G}_{IJ}\dot\varphi^I\dot\varphi^J + V(\varphi^I)\right], \\ \dot H &= -\frac{1}{2M_{\rm pl}^2}\mathcal{G}_{IJ}\dot\varphi^I\dot\varphi^J, \end{aligned} \tag{3.15}$$

where as usual the Hubble parameter H is defined by $H \equiv \frac{\dot a}{a}$.

Later, we will consider the full quantum theory of the inflaton and decompose the field as quantum fluctuations around the classical (background) value

$$\phi^I(x^\mu) = \varphi^I(t) + \delta\phi^I(x^\mu) \tag{3.16}$$

In Section4, we will elaborate on refinements of this decomposition which respect field-space covariance. Fluctuations in the fields will necessary lead to fluctuations in the stress-energy tensor, which in turn, through the Einstein field equations, result in perturbations of the spacetime metric away from a perfect FRW universe. We parametrize these spacetime fluctuations as

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\ &= -\left(1+2A\right)dt^2 + 2a\left(\partial_i B\right)dx^i dt + a^2\left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j \end{aligned} \tag{3.17}$$

It is important to note that A, E, B and ψ are not all independent, and even more importantly, are gauge dependent quantities (in the sense they depend on the chosen coordinate chart of the spacetime manifold$\mathcal{M}$).

3.4 Our Concrete Model

For concreteness, we now specify the particular form of the potential and the number of fields we will work with in our subsequent calculations. The simplest multi-field model is of course a two-field model. Recalling the general renormalizability considerations for the potential in the Jordan frame as outlined in Section3.2, we take

$$\tilde{V}(\phi,\chi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}g\phi^2\chi^2 + \frac{\lambda_\phi}{4}\phi^4 + \frac{\lambda_\chi}{4}\chi^4 \tag{3.18}$$

We also write the non-minimal coupling function as

$$f(\phi,\chi) = \frac{1}{2}\left[M_{\rm pl}^2 + \xi_\phi\phi^2 + \xi_\chi\chi^2\right] \tag{3.19}$$

so that the potential in the Einstein frame becomes

$$V(\phi,\chi) = \frac{M_{\rm pl}^4}{4}\frac{\left(2m_\phi^2\phi^2 + 2m_\chi^2\chi^2 + 2g\phi^2\chi^2 + \lambda_\phi\phi^4 + \lambda_\chi\chi^4\right)}{\left[M_{\rm pl}^2 + \xi_\phi\phi^2 + \xi_\chi\chi^2\right]^2}. \tag{3.20}$$

We would like to bring to the attention of the reader that the non-minimal coupling constants ξ_I need not necessarily be equal, unless protected by some underlying symmetry. One particular instance of such a restriction is illustrated in the case of Higgs inflation, where the fields form part of a complex doublet, so that all component fields interact with the Ricci scalar equally (modulo gauge choices)[23, 21]. Unequal ξ_I result in yet another interesting feature of our models, namely the existence of bumps and ridges in the potential $V(\phi)$. This can be seen most easily by noting that the potential asymptotes roughly to $\frac{1}{\xi_J}$ as we take the J^{th} field to take arbitrarily large values. Hence, for unequal non-minimal coupling constants, the different asymptotic plateaus lead to distinctive valley/mountain-like characteristics. The potential in the

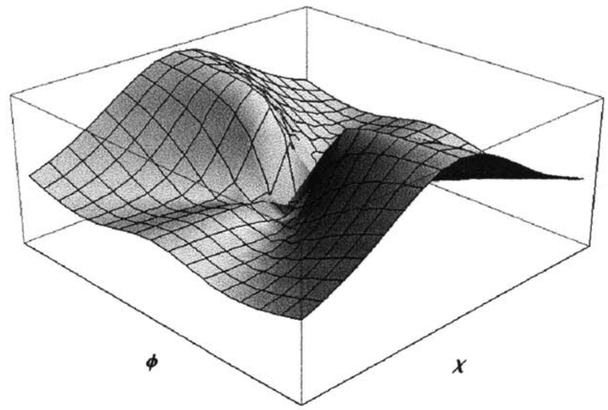

Figure 3-1: The Einstein-frame effective potential, Eq. (3.20), for a two-field model. The potential shown here corresponds to the couplings $\xi_\chi/\xi_\phi = 0.8$, $\lambda_\chi/\lambda_\phi = 0.3$, $g/\lambda_\phi = 0.1$, and $m_\phi^2 = m_\chi^2 = 10^{-2}\,\lambda_\phi M_{\rm pl}^2$.

Jordan frame given in Eq.(3.20) is plotted in Figure 3-1.

Recent reviews have emphasized the correlation between divergent classical field-space trajectories and sizable non-gaussianities (i.e. non-negligible three-point correlation functions) in the cosmic microwave background. The bumpiness of the potential in the Einstein frame is perfectly suited to give such dynamically rich behavior. However, we also see that including masses for the fields can tame these divergences in the sense that small dents can develop at the top of the ridges (for example, along the $\chi \sim 0$ axis in Figure 3-1). We can make these statements more quantitative by looking at the second derivative of the potential in Eq.(3.20) along $\chi = 0$,

$$\left(\partial_\chi^2 V\right)_{|\chi=0} = \frac{1}{\left[M_{\rm pl}^2 + \xi_\phi \phi^2\right]^3}\left[\left(g\xi_\phi - \lambda_\phi \xi_\chi\right)\phi^4 + \left(\xi_\phi m_\chi^2 - 2\xi_\chi m_\phi^2 + gM_{\rm pl}^2\right)\phi^2 + m_\chi^2 M_{\rm pl}^2\right]. \tag{3.21}$$

For subplanckian field masses, $m_\phi^2, m_\chi^2 \ll M_{\rm pl}^2$, yet at energy scales high enough for inflation to occur where $\xi_\phi \phi^2 \gg M_{\rm pl}^2$, we find that the ridge along $\chi = 0$ remains a local maximum as long as:

$$g\xi_\phi < \lambda_\phi \xi_\chi. \tag{3.22}$$

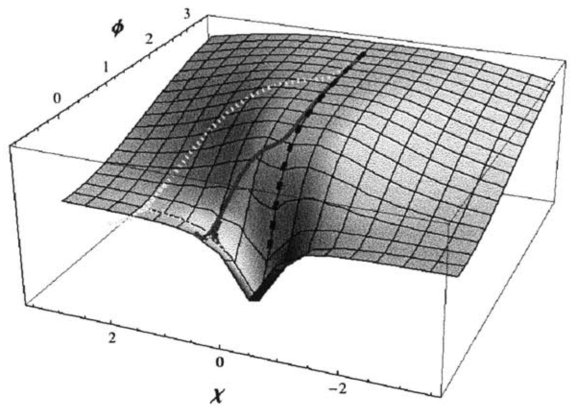

Figure 3-2: Parametric plot of the fields' evolution superimposed on the Einstein-frame potential. Trajectories for the fields ϕ and χ that begin near the top of a ridge will diverge. In this case, the couplings of the potential are $\xi_\phi = 10$, $\xi_\chi = 10.02$, $\lambda_\chi/\lambda_\phi = 0.5$, $g/\lambda_\phi = 1$, and $m_\phi = m_\chi = 0$. (We use a dimensionless time variable, $\tau \equiv \sqrt{\lambda_\phi} M_{\rm pl} t$, so that the Jordan-frame couplings are measured in units of λ_ϕ.) The trajectories shown here each have the initial condition $\phi(\tau_0) = 3.1$ (in units of $M_{\rm pl}$) and different values of $\chi(\tau_0)$: $\chi(\tau_0) = 1.1 \times 10^{-2}$ ("trajectory 1,"yellow dotted line); $\chi(\tau_0) = 1.1 \times 10^{-3}$ ("trajectory 2," red solid line); and $\chi(\tau_0) = 1.1 \times 10^{-4}$ ("trajectory 3," black dashed line).

In this scenario, we note that our potential reproduces many of the interesting features of the well-studied product potential $V = m^2 e^{-\lambda\phi^2}\chi^2$ [18, 16] , yet arises completely naturally in the Einstein frame, assuming nothing more than a simple, renormalizable polynomial potential in the Jordan frame.

As long as the non-minimal coupling constants are not exactly equal, the potential in Eq.(3.20) can easily lead to sharply diverging trajectories, as highlighted in Figure 3-2

The three trajectories plotted in Figure 3-2 are excellent representatives of the three relevant types of field dynamics we wish to study. We showed in [13] how trajectory 2 produces a sizable bispectrum.

We note two additional strengths of our model. Unlike for example the product potential $V = m^2 e^{-\lambda\phi^2}\chi^2$, even once the fields fall off the ridge, sufficient inflation can still occur thanks to the valleys which focus all trajectories towards the global minimum at $\phi = \chi = 0$. In fact, the close-to-quadratic shape of the valley allows for oscillatory motion, such that we might even detect observable "ringing" in the scale factor $a(t)$, which has been proposed as a novel observational indicator for multifield

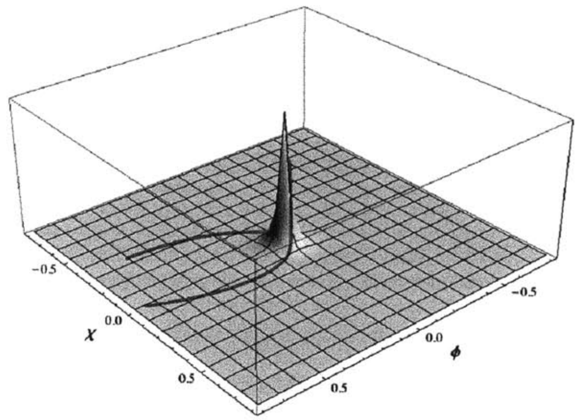

Figure 3-3: Parametric plot of the evolution of the fields ϕ and χ superimposed on the Ricci curvature scalar for the field-space manifold, $\mathcal{R}$, in the absence of a Jordan-frame potential. The fields' geodesic motion is nontrivial because of the non-vanishing curvature. Shown here is the case $\xi_\phi = 10$, $\xi_\chi = 10.02$, $\phi(\tau_0) = 0.75$, $\chi(\tau_0) = 0.01$, $\phi'(\tau_0) = -0.01$, and $\chi'(\tau_0) = 0.005$.

inflation [24]. Secondly, the global minimum of the potential can just as well support a prolonged period of dynamic reheating.

