Affine Quantum Algebras, Weyl Groups and Constructible Functions

by

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Abstract

This thesis consists of two parts. In the first part we study the affine quantum group of type A, giving a geometric description of its natural inner product, and studying the theory of cells attached to the canonical basis.

In the second part we study a realization of the group algebra of the Weyl group in a convolution algebra of constructible functions on the Steinberg variety, and examine how this may be used to see Springer representations.

Thesis Supervisor: George Lusztig
Title: Professor of Mathematics

To my grandparents, Denis K. and Phyllis O'Donovan.

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Through his lectures, his papers and our many conversations, George Lusztig has shown me an amazing mathematical world. I am glad to be able to thank him here for this, and for all his encouragement and support (financial and intellectual) throughout my time as a graduate student.

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Finally I thank my family for being understanding of their son's perhaps bewildering interests. They have always supported me, however unexpectedly I turned out.

Contents

1 Inner Products **11**

1.1 Background 11

1.2 Inner product on $\mathbf{U}_D$ 14

1.3 Inner product on $\dot{\mathbf{U}}$ 16

1.4 Comparison of inner products 17

1.5 Geometric Interpretation 21

1.6 Applications 22

1.7 Affine $\mathfrak{gl}_n$ 24

2 Cells in affine algebras **25**

2.1 Schur-Weyl Duality 25

2.2 Distinguished elements in $\mathfrak{A}_D$ 27

2.3 Cells in $\mathfrak{A}_D$ 30

2.4 Cells in $\dot{\mathbf{U}}$ 34

3 Weyl groups and constructible functions **37**

3.1 Background 37

3.2 Springer representations. 39

3.3 Specialization and microlocalization 40

3.4 Microlocal construction of $\mathcal{W}$ 43

3.5 Lagrangian cycles and intersection multiplicities. 45

Chapter 1

Inner Products

In [L92], motivated by the results of [BLM], Lusztig defined a variant of the quantized enveloping algebra known as the modified quantum group. This algebra can be given a canonical basis $\dot{\mathbf{B}}$ which generalizes the canonical basis $\mathbf{B}$ of the minus part of the ordinary quantum group. Just as for $\mathbf{B}$, (c.f. [GL], [K91]) it is possible to characterize this basis, up to sign, in terms of an involution and an inner product.

In [BLM] the quantized enveloping algebra of $\mathfrak{gl}_n$ was constructed geometrically. Subsequently Lusztig [L99], and independently Ginzburg and Vasserot [GV], observed that this construction could be extended to the case of quantum affine $\mathfrak{gl}_n$. In this chapter we show that the inner product on the modified quantum group $\dot{\mathbf{U}}$ of affine $\mathfrak{sl}_n$ may be obtained via this method, and establish a positivity property for this case which is conjectured to hold in general.

1.1 Background

We begin by recalling the setup of [L99]. Fix a positive integer n. Let D be a positive integer, ϵ an indeterminate, $\mathbf{k}$ a finite field with q elements and v a square root of q. Given V a free $\mathbf{k}[\epsilon, \epsilon^{-1}]$-module of rank D, a *lattice* in V is a free $\mathbf{k}[\epsilon]$-submodule of V, of rank D. Let $\mathcal{F}^n$ denote the set of *n-step periodic lattices* in V, that is, $\mathcal{F}^n$ consists of sequences of lattices $\mathbf{L} = (L_i)_{i\in\mathbb{Z}}$ where $L_{i-1} \subset L_i$, and $L_{i-n} = \epsilon L_i$ for all $i \in \mathbb{Z}$. The group G of automorphisms of V acts on $\mathcal{F}^n$ in the natural way. We shall be interested in functions supported on $\mathcal{F}^n$ and its square which are invariant with respect to the action of G (where G acts diagonally on $\mathcal{F}^n \times \mathcal{F}^n$). Thus we first describe the orbits of G on these spaces. Let $\mathfrak{S}_{D,n}$ be the finite set of all $\mathbf{a} = (a_i)_{i\in\mathbb{Z}}$ such that

- $a_i \in \mathbb{N}$;
- $a_i = a_{i+n}$ for all $i \in \mathbb{Z}$;
- for all $i \in \mathbb{Z}, a_i + a_{i+1} + \cdots + a_{i+n-1} = D$.

The G orbits on $\mathcal{F}^n$ are indexed by their graded dimension in the following sense: For $\mathbf{L} \in \mathcal{F}^n$, let $|\mathbf{L}| \in \mathfrak{S}_{D,n}$ be given by $|\mathbf{L}|_i = \dim(L_i/L_{i-1})$. For $\mathbf{a} \in \mathfrak{S}_{D,n}$ set $\mathcal{F}_\mathbf{a} = \{\mathbf{L} \in \mathcal{F}^n \colon |\mathbf{L}| = \mathbf{a}\}$; the $\mathcal{F}_\mathbf{a}$ are precisely the G-orbits on $\mathcal{F}^n$. The G orbits on $\mathcal{F}^n \times \mathcal{F}^n$ are indexed, slightly more elaborately, by the set of matrices $\mathfrak{S}_{D,n,n}$, where $A = (a_{i,j})_{i,j\in\mathbb{Z}}$, is in $\mathfrak{S}_{D,n,n}$ if

- $a_{i,j} \in \mathbb{N}$;
- $a_{i,j} = a_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$;
- for any $i \in \mathbb{Z}, a_{i,*} + a_{i+1,*} + \cdots + a_{i+n-1,*} = D$;
- for any $j \in \mathbb{Z}, a_{*,j} + a_{*,j+1} + \cdots + a_{*,j+n-1} = D$.

Here

$$a_{i,*} = \sum_{j\in\mathbb{Z}} a_{i,j}; \qquad a_{*,j} = \sum_{i\in\mathbb{Z}} a_{i,j}.$$

For $A \in \mathfrak{S}_{D,n,n}$ set

$$r(A) = (a_{i,*})_{i\in\mathbb{Z}} \in \mathfrak{S}_{D,n} \qquad c(A) = (a_{*,j})_{j\in\mathbb{Z}} \in \mathfrak{S}_{D,n}.$$

For $A \in \mathfrak{S}_{D,n,n}$ the corresponding G-orbit $\mathcal{O}_A$ consists of pairs $(\mathbf{L}, \mathbf{L}')$ such that

$$a_{i,j} = \dim\left(\frac{L_i \cap L'_j}{(L_{i-1}\cap L'_j) + (L_i \cap L'_{j-1})}\right),$$

so $\mathbf{L} \in \mathcal{F}_{r(A)}$ and $\mathbf{L}' \in \mathcal{F}_{c(A)}$.

Let $\mathfrak{A}_{D;q}$ be the space of integer-valued G-invariant functions on $\mathcal{F}^n \times \mathcal{F}^n$ supported on a finite number of orbits. If e_A denotes the characteristic function of an orbit $\mathcal{O}_A$, the set $\{e_A : A \in \mathfrak{S}_{D,n,n}\}$ is a basis of $\mathfrak{A}_{D;q}$. The space $\mathfrak{A}_{D;q}$ has a natural convolution product which gives it the structure of an associative algebra. With respect to the basis of characteristic functions the structure constants are given as follows. For $A, B, C \in \mathfrak{S}_{D,n,n}$, let $\nu_{A,B,C}$ be the coefficient of e_C in the product $e_A e_B$. Then $\nu_{A,B,C}$ is zero unless $c(A) = r(B)$, $r(A) = r(C)$ and $c(B) = c(C)$. Now suppose these conditions are satisfied and fix $(\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_C$. Then $\nu_{A,B,C}$ is the number of points in the set

$$\{\mathbf{L}' \in \mathcal{F}_{c(A))} : (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_A, (\mathbf{L}', \mathbf{L}'') \in \mathcal{O}_B\}.$$

Clearly this is independent of the choice of $(\mathbf{L}, \mathbf{L}'')$, and moreover it can be shown that these structure constants are polynomial in q, allowing us to construct an algebra $\mathfrak{A}_D$ over $\mathbb{Q}(v)$ (we will, by deliberate misuse, treat v as both an indeterminate and a square root of q, depending on the context). This algebra is sometimes known as the affine q-Schur algebra. It is more convenient to use a rescaled version of the basis $\{e_A\}$ of $\mathfrak{A}_D$, with elements $[A] = v^{-d_A} e_A$ where

$$d_A = \sum_{i\geq k, j<l, 1\leq i\leq n} a_{ij} a_{kl}.$$

Note that if we define $\Psi([A]) = [A^t]$ then it is easy to check that Ψ is an algebra anti-automorphism, which we will sometimes call the transpose anti-automorphism.

Next we introduce quantum groups. In order to do this we recall the notion of a root datum.

Definition. A *Cartan datum* is a pair $(I, \cdot)$ consisting of a finite set I and a $\mathbb{Z}$-valued symmetric bilinear pairing on the free Abelian group $\mathbb{Z}[I]$, such that

- $i \cdot i \in \{2, 4, 6, \ldots\}$
- $2\frac{i\cdot j}{i\cdot i} \in \{0, -1, -2, \ldots\}$, for $i \neq j$.

A *root datum* of type $(I, \cdot)$ is a pair Y, X of finitely-generated free Abelian groups and a perfect pairing $\langle , \rangle : Y \times X \to \mathbb{Z}$, together with imbeddings $I \subset X$, $(i \mapsto i)$ and $I \subset Y$, $(i \mapsto i')$ such that $\langle i, j' \rangle = 2\frac{i\cdot j}{i\cdot i}$.

Given a root datum, we may define an associated quantum group $\mathbf{U}$. Since it is the only case we need, we will assume that our datum is symmetric and simply laced so that $i \cdot i = 2$ for each $i \in I$, and $i \cdot j \in \{0, -1\}$ if $i \neq j$. In this case, $\mathbf{U}$ is generated as an algebra over $\mathbb{Q}(v)$ by symbols E_i, F_i, K_μ, $i \in I$, $\mu \in Y$, subject to the following relations.

- $K_0 = 1$, $K_{\mu_1} K_{\mu_2} = K_{\mu_1+\mu_2}$ for $\mu_1, \mu_2 \in Y$;
- $K_\mu E_i K_\mu^{-1} = v^{\langle \mu, i' \rangle} E_i, \quad K_\mu F_i K_\mu^{-1} = v^{-\langle \mu, i' \rangle} F_i$ for all $i \in I$, $\mu \in Y$;
- $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}}$;

- $E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i$, for $i, j \in I$ with $i \cdot j = 0$;
- $E_i^2E_j + (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$;
- $F_i^2F_j + (v + v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ for $i, j \in I$ with $i \cdot j = -1$.

The other object we need is the modified quantum group $\dot{\mathbf{U}}$. Let $\mathbf{Mod}_X$ denote the category of left $\mathbf{U}$-modules V with a weight decomposition, that is

$$V = \bigoplus_{\lambda \in X} V_\lambda,$$

where

$$V_\lambda = \{v \in V \colon K_\mu v = v^{\langle \mu, \lambda \rangle} v, \forall \mu \in Y\}.$$

The forgetful functor to the category of vector spaces has an endomorphism ring R. Thus an element of a of R associates to each $V \in \mathrm{Ob}(\mathbf{Mod}_X)$ an endomorphism a_V, such that for any morphism $f \colon V \to W, a_W \circ f = f \circ a_V$. Thus any element of $\mathbf{U}$ clearly determines an element of R. For each $\lambda \in X$, let $1_\lambda \in R$ be the projection to the λ weight space. Then R is isomorphic to the direct product $\prod_{\lambda \in X} \mathbf{U}1_\lambda$, and we set

$$\dot{\mathbf{U}} = \bigoplus_{\lambda \in X} \mathbf{U}1_\lambda.$$

To see the connection between our convolution algebra and quantum groups, we will need the following notation. For $\mathbf{a} \in \mathfrak{S}_{D,n}$ let $\mathbf{i_a} \in \mathfrak{S}_{D,n,n}$ be the diagonal matrix with $(\mathbf{i_a})_{i,j} = \delta_{i,j}a_i$. Let $E^{i,j} \in \mathfrak{S}_{1,n,n}$ be the matrix with $(E^{i,j})_{k,l} = 1$ if $k = i + sn, l = j + sn$, some $s \in \mathbb{Z}$, and 0 otherwise. Let $\mathfrak{S}^n$ be the set of all $\mathbf{b} = (b_i)_{i \in \mathbb{Z}}$ such that $b_i = b_{i+n}$ for all $i \in \mathbb{Z}$. Let $\mathfrak{S}^{n,n}$ denote the set of all matrices $A = (a_{i,j}), i, j \in \mathbb{Z}$, with entries in $\mathbb{Z}$ such that

- $a_{i,j} \geq 0$ for all $i \neq j$;
- $a_{i,j} = a_{i+n,j+n}$, for all $i, j \in \mathbb{Z}$;
- For any $i \in \mathbb{Z}$ the set $\{j \in \mathbb{Z} \colon a_{i,j} \neq 0\}$ is finite;
- For any $j \in \mathbb{Z}$ the set $\{i \in \mathbb{Z} \colon a_{i,j} \neq 0\}$ is finite.

Thus we have $\mathfrak{S}_{D,n,n} \subset \mathfrak{S}^{n,n}$ for all D. For $i \in \mathbb{Z}/n\mathbb{Z}$ let $\mathbf{i} \in \mathfrak{S}^n$ be given by $\mathbf{i}_k = 1$ if $k = i \bmod n$, $\mathbf{i}_k = -1$ if $k = i + 1 \bmod n$, and $\mathbf{i}_k = 0$ otherwise. We write $\mathbf{a} \cup_i \mathbf{a}'$ if $\mathbf{a} = \mathbf{a}' + \mathbf{i}$. For such $\mathbf{a}, \mathbf{a}'$ set ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'} \in \mathfrak{S}^{n,n}$ to be $\mathbf{i_a} - E^{i,i} + E^{i,i+1}$, and ${}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}} \in \mathfrak{S}^{n,n}$ to be $\mathbf{i}_{\mathbf{a}'} - E^{i+1,i+1} + E^{i+1,i}$. Note if $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}_{D,n}$ then ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}, {}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}} \in \mathfrak{S}_{D,n,n}$. For $i \in \mathbb{Z}/n\mathbb{Z}$ set

$$E_i(D) = \sum [{}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}], \qquad F_i(D) = \sum [{}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}],$$

where the sum is taken over all $\mathbf{a}, \mathbf{a}'$ in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$. For $\mathbf{a} \in \mathfrak{S}^n$ set

$$K_{\mathbf{a}}(D) = \sum_{\mathbf{b} \in \mathfrak{S}_{D,n}} v^{\mathbf{a} \cdot \mathbf{b}} [\mathbf{i_b}]$$

where, for any $\mathbf{a}, \mathbf{b} \in \mathfrak{S}^n$, $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i \in \mathbb{Z}$. If we let $X = Y = \mathfrak{S}^n$, and $I = \mathbb{Z}/n\mathbb{Z}$, with the embedding of $I \subset X = Y$ and pairing as given above, we obtain a symmetric simply-laced root datum. We call the quantum group associated to it $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$. It can be shown [L99] that the elements $E_i(D), F_i(D), K_{\mathbf{a}}(D)$, generate a subalgebra $\mathbf{U}_D$ which is a quotient of the quantum group $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, via map the notation suggests. Note that this gives the algebra $\mathfrak{A}_D$ the structure of a $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$-module.

1.2 Inner product on $\mathbf{U}_D$

Definition. We define a bilinear form

$$(,)_D \colon \mathfrak{A}_{D;q} \times \mathfrak{A}_{D;q} \to \bar{\mathbb{Q}}_l$$

by

$$(f, \tilde{f})_D = \sum_{\mathbf{L},\mathbf{L}'} v^{\sum |\mathbf{L}|_i^2 - \sum |\mathbf{L}'|_i^2} f(\mathbf{L},\mathbf{L}')\tilde{f}(\mathbf{L},\mathbf{L}'),$$

for f and $\tilde{f}$ in $\mathfrak{A}_{D,q}$, where $\mathbf{L}$ runs over $\mathcal{F}^n$ and $\mathbf{L}'$ runs over a set of representatives for the G-orbits on $\mathcal{F}^n$.

Let $\mathcal{O}_A$ be a G-orbit on $\mathcal{F}^n \times \mathcal{F}^n$, and let

$$X_A^{\mathbf{L}} = \{\mathbf{L}' \in \mathcal{F}^n \colon (\mathbf{L},\mathbf{L}') \in \mathcal{O}_A\}.$$

It is easy to check that

$$2d_A - 2d_{A^t} = \sum_{i=1}^n a_{i,*}^2 - \sum_{j=1}^n a_{*,j}^2. \tag{1.2.1}$$

Thus if A, A' are in $\mathfrak{S}_{D,n,n}$ we find that

$$(e_A, e_{A'})_D = \delta_{A,A'} q^{d_A - d_{A^t}} \#|X_{A^t}^{\mathbf{L}'}|,$$

where $\mathbf{L}'$ is any lattice in $\mathcal{F}_{c(A)}$. Note that this makes it clear that the bilinear form is symmetric, which is not immediate from the initial definition. If $\{\eta_{A,B;q}^C\}$ are the structure constants of $\mathfrak{A}_{D;q}$ with respect to the basis $\{[A] \colon A \in \mathfrak{S}_{D,n,n}\}$, then we have

$$([A],[A'])_D = \delta_{A,A'} v^{d_A - d_{A^t}} \eta_{A^t,A;q}^{\mathbf{i}_{c(A)}}. \tag{1.2.2}$$

We therefore obtain an inner product on $\mathfrak{A}_D$ taking values in $\mathbb{Q}(v)$ by defining

$$([A],[A'])_D = \delta_{A,A'} v^{d_A - d_{A'}} \eta_{A^t,A,\mathbf{i}_{c(A)}} \in \mathbb{Z}[v, v^{-1}] \tag{1.2.3}$$

We now give some basic properties of this inner product:

Proposition 1.2.1. *Let $A \in \mathfrak{S}_{D,n}$, and let $f, \tilde{f} \in \mathfrak{A}_D$. Then we have*

$$([A]f, \tilde{f})_D = v^{d_A - d_{A^t}} (f, [A^t]\tilde{f})$$

Proof. Clearly it suffices to establish this equation in the algebra $\mathfrak{A}_{D;q}$. Since the characteristic functions of G-orbits form a basis of $\mathfrak{A}_{D;q}$, we may assume that $f = e_B$ and $\tilde{f} = e_C$, moreover we may assume that

$$r(A) = r(C), \quad c(A) = r(B), \quad c(B) = c(C). \tag{1.2.4}$$

as both sides are zero otherwise. It follows immediately that

$$[A] \cdot e_B = v^{-d_A} e_A \cdot e_B, \qquad v^{d_A - d_{A^t}} [A^t] \cdot e_C = v^{d_A - 2d_{A^t}} e_{A^t} \cdot e_C.$$

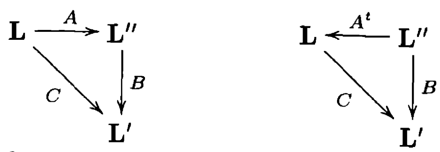
Hence if $(\tilde{\mathbf{L}}, \mathbf{L}') \in \mathcal{O}_C$ is fixed,

$$\begin{aligned} ([A] \cdot e_B, e_C)_D &= q^{d_C - d_{C^t}} \#|X_{C^t}^{\mathbf{L}'}| \cdot v^{-d_A} \#\{\mathbf{L}'' \colon (\tilde{\mathbf{L}}, \mathbf{L}'') \in \mathcal{O}_A, (\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_B\} \\ &= v^{\alpha} \#\{\mathbf{L}, \mathbf{L}'' \colon (\mathbf{L}, \mathbf{L}'') \in \mathcal{O}_A, (\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_B, (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_C\}, \end{aligned} \tag{1.2.5}$$

where $\alpha = 2d_C - 2d_{C^t} - d_A$. Similarly, if $(\tilde{\mathbf{L}}'', \mathbf{L}') \in \mathcal{O}_A$ is fixed

$$\begin{aligned} v^{d_A - d_{A^t}}(e_B, [A^t] \cdot e_C)_D &= q^{d_B - d_{B^t}} \#|X_{B^t}^{\mathbf{L}'}| \cdot v^{d_A - 2d_{A^t}} \#\{\mathbf{L} \colon (\tilde{\mathbf{L}}'', \mathbf{L}) \in \mathcal{O}_F, (\mathbf{L}, \mathbf{L}') \in \mathcal{O}_B\} \\ &= v^{\beta} \#\{\mathbf{L}, \mathbf{L}'' \colon (\mathbf{L}'', \mathbf{L}) \in \mathcal{O}_A^t, (\mathbf{L}'', \mathbf{L}') \in \mathcal{O}_B, (\mathbf{L}, \mathbf{L}' \in \mathcal{O}_C)\}, \end{aligned} \tag{1.2.6}$$

where $\beta = 2d_B - 2d_{B^t} + d_A - 2d_{A^t}$.



