8.044 Statistical Physics I
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044: Statistical Physics I — Spring Term 2008

Problem Set #3

Problem 1: Energy in an Ideal Gas

For an ideal gas of classical non-interacting atoms in thermal equilibrium the cartesian components of the velocity are statistically independent. In three dimensions

$$p(v_x, v_y, v_z) = (2\pi\sigma^2)^{-3/2} \exp[-\frac{v_x^2 + v_y^2 + v_z^2}{2\sigma^2}]$$

where $\sigma^2 = kT/m$. The energy of a given atom is $E = \frac{1}{2}m|\vec{v}|^2$.

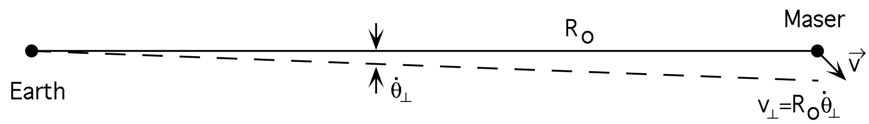
a) Find the probability density for the energy of an atom in the three dimensional gas, $p(E)$.

It is possible to create an ideal two dimensional gas. For example, noble gas atoms might be physically adsorbed on a microscopically smooth substrate; they would be bound in the perpendicular direction but free to move parallel to the surface.

b) Find $p(E)$ for the two dimensional gas.

Problem 2: Distance to the Center of the Galaxy

A few years ago the radio astronomer Dr. James Moran and his colleagues at the Harvard-Smithsonian Center for Astrophysics developed a new method for estimating the distance from the Earth to the center of the galaxy, R_0. Their method involves the observation of water vapor masers, small highly localized clouds of interstellar water vapor whose molecules emit radiation in a cooperative fashion in an extremely narrow and stable spectral line (the frequency is 22.3 Gigahertz which corresponds to a wavelength of 1.3 cm.). A group of such masers is located in a region known to be very close to the galactic center. The CfA group made extensive measurements on about 20 of these. From the Doppler shifts of the lines they found the velocity of each, $v_\parallel$, along the line of sight from the earth to the galactic center. By following the angular position in the sky over a period of 2 years they found the angular velocity, $\dot{\theta}_\perp$, of each object perpendicular to the line of sight.



a) The CfA group assumed that the velocities of the radiating clouds were distributed in a random but isotropic fashion in space, that is, $p(\vec{v})$ had spherical symmetry. Under these conditions the standard deviation of the probability density for $v_{\|}$ should be R_0 times the standard deviation for $\dot{\theta}_{\perp}$. The measurements gave $\sigma_{v_{\|}} = 20$ kilometers/second and $\sigma_{\dot{\theta}_{\perp}} = 9.1 \times 10^{-17}$ radians/second. What is R_0 in kilometers and kiloparsecs? [1 parsec = 3.09×10^{13} kilometers and a rough estimate of the diameter of our entire galaxy is 25 kiloparsecs.]

b) Assume a model in which the observed radiating clouds are randomly distributed on a spherical shell of radius r. The center of the sphere might be a newly formed star which spawned the clouds during its birth. Assume also that their velocities are due only to the expansion of the radius of the shell: $\vec{v} = \dot{r}\hat{r}$ where $\dot{r}$ is a constant and $\hat{r}$ is a unit vector in the radial direction. Use functions of a random variable to find the probability density for $v_{\|}$ that would result.

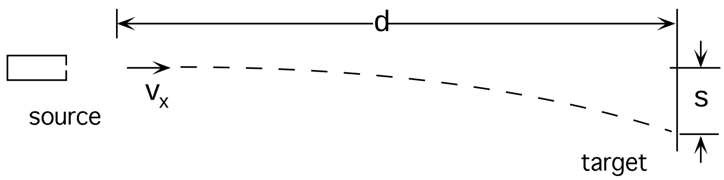
Problem 3: Sizes of Random Window Panes

In a window factory, a machine is used to cut glass sheets into *rectangular* window panes. The machine has a glitch which causes the finished dimensions of each pane to come out randomly. Specifically, the horizontal lengths (x) of the panes are uniformly distributed between 2 ft and 4 ft. Also, the vertical lengths (y) are uniformly distributed between 2 ft and 4 ft.

a) Find an expression for $p(\text{Area})$, the probability density for the area of a finished pane. [*Hint:* Draw the x, y-plane and carefully consider the limits of integration. It may be useful to break up the calculation into two separate integrals.]

b) What is the most probable area of a pane?