So far, we have only addressed the effects of the potential in the Einstein frame. Recall that the conformal transformation between the Jordan and Einstein frames also introduced non-canonical kinetic terms, which we interpreted as inducing curvature in the target field-space. We may view the field evolutions as trajectories in this curved field-space. Indeed, by homogeneity and isotropy, we have $\phi(x^\mu) = \phi(t)$, so that we can consider t as some affine parameter along the path traced out in field-space. Akin to geodesics in general relativity, field trajectories in our model can diverge in field space even in the absence a potential. Indeed, the curvature of the field-manifold $\mathcal{F}$ alone, parametrized by the corresponding Ricci scalar $\mathcal{R}$, can source geodesic deviation. The Ricci scalar for the two field case is listed in Eq.(9). We plot the field trajectories in the absence of a potential in the Jordan frame - and thus also in the Einstein frame - superimposed upon the Ricci scalar $\mathcal{R}$. We see that the curvature of $\mathcal{F}$ vanishes for larges field values but rises sharply as $\phi, \chi \to 0$. $\mathcal{R}$ will thus become particularly relevant in studies of preheating.

In Section 7, we find a formalism which reduces even those models which do include a potential to the problem of finding geodesics of a suitably related manifold. This geometric picture of multifield inflation motivates us to push the analogy with general relativity even further, and construct a manifestly field-space covariant framework. This is the goal of the next section

Chapter 4

Covariant Formalism

4.1 The Necessity of Field-Space Covariance

Relativity was born from the idea that physics cannot on our choice of coordinates. We have again and again learned to "filter out" unphysical degrees of freedom in our description of physical systems, from gauge artifacts in particle gauge theories to apparent spacetime singularities which reflect nothing more than a poor choice of spacetime coordinates.

The action in Eq.(3.7) is invariant under arbitrary field redefinitions. Indeed, under a field-space coordinate transformation $\phi^K \to \phi'^K(\phi^L)$, we have

$$\mathcal{G}'_{IJ}(\phi'^K) = \frac{M_{\rm pl}^2}{2f(\phi'^I)}\left[\delta'_{IJ} + \frac{3}{f(\phi'^I)}f'_{,I}f'_{,J}\right] \tag{4.1}$$

$$\to \frac{M_{\rm pl}^2}{2f(\phi'^I(\phi^L))}\left[\frac{\partial\phi^K}{\partial\phi'^I}\frac{\partial\phi^L}{\partial\phi'^J}\delta_{KL} + \frac{3}{f(\phi'^I(\phi^L))}\frac{\partial\phi^K}{\partial\phi'^I}f_{,K}\frac{\partial\phi^L}{\partial\phi'^J}f_{,L}\right] \tag{4.2}$$

$$= \frac{\partial\phi^K}{\partial\phi'^I}\frac{\partial\phi^L}{\partial\phi'^J}\mathcal{G}_{KL} \tag{4.3}$$

Similarly,

$$g^{\mu\nu}\partial_\mu\phi'^I\partial_\nu\phi'^J = g^{\mu\nu}\frac{\partial\phi'^I}{\partial\phi^K}\partial_\mu\phi^K\frac{\partial\phi'^J}{\partial\phi^L}\partial_\nu\phi^J \tag{4.4}$$

$$= g^{\mu\nu}\partial_\mu\phi^K\partial_\nu\phi^L\frac{\partial\phi'^I}{\partial\phi^K}\frac{\partial\phi'^J}{\partial\phi^L} \tag{4.5}$$

As the two terms transform with the inverse transformation, the combination $\mathcal{G}_{IJ} g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J$ is thus invariant, so that

$$S_E[\phi'] = S_E[\phi] \tag{4.6}$$

The physical content of our theory thus cannot depend on our choice of the fields and explains the need for a field-space covariant formalism.

4.2 Field-space Covariant Derivatives & Equations of Motion

Our geometrization of multifield inflation was heavily influenced by [18]. In their work, the authors of [18] constructed what they called the kinematic basis. The kinematic basis is essentially a tetrad basis, locally labeling the field-manifold along any inflationary trajectory, akin to the Frenet-Serret frame familiar from classical mechanics. While several kinematic quantities greatly simply in this framework, the construction is inevitably local and dependent on the nature of the field trajectory.

Our formalism instead constructs a globally defined set of coordinates, which covers the entire field manifold $\mathcal{F}$. On one hand, this simplifies the study of $\mathcal{F}$'s global geometry, which will determine the general nature of field-space trajectories as we will see in Section 7. On the other hand, it also provides greater insight as to how inflationary dynamics relate directly to the field-content specified in the Lagrangian. We also leave all indices explicit.

As we have already pointed out, we view the $\mathcal{N}$ fields ϕ^I as coordinates for the entire field-manifold, since they provide a map from $\mathbb{R}^n$ to $\mathcal{F}$. Note that ϕ^I is *not* a vector, the same way that x^μ is not a vector in general relativity.

The inner product defined on $T_p\mathcal{F}$, the tangent space of $\mathcal{F}$ at a point P labeled by coordinates ϕ^K_p, is computed using the metric $\mathcal{G}_{IJ}(\phi^L_p)$. In other words, for any two vectors $A^I, B^J \in T_p\mathcal{F}$, their inner product is defined as

$$A \cdot B = \mathcal{G}_{IJ}(\phi_p^L) A^I B^J \tag{4.7}$$

The metric, being a symmetric rank-two tensor, is invertible. We denote its inverse by $\mathcal{G}^{IJ}$ such that

$$\mathcal{G}_{IJ}\mathcal{G}^{JK} = \delta_I^K \tag{4.8}$$

The metric also allows us to raise and lower field-space indices. More formally, it provides a map from vectors to 1-forms and the inverse metric maps 1-forms to vectors:

$$A_I = \mathcal{G}_{IJ} A^J \tag{4.9}$$

$$A^I = \mathcal{G}^{IJ} A_J \tag{4.10}$$

To define the physical rate of change of a vector along some path, we define the field-space covariant derivative ∇_I by its action on a vector A^J and a 1-form B_J such that

$$\nabla_I A^J = \partial_I A^J + \Gamma^J_{IK} A^K \tag{4.11}$$

$$\nabla_I B_J = \partial_I B_J - \Gamma^K_{IJ} A_K \tag{4.12}$$

where Γ^A_{BC} is the field-space Levi-Civita connection, which is the unique metric-compatible connection such that $\nabla_I \mathcal{G}_{JK} = 0$ [26]. The field-space connection can perhaps best be understood in terms of its actions on the holonomic basis vectors $\hat{e}_I \equiv \frac{\partial}{\partial \phi^I}$. Indeed, we can write any vector as a linear superposition of basis vectors:

$$V = V^I \hat{e}_I \tag{4.13}$$

The covariant derivative then acts as

$$\begin{aligned} \nabla_J V = \nabla_J(V^I \hat{e}_I) &= (\nabla_J V^I)\hat{e_I} + V^I \nabla_J \hat{e}_I && (4.14) \\ &= (\partial_J V^I)\hat{e}_I + V^I \Gamma^K_{IJ} \hat{e}_K && (4.15) \\ (\nabla_J V)^K \hat{e}_K &= (\partial_J V^K + V^I \Gamma^K_{IJ})\hat{e}_K && (4.16) \end{aligned}$$

The affine connection thus satisfies $\nabla_A \hat{e}_B = \Gamma^C_{AB} \hat{e}_C$ [1]. Note how we replaced $(\nabla_J V^I)$ by $\partial_J V^I$ in going from the first to second line. This comes from the fact that we mean the derivative is operating explicitly on the component V^I. When we defined the covariant derivative in Eq.(4.11), we technically really meant $(\nabla_J V)^I$.

As we wish to study the dynamics of our field theory, we of course need to define what is meant by the time-derivative. The "ordinary" definition from calculus holds in the sense that we can write the covariant time-derivative as:

$$\mathcal{D}_t \phi^I(t) \equiv \lim_{\epsilon \to 0} \frac{\phi^I(t+\epsilon) - \phi^I(t)}{\epsilon} \tag{4.17}$$

Recall that $\phi^I(t)$ is, from a geometric standpoint, a path traced out on $\mathcal{F}$ parametrized by an affine parameter t. The subtlety in the definition of the time derivative comes from the fact that the tangent spaces at $t+\epsilon$ and t are not the same, which is why we need to resort to the field-space covariant derivative we defined above. We thus define

$$\mathcal{D}_t \phi^I(t) = \frac{d\phi^K(t)}{dt} \nabla_K \phi^I(t) \tag{4.18}$$

$\mathcal{D}_t$ is therefore a directional derivative, obtained by taking the inner product of the field-space covariant rate of change $\nabla_K \phi^I(t)$ with the field velocity vector $\frac{d\phi^K(t)}{dt}$. From here on, we will abbreviate the velocity vector as $\frac{d\phi^K(t)}{dt} = \dot{\phi}^K(t)$.

Recall now the equations of motions for the fields we derived earlier in Section 3.3

$$\Box \phi^I + g^{\mu\nu} \Gamma^I_{JK} \partial_\mu \phi^J \partial_\nu \phi^K - \mathcal{G}^{IK} V_{,K} = 0, \tag{4.19}$$

[1]This can in fact be used as definition of the Christoffel symbol.