As the diagram clearly shows, the last line of equation (1.2.5) is the same as the last line of equation (1.2.6) if $\alpha = \beta$, that is, if

$$2d_C - 2d_{C^t} - d_A = 2d_B - 2d_{B^t} + d_A - 2d_{A^t} \tag{1.2.7}$$

But this follows directly from equation (1.2.1) and equation (1.2.4). □

We have the following easy consequence:

Corollary 1.2.2. *Let $i \in \mathbb{Z}$, and let $f, \tilde{f} \in \mathfrak{A}_D$ and $\mathbf{c} \in \mathfrak{S}^n$. Then we have*

1. $(E_i(f), \tilde{f})_D = (f, vK_{\mathbf{i}}F_i(\tilde{f}))_D$
2. $(F_i(f), \tilde{f})_D = (f, vK_{-\mathbf{i}}E_i(\tilde{f}))_D$
3. $(K_{\mathbf{c}}(f), \tilde{f})_D = (f, K_{\mathbf{c}}(\tilde{f}))_D$

Proof. We may assume that $f = e_A$ and $\tilde{f} = e_B$. The third equation can then be checked immediately from the formulas above. The second equation follows from the other two, so it only remains to prove the first. We may assume that $r(A) = r(B) - \mathbf{i}$ and $c(A) = c(B)$, as both sides are zero otherwise. Set $\mathbf{a} = r(A), \mathbf{b} = r(B)$ (see section 1.1).

Then from the definitions we have

$$E_i(e_A) = [{}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}] \cdot e_A, \qquad vK_{\mathbf{i}}F_i(e_B) = v^{1+\mathbf{i}\cdot\mathbf{a}}[{}_{\mathbf{a}}\mathbf{f}_{\mathbf{b}}] \cdot e_B.$$

Since ${}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}} = {}_{\mathbf{a}}\mathbf{f}_{\mathbf{b}}^t$, and $d_{{}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}} - d_{{}_{\mathbf{a}}\mathbf{f}_{\mathbf{b}}} = 1 + \mathbf{i} \cdot \mathbf{a}$ the result now follows immediately from the previous proposition. □

Remark. There is a unique algebra anti-automorphism $\rho \colon \mathbf{U}\left(\widehat{\mathfrak{gl}}_n\right) \to \mathbf{U}\left(\widehat{\mathfrak{gl}}_n\right)$ such that

$$\rho(E_i) = vK_{\mathbf{i}}F_i, \quad \rho(F_i) = vK_{-\mathbf{i}}E_i \quad \rho(K_i) = K_i$$

With this we may state the result of the previous corollary in the form

$$(u(f), \tilde{f})_D = (f, \rho(u)\tilde{f})_D, \qquad u \in \mathbf{U}\left(\widehat{\mathfrak{gl}}_n\right), \quad f, \tilde{f} \in \mathfrak{A}_D.$$

Lemma 1.2.3. 1. *For $A \in \mathfrak{S}_{D,n,n}$, $([A], [A])_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$*

2. *For $A, A' \in \mathfrak{S}_{D,n,n}$ and $A \neq A'$, $([A], [A'])_D = 0$*

Proof. The second part of the statement is obvious. For the first, note that $X_{A^t}^{\mathbf{L}'}$ is an irreducible variety of dimension d_{A^t}, (see [L99, 4.3]). Since we have

$$([A], [A'])_D = \delta_{A,A'} q^{-d_{A^t}} \#|X_{A^t}^{\mathbf{L}'}|,$$

the Lang-Weil estimates [LW] then show that $([A], [A])_D \in 1 + v^{-1}\mathbb{Z}[v^{-1}]$, as required. □

Remark. The results of this section are almost identical to the results of [L99, section 7]; however, as our inner product is not quite the same as that of [L99, 7.1], the proofs seem somewhat simpler.

1.3 Inner product on $\dot{\mathbf{U}}$

Notice that if $\mathbf{a} \in \mathfrak{S}^n$ then the sum $a_{i_0} + \cdots + a_{i_0+n-1}$ is independent of $i_0 \in \mathbb{Z}$; denote it by $\nabla_{\mathbf{a}}$. Let $Y = \{\mathbf{a} \in \mathfrak{S}^n \colon \nabla_{\mathbf{a}} = 0\}$. Let X be the quotient of $\mathfrak{S}^n$ by the subgroup generated by $\mathbf{b}_0$, the element with all entries equal to 1. Clearly the pairing on $\mathfrak{S}^n$ given in section 1 induces a non-singular pairing $Y \times X \to \mathbb{Z}$.

Let $I = \mathbb{Z}/n\mathbb{Z}$, and define maps $I \to X$, $I \to Y$ sending i to $\mathbf{i} \in \mathfrak{S}^n$ (see the end of section 1.1), taking the appropriate coset in X. This is the root datum of $\widehat{\mathfrak{sl}}_n$. Let $\mathbf{U}$ be the quantized enveloping algebra associated to this datum, and let $\dot{\mathbf{U}}$ be the modified algebra corresponding to $\mathbf{U}$. We wish to obtain an inner product on $\dot{\mathbf{U}}$ using those on $\mathbf{U}_D$.

We begin with some technical lemmas. Given $A \in \mathfrak{S}^{n,n}$ let $a_{i,\geq s} = \sum_{j\geq s} a_{i,j}$, and $a_{i,>s}$, $a_{i,\leq s}$, etc. similarly.

Lemma 1.3.1. *a) Let $A \in \mathfrak{S}_{D,n,n}$ and $\mathbf{a}' = r(A)$. If there is an $\mathbf{a} \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$ (i.e. if $a'_{i+1} > 0$) then we have*

$$[{}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}][A] = \sum_{s\in\mathbb{Z}, a_{i+1,s}\geq 1} v^{a_{i,\geq s} - a_{i+1,>s}} \left(\frac{1 - v^{-2(a_{i,s}+1)}}{1 - v^{-2}}\right) [A + E^{i,s} - E^{i+1,s}], \tag{1.3.1}$$

where $A = (a_{i,j})$.

b) Let $A' \in \mathfrak{S}_{D,n,n}$ and $\mathbf{a} = r(A')$. If there is an $\mathbf{a}' \in \mathfrak{S}_{D,n}$ such that $\mathbf{a} \cup_i \mathbf{a}'$ (i.e. if $a_i > 0$) then we have

$$[{}_{\mathbf{a}'}\mathbf{f}_{\mathbf{a}}][A'] = \sum_{s\in\mathbb{Z}, a_{i,s}\geq 1} v^{a'_{i+1,\leq s} - a'_{i,<s}} \left(\frac{1 - v^{-2(a'_{i+1,s}+1)}}{1 - v^{-2}}\right) [A' - E^{i,s} + E^{i+1,s}], \tag{1.3.2}$$

where $A' = (a'_{i,j})$.

Proof. This follows by rescaling the statement of Proposition 3.5 in [L99]. □

Let $\mathcal{R}$ be the subring of $\mathbb{Q}(v)[u]$ generated by $\{v^j : j \in \mathbb{Z}\}$, and

$$\prod_{i=1}^{t} (v^{-2(a-i)}u^2 - 1)/(v^{-2i} - 1); \qquad a \in \mathbb{Z}, t \geq 1.$$

For $A \in \mathfrak{S}^{n,n}$ let ${}_pA$ be the matrix with $({}_pA)_{i,j} = a_{i,j} + p\delta_{i,j}$. We have the following partial analogue of [BLM, 4.2].

Lemma 1.3.2. *Let $A_1, A_2, \ldots, A_k$ be matrices of the form ${}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}'}$ or ${}_{\mathbf{a}}\mathbf{f}_{\mathbf{a}'}$, for $\mathbf{a}, \mathbf{a}' \in \mathfrak{S}^n$, and A any element of $\mathfrak{S}^{n,n}$. Then there exist matrices $Z_1, Z_2, \ldots, Z_m \in \mathfrak{S}^{n,n}$ and $p_0 \in \mathbb{Z}$ such that*

$$[{}_pA_1][{}_pA_2]\ldots[{}_pA_k][{}_pA] = \sum_{i=1}^{m} G_i(v, v^{-p})[{}_pZ_i], \qquad G_i \in \mathcal{R} \tag{1.3.3}$$

for all $p \geq p_0$.

Proof. Use induction on k. When $k = 1$ the result follows from the previous lemma, once we note that both $a_{i,\geq s} - a_{i+1,>s}$ and $a_{i+1,\leq s} - a_{i,<s}$ are unchanged when A is replaced with $A + pI$. □

For $\lambda \in X$ let 1_λ be the idempotent in $\dot{\mathbf{U}}$ defined in [L92, 23.1.1]. There is a surjective homomorphism

$$\phi_D \colon \dot{\mathbf{U}} \to \mathbf{U}_D$$

which, for $\lambda \in X$, sends $E_i 1_\lambda \mapsto E_i(D)[\mathbf{i_a}]$ and $F_i 1_\lambda \mapsto F_i(D)[\mathbf{i_a}]$ if there is an $\mathbf{a}$ in $\mathfrak{S}_{D,n}$ such that $\mathbf{a} = \lambda \bmod \mathbb{Z}\mathbf{b}_0$, otherwise both $E_i 1_\lambda$, $F_i 1_\lambda$ are sent to zero.

Let $\mathbf{f}$ be the algebra attached to the root datum described above (see [L92, chapter 3]). Pick a monomial basis of $\mathbf{f}$, $\{\zeta_i : i \in J\}$ say. Then the triangular decomposition for $\dot{\mathbf{U}}$ [L92, 23.2.1] shows that $\mathfrak{B} = \{\zeta_i^+ \zeta_j^- 1_\lambda : i, j \in J, \lambda \in X\}$ is a basis of $\dot{\mathbf{U}}$, where $+ : \mathbf{f} \to \mathbf{U}^+$, and $- : \mathbf{f} \to \mathbf{U}^-$ are the standard maps given in [L92, 3.1.1]. Define a bilinear pairing $\langle , \rangle_D$ on $\dot{\mathbf{U}}$ via ϕ_D as follows:

$$\langle x, y \rangle_D = (\phi_D(x), \phi_D(y))_D$$

Proposition 1.3.3. *Let $k \in \{0, 1, \dots, n-1\}$, then if $x, y \in \dot{\mathbf{U}}$*

$$\langle x, y \rangle_{k+pn}$$

converges in $\mathbb{Q}((v^{-1}))$, as $p \to \infty$, to an element of $\mathbb{Q}(v)$.

Proof. We may assume that x, y are elements of $\mathfrak{B}$. Then we need to show that

$$\langle \zeta_{i_1}^+ \zeta_{j_1}^- 1_\lambda, \zeta_{i_2}^+ \zeta_{j_2}^- 1_\mu \rangle_{k+pn} \qquad i_1, i_2, j_1, j_2 \in J; \lambda, \mu \in X$$

converges as $p \to \infty$. Let $\iota : \mathbf{f} \to \mathbf{f}$ is the $\mathbb{Q}(v)$-algebra anti-automorphism fixing the generators $\theta_i, 1 \le i \le n$. Using Proposition 1.2.2, it is easy to see that this inner product differs from

$$\langle 1_\lambda, \iota(\zeta_{j_1})^+ \iota(\zeta_{i_1})^- \zeta_{i_2}^+ \zeta_{j_2}^- 1_\mu \rangle_{k+pn} \tag{1.3.4}$$

by a power of v which is independent of p. But then the definition of the inner product and the previous proposition show that (1.3.4) may be written as $G(v, v^{-p})$ for some $G \in \mathcal{R}$. The result then follows immediately from the definition of $\mathcal{R}$. □

Definition. We define

$$(,) : \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbb{Q}(v),$$

a symmetric bilinear form on $\dot{\mathbf{U}}$ given by

$$(x, y) = \sum_{k=0}^{n-1} \lim_{p \to \infty} \langle x, y \rangle_{k+pn}.$$

Remark. Note that the proof of the last proposition actually allows us to conclude that

$$(\phi_D(\zeta_i^+ \zeta_j^- 1_\lambda), [{}_pA])_{k+pn}$$

converges to an element of $\mathbb{Q}(v)$, as $p \to \infty$, for any $A \in \mathfrak{S}^{n,n}$. We will need this in the next section.

1.4 Comparison of inner products

There is a natural definition of an inner product on $\dot{\mathbf{U}}$ in the algebraic setting.

Theorem 1.4.1. *There exists a unique $\mathbb{Q}(v)$ bilinear pairing $\langle , \rangle : \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbb{Q}(v)$ such that*

1. $\langle 1_{\lambda_1} x 1_{\lambda_2}, 1_{\mu_1} y 1_{\mu_2} \rangle = 0 \qquad \forall x, y \in \dot{\mathbf{U}}$ *unless* $\lambda_1 = \mu_1, \lambda_2 = \mu_2$*;*
2. $\langle ux, y \rangle = \langle x, \rho(u) y \rangle \qquad \forall x, y \in \dot{\mathbf{U}}, u \in \mathbf{U}$*; and*
3. $\langle x^- 1_\lambda, y^- 1_\lambda \rangle = (x, y), \qquad \forall x, y \in \mathbf{f}, \lambda \in X.$

Here (x, y) is the standard inner product on $\mathbf{f}$, (see [L93, 1.2.5]). The resulting inner product is automatically symmetric.

Proof. See [L93, 26.1.2]. □

Theorem 1.4.2. *The inner products $(,)$ of section 1.3 and $\langle,\rangle$ of Theorem 1.4.1 coincide.*

The remainder of this section is devoted to the proof of this theorem. The first property listed in Theorem 1.4.1 clearly holds for $(,)$, as the representatives for elements of X in $\mathfrak{S}_{D,n}$ are distinct when they exist. The second follows from Proposition 1.2.2; thus it only remains to verify the third. Fix $\lambda \in X$.

The algebra $\mathbf{f}$ is naturally graded: $\mathbf{f} = \bigoplus_{\nu \in \mathbb{N}I} \mathbf{f}_\nu$. For $\nu \in \mathbb{Z}[I]$, with $\nu = \sum_{i \in I} \nu_i i$ let $tr(\nu) = \sum_{i \in I} \nu_i$. If z is homogeneous we set $|z| = \nu$, where $z \in \mathbf{f}_\nu$. Thus for the third property we may assume that $x, y \in \mathbf{f}$ are homogeneous, i.e. $x, y \in \mathbf{f}_\nu$ for some ν, and proceed by induction on $N = \mathrm{tr}(\nu)$. If $N = 0$ then we are reduced to the equation

$$(1_\lambda, 1_\lambda) = 1,$$

which is trivial. Now suppose that $N > 0$ and the result is known for $x, y \in \mathbf{f}_\nu$ when $\mathrm{tr}(\nu) < N$. If x, y are in $\mathbf{f}_\nu$, $\mathrm{tr}(\nu) = N$, then we may assume that they are monomials, and $y = \theta_i z$ for some $z \in \mathbf{f}_{\nu - \mathbf{i}}$. Then we have

$$\begin{aligned}(x^- 1_\lambda, y^- 1_\lambda) &= (x^- 1_\lambda, F_i z^- 1_\lambda) \\ &= (v K_{-\mathbf{i}} E_i x^- 1_\lambda, z^- 1_\lambda).\end{aligned}$$

Using standard commutation formulas (see [L93, 3.1.6]) this becomes

$$(v K_{-\mathbf{i}} x^- E_i 1_\lambda, z^- 1_\lambda) + \frac{1}{1 - v^{-2}} \left(({}_i r(x)^- - v K_{-\mathbf{i}} r_i(x)^- K_{-\mathbf{i}}) 1_\lambda, z^- 1_\lambda \right)$$

and tidying this up we get

$$\frac{1}{1 - v^{-2}} ({}_i r(x)^- 1_\lambda, z^- 1_\lambda) + \left(v^{\mathbf{i} \cdot |x| - \mathbf{i} \cdot \lambda - 1} \left(x^- E_i - \frac{v^{-\mathbf{i} \cdot \lambda}}{v - v^{-1}} r_i(x)^- \right) 1_\lambda, z^- 1_\lambda \right)$$

The properties of $(,)$ on $\mathbf{f}$ show that $\frac{1}{1 - v^{-2}} ({}_i r(x), z) = (x, \theta_i z)$, thus we are done by induction if we can show that

$$\left(x^- E_i - \frac{v^{-\mathbf{i} \cdot \lambda}}{v - v^{-1}} r_i(x)^- \right) 1_\lambda$$

annihilates $\mathbf{U}^- 1_\lambda$. To see this we need an explicit result about multiplication in $\mathfrak{A}_D$.

Lemma 1.4.3. *Let $A \in \mathfrak{S}^{n,n}$ be such that $a_{r,s} = 0$ for $r < s$ unless $r = s - 1$ and $r = i$ mod n, when $a_{r,r+1} \in \{0, 1\}$; then the following hold for p sufficiently large.*

1. *For $j \neq i$ we have*

$$F_j [{}_p A] = \sum_{k=1}^{m} g_k(v) [{}_p Z_k]$$

where $g_k(v) \in \mathbb{Z}[v, v^{-1}]$ are independent of $\{a_{r,s} \colon r \leq s\}$, and $Z_k \in \mathfrak{S}^{n,n}$ have $(Z_k)_{r,s} = a_{r,s}$ for $r < s$.

2.

$$\begin{aligned}F_i [{}_p A] = &\sum_{k=1} g_k(v) [{}_p Z_k] \\ &+ v^{1 - \mathbf{i} \cdot r(A)} \left(\frac{1 - v^{-2(a_{i+1,i+1} + 1 + p)}}{1 - v^{-2}} \right) [{}_p (A + E^{i+1,i+1} - E^{i,i+1})]\end{aligned}$$

where $g_k(v) \in \mathbb{Z}[v, v^{-1}]$ are independent of $\{a_{r,s} : r \leq s\}$, and $Z_k \in \mathfrak{S}^{n,n}$ have $(Z_k)_{r,s} = a_{r,s}$ for $r < s$, and the final term occurs only if $a_{i,i+1} = 1$.

Proof. Both of these formulas are consequences of the following, which is valid for any A (see Lemma 1.3.1).

$$F_j[{}_pA] = \sum_{k\colon ({}_pA)_{j,k} \geq 1} v^{a_{j+1,\leq k} - a_{j,<k}} \left(\frac{1 - v^{-2(a_{j+1,k} + p\delta_{j+1,k} + 1)}}{1 - v^{-2}} \right) [{}_pA + E^{j+1,k} - E^{j,k}].$$

□

Let $\sum_{j=1}^{n} \lambda_j = k \bmod n$, where $k \in \{0, 1, \dots, n-1\}$, and suppose that $D = k + pn$ for some p. Let $\mathfrak{A}_D^- = \text{span}\{[A] \colon a_{r,s} = 0, \forall r < s\}$, and note that Lemma 1.4.3 shows that $\phi_D(x^- 1_\lambda) \in \mathfrak{A}_D^-$ for any $x \in \mathbf{f}$. In fact, it is also clear that

$$\phi_D(x^- E_i 1_\lambda) = \sum_{k=1}^{m_1} a_k(v)[{}_pB_k] + \sum_{k=1}^{m_2} g_k(v)[{}_pH_k]$$

where $(B_k)_{i,i+1} = 1$ and $(H_k)_{i,i+1} = 0$, and a_k, g_k are independent of λ and p. Moreover from the formula in the proof of the Lemma 1.4.3 it is easy to see that

$$\phi_D(x^- 1_\lambda) = \sum_{k=1}^{m_1} a_k(v)[{}_pB_k + E^{i+1,i+1} - E^{i,i+1}].$$

We are now ready to set up the key step in the proof of Theorem 1.4.2: Let $\pi_D \colon \mathfrak{A}_D \to \mathfrak{A}_D^-$ be the orthogonal projection. Define $s_D \colon \mathbf{f} \to \mathfrak{A}_D^-$ by setting

$$x \mapsto \pi_D(\phi_D(x^- E_i 1_\lambda))$$

and define $r_D \colon \mathbf{f} \to \mathfrak{A}_D^-$ by setting

$$x \mapsto \frac{v^{-\mathbf{i}\cdot\lambda}}{v - v^{-1}} \phi_D(r_i(x)^- 1_\lambda)$$

Proposition 1.4.4. *Let $x \in \mathbf{f}$.*

$$s_D(x) - r_D(x) = v^{-2p} \left(\sum_{k=1}^{m} c_k(v)[{}_pZ_k] \right)$$

for some $Z_k \in \mathfrak{S}^{n,n}$, independent of p.

