# Equivariant Cohomology, Homogeneous Spaces and Graphs

by

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## Abstract

The focus of this thesis is manifolds with group actions, in particular symplectic manifolds with Hamiltonian torus actions. We investigate the relationship between the equivariant cohomology of the manifold $M$ and the fixed point data of the torus action. We are interested in understanding the topology of the space of $T$-orbits in $M$. In particular, we explore aspects of this topology which are determined by data from the image of a moment map $\Phi : M \to \mathfrak{t}^*$ associated to the Hamiltonian action. To better understand the orbit space, we apply the algebraic techniques of equivariant cohomology to the study these systems further. Equivariant cohomology associates to a manifold with a $G$-action a ring $H_G^*(M)$. Much of the topology of the orbit space is encoded in the equivariant cohomology ring $H_G^*(M)$. In 1998, Goresky, Kottwitz and MacPherson provided a new method for computing this ring. Their method associates to this orbit space a graph $\Gamma$ whose vertices are the zero-dimensional orbits and edges the connected components of the set of one-dimensional orbits. The ring $H_T^*(M)$ can then be computed combinatorially in terms of the data incorporated in $\Gamma$. The strength of this construction is that it makes the computation of equivariant cohomology into a combinatorial computation, rather than a topological one.

In the projects described herein, we apply the GKM theory to the case of homogeneous spaces by studying the combinatorics of their associated graphs. We exploit this theory to understand the geometry of homogeneous spaces with non-zero Euler characteristic. Next, we describe how to weaken the hypotheses of the GKM theorem. The spaces to which the GKM theorem applies must satisfy certain dimension conditions; however, there are many manifolds $M$ with naturally arising $T$-actions that do not satisfy these conditions. We allow a more general situation, which includes some of these cases. Finally, we find a theory identical to the GKM theory in a setting suggested by work of Duistermaat. As in the GKM situation, this theory applies only when the spaces involved satisfy certain dimension conditions.

Thesis Supervisor: Victor Guillemin
Title: Professor of Mathematics

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My family has been a tremendous support network during the past years. I am forever grateful to my mother Suzanne, my father Eric, and my two brothers Jim and Tim for believing in me whenever I had doubts. Without them, I never would have made it as far as MIT. I am also indebted to my Uncle Charlie and cousins Anna and Alex, for allowing me to escape MIT and Cambridge on occasional Friday evenings for pizza and good cheer.

I owe much of my knowledge of symplectic geometry to my mathematical siblings. Most notably, Rebecca Goldin has been a wonderful friend and colleague during my years in graduate school I have also benefited from the mathematical wisdom of Phil Bradley, Allen Knutson, Misha Kogan, and Catalin Zara. Finally, in the last year, I have glimpsed a new view of symplectic geometry from Megumi Harada. I look forward to our collaboration in the years to come.

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I am sure that in my thesis haze, I have forgotten someone, and I do apologize! I thank you, and will miss you all. Do visit in California!

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# Chapter 1

# Introduction

The focus of this thesis is on manifolds with torus actions, and the relation between the equivariant cohomology of these spaces and the fixed point data of the torus action. Of particular interest are symplectic manifolds with Hamiltonian actions. The goals of the thesis are two-fold. First, we use the theory of Goresky, Kottwitz, and MacPherson to understand the geometry of homogeneous spaces with non-zero Euler characteristic. Second, we present several results enlarging the class of manifolds to which this theory applies.

Let $M$ be a compact symplectic manifold equipped with a Hamiltonian action of a torus $T = (S^1)^n$, and let $\Phi : M \to \mathfrak{t}^*$ denote the moment map. The Atiyah Guillemin-Sternberg convexity theorem ([A],[GS1]) states that the image of the moment map $\Phi$ is the convex hull of the image of the fixed points, $\Phi(M^T)$. The image of the moment map is closely related to the space of $T$-orbits in $M$, and we are interested in understanding the topology of this orbit space. In particular, we would like to understand aspects of this topology which are determined by "moment data," such as the image $\Delta = \Phi(M)$.

We can obtain only partial results in the classification of the orbit space using purely geometric techniques. If we apply the algebraic techniques of equivariant cohomology to the study these systems further, we can obtain stronger results. Equivariant cohomology associates to a topological space $M$ with a $G$-action a ring $H^*_G(M)$. Much of the topology of the orbit space is encoded in the equivariant cohomology ring $H^*_G(M)$, and it is hence of great importance to have means of calculating $H^*_G(M)$. Over the past 50 years, several techniques have been proposed to compute $H^*_G(M)$. Nonetheless, the general computation of this ring, even for a Hamiltonian $T$-space, has not been achieved.

In 1998, Goresky, Kottwitz and MacPherson provided a new method for computing this ring [GKM]. Their results apply to spaces which are equivariantly formal and the class of these spaces includes symplectic manifolds with Hamiltonian torus actions. However, the GKM theory works only when the orbit space satisfies certain dimension conditions: that the set of zero-dimensional orbits in the orbit space is zero-dimensional and that the set of one-dimensional orbits is one-dimensional. Their method associates to this orbit

space a graph $\Gamma$ whose vertices are the zero-dimensional orbits and edges the connected components of the set of one-dimensional orbits. The strength of their construction is that it makes the computation of the equivariant cohomology into a combinatorial computation, rather than a geometric one. The methods described below enable us to extend the GKM theory to situations in which hypergraphs, not graphs, are the paramount objects. There are also indications (see [BoGH]) that the graph and hypergraph techniques involved in this research will have interesting applications to combinatorics and representation theory.

In this thesis, we present several results applying and generalizing this theory. In Chapter 2, we make use of the GKM theory in studying homogeneous spaces by examining the combinatorics of their associated graphs. In Chapter 3, we give graph theoretic definitions motivated by the GKM theory, and describe several purely combinatorial results and constructions. In Chapter 4, we describe how to weaken the hypotheses of the GKM theorem. The GKM spaces must satisfy the dimension conditions above; however, there are many manifolds $M$ with naturally arising $T$ actions that do not satisfy these conditions. We allow a more general situation, which includes some of these cases. Finally, in Chapter 5, we describe a mod 2 version of the GKM theory for the *real loci* of symplectic manifolds. As above, this theory applies only when the orbit space of the action satisfies certain dimension conditions. In the rest of this chapter, we will present the background necessary for the remainder of the thesis, and set up the appropriate notation.