Problem 4: Measuring an Atomic Velocity Profile

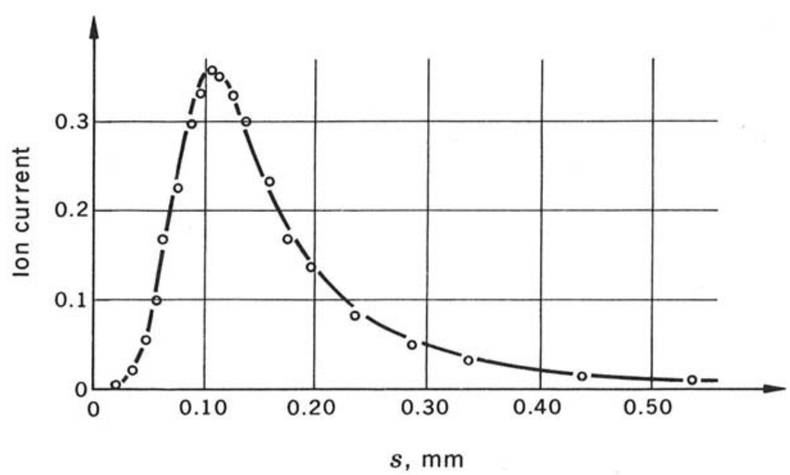


Atoms emerge from a source in a well collimated beam with velocities $\vec{v} = v_x \hat{x}$ directed horizontally. They have fallen a distance s under the influence of gravity by the time they hit a vertical target located a distance d from the source.

$$s = \frac{A}{v_x^2}$$

where $A = \frac{1}{2} g d^2$ is a constant. An empirical fit to measurements at the target give the following probability density $p_x(\zeta)$ for an atom striking the target at position s.

$$\begin{aligned} p_s(\zeta) &= \frac{A^{3/2}}{\sigma^2 \sqrt{2\pi\sigma^2}} \frac{1}{\zeta^{5/2}} \exp[-\frac{A}{2\sigma^2 \zeta}] && \zeta \geq 0 \\ &= 0 && \zeta < 0 \end{aligned}$$



Find the probability density $p_{v_x}(\eta)$ for the velocity at the source. Sketch the result.

Problem 5: Planetary Nebulae

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NGC 7293, the Helix Nebula in Aquarius

Planetary Nebula Shapley 1

Pictures from the archive at **Astronomy Picture of the Day**
http://antwrp.gsfc.nasa.gov/apod/

During stellar evolution, a low mass star in the "red supergiant" phase (when fusion has nearly run its course) may blow off a sizable fraction of its mass in the form of an expanding spherical shell of hot gas. The shell continues to be excited by radiation from the hot star at the center. This glowing shell is referred to by astronomers as a planetary nebula (a historical misnomer) and often appears as a bright *ring*. We will investigate how a shell becomes a ring. One could equally ask "What is the shadow cast by a semitransparent balloon?"

A photograph of a planetary nebula records the amount of matter having a given radial distance $r_\perp$ from the line of sight joining the central star and the observer. Assume that the gas atoms are uniformly distributed over a spherical shell of radius R centered on the parent star. What is the probability density $p(r_\perp)$ that a given atom will be located a perpendicular distance $r_\perp$ from the line of sight? [Hint: If one uses a spherical coordinate system (r, θ, ϕ) where $\theta = 0$ indicates the line of sight, then $r_\perp = r \sin\theta$. $P(r_\perp)$ corresponds to that fraction of the sphere with $R \sin\theta < r_\perp$.]