Again using the fact that $\phi^I(x^\mu) = \phi^I(t)$ by isotropy and homogeneity, along with the identity that $\Box = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi^I)$, we can simplify this to

$$\begin{aligned}
\partial_t(\dot{\phi}^I) + \Gamma^I_{JK}\dot{\phi}^J\dot{\phi}^K + 3H\dot{\phi}^I + \mathcal{G}^{IK}V_{,K} &= 0 \qquad (4.20)\\
\frac{\partial\phi^L}{\partial t}\left(\frac{\partial}{\partial\phi^L}\dot{\phi}^I + \Gamma^I_{JL}\dot{\phi}^J\right) + 3H\dot{\phi}^I + \mathcal{G}^{IK}V_{,K} &= 0 \qquad (4.21)\\
\mathcal{D}_t\dot{\phi}^I + 3H\dot{\phi}^I + \mathcal{G}^{IK}V_{,K} &= 0 \qquad (4.22)
\end{aligned}$$

Since the action defined in Eq.(3.7) was a field-space invariant, the equations of motion had to be field-space covariant; however, we have now put them in a manifestly covariant form. In other words, if we redefined our fields to some new set ϕ'^I, then the equations would simply be given by $\mathcal{D}_t\dot{\phi}'^I + 3H\dot{\phi}'^I + \mathcal{G}'^{IK}V'_{,K} = 0$. Note how $\mathcal{D}_t = \dot{\phi}^A\nabla_A = \dot{\phi}'^B\nabla'_B$ is a field-space invariant, with all field-space indices contracted. This makes sense as what we mean by a time derivative should not depend on our choice of field-space coordinates.

The construction of the field-space invariant time-derivative is an excellent example of what we referred to at the beginning of this section as counting only the physical degrees of freedom. To calculate the rate of change of a field-space vector in a covariant way, we saw that we needed to consider both the contributions from how the vector changes with respect to fixed basis-vectors as well as how the basis-vectors themselves change when we compare two tangent spaces. In the form of Eq.(4.22), we see our equations of motion are simply $\mathcal{N}$ copies of the familiar single-field dynamical equation $\ddot{\phi}+3H\dot{\phi}+V_{,\phi} = 0$, made suitably covariant to reflect the non-trivial field-space metric $\mathcal{G}_{IJ}$.

Chapter 5

Density Perturbations

So far, we have only considered the classical field trajectories. However, inflation is a *quantum* field theory, and some of its greatest successes, such as the prediction of the statistical properties of temperature fluctuations in the cosmic microwave background radiation, rely crucially on its inherent quantum nature. In this section, we provide a geometric view of the so-called background-field method which decomposes the full field dynamics into quantum perturbations upon a classical trajectory.

5.1 The Background-Field Method

A field which obeys the rules of quantum mechanics does not follow the classical equations of motion derived from the action. The quantum theory is indeed not defined simply through the action S but through the path integral[1]

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}+J\phi)} \tag{5.1}$$

However, as hinted at earlier, we may decompose the full field behavior as follows:

$$\phi(x^\mu) = \phi_{cl}(t) + \delta\phi(x^\mu) \tag{5.2}$$

[1]We consider here only the case of a single-field model for pedagogical purposes, but the analysis holds over just as well for multifield theories.

Note that the classical field evolution dominates the path integral, since setting $\frac{\delta S}{\delta \phi} = 0$ is nothing more than making a "steepest-descent" approximation. A good way to approximate the full quantum theory is thus to Taylor expand the action in the path integral around the classical background, keeping only a finite number of terms containing the small quantum fluctuations $\delta\phi$. We write

$$S[\phi] = S[\phi_{cl} + \delta\phi] \tag{5.3}$$

$$= S[\phi_{cl}] + \int d^4x\sqrt{-g}\frac{\delta\mathcal{L}(\phi_{cl})}{\delta\phi}\delta\phi \tag{5.4}$$

$$+ \int d^4x\sqrt{-g}\frac{\delta^2\mathcal{L}(\phi_{cl})}{\delta\phi\delta\phi}\delta\phi\delta\phi + \mathcal{O}(\delta\phi^3) \tag{5.5}$$

$\int d^4x\sqrt{-g}\frac{\delta\mathcal{L}}{\delta\phi}|_{\phi=\phi_{cl}}\delta\phi$ vanishes since the classical equations of motion are

$$\int d^4x\sqrt{-g}\frac{\delta\mathcal{L}(\phi_{cl})}{\delta\phi}\delta\phi = \frac{\delta S}{\delta\phi}|_{\phi=\phi_{cl}} = 0,$$

leaving us only with the last term $\int d^4x\sqrt{-g}\frac{\delta^2\mathcal{L}(\phi_{cl})}{\delta\phi\delta\phi}\delta\phi\delta\phi$ up to $\mathcal{O}(\delta\phi^3)$.

Ignoring the $\mathcal{O}(\delta\phi^3)$ contributions, the path integral reduces to a functional Gaussian integral, and can in fact be computed explicitly. The analysis above carries over to the multifield case by essentially replacing $\delta\phi \to \delta\phi^I$. The computation of the path integral shall not concern us for the moment, as we instead turn our attention to the non-covariant nature of $\delta\phi^I$. Our subsequent discussion is particularly indebted to [19].

5.2 Geodesic Construction of $\mathcal{Q}^I$

Consider Figure 5-1 depicting the classical trajectory $\phi^I_{cl}(t)$ traced out in field-space. At some time t, the full field configuration is given by $\phi^I(x^\mu) \equiv \phi^I(t,\vec{x})$. The difference $\phi^I(t,\vec{x}) - \phi^I_{cl}(t) \equiv \delta\phi^I(t,\vec{x})$ represents a finite coordinate displacement, and as such does not transform covariantly like a vector. This is akin to the situation in general relativity where the quantity $\Delta x^\mu = x^\mu_A - x^\mu_B$, which represents the difference in spacetime coordinate of two events A and B, does not properly parametrize the physical distance between them, and will be a frame-dependent quantity.

We assume the neighborhood containing $\phi^I_{cl}(t)$ and $\phi^I(t,\vec{x})$ is geodesically complete, which should be the case if the quantum fluctuations can be treated perturbatively. This is tantamount to considering small (yet not infinitesimal) neighborhoods of $\phi_{cl}(t)$ and there is no reason to expect any topological obstructions. In this case, we can connect the points $\phi^I_{cl}(t)$ and $\phi^I(t,\vec{x})$ by a geodesic parametrized by λ, which we denote by $\phi^I(t,\vec{x},\lambda)$. We choose λ such that $\phi^I(t,\vec{x},\lambda=0)=\phi^I(t)$ and $\phi^I(t,\vec{x},\lambda=\epsilon)=\phi^I(t,\vec{x})$. ϵ will simply be a book-keeping tool, so that we can keep track of the perturbative expansion at each order in ϵ. At the end of our calculation, we in fact set $\epsilon=1$.

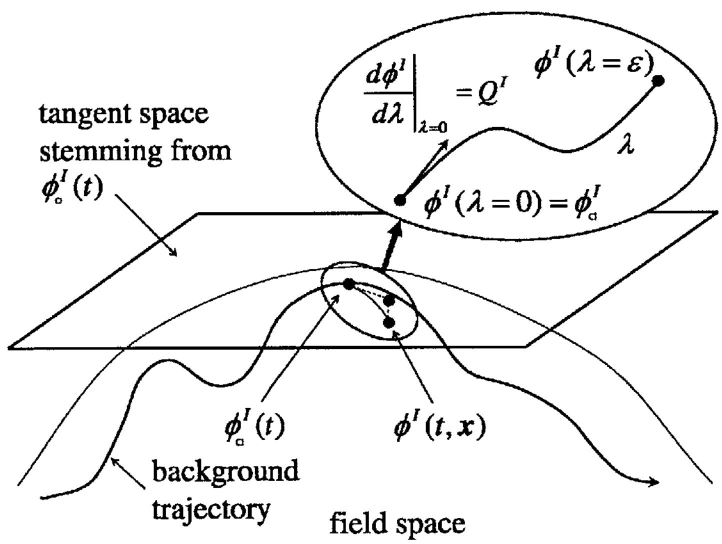

Figure 5-1: For any value of time t, the background field trajectory $\phi^I_{cl}(t)$ can be connected to the real field configuration $\phi^I(t,\vec{x})$ by a geodesic parametrized by λ. The vector tangent to the path at $\lambda=0$, $Q^I \equiv \frac{d\phi^I}{d\lambda}|_{\lambda=0}$, serves as a suitably covariant replacement for $\delta\phi^I$. (*Graphic courtesy of* [19], *and edited to reflect differences in notation*)

We define a covariantized version of $\delta\phi^I(t,\vec{x})$ by

$$Q^I(t,\vec{x}) \equiv \frac{d\phi^I(t,\vec{x},\lambda)}{d\lambda}|_{\lambda=0} \tag{5.6}$$

Note that Q^I is the vector, at the point with coordinates $\phi^I_{cl}(t)$, which is tangent to the geodesic curve connecting $\phi^I_{cl}(t)$ and $\phi^I(t,\vec{x})$. We can construct a perturbative expansion around $\mathcal{Q}^I$ thanks to a tool perhaps most familiar from Lie Theory: the exponential map.

The exponential map at a point P with coordinates ϕ^I_p of a manifold $\mathcal{F}$ is a map from the tangent space of $\mathcal{F}$ at P to $\mathcal{F}$: $exp : T_P\mathcal{F} \to \mathcal{F}$. Let A^I be a vector in the tangent space at P, $A \in T_P\mathcal{F}$. Under the assumption of geodesic completeness, there exists a unique geodesic through P with affine parameter λ, denoted by$\gamma^I(\lambda)$, such that $\gamma^I(0) = \phi^I_p$ and $\frac{d\gamma^I}{d\lambda}|_{at\,P} = A^I$. Then $exp_P(\epsilon A) = \gamma(\epsilon)$.