Proof. We may assume that x is a monomial, and proceed by induction on $\text{tr}(|x|)$. It is easy to check that $s_D(1) = r_D(1) = 0$, so we may assume that $x \in \mathbf{f}_\nu$, $\text{tr}(\nu) > 0$, and that $x = \theta_j z$ where $z \in \mathbf{f}_{\nu - \mathbf{j}}$. Now as above we have

$$\phi_D(z^- E_i 1_\lambda) = \sum_{k=1}^{m_1} a_k(v)[{}_pB_k] + \sum_{k=1}^{m_2} g_k(v)[{}_pH_k] \qquad (B_k)_{i,i+1} = 1, (H_k)_{i,i+1} = 0,$$

and so $s_D(z) = \sum_{k=1}^{m_2} g_k(v)[{}_pH_k]$. Let $E = E^{i+1,i+1} - E^{i,i+1} \in \mathfrak{S}^{n,n}$. Using the lemma we see that since

$$\phi_D(x^- E_i 1_\lambda) = F_j \phi_D(z^- E_i 1_\lambda),$$

we have

$$s_D(x) = \delta_{i,j} \sum_k a_k(v) v^{1-\mathbf{i}\cdot r(B_k)} \left(\frac{1 - v^{-2((B_k)_{i+1,i+1}+p+1)}}{1-v^{-2}} \right) [_p(B_k + E)] \\ + F_j s_D(z). \tag{1.4.1}$$

Now $r(B_k) = \lambda + \mathbf{i} - |x|$, hence $1 - \mathbf{i} \cdot r(B_k) = \mathbf{i} \cdot (|x| - \lambda) - 1$, so

$$v^{1-\mathbf{i}\cdot r(B_k)} \left(\frac{1 - v^{-2((B_k)_{i+1,i+1}+p+1)}}{1-v^{-2}} \right) = \left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v - v^{-1}} \right) (1 - v^{-2p} v^{-2((B_k)_{i+1,i+1}+1)}).$$

The definition of r_i shows that

$$\begin{aligned} r_D(x) &= r_D(\theta_j z) \\ &= \delta_{i,j} \left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v - v^{-1}} \right) \phi_D(z^- 1_\lambda) + F_j r_D(x) \\ &= \delta_{i,j} \left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v - v^{-1}} \right) \sum_k a_k(v) [_p(B_k + E)] + F_j r_D(x), \end{aligned}$$

so we see that

$$\begin{aligned} s_D(x) - r_D(x) = {} & F_j(s_D(z) - r_D(z)) \\ & - \delta_{i,j} v^{-2p} \sum_k a_k(v) \left(\frac{v^{\mathbf{i}\cdot(|x|-\lambda)}}{v - v^{-1}} \right) v^{-2((B_k)_{i+1,i+1}+1)} [_p(B_k + E)] \end{aligned}$$

and so using induction and the lemma again, we are done. □

Corollary 1.4.5. *Let $x \in \mathbf{f}$, then*

$$u = \left(x^- E_i - \frac{v^{-\mathbf{i}\cdot\lambda}}{v - v^{-1}} r_i(x)^- \right) 1_\lambda$$

is orthogonal to $\mathbf{U}^- 1_\lambda$.

Proof. Let $y \in \mathbf{f}$ be a monomial. Then we have

$$(u, y^- 1_\lambda) = \lim_{p\to\infty} \langle u, y^- 1_\lambda \rangle_{k+pn},$$

and by definition

$$\langle u, y^- 1_\lambda \rangle_{k+pn} = (s_{k+pn}(x) - r_{k+pn}(x), \phi_{k+pn}(y^- 1_\lambda))_{k+pn}. \tag{1.4.2}$$

By the previous proposition,

$$s_{k+pn}(x) - r_{k+pn}(x) = v^{-2p} \left(\sum_{j=1}^m c_j(v) [_p Z_j] \right), \qquad Z_j \in \mathfrak{S}^{n,n},$$

and by the remark at the end of section 2, we know that $([_p Z_j], \phi_{k+pn}(y^- 1_\lambda))_{k+pn}$ converges in $\mathbb{Q}((v^{-1}))$ as $p \to \infty$. Thus the right-hand side of Equation 1.4.2 tends to zero as required. □

This completes the proof of Theorem 1.4.2.

1.5 Geometric Interpretation

Recall from [L99, section 4] that $\mathfrak{A}_D$ possesses a canonical basis $\mathfrak{B}_D$ consisting of elements $\{A\}$, $A \in \mathfrak{S}_{D,n,n}$. To define these elements we must assume $\mathbf{k}$ is algebraically closed (either the algebraic closure of $\mathbb{F}_q$, in which case we must use sheaves in the étale topology, or $\mathbb{C}$ in which case we use the analytic topology). Fix $A \in \mathfrak{S}_{D,n}$, and $\mathbf{L} \in \mathcal{F}_{r(A)}$. The space $\mathcal{F}_D$ can be given the structure of an ind-scheme such that the set $X_A^{\mathbf{L}}$ (see section 1.2) lies naturally in a projective algebraic variety. Thus it makes sense to consider its closure $\bar{X}_A^{\mathbf{L}}$. Let $\mathbf{A}$ be the simple perverse sheaf on $\bar{X}_A^{\mathbf{L}}$ whose restriction to $X_A^{\mathbf{L}}$ is $\mathbb{C}[d_A]$. Let $\mathcal{H}^s(\mathbf{A})$ to be the s-th cohomology sheaf of $\mathbf{A}$. For $A_1 \in \mathfrak{S}_{D,n,n}$ such that $X_{A_1}^{\mathbf{L}} \subset \bar{X}_A^{\mathbf{L}}$ we write $A_1 \leq A$, and set

$$\Pi_{A_1,A} = \sum_{s\in\mathbb{Z}} \dim(\mathcal{H}_y^{s-d_{A_1}}(\mathbf{A}))v^s \in \mathcal{A},$$

where $\mathcal{H}_y^{s-d_{A_1}}(\mathbf{A})$ is the stalk of $\mathcal{H}^{s-d_{A_1}}(\mathbf{A})$ at a point $y \in \mathbf{X}_{A_1}^{\mathbf{L}}$ (since $\mathbf{A}$ is constructible with respect to the stratification of $\bar{X}_A^{\mathbf{L}}$ given by $\{X_{A_1}^{\mathbf{L}} : A_1 < A\}$, this is independent of the choice of y). We have

$$\{A\} = \sum_{A_1; A_1 \leq A} \Pi_{A_1,A}[A_1].$$

Note that the following is an immediate consequence of the definitions and Lemma 1.2.3.

Lemma 1.5.1. *Let $A, A' \in \mathfrak{S}_{D,n,n}$, then,*

$$(\{A\}, \{A'\})_D \in \delta_{A,A'} + v^{-1}\mathbb{Z}[v^{-1}].$$

□

The algebra $\mathfrak{A}_D$ may be viewed as a convolution algebra of (equivariant) complexes on $\mathcal{F}^n$. We wish to give an interpretation of the inner product of section 1.2 in this context. Suppose that $A, B \in \mathfrak{S}_{D,n,n}$. We want to describe $(\{A\}, \{B\})$. We may assume that $r(A) = r(B) = \mathbf{a}$ and $c(A) = c(B) = \mathbf{b}$. Let $\mathbf{L}' \in \mathcal{F}_{\mathbf{b}}$. Let $\mathbf{A}^{\mathbf{t}}$ and $\mathbf{B}^{\mathbf{t}}$ denote the simple perverse sheaves on $\bar{\mathbf{X}}_{A^t}^{\mathbf{L}'}$ and $\bar{\mathbf{X}}_{B^t}^{\mathbf{L}'}$ respectively. Then define

$$\langle \{A\}, \{B\} \rangle^D = \sum_{i\in\mathbb{Z}} \dim(H_c^i(\mathcal{F}_{\mathbf{a}}, \mathbf{A}^{\mathbf{t}} \otimes \mathbf{B}^{\mathbf{t}}))v^i. \tag{1.5.1}$$

$\langle , \rangle^D$ extends to an inner product on the whole of $\mathfrak{A}_D$ (viewed as an algebra of equivariant complexes on $\mathcal{F}^n$). We want to show that it is the same as the inner product $(,)_D$ of section 1.2 , at least on the subalgebra $\mathbf{U}_D$. We start by showing that $\langle , \rangle$ satisfies the properties of Proposition 1.2.2.

Lemma 1.5.2. *Let $A, B, C \in \mathfrak{S}_{D,n,n}$, and suppose that $X_A^{\mathbf{L}}$ is a closed orbit. Then*

$$\langle \{A\}\{B\}, \{C\} \rangle^D = v^{d_A - d_{A^t}} \langle \{B\}, \{A^t\}\{C\} \rangle^D.$$

Proof. We need to recall the definition of the convolution product. Let

$$Z = \mathcal{O}_A \subset \mathcal{F}_{\mathbf{a}} \times \mathcal{F}_{\mathbf{b}},$$

where $r(A) = \mathbf{a}$, $c(A) = \mathbf{b}$. We have maps $p_1 : Z \to \mathcal{F}_{\mathbf{a}}$ and $p_2 : Z \to \mathcal{F}_{\mathbf{b}}$, the first and second projections respectively. They are clearly surjective, and the dimension of a fibre of p_1 is d_A, and the dimension of a fibre of p_2 is d_{A^t}.

Then

$$(\mathbf{A} * \mathbf{B})^t = (p_1)_! p_2^*(\mathbf{B}^{\mathbf{t}})[d_{A^t}], \qquad (\mathbf{A}^t * \mathbf{C})^t = (p_2)_! p_1^*(\mathbf{C}^{\mathbf{t}})[d_A].$$

Now

$$
\begin{aligned}
(\mathbf{A} * \mathbf{B})^t \otimes \mathbf{C}^\mathbf{t} &= (p_1)_! p_2^*(\mathbf{B}^\mathbf{t})[d_{A^t}] \otimes \mathbf{C}^\mathbf{t} \\
&= (p_1)_!(p_2^*(\mathbf{B}^\mathbf{t}) \otimes p_1^*(\mathbf{C}^\mathbf{t})[d_{A^t}]),
\end{aligned} \tag{1.5.2}
$$

$$
\begin{aligned}
\mathbf{B}^\mathbf{t} \otimes (\mathbf{A}^t * \mathbf{C})^t &= \mathbf{B}^\mathbf{t} \otimes (p_2)_! p_1^*(\mathbf{C}^\mathbf{t})[d_A] \\
&= (p_2)_!(p_2^*(\mathbf{B}^\mathbf{t}) \otimes p_1^*(\mathbf{C}^\mathbf{t})[d_A]),
\end{aligned} \tag{1.5.3}
$$

where the second equality in each case uses the "projection formula". It is thus clear that the compactly supported cohomologies will be the same up to shifts, with the difference in shift being $d_A - d_{A^t}$ as required. □

Lemma 1.5.3. *Let $A, B \in \mathfrak{S}_{D,n,n}$, and $\mathbf{c} \in \mathfrak{S}^n$. Then*

1. $\langle E_i\{A\}, \{B\}\rangle^D = \langle \{A\}, vK_\mathbf{i}F_i\{B\}\rangle^D$.
2. $\langle F_i\{A\}, \{B\}\rangle^D = \langle \{A\}, vK_{-\mathbf{i}}E_i\{B\}\rangle^D$
3. $\langle K_\mathbf{c}\{A\}, \{B\}\rangle^D = \langle \{A\}, K_\mathbf{c}\{B\}\rangle^D$

Proof. This follows from the previous lemma exactly as in the proof of corollary 1.2.2, since the varieties $X^\mathbf{L}_{\mathbf{a}+\mathbf{i}\mathbf{e}_\mathbf{a}}$ are closed. □

The algebra $\mathbf{U}_D$ is spanned by elements of the form $T_1T_2\dots T_N[\mathbf{i}_\mathbf{a}]$ where T_s is either E_i or F_i for some i. Thus the previous lemma shows we need only check that

$$\langle T_1T_2\dots T_N[\mathbf{i}_\mathbf{a}], [\mathbf{i}_\mathbf{a}]\rangle^D = (T_1T_2\dots T_N[\mathbf{i}_\mathbf{a}], [\mathbf{i}_\mathbf{a}])_D$$

But this will follow if we can show that

$$\langle \{A\}, [\mathbf{i}_\mathbf{a}]\rangle^D = (\{A\}, [\mathbf{i}_\mathbf{a}])_D$$

for all $A \in \mathfrak{S}_{D,n,n}$, as $\{\{A\} \colon A \in \mathfrak{S}_{D,n,n}\}$ is a basis of $\mathfrak{A}_D$. But as the simple perverse sheaf corresponding to $\{\mathbf{i}_\mathbf{a}\} = [\mathbf{i}_\mathbf{a}]$ is just the skyscraper sheaf at the point $\mathbf{L}'$, this last equality follow from directly from the definitions. We have therefore shown the following result.

Proposition 1.5.4. *On the algebra $\mathbf{U}_D$ the inner products $\langle , \rangle^D$ and $(,)_D$ coincide.* □

Remark. It can be shown that the algebra $\mathfrak{A}_D$ is generated by the elements $\{A\}$ for which $X^\mathbf{L}_A$ is closed, and so the above argument adapts to show that the inner products in fact agree on the whole of $\mathfrak{A}_D$. Henceforth we will use the notation $(,)_D$ when referring to the inner product on $\mathfrak{A}_D$ in either of its incarnations.

1.6 Applications

We now give some applications of our results. Theorem 1.4.2 allows us to give an alternative proof of an injectivity result due to Lusztig [L99a]. In that paper he defines "transfer maps"

$$\psi_D \colon \mathfrak{A}_D \to \mathfrak{A}_{D-n},$$

which are characterized , at least on $\mathbf{U}_D$, by the following,

- $\psi_D(E_i(D)) = E_i(D-n)$;
- $\psi_D(F_i(D)) = F_i(D-n)$;
- $\psi_D(K_\mathbf{a}(D)) = v^{\mathbf{a}\cdot\mathbf{b}_0}K_\mathbf{a}(D-n)$.

Let $\hat{\mathbf{U}} = \varprojlim_D \mathfrak{A}_D$ where the limit is taken over the projective system given by the maps ψ_D, for $D \geq 0$. Since the maps ϕ_D are compatible with this system, that is, $\psi_{D+n}\phi_{D+n} = \phi_D$, there is a unique map $\phi\colon \dot{\mathbf{U}} \to \hat{\mathbf{U}}$, which factors the maps ϕ_D through the canonical map $\hat{\mathbf{U}} \to \mathfrak{A}_D$. Then we have the following result.

Proposition 1.6.1. *The homomorphism ϕ is injective.*

Proof. Let u be in the kernel of ϕ. Then for every D we have $\phi_D(u) = 0$, and hence by Theorem 1.4.2 we see that u is in the radical of the inner product on $\dot{\mathbf{U}}$. Since this inner product is nondegenerate it follows that $u = 0$. □

The modified quantum group $\dot{\mathbf{U}}$ is equipped with a canonical basis $\dot{\mathbf{B}}$ which generalizes the canonical basis of $\mathbf{U}^-$. We can use the compatibility of the inner products to show a kind of "asymptotic" compatibility of the canonical bases of $\dot{\mathbf{U}}$ and $\mathfrak{A}_D$.

Proposition 1.6.2. *Let $b \in \dot{\mathbf{B}}$. Then there exists λ such that $b \in \dot{\mathbf{U}}1_\lambda$. Set k to be the residue of $\sum_{i=1}^n \lambda_i$* mod n. Then there is a $p_0 > 0$ such that for all $p > p_0$ we have $\phi_{k+pn}(b) \in \mathfrak{B}_D$.

Proof. The canonical basis of $\dot{\mathbf{U}}$ is characterized (up to sign) by the properties of being invariant under the bar involution, lying in the integral form $\dot{\mathbf{U}}_{\mathcal{A}}$, and having self inner product 1 modulo $v^{-1}\mathbb{Z}[[v^{-1}]]$ (for a precise statement see [L93, Theorem 26.3.1]). Each of these ingredients has a counterpart for $\mathfrak{A}_D$, and using Lemma 1.5.1 it is easy to see that $\mathfrak{B}_D$ is also characterized in this way. Since the inner product $\dot{\mathbf{U}}$ is obtained as a limit from the inner products on $\mathfrak{A}_D$, it follows that for large p we have

$$(b, b)_{k+pn} = 1 \bmod v^{-1}\mathbb{Z}[v^{-1}].$$

The bar involutions on $\dot{\mathbf{U}}$ and $\mathfrak{A}_D$ are compatible, as can be easily checked on generators, and the maps ϕ_D are compatible with the integral forms. Therefore at least for large p we have $\phi_D(b)$ is, up to sign, an element of $\mathfrak{B}_D$, the canonical basis of $\mathfrak{A}_D$. The issue of sign can be resolved by using induction, and the fact that for an element of $\mathbf{U}^{\pm}1_\lambda$ the compatibility can be obtained directly from geometry (see [L99a, section 3]), so that in fact $\phi_D(b)$ is an element of $\mathfrak{B}_D$. □

We can also combine Theorem 1.4.2 and Proposition 1.5.4 to prove a positivity result for the inner product of two elements of $\dot{\mathbf{B}}$.

Theorem 1.6.3. *Let $b_1, b_2 \in \dot{\mathbf{B}}$.*

$$(b_1, b_2) \in \mathbb{N}[[v^{-1}]] \cap \mathbb{Q}(v).$$

Proof. We may assume that there is a $\lambda \in X$ such that $b_1 1_\lambda = b_1$, and $b_2 1_\lambda = b_2$. Let $k \in \{0, 1, \ldots, n-1\}$ be such that $\sum_{j=1}^n \lambda_j = k \bmod n$. Then

$$(b_1, b_2) = \lim_{p\to\infty} (\phi_{k+pn}(b_1), \phi_{k+pn}(b_2))_{k+pn}$$

By Proposition 1.6.2 we know that for all large enough p, $\phi_D(b_1), \phi_D(b_2)$ are in $\mathfrak{B}_D$, hence it is clear from Equation (1.5.1) that

$$(\phi_{k+pn}(b_1), \phi_{k+pn}(b_2))_{k+pn} \in \mathbb{N}[v, v^{-1}].$$

However, it follows also from Lemma 1.5.1 that the left-hand side is in fact in $\mathbb{N}[v^{-1}]$ (this can also be seen directly, using the definition of intersection cohomology sheaves). Hence (b_1, b_2) is the limit of elements of $\mathbb{N}[v^{-1}]$, and the statement follows. □

Remark. The result of Proposition 1.6.2 is actually true without restriction on p, as was shown by Schiffmann and Vasserot [ScV]. We included this asymptotic version to keep the chapter self-contained, and because it can be used to give an alternative proof of the full result (see Theorem 2.4.1).

1.7 Affine $\mathfrak{gl}_n$

The results of this chapter are so far restricted to the modified algebra of quantum affine $\mathfrak{sl}_n$. We now show how to extend the constructions to the case of quantum affine $\mathfrak{gl}_n$. First we define an extension of the algebras $\mathfrak{A}_D$. Let

$$\tilde{\mathfrak{A}}_D = \bigoplus_{m \in \mathbb{Z}} \mathfrak{A}_D \alpha_m,$$

where the α_ms are orthogonal idempotents. Thus $\tilde{\mathfrak{A}}_D$ consists of $\mathbb{Z}$ copies of the algebra $\mathfrak{A}_D$. This algebra receives a homomorphism $\tilde{\phi}_D$ from the modified quantum group of $\widehat{\mathfrak{gl}}_n$ defined as follows: for $\lambda \in \mathfrak{S}^n$, we have $E_i 1_\lambda \mapsto E_i(D)[\mathbf{i_a}]\alpha_p$ and $F_i 1_\lambda \mapsto F_i(D)[\mathbf{i_a}]\alpha_p$ if there is an $\mathbf{a}$ in $\mathfrak{S}_{D,n}$ such that $\lambda + p\mathbf{b}_0 = \mathbf{a}$, otherwise both $E_i 1_\lambda$, $F_i 1_\lambda$ are sent to zero. It is easy to check that this is a well defined homomorphism using the results of [L99].

