8.044 Statistical Physics I
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044: Statistical Physics I — Spring Term 2008

Problem Set #10

Problem 1: Lattice Heat Capacity of Solids

This problem examines the lattice contribution to the heat capacity of solids. Other contributions may be present such as terms due to mobile electrons in metals or magnetic moments in magnetic materials.

A crystalline solid is composed of N primitive unit cells, each containing J atoms. A primitive unit cell is the smallest part of the solid which, through translational motions alone, could reproduce the entire crystal. The atoms in the unit cell could be the same or they could differ: diamond has two carbon atoms per primitive unit cell, sodium chloride has one Na and one Cl. J could be as small as one, as in a crystal of aluminum, or it could be tens of thousands as in the crystal of a large biological molecule.

a) *The Classical Model* Assume each atom in the crystal is statistically independent of all the others, and that it can vibrate about its equilibrium position as a harmonic oscillator in each of 3 orthogonal directions. In principle there could be $3J$ different frequencies of vibration in such a model; in fact, symmetry conditions usually introduce degeneracies, reducing the number of frequencies (but not the number of modes). On the basis of classical mechanics, find the heat capacity at constant volume (i.e. constant lattice spacing) for this model.

b) *The Einstein Model* The result of the classical model does not agree with observation. The heat capacity of the lattice varies with temperature and goes to zero at $T = 0$. Again assume that the atoms are statistically independent and execute harmonic motion about their mean positions. This time find the heat capacity using quantum mechanics. For simplicity, assume that the $3J$ frequencies are identical and equal to ν. What is the limiting behavior of C_V for $kT << h\nu$ and for $kT >> h\nu$?

c) *Phonons* The result of the Einstein model is in better agreement with measured heat capacities, but it is still not completely correct. In particular, the lattice contribution to C_V approaches $T = 0$ as T^3, a more gradual temperature dependence than found in the Einstein model (using $3J$ different frequencies does not help). The remaining flaw in the model is that the atomic motions are <u>not</u> independent. Pluck one atom and the energy introduced will soon spread throughout the crystal. Rather, the crystal has $3JN$ normal modes of vibration, called phonons, each of which involves all of the atoms in the solid. The amplitude of each normal mode behaves as a harmonic oscillator, but the frequencies of the the normal modes span a wide range from almost zero up to the frequency one might expect when one atom vibrates with respect to fixed neighbors. A phonon of radian frequency ω is represented by a quantum mechanical harmonic oscillator of the same frequency. The density of frequencies $D(\omega)$ is defined such that $D(\omega_0)\,d\omega$ is the number of phonons in the crystal with frequencies between ω_0 and $\omega_0 + d\omega$. Normalization requires that

$$\int_0^\infty D(\omega)\,d\omega = 3JN.$$

The thermodynamic internal energy of the lattice is

$$E(T) = \int_0^\infty < \epsilon(\omega, T) > D(\omega)\,d\omega$$

where $< \epsilon(\omega, T) >$ is the mean energy of a quantum oscillator with radian frequency ω at a temperature T.

i) Write down the full integral expression for $E(T)$. Evaluate the expression in the limit $kT >> \hbar\omega_{\rm max}$ where $\omega_{\rm max}$ is the highest phonon frequency in the solid. What is the heat capacity in this limit? You should get the classical result.

ii) It can be shown that near $\omega = 0$,

$$D(\omega) \to \frac{3V}{2\pi^2 < v >^3}\,\omega^2,$$

where V is the volume of the crystal and $< v >$ is an average sound velocity in the solid. Find the heat capacity of the lattice for temperatures so low that only those phonons in the quadratic region of $D(\omega)$ are excited. Use the fact that

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}.$$

Problem 2: Thermal Noise in Circuits I, Mean-Square Voltages and Currents

An arbitrary network of passive electronic components is in thermal equilibrium with a reservoir at temperature T. It contains no sources.

a) Find the probability density $p(v)$ that a voltage v will exist on a capacitor of capacitance C. [Hint: consider the capacitor alone as a subsystem.] Find an expression for the root-mean-square voltage $\sqrt{< v^2 >}$. What is this in microvolts when $T = 300$K and $C = 100$pF?

b) Find $p(i)$ and $\sqrt{< i^2 >}$ for the current i through an inductor of inductance L. What is the root-mean-square current in nanoamps when $T = 300$K and $L = 1$mH?

c) Why does this method not work for the voltage on a resistor?

