3 Conservation of volume in phase space

We show (via the example of the pendulum) that frictionless systems conserve volumes (or areas) in phase space.

Conversely, we shall see, dissipative systems contract volumes.

Suppose we have a 3-D phase space, such that

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, x_3) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
\dot{x}_3 &= f_3(x_1, x_2, x_3)
\end{align*}
\]

or

\[
\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})
\]

The equations describe a “flow,” where \(d\vec{x}/dt\) is the velocity.

A set of initial conditions enclosed in a volume \(V\) flows to another position in phase space, where it occupies a volume \(V'\), neither necessarily the same shape nor size:

Assume the volume \(V\) has surface \(S\).
Let

- $\rho = \text{density of initial conditions in } V$;
- $\rho \vec{f} = \text{rate of flow of points (trajectories emanating from initial conditions) through unit area perpendicular to the direction of flow}$;
- $\text{ds} = \text{a small region of } S$; and
- $\vec{n} = \text{the unit normal (outward) to } \text{ds}$.

Then

\[
\text{net flux of points out of } S = \int_S (\rho \vec{f} \cdot \vec{n}) \text{ds}
\]

or

\[
\int_V \frac{\partial \rho}{\partial t} \text{dV} = - \int_S (\rho \vec{f} \cdot \vec{n}) \text{ds}
\]

i.e., a positive flux $\implies$ a loss of “mass.”

Now we apply the divergence theorem to convert the integral of the vector field $\rho \vec{f}$ on the surface $S$ to a volume integral:

\[
\int_V \frac{\partial \rho}{\partial t} \text{dV} = - \int_V [\nabla \cdot (\rho \vec{f})] \text{dV}
\]

Letting the volume $V$ shrink, we have

\[
\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{f})
\]

Now follow the motion of $V$ to $V'$ in time $\delta t$:

The boundary deforms, but it always contains the same points.
We wish to calculate \( \frac{d\rho}{dt} \), which is the rate of change of \( \rho \) as the volume moves:

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial \rho}{\partial x_2} \frac{dx_2}{dt} + \frac{\partial \rho}{\partial x_3} \frac{dx_3}{dt}
\]

\[
= -\nabla \cdot (\rho \vec{f}) + (\nabla \rho) \cdot \vec{f}
\]

\[
= -(\nabla \rho) \cdot \vec{f} - \rho \nabla \cdot \vec{f} + (\nabla \rho) \cdot \vec{f}
\]

\[
= -\rho \nabla \cdot \vec{f}
\]

Note that the number of points in \( V \) is

\[
N = \rho V
\]

Since points are neither created nor destroyed we must have

\[
\frac{dN}{dt} = V \frac{d\rho}{dt} + \rho \frac{dV}{dt} = 0.
\]

Thus, by our previous result,

\[
-\rho V \nabla \cdot \vec{f} = -\rho \frac{dV}{dt}
\]

or

\[
\frac{1}{V} \frac{dV}{dt} = \nabla \cdot \vec{f}
\]

This is called the *Lie derivative*.

We shall next arrive at the following main results by example:

- \( \nabla \cdot \vec{f} = 0 \Rightarrow \) volumes in phase space are conserved. Characteristic of conservative or Hamiltonian systems.

- \( \nabla \cdot \vec{f} < 0 \Rightarrow \frac{dV}{dt} < 0 \Rightarrow \) volumes in phase space contract. Characteristic of dissipative systems.

We use the example of the pendulum:

\[
\dot{x}_1 = x_2 = f_1(x_1, x_2)
\]

\[
\dot{x}_2 = -\frac{g}{l} \sin x_1 = f_2(x_1, x_2)
\]
Calculate
\[ \nabla \cdot \vec{f} = \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} = 0 + 0 \]

Pictorially

Note that the area is conserved.

Conservation of areas holds for all conserved systems. This is conventionally derived from Hamiltonian mechanics and the canonical form of equations of motion.

In conservative systems, the conservation of volumes in phase space is known as Liouville’s theorem.

4 Damped oscillators and dissipative systems

4.1 General remarks

We have seen how conservative systems behave in phase space. What about dissipative systems?

What is a fundamental difference between dissipative systems and conservative systems, aside from volume contraction and energy dissipation?

- Conservative systems are invariant under time reversal.
- Dissipative systems are not; they are irreversible.