The algebras $\tilde{\mathfrak{A}}_D$ also have an $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ module structure (in fact a bi-module structure). The action of E_i and F_i is just as for $\mathbf{U}$, so we only need to define the action of $K_\mathbf{a}$ for $\mathbf{a} \in \mathfrak{S}^n$. This is given by

$$K_\mathbf{a}(x\alpha_m) = v^{-m\mathbf{a}\cdot\mathbf{b}_0}(K_\mathbf{a}(D)x)\alpha_m.$$

The transfer maps of [L99a] can be extended to this situation also. We set $\tilde{\psi}_D \colon \tilde{\mathfrak{A}}_D \to \tilde{\mathfrak{A}}_{D-n}$ where $\tilde{\psi}_D(x\alpha_m) = \psi_D(x)\alpha_{m-1}$. It is then easy to check that, in contrast to the case of the algebras $\mathfrak{A}_D$ with maps ψ_D, the maps $\tilde{\psi}_D$ are $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$-module maps, and are compatible with the maps $\tilde{\phi}_D$. Finally we extend the inner product on $\mathfrak{A}_D$ to $\tilde{\mathfrak{A}}_D$ by making each of the copies of $\mathfrak{A}_D$ orthogonal, that is

$$(,)_D \colon \tilde{\mathfrak{A}}_D \times \tilde{\mathfrak{A}}_D \to \mathbb{Q}(v).$$

is given by $(x\alpha_p, y\alpha_q)_D = \delta_{p,q}(x,y)_D$, for $x, y \in \mathfrak{A}_D$ (we use the same notation for the inner product on $\tilde{\mathfrak{A}}_D$ as for that on $\mathfrak{A}_D$).

Using exactly the same techniques as above, we recover the inner product on $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, and therefore we get an analogue of the injectivity result of section 1.6.

Proposition 1.7.1. *Let $\hat{\mathfrak{A}}$ denote the projective limit of the algebras $(\tilde{\mathfrak{A}}_D, \tilde{\psi}_D)$. Then the natural map $\tilde{\psi}$ from $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ to $\hat{\mathfrak{A}}$, is an injection.* □

Remark. All the results of this chapter have analogues for the nonaffine case, which can be proved in exactly the same way. The module V is replaced by a D-dimensional vector space over $\mathbf{k}$, and the space $\mathcal{F}^n$ of n-step periodic lattices should be replaced by the space of n-step flags in that vector space. In this case it is the algebra corresponding to $\mathbf{U}_D$ is actually equal to the algebra analogous to $\mathfrak{A}_D$, hence the results are in sometimes easier in this case.

Chapter 2

Cells in affine algebras

Given any algebra with a specified basis it is possible to define a notion of cells— left, right and two-sided. Of course, if one picks the basis of the algebra arbitrarily, it is unlikely that these objects will contain any interesting information about the algebra in question. However, if the algebra has a natural choice of basis, the situation can be quite different. Examples of this arise in a number of places: The Kazhdan–Lusztig basis of a Hecke algebra gives rise to a notion of cells which in the case of finite Weyl groups is essential in the classification of the characters of finite groups of Lie type. On the other hand, although the plus part of the quantum group possesses a canonical basis, the theory of cells there is trivial. If we extend the canonical basis to one for the modified quantum group $\dot{\mathbf{U}}$ however, the theory of cells is once again interesting.

In the case of quantum groups of finite type, work of Lusztig [L95] completely describes the cells. In contrast, when one considers the case of affine quantum groups, though the theory is conjecturally richer, there is almost nothing known about it. In this chapter we show that the geometric construction of $\dot{\mathbf{U}}$ gives us complete information about the cell structure of quantum affine $\mathfrak{sl}_n$. Just as in the case of affine Weyl groups, the cells seem closely related to the finite dimensional representation theory of the algebra. We will first investigate the structure of cells in $\mathfrak{A}_D$ and then show how this can be used to obtain the cell structure of $\dot{\mathbf{U}}$.

We begin by recalling the definition of cells. Suppose R is a ring, and A an associative algebra over R, with an R–basis B. We say that a left ideal is *based* if it is the span of a subset of the basis B. We define a preorder on the elements of B as follows. Let $x \preceq_L y$ for $x, y \in B$ if x lies in every based left ideal which contains y. The equivalence classes of this preorder are precisely the left cells of A. If we replace "left ideal" with "right ideal" or "two–sided ideal" we get the corresponding notion of right cells or two–sided cells.

2.1 Schur-Weyl Duality

We first need to give another description of the algebra $\mathfrak{A}_D$, as a commutator algebra. Recall that if we specialize $v = \sqrt{q}$, then $\mathfrak{A}_D$ becomes an algebra of functions on $\mathcal{F}^n \times \mathcal{F}^n$, where $\mathcal{F}^n$ is the space of n-step periodic lattices, see section 1.1. Consider now the space of complete periodic lattices $\mathcal{B}^D$, that is, sequences of lattices $\mathbf{L} = (L_i)$ in our free module V such that $L_i \subset L_{i+1}$, $L_{i-D} = \epsilon L_i$, and $\dim_{\mathbf{k}}(\mathrm{L_i}/\mathrm{L_{i-1}}) = 1$ for all $i \in \mathbb{Z}$.

Let $\mathcal{H}_D$ be the affine Hecke algebra of GL_D, thus $\mathcal{H}_D$ is an algebra over $\mathbb{Z}[v, v^{-1}]$ generated by symbols T_i, X_j, X_j^{-1}, where $i \in \{1, 2, \ldots, D-1\}$, and $j \in \{1, 2, \ldots, D\}$, subject to the relations

- $(T_i - v)(T_i + v) = 0, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $i = 1, 2, \ldots, D-1$;
- $T_iT_j = T_jT_i$ if $|i-j| \geq 2$;
- $X_iX_i^{-1} = X_i^{-1}X_i = 1, \qquad X_iX_j = X_jX_i$, for all i, j;
- $T_i^{-1}X_iT_i^{-1} = X_{i+1}$ for $i = 1, 2, \ldots, D-1; \qquad T_iX_j = X_jT_i$ for $j \neq i, i+1$.

This is the "Bernstein presentation". Let W be the affine Weyl group of type GL_D (when we wish to specify D, we will use the notation $\hat{A}_{D-1}$), that is, W is the semidirect product of the symmetric group S_D with $\mathbb{Z}^D$. It is an extension by $\mathbb{Z}$ of a Coxeter group, thus the usual yoga can be used to extend the Kazhdan-Lusztig theory. Let the set of simple reflections of the Coxeter group be $S = \{s_i : i = 0, 1, \ldots, D\}$. The "Iwahori" presentation of $\mathcal{H}_D$ yields a basis $\{T_w : w \in W\}$. Lusztig observed that W has a natural incarnation as a permutation group on the integers. Indeed W is isomorphic to the set of all permutations σ of the integers such that $\sigma(i + D) = \sigma(i) + D$. See for example [Xi] for more details. Thus an element of W obviously corresponds to an infinite permutation matrix, which we denote A_w when we wish to make the distinction between the group element and the matrix.

Let $\mathfrak{H}_{D,q}$ be the space of functions on $\mathcal{B}^D \times \mathcal{B}^D$ which are invariant under the diagonal action of $G = Aut(V)$, and which are supported on finitely many G orbits. Just as for $\mathfrak{A}_D$, this is an algebra under convolution. Let $\mathbf{b}_0 = (\ldots, 1, 1, \ldots) \in \mathfrak{S}^D$ (see 1.1). The orbits of G on $\mathcal{B}^D \times \mathcal{B}^D$ are indexed by matrices $A = (a_{i,j})$ where the matrix A must have $r(A) = c(A) = \mathbf{b}_0$, and $a_{i,j} = a_{i+D,j+D}$, for all $i, j \in \mathbb{Z}$. Thus these are precisely the permutation matrices $\{A_w : w \in W\}$ discussed above. We use a basis $\{[A_w] : w \in W\}$ of $\mathfrak{H}_{D,q}$ which corresponding to the indicator functions of the G-orbits, scaled by $v^{-l(w)}$ (where $l(w)$ is the dimension of the corresponding affine Schubert variety).

Proposition 2.1.1. *The map $\mathcal{H}_{v=\sqrt{q}} \to \mathfrak{H}_{D,q}$ which sends $T_w \mapsto [A_w]$ is an algebra isomorphism.* □

Now if $\mathfrak{T}_{D,q}$ is the space of functions spanned by the indicator functions of G-orbits on $\mathcal{F}^n \times \mathcal{B}^D$ then convolution makes $\mathfrak{T}_D$ into a left module for $\mathfrak{A}_{D,q}$, and a right module for $\mathfrak{H}_{D,q}$, and moreover this is the specialization of a $\mathfrak{A}_D - \mathcal{H}_D$ bimodule $\mathfrak{T}_D$. Moreover, it is clear that the affine q-Schur algebra $\mathfrak{A}_D$ is the commutator algebra of the right $\mathcal{H}_D$ module $\mathfrak{T}_D$, the description we were seeking.

We can describe $\mathfrak{T}_D$ algebraically as follows: To each element $\mathbf{a} \in \mathfrak{S}_{D,n}$ we can associate a parabolic subgroup of the symmetric group $S_{\mathbf{a}}$ — it is the subgroup preserving the subsets $\{1, 2, \ldots, a_1\}, \{a_1{+}1, \ldots, a_1{+}a{+}2\}, \ldots, \{D{-}a_n{+}1, \ldots, D\}$ of $\{1, 2, \ldots, D\}$. Set $T_{\mathbf{a}} = \sum_{w \in S_{\mathbf{a}}} v^{l(w)} T_w$ (here we are viewing $S_{\mathbf{a}}$ as a subgroup of W in the obvious way). Then as a module for the Hecke algebra, $\mathfrak{T}_D$ is isomorphic to

$$\bigoplus_{\mathbf{a} \in \mathfrak{S}_{D,n}} T_{\mathbf{a}} \mathcal{H}_D$$

Similarly we see that we can describe an element $[A]$ of $\mathfrak{A}_D$ uniquely by a triple consisting of an element $w_A \in W$ together with a pair $\mathbf{a}, \mathbf{b} \in \mathfrak{S}_{D,n}$. Indeed $\mathbf{a}, \mathbf{b}$ are just $r(A)$ and $c(A)$ respectively, and w_A is the element of maximal length in the (finite) double coset of $S_{\mathbf{a}} \backslash W / S_{\mathbf{b}}$ determined by the matrix A. This also allows us to describe the structure constants for $\mathfrak{A}_D$ with respect to the basis $\{[A] : A \in \mathfrak{S}_{D,n}\}$ in terms of those for $\mathcal{H}_D$ with respect to the basis $\{T_w : w \in W\}$. In fact simple algebraic considerations (or an analogous discussion of the geometry involved) shows that the same holds for the structure constant with respect to the bases coming from intersection cohomology.

More precisely, suppose that we denote the various structure constants for $\mathcal{H}_D$ and $\mathfrak{A}_D$ as follows: Let $A, B \in \mathfrak{S}_{D,n}$, let $v, w \in W$, and let $\{C_w : w \in W\}$ be the Kazhdan-Lusztig basis of the Hecke algebra.

- $[A][B] = \sum_C \eta^C_{A,B} [C]$;
- $\{A\}\{B\} = \sum_C \nu^C_{A,B} \{C\}$;
- $T_v T_w = \sum_z f^z_{v,w} C_z$;
- $C_v C_w = \sum_z h^z_{v,w} C_z$.

Then we have the following relationships between them.

Lemma 2.1.2. *Let $A, B, C \in \mathfrak{S}_{D,n}$, and let $w_A, w_B, w_C \in W$ be the corresponding element of the Weyl group. Suppose that $c(A) = r(B) = \mathbf{c}$. Let $w_{\mathbf{c}}$ be the longest element of $S_{\mathbf{c}}$ and let*

$$p_{\mathbf{c}} = v^{-l(w_{\mathbf{c}})} \sum_{x \in S_{\mathbf{c}}} v^{2l(x)}$$

be the shifted Poincaré polynomial of $S_{\mathbf{c}}$. *We have*

$$p_{\mathbf{c}}\eta^{C}_{A,B} = f^{w_C}_{w_A,w_B}.$$

$$p_{\mathbf{c}}\nu^{C}_{A,B} = h^{w_C}_{w_A,w_B};$$

Proof. Using the algebraic description of $\mathfrak{T}_D$, the first statement can be proved algebraically, by interpreting the basis elements $[A], [B], [C]$ as sums of elements in double cosets of the Hecke algebra. Similarly one can show the second statement entirely algebraically, but it is perhaps more enlightening to use the interpretation of the multiplication in terms of perverse sheaves (see [L99] for a discussion of this). The affine Schubert variety for w_A fibres over the varieties $\bar{X}^{\mathbf{L}}_{A}$ with fibre given by a partial flag variety corresponding to $c(A)$, and the polynomial $p_{\mathbf{c}}$ arises from the cohomology of this fibre. □

This will be crucial in describing the cell structure of the affine q-Schur algebra.

2.2 Distinguished elements in $\mathfrak{A}_D$

Our first step in understanding the theory of cells in $\dot{\mathbf{U}}$ is to understand the corresponding theory for the affine q-Schur algebra. This is essentially an exercise in transferring the information known about the Hecke algebra $\mathcal{H}_D$ to our case. The key ingredient in our approach is to use Lusztig's notion of distinguished elements.

We begin by defining a somewhat mysterious integer-valued function a'_D, which together with certain variants, play a crucial rôle in our study of cells.

For $\{C\} \in \mathfrak{B}_D$ we set $n^{C}_{A,B}$ to be the largest power of v occurring in the structure constant $\nu^{C}_{A,B}$, and for $\mathbf{a} \in \mathfrak{S}_{D,n}$ set $|\mathbf{a}|^2 = \sum_{i=1}^{n} a_i^2$.

Definition. For $\{A\} \in \mathfrak{B}_D$, such that $\{A\} = \{A\}[\mathbf{i_a}]$, consider the set of positive integers

$$\{n^{A}_{B,C} + |\mathbf{b}|^2 - |\mathbf{a}|^2 \colon \{B\}, \{C\} \in \mathfrak{B}_D; \{C\} \in [\mathbf{i_b}]\mathfrak{A}_D[\mathbf{i_a}]\}.$$

If it has a largest element d we set $a'_D = d$, otherwise we set $a'_D = \infty$.

At first sight it would seem that we elided by saying that a'_D is "integer-valued" above, however the following lemma shows this is not the case.

Lemma 2.2.1. *The function* a'_D *is finite for every* $\{A\} \in \mathfrak{B}_D$.

Proof. To show this we wish to use the fact that we can interpret the structure constants in terms of those for the affine Hecke algebra, and then use the result of Lusztig [L85], which shows that the corresponding function on the Kazhdan-Lusztig basis is finite. Indeed, using Lemma 2.1.2 we see that for $\{A\}, \{B\}, \{C\} \in \mathfrak{B}_D$ we have $\nu^{C}_{A,B} = p^{-1}_{c(A)} h^{z}_{x,y}$ where $x, y, z \in W$ are the corresponding element of the affine Weyl group of type $\hat{A}_{D-1}$. Now by Theorem 7.2 in [L85] we have $v^{-l(w_0)} h^{z}_{x,y} \in \mathbb{Z}[v^{-1}]$ for any $x, y, z \in W$, where w_0 is the longest element in S_D, the finite Weyl group. The result follows. □

Remark. Note that we have shown that a'_D is not only finite, but in fact bounded. This appears, at least to the author, to be quite a deep result, as he has no real understanding of why the proof in [L95] works. It is possible the recent work of Bezrukavnikov on the anti-spherical module will make this more transparent.

We also set $\gamma^{C}_{A,B}$ to be the coefficient of $v^{a'_D(C)-|\mathbf{b}|^2+|\mathbf{a}|^2}$ in $\nu^{C}_{A,B}$ (which in general may be zero).

Definition. Let $\mathbf{a} \in \mathfrak{S}_{D,n}$. For $\{A\} \in [\mathbf{i_a}]\mathfrak{A}_D[\mathbf{i_a}]$ set $\Delta(A)$ to be the integer $d \geq 0$ such that

$$([\mathbf{i_a}], \{A\})_D = a_d v^{-d} + a_{d+1} v^{-d-1} + \dots,$$

where $a_d \neq 0$. Set $n_A = a_d$.

Lemma 2.2.2. *Let* $\mathbf{a} \in \mathfrak{S}_{D,n}$. *For any* $\{A\} \in \mathfrak{B}_D$ *with* $[\mathbf{i_a}]\{A\}[\mathbf{i_a}] = \{A\}$ *we have* $a'_D(A) \leq \Delta(A)$.

Proof. This follows an idea of Springer in the Hecke algebra case. Suppose that $\{B\}, \{C\}$ are in $\mathfrak{B}_D$, and consider the product $\{B\}\{C\}$. We may write this as a sum $\sum_{\{E\}\in\mathfrak{B}_D} \nu^E_{B,C}\{E\}$. Chose $\{B\} \in [\mathbf{i_a}]\mathfrak{A}_D[\mathbf{i_b}]$, and $\{C\} \in [\mathbf{i_b}]\mathfrak{A}_D[\mathbf{i_a}]$ so that

$$\nu^A_{B,C} = \gamma^A_{B,C} v^{a'_D(A)-|\mathbf{b}|^2+|\mathbf{a}|^2} + \dots,$$

where $\gamma^A_{B,C} \neq 0$ and the remaining terms are of lower degree. We have the inner product

$$(\{B\}\{C\}, [\mathbf{i_a}])_D = \sum_{\{E\}\in\mathfrak{B}_D} \nu^E_{B,C}(\{E\}, [\mathbf{i_a}])_D. \tag{2.2.1}$$

All the terms here have nonnegative integer coefficients when written as power series (using the positivity of the inner product). The properties of the inner product $(,)_D$ show that this is also equal to

$$v^{|\mathbf{a}|^2-|\mathbf{b}|^2}(\{C\}, \{B^t\})_D$$

Now since the canonical basis is almost orthogonal with respect to the inner product, we see that all the terms on the left-hand side of Equation 2.2.1 lie in $v^{|\mathbf{a}|^2-|\mathbf{b}|^2}\mathbb{N}[[v^{-1}]]$. In particular, taking $\{E\} = \{A\}$ we get that

$$\nu^A_{B,C}(\{A\}, [\mathbf{i_a}])_D = n_A \gamma^A_{B,C} v^{a'_D(A)-|\mathbf{b}|^2+|\mathbf{a}|^2-\Delta(\{A\})} + \dots \in v^{|\mathbf{a}|^2-|\mathbf{b}|^2}\mathbb{N}[[v^{-1}]],$$

and hence the result. □

Motivated by this, we define the set of *distinguished elements* of $\mathfrak{B}_D$ as follows.

Definition. Let $\mathcal{D}_D$ be the set of elements $\{A\}$ in $\mathfrak{B}_D$ such that there is a $\mathbf{a} \in \mathfrak{S}_{D,n}$ with $\{A\} \in [\mathbf{i_a}]\mathfrak{A}_D[\mathbf{i_a}]$ and $a'_D(A) = \Delta(A)$.

The distinguished elements $\mathcal{D}_D$ are defined by analogy with the Hecke algebra case due to Lusztig [L87]. We note the some consequences of the above proof.

Corollary 2.2.3. *We have the following properties:*

1. *If* $\{A\} \in \mathcal{D}_D$ *and* $\{B\}, \{C\} \in \mathfrak{B}_D$ *are such that* $\gamma^A_{B,C} \neq 0$ *then* $\{C\} = \{B^t\}$.
2. *For each* $\{B\} \in \mathfrak{B}_D$, *there is a unique* $\{A\} \in \mathcal{D}_D$ *with* $\gamma^A_{B,B^t} \neq 0$
3. *If* $\{A\} \in \mathcal{D}_D$ *then* $\{A\} = \{A^t\}$.

Proof. For the first, note that in the above proof, the almost orthogonality of $\mathfrak{B}_D$ with respect to the inner product implies that it is necessary and sufficient to have $\{C\} = \{B^t\}$. That the product contains just one element of $\mathcal{D}_D$ is also immediate. For the last statement, pick $\{B\}, \{C\}$ such that $\gamma^A_{B,C} \neq 0$. By the first statement, we see that $\{C\} = \{B^t\}$. Since the product $\{B\}\{B^t\}$ is preserved by the transpose anti-automorphism Ψ, we see that $\gamma^{A^t}_{B,B^t} \neq 0$, and so by the second statement, $\{A\} = \{A^t\}$. □

Recall that to each element of $\mathfrak{B}_D$ we have attached an element of the affine Weyl group. We will show that in this way, the distinguished elements of $\mathfrak{B}_D$ actually correspond to distinguished elements of W.