Since $\gamma^I(\lambda)$ is simply a differentiable function, we can also resort to Taylor's theorem and relate $\gamma^I(\epsilon)$ to $\gamma^I(0)$ through the following series expansion in ϵ:

$$\gamma^I(\epsilon) = \gamma^I(0) + \frac{d\gamma^I}{d\lambda}|_{\lambda=0}\epsilon + \frac{1}{2!}\frac{d^2\gamma^I}{d\lambda^2}|_{\lambda=0}\epsilon^2 + ... + \frac{1}{n!}\frac{d^n\gamma^I}{d\lambda^n}|_{\lambda=0}\epsilon^n + ... \tag{5.7}$$

In the case of multifield inflation, the geodesic curve in question is given by $\phi^I(t,\vec{x},\lambda)$ while the point P is the field value along the classical trajectory at time t with coordinates $\phi^I(t)$. We can thus similarly write

$$\phi^I(t,\vec{x},\lambda=\epsilon) = \phi^I_{cl} + \frac{d\phi^I(t,\vec{x},\lambda)}{d\lambda}|_{\lambda=0}\epsilon + \frac{1}{2!}\frac{d^2\phi^I(t,\vec{x},\lambda)}{d\lambda^2}|_{\lambda=0}\epsilon^2 + ... + \frac{1}{n!}\frac{d^n\phi^I(t,\vec{x},\lambda)}{d\lambda^n}|_{\lambda=0}\epsilon^n + ... \tag{5.8}$$

It is important to emphasize that the derivatives in Eq.(5.8) are "ordinary" derivatives, *not* covariant ones. However, we can trade higher-power derivatives for a product of lower ones through the geodesic equation:

$$0 = \mathcal{D}_\lambda \frac{d\phi^I(t,\vec{x},\lambda)}{d\lambda} \equiv \frac{d\phi^J(t,\vec{x},\lambda)}{d\lambda}\nabla_J \frac{d\phi^I(t,\vec{x},\lambda)}{d\lambda} \tag{5.9}$$

$$0 = \frac{d^2\phi^I(t,\vec{x},\lambda)}{d\lambda^2} + \Gamma^I_{JK}\frac{d\phi^J(t,\vec{x},\lambda)}{d\lambda}\frac{d\phi^K(t,\vec{x},\lambda)}{d\lambda} \tag{5.10}$$

Evaluating Eq.(5.10) at $\lambda = 0$, and using the fact that $\mathcal{Q}^I = \frac{d\phi^I(t,\vec{x},\lambda)}{d\lambda}|_{\lambda=0}$, we have

$$\frac{d^2\phi^I(t,\vec{x},\lambda)}{d\lambda^2}|_{\lambda=0} = -\Gamma^I_{JK}(\phi_{cl})\mathcal{Q}^J\mathcal{Q}^K \tag{5.11}$$

We can thus exchange n^{th} order derivatives like $\frac{d^n\phi^I(t,\vec{x},\lambda)}{d\lambda^n}|_{\lambda=0}$ for products of n $\mathcal{Q}^I$'s. To $\mathcal{O}(\mathcal{Q}^3)$, we have [19]

$$\phi^I(t,\vec{x},\lambda 1) - \phi^I_{cl}(t) \equiv \delta\phi^I(t,\vec{x}) \tag{5.12}$$

$$= \mathcal{Q}^I - \frac{1}{2!}\Gamma^I_{JK}\mathcal{Q}^J\mathcal{Q}^K + \frac{1}{3!}\left(\Gamma^I_{LM}\Gamma^M_{JK} - \nabla_L\Gamma^I_{JK}\right)\mathcal{Q}^J\mathcal{Q}^K\mathcal{Q}^L + \mathcal{O}(\mathcal{Q}^3) \tag{5.13}$$

The vector $\mathcal{Q}^I$ transforms, by definition, covariantly and provides the covariant perturbative formalism we set out to construct, thus concluding this section.

Chapter 6

Non-Gaussianities, the δN formalism and Hamiltonian Mechanics

6.1 Beyond Slow-roll: The full Hamiltonian Phase Space

We extend the covariant perturbative framework set out in Sec.5 by considering not only perturbations of the fields but also of their momentum. We can thus no longer work on the $\mathcal{N}$ dimensional configuration space (which we have called field-space so far), but instead must consider the entire $2\mathcal{N}$-dimensional phase space. In geometric terms, the phase space is the union of the field manifold $\mathcal{F}$ and its tangent bundle. Note this is technically not equivalent to the Hamiltonian phase space which is the union of $\mathcal{F}$ and its cotangent bundle. In other words, we are considering $\left(\phi^I, \dot{\phi}^I\right)$ as our two sets of coordinates instead of $\left(\phi^I, \mathcal{G}_{IJ}\dot{\phi}^J\right)$. We will abuse notation and continue to refer to our treatment as using the Hamiltonian formalism.

Prior work [17, 27, 20] which had considered variations of the momenta often invoked the slow-roll approximation which relates the field velocity to the field configuration by neglecting the acceleration term:

$$\mathcal{D}_t\dot{\phi}^I \approx 0 \quad (6.1)$$

$$\rightarrow 3H\dot{\phi}^I + \mathcal{G}^{IK}V_{,K} \approx 0 \quad (6.2)$$

$$\dot{\phi}^I \approx -\frac{\mathcal{G}^{IK}V_{,K}}{3H} \quad (6.3)$$

This constraint halves the degrees of freedom and thus allowed them to again work on the field configuration space.

In this section, we derive equations up to second order in the field and momentum perturbations without invoking the slow roll approximation. The main idea behind our formalism, inspired by [20], is that we can find the time-evolution of the perturbations by Taylor expanding the classical equations of motion along some path.

First, we rewrite the equations of motion in terms of the number of e-folds N (not to be confused with a field-space index), and treat the momentum $p^I \equiv \mathcal{D}_N\phi^I$ as independent from ϕ^I. The background equation of motion becomes

$$\mathcal{D}_N p^I + (3-\epsilon(p))p^I + \frac{V^{,I}}{H^2(\phi,p)} = 0 \quad (6.4)$$

where $V^{,I} \equiv \mathcal{G}^{IJ}\nabla_J V$ and the slow-roll parameter ϵ is a function of the p^I only[1]

$$\epsilon(p) = \frac{\mathcal{G}_{IJ}p^I p^J}{2M_{pl}^2} \quad (6.5)$$

and the Hubble parameter is given by

$$H^2(p,\phi) = \frac{1}{M_{pl}^2}\left(\frac{V(\phi)}{3-\epsilon(p)}\right) \quad (6.6)$$

The advantage of a Hamiltonian formulation means that we trade $\mathcal{N}$ second order equations of motion for $2\mathcal{N}$ first order differential equations, which makes certain calculations often easier to solve numerically. Indeed, we can write Eq.(6.4) as a

[1]When indicating the functional dependence of some quantity, we suppress the indices so that we write $f(\phi,p)$ instead of $f(\phi^I,p^I)$.

system of first order equations:

$$\mathcal{D}_N \begin{pmatrix} \phi^I \\ p^I \end{pmatrix} = \begin{pmatrix} p^I \\ -\frac{V}{H^2 M_{pl}^2} p^I - \frac{V_{,I}}{H^2} \end{pmatrix} \equiv \begin{pmatrix} F_1^I \\ F_2^I \end{pmatrix} \tag{6.7}$$

where we have used the fact that $3 - \epsilon = \frac{V}{H^2 M_{pl}^2}$.

To avoid writing the field-momentum multiplet in Eq. (6.1), we introduce an additional lowercase latin index $i, j ... \in \{1, 2\}$ such that $\phi_j^I \equiv \begin{pmatrix} \phi^I \\ p^I \end{pmatrix}$. In other words, $\phi_1^I = \phi^I$ and $\phi_2^I = p^I$, so that j refers to whether we are speaking of the field configuration or the field momentum. At times, we adopt the even more compact notation $\phi^\alpha \equiv \phi_j^I$, with the greek indices now running from 1 to $2\mathcal{N}$.

We can then write Eq.(6.1) succintly as

$$\mathcal{D}_N \phi^\alpha = F^\alpha(\phi^\beta) \tag{6.8}$$

where $F_1^I(\phi^\beta) = p^I$ and $F_2^I(\phi^\beta) = -\frac{V}{H^2 M_{pl}^2} p^I - \frac{V_{,I}}{H^2}$. Before deriving the appropriate perturbation equations, we briefly outline the δN formalism which motivates these results.

6.2 δN and diverging trajectories

The δN formalism, introduced in [14, 33, 30, 28, 31], provides an elegant method to compute non-gaussianities in the density perturbation spectrum produced by inflation. δN refers to the change in the number of e-folds relative to a change in the field's positions and velocities at some fiduciary scale which we normally take to be when momentum modes last crossed the Hubble scale, $k_* \approx aH$. We write such ratios as $\frac{\partial N}{\partial \phi_*}$; however, it must be made clear that these are not ordinary derivatives, and instead are non-local by construction. The δN formalism is particularly useful since the non-gaussianity parameter f_{NL}, which provides a powerful observational discriminator for multifield models, can be expressed entirely in terms of such derivatives of

N. The construction below was heavily inspired by [20, 15].

Solving the $2\mathcal{N}$ first-order equations of motion of Eq. (6.8) gives us the classical background trajectories, which we can label with the $2\mathcal{N}$ integration constants we denote by λ^α. For example, one of the integration constants is the number of e-folds N. The λ^α should not be confused with the affine parameter λ introduced in Section 5.