## 1.1 Equivariant cohomology

The results in this thesis are, in large part, concerned with computing the equivariant cohomology of $G$-spaces. A standard definition of equivariant cohomology is due to Borel [Bo]. Let $M$ be a topological space with a continuous action of a group $G$. We will be interested in the situation when $M$ is a symplectic manifold, $G$ a compact, connected Lie group, and the action Hamiltonian. In this situation, if $G$ acts freely on $M$, then we would like the equivariant cohomology of $M$ to satisfy

$$H^*_G(M) = H^*(M/G),$$

since in this case, the quotient $M/G$ is again a manifold. When $M$ is a compact symplectic manifold with a Hamiltonian torus action, however, $M$ necessarily has fixed points, and so the action is not free. In general, if $G$ does not act freely, then $M/G$ is not necessarily Hausdorff, and often not even a topological manifold, so we cannot hope to define equivariant cohomology in this way. However, the Borel construction produces a manifold which is homotopy equivalent to $M$, on which $G$ does act freely. Let $EG$ be the classifying bundle of $G$. This space is contractible, and $G$ acts on $EG$ freely. Moreover, $M \times EG$ is homotopy

equivalent to $M$, and since $G$ acts freely on $EG$, the diagonal action of $G$ on $M \times EG$ is also free. The classifying space of $G$ is $BG = EG/G$. For constructions of the classifying bundle and space, see [GS2, pp. 5–6], [M1], and [M2].

**Definition 1.1.1.** *The* Borel space *of a space $M$ with a group action $G$ is*

$$M_G := (M \times EG)/G = M \times_G EG.$$

We use the Borel space to define the equivariant cohomology of $M$.

**Definition 1.1.2.** *The* equivariant cohomology *of a space $M$ with a group action $G$ is the ordinary cohomology of the Borel space,*

$$H^*_G(M) := H^*(M_G).$$

The ordinary cohomology on the right hand side can be thought of as deRham cohomology or as singular cohomology. In some cases, we will use $\mathbb{Z}_2$ coefficients, and in these cases, it will be necessary to interpret the right hand side as singular cohomology.

It is easy to check that using this definition of equivariant cohomology still yields the identity

$$H^*_G(M) = H^*(M/G)$$

when the $G$ action is free.

In ordinary cohomology, the cohomology of a point is just a copy of the coefficient ring. On the other hand, the equivariant cohomology of a point is

$$H^*_G(pt) = H^*(pt \times_G EG) = H^*(EG/G) = H^*(BG).$$

There are two groups $G$ which we will study in this thesis. The first is the circle $G = S^1$, or more generally the compact torus $G = S^1 \times \cdots \times S^1$. To determine $H^*_{S^1}(pt)$, we must determine $BS^1$. In this case, we observe $S^1$ acts freely on $S^{2n-1}$, but $S^{2n-1}$ is not contractible. However, if we take the unit sphere inside $\mathbb{C}^\infty$, $S^1$ acts freely on this space, and it *is* contractible. Thus, we may take $ES^1 = S^{2\infty-1}$ and so $BS^1 = \mathbb{C}P^\infty$. As a result,

$$H^*_{S^1}(pt) = H^*(BS^1) = \mathbb{C}[x],$$

where the cohomology class $x$ has degree two. Moreover, since $T^n = S^1 \times \cdots \times S^1$, the classifying space $BT^n = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty$, and so

$$H^*_{T^n}(pt) = H^*(BT^n) = \mathbb{C}[x_1, \ldots, x_n],$$

where each class $x_i$ has degree two.

The other group we are interested in is the 2-primary torus $G = \mathbb{Z}_2$ or $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. In this case, it is not hard to see that $\mathbb{Z}_2$ acts freely on the unit sphere in $\mathbb{R}^n$ by the antipodal map. If we consider the unit sphere in $\mathbb{R}^\infty$, $\mathbb{Z}_2$ acts freely on this space and it is contractible, and hence $E\mathbb{Z}_2 = S^\infty$ and $B\mathbb{Z}_2 = \mathbb{R}P^\infty$. When we are studying $\mathbb{Z}_2$-actions, we will be interested in cohomology with $\mathbb{Z}_2$ coefficients, wherefore

$$H^*_{\mathbb{Z}_2}(pt; \mathbb{Z}_2) = H^*(B\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2[x],$$

where the class $x$ has degree one. Moreover, since $T^n_{\mathbb{R}} = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, the classifying space is $BT^n_{\mathbb{R}} = \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$, and so

$$H^*_{T^n_{\mathbb{R}}}(pt; \mathbb{Z}_2) = H^*(BT^n_{\mathbb{R}}; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_n],$$

where each class $x_i$ has degree one.

One can also define equivariant de Rham theory, in which one defines equivariant forms $\Omega^*_G(M)$ and an equivariant operator

$$d_G : \Omega^i_G(M) \to \Omega^{i+1}_G(M).$$

The equivariant cohomology is then defined to be the kernel of $d_G$ modulo the image. This is known as the Cartan model for equivariant cohomology, and is detailed in [GS2]. We will use the Borel model here because we will need to use $\mathbb{Z}_2$ coefficients, which we cannot do using the Cartan model.

## 1.2 Equivariant formality

The Borel model for equivariant cohomology allows us easily to see that $H^*_G(M)$ is a module over $H^*_G(pt)$. There is a natural fibration

$$\begin{array}{ccc} M \hookrightarrow & M_G = M \times_G EG & \\ & \downarrow \pi & \\ & BG & \end{array}$$

with fibre $M$. The map $\pi$ induces a map in cohomology,

$$\pi^* : H^*(BG) \to H^*(M_G),$$

defining the module structure. We would like to understand the module structure of $H^*_G(M)$, and in particular we would like to know when $H^*_G(M)$ is a free module over $H^*_G(pt)$.

We determine the module structure of $H_G^*(M)$ by calculating the Leray-Serre spectral sequence converging to $H_G^*(M_G)$. Given the fibration $\pi$ above, let $\mathcal{H}^*(M)$ denote the local coefficient system on $BG$ associated to this fibration. Then the $E_2$-term of the spectral sequence we would like to compute is

$$E_2^{p,q} = H^p(BG; \mathcal{H}^q(M)).$$

**Definition 1.2.1.** *We say that $M$ is* equivariantly formal *if this spectral sequence collapses at the $E_2$ level; that is if*

$$E_2^{p,q} = E_\infty^{p,q}.$$

When $M$ is equivariantly formal, we have the identity

$$H_G^*(M) \cong H^*(M) \otimes H_G^*(pt)$$

as $H_G^*(pt)$-modules. In particular, when $M$ is equivariantly formal, $H_G^*(M)$ is a *free* $H_G^*(pt)$-module.

