12 Hénon attractor

The chaotic phenomena of the Lorenz equations may be exhibited by even simpler systems.

We now consider a discrete-time, 2-D mapping of the plane into itself. The points in $\mathbb{R}^2$ are considered to be the Poincaré section of a flow in higher dimensions, say, $\mathbb{R}^3$.

The restriction that $d > 2$ for a strange attractor does not apply, because maps generate discrete points; thus the flow is not restricted by continuity (i.e., lines of points need not be parallel).

12.1 The Hénon map

The discrete time, 2-D mapping of Hénon is

$$X_{k+1} = Y_k + 1 - \alpha X_k^2$$

$$Y_{k+1} = \beta X_k$$

- $\alpha$ controls the nonlinearity.
- $\beta$ controls the dissipation.

Pictorially, we may consider a set of initial conditions given by an ellipse:
Now bend the ellipse, but preserve the area inside it (we shall soon quantify area preservation):

\[
\text{Map } T_1 : \quad X' = X \\
Y' = 1 - \alpha X^2 + Y
\]

Next, contract in the \(x\)-direction (\(|\beta| < 1\))

\[
\text{Map } T_2 : \quad X'' = \beta X' \\
Y'' = Y'
\]

Finally, reorient along the \(x\) axis (i.e. flip across the diagonal).

\[
\text{Map } T_3 : \quad X''' = Y'' \\
Y''' = X''
\]

The composite of these maps is

\[ T = T_3 \circ T_2 \circ T_1. \]

We readily find that \(T\) is the Hénon map:

\[
X''' = 1 - \alpha X^2 + Y \\
Y''' = \beta X
\]
12.2 Dissipation

The rate of dissipation may be quantified directly from the mapping via the Jacobian.

We write the map as

\[ X_{k+1} = f(X_k, Y_k) \]
\[ Y_{k+1} = g(X_k, Y_k) \]

Infinitesimal changes in mapped quantities as a function of infinitesimal changes in inputs follow

\[ df = \frac{\partial f}{\partial X_k} dX_k + \frac{\partial f}{\partial Y_k} dY_k \]

We may approximate, to first order, the increment \( \Delta X_{k+1} \) due to small increments \( (\Delta X_k, \Delta Y_k) \) as

\[ \Delta X_{k+1} \approx \frac{\partial f}{\partial X_k} \Delta X_k + \frac{\partial f}{\partial Y_k} \Delta Y_k \]

When \( (\Delta X_k, \Delta Y_k) \) are perturbations about a point \((x_0, y_0)\), we have, to first order,

\[
\begin{bmatrix}
\Delta X_{k+1} \\
\Delta Y_{k+1}
\end{bmatrix} =
\begin{bmatrix}
f'_{X_k}(x_0, y_0) & f'_{Y_k}(x_0, y_0) \\
g'_{X_k}(x_0, y_0) & g'_{Y_k}(x_0, y_0)
\end{bmatrix}
\begin{bmatrix}
\Delta X_k \\
\Delta Y_k
\end{bmatrix}.
\]

Rewrite simply as

\[
\begin{bmatrix}
\Delta x' \\
\Delta y'
\end{bmatrix} =
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}.
\]

Geometrically, this system describes the transformation of a rectangular area determined by the vertex \((\Delta x, \Delta y)\) to a parallelogram as follows:
Here we have taken account of transformations like

\[(\Delta x, 0) \rightarrow (a\Delta x, c\Delta x)\]
\[(0, \Delta y) \rightarrow (b\Delta y, d\Delta y)\]

If the original rectangle has unit area (i.e., \(\Delta x\Delta y = 1\)), then the area of the parallelogram is given by the magnitude of the cross product of \((a, c)\) and \((b, d)\), or, in general, the Jacobian determinant

\[
J = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} \frac{\partial X_{k+1}}{\partial X_k} & \frac{\partial X_{k+1}}{\partial Y_k} \\ \frac{\partial Y_{k+1}}{\partial X_k} & \frac{\partial Y_{k+1}}{\partial Y_k} \end{vmatrix}_{(x_0, y_0)}
\]

Therefore

\[|J| > 1 \implies \text{dilation}\]
\[|J| < 1 \implies \text{contraction}\]

For the Hénon map,

\[
J = \begin{vmatrix} -2\alpha X_k & 1 \\ \beta & 0 \end{vmatrix} = -\beta
\]

Thus areas are multiplied at each iteration by \(|\beta|\).

After \(k\) iterations of the map, an initial area \(a_0\) becomes

\[a_k = a_0|\beta|^k\]

### 12.3 Numerical simulations

Hénon chose \(\alpha = 1.4, \beta = 0.3\). The dissipation is thus considerably less than the factor of \(10^{-6}\) in the Lorenz model.

An illustration of the attractor is given by

[BPV, Figure VI.19]
Numerical simulations show the basin of attraction to be quite complex.

Sensitivity to initial conditions is confirmed by

BPV, Figure VI.20

The weak dissipation allows one to see the fractal structure induced by the repetitive folding

BPV, Figure VI.21

Note the apparent scale-invariance: at each magnification of scale, we see that the upper line is composed of 3 separate lines.

The fractal dimension $D = 1.26$. (We shall soon discuss how this is computed.)

The action of the Hénon map near the attractor is evident in the deformation of a small circle of initial conditions:

BPV, Figure VI.22

At the scale of the attractor we can see the combined effects of stretching and folding:

BPV, Figure VI.23