We are interested in the behavior of trajectories upon varying the integration constants $\lambda^\alpha \to \lambda^\alpha + \delta\lambda^\alpha$ along $N = constant$ slices of the phase space [33][2].

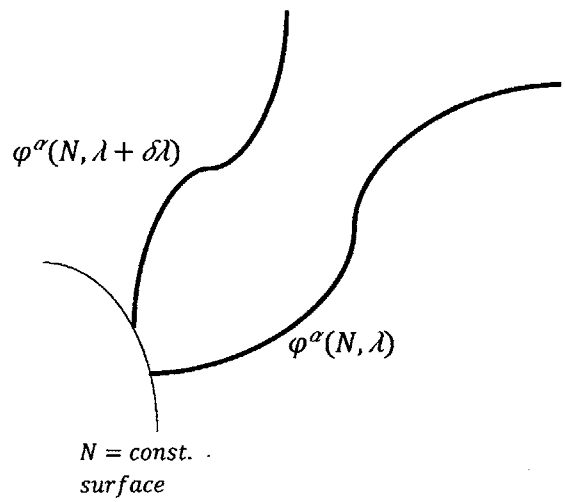

Figure 6-1: The classical field trajectories in phase space $\phi^\alpha(N, \lambda)$ are parametrized by the number of e-folds N and labeled by the constants of motion λ. We consider how variations of λ along the $N = const.$ surface alters the subsequent dynamics by finding the equations of motion for$\delta\phi^\alpha(N) \equiv \phi^\alpha(N, \lambda + \delta\lambda) - \phi^\alpha(N, \lambda)$.

We thus define the perturbation as

$$\phi^\alpha(N, \lambda^\beta + \delta\lambda^\beta) - \phi^\alpha_{cl}(N, \lambda^\beta) \equiv \delta\phi^\alpha(N) \tag{6.9}$$

Note the perturbation $\delta\phi^\alpha(N)$ defined above is completely different from the $\delta\phi^I$ we introduced in Sec.5 . Here, we are considering how the classical field theory behaves under variation of the constants of motion λ^α instead of trying to incorporate quantum effects.

[2]Since N is one of the $2\mathcal{N}$ integration constants, we are varying only $2\mathcal{N} - 1$ of the λ^α.

We can think of tracing out a path, which is *not* a geodesic, connecting the two phase space points $\phi^\alpha(N, \lambda^\beta + \delta\lambda^\beta)$ and $\phi^\alpha_{cl}(N, \lambda^\beta)$ by continuously varying λ to $\lambda + \delta\lambda$. In fact, our analysis will not depend on the exact nature of the path: it is there simply so that we can define the difference $\delta\phi^\alpha(N)$ perturbatively, by taking directional derivatives along the path.

In the end, we only need consider the evolution of statistical distributions in phase space, i.e. we care only about how the correlation between neighboring trajectories evolves over time [27, 17]. This is the underlying reason we can neglect to specify the exact path in phase space connecting the two points.

Finally, we Taylor expand $\delta\phi^\alpha(N)$ to second order as follows:

$$\delta\phi^\alpha = \delta^{(1)}\phi^\alpha + \frac{1}{2!}\delta^{(2)}\phi^\alpha + \mathcal{O}(\delta^{(3)}\phi^\alpha) \tag{6.10}$$

Note that the upper index in parentheses refers to the order of the expansion: the n^{th} -order contribution to $\delta\phi^\alpha$ would be written as $\frac{1}{n!}\delta^{(n)}\phi^\alpha$.

Concretely, the field configuration and momentum perturbations are given by

$$\delta^{(1)}\phi^I = \frac{d\phi^I}{d\lambda}\delta\lambda \equiv Q^I\delta\lambda \tag{6.11}$$

$$\delta^{(2)}\phi^I = \mathcal{D}_\lambda \frac{d\phi^I}{d\lambda}(\delta\lambda)^2 \tag{6.12}$$

$$\delta^{(1)}p^I = \mathcal{D}_\lambda p^I \delta\lambda \tag{6.13}$$

$$\delta^{(2)}p^I = \mathcal{D}_\lambda \mathcal{D}_\lambda p^I (\delta\lambda)^2 \tag{6.14}$$

where $\mathcal{D}_\lambda = \frac{d\phi^K}{d\lambda}\nabla_K$ is the phase-space covariant directional derivative along the path parametrized by varying λ (the labeling-indices on λ are suppressed). The vector Q^I, though defined analogously to the geodesic construction of Section 5 should not be confused with $\mathcal{Q}^I$ which obeys the geodesic equation $\mathcal{D}_\lambda \mathcal{Q}^I = 0$ - which Q^I obviously does not.

6.3 First Order Results

We Taylor expand the background field equations of motion along λ to first order by applying the operator $\mathcal{D}_\lambda$ to Eq.(6.4)

$$\mathcal{D}_\lambda\left(\mathcal{D}_N p^I + (3-\epsilon(p))p^I + \frac{V^{,I}}{H^2(\phi,p)}\right) = 0 \quad (6.15)$$

$$\mathcal{D}_N\mathcal{D}_\lambda p^I + [\mathcal{D}_\lambda,\mathcal{D}_N]p^I - (\mathcal{D}_\lambda\epsilon)p^I + (3-\epsilon)\mathcal{D}_\lambda p^I + \mathcal{D}_\lambda\left(\frac{V^{,I}}{H^2(\phi,p)}\right) = 0??(6.16)$$

We now note two important facts. First, we remark that

$$\begin{aligned}\mathcal{D}_\lambda p^I &= \mathcal{D}_\lambda\frac{d\phi^I}{dN} && (6.17)\\ &= \frac{d}{d\lambda}\frac{d\phi^I}{dN} + \Gamma^I_{JK}\frac{d\phi^J}{d\lambda}\frac{d\phi^K}{dN} && (6.18)\\ &= \frac{d\phi^J}{dN}\nabla_J\frac{d\phi^I}{d\lambda} && (6.19)\\ &= \mathcal{D}_N Q^I && (6.20)\end{aligned}$$

where we used the fact that ordinary derivatives commute and we recall that $Q^I = \frac{d\phi^I}{d\lambda}$. Secondly, we use the following identity which holds for any vector V^I

$$[\mathcal{D}_\lambda,\mathcal{D}_N]V^I = R^I_{JKL}V^J Q^K p^L \quad (6.21)$$

where R^I_{JKL} is the Riemann tensor of the phase space.

We can now simplify our expression for the first order equation of motion. We find that

$$\mathcal{D}_\lambda\epsilon = \frac{1}{M_{pl}^2}(D_\lambda p^I)p^J\mathcal{G}_{IJ} \quad (6.22)$$

and

$$\begin{aligned}\mathcal{D}_\lambda\left(\frac{V^{,I}}{H^2}\right) &= Q^J\left[\frac{\nabla_J\nabla^I V}{H^2} - \frac{1}{H^2 V}\left(\nabla^I V\right)\left(\nabla_J V\right)\right] && (6.23)\\ &- \frac{\nabla^I V}{V}\left(\mathcal{D}_\lambda p^J\right)p^B\mathcal{G}_{JB} && (6.24)\end{aligned}$$

Putting everything together, we obtain our final result

$$0 = \mathcal{D}_N \mathcal{D}_\lambda p^I + R^I_{JKL} p^J Q^K p^L - \frac{1}{M_{pl}^2} \left(\mathcal{D}_\lambda p^A\right) p^B \mathcal{G}_{AB} p^I + (3-\epsilon) \mathcal{D}_\lambda p^I \quad (6.25)$$

$$+ \; Q^J \left[\frac{\nabla_J \nabla^I V}{H^2} - \frac{1}{H^2 V} \left(\nabla^I V\right) \left(\nabla_J V\right)\right] - \frac{\nabla^I V}{V} \left(\mathcal{D}_\lambda p^J\right) p^B \mathcal{G}_{JB} \quad (6.26)$$

We can recast our expression in the form

$$\mathcal{D}_N \delta^{(1)} \phi^\alpha = P^\alpha_\beta \delta^{(1)} \phi^\beta \quad (6.27)$$

With all indices explicit, we write

$$\mathcal{D}_N \delta^{(1)} \phi^I_j = P^{I\,m}_{j\,K} \delta^{(1)} \phi^K_m \quad (6.28)$$

Our Eq. (6.20) indicates that $P^{I\,1}_{1\,J} = 0$ and $P^{I\,2}_{1\,J} = \delta^I_J$. From Eq. (6.26), we find that

$$P^{I\,1}_{2\,J} = -R^I_{LJK} p^L p^K - \frac{V}{H^2} \left[\frac{\nabla_J \nabla^I V}{V} - \frac{\left(\nabla^I V\right)\left(\nabla_J V\right)}{V^2}\right] \quad (6.29)$$

$$P^{I\,2}_{2\,J} = \frac{1}{M_{pl}^2} p^I p_J - \frac{V}{H^2 M_{pl}^2} \delta^I_J + \frac{\nabla^I V}{V} p_J \quad (6.30)$$

Our results thus agree with [15] if we also adopt units such that $M_{pl} = 1$.