F. Kirwan [Ki] and V. Ginzburg [Gi] independently proved that in the symplectic setting, one often has equivariant formality.

**Theorem 1.2.2 (Kirwan, Ginzburg).** *Suppose $M$ is a compact symplectic manifold with a Hamiltonian $G$-action. Suppose further that $M$ admits an equivariant symplectic form. Then $M$ is equivariantly formal.*

## 1.3 Localization

The main difference between equivariant cohomology and ordinary cohomology is that it has a much larger coefficient ring, $H_G^*(pt)$. Thus equivariant cohomology is a richer theory than ordinary cohomology, and this extra structure is given in part by the orbit space of the $G$-action. In this section, we explore the module structure of $H_G^*(M)$ over $H_G^*(pt)$. We will closely follow [AB] in terms of exposition. Atiyah and Bott take an algebraic approach to the localization theorem. This same view is given in [GS2]. In the next section, we will state the localization theorem in the symplectic setting, as proved by Kirwan in [Ki]. Kirwan's approach is more geometric, using Morse theoretic techniques.

We will first restrict our attention to the case where $G = T$ is a (compact) torus. Then

$$H_T^*(pt) = \mathbb{C}[x_1, \ldots, x_n] \cong S(\mathfrak{t}^*),$$

and the variables $x_i$ should be viewed as coordinates on the Lie algebra $\mathfrak{t}$ or its complexification $\mathfrak{t}_\mathbb{C}$. The support of a module over this ring is naturally a subset of $\mathfrak{t}_\mathbb{C}$.

Now let $M^T \subseteq M$ be the set of fixed points of the $T$-action. The natural inclusion $r : M^T \hookrightarrow M$ induces a map in equivariant cohomology,

$$r^* : H_T^*(M) \to H_T^*(M^T).$$

The localization theorem describes the support of the kernel and cokernel of this map, and can be stated as follows for tori.

**Theorem 1.3.1 (Localization).** *The kernel and cokernel of the map*

$$r^* : H_T^*(M) \to H_T^*(M^T)$$

*have support in $\bigcup_K \mathfrak{k}_{\mathbb{C}}$, where $K$ runs over the (finite) set of all stabilizer groups not equal to $T$. In particular, both modules have the same rank.*

This says that the kernel and cokernel of $r^*$ are torsion submodules. In particular, if $M$ is equivariantly formal, then $H_T^*(M)$ is a free $H_T^*(pt)$-module, and so in this case, $r^*$ is an injection. In symplectic geometry, this result is Kirwan's injectivity theorem, which we discuss in the next section.

From the localization theorem, one can derive an integration formula. If $F$ is a connected component of $M^T$, let $r_F : F \hookrightarrow M$ be the natural inclusion and $\pi^F$ the projection $\pi^F : F \times BT \to BT$. Recall that both $\pi$ and $\pi^F$ have natural push-forward maps,

$$\pi_* : H_T^*(M) \to H_T^*(pt),$$

and

$$\pi_*^F : H_T^*(F) \to H_T^*(pt).$$

When we are using deRham cohomology, the pushforward $\pi_*$ can be thought of as integration along the fibre. Moreover, let $\nu_F$ denote the normal bundle to $F$ and $E(\nu_F)$ the equivariant Euler class of the normal bundle. Then we have the following formula relating the pushforward of a cohomology class to the restriction to the fixed points.

**Theorem 1.3.2 (Integration formula).** *If $\omega \in H_T^*(M)$, then after localizing,*

$$\pi_*\omega = \int_M \omega = \sum_{F \subseteq M^T} \pi_*^F \left\{ \frac{r_F^*\omega}{E(\nu_F)} \right\}. \qquad \text{(ABBV)}$$

*Notice that the left hand side is in $H_T^*(pt)$. Thus, the sum on the right hand side is a polynomial.*

The integration formula has been stated as such by Atiyah and Bott [AB] and Berline and Vergne [BeV], and is referred to as the Atiyah-Bott Berline-Vergne (ABBV) localization formula.

Thus far, we have considered the case when $G = T$ is a torus. The second case we will be interested in is the case when $G = \mathbb{Z}_2^n$ is a $\mathbb{Z}_2$-torus. This is discussed more thoroughly in [AP, Chapter 3]. We will state the mod 2 localization theorem here for completeness. The $\mathbb{Z}_2$ coefficients are essential.

**Theorem 1.3.3 (** mod 2 **Localization).** *The kernel and cokernel of the map*

$$r^* : H^*_{\mathbb{Z}_2^n}(M; \mathbb{Z}_2) \to H^*_{\mathbb{Z}_2^n}(M^{\mathbb{Z}_2^n}; \mathbb{Z}_2)$$

*are torsion submodules. In particular, both modules have the same rank.*

In particular, if $M$ is equivariantly formal, then $H^*_{\mathbb{Z}_2^n}(M)$ is a free $H^*_{\mathbb{Z}_2^n}(pt)$-module, and so in this case, $r^*$ is an injection.

From the mod 2 localization theorem, one can derive a push forward formula. If $F$ is a connected component of $M^{\mathbb{Z}_2^n}$, let $r_F : F \hookrightarrow M$ be the natural inclusion and $\pi^F$ the projection $\pi^F : F \times B\mathbb{Z}_2^n \to B\mathbb{Z}_2^n$. Both $\pi$ and $\pi^F$ have natural push-forward maps, $\pi_* : H^*_{\mathbb{Z}_2^n}(M; \mathbb{Z}_2) \to H^*_{\mathbb{Z}_2^n}(pt; \mathbb{Z}_2)$ and $\pi^F_* : H^*_{\mathbb{Z}_2^n}(F; \mathbb{Z}_2) \to H^*_{\mathbb{Z}_2^n}(pt; \mathbb{Z}_2)$. In the $\mathbb{Z}_2$ setting, we must use singular cohomology, and so the push-forward can no longer be thought of as an integration. Let $\nu_F$ denote the normal bundle to $F$ and $E(\nu_F)$ the equivariant Euler class of the normal bundle. Then we have the following formula relating the pushforward of a cohomology class to the restriction to the fixed points.