Lemma 2.2.4. *Let* $\{A\} \in \mathcal{D}_D$, *then the Weyl group element* w_A *is distinguished and conversely.*

Proof. Let $\mathbf{a} \in \mathfrak{S}_{D,n}$ be such that $[\mathbf{i_a}]\{A\} = \{A\}$. By definition we see that

$$(\{A\}, [\mathbf{i_a}])_D = \Pi_{\mathbf{i_a},E},$$

the stalk of the intersection cohomology sheaf on $\bar{X}_E^{\mathbf{L}}$ at the point corresponding to $[\mathbf{i_a}]$. But this is equal to $v^{l(w_\mathbf{a})} p_{1,w_E}$, where p_{1,w_E} is the affine Kazhdan-Lusztig polynomial attached to $1, w_E$, and $w_\mathbf{a}$ is the longest element of the parabolic subgroup attached to $\mathbf{a}$ (the intersection cohomology sheaves are related by a smooth pullback with fibre dimension $l(w_\mathbf{a})$. Thus we see that if $\Delta(w_E)$ is the lowest power of v^{-1} occurring in p_{1,w_E},

$$a'_D(E) \leq a'(w_E) - l(w_\mathbf{a}) \leq \Delta(w_E) - l(w_\mathbf{a}) = a'_D(E).$$

Here the function a' on the Kazhdan-Lusztig basis is the one defined in [L87] (there denoted simply a). For $z \in W$, we set $a'(z)$ to be the highest power of v appearing in a structure constant $h^z_{x,y}$ as x, y vary over W. The first inequality follows directly from the definitions of a', a'_D, and the second from analog of Lemma 2.2.2 for $\mathcal{H}_D$. It follows immediately that w_E is distinguished. To establish the converse, it is necessary to note that if one picks any element x of the left cell containing the distinguished element, then by [L87] the structure constant $h^{w_E}_{x^{-1},x}$ has $a'(w_E)$ as its highest power of v, and so $a'_D(E) = a'(w_E) - l(w_\mathbf{a})$. □

We now show that all the notions of cells for $\mathfrak{A}_D$ can be deduced from those for the Hecke algebra. More precisely we have the following result.

Proposition 2.2.5. *Let* $\{A\}, \{B\} \in \mathfrak{B}_D$.

1. $\{A\} \sim_L \{B\}$ *if and only if* $w_A \sim_L w_B$ *and* $c(A) = c(B)$;
2. $\{A\} \sim_R \{B\}$ *if and only if* $w_A \sim_R w_B$ *and* $r(A) = r(B)$;
3. $\{A\} \sim_{LR} \{B\}$ *if and only if* $w_A \sim_{LR} w_B$;
4. $a'_D(\{A\}) = a'(w_A) - l\left(w_{c(A)}\right)$;
5. *Each left cell contains precisely one distinguished element.*

Proof. For the first claim, note that since the notion of cell in $\mathfrak{A}_D$ is defined essentially by using a subset of the Kazhdan-Lusztig basis consisting of those elements which are of maximal length in certain double cosets, it is clear that if $\{A\} \sim_L \{B\}$ then $w_A \sim_L w_B$. Moreover, certainly we have $c(A) = c(B)$. For the converse, we need to use the distinguished elements. Suppose that $w_A \sim_L w_B$ and $c(A) = c(B)$. Then if d is the unique distinguished element in the left cell Γ containing w_A, w_B (which exists by [L87]), d determines a distinguished element of $\mathfrak{B}_D$, $\{E\}$ say, where $c(E) = c(A)$.

For $x, y, z \in W$ let $\gamma^z_{x,y}$ denote the coefficient of $v^{a'(z)}$ in $h^z_{x,y}$. Then by [L87, Theorem 1.8] we know that $\gamma^z_{x,y} = \gamma^{x^{-1}}_{y,z^{-1}} = \gamma^{y^{-1}}_{z^{-1},x}$, and so as $\gamma^d_{w_A^{-1},w_A} \neq 0$ we see that

$$h^d_{w_A^{-1},w_A}, h^{w_A}_{w_A^{-1},d}, h^d_{w_B^{-1},w_B}, h^{w_B}_{w_B^{-1},d}$$

are all nonzero, and hence the same is true of

$$\nu^E_{A^t,A}, \nu^A_{A^t,E}, \nu^E_{B^t,B}, \nu^B_{B^t,E}.$$

It follows that $\{A\} \sim_L \{E\}$ and $\{B\} \sim_L \{E\}$, and hence $\{A\} \sim_L \{B\}$.

The second claim either follows in the same way, or by taking inverses in W, which corresponds to applying the transpose map Ψ in $\mathfrak{A}_D$.

For the third, the forward implication is again clear. If $w_A \sim_{LR} w_B$, then it follows that the left cell containing w_A and the right cell containing w_B intersect (since this is true of any left and right cell in the same two-sided cell of an affine Hecke algebra). As any element in this intersection will give rise to an element $\{C\}$ of the q-Schur algebra with $r(C) = r(B)$ and $c(C) = c(A)$, we obtain the result using the first two parts of the proposition. The fourth claim follows from Corollary 2.2.3 (see the end of the proof of Lemma 2.2.4 . Since each left cell of the Hecke algebra contains a unique distinguished element, the fifth claim follows from the Lemma 2.2.4 and the first claim. □

2.3 Cells in $\mathfrak{A}_D$

We saw at the end of the last section that the theory of cells of the affine q-Schur algebra is determined by that for the type A affine Hecke algebra. This allows us to describe explicitly the number of two-sided cells in the affine q-Schur algebra, and also the number of left cells (and hence right cells) in a given two-sided cell. To do this we recall the combinatorial definitions which describe the bijection between cells for the Hecke algebra and partitions.

Definition. Suppose $w \in W$ the affine Weyl group of type $\tilde{A}_{D-1}$. Then we may view w as a permutation of $\mathbb{Z}$. A sequence $(i_1, i_2, \ldots, i_r)$ is called a *d-chain* if $i_1 < i_2 < \ldots < i_r$ and $(i_1)w > (i_2)w > \ldots > (i_r)w$, and the $\{i_j, j = 1, \ldots, r\}$ are all incongruent modulo D.

Let $\mathcal{P}_D$ be the set of partitions of D. We define a map $\sigma\colon W \to \mathcal{P}_D$ as follows. For $w \in W$, let d_j be the maximal size of a set of j d-chains, the union of whose elements are all incongruent modulo D. Then it is know that $\lambda = (d_1, d_2 - d_1, d_3 - d_2, \ldots, d_{D-1} - d_D)$ is a partition of D. Set $\sigma(w) = \lambda$.

The following result is due to Lusztig, based on the work of Shi, see [L85a], [Sh].

Theorem 2.3.1. *The fibres of σ are precisely the two-sided cells of W.* □

We may use this to give a description of the two-sided cells in the affine q-Schur algebra as follows:

Definition. Let $A \in \mathfrak{S}_{D,n}$. An *anti-diagonal path* in A is an infinite strip of entries $(a_{i_k,j_k} : k \in \mathbb{Z})$ such that (i_k, j_k) is either equal to $(i_{k-1} - 1, j_{k-1})$ or $(i_{k-1}, j_{k-1} + 1)$ with the latter being the case for all but finitely many k. Thus visually if you draw the matrix with rows increasing from top to bottom, and columns from left to right, (as we will do) then path starts and ends with infinite vertical strips, and takes finitely many right or vertical turns.

Let d_j be the maximal size of the sum of entries in the union of j anti-diagonal paths. Then we define a map $\rho\colon \mathfrak{S}_{D,n} \to \mathcal{P}_D^n$ where $\mathcal{P}_D^n$ is the set of partitions of D with at most n parts, by setting $\rho(A) = (d_1, d_2 - d_1, \ldots, d_n - d_{n-1})$. As above, it follows from general results on posets that $\rho(A)$ is indeed a partition. The fact that it can have at most n parts is obvious. We will sometimes view ρ as a map from $\mathfrak{B}_D$ in the obvious way.

Proposition 2.3.2. *The fibres of the map $\rho\colon \mathfrak{B}_D \to \mathcal{P}_D^n$ are the two-sided cells of $\mathfrak{A}_D$.*

Proof. This is a simple combination of the statements of Proposition 2.2 and Theorem 2.3.1. □

Example. Suppose that $n = 2$ and $D = 5$. Consider the element $\{A\}$ of $\mathfrak{B}_5$ corresponding to

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 0 & \boxed{1} & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \boxed{1} & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{1} & \boxed{1} & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{0} & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

where the top left entry shown is in the $(1,1)$ entry of A. Then $\{A\}$ lies in the two-sided cell corresponding to the partition $(4,1)$. The boxed entries give part of an anti-diagonal path which has entry sum 4. Note that it is not unique.

Given a partition $\lambda \in \mathcal{P}_D^n$ we denote the two-sided cell $\rho^{-1}(\lambda)$ by $\mathbf{c}_\lambda$. We will often use the same notion for a partition in $\mathcal{P}_D$ and a two-sided cell of $\mathcal{H}_D$.

Somewhat more elaborate is a description of the number of left cells in a two-sided cell of the affine q-Schur algebra. Notice that Proposition 2.2 shows that each left cell of the affine Hecke algebra gives rise to a number of left cells of the affine q-Schur algebra, with the number depending on the set of simple reflections of the symmetric group S_D which decrease the length of any element of the left cell when multiplied on the right.

Definition. For $w \in W$ let $\mathcal{R}(w) = \{s \in S \colon l(ws) < l(w)\}$ and $\mathcal{L}(w) = \{s \in S \colon l(sw) < l(w)\}$.

It is known that the functions $\mathcal{R}, \mathcal{L}$ are constant on right and left cells respectively. Thus for Γ a left cell, we may write $\mathcal{R}(\Gamma)$ for the set $\mathcal{R}(w)$, where w is any element of Γ. The left cells of $\mathcal{H}_D$ have been described by Shi ([Sh], chapter 14) as the fibres of a map to a set of tableaux, such that the shape of the tableau associated to a left cell is given by the partition of the two-sided cell it lies in, and the entries must increase down the columns.

In order to describe this map in more detail we need some to make some definitions. We use the description of W as a group of permutations of the integers, and in particular the associated infinite matrices. Let $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ be such a matrix. A *block* is a set of consecutive rows of A. For a block of m rows $i+1, i+2, \ldots, i+m$, let the nonzero entries be $\{a_{i+1,j_1}, a_{i+2,j_2}, \ldots, a_{i+m,j_m}\}$. We say the block is a *descending chain* if $j_1 > j_2 > \ldots > j_m$. A block is a *maximal descending chain* (MDC) if it cannot be imbedded in a larger such block.

Say that an element of $w \in W$ has *full MDC form* at i, if there exist consecutive MDC blocks $(A_l, A_{l-1}, \ldots, A_1)$ of A_w of size m_t, for $t = 1, \ldots, l$, with $\sum_{t=1}^{l} m_t = D$, and $i+1$ the first row of A_l (so $i + \sum_{j=1}^{k} m_j + 1$ is the first row of A_{l-k}). Suppose a full MDC form has blocks which are of (weakly) increasing size (so A_i has at most as many rows as A_{i-1}). Let j_t^u be the column of the nonzero entry in the u-th row of A_t. Then we say the form is *normal* if $j_1^u - n < j_l^u < j_{l-1}^u < \ldots < j_1^u$, for each u (where we ignore terms in this sequence which do not exist).

Recall that the two-sided cells of $\mathcal{H}_D$ are indexed by partitions of D. For such a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$ we let N_λ be the set of elements of the two-sided cell $\mathbf{c}_\lambda$ corresponding to λ which have normal MDC form $(A_r, A_{r-1}, \ldots, A_1)$ at i for some $i \in \mathbb{Z}$, where λ_j is the number of rows in A_j.

Theorem 2.3.3. *[Sh] Let $w \in \mathbf{c}_\lambda$. Then there is a $y \in N_\lambda$ with $y \sim_L w$.* □

Let $\mathcal{C}_\lambda$ be the set of Young diagrams of shape λ with entries $\{1, 2, \ldots, D\}$ which decrease down columns. In [Sh, chapter 14] Shi defines a map T from the left cells in $\mathbf{c}_\lambda$ to $\mathcal{C}_\lambda$. Let Γ be a left cell in $\mathbf{c}_\lambda$. Choose $y \in N_\lambda \cap \Gamma$ and then set the entries of column u of $T(\Gamma)$ to be the residues modulo D of the numbers $\{j_t^u \colon 1 \leq t \leq \mu_u\}$ where μ is the partition dual to λ. Shi shows this is independent of the choice of y, and that it gives a bijection.

Example. Consider the matrix A_w in $\hat{A}_4$ given as follows

$$\begin{pmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

where the first column shown is column 1. Then it is easy check that $w \in N_{(3,2)}$ and the associated tableau is given below.

5	4	1
3	2	

It follows directly from this construction (though this is not explicitly described in [Sh]) that the set of simple reflections in $\mathcal{R}(\Gamma)$ is determined by this tableau. Indeed since $\mathcal{R}(\Gamma)$ is given by

$\mathcal{R}(w)$ for any $w \in \Gamma$, we may use an element of N_λ as above. Then it is easy to see that the simple reflection s_i is in $\mathcal{R}(\Gamma)$ precisely when i appears to the right of $i+1$ in the tableau (where one reads modulo D for s_0).

We now consider the left cells of $\mathfrak{A}_D$. Each such cell correspond to a left cell Γ of $\mathcal{H}_D$ and an element $\mathbf{a}$ of $\mathfrak{S}_{D,n}$ such that the simple reflections J of $S_{\mathbf{a}}$ are a subset of $\mathcal{R}(\Gamma)\backslash\{s_0\}$. Of course in general such a subset may not exist.

Definition. For $\lambda \in \mathcal{P}_D$, let $\mathcal{C}^n_\lambda$ be the the set of tableaux of shape λ with entries from $\{1, 2, \dots, n\}$ strictly decreasing down columns. Thus $\mathcal{C}^n_\lambda$ is empty if λ has more than n parts.

If we fix a two-sided cell $\mathbf{c}_\lambda$, where $\lambda \in \mathcal{P}^n_D$, using the description of $\mathcal{R}(\Gamma)$ in terms of the tableau $T(\Gamma)$, it is easy to see that the left cells in $\mathbf{c}_\lambda$ are indexed by the elements of $\mathcal{C}^n_\lambda$. Indeed to each tableau $T \in \mathcal{C}^n_\lambda$ there is a well-defined element $h(T)$ of $\mathcal{C}_\lambda$ given as follows. Order the boxes of T by listing those labelled 1 first, then 2, and so on, always reading from right to left. Then construct $h(T) \in \mathcal{C}_\lambda$ by labelling each box with its position in the order just described. This gives the left cell of W. The element $\mathbf{a}$ is determined by letting a_i be the number of boxes of T labelled i, for $i \in \{1, 2, \dots, n\}$. The following example makes the correspondence clear.

Example. Let $D = 5$ and $n = 3$. Suppose that we consider the tableau

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

in $\mathcal{C}^3_{(2,2,1)}$. Then the tableau corresponding to it in $\mathcal{C}_{(2,2,1)}$ is

$$\begin{array}{|c|c|}\hline 5 & 3 \\ \hline 4 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$$

and the sequence $\mathbf{a}$ is $(2, 2, 1)$ (repeated periodically).

This allows us to count the number of left cells in a two-sided cell of $\mathfrak{A}_D$.

Proposition 2.3.4. *Let* $\mathbf{c}$ *be a two-sided cell of* $\mathfrak{A}_D$*. If* λ *is the partition of* D *associated to* $\mathbf{c}$*, and* $\lambda(i) := \lambda_i - \lambda_{i+1}$*, then the number of left cells in* $\mathbf{c}$ *is*

$$\prod_{i=1}^{n-1} \binom{n}{i}^{\lambda(i)}$$

□

Finally we wish to construct the asymptotic algebra associated to a two-sided cell. We will need a variant of the function a'_D.

Definition. Let $\{A\} \in \mathfrak{B}_D$. If there is an integer $d \geq 0$ such that $v^{-d}\nu^C_{A,B} \in \mathbb{Z}[v^{-1}]$ for all $\{B\}, \{C\} \in \mathfrak{B}_D$ then let $a(A)$ be the smallest such. Otherwise set $a(A) = \infty$.

Note that the proof of Lemma 2.2.1 shows that a_D is always finite. More interestingly we have the following result.

Lemma 2.3.5. *The functions* a'_D *and* a_D *agree. Moreover the function* a_D *is constant on* $\mathbf{c}[\mathbf{i_a}]$ *for any two-sided cell* $\mathbf{c}$ *and any* $\mathbf{a} \in \mathfrak{S}_{D,n}$.

Proof. Both of these follow from facts about the Hecke algebra: If $\gamma^z_{x,y}$ denotes the coefficient of $v^{a'(z)}$ in $h^z_{x,y}$, then it follows from the results above that for $\{A\}, \{B\}, \{C\} \in \mathfrak{B}_D$ we have $\gamma^C_{A,B} = \gamma^{w_C}_{w_A,w_B}$. Moreover, by the results of [L87] (see Lemma 2.2) we know that $\gamma^z_{x,y} = \gamma^{x^{-1}}_{y,z^{-1}} = \gamma^{y^{-1}}_{z^{-1},x}$ and hence $\gamma^C_{A,B} = \gamma^{A^t}_{B,C^t} = \gamma^{B^t}_{C^t,A}$. It is now easy to see that $a'_D = a_D$. Since a' is constant on two-sided cells of the Hecke algebra, the second statement is clear. □

We now rescale the canonical basis of $\mathfrak{A}_D$. Set $\langle A\rangle = v^{-a_D(A)}\{A\}$. Let $\mathfrak{A}_\mathbf{c}$ denote the span of the elements in $\mathbf{c}$ a two-sided cell of $\mathfrak{B}_D$. This becomes an algebra by identifying it with a subquotient of $\mathfrak{A}_D$ in the obvious way. The structure constants of $\mathfrak{A}_\mathbf{c}$ with respect to the new basis lie in $\mathbb{Z}[v^{-1}]$. Indeed the product $\langle A\rangle\langle B\rangle = \sum_{\langle C\rangle} v^{-a_D(A)}\nu^C_{A,B}\langle C\rangle$, since $a_D(A) = a_D(C)$, and the coefficients $v^{-a_D(A)}\nu^C_{A,B}$ all lie in $\mathbb{Z}[v^{-1}]$. Thus if $\mathcal{L}_\mathbf{c}$ is the $\mathbb{Z}[v^{-1}]$ span of the $\{\langle A\rangle \colon \{A\} \in \mathbf{c}\}$, $\mathcal{L}_\mathbf{c}$ has the structure of a $\mathbb{Z}[v^{-1}]$ algebra. The quotient $J_\mathbf{c} = \mathcal{L}_\mathbf{c}/v^{-1}\mathcal{L}_\mathbf{c}$ is then a $\mathbb{Z}$ algebra, where if t_A is the image of $\langle A\rangle$, the multiplication in $J_\mathbf{c}$ is given by

$$t_A t_B = \sum_C \gamma^C_{A,B} t_C.$$

Let $\mathcal{D}_\mathbf{c} = \mathcal{D}_D \cap \mathbf{c}$. It follows from the above that the set $\{t_E \colon \{E\} \in \mathcal{D}_\mathbf{c}\}$ gives a decomposition of the identity into orthogonal idempotents.

By using the results of [Xi] or [BO] we can also give an explicit description of this asymptotic algebra. For $\lambda \in \mathcal{P}^n_D$ and $i \in \{1, 2, \ldots, n\}$, let $\lambda(i) = \lambda_i - \lambda_{i+1}$, (where $\lambda_{n+1} = 0$). Let G_λ be the reductive group $\prod_{i=1}^n GL_{\lambda(i)}(\mathbb{C})$ and let R_λ be the K-group of its representations, so that the irreducible representations $\widehat{G}_\lambda$ form a $\mathbb{Z}$-basis of R_λ. Let T_λ be the set of triples (E_1, E_2, κ) where $\{E_1\}, \{E_2\} \in \mathcal{D}_\lambda$, and $\kappa \in \widehat{G}_\lambda$. Let $\mathcal{J}_\lambda$ be the free Abelian group on T_λ. Define a ring structure on $\mathcal{J}_\lambda$ by

$$(E_1, E_2, \kappa)(E_1' E_2', \kappa') = \sum c^{\kappa''}_{\kappa,\kappa'} \delta_{E_2, E_1'} (E_1, E_2', \kappa'')$$

where the sum is over $\kappa'' \in \widehat{G}_\lambda$ and $c^{\kappa''}_{\kappa,\kappa'}$ is the multiplicity of κ'' in the G_λ-module $\kappa \otimes \kappa'$. Thus $\mathcal{J}_\lambda$ is a matrix ring of rank N over the representation ring R_λ, where N is the number of left cells in $\mathbf{c}_\lambda$ given in Proposition 2.3.4.