6.4 Second Order Results

We can find the second order perturbation equations by applying $\mathcal{D}_\lambda$ to the first order equation of motion

$$\mathcal{D}_\lambda \left(\mathcal{D}_\lambda \left(\mathcal{D}_N p^I + (3 - \epsilon(p)) p^I + \frac{V^{,I}}{H^2(\phi, p)}\right)\right) = 0 \quad (6.31)$$

The manipulations are tedious yet very similar those in the first-order calculation which is why we simply quote the end result

$$
\begin{aligned}
0 =&\mathcal{D}_\lambda\left(\mathcal{D}_\lambda\left(\mathcal{D}_N p^I + (3-\epsilon(p))p^I + \frac{V^{,I}}{H^2(\phi,p)}\right)\right)\\
=&\mathcal{D}_N\left(\mathcal{D}_\lambda\mathcal{D}_\lambda p^I\right) + R^I_{JKL}\left(\mathcal{D}_\lambda p^J\right)Q^K p^L + \left(\nabla_A R^I_{LJK}\right)p^L p^K Q^J Q^A + R^I_{LJK}\left(\mathcal{D}_\lambda p^L\right)p^K Q^J\\
&+R^I_{LJK}p^L\left(\mathcal{D}_\lambda p^K\right)Q^J + R^I_{LJK}p^L p^K\left(\mathcal{D}_\lambda Q^J\right) - \frac{1}{M_{pl}^2}\left(\mathcal{D}_\lambda\mathcal{D}_\lambda p^J\right)p_J p^I\\
&-\frac{1}{M_{pl}^2}\left(\mathcal{D}_\lambda p^K\right)\left(\mathcal{D}_\lambda p_K\right)p^I - \frac{2}{M_{pl}^2}\left(\mathcal{D}_\lambda p^J\right)p_J\left(\mathcal{D}_\lambda p^I\right) + (3-\epsilon)\,\mathcal{D}_\lambda\mathcal{D}_\lambda p^I\\
&+\frac{1}{H^2}\left(\nabla_A\nabla_J\nabla^I V\right)Q^J Q^A + \frac{1}{H^2}\left(\nabla_J\nabla^I V\right)\left(\mathcal{D}_\lambda Q^J\right) - \frac{\left(\nabla_J\nabla^I V\right)}{V}Q^J\left(\mathcal{D}_\lambda p^A\right)p_A\\
&-\frac{1}{H^2 V}\left(\nabla_J\nabla^I V\right)Q^J Q^A\left(\nabla_A V\right) - \frac{1}{H^2 V}\left(\nabla_A\nabla^I V\right)Q^A Q^J\left(\nabla_J V\right)\\
&-\frac{1}{H^2 V}\left(\nabla^I V\right)\left(\nabla_A\nabla_J V\right)Q^J Q^A - \frac{1}{H^2 V}\left(\nabla^I V\right)\left(\nabla_J V\right)\left(\mathcal{D}_\lambda Q^J\right)\\
&+\frac{2}{H^2 V}\left(\nabla^I V\right)\left(\nabla_J V\right)\left(\nabla_A V\right)Q^A Q^J + \frac{1}{V^2}\left(\nabla^I V\right)\left(\nabla_J V\right)Q^J\left(\mathcal{D}_\lambda p^A\right)p_A\\
&-\frac{1}{V}\left(\nabla_A\nabla^I V\right)p_J Q^A\left(\mathcal{D}_\lambda p^J\right) + \frac{1}{V^2}\left(\nabla^I V\right)\left(\nabla_A V\right)p_J Q^A\left(\mathcal{D}_\lambda p^J\right)\\
&-\frac{\left(\nabla^I V\right)}{V}\left(\mathcal{D}_\lambda p_J\right)\left(\mathcal{D}_\lambda p^J\right) - \frac{\left(\nabla^I V\right)}{V}p_J\left(\mathcal{D}_\lambda\mathcal{D}_\lambda p^J\right)
\end{aligned}
$$

We recast this result in the form

$$
\mathcal{D}_N\delta^{(2)}\phi^I_j = P^{I\,m}_{j\,K}\delta^{(2)}\phi^K_m + Q^{I\,a\,b}_{j\,K\,L}\delta^{(1)}\phi^K_a\delta^{(1)}\phi^L_b \tag{6.32}
$$

and compare our expression with those of [15]. Fortunately, we find perfect agreement. We will not got into more detail as to how these evolution equations can be used to numerically compute the non-gaussianity parameter f_{NL}. Instead, we refer the interested reader to [17, 15].

Chapter 7

Diverging Trajectories in Superspace

In this section, we find yet another way to understand when classical field trajectories diverge in field-space based on a technique borrowed from classical mechanics.

7.1 The Mini-Superspace Approximation

We take the general action in the Einstein frame to be given by:

$$S_E = \int d^4x\sqrt{-g}\left(\frac{1}{2}M_{pl}^2 R + \frac{1}{2}\mathcal{G}_{IJ}\dot{\phi}^I\dot{\phi}^J - V_E(\phi)\right) \tag{7.1}$$

where $M_{pl} = 1/\sqrt{8\pi G}$ and R is the Ricci scalar of a flat FRW spacetime, given by $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$. We say the fields are maps from the FRW spacetime to a target space $\mathcal{F}$ (the field manifold) with metric $\mathcal{G}_{IJ}$. In Section 3.2, we showed how renormalization of quantum fields in curved spacetime necessarily lead to such non-canonical field kinetic terms after performing the conformal transformation from the Jordan to the Einstein frame. Note that, by assumption of isotropy and homogeneity, we must have $\phi(x^\mu) = \phi(t)$. We thus write $\phi : \mathbb{R} \to \mathcal{F}$. The subscript E for the Einstein frame will be omitted throughout the rest of this section.

Now, R is given by [38]

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \tag{7.2}$$

Using the fact that $\sqrt{-g} = a^3(t)$, we rewrite the gravity part of the action as

$$\begin{aligned}
S_E &= \int d^4x\sqrt{-g}\left(\frac{1}{2}M_{pl}^2 R\right) &(7.3)\\
&= \int d^4x a^3(t)\frac{1}{2}M_{pl}^2 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) &(7.4)\\
&= 3M_{pl}^2\int d^4x\left((\frac{d}{dt}\dot{a})a^2 + \dot{a}^2 a\right) &(7.5)\\
&= \int d^4x\left(-3M_{pl}^2\dot{a}^2 a\right) &(7.6)\\
&= \int d^4x\sqrt{-g}\left(-3M_{pl}^2\frac{\dot{a}^2}{a^2}\right) &(7.7)
\end{aligned}$$

where we integrated by parts in Eq.(7.6). The entire action is independent of space and we can perform the space part of the integral, writing $\int d^3x \equiv \mathcal{V}$, i.e. $\mathcal{V}$ is the spatial volume of the FRW universe. Our action thus becomes:

$$\begin{aligned}
S &= \int d^4x a^3\left(-3M_{pl}^2\frac{\dot{a}^2}{a^2} + \frac{1}{2}\mathcal{G}_{IJ}\dot{\phi}^I\dot{\phi}^J - V(\phi)\right) &(7.8)\\
&= \mathcal{V}\int dt\left(-3M_{pl}^2\dot{a}^2 a + a^3\frac{1}{2}\mathcal{G}_{IJ}\dot{\phi}^I\dot{\phi}^J - a^3 V(\phi)\right) &(7.9)
\end{aligned}$$

We will work entirely in terms of $\frac{S}{\mathcal{V}}$, neglecting the total spatial volume dependence, which while infinite does not affect any of the subsequent calculations. We will abuse notation and still just write S for this action-density.

7.2 Augmenting the Dimensionality of the Target Space $\mathcal{F}$.

We now want to write this in the form which manifestly resembles $L = Kinetic - Potential$. We thus define:

$$\mathcal{M}_{ij} = \begin{pmatrix} -3M_{pl}^2 a & 0 \\ 0 & a^3 \mathcal{G}_{IJ} \end{pmatrix} \tag{7.10}$$

$$\psi^i = \begin{pmatrix} a \\ \phi^I \end{pmatrix} \tag{7.11}$$

as well as

$$V(a, \phi) = a^3 V(\phi) \tag{7.12}$$

The small Latin indices will denote the composite objects which incorporate a and the $\mathcal{N}$ fields ϕ^I. In short, we have $i, j = \{0, 1, 2, ..., \mathcal{N}\}$, while the capital Latin indices denote only the ϕ-field part, i.e. $I, J = \{1, 2, ..., \mathcal{N}\}$. The action now becomes:

$$S = \int dt \, \frac{1}{2} \mathcal{M}_{ij} \dot{\psi}^i \dot{\psi}^j - V(a, \phi) \tag{7.13}$$

This is just the action for a point particle moving on a $\mathcal{N}+1$ dimensional manifold with Lorentzian metric given by $\mathcal{M}_{ij}$ and coordinates ψ^i. a is now like a time-component, since $\mathcal{M}_{00} < 0$. We will denote this augmented target space - of dimension $\mathcal{N}+1$ - as $\mathcal{A}$. These results generalize prior work which either did not consider non-minimal couplings [40] or otherwise, did not perform the conformal transformation from the Jordan to the Einstein frame [39].

7.3 The Jacobi Metric

We now employ a technique borrowed from classical mechanics called the Jacobi metric. In essence, it recasts the problem of a particle moving on a manifold $\mathcal{B}$

under the influence of a potential V into a problem of finding geodesics on a suitably related manifold $\mathcal{B}'$. The metric on $\mathcal{B}$ is related to the metric on $\mathcal{B}'$ by a conformal transformation [41].

In general, if a particle has a conserved energy E, and is moving in a potential V, the metrics are related by:

$$\mathcal{B}'_{\alpha\beta} = 2(E - V)\mathcal{B}_{\alpha\beta} \tag{7.14}$$

We emphasize again that such a conformal transformation is *not* a diffeomorphism. Instead, the term $2(E - V)$ essentially acts as a local stretching of the metric which reproduces those effects we would otherwise assign to gradients of the potential. In our case, we have an additional constraint concerning the total energy E. Indeed, when we considered our action, we parametrized everything in terms of t, but we could have re-parametrized it in terms of another parameter, say τ. We can incorporate this redundancy by writing the flat FRW metric as $ds^2 = -N^2(t)dt^2 + a^2(t)d\vec{x}^2$. Since N is not a dynamical variable, the variation of the action with respect to N must vanish. As long as this constraint is taken into consideration, we can then set $N = 1$ for the rest of our calculations[42]. The resulting constraint equation is that the total energy vanishes:

$$E = \frac{1}{2}\mathcal{M}_{ij}\dot{\psi}^i\dot{\psi}^j + V(a, \phi) = 0 \tag{7.15}$$

where we have already set $N = 1$ in the constraint equation. This remarkable equation tells us that the total energy of the universe vanishes at all times within the mini-superpsace approximation. Interestingly, one can show that Eq.(7.15) is nothing other than the Friedmann equation in Eq.(3.15), rewritten in a different form.