**Theorem 1.3.4 (** mod 2 **integration formula).** *If $\omega \in H^*_{\mathbb{Z}_2^n}(M; \mathbb{Z}_2)$, then after localizing,*

$$\pi_*\omega = \sum_{F \subseteq M^T} \pi^F_* \left\{ \frac{r_F^*\omega}{E(\nu_F)} \right\}.$$

*Notice that the left hand side is in $H^*_{\mathbb{Z}_2^n}(pt; \mathbb{Z}_2)$. Thus, the sum on the right hand side is a polynomial.*

We will also refer to this mod 2 integration formula as the Atiyah-Bott Berline-Vergne (ABBV) localization formula.

## 1.4 Kirwan's injectivity and surjectivity

In the 1980's, Frances Kirwan made a fundamental contribution to the study of Hamiltonian torus actions on symplectic manifolds. In [Ki], she proved the following two theorems. The first theorem relates the equivariant cohomology of $M$ to the equivariant cohomology of the fixed point sets. In the case of isolated fixed points, this lays the groundwork for the GKM theorem. When $M$ is a compact symplectic manifold with an equivariant symplectic form, Kirwan showed that $M$ is equivariantly formal. Thus, $H^*_G(M)$ is a free

$H^*_G(pt)$-module, and so the following theorem is simply the localization theorem in that setting.

**Theorem 1.4.1 (Kirwan).** *Let a torus $T$ act on a compact symplectic manifold $M$ in a Hamiltonian fashion, and let $F$ denote the set of points fixed by $T$ and $r : F \hookrightarrow M$ the natural inclusion. Then the induced map in equivariant cohomology*

$$r^* : H^*_T(M) \to H^*_T(F)$$

*is an injection.*

Kirwan's proof of injectivity studies components $\Phi^\xi$ of the moment map, and analyzes the critical sets of these functions. These are perfect, in fact equivariantly perfect, Morse-Bott functions. This result is particularly useful when $F$ consists of finitely many isolated points. In this case,

$$H^*_T(F) = \bigoplus_{p \in F} H^*_T(pt),$$

which is simply a sum of polynomial rings.

A second result of Kirwan's relates the equivariant cohomology of $M$ to the ordinary cohomology of the symplectic quotient $M//T(\lambda)$. Suppose that $\Phi : M \to \mathfrak{t}^*$ is a moment map, and that $\lambda$ is a regular value of $\Phi$. Suppose further that $T$ acts freely on $\Phi^{-1}(\lambda)$. Then $M//T(\lambda) := \Phi^{-1}(\lambda)/T$ is a manifold, and in fact $M//T(\lambda)$ inherits a natural symplectic form. The inclusion

$$\kappa : \Phi^{-1}(\lambda) \hookrightarrow M$$

induces a map in equivariant cohomology

$$\kappa^* : H^*_T(M) \to H^*_T(\Phi^{-1}(\lambda)) = H^*(M//T(\lambda)).$$

Kirwan's theorem states that this map is surjective.

**Theorem 1.4.2 (Kirwan).** *Let a torus $T$ act on a compact symplectic manifold $M$ in a Hamiltonian fashion, and let $\lambda$ be a regular value of the moment map. Suppose further that $T$ acts freely on $\Phi^{-1}(\lambda)$, and let $\kappa : \Phi^{-1}(\lambda) \hookrightarrow M$ denote the inclusion. Then the induced map in equivariant cohomology*

$$\kappa^* : H^*_T(M) \to H^*_T(\Phi^{-1}(\lambda)) = H^*(M//T(\lambda))$$

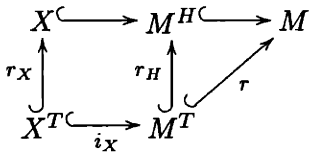
*is a surjection.*

The hypothesis that $T$ acts freely on $\Phi^{-1}(\lambda)$ can be dropped, but in this case, the symplectic reduction $M//T(\lambda)$ is an orbifold. The proof of the surjectivity theorem involves analyzing the critical sets of the map $\|\Phi\|^2$ and using this function as a Morse-Kirwan func-

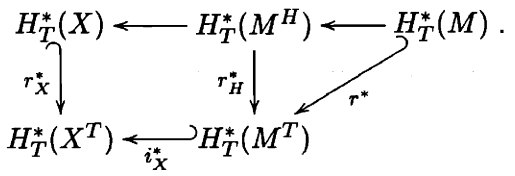
tion. S. Tolman and J. Weitsman have computed the kernel of the map of $\kappa^*$ [TW1]. R. Goldin refined this computation [G].

## 1.5 The Chang-Skjelbred theorem

Suppose $M$ is a compact, connected symplectic manifold with a Hamiltonian torus action of $T = T^n$. Let $H$ be a codimension one subtorus of $T$ and let $X$ be a connected component of $M^H$. Then there are natural inclusion maps




inducing a commutative diagram in equivariant cohomology




We use the notation $r$ for the inclusion $M^T \hookrightarrow M$ because the map $r^*$ is the *restriction* of a class on the manifold to the fixed points. We use the notation $i_X$ for the inclusion $X^T \hookrightarrow M^T$ because the map $i_X^*$ *ignores* the fixed points not in $X^T$. This differs from the standard notation, but will be consistent throughout this thesis.

The content of the Chang-Skjelbred theorem is that the image of $r^*$ is the same as the image of $r_H^*$.

We will make us of the Chang-Skjelbred theorem stated in several different ways. The first way is the standard statement from [CS].

**Theorem 1.5.1 (Chang-Skjelbred).** *The image of $r^* : H_T^*(M) \to H_T^*(M^T)$ is the set*

$$\bigcap_H r_H^*(H_T^*(M^H)),$$

*where the intersection is taken over all codimension-one subtori $H$ of $T$.*

**Remark 1.5.2.** *In fact, the only nontrivial contributions to this intersection are those codimension-one subtori $H$ which appear as isotropy groups of elements of $M$. Since $M$ is compact, there are only finitely many such isotropy groups.*

A proof of the Theorem 1.5.1 can be found in [BrV1] or in [GS2]. These proofs are entirely algebraic, and derive the result from localization, Theorem 1.3.1. There is a Morse

theoretic proof of a second statement of the Chang-Skelbred theorem in [TW2]. This statement uses the notion of the $k$-skeleton of a Hamiltonian $T$-space.