Proposition 2.3.6. *1. There is a ring isomorphism $J_{\mathbf{c}_\lambda} \to \mathcal{J}_\lambda$ which restricts to a bijection between the canonical basis of $J_{\mathbf{c}_\lambda}$ and T_λ.*

2. *For any $\{E\} \in \mathcal{D}_{\mathbf{c}_\lambda}$, the subset of $\mathbf{c}_\lambda$ corresponding to $\{(E_1, E_2, \kappa) \in T_\lambda \colon E_2 = E\}$ under the bijection is a left cell.*

3. *For any $\{E\} \in \mathcal{D}_{\mathbf{c}_\lambda}$, the subset of $\mathbf{c}_\lambda$ corresponding to $\{(E_1, E_2, \kappa) \in T_\lambda \colon E_1 = E\}$ under the bijection is a right cell.*

□

This shows that all the simple modules of the $\mathbb{C}$-algebra $\mathbb{C}\otimes J_{\mathbf{c}_\lambda}$ are N-dimensional and that the set of isomorphism classes of such modules is in bijection with the semisimple conjugacy classes of G_λ.

The asymptotic algebra also receives a homomorphism from the original algebra, once we tensor with $\mathbb{Q}(v)$. Define a map $\Phi_{\mathbf{c}_\lambda} : \mathfrak{A}_D \to \mathbb{Q}(v) \otimes J_{\mathbf{c}_\lambda}$ as follows:

$$\Phi_{\mathbf{c}_\lambda}(\{A\}) = \sum_{\{E\}\in\mathcal{D}_{\mathbf{c}_\lambda}, \{B\}\in\mathbf{c}_\lambda} \nu^B_{A,E} t_B.$$

One shows it is a homomorphism as in [L95, Proposition 1.9], where the property of the structure constants which is needed follows from the Hecke algebra case. This allows one to pull back representations of $J_{\mathbf{c}_\lambda}$ to representations of $\mathfrak{A}_D$. It would be interesting to understand which representations of $\mathfrak{A}_D$ arise in this way.

2.4 Cells in $\dot{\mathbf{U}}$

Let $\dot{\mathbf{U}}$ be the modified quantum group of affine $\mathfrak{sl}_n$, and let $\dot{\mathbf{B}}$ be its canonical basis (see section 1.3, or [L93]). We know show how we can lift information about the cell structure of the affine q-Schur algebra to the modified quantum group. The following theorem relating the canonical bases $\dot{\mathbf{B}}$ and $\mathfrak{B}_D$ was conjectured by Lusztig, and proved in [ScV]

Theorem 2.4.1. *For all $b \in \dot{\mathbf{B}}$ we have $\phi_D(b) \in \{0\} \cup \mathfrak{B}_D$. Moreover the kernel of ϕ_D is spanned by the elements $b \in \dot{\mathbf{B}}$ such that $\phi_D(\mathbf{b}) = 0$.*

Proof. Proposition 1.6.2 gives an asymptotic version of this result, however we will need its full strength here. One can give a new proof of the theorem, using the asymptotic version as a first step. What is left to show is that the "transfer maps" between the algebras $\mathfrak{A}_D$ preserve the basis of $\mathbf{U}_D$. This can be shown using a sheaf-theoretic description of the transfer map: One should interpret the coproduct as a "hyperbolic localization" to the fixed points of a suitable $\mathbb{C}^\times$ action, and the "sign representation" of $\mathfrak{A}_n$ as taking vanishing cycles with respect to a suitably generic function (this is essentially in the paper of Kashiwara-Tanisaki [KT]). Then purity arguments show that the coproduct takes semisimple complexes to semisimple complexes, and moreover equivariance shows that the complexes thus obtained split as appropriate tensor products. To check compatibility of bases one should use the fact that stalk Euler characteristics are preserved by localization, and the fact that the complexes in $\mathbf{U}_D$ have "nilpotent" singular support in a suitable sense. The details have yet to be fully worked out. □

It follows that the image $\mathbf{U}_D$ is a union of two-sided cells of $\dot{\mathbf{U}}$. Moreover the injectivity result of section 1.6 shows that any two-sided cell will eventually lie in some $\mathbf{U}_D$. Given $A \in \mathfrak{S}_{D,n}$ we say that A is *aperiodic* if, for any integer $k \neq 0$ there is an integer p with $a_{p,p+k} = 0$. Thus only the main diagonal of A can consist entirely of nonzero entries. In [L99], Lusztig showed that $\mathbf{U}_D$ is spanned by a subset $\mathbf{B}_D$ of $\mathfrak{B}_D$ consisting of those $\{A\}$ for which A is aperiodic.

Now if ϕ_D were surjective this would be a straightforward consequence of the previous section. Indeed in the finite type case, this is true, and the results of [L95] in the case of $\mathfrak{sl}_n$ can be recovered in this way, as was essentially done by Du in [Du] (note however that [L95] is much more general, classifying the cell structure for any finite type quantum group).

In the affine case it is no longer true that we have a surjective homomorphism from the quantum group. Thus we need to be more careful in lifting information from $\mathfrak{A}_D$ to $\dot{\mathbf{U}}$. We begin with the analogue of the a_D function.

Let $c_{b,b'}^{b''}$ be the structure constants of $\dot{\mathbf{U}}$ with respect to $\dot{\mathbf{B}}$. For a two-sided cell $\mathbf{c}$ in $\dot{\mathbf{U}}$ let $\dot{\mathbf{U}}_\mathbf{c}$ be the subspace of $\dot{\mathbf{U}}$ spanned by the elements of $\mathbf{c}$. We endow $\dot{\mathbf{U}}_\mathbf{c}$ with an algebra structure by identifying it with a subquotient of $\dot{\mathbf{U}}$, so that for $b, b' \in \mathbf{c}$ the product is given by

$$bb' = \sum_{b'' \in \dot{\mathbf{B}}_\mathbf{c}} c_{b,b'}^{b''} b''.$$

Definition. Let $b \in \dot{\mathbf{B}}$. If there is an integer $n \geq 0$ such that $v^{-n} c_{b,b'}^{b''} \in \mathbb{Z}[v^{-1}]$ for all $b', b'' \in \mathbf{c}$ then let $a(b)$ be the smallest such. Otherwise set $a(b) = \infty$.

The following observation simple observation tells us about the left cells in $\dot{\mathbf{U}}$.

Lemma 2.4.2. *Let Γ be a left cell of $\mathfrak{A}_D$, and let $\{E\} \in \mathcal{D}_D$ be the unique distinguished element in Γ. Then if $\mathbf{B}_D \cap \Gamma \neq \emptyset$ we must have $\{E\} \in \mathbf{B}_D$. Moreover $\mathbf{B}_D \cap \Gamma$ is a single left cell of $\mathbf{U}_D$*

Proof. Pick $\{A\} \in \mathbf{B}_D \cap \Gamma$. Then we know that

$$\{A^t\}\{A\} = \nu_{A^t,A}^E \{E\} + \dots,$$

where $\nu_{A^t,A}^E \neq 0$ since $\gamma_{A^t,A}^E \neq 0$ (the unique distinguished element for which $\gamma_{A^t,A}^E$ must clearly be the one in the left cell containing $\{A\}$). This implies that $\{E\} \in \mathbf{B}_D$. By arguing as in the proof of the first claim in Proposition 2.2) we see that the intersection $\mathbf{B}_D \cap \Gamma$ is a single left cell. □

This has some important corollaries which we now record.

Corollary 2.4.3. *We have the following properties of a functions.*

1. *The functions a_D, a'_D coincide with the analogous functions defined in terms of $\mathbf{U}_D$ instead of $\mathfrak{A}_D$.*

2. *For $b \in \dot{\mathbf{B}}$ if $\phi_D(b) \neq 0$ then $a(b) = a_D(\phi_D(b))$. In particular, $a(b)$ is finite.*

Proof. The claims are easy consequences of the above lemma, using distinguished elements. □

We now know each left cell in $\dot{\mathbf{U}}_D$ contains a unique distinguished element. We wish to show that the same is true of the left cells in $\dot{\mathbf{U}}$. Since any left cell will occur as a left cell of $\dot{\mathbf{U}}_D$ for sufficiently large D, it suffices to show that the distinguished element we obtain is independent of D. Since the distinguished element is characterized as the idempotent element in the asymptotic algebra, which is determined by the two-sided cell, it is independent of the algebra U_D we choose. Moreover, since by Theorem 1.4.2 the inner product on $\dot{\mathbf{U}}$ is obtained as a limit from those on $\mathfrak{A}_D$ we may give an intrinsic characterization of the set $\mathcal{D}_{\mathbf{c}}$ of distinguished elements in a two-sided cell $\mathbf{c}$. Recall from [L95, 3.7] that $\dot{\mathbf{U}}$ possesses an anti-automorphism $\sharp\colon \dot{\mathbf{U}} \to \dot{\mathbf{U}}$, which is such that $\phi_D(x^\sharp) = \Psi(\phi_D(x))$, for any $x \in \dot{\mathbf{U}}$.

Proposition 2.4.4. *Let $b \in \mathbf{c}$ and $\lambda \in X$ be such that $b \in \dot{\mathbf{U}}1_\lambda$. Then $v^{a(b)}(1_\lambda, b) \in \mathbb{Z}[v^{-1}]$ with nonzero constant term for $b \in \mathcal{D}_{\mathbf{c}}$ and $v^{a(b)}(1_\lambda, b) \in v^{-1}\mathbb{Z}[v^{-1}]$ otherwise. Moreover $b = b^\sharp$.* □

It remains to investigate the structure of the two-sided cells of $\mathbf{U}_D$. This again lifts directly from $\mathfrak{A}_D$, as the following simple observations show.

The transfer map $\psi_D\colon \mathbf{U}_D \to \mathbf{U}_{D-n}$ is such that $\psi_D(\{A\}) = \{A - I\}$ if the entries of $A - I$ are nonnegative and $\psi_D(\{A\}) = 0$ otherwise ($I = (\delta_{ij})$ is the identity matrix). Using this along with our combinatorial description of two-sided cells in $\mathfrak{A}_D$, we obtain the following statement. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$ be in $\mathcal{P}_D^n$, and let $\mathbf{k}_\lambda$ denote the intersection $\mathbf{B}_D \cap \mathbf{c}_\lambda$. Then $\mathbf{k}_\lambda$ is a union of two-sided cells of $\mathbf{U}_D$ and moreover it follows from the above discussion that $\mathbf{k}_\lambda$ maps to 0 under ψ_D unless $\lambda_n > 0$, when it maps to $\mathbf{k}_{\lambda'}$ in $\mathbf{U}_{D-n}$ where $\lambda' = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1)$ (i.e.λ' is obtained by removing the first column of the Young diagram for λ).

However, the following observation which follows easily from Proposition 2.3.2 now shows that we are almost done.

Lemma 2.4.5. *Let A be an element of $\mathfrak{S}_{D,n}$. Then $\rho(A)$ has strictly less than n parts, precisely when A has no completely nonzero diagonal (i.e. for each $k \in \mathbb{Z}$ there is some $p \in \mathbb{Z}$ with $a_{p,p+k} = 0$). In particular, if $\lambda \in \mathcal{P}_D^n$ has fewer than n parts $\mathbf{c}_\lambda$ consists entirely of aperiodic elements.* □

Thus for such λ we see that $\mathbf{k}_\lambda = \mathbf{c}_\lambda$, and it consists of a single two-sided cell of $\mathbf{U}_D$, or $\dot{\mathbf{U}}$.

Recall the group X of the root datum of $\dot{\mathbf{U}}$ from section 1.3. For convenience, here we will view it as a quotient of $\mathbb{Z}^n$ (by taking the entries $a_1, a_2, \ldots, a_n$). We define X^+ to be the"dominant weights" in X. Let $I_0 = \{i \in \mathbb{Z}/n\mathbb{Z}\colon i \not\equiv 0 \bmod n\}$. The set X^+ consists of those $\mu \in X$ with $\mu(i)\colon = \langle i, \mu\rangle \geq 0$ for $i \in I_0$.

Proposition 2.4.6. *The two-sided cells of $\dot{\mathbf{U}}$ are naturally parameterized by X^+.*

Proof. First note that each partition λ with at most n parts determines an element μ in X^+ by taking the coset of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in X, and the previous paragraph shows that this gives a natural bijection between X^+ and the two-sided cells of $\dot{\mathbf{U}}$. Indeed each μ in X^+ has a unique representative λ in $\mathbb{Z}^n$ with final entry 0. The cell corresponding to μ is $\mathbf{c}_\lambda$, thought of as a cell of $\dot{\mathbf{U}}$. (It is actually a cell of $\dot{\mathbf{U}}$, $\mathbf{U}_D$, and $\mathfrak{A}_D$!) □

Note that this classification has an interesting consequence: The number of left cells in a two-sided cell $\mathbf{c}_\lambda$ of $\mathfrak{A}_D$ depends only on the element of X^+ it determines, as can be seen from the formula in Proposition 2.3.4. Thus since each left cell of $\mathfrak{A}_D$ intersects $\mathbf{U}_D$ in at most one left cell, and the two algebras have the same number of left cells, this intersection is always nonempty.

We may also give an explicit formula for the value of the a function, using the fact that we know the value of the corresponding function on the Hecke algebra. Indeed if $w \in \hat{A}_{D-1}$ lies in the cell $\mathbf{c}_\lambda$ then $a(w) = (D - \sum \lambda_i^2)/2$.

Lemma 2.4.7. *Let $\mu \in X^+$ and let $\lambda \in \mathbb{Z}^n$ be its representative with 0 in the final entry. Then $b \in \dot{\mathbf{B}}$ lies in the cell $\mathbf{c}_\mu$ corresponding to μ, and $b1_\nu = b$, for some $\nu \in X$. Pick the unique representative υ of ν in $\mathbb{Z}^n$ such that $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \upsilon_i$. Then we have we have $a(b) = \sum_{i=1}^n (\lambda_i^2 - \upsilon_i^2)$.* □

We have also already constructed the asymptotic algebra A_μ for each $\mu \in X^+$. This is just the ring $J_{\mathbf{c}_\lambda}$ constructed in the previous section, where λ is the representative of μ described above and $\mathbf{c}_\lambda$ is the two-sided cell of some $\mathfrak{A}_D$ corresponding to μ.

Let $G_\mu := \prod_{i=1}^{n-1} GL_{\lambda(i)}(\mathbb{C})$, and let R_μ be its representation ring. Combining the above with Proposition 2.3.6 we find that the asymptotic ring A_μ is isomorphic to a matrix ring over R_μ of size $\prod_{i=1}^{n-1} \binom{n}{i}^{\mu(i)}$. Thus we may pull back modules of this matrix ring to obtain modules for $\dot{\mathbf{U}}$. These would appear to be closely related to the "extremal weight modules" of Kashiwara [K02], which are in turn related to the universal standard modules defined by Nakjima in his geometric classification of simple modules for quantum affine algebras. Indeed Kashiwara has a number of conjectures about the structure of these modules which he suggests should be closely related to the conjectures of Lusztig that we establish here for $\widehat{\mathfrak{sl}_n}$ (see the remark below). (Some of Kashiwara's conjectures have recently been proved in the simply-laced case by Nakajima [N].)

Remark. The results of this section establish (in the case of $\widehat{\mathfrak{sl}}_n$) all the conjectures in [L95, section 5]. It should be noted that paragraph 5.4 of this section contains a misprint. Given $\lambda \in X^+$ the numbers $\lambda(i)$, for $i \in I_0$, should be given by the formula $\lambda(i) = \langle i, \lambda \rangle$.

Chapter 3

Weyl groups and constructible functions

One of the fundamental results in geometric representation theory is Springer's construction ([S76], [S78]) of representations of the Weyl group W in the homology of certain subvarieties of the associated flag variety. This result, first obtained using positive characteristic methods, was investigated by many people, and a vast array of interpretations was subsequently obtained. One of these, due to Kazhdan and Lusztig [KL80b], gave a construction of the group algebra $\mathbb{Z}[W]$ in the homology of the Steinberg variety $\mathcal{Z}$. Combined with their work on Hecke algebras, this made it clear that the group algebra $\mathbb{Z}[W]$ comes equipped with two natural, but distinct, bases. The first of these, the Kazhdan-Lusztig basis ([KL79], [KL80a]), comes from the intersection cohomology sheaves of Schubert varieties, and the second from the fundamental classes of the irreducible components of $\mathcal{Z}$. A connection between the two was made by Kashiwara and Tanisaki [KT] using the characteristic cycle construction (though they worked in the context of $\mathcal{D}$-modules).

More recently, Lusztig [L97] gave a new elementary construction of the group algebra of the Weyl group as a convolution algebra of constructible functions on $\mathcal{Z}$, motivated by the convolution in K-theory described in [KT], and his own construction of the enveloping algebra of the negative part of a semisimple Lie algebra [L91]. He also produced a "semicanonical" basis of the group algebra in this way and conjectured that it coincides with the second of the two natural bases mentioned above. This chapter consists of an attempt to understand the connection between this construction and those mentioned above — in particular to give an explicit description of the "semicanonical" basis functions which allows one to verify Lusztig's conjecture, and also to construct from them representations of the Weyl group in constructible functions on Springer fibres.

3.1 Background

Let G be a reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, let $\mathcal{B}$ be the flag variety of Borel subalgebras in $\mathfrak{g}$, and let W be the Weyl group of G. It is well known that the cotangent bundle of the flag variety can be described explicitly as

$$T^*\mathcal{B} = \{(e, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \colon e \in \mathfrak{b}\},$$

where $\mathcal{N}$ is the nilpotent cone in $\mathfrak{g}$. This description makes it clear that we have maps $\pi \colon T^*\mathcal{B} \to \mathcal{B}$ and $\mu \colon T^*\mathcal{B} \to \mathcal{N}$, which are the restrictions of the two projections from $\mathcal{N} \times \mathcal{B}$. Since a regular nilpotent element is contained in a unique Borel subalgebra, and the regular nilpotents constitute the dense open orbit of the adjoint action of G on $\mathcal{N}$, the map μ is birational. Moreover, as the cotangent bundle $T^*\mathcal{B}$ is smooth, and μ is clearly proper, we have obtained a resolution of singularities, known as the Springer resolution. The fibre of μ over a point $e \in T^*\mathcal{B}$ (the Springer fibre) is

$$\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} \colon e \in \mathfrak{b}\},$$

hence it may also be described as the zero-set of the corresponding "nilpotent" vector field on $\mathcal{B}$. It is these varieties on whose cohomology Springer constructed representations of the Weyl group. I wish to describe a new way to obtain representations of W from the varieties $\mathcal{B}_e$.

We need one more variety, which is closely related to $T^*\mathcal{B}$ — the Steinberg variety $\mathcal{Z}$. This is simply the product of $T^*\mathcal{B}$ with itself over $\mathcal{N}$, that is,

$$\mathcal{Z} = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B} = \{(e, \mathfrak{b}, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \colon e \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

There is another description of $\mathcal{Z}$ coming from the Bruhat decomposition. The group G acts diagonally on $\mathcal{B} \times \mathcal{B}$ with $|W|$ orbits which we denote Y_w, $w \in W$. If we imbed $\mathcal{Z}$ in $T^*\mathcal{B} \times T^*\mathcal{B}$ by the map

$$(e, \mathfrak{b}, \mathfrak{b}') \mapsto (e, \mathfrak{b}, -e, \mathfrak{b}'),$$

then the image is precisely the union of the conormal bundles to the G-orbits Y_w:

$$\mathcal{Z} = \coprod_{w \in W} T^*_{Y_w}(\mathcal{B} \times \mathcal{B}).$$

Hence we see that $\mathcal{Z}$ is a variety of pure dimension $2d$, where $d = \dim_{\mathbb{C}} \mathcal{B}$, with $|W|$ components. Let $\mathcal{Z}_w$ be the conormal bundle of Y_w, so the closure of $\mathcal{Z}_w$ is a component of $\mathcal{Z}$.

Lusztig [L97] constructs an algebra of functions on $\mathcal{Z}$ which is isomorphic to the group algebra $\mathbb{Z}[W]$. In order to describe his construction, we first introduce some general notions. For a variety X, let $\mathcal{F}(X)$ denote the algebra of constructible functions on X, that is, the algebra generated over $\mathbb{Z}$ by the characteristic functions of closed subvarieties. These functions come with a natural notion of pull-back and push-forward (see for example [M]). If $\pi \colon Y \to X$ is a morphism, pull-back is defined by

$$\pi^*(f)(y) = f(\pi(y)),$$

for $f \in \mathcal{F}(X)$.

Push-forward is defined as "integration along the fibres" using the Euler characteristic,

$$\pi_!(f)(x) = \int_{\pi^{-1}(x)} f = \sum_{a \in \mathbb{Z}} a\chi(\pi^{-1}(x) \cap f^{-1}(a))$$

where the second equality is by definition, and $f \in \mathcal{F}(Y)$. Note that in the definition of push-forward, to ensure the appropriate functoriality, we should take Euler characteristics with compact supports, but in the complex case, this is the same as taking ordinary Euler characteristics (see [Su]).