Recall from Section 2.3 that the energy density ρ remains constant during inflation while the total spatial volume increases exponentially. How then can $E = 0$ at all times? The answer comes from the fact that the gravitational potential energy is negative and exactly cancels the increase in energy sourced by the inflaton field. [1]

[1] Alan Guth, one of the founding fathers of inflation, has often to referred to the inflationary

Rewriting Eq.(7.15) as

$$\mathcal{M}_{ij}\dot{\psi}^i\dot{\psi}^j = -2V(a,\phi) \tag{7.16}$$

we see that $\dot{\psi}$ is timelike for $V(a,\phi) = a^3V(\phi) > 0$ (since we assume $V(\phi) > 0$ for all ϕ in our cases of interest). Note that a vector being timelike is a coordinate independent statement. In particular we can change coordinates $\psi^i \to \tilde{\psi}^i(\psi)$ where the metric is diagonal. In these coordinates, there are no cross terms, and we can write

$$\tilde{\mathcal{M}}_{00}\dot{\tilde{\psi}}^0\dot{\tilde{\psi}}^0 + \tilde{\mathcal{M}}_{IJ}\dot{\tilde{\psi}}^I\dot{\tilde{\psi}}^J < 0 \tag{7.17}$$

which shows that

$$||\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| > ||\tilde{\mathcal{M}}_{IJ}\dot{\psi}^I\dot{\psi}^J|| \tag{7.18}$$

namely, the time-component part of the vector dominates the space part. This is the true definition of a vector being "time-like".

Now consider any function $f(\psi) > 0$ - it is important that f does not switch sign on the entire interval under consideration. We now show that multiplying the metric by $-f$ or $+f$ does not change the time-like nature of $\dot{\psi}^i$. Indeed, note that:

$$||-f\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| = ||+f\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| = f||\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| \tag{7.19}$$

Hence, using Eq.(7.18), we can always write:

$$||-f\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| = f||\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| > f||\tilde{\mathcal{M}}_{IJ}\dot{\psi}^I\dot{\psi}^J|| = ||-f\tilde{\mathcal{M}}_{IJ}\dot{\psi}^I\dot{\psi}^J|| \tag{7.20}$$

and trivially:

$$||f\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| = f||\tilde{\mathcal{M}}_{00}\dot{\psi}^0\dot{\psi}^0|| > f||\tilde{\mathcal{M}}_{IJ}\dot{\psi}^I\dot{\psi}^J|| = ||f\tilde{\mathcal{M}}_{IJ}\dot{\psi}^I\dot{\psi}^J|| \tag{7.21}$$

mechanism as 'the ultimate free lunch' for precisely this reason.

Again, this is a coordinate independent statement since Eq.(7.18) was also a coordinate independent statement.

It is thus convenient to choose $+f$ if the decision need be made, since it will then not change the signature of the metric and a vector V^i being timelike can still be understood as $\mathcal{M}_{ij}V^iV^j < 0$.

This short digression was relevant since in the case where $E = 0$ we have $2(E - V) = -2V$. So that the Jacobi metric for our field theory would naively look like:

$$\mathcal{J}_{ij}^{first\,guess} = 2(E - V)\mathcal{M}_{ij} = -2V\mathcal{M}_{ij} \tag{7.22}$$

The problem is that we cannot write this as a conformal transformation because $-2V < 0$ in our case. We can thus multiply the metric by $+2V$ instead - by the argument presented above - so that we can define the Jacobi metric as:

$$\mathcal{J}_{ij} = +2V\mathcal{M}_{ij} = e^{ln(+2V)}\mathcal{M}_{ij} \tag{7.23}$$

Define now a new parameter with which we will parametrize our trajectories:

$$s(t) = 2\int_0^t V(a(t'), \phi(t'))dt' \tag{7.24}$$

While we will not go through the derivation here, it turns out that the equations of motion for $\psi(s)$ can now be written as geodesic of $\mathcal{J}_{ij}$ [41]:

$$\frac{d^2\psi^i}{ds^2} + \Gamma^i_{jk}\frac{d\psi^j}{ds}\frac{d\psi^k}{ds} = \nabla_{\psi'(s)}\psi'^i(s) = 0 \tag{7.25}$$

where the covariant derivative is defined through the metric $\mathcal{J}_{ij}$,$\psi'^i \equiv \frac{d\psi^i}{ds}$ and $\nabla_{\psi'} \equiv \frac{d\psi^k}{ds}\nabla_k$.

We have thus completely recast a classical field theory with non-minimal couplings in an expanding FRW metric to studying geodesics on $\mathcal{J}_{ij}$.

Chapter 8

Conclusion

Narrowing in on the exact workings of the inflationary mechanism, one of the central questions remains whether inflation is driven by one or several scalar fields. The most promising indicator is given by the parametrization of non-gaussianities in the density perturbation spectrum we observe in the cosmic microwave background. As sizable non-gaussianities correlate to diverging inflationary trajectories in field-space, tackling this important question requires understanding the underlying dynamics of the theory. We believe this is best approached from a geometric standpoint.

First, we constructed a covariant formalism to accommodate the field-space diffeomorphism invariance of cosmologically relevant multifield actions in the Einstein frame. Furthermore, borrowing techniques from differential geometry and advanced classical mechanics, we recast both classical and quantum aspects of multifield inflation, with complicated non-canonical kinetic terms and arbitrary interactions, into the simpler problem of studying geodesic motion on a suitably related manifold.

In [37], we showed how our covariant formalism can be used to calculate the non-gaussianity parameter f_{NL} for the general class of multifield, non-minimally coupled inflation models. It remains an exciting subject for research to now apply the techniques of Sections 6 and 7 to the specific case of cosmological non-gaussianities.

Chapter 9

Appendix A: Field-Space Metric and Related Quantities

Given $f(\phi^I)$ in Eq. (3.19) for a two-field model, the field-space metric in the Einstein frame, Eq. (??), takes the form

$$\begin{aligned} \mathcal{G}_{\phi\phi} &= \left(\frac{M_{\rm pl}^2}{2f}\right)\left[1+\frac{3\xi_\phi^2\phi^2}{f}\right], \\ \mathcal{G}_{\phi\chi} = \mathcal{G}_{\chi\phi} &= \left(\frac{M_{\rm pl}^2}{2f}\right)\left[\frac{3\xi_\phi\xi_\chi\phi\chi}{f}\right], \\ \mathcal{G}_{\chi\chi} &= \left(\frac{M_{\rm pl}^2}{2f}\right)\left[1+\frac{3\xi_\chi^2\chi^2}{f}\right]. \end{aligned} \tag{9.1}$$

The components of the inverse metric are

$$\begin{aligned} \mathcal{G}^{\phi\phi} &= \left(\frac{2f}{M_{\rm pl}^2}\right)\left[\frac{2f+6\xi_\chi^2\chi^2}{C}\right], \\ \mathcal{G}^{\phi\chi} = \mathcal{G}^{\chi\phi} &= -\left(\frac{2f}{M_{\rm pl}^2}\right)\left[\frac{6\xi_\phi\xi_\chi\phi\chi}{C}\right], \\ \mathcal{G}^{\chi\chi} &= \left(\frac{2f}{M_{\rm pl}^2}\right)\left[\frac{2f+6\xi_\phi^2\phi^2}{C}\right], \end{aligned} \tag{9.2}$$

where we have defined the convenient combination

$$\begin{aligned} C(\phi,\chi) &\equiv M_{\rm pl}^2 + \xi_\phi(1+6\xi_\phi)\phi^2 + \xi_\chi(1+6\xi_\chi)\chi^2 \\ &= 2f + 6\xi_\phi^2\phi^2 + 6\xi_\chi^2\chi^2. \end{aligned} \tag{9.3}$$

The Christoffel symbols for our field space take the form

$$\begin{aligned} \Gamma^{\phi}{}_{\phi\phi} &= \frac{\xi_\phi(1+6\xi_\phi)\phi}{C} - \frac{\xi_\phi\phi}{f}, \\ \Gamma^{\phi}{}_{\chi\phi} = \Gamma^{\phi}{}_{\phi\chi} &= -\frac{\xi_\chi\chi}{2f}, \\ \Gamma^{\phi}{}_{\chi\chi} &= \frac{\xi_\phi(1+6\xi_\chi)\phi}{C}, \\ \Gamma^{\chi}{}_{\phi\phi} &= \frac{\xi_\chi(1+6\xi_\phi)\chi}{C}, \\ \Gamma^{\chi}{}_{\phi\chi} = \Gamma^{\chi}{}_{\chi\phi} &= -\frac{\xi_\phi\phi}{2f}, \\ \Gamma^{\chi}{}_{\chi\chi} &= \frac{\xi_\chi(1+6\xi_\chi)\chi}{C} - \frac{\xi_\chi\chi}{f} \end{aligned} \tag{9.4}$$

For two-dimensional manifolds we may always write the Riemann tensor in the form

$$\mathcal{R}_{ABCD} = \mathcal{K}(\phi^I)\left[\mathcal{G}_{AC}\mathcal{G}_{BD} - \mathcal{G}_{AD}\mathcal{G}_{BC}\right], \tag{9.5}$$

where $\mathcal{K}(\phi^I)$ is the Gaussian curvature. In two dimensions, $\mathcal{K}(\phi^I) = \frac{1}{2}\mathcal{R}(\phi^I)$, where $\mathcal{R}(\phi^I)$ is the Ricci scalar. Given the field-space metric of Eq. (9.1), we find

$$\mathcal{R}(\phi^I) = 2\mathcal{K}(\phi^I) = \frac{2}{3M_{\rm pl}^2C^2}\left[(1+6\xi_\phi)(1+6\xi_\chi)(4f^2) - C^2\right]. \tag{9.6}$$

Bibliography

[1] A. H. Guth and D. I. Kaiser, "Inflationary cosmology: Exploring the universe from the smallest to the largest scales," Science **307**,884 (2005) [arXiv:astro-ph/0502328].