Suppose $M$ is a compact, connected symplectic manifold with a Hamiltonian torus action of $T^n$. We will be interested in the orbits of $T$ inside $M$, and we will refer to them as follows.

**Definition 1.5.3.** *The $k$-skeleton $M^{(k)}$ of $M$ is the set*

$$M^{(k)} = \{x \in M \mid \dim(T \cdot x) \leq k\}$$

*The points of the $k$-skeleton have stabilizer of dimension at least $n - k$.*

S. Tolman and J. Weitsman prove a version of the Chang-Skjelbred theorem using Morse-Kirwan theory to relate the equivariant cohomology of the one-skeleton to the equivariant cohomology of $M$.

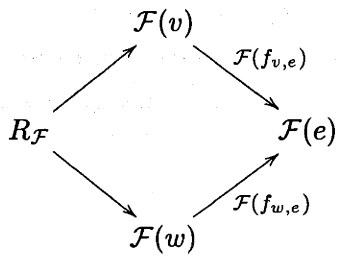
**Theorem 1.5.4 (Tolman-Weitsman).** *There is a natural inclusion $j : M^T \hookrightarrow M^{(1)}$ and combining this with $r : M^T \hookrightarrow M$ we get the following maps in equivariant cohomology*

$$\begin{array}{ccc} H_T^*(M^{(1)}) & \longleftarrow & H_T^*(M) \\ {\scriptstyle j^*}\downarrow & \swarrow_{r^*} & \\ H_T^*(M^T) & & \end{array} .$$

*The images of $r^*$ and $j^*$ are the same.*

The final statement of the Chang-Skjelbred theorem is more algebraic, and is a more general way to think about one-skeleta and graphs. This statement is due to V. Puppe. Suppose $M$ is a $T$-manifold with one-skeleton corresponding to a (hyper)graph $\Gamma = (V, E)$. Associate to this a small category $\mathcal{C}$ whose objects $\mathcal{O}$ are elements of $V \cup E$, and whose morphisms are $f_{v,e}$ between a vertex $v$ and a (hyper)edge $e$ to which it is adjacent.

For every functor $\mathcal{F} : \mathcal{C} \to \mathit{Rings}$ taking $\mathcal{C}$ into the category of graded rings, there is a universal ring $R_{\mathcal{F}}$ with maps $R_{\mathcal{F}} \to \mathcal{F}(v)$ for every vertex such that the following diagrams commute




In this situation, the Chang-Skjelbred Theorem can be stated as follows.

**Theorem 1.5.5 (Puppe).** *If $\mathcal{F}$ is the functor taking $\mathcal{F}(v) = H_T^*(X_v)$, $\mathcal{F}(e) = H_K^*(X_e)$, and $\mathcal{F}(f_{v,e})$ the restriction map $\pi_K : S(\mathfrak{t}^*) \to S(\mathfrak{k}^*)$, where $X_v$ is the connected component of $M^T$ corresponding to $v \in V$ and $X_e$ the connected component of $M^K$ corresponding to $e \in E$ for some codimension 1 subtorus $K$, then $H_T^*(M)$ is the universal ring $R_{\mathcal{F}}$ in this construction.*

## 1.6 Moment maps and graphs

Suppose that $(M, \omega)$ is a symplectic manifold, and that $T$ acts on $M$ in a Hamiltonian fashion. Let $\xi \in \mathfrak{t}$, and let $X_\xi$ be the symplectic vector field on $M$ generated by the infinitesimal action of $\mathfrak{t}$ on $M$. Then because the action is Hamiltonian, we have

$$\iota_{X_\xi}\omega = -d\phi^\xi,$$

and the map $\Phi : M \to \mathfrak{t}^*$ with components $\phi^\xi$ is the moment map. This map is determined up to a constant. In the 1980's, Atiyah [A] and Guillemin-Sternberg [GS1] independently proved the following theorem computing the image of $\Phi$.

**Theorem 1.6.1 (Atiyah,Guillemin-Sternberg).** *Suppose $M$ is a compact symplectic manifold with a Hamiltonian torus action of a torus $T$ with moment map $\Phi$. Then $\Phi(M)$ is a convex polytope, and is the convex hull of the points $\Phi(M^T)$ which are the images of the fixed points of the $T$-action.*

A version of this theorem has also been proved for nonabelian groups, but in the interest of brevity we do not include it here.

**Definition 1.6.2.** *The polytope $\Delta = \Phi(M)$ is called the* moment polytope *of $M$. The $k$-*skeleton *$\Delta^{(k)}$ of $\Delta$ is the image $\Phi(M^{(k)})$ under $\Phi$ of the $k$-skeleton of $M$.*

Notice that the 0-skeleton of $M$ consists of the fixed points $M^T$, and the 0-skeleton of $\Delta$ consists of isolated points in $\mathfrak{t}^*$ corresponding to the connected components of $M^T$. Furthermore, the $k$-skeleton $\Delta^{(k)}$ consists of convex subsets of intersections of hyperplanes of dimension at most $k$. The 1-skeleton of $M$ naturally has the structure of a hypergraph.

**Definition 1.6.3.** *A* hypergraph *$\Gamma = (V, E)$ consists of a set $V$ of vertices and a set $E \subseteq \mathcal{P}(V)$ of hyperedges, which are subsets of $V$.*

The hypergraph associated to a moment polytope has vertices corresponding to the components of the fixed point set $M^T$ and hyperedges corresponding to subsets of $V$ which lie on a closed submanifolds contained in the 1-skeleton $M^{(1)}$.

**Definition 1.6.4.** *We say that $M$ is a* GKM manifold *if*

$$\#M^T < \infty$$

*and*

$$\dim(M^{(1)}) \leq 2.$$

The assumption that $\dim(M^{(1)}) \leq 2$ is a strong one. Let $X_H$ be a component of the one-skeleton which is fixed by a codimension 1 subtorus $H$. If $\dim X_H = 2$, then $X_H$ is symplectomorphic to $S^2$ with a Hamiltonian $S^1 \cong T/H$ action with fixed points denoted $\{N, S\}$. See, for example, [GS2]. In this case, the hypergraph associated to $\Delta$ is particularly nice: it is a *graph*. Each hyperedge consists of exactly 2 points. Let $\Gamma = (V, E)$ be the graph associated to $\Delta$.

The GKM conditions have a very simple and elegant interpretation in terms of the isotropy representations of $T$ at fixed points of $M$.