Using these operations, the algebra $\mathcal{F}(\mathcal{Z})$ becomes a convolution algebra as follows: for $f, g \in \mathcal{F}(\mathcal{Z})$ set

$$(f * g)(e, \mathfrak{b}, \mathfrak{b}') = \int_{\mathfrak{b}_1 \in \mathcal{B}_e} f(e, \mathfrak{b}, \mathfrak{b}_1) g(e, \mathfrak{b}_1, \mathfrak{b}').$$

Alternatively, we may write this as a composition of pullbacks and pushforwards in analogy with the construction of the Hecke algebra in [S82]. Observe that $(\mathcal{F}(\mathcal{Z}), *)$ has a unit given by $\mathbf{1} = 1_{\mathcal{Z}_1}$.

Let $\leq$ denote the usual Bruhat order on W, l the length function. Then $S = \{s \in W \colon l(s) = 1\}$ is the set of simple reflections in W. For $s \in S$ set $f_s = 1_{\overline{\mathcal{Z}}_s}$, the characteristic function of the corresponding component of $\mathcal{Z}$. Let $\mathcal{W}$ be the algebra of functions generated under convolution by the f_s, for $s \in S$. We have the following result.

Theorem 3.1.1. *[L97] The algebra $\mathcal{W}$ is isomorphic to the group algebra of the Weyl group via the map $\phi \colon \mathbb{Z}[W] \to \mathcal{W}$ defined by $\phi(s) = \mathbf{1} - f_s$, $s \in S$.*

Moreover, $\mathcal{W}$ has a distinguished basis $\{f_w \colon w \in W\}$ which is characterized by the following properties:

1. *f_w is equal to 1 on $\mathcal{Z}_w$;*

2. *for any w' with $w' \not\leq w$, f_w is 0 on $\mathcal{Z}_{w'}$;*

3. *for any w' with $w' < w$, f_w is 0 on a dense open subset of $\mathcal{Z}_{w'}$.*

Proof. We briefly sketch the argument for the convenience of the reader. One first checks that the functions f_s obey the relation implied by the fact that s is an involution. Then in order to show the algebra they generate is a quotient of $\mathbb{Z}[W]$ it only remains to check the braid relations. This is a rank two calculation, which can be carried out by hand, and is the bulk of the work in [L97]— one needs an explicit description of the Springer fibres in each rank two group. Thus the algebra $\mathcal{W}$ is a quotient of $\mathbb{Z}[W]$. To show that ϕ is actually an isomorphism, one notes that restricting functions to the zero-section, i.e. the component of $\mathcal{Z}$ corresponding to the longest element of W, is an algebra homomorphism. The composite of ϕ with this map can be seen to be an isomorphism, as an easy consequence of the Bruhat decomposition.

The construction of the basis $\{f_w : w \in W\}$ is inductive, with f_s given by the above formula. One shows that if $w = s_1 s_2 \dots s_r$ is a reduced expression of w then the product $f_{s_1} f_{s_2} \dots f_{s_r}$ is equal to 1 on $\mathcal{Z}_w$ and supported on the union of the $\mathcal{Z}_v$ for $v < w$ in the Bruhat order. Hence if f_v is known for all $v < w$ we may subtract appropriate multiples of them from $f_{s_1} f_{s_2} \dots f_{s_r}$ to obtain f_w. □

Remark. The morphism ϕ we have described differs from the one given in [L97] by the sign character. We do this so that the representations we obtain from Springer fibres match with the usual conventions.

We will sometimes refer to the basis $\{f_w : w \in W\}$ or its preimage in $\mathbb{Z}[W]$ as the *semicanonical basis* of $\mathcal{W}$ or W respectively. In [L97], Lusztig makes a number of conjectures about the functions $\{f_w : w \in W\}$. Recall that $\mathbb{Z}[W]$ has a natural basis $\{b_w : w \in W\}$, defined by Kashiwara and Tanisaki in terms of characteristic cycles [KT], or equivalently, in terms of the top homology of $\mathcal{Z}$ as in [KL80b].

Conjecture 3.1.2. *[L97, 4.17] Under the isomorphism ϕ, the bases $\{b_w : w \in W\}$ and $\{f_w : w \in W\}$ correspond, that is, $\phi(b_w) = f_w$.*

We define the support of a function $f \in \mathcal{F}(\mathcal{Z})$ as the set $\text{supp}(F) = \{x \in \mathcal{Z} : f(x) \neq 0\}$ (here we follow Lusztig [L97], it is perhaps more standard to take the closure of this set).

Conjecture 3.1.3. *[L97, 4.18] Let $w \in W$. The support of the function f_w is contained in $\overline{\mathcal{Z}}_w$.*

3.2 Springer representations.

We now show how one can use Lusztig's construction of the algebra $\mathcal{W}$ to obtain Springer representations, given the conjectures stated above. We first need to recall the notion of geometric cells, due to Spaltenstein and Steinberg [St]. There is an obvious map $\rho \colon \mathcal{Z} = T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B} \to \mathcal{N}$ given by $(e, \mathfrak{b}, \mathfrak{b}') \mapsto e$. The following result is essentially contained in [St] and [Sp].

Theorem 3.2.1. *Let $\mathcal{O}$ be a nilpotent orbit. The inverse image $\rho^{-1}(\mathcal{O})$ is pure dimensional, and its dimension is equal to the dimension of $\mathcal{Z}$. Thus the closure $\overline{\rho^{-1}(\mathcal{O})}$ is a union of components of $\mathcal{Z}$.*

Proof. The key point is to show that if $\mathfrak{n}$ is the nilpotent radical of a fixed Borel subalgebra, then all components of $\mathcal{O} \cap \mathfrak{n}$ have dimension $1/2 \cdot \dim(\mathcal{O})$. In fact, using the Killing form, $\mathcal{O}$ can be given a natural symplectic form with respect to which $\mathcal{O} \cap \mathfrak{n}$ is Lagrangian. □

Since the components of $\mathcal{Z}$ are indexed by elements of W, this allows us to make the following definition:

Definition. Given a nilpotent orbit $\mathcal{O}$, we attach to it the subset of the Weyl group

$$\mathcal{C}_{\mathcal{O}} = \{w \in W : \mathcal{Z}_w \subset \overline{\rho^{-1}(\mathcal{O})}\}.$$

The sets $C_{\mathcal{O}}$ are called *geometric cells* (or sometimes *S-cells*). They partition W.

Remark. This notion of cell is distinct from that given by the Kazhdan-Lusztig basis— the two-sided cells for a finite Weyl group are indexed by the special nilpotent orbits, which in general are a proper subset of the set of nilpotent orbits (in type A, all nilpotent orbits are special). Note however that if we equip the group algebra $\mathbb{Z}[W]$, with the basis $\{b_w : w \in W\}$ given in terms of the characteristic cycle map, we can define a different notion of cells. It is conjectured that the geometric cells are precisely the two-sided cells with respect to this basis. It is also worth remarking that even in type A, the Kazhdan-Lusztig basis does not coincide with the basis $\{b_w : w \in W\}$ defined above (see the paper of Kashiwara and Saito [KSa]).

Let $e \in \mathcal{O}$ be a nilpotent. The definition of convolution given in the previous section clearly extends to make $\mathcal{F}(\mathcal{B}_e \times \mathcal{B}_e)$ a bimodule for $\mathcal{W}$, and hence by Theorem 3.1.1, a bimodule for $\mathbb{Z}[W]$. However $\mathcal{F}(\mathcal{B}_e \times \mathcal{B}_e)$ is, for our purposes, hugely infinite dimensional. The subtlety in this approach to Springer representations is to find inside this bimodule the appropriate finite-dimensional submodule. We do this as follows. For each $w \in C_{\mathcal{O}}$ let $g_w = f_w|_{\rho^{-1}(e)}$, the restriction of the f_w to $\rho^{-1}(e)$, which is just $\mathcal{B}_e \times \mathcal{B}_e$. Let M_e be the $\mathbb{Z}$-submodule of $\mathcal{F}(\mathcal{B}_e \times \mathcal{B}_e)$ spanned by $\{g_w : w \in C_{\mathcal{O}}\}$.

Lemma 3.2.2. *Conjecture 3.1.3 implies that M_e is a bimodule for $\mathbb{Z}[W]$.*

Proof. In fact we need strictly less than the Conjecture. What is crucial is that the image of the support of f_w for $w \in C_{\mathcal{O}}$ under the map ρ lies in the closure $\overline{\mathcal{O}}$. To see that this implies the lemma, note that it is clear from the definition of convolution that if $\mathcal{U} \subset \mathcal{N}$ is an open subset of the nilpotent cone, then the set of functions $\{f \in \mathcal{W} : f|_{\rho^{-1}(\mathcal{U})} = 0\}$ forms an ideal in $\mathcal{W}$. Hence by considering the complement of $\overline{\mathcal{O}} \backslash \mathcal{O}$, we see that the condition on image of the supports of the f_w yields the result. □

Moreover Conjecture 3.1.2 implies that the bimodules one obtains are precisely the endomorphism modules of the Springer representations in the top homology of $\mathcal{B}_e$, as a consequence of the results of Kashiwara-Tanisaki [KT]. One may also obtain (left) representations of W in $\mathcal{F}(\mathcal{B}_e)$ by fixing the Borel in the second factor. It is entertaining to compute the functions in M_e directly in the rank two cases. In the approaches to Springer representations using homology, one gets an action of the component group of the centralizer $Z_G(e)$ which commutes with the action of W. This allows one to split representations up into their irreducible components. It would be nice to understand this action in the context of constructible functions also.

Thus we must verify Lusztig's conjectures. Let us briefly describe our approach (the terms used will be defined in the following sections). We attempt to "microlocalize" the construction of the Weyl group in the K-group of G-equivariant constructible sheaves on $\mathcal{B} \times \mathcal{B}$. Recall that the Hecke algebra $\mathcal{H}$ associated to W can be realized as a convolution algebra of perverse sheaves on $\mathcal{B} \times \mathcal{B}$ by a convolution construction similar to the one we defined above for functions (smooth pullbacks and proper push-forwards preserve perverse sheaves up to shift). If we pass to the K-group (or equivalently take stalk Euler characteristics), we obtain the group algebra $\mathbb{Z}[W]$, realized as a convolution algebra $\mathcal{H}$ (see for example [S82], or in the context of $\mathcal{D}$-modules [KT]). If we accept Conjecture 3.1.2, then the Index theorem of Kashiwara [KSc, 9.5] shows that the functions in $\mathcal{W}$ give the stalk Euler characteristic of the corresponding perverse sheaf when they are restricted to the zero-section of $T^*(\mathcal{B} \times \mathcal{B})$. Our idea is that, in general, the functions in the algebra $\mathcal{W}$ give the "microlocal stalk Euler characteristics" of the perverse sheaves in $\mathcal{H}$. What follows is a construction which makes this precise.

3.3 Specialization and microlocalization

For a smooth variety X, let $\mathbb{D}(X)$ denote the bounded derived category of sheaves on X (in the analytic topology). Let $\mathbf{D}_c(X)$ denote the full subcategory of complexes with constructible cohomology sheaves. Let $\mathcal{P}(X)$ denote the category of perverse sheaves, the heart of the perverse t-structure on $\mathbf{D}_c(X)$. For more details see [BBD] or [KSc]. Following a grand tradition of notational abuse, we will usually say "sheaf" when referring to an object of $\mathbf{D}(X), \mathbf{D}_c(X)$ or $\mathcal{P}(X)$.

A fundamental construction in intersection theory is the deformation to the normal cone. Suppose we have a smooth variety M, and a smooth subvariety $N \subset M$. Let $T_N M$ denote the normal bundle to N in M. If d is the dimension of M, there exists a $d+1$ dimensional smooth variety $\tilde{M}_N$ which possesses maps p and t to M and $\mathbb{C}$ respectively. The fibres of t over $\mathbb{C}$ are isomorphic to M over nonzero complex numbers, and to $T_N M$ over 0. More precisely we have commutative diagrams:

$$\begin{array}{ccccc} T_N M & \xrightarrow{s} & \tilde{M}_N & \xleftarrow{j} & M \times \mathbb{C}^\times \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \sigma} & \\ N & \xrightarrow{i} & M & & \end{array}$$

and,

$$\begin{array}{ccccc} T_N M & \xrightarrow{s} & \tilde{M}_N & \xleftarrow{j} & M \times \mathbb{C}^\times \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ \{0\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^\times \end{array}$$

where all the horizontal maps are inclusions. Later we will need the following construction: Given a cycle Z in M, we set the normal cone of Z to N to be the intersection (counting multiplicities) of $t^{-1}(0)$ with $\overline{\sigma^{-1}(Z)}$.

There are a number of ways to construct the variety $\tilde{M}_N$, with perhaps the nicest given by blowing up $M \times \mathbb{C}$ along $N \times \{0\}$. The exceptional divisor can be shown to be the union of two pieces, one isomorphic to $T_N M$, the other to $\mathbb{P}T_N M$. We set $\tilde{M}_N$ to be the open subset of the blow-up given by the complement of $\mathbb{P}T_N M$. The maps p and t are then obtained from the restriction of the natural map of the blow-up to $M \times \mathbb{C}$. For a nice discussion of this construction and its uses in intersection theory see the survey of Fulton [F].

Note that we $T_N M$ sits in $\tilde{M}_N$ as a hypersurface defined by the equation t. In particular, we may apply the functors of nearby and vanishing cycles on $\tilde{M}_N$ to produce sheaves on $T_N M$. This is the key point in describing specialization of sheaves.

We now recall the definition of nearby and vanishing cycles. We use the conventions of [KSc]. Suppose that X is a smooth variety and $f\colon X \to \mathbb{C}$ a holomorphic function. Set $Y = f^{-1}(0)$, and let $i\colon Y \to X$ denote the inclusion. Let $\tilde{\mathbb{C}}^\times$ be the universal cover of $\mathbb{C}$, and $p\colon \tilde{\mathbb{C}}^\times \to \mathbb{C}$ the covering map (e.g. we may take $\tilde{\mathbb{C}}^\times = \mathbb{C}$, and $p(z) = \exp(2\pi\sqrt{-1}z)$). Then consider the diagram:

$$\begin{array}{ccccc} & & \tilde{X}^\times & \longrightarrow & \tilde{\mathbb{C}}^\times \\ & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & X & \xrightarrow[f]{} & \mathbb{C}, \end{array}$$

where the square is Cartesian, that is $\tilde{X}^\times = X \times_{\mathbb{C}} \tilde{\mathbb{C}}^\times$.

Definition. For $F \in \mathbf{D}(X)$ we define $\psi_f(F) = i^* R\tilde{p}_* \tilde{p}^*(F)$.

ψ_f is called the functor of *nearby cycles*. It depends only on the $F|_{X \setminus Y}$. Note that since $(\tilde{p}^*, R\tilde{p}_*)$ are adjoints, we have a natural morphism of sheaves $i^*(F) \to \psi_f(F)$. The *vanishing cycles* of F, $\phi_f(F)$ is the shifted cone of this morphism, that is, we have a distinguished triangle in $\mathbf{D}_c(X)$

$$i^*(F) \to \psi_f(F) \to \phi_f(F)[1] \to .$$

Goresky and MacPherson have given more concrete geometric description of these functors using a stratified contraction to the special fibre, see [GM].

Having constructed the functor of nearby cycles and the deformation to the normal cone, we may define the functor of *specialization*.

Definition. Let N be a smooth subvariety of M, and let $F \in \mathbf{D}_c(M)$. Define $\nu_N\colon \mathbf{D}_c(M) \to$

$\mathbf{D}_c(T_N M)$ by

$$\nu_N(F) = \psi_t(p^{-1}(F))$$

where the maps are as in the diagram at the start of the section. The functor of *microlocalization*, μ_N is obtained from that of specialization by applying the Fourier transform, which maps conic sheaves on a vector bundle to conic sheaves on the dual bundle, thus $\mu_N\colon \mathbf{D}_c(M) \to \mathbf{D}_c(T_N^*M)$. For a discussion of the Fourier transform we refer the reader to [KSc, chapter 3].

The following theorem shows that these operations behave well with respect to perversity.

Theorem 3.3.1. *Let M be a smooth variety.*

- *If f is a holomorphic function on M, then the functors $\psi_f[-1]$ and ϕ_f are t-exact.*
- *Let N be a smooth subvariety of codimension d in a smooth variety M. Then the functors ν_N and $\mu_N[d]$ are t-exact.*

Proof. See for example Corollary 10.3.13 and Proposition 10.3.19 of [KSc]. □

We will denote $\mu_Y[\mathrm{codim}(Y)]$ by $\tilde{\mu}_Y$, thus $\tilde{\mu}_Y$ preserves perverse sheaves. We will need to understand how microlocalization behaves with respect to push-forward and pull-back. Let $f\colon Y \to X$ be a morphism of complex manifolds, and let N be a closed submanifold of codimension k in Y with $f(N) \subset M$, a closed submanifold of X of codimension l. Consider the diagrams

$$T^*Y \xleftarrow[\rho_f]{} Y \times_X T^*X \xrightarrow[\varpi_f]{} T^*X$$

and

$$T_N^*Y \xleftarrow[\rho_f]{} N \times_M T_M^*X \xrightarrow[\varpi_f]{} T_M^*X$$

Recall that f is said to be *clean* with respect to M if $f^{-1}(M)$ is a submanifold of Y, and the map $\rho_f\colon N \times_M T_M^*X \to T_N^*Y$ is surjective.

Theorem 3.3.2. *Let $G \in \mathbf{D}_c(Y)$. Then there exists a canonical morphism*

$$R\varpi_{f!}\rho_f^*\mu_N(G) \longrightarrow \mu_M(Rf_!G).$$

This map is an isomorphism if $f^{-1}(M) = N$, f is clean with respect to M, and proper on the support of G.

Proof. See Theorem 4.3.4 of [KSc]. □

Theorem 3.3.3. *Let $F \in \mathbf{D}_c(X)$. There is a canonical morphism*

$$\mu_N(f^!F) \longrightarrow R\rho_{f*}\varpi_f^!\mu_M(F).$$

This map is an isomorphism if f and $f|_N$ are smooth.

Proof. See Theorem 4.3.5. of [KSc]. □

We now describe the characteristic cycle construction. We first introduce the slightly cruder notion of *singular support*. Intuitively, this is supposed to record the directions in which the cohomology sheaves of $F \in \mathbf{D}_c(X)$ propagate.

Definition. Let $F \in \mathbf{D}_c(X)$. Define $SS(F) \subset T^*X$ by the condition that for $p \in T^*X$, we have $p \notin SS(F)$ if there is an open neighbourhood U of p such that for any $x \in X$ and any holomorphic function f defined near x with $f(x) = 0$ and $df(x) \in U$, we have $\phi_f(F)_x = 0$.

Recall that T^*X carries a natural holomorphic symplectic structure. It is known that for $F \in \mathbf{D}_c(X)$ the singular support $SS(F)$ is a Lagrangian subvariety of T^*X. More precisely, let $\mathcal{S}$ be a stratification of X, (say a Whitney stratification). Let $\Lambda_\mathcal{S}$ be the conormal variety of the stratification, $\Lambda_\mathcal{S} = \bigsqcup_{S\in\mathcal{S}} T_S^*X$. This is a closed conic Lagrangian subvariety of T^*X, with components $\Lambda_S =$

$\overline{T^*_S(X)}$. If F is constructible with respect to $\mathcal{S}$, then $SS(F)$ is Lagrangian and lies in $\Lambda_{\mathcal{S}}$ and hence is a union of some components of $\Lambda_{\mathcal{S}}$.

For a constructible sheaf, it is possible to refine the $SS(F)$ to obtain a Lagrangian cycle. Essentially the issue is to attach a multiplicity to each component of the singular support. Let $\tilde{\Lambda}_{\mathcal{S}}$ be the set of *nondegenerate* covectors in $\Lambda_{\mathcal{S}}$, that is,

$$\tilde{\Lambda}_{\mathcal{S}} = \Lambda_{\mathcal{S}} \backslash \bigcup_{S_1 \neq S_2} \Lambda_{S_1} \cap \Lambda_{S_2}.$$

Similarly set $\tilde{\Lambda}_S = \Lambda_S \cap \tilde{\Lambda}_{\mathcal{S}}$. For a holomorphic function f on X let Λ_f denote the graph of df in T^*X. It is a Lagrangian submanifold. For any F constructible with respect to $\mathcal{S}$ we construct a Lagrangian cycle as follows. Let Λ_S be a component of $\Lambda_{\mathcal{S}}$. Then for $\xi \in \tilde{\Lambda}_S$ pick a holomorphic function f with $f(x) = 0$ and $df(x) = \xi$, where $\xi \in T^*_xX$, and such that Λ_f intersects Λ_S transversally at ξ. Let $m_S = (-1)^{\dim(X)}\chi(\phi_f(F))_x$. Then it can be shown that this integer is independent of all the choices involved, and we set

$$CC(F) = \sum_{S \in \mathcal{S}} m_S \Lambda_S.$$

Note that m_S is clearly zero for any component of Λ_S which is not contained in $SS(F)$. The support of $\phi_f(F)$ is contained in the projection to X of the intersection $\Lambda_f \cap SS(F)$, hence we see that locally $\phi_f(F)$ is concentrated at x. If F is perverse, then since the vanishing cycle functor preserves perversity, this forces $\phi_f(F)$ to be concentrated in degree 0, and hence the coefficients m_S are nonzero whenever $\phi_f(F)$ is nonzero, thus $SS(F)$ is precisely the support of the cycle $CC(F)$.

