8.044 Statistical Physics I
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044: Statistical Physics I — Spring Term 2007

Final Exam

Problem 1: (35 points) An engine using thermal radiation

When discussing heat engines in class, we considered taking a "substance" around a cycle. In this problem, let's use thermal radiation as the substance. For radiation in thermal equilibrium in a cavity, we found expressions for the internal energy $U = AVT^4$, pressure $P = \frac{1}{3}AT^4$, and entropy $S = \frac{4}{3}AVT^3$, where A is a constant. Let's consider the following four-step cycle:

$1 \rightarrow 2$: Expansion from V_1 to V_2 at constant temperature T_H.

$2 \rightarrow 3$: Expansion from V_2 to V_3 adiabatically.

$3 \rightarrow 4$: Compression from V_3 to V_4 at constant temperature T_C.

$4 \rightarrow 1$: Compression from V_4 to V_1 adiabatically.

a) Make a qualitatively correct sketch of this cycle on a $P - V$ diagram (with V on the horizontal axis).

b) For the isothermal expansion from V_1 to V_2, calculate the work done *by* the thermal radiation.

c) For the isothermal expansion from V_1 to V_2, calculate Q_H, the heat transferred *from* the reservoir with temperature T_H.

d) For the adiabatic expansion from V_2 to V_3, calculate the work done *by* the thermal radiation. Find an expression for this work in terms of V_2, T_H, and T_C only.

e) What is the efficiency $\eta \equiv W_{out}/Q_H$ of this engine? You need *not* perform a calculation for this part, just a one or two sentence explanation would suffice.

Problem 2: (35 points) One dimensional transmission line

A coaxial transmission line of length L is terminated by a short circuit at each end. It supports standing electromagnetic waves with the dispersion relation $\omega = ck$, where c is the speed of light and k is the wavevector. We can treat the transmission line as a one dimensional analog of thermal (black body) radiation.

a) What are the allowed wave vectors k_n on the transmission line described above?

b) Assuming that n is a large number, find the density of modes in frequency, $D(\omega)$. Note that there is only one polarization direction in this case.

c) Find $u(\omega, T)$, the energy per unit length per unit frequency interval, when the line is in thermal equilibrium at temperature T.

Problem 3: (25 points) Heat Pump

Recall that one may make an ideal refrigerator or heat pump by running a Carnot engine in reverse. A building at a temperature T is heated by an ideal heat pump which uses the outside atmosphere at T_0 as a heat source (assume that T_0 remains constant). The pump consumes power W (in units of energy per time) and the building loses heat at a rate $\alpha(T - T_0)$ (also in units of energy per time). What is the equilibrium temperature T_e of the building?

(Hint: Recall, for a quadratic equation of the form $ax^2 + bx + c = 0$, the roots are $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.)

Problem 4: (35 points) Fermions in two-dimensions

A collection of N non-interacting, spin-1/2 fermions are confined to move in two dimensions. They are confined within a rectangular area with dimensions L_x and L_y. Here, the wavevectors allowed by periodic boundary conditions are $\vec{k} = (2\pi/L_x)n_x\hat{x} + (2\pi/L_y)n_y\hat{y}$ where n_x and n_y can take on all positive and negative integer values. The energy for a single fermion is given by $\epsilon = \hbar^2 k^2/2m$.

a) Calculate $N(k)$, the number of single-particle states with wavevector magnitude smaller than k.

b) Calculate $D(\epsilon)$, the density of single-particle states as a function of their energy ϵ. Make a carefully labeled sketch of your result.

c) Find an expression for the total energy E of the collection of fermions at $T = 0$.

Problem 5: (35 points) An S=1/2 paramagnet

Consider a system composed of N distinguishable, non-interacting spins in a magnetic field H applied along the z-direction. The system is in equilibrium at temperature T. For each spin, there are two allowed values for the z-component of the magnetic moment: $\mu_z = \mu_0$ and $-\mu_0$. The energy for a spin is given by $\epsilon = -\mu_z H$.

a) Calculate the partition function for a single spin $Z_1(T, H)$.

b) Calculate the net magnetic moment along the z-direction, $M(T, H)$, for the collection of N spins.

c) What are the limiting values of M in the high-T and low-T limits?

d) Make a qualitatively correct sketch of M versus T. Place labels on the axes to indicate a scale.

e) Make a qualitatively correct sketch of M versus H. Place labels on the axes to indicate a scale.

Problem 6: (35 points) Non-interacting Bosons

Consider a collection of N non-interacting bosons. We'll consider these bosons to have no spin degrees of freedom. They are distributed among 3 single particle states: ψ_0 and ψ_1 both with energy $\epsilon = 0$, and ψ_2 with energy $\epsilon = \Delta$.

We can label the possible N-particle energy states in terms of the occupation numbers as $|n_0, n_1, n_2\rangle$, where $n_0 + n_1 + n_2 = N$.

a) Find an expression for the allowed energies of the N-particle states in terms of Δ and any relevant quantum numbers. (Hint: This part has a simple answer.)

b) For n_2 particles in ψ_2, how many ways can you arrange the remaining particles among the two other states?

c) Write down an expression for the partition function $Z(T)$ for the collection of N bosons. Find a closed form expression for $Z(T)$. The following formulae may be useful:

$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}$$

$$\sum_{n=0}^{N} (N - n + 1)x^n = \frac{(N + 1) - (N + 2)x + x^{N+2}}{(1 - x)^2}$$

d) Consider the case for large N at low temperatures ($kT << N\Delta$). Calculate the internal energy E in this limit.

e) Early in the course, we classified thermodynamic variables of macroscopic systems as being either extensive or intensive. There is something unusual about the expression for E. Comment briefly on what is unusual and the reasons for this (in one or two sentences).