**Theorem 1.6.5.** *The conditions $\#M^T < \infty$ and $\dim(M^{(1)}) \leq 2$ are satisfied if and only if, for every $p \in M^T$, the weights $\alpha_{i,p}$, $i = 1, \ldots, d$ of the isotropy representation of $T$ on $T_pM$ are pair-wise linearly independent, that is for $i \neq j$, $\alpha_{i,p}$ is not a multiple of $\alpha_{j,p}$.*

For the proof of this, see [GZ1]. This description gives us some additional structure on our graph $\Gamma = (V, E)$. Each edge $e$ corresponds to a sphere fixed by some codimension 1 subtorus $H_e$. We label each edge $e$ with a weight $\alpha_e \in \mathfrak{t}^*$ corresponding to how fast $T/H_e$ is spinning the associated sphere. We now make this precise. We actually think of $E$ as oriented edges, with each undirected edge appearing twice, one with each orientation. If $e = (x, y) \in E$, then we will let $e^{-1} = (y, x)$ be the edge with the reverse orientation. Finally, for every $x \in V$, we let

$$St(x) := \{(x, y) | y \in V \text{ and } (x, y) \in E\}.$$

**Definition 1.6.6.** *An* axial function *is a map*

$$\alpha : E \to \mathfrak{t}^*$$

*satisfying*

1. *For every $x \in V$, the weights $\alpha(e)$ for $e \in St(x)$ are pairwise linearly independent.*
2. *The function $\alpha$ is antisymmetric: $\alpha(e) = -\alpha(e^{-1})$.*
3. *For every edge $e = (x, y) \in E$, there is a map $\nabla_e : St(x) \to St(y)$ such that $(\nabla_e)^{-1} = \nabla_{e^{-1}}$.*
4. *Each map $\nabla_e$ satisfies $\nabla_e(e) = e^{-1}$.*
5. *Let $e = (x, y)$, $d \in St(x)$ and $f \in St(y)$ such that $f = \nabla_e(d)$. For every such triple,*

$$\alpha(f) - \alpha(d) = c \cdot \alpha(e), \tag{1.1}$$

*for some constant $c = c_{d,e,f}$ depending on $d$, $e$, and $f$.*

*When the constants $c = c_{d,e,f}$ are all integers, we say that $\alpha$ is a* GKM axial function.

We will explore the properties of the maps $\nabla$ more closely in Chapter 3. The integrality condition in the last condition on an axial function is interesting combinatorially for the following reason.

**Theorem 1.6.7 (Guillemin-Zara).** *Suppose $\Gamma = (V, E)$ is a graph with a GKM axial function $\alpha$. Then there is an open manifold $M$ with a torus action such that the corresponding GKM graph is $\Gamma$.*

We can associate a ring to the data $\Gamma$ and $\alpha$ as follows. We define the cohomology of the pair $(\Gamma, \alpha)$ to be

$$H^*(\Gamma, \alpha) = \left\{ f : V \to S(\mathfrak{t}^*) \;\middle|\; f(x) - f(y) \in \alpha(e) \cdot S(\mathfrak{t}^*) \quad \forall (x, y) = e \in E \right\}.$$

## 1.7 The Goresky-Kottwitz-MacPherson theorem

Again, there are a few statements of this result, and we will want to use this result in its various guises. The idea is to derive the GKM theorem from the Chang-Skjelbred theorem when the connected components of the one-skeleton are two-dimensional. In this case, it is necessarily true that these components are in fact 2-spheres.

Consider the case in which $G$ acts with isolated fixed points, and $\dim X_H \leq 2$ for all $X_H$. Then $X_H$ is symplectomorphic to $S^2$ with a Hamiltonian $S^1 \cong T/H$ action with fixed points denoted $\{N, S\}$. Theorem 4.2.1 gives an explicit description of $r^*$, which was proved in significant generality in [GKM]. First we find the cohomology of each component $X_H$. Suppose first that $G \cong S^1$. In that case,

$$r^*_{X_H} : H^*_{S^1}(S^2) \hookrightarrow H^*_{S^1}(\{N, S\})$$

is the inclusion induced by $\{N, S\} \subset S^2$. It is clear that constant functions are equivariant classes in degree zero. But $\dim H^0_{S^1}(S^2) = 1$, so these are the only equivariant classes in degree zero. Moreover, $\dim H^{2i}_{S^1}(S^2) = 2$ for $i > 0$, so this is the only restriction in equivariant cohomology. Thus, an equivariant class $f$ must satisfy

$$\frac{f_N}{c \cdot x} + \frac{f_S}{-c \cdot x} \in \mathbb{C}[x], \tag{1.2}$$

where $f_N$ and $f_S$ are the restrictions of $f$ to the points $N$ and $S$, respectively, $c$ is a constant, and $c \cdot x$ is the weight of the $S^1$ action at $T_N S^2$. We have identified the equivariant cohomology of a point $H^*_{S^1}(pt)$ with $\mathbb{C}[x]$. We can think of $c$ as the speed at which $S^1$ is spinning $S^2$.

Let $R$ be the graded ring $H^*_{S^1}(\{N,S\})$ subject to the above restriction. The dimension check above shows that as modules over $H^*_{S^1}(pt)$, $H^*_{S^1}(S^2) = R$. However, the module structure forces the rings to be equal, so that condition (1.2) is the only condition for $f \in im(r^*_{X_H})$.

Suppose now that $G \cong T^n$.

**Proposition 1.7.1.** *Suppose that $S^2$ is a Hamiltonian $G$-spaces for $G \cong T^n$. Let $H$ be a codimension 1 subtorus which acts trivially. Then a function $f = (f_N, f_S) \in S(\mathfrak{t}^*) \oplus S(\mathfrak{t}^*)$ is in the image of $r^* : H^*_G(S^2) \to H^*_G(\{N,S\})$ if and only if*

$$f_N - f_S \in \ker(\pi_H),$$

*where $\pi_H : S(\mathfrak{t}^*) \to S(\mathfrak{h}^*)$ is induced by the projection $\mathfrak{t}^* \to \mathfrak{h}^*$.*

*Proof.* Because $H$ acts trivially on $S^2$,

$$H^*_H(S^2) = H^*(S^2) \otimes S(\mathfrak{h}^*),$$

and thus

$$H^*_T(S^2) = H^*_{S^1}(S^2) \otimes S(\mathfrak{h}^*).$$

But $H^*_{S^1}(S^2) \otimes S(\mathfrak{h}^*)$ is precisely the kernel of $\pi_H$. □

Using this description of $H^*_T(X_H)$, we have the following corollary due to Goresky, Kottwitz and MacPherson [GKM].