Problem 3: Thermal Noise in Circuits II, Johnson Noise of a Resistor

When we discussed jointly Gaussian random variables in the first part of this course, we learned that the noise voltage in a circuit is a random process, a signal which evolves in time. It will be composed of a variety of different frequency components. The noise power in a unit frequency interval centered at radian frequency ω, $S_v(\omega)$, is referred to as the power spectrum, or simply the spectrum, of the voltage fluctuations. The mean square fluctuation on the voltage $< v^2 >$ is obtained by integrating $S_v(\omega)$ over all ω.

The advantage of the approach to circuit noise introduced in Problem 2 is that mean square voltages and currents in individual lossless components can be found immediately, with out reference to the remainder of the circuit. The disadvantage is that it does not allow one to find the spectrum of the voltage or current fluctuations in those components.

There is another approach to determining the noise in circuits which we will introduce here. It has the advantage of allowing the spectrum of the fluctuations to be found anywhere in a circuit. The disadvantage is that one must be able to find the AC transmission function from one part of the circuit to another. *This method assumes that the noise power entering the circuit emanates from each of the dissipative components (resistors).* Thus, one must replace each real resistor with an ideal resistor plus a noise source. We will determine in this problem what the characteristics of that noise source must be.

We will find the noise power emanating from a resistor by connecting it to a lossless transmission line, assuming thermal equilibrium, and using the principle of detailed balance. A coaxial transmission line which is excited only in the TEM modes behaves like a one dimensional system. A vacuum filled line of length L, terminated by a short circuit at each end, supports standing waves of voltage with the dispersion relation $\omega = ck$, where c is the speed of light and k is the wavevector. We can treat the transmission line as a one dimensional analog of thermal (black body) radiation.

a) What are the allowed wave vectors k_n on the transmission line described above?

b) Assuming that n is a large number, find the density of modes on the line $D(\omega)$. As in 3 dimensions, one need only consider positive ω and notice that there is only one "polarization" direction for the voltage in this case.

c) Find $u_1(\omega, T)$, the energy per unit length per unit frequency interval, when the line is in thermal equilibrium at temperature T.

d) Usually transmission lines operate under conditions where $k_B T >> \hbar\omega$. Find the limiting form of $u_1(\omega, T)$ under these conditions.

The energy density on the line in part d) was calculated for standing waves, but it can be regarded as composed of running waves traveling in two directions. If the line is cut at some point and terminated with a resistor having the characteristic impedance of the line, waves traveling toward the resistor behave as if the line were still infinitely long in that direction and they will never return; that is, they are completely absorbed. If the resistor is at the same temperature as the line, it must send power to the line equal to the power which flows to it.

e) Find the thermal energy per unit frequency interval flowing out of the resistor, $P(\omega)$. Thus $P(\omega)\, d\omega$ is the thermal power in the bandwidth $d\omega$. Note that it is "white" noise in that it is independent of frequency (a flat frequency spectrum) and that it does not depend on the value of the resistance. This power is referred to as the "Johnson noise" associated with the resistor.

f) What is the noise power from a resistor at room temperature in a 10MHz bandwidth (real frequency as opposed to radian frequency)?

Problem 4: Isothermal Atmosphere

Consider a tall column filled with nitrogen gas standing in the Earth's gravitational field (assume its height is, say, several kilometers). The gas is in thermal equilibrium at a uniform temperature T, but the local number density $n(z)$, the number of molecules per cubic meter at height z, will be greater near the bottom.

a) Let $P(z)$ be the pressure at height z. Explain why the equation

$$P(z) = P(z + \Delta z) + mg\, n(z)\Delta z$$

should hold. Start with Newton's second law applied to the gas in the height interval bewteen z and $z + \Delta z$.

b) Convert the equation to a differential equation and solve it for $P(z)$, given that the pressure is $P(0)$ at the base. Recall that the ideal gas law may be used to re-express $n(z)$ in terms of more convenient quantities. Take g to be a constant.

c) Describe the behavior with height of the local number density. At what height has the number density dropped to one-half of its value at the surface? Provide both algebraic and numerical answers.

d) Now let's examine this problem using the chemical potential. Write down an expression for the chemical potential $\mu(z)$. How does the chemical potential vary with concentration $n(z)$?

e) At equilibrium, how must $n(z)$ vary with z? Explain your reasoning.