[2] D. H. Lyth and A. Riotto, "Particle physics models of inflation and the cosmological density perturbation," Phys. Rept. **314**, 1 (1999) [arXiv:hep-ph/9807278].

[3] B. A. Bassett, S. Tsujikawa, and D. Wands, "Inflation dynamics and reheating," Rev. Mod. Phys. 78 (2006): 537 [arXiv:astro-ph/0507632].

[4] D. Wands, "Multiple field inflation," Lect. Notes. Phys. **738**, 275 (2008) [arXiv:astro-ph/0702187].

[5] N. Bartolo et al., "Non-Gaussianity from inflation: Theory and observations," Phys. Rept. **402**, 103 (2004) [arXiv:astro-ph/0406398].

[6] X. Chen, "Primordial non-Gaussianities from inflation models," Adv. Astron. **2010**, 638979 (2010) [arXiv:1002.1416 [astro-ph]].

[7] C. T. Byrnes and K.-Y. Choi, "Review of local non-Gaussianity from multi-field inflation," Adv. Astron. 2010, 724525 (2010) [arXiv:1002.3110 [astro-ph.CO]].

[8] E. Komatsu et al., "Non-Gaussianity as a probe of the physics of the primordial universe and the astrophysics of the low redshift universe," arXiv:0902:4759 [astro-ph.CO].

[9] C. G. Callan, Jr., S. R. Coleman, and R. Jackiw, "A new improved energy-momentum tensor," Annals Phys. 59 (1970): 42-73.

[10] T. S. Bunch, P. Panangaden, and L. Parker, "On renormalisation of $\lambda\phi^4$ field theory in curved space-time, I," J. Phys. A 13 (1980): 901-918; T. S. Bunch and P. Panangaden, "On renormalisation of $\lambda\phi^4$ field theory in curved space-time, II," J. Phys. A (1980): 919-932.

[11] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (New York: Cambridge University Press, 1982).

[12] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (New York: Taylor and Francis, 1992).

[13] D. I. Kaiser, "Conformal transformations with multiple scalar fields," Phys. Rev. **D81**, 084044 (2010) [arXiv:1003.1159 [gr-qc]].

[14] D. Seery and J. E. Lidsey, "Primordial non-Gaussianities from multiple-field inflation," JCAP 0509 (2005): 011 [arXiv:astro-ph/0506056].

[15] S. Yokoyama, T. Suyama, and T. Tanaka, "Primordial non-Gaussianity in multi-scalar slow-roll inflation," JCAP 0707 (2007): 013 [arXiv:0705.3178 [astro-ph]; S. Yokoyama, T. Suyama, and T. Tanaka, "Primordial non-Gaussianity in multi-scalar inflation," Phys. Rev. D77 (2008): 083511 [arXiv:0711.2920 [astro-ph]].

[16] C. T. Byrnes, K-Y Choi, and L. M. H. Hall, "Conditions for large non-Gaussianity in two-field slow-roll inflation," JCAP 0810 (2008): 008 [arXiv:0807.1101 [astro-ph]].

[17] J. Ellison, D. J. Mulryne, D. Seery, and R. Tavakol, "Evolution of f_{NL} to the adiabatic limit," JCAP 1111 (2011): 005 [arXiv:1106.2153 [astro-ph.CO]]; D. Seery, D. J. Mulryne, J. Frazer, and R. H. Ribeiro, "Inflationary perturbation theory is geometrical optics in phase space," JCAP 09 (2012): 010 [arXiv:1203.2635 [astro-ph.CO]].

[18] C. M. Peterson and M. Tegmark, "Testing two-field inflation," Phys. Rev. **D83**, 023522 (2011) [arXiv:1005.4056 [astro-ph.CO]]; "Non-Gaussianity in two-field in-

flation," Phys. Rev. **D84**, 023520 (2011) [arXiv:1011.6675 [astro-ph.CO]]; "Testing multi-field inflation: A geometric approach," arXiv:1111.0927 [astro-ph.CO].

[19] J.-O. Gong and T. Tanaka, "A covariant approach to general field space metric in multi-field inflation," JCAP 1103 (2011): 015 [arXiv:1101.4809 [astro-ph.CO]].

[20] J. Elliston, D. Seery, and R. Tavakol, "The inflationary bispectrum with curved field-space," arXiv:1208.6011 [astro-ph.CO].

[21] F. L. Bezrukov and M. E. Shaposhnikov, "The Standard Model Higgs boson as the inflaton," Phys. Lett. **B659**, 703 (2008) [arXiv:0710.3755 [hep-th]].

[22] A. de Simone, M. P. Hertzberg, and F. Wilczek, "Running inflation in the Standard Model," Phys. Lett. B678 (2009): 1-8 [arXiv:0812.4946 [hep-ph]]; F. L. Bezrukov, A. Magnin, and M. E. Shaposhnikov, "Standard Model Higgs boson mass from inflation," Phys. Lett. B675 (2009): 88-92 [arXiv:0812.4950 [hep-ph]]; F. L. Bezrukov and M. E. Shaposhnikov, "Standard Model Higgs boson mass from inflation: two loop analysis," JHEP 0907 (2009): 089 [arXiv:0904.1537 [hep-ph]]; A. O. Barvinsky, A. Yu. Kamenshchik, C. Kiefer, A. A. Starobinsky, C. Steinwachs, "Asymptotic freedom in inflationary cosmology with a non-minimally coupled Higgs field," JCAP 0912 (2009): 003 [arXiv:0904.1698 [hep-ph]].

[23] R. N. Greenwood, D. I. Kaiser, and E. I. Sfakianakis, "Multifield dynamics in Higgs inflation," Physical Review D87 (2013): 064021 [arXiv:hep-ph/1210.8190]

[24] X. Chen, "Primordial features as evidence for inflation," arXiv:1104.1323 [hep-th]; X. Chen, "Fingerprints of primordial universe paradigms as features in density perturbations," arXiv:1106.1635 [astro-ph.CO].

[25] J. White, M. Minamitsuji, and M. Sasaki, "Cosmological perturbation in multi-field inflation with non-minimal coupling," arXiv:1205.0656 [astro-ph.CO].

[26] R. W. Wald, *General Relativity* (Chicago: University of Chicago Press, 1984).

[27] G. J. Anderson, D. J. Mulryne, and D. Seery, "Transport equations for the inflationary trispectrum," arXiv:1205.0024 [astro-ph.CO].

[28] M. Sasaki and E. D. Stewart, "A general analytic formula for the spectral index of the density perturbations produced during inflation," Prog. Theo. Phys. **95**, 71 (1996) [arXiv:astro-ph/9507001].

[29] D. H. Lyth, K. A. Malik, and M. Sasaki, "A general proof of the conservation of the curvature perturbation," JCAP 0505 (2005): 004 [arXiv:astro-ph/0411220].

[30] H.-C. Lee, M. Sasaki, E. D. Stewart, T. Tanaka, and S. Yokoyama, "A new delta N formalism for multi-component inflation," JCAP 0510 (2005): 004 [arXiv:astro-ph/0506262].

[31] D. H. Lyth and Y. Rodriguez, "The inflationary prediction for primordial non-Gaussianity," Phys. Rev. Lett. 95 (2005): 121302 [arXiv:astro-ph/0504045].

[32] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, "Theory of cosmological perturbations," Phys. Rept. **215**, 203 (1992).

[33] M. Sasaki and T. Tanaka, "Super-Horizon Scale Dynamics of Multi-Scalar Inflation", Prog. Theor. Phys. **95**, 71 (1996) [arXiv:astro-ph/9801017]

[34] P.A.R. Ade et al. [Planck Collaboration], "Planck 2013 results. XVI. Cosmological parameters," (2013) [arXiv:astro-ph/1303.5076].

[35] ATLAS Collaboration, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," Phys. Lett. B 716 (2012) [arXiv: hep-ex/1207.7214]

[36] CMS Collaboration, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," Phys. Lett. B 716 (2012) [arXiv: hep-ex/1207.7214]

[37] D. Kaiser, E. Mazenc, E. Sfakianakis, "Primordial Bispectrum from Multifield Inflation with Nonminimal Couplings," Phys. Rev. **D87** 064004 (2013) [arXiv: astro-ph.CO/1210.7487]

[38] D. Kaiser, "FRW Spacetime: An Informal Primer" (2011), MIT CTP internal document

[39] L. A. Elias, A. Saa, "Non-Minimally Coupled Cosmology as Geodesic Motion," PoSIC2006:039 (2006) [arXiv:gr-qc/0702037]

[40] P. K. Townsend, M. N.R. Wohlfarth, "Cosmology as Geodesic Motion," Class.Quant.Grav. 21 (2004) 5375 [arXiv:hep-th/0404241]

[41] J. Awrejcewicz, *Classical Mechanics: Dynamics*, vol. 2 (Springer: Advances in Mechanics and Mathematics, 2012)

[42] B. Vakili, F. Khazaie, "Noether Symmetric Classical and Quantum Scalar Field Cosmology," Class. Quantum Grav. **29** 035015 (2012) [arXiv:gr-qc/1109.3352]