**Corollary 1.7.2 (GKM).** *Let $M$ be a compact, symplectic manifold with a Hamiltonian action of a compact torus $T$. Assume that $M^T$ consists of isolated fixed points $\{p_1, \ldots, p_d\}$ and that each component $X_H$ of $M^H$ has dimension 0 or 2 for $H \subset T$ a codimension-1 torus. Let $f_i$ be the restriction of a class $f \in H^*_T(M)$ to the fixed point $p_i$. Let $\pi_H : \mathfrak{g}^* \to \mathfrak{h}^*$ be the projection induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Then the map*

$$r^* : H^*_T(M) \longrightarrow H^*_T(M^T) = \bigoplus_{p \in M^T} H^*_T(pt)$$

*has image $(f_1, \ldots, f_d)$ such that*

$$\pi_H(f_i) = \pi_H(f_j)$$

*whenever $\{p_i, p_j\} = X_H \cap M^T$, where $\pi_H : S(\mathfrak{t}^*) \to S(\mathfrak{h}^*)$ is induced by the projection $\mathfrak{t}^* \to \mathfrak{h}^*$.*

This theorem can also be stated in terms of graphs and the cohomology ring we defined above. This is the most combinatorial statement of the GKM theorem, and this is the description we will use most often throughout this thesis.

**Theorem 1.7.3 (GKM).** *Let $(\Gamma, \alpha)$ be the GKM graph and fixed point data for the Hamiltonian torus action of $T$ on $M$. Then $H_T^*(M)$ injects into $Maps(V, S(\mathfrak{t}^*))$ with image $H^*(\Gamma, \alpha)$.*

## 1.8 Summary of main results

In Chapter 2, we will apply the GKM theory to homogeneous spaces. We will compare the Borel description and GKM description of the equivariant cohomology of a homogeneous space $M = G/K$, and we will compute an explicit isomorphism between the two rings. Then we will explore some additional properties of the GKM theory in the specific case of homogeneous spaces.

In Chapter 3, we will look at the combinatorics of the GKM theory. We will define abstract notions such as *connections*, *axial functions*, *Betti numbers*, and *cohomology* on regular graphs. We will use the connection to compute generators for the cohomology.

In Chapter 4, we will extend the GKM theory to a situation where the one-skeleton is four-dimensional rather than two-dimensional. In this setting, rather than a graph, the one-skeleton of the moment polytope is a hypergraph. We will also explore some of the notions discussed in Chapter 3 for hypergraphs, and discuss the relationship of these with symplectic geometry.

Finally, in Chapter 5, we will extend the GKM theory to the real loci of symplectic manifolds. Duistermaat introduced real loci and proved several results relating the topology of the real locus of a symplectic manifold to the topology of the manifold itself. We show that similar results hold equivariantly, and using equivariant cohomology, we are able to strengthen some of Duistermaat's original results.

Throughout, we will work out several examples in detail.

# Chapter 2

# Homogeneous spaces as GKM manifolds

The fundamental theme in exploiting and generalizing the GKM theory is the study of graphs and how they correspond to manifolds with group actions. The first goal in this thesis is to refine the GKM theory in the case when $M$ is a homogeneous space. The second goal along these lines is to study the related Cayley graphs, which is discussed in Chapter 3.

## 2.1 Preliminaries

Let $T$ be a torus of dimension $n > 1$, $M$ a compact manifold,

$$\tau : T \times M \to M$$

a faithful action of $T$ on $M$, and $M/T$ the orbit space of $\tau$. $M$ is called a *GKM manifold* if the set of zero dimensional orbits in the orbit space $M/T$ is zero dimensional and the set of one dimensional orbits in $M/T$ is one dimensional. Under these hypotheses, the union, $\Gamma \subset M/T$, of the set of zero and one dimensional orbits has the structure of a graph: Each connected component of the set of one-dimensional orbits has at most two zero-dimensional orbits in its closure; so these components can be taken to be the edges of a graph and the zero-dimensional orbits to be the vertices. Moreover, each edge, $e$, of $\Gamma$ consists of orbits of the same orbitype: namely, orbits of the form $\mathcal{O}_e = T/H_e$, where $H_e$ is a codimension one subgroup of $T$. Hence one has a labeling

$$e \to H_e \tag{2.1}$$

of the edges of $\Gamma$ by codimension one subgroups of $T$. When the action of $T$ is a Hamiltonian action on a symplectic GKM manifold $M$, then this graph inside the orbit space is the GKM graph, and the labeling (2.1) is related to the axial function, as discussed in Section 1.6.

It has recently been discovered that if $M$ has either a $T-$invariant complex structure or a $T-$invariant symplectic structure, the data above - the graph $\Gamma$ and the labeling (2.1) - contain a surprisingly large amount of information about the global topology of $M$, namely the equivariant cohomology ring of $M$. Knutson and Rosu have shown that the the ring $K_T(M) \otimes \mathbb{C}$ is also determined by the above data.

The manifolds $M$ which we will be considering below will be neither complex nor symplectic; however we will make an assumption about them which is in some sense much stronger then either of these assumptions. Namely, we will assume that $M$ is a *homogeneous space*. Let $M$ be a $G$ space, where $G$ is a compact, semisimple, connected Lie group with Cartan subgroup $T$. We will assume that $G$ acts transitively on a manifold $M$. Then there is a simple criterion to determine when $M$ is a GKM manifold with respect to the induced $T$-action.

**Theorem 2.1.1.** *Suppose $M$ is a $G$-homogeneous manifold. Then the following are equivalent.*

1. *The action of $T$ on $M$ is a GKM action;*

2. *The Euler characteristic of $M$ is non-zero;*

3. *$M$ is of the form $M = G/K$, where $K$ is a closed subgroup of $G$ containing $T$.*

As we mentioned above, the data (2.1) determine the ring structure of $H_T^*(M)$ if $M$ is either complex or symplectic. This result is, in fact, true modulo an assumption which is weaker than either of these assumptions; and this assumption - equivariant formality - is satisfied by homogeneous spaces which satisfy the hypotheses of the theorem. Hence, for these spaces, one has two completely different descriptions of the ring $H_T^*(M)$: the graph theoretical description above and the classical Borel description, of which we will give an account in Section 2.2.1. In Section 2.2.2, we will compute the graph $\Gamma$ of a space $M$ of the form $G/K$, with $T \subset K$, and show that it is a *homogeneous* graph, i.e. we will show that the Weyl group of $G$, $W_G$, acts transitively on the vertices of $\Gamma$ and that this action preserves the labeling (2.1). We will then use this result to compare the two descriptions of $H_T^*(M)$.