Our construction of the characteristic cycle follows [KSc]. In the language of $\mathcal{D}$-modules one naturally obtains the characteristic cycle of a holonomic module as a positive Lagrangian cycle via a good filtration. This ammounts to a different normalization of the characteristic cycle. Thus for us $CC(\mathbb{C}_X) = [T^*_XX]$, whereas in say [KT] they normalize so that $CC(\mathbb{C}_X) = (-1)^{\dim(X)}[T^*_XX]$.

3.4 Microlocal construction of $\mathcal{W}$

The group algebra of the Weyl group can be realized as the K-group of a certain category of perverse sheaves on $\mathcal{B} \times \mathcal{B}$, equipped with a convolution product. Let $\mathcal{H}$ be the category of semisimple complexes (that is, complexes which are direct sums of simple perverse sheaves with shifts) on $\mathcal{B} \times \mathcal{B}$ which are constructible with respect to the G-orbit stratification. Thus this category has simple objects $\{\mathcal{L}_w : w \in W\}$ where $\mathcal{L}_w$ is the intersection cohomology complex attached to the orbit Y_w (see section 3.1), and hence the K-group has a basis given by the classes $[\mathcal{L}_w]$. The convolution product is given by a triple diagram in the normal way: Let $d = \dim(\mathcal{B})$. The threefold product of $\mathcal{B}$ with itself admits three maps to $\mathcal{B} \times \mathcal{B}$, denoted p_{ij} where i and j are distinct elements of $\{1, 2, 3\}$. For $\mathcal{A}, \mathcal{B} \in \mathcal{H}$ we set $\mathbf{A} * \mathbf{B}$ to be the complex $(p_{13})_!(p_{12}^*(\mathbf{A})[d] \otimes^L p_{23}^*(\mathbf{B})[d])[-d]$. Since all the morphisms p_{ij} are smooth and proper, standard facts show that $\mathbf{A} * \mathbf{B}$ lies in $\mathcal{H}$. Thus $\mathcal{H}$ gives a "categorical" realization of the Hecke algebra associated to W, and the corresponding K-group is therefore isomorphic to $\mathbb{Z}[W]$. For details of this construction see, for example, [S82]. We make explicit our isomorphism of the K-group with $\mathbb{Z}[W]$ as follows. Given $\mathbf{A} \in \mathcal{H}$ set

$$e(\mathbf{A}) = \sum_{w \in W} \left(\sum_i (-1)^{i-l(w)} \dim \mathcal{H}^i_w(\mathbf{A}) \right) w,$$

where $\mathcal{H}^i_w(\mathbf{A})$ is the stalk cohomology of $\mathbf{A}$ at any point of Y_w. Then e is an algebra isomorphism sending, for example, $[\mathcal{L}_s]$ to $s-1$. (Note that our map corresponds to setting $v = -1$ in the algebraic context. This is slightly different to the normal specialization, which sets $v = 1$ (although $q = v^2$ still specializes to 1).

Kashiwara and Tanisaki [KT] related $\mathcal{H}$ to the construction of $\mathbb{Z}[W]$ in the top homology of $\mathcal{Z}$

[KL80b] by showing that the characteristic cycle map is actually an algebra isomorphism between the two realizations of $\mathbb{Z}[W]$. Indeed as all the sheaves in $\mathcal{H}$ are constructible with respect to the orbit stratification, it follows from the above that for $\mathbf{A} \in \mathcal{H}$, $CC(\mathbf{A})$ is a union of some components of $\mathcal{Z}$. They defined the basis $\{b_w : w \in W\}$ by setting b_w to be the preimage of the component $\overline{\mathcal{Z}}_w$.

We wish to obtain the algebra $\mathcal{W}$ of functions on $\mathcal{Z}$ via some sort of microlocalization of the sheaves in $\mathcal{H}$. Let $\pi : T^*(\mathcal{B} \times \mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ denote the bundle map. We define a map ϖ from perverse sheaves on $\mathcal{B} \times \mathcal{B}$ to $\mathbb{Z}$-valued functions on $T^*(\mathcal{B} \times \mathcal{B})$ as follows:

$$\varpi(\mathcal{P})(\xi) = \chi(\mu_{\pi(\xi)}(\mathcal{P}))_\xi,$$

where $\mathcal{P}$ is a perverse sheaf on $\mathcal{B} \times \mathcal{B}$, and $\xi \in T^*(\mathcal{B} \times \mathcal{B})$. It follows that $\varpi(\mathcal{P})$ is constructible when restricted to the cotangent space of any point, and moreover the equivariance of the sheaves in $\mathcal{H}$ ensures that the image of $\mathcal{H}$ consists of constructible functions.

The following is the key assertion in this approach.

Conjecture 3.4.1. *The map ϖ gives an algebra isomorphism from $K(\mathcal{H})$ to $\mathcal{W}$, where $K(\mathcal{H})$ is the Grothedieck group of $\mathcal{H}$.*

We now show that the conjecture is implied by the equality of the stalk Euler characteristics of two different microlocalizations. Given a point $\xi \in T^*(\mathcal{B} \times \mathcal{B})$ and a simple reflection s, if $\pi(\xi) = (b_1, b_2)$ then let $X_s(\xi) = \{(b, b_2) : (b, b_1) \in \bar{Y}_s\}$.

Lemma 3.4.2. *Suppose that for each perverse sheaf $\mathbf{A}$ in $\mathcal{H}$, and $\xi \in T^*_{X_s(\xi)}(\mathcal{B} \times \mathcal{B})$ we have*

$$\chi(\mu_{\pi(\xi)}(\mathbf{A}))_\xi = \chi(\mu_{X_s(\xi)}(\mathbf{A}))_\xi, \tag{3.4.1}$$

Then the above conjecture holds.

Proof. To show that ϖ is an algebra homomorphism it is enough to compare the actions of a simple reflection $s \in S$, since these generate the group algebra. Thus we begin by describing the action of convolving with $[\mathcal{L}_s]$ for s a simple reflection. Note that since $\bar{Y}_s$ is smooth (it is $\mathbb{P}^1$ bundle over $\mathcal{B}$), the characteristic cycle of $\mathcal{L}_s$ is $-[\Lambda_s]$. Using functoriality of microlocalization, an easy SL_2 calculation shows that $\varpi(\mathcal{L}_s) = -f_s$. Hence the conjecture predicts that convolving with $\mathcal{L}_s$ corresponds to multiplication by f_s. Let Z_s be the variety $\{(b_1, b_2, b_3) \in \mathcal{B} \times \mathcal{B} \times \mathcal{B} : (b_1, b_2) \in \bar{Y}_s\}$. Then let $p : Z_s \to \mathcal{B} \times \mathcal{B}$ denote the map $(b_1, b_2, b_3) \mapsto (b_1, b_3)$ and $q : Z_s \to \mathcal{B} \times \mathcal{B}$ denote the map $(b_1, b_2, b_3) \mapsto (b_2, b_3)$. It is easy to see that

$$\mathcal{L}_s * \mathbf{A} = p_! q^*(\mathbf{A})[1].$$

By the above discussion, we wish to compare the values of $\varpi(\mathcal{L}_s * \mathbf{A})(\xi)$ and $f_s * \varpi(\mathbf{A})(\xi)$ for $\xi \in \mathcal{Z} \subset T^*(\mathcal{B} \times \mathcal{B})$. Now let $x = \pi(\xi)$, thus $x = (b_1, b_2)$ for some pair of Borel subalgebras $b_1, b_2 \in \mathcal{B}$. The projective line $X_s(\xi)$ in the statement of the lemma is just the set $p(q^{-1}(x))$. Thus $X_s(\xi)$ consists of all (b, b_2) with $(b, b_1) \in \bar{Y}_s$. Now observe that both of the maps p and q are smooth and proper (each makes Z_s into a $\mathbb{P}^1$-bundle over $\mathcal{B} \times \mathcal{B}$. Thus we may straightforwardly apply the theorems in the previous section to calculate $\mu_x(p_! q^*(\mathbf{A})$.

$$\begin{aligned} \mu_x(p_! q^*(\mathbf{A}))[1] &= R\varpi_{p!}\rho_p^* \mu_{p(x)}(q^*(\mathbf{A})[1]) \\ &= R\varpi_{p!}\rho_p^* \mu_{p(x)}(q^!(\mathbf{A}))[-1] \\ &= R\varpi_{p!}\rho_p^* R\rho_{q_*}\varpi_q^!(\mu_Y(\mathbf{A}))[-1]. \end{aligned} \tag{3.4.2}$$

Now since Verdier duality acts trivially on stalk Euler characteristics in the complex setting (see for example [KSc, Exercise IX.12] and [Su]), we see that ! and * are interchangeable once we take Euler characteristics. It is then easy to check that if we assume the equality in the statement of the lemma, we have $\varpi(\mathcal{L}_s * \mathbf{A})(\xi) = -f_s * \varpi(\mathbf{A})(\xi)$ as required. □

3.5 Lagrangian cycles and intersection multiplicities.

For the moment, we will assume that equation (3.4.1) holds, and so ϖ is assumed to be a homomorphism. We now show that this implies Theorem 3.1.1, both of Lusztig's conjectures and hence our construction of the Springer representations. The crucial tool in establishing these claims is the index theorem of Byrlinski, Dubson and Kashiwara [BDK], and its variants as described in [G86]. Let us briefly recall their statements. The point is essentially that the Euler characteristic information of a constructible sheaf is encoded in the characteristic cycle, and it may be recovered via suitable intersection multiplicities.

The first problem here is that Lagrangian cycles are not compact, hence if we are to use some kind of local intersection multiplicity to describe stalk Euler characteristics, we must be careful what we mean. We follow the approach of Kashiwara [K85] as expanded by Ginzburg — though the reader should be warned that the discussion of local intersection multiplicities in section 11 of [G86] contains an error [G01] which we fix here. All objects in this section are at least real (sub)analytic

Suppose that X is a complex manifold, and T^*X is its cotangent bundle. For a Lagrangian cycle $\Lambda \subset T^*X$ we first define, for $x \in X$ the local intersection multiplicity $I_x([X], \Lambda)$. Let f be a real-analytic function on X. If V is a complex vector space, the operation of taking real parts gives an isomorphism from $Hom_{\mathbb{C}}(V, \mathbb{C})$ to $Hom_{\mathbb{R}}(V, \mathbb{R})$, hence we may naturally identify the cotangent bundles $T^*(X^{\mathbb{R}})$ and T^*X, where $X^{\mathbb{R}}$ is the underlying real manifold. Thus we may view the differential of f as a Lagrangian cycle in T^*X. As in section 3.3 we will denote this cycle by Λ_f. Equip X with an analytic Riemannian metric, and let $\rho_x : X \to \mathbb{R}$ denote the squared distance from x. Then the following observation follows immediately from the curve selection lemma (see [K85] for details).

Lemma 3.5.1. *Let Λ be a conic Lagrangian cycle in T^*X. Then the (set-theoretic) intersection of Λ with Λ_{ρ_x} is isolated at x.* □

Standard perturb-and-count methods now allow us to define the intersection multiplicity at x of $\Lambda_{\rho_x} \cap \Lambda$ for any conic Lagrangian Λ.

Definition. Given Λ a conic Lagrangian cycle, we set

$$I_x([X], \Lambda) = \Lambda_{\rho_x} \cdot \Lambda.$$

We are now able to state the first of the Index theorems that we need. (Indeed all the others follow from it, once we know how to manipulate characteristic cycles). A readable account of a version of this theorem (in fact a slight generalization of it) is given in [GrM].

Theorem 3.5.2 ([BDK],[D84]). *Let X be a complex manifold, and $F \in \mathbf{D}_c(X)$. For any $x \in X$ we have*

$$\chi_x(F) = I_x([X], CC(F)).$$

□

The reader should note that we use this result on varieties of dimension $2\dim_{\mathbb{C}}(\mathcal{B})$ and hence the sign will always be positive. Next we wish to show how to extend the definition of intersection multiplicities to arbitrary conic Lagrangian cycles, Λ_1, Λ_2. First assume that $\Lambda_1 = T^*_Y X$ for some closed submanifold $Y \subset X$. Then we may use the deformation to the normal cone of $T^*_Y X$ as described in section 3.3.

Because $T^*_Y X$ is Lagrangian, the symplectic form allows us to identify $T_{T^*_Y X} T^*X \simeq T^*(T^*_Y X) = T^*\Lambda_1$. Moreover if we replace Λ_2 with the normal cone $C_{\Lambda_1}(\Lambda_2)$ (see section 3.3) then it is known that this cycle is again conic Lagrangian in $T^*\Lambda_1$ [KSc, Theorem 8.3.17]. Thus for $\xi \in \Lambda_1$ we define

$$I_\xi(\Lambda_1, \Lambda_2) = I_\xi(\Lambda_1, C_{\Lambda_1}(\Lambda_2)).$$

Finally (we include this for completeness only— we never need this case in what follows) for Λ_1, Λ_2 arbitrary, we define the intersection multiplicity by reduction to the diagonal. Let Δ denote the diagonal in $X \times X$. We set

$$I_\xi(\Lambda_1, \Lambda_2) = I_\xi(T^*_\Delta(X \times X), \Lambda_1 \times \Lambda_2).$$

Finally, we can describe the Euler characteristics of the microlocalization of a sheaf along a closed submanifold in terms of the characteristic cycle of the sheaf. To do this we need the following theorem, which follows from [Sa, 4.4] and the fact that the Fourier transform preserves characteristic cycles (once suitable identifications are made, see [KSc, Exercise IX.7]). The corresponding statement for $\mathcal{D}$-modules is given in [G86, section 7].

Theorem 3.5.3. *Let Y be a closed submanifold of X, and $F \in \mathbf{D}_c(X)$. Then using the identification $T_{T^*_Y X}T^*X \simeq T^*(T^*_Y X)$ we have*

$$CC(\mu_Y(F)) = C_{T^*_Y X}(CC(F)).$$

□

Combining this with Theorem 3.5.2 we obtain the following consequence.

Corollary 3.5.4. *Let Y be a closed submanifold of X, let $F \in \mathbf{D}_c(X)$, and let $\xi \in T^*_Y X$. We have*

$$\chi_\xi(\bar{\mu}_Y(F)) = I_\xi([T^*_Y X], CC(F)).$$

In particular, we obtain the formula,

$$\varpi(\mathbf{A})(\xi) = I_\xi([T^*_{\pi(\xi)}\mathcal{B} \times \mathcal{B}], CC(\mathbf{A})),$$

for $\mathbf{A} \in \mathcal{H}$. □

It is this corollary that immediately implies both of Lusztig's conjectures, and gives an explicit, though not really computable, formula for the semicanonical basis functions.

Theorem 3.5.5. *Assume that ϖ is a homomorphism. Then Lusztig's conjectures follow and moreover we have*

$$f_w(\xi) = I_\xi([T^*_{\pi(\xi)}(\mathcal{B} \times \mathcal{B})], [\overline{\mathcal{Z}}_w]).$$

□

We now wish to address equation (3.4.1), which have so far been assuming. The above discussion shows that it is equivalent to a statement about intersection multiplicities. Indeed in this language we must show that

$$I_\xi([T^*_{\pi(\xi)}(\mathcal{B} \times \mathcal{B})], [\overline{\mathcal{Z}}_w]) = I_\xi([T^*_{X_s(\xi)}(\mathcal{B} \times \mathcal{B})], [\overline{\mathcal{Z}}_w]), \tag{3.5.1}$$

for each $w \in W$.

We outline an approach to establishing this equation, which depends on the equality of our previous description of local intersection multiplicities, with an alternative one given by Dubson [D89]. The advantage of his approach is that it does not need to deform to the normal cone, and hence it is easier to compute with. The idea is essentially an elaboration of Lemma 3.5.1. Let Λ_1, Λ_2 be Lagrangian cycles in T^*X as above (though in fact here we do not need them to be conic, only subanalytic) and let ξ be a point in their intersection. Pick an analytic Riemannian metric on T^*X, and consider the resulting function d_x measuring the distance from ξ. Then if $f_t \colon T^*X \to T^*X$ denotes the Hamiltonian flow induced by d_x, Dubson shows that for sufficiently small closed balls $B_\epsilon(\xi)$ about ξ (defined using our fixed Riemannian metric), there is an open interval $(0, c(\epsilon))$ such that for all $t \in (0, c(\epsilon))$ we have $\partial B_\epsilon(\xi) \cap f_t(\Lambda_1) \cap \Lambda_2 = \emptyset$ and hence at least in $B_\epsilon(\xi)$ these cycles intersect properly, and we may define the degree of intersection in the normal way. Let $\mathrm{mult}_\xi(\Lambda_1, \Lambda_2)$ be this degree, where the multiplicity is independent of all choices made (this is essentially the main result of [D89]).

It is easy to check that for any point $x \in X \subset T^*X$, and for a conic Lagrangian Λ, we have $I_x([X], \Lambda) = \mathrm{mult}_x([X], \Lambda)$, and so Theorem 3.5.2 remains true if we replace our original definition of local multiplicity with Dubson's one. However what we need is that the two multiplicities always agree.

Conjecture 3.5.6. *For any conic Lagrangian cycles Λ_1,Λ_2 we have*

$$\mathrm{mult}_\xi(\Lambda_1, \Lambda_2) = I_\xi(\Lambda_1, \Lambda_2).$$

We now show how this assumption establishes equation 3.5.1.

Lemma 3.5.7. *For $\xi \in \mathcal{Z}_w$ we have*

$$\mathrm{mult}_\xi([T^*_{\pi(\xi)}(\mathcal{B} \times \mathcal{B})], [\overline{\mathcal{Z}_w}]) = \mathrm{mult}_\xi([T^*_{X_s(\xi)}(\mathcal{B} \times \mathcal{B})], [\overline{\mathcal{Z}_w}]).$$

Hence Conjecture 3.5.6 implies Conjecture 3.4.1.

Proof. The crucial point of the lemma is that the lines $X_s(\xi)$ sit nicely in the orbit stratification of $\mathcal{B} \times \mathcal{B}$. More precisely, it follows from the Bruhat decomposition that all but one point of $X_s(\xi)$ lies in Y_u for some $u \in W$. The case where $X_s(\xi) \cap \bar{Y}_w = \pi(\xi)$ is easy and can be one separately. Otherwise we have $X_s(\xi) \subset \bar{Y}_w$, and so we know, by Whitney's condition (a) (see [KSc, Exercise VIII.12]), that

$$\overline{\mathcal{Z}}_w|_{X_s(\xi)} \subset T^*_{X_s(\xi)}, \tag{3.5.2}$$

except perhaps at the point $X_s(\xi)\backslash Y_u$. If $\pi(\xi)$ is not this point, since everything is local, we can easily see that the two Lagrangian cycles $T^*_{\pi(\xi}(\mathcal{B} \times \mathcal{B})$ and $T^*_{X_s(\xi)}(\mathcal{B} \times \mathcal{B})$ can be moved into each other in a fashion which preserves the intersection multiplicity. If $\pi(\xi)$ is this point, then we must be slightly more careful. The point is to "rotate" the cycle $T^*_{X_s(\xi)}(\mathcal{B} \times \mathcal{B})$ into $T^*_{\pi(\xi}(\mathcal{B} \times \mathcal{B})$ in such a way that one never creates extra intersection. This can be done straightforwardly in coordinates using (3.5.2). □

Hence we see that the equality of our two definitions of intersection multiplicity would immediately imply equation (3.5.1), and hence the conjectures of Lusztig.

Remark. There are a number of situations in which an algebra has been constructed from a variety via convolution in both Borel-Moore homology and constructible functions. The idea of this chapter is to show that in the case when the underlying variety is the conormal space to a stratification, the two constructions can be derived from a single construction on the base manifold by some sort of microlocalization. However, there are cases when these two constructions can be carried out (for example on Nakajima's quiver varieties [N94], [N98], [L]) where there is no base manifold—the underlying variety is a Lagrangian subvariety of a symplectic manifold which is no longer a cotangent bundle. It would be interesting to give a purely symplectic way of comparing the two constructions which would apply in these cases, and would give an expression for the corresponding semicanonical functions.

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