One of the main goals in this chapter is to show that for homogeneous manifolds $M$ of GKM type, some important features of the geometry of $M$ can be discerned from the graph $\Gamma$ and the labeling (2.1). One such feature is the existence of a $G-$invariant almost complex structure. The subgroups, $H_e$, labeling the edges of $\Gamma$ are of codimension one in $T$; so, up to sign, they correspond to weights, $\alpha_e$, of the group $T$. It is known that the $W_K-$invariant

labeling (2.1) can be lifted to a $W_K-$invariant labeling

$$e \to \alpha_e \tag{2.2}$$

if $M$ is a coadjoint orbit of $G$ (hence, in particular, a complex $G-$manifold). Moreover, this labeling satisfies the conditions of an axial function (see Section 1.6). In Section 2.3 we prove the following result.

**Theorem 2.1.2.** *The homogeneous space $M$ admits a $G-$invariant almost complex structure if and only if $\Gamma$ possesses a $W_K-$invariant axial function* (2.2) *compatible with* (2.1).

This raises the issue: Is it possible to detect from the graph theoretic properties of the axial function (2.2) whether or not $M$ admits a $G-$invariant complex structure? Fix a vector $\xi \in \mathfrak{t}$ such that $\alpha_e(\xi) \neq 0$ for all oriented edges, $e$, of $\Gamma$, and orient these edges by requiring that $\alpha_e(\xi) > 0$. We prove in Section 2.4 the following theorem.

**Theorem 2.1.3.** *A necessary and sufficient condition for $M$ to admit a $G-$invariant complex structure is that there exist no oriented cycles in $\Gamma$.*

**Remarks:**

1. $M$ admits a $G-$invariant complex structure if and only if it admits a $G-$invariant symplectic structure; and, by the Kostant-Kirillov theorem, it has either (and hence both) of these properties if and only if it is a coadjoint orbit of $G$.

2. By the Goresky-Kottwitz-MacPherson theorem, the graph $\Gamma$ and the axial function (2.2) determine the cohomology ring structure of $M$. The *additive* cohomology of $M$, i.e. its Betti numbers, $\beta_i$, can be computed by the following simple recipe: For each vertex, $p$, of the graph $\Gamma$, let $\sigma_p$ be the number of oriented edges issuing from $p$ with the property that $\alpha_e(\xi) < 0$. Then

$$\beta_i = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ \#\{p; \sigma_p = i/2\}, & \text{if } i \text{ is even.} \end{cases}$$

One question we do not address in this chapter is the question: When is a labeled graph the GKM graph of a homogeneous space of the form $G/K$ with $T \subset K$? We will provide some partial answers to this question in Chapter 3.

## 2.2 Equivariant cohomology

### 2.2.1 The Borel Construction

Let $G$ be a compact semi-simple Lie group, $T$ a Cartan subgroup of $G$, $K$ a closed subgroup of $G$ such that

$$T \subset K \subset G \,,$$

and let $\mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ be the Lie algebras of $T$, $K$, and $G$.

Let $\Delta_K \subset \Delta_G$ be the roots of $K$ and $G$, with $\Delta_K^+ \subset \Delta_G^+$ sets of positive roots, let

$$\Delta_{G,K} = \Delta_G - \Delta_K \,,$$

and let $W_K \subset W_G$ be the Weyl groups of $K$ and $G$. We will regard an element of $W_G$ both as an element of $N(T)/T$ and as a transformation of the dual Lie algebra $\mathfrak{t}^*$ (or as a transformation of $\mathfrak{t}$, via the isomorphism $\mathfrak{t}^* \simeq \mathfrak{t}$ given by the Killing form). Also, we will assume for simplicity that $G$ is simply connected and that the homogeneous space $G/K$ is oriented.

Now suppose $M$ is a $G$-manifold. Then the equivariant cohomology ring $H_T^*(M)$ is related to the cohomology ring $H_G^*(M)$ by

$$H_T^*(M) = H_G^*(M) \otimes_{S(\mathfrak{t}^*)^{W_G}} S(\mathfrak{t}^*) \,.$$

(see [GS2, Chap. 6]), where $S(\mathfrak{t}^*)$ is the symmetric algebra of $\mathfrak{t}^*$. In particular, let $M = G/K$, where $K$ acts on $G$ by *right* multiplication. Then $G$ acts on $M$ by *left* multiplication and

$$H_G^*(M) = H_G^*(G/K) = S(\mathfrak{k}^*)^K = S(\mathfrak{t}^*)^{W_K} \,,$$

hence

$$H_T^*(G/K) = S(\mathfrak{t}^*)^{W_K} \otimes_{S(\mathfrak{t}^*)^{W_G}} S(\mathfrak{t}^*) \,. \tag{2.3}$$

This is the Borel description of $H_T^*(G/K)$. Throughout this paper, unless stated otherwise, $M$ is the homogeneous space $G/K$.

### 2.2.2 The GKM Description

In the following sections, we will show that homogeneous space $M = G/K$ satisfying the hypotheses of Theorem 2.1.1 is a GKM space, and we will compute its GKM graph. We will then relate the the GKM description of the equivariant cohomology ring of $M$ to the Borel description given above.

### Equivariant formality

The $S(\mathfrak{t}^*)$-module structure of the equivariant cohomology ring $H_T^*(M)$ can be computed by a spectral sequence, as discussed in Chapter 1. We want to show that this spectral sequence collapses at the $E^2$ level. Indeed, if $M = G/K$, with $T \subset K$, then,

$$H^{odd}(M) = 0, \tag{2.4}$$

(see [GHV, p. 467]), and from this it is easy to see that all the higher order coboundary operators in this spectral sequence have to vanish by simple degree considerations. Hence $M$ is equivariantly formal. One implication of equivariant formality is a version of Kirwan's injectivity theorem for homogeneous spaces. We will prove this here, as Kirwan's theorem only applies to symplectic manifolds with Hamiltonian actions.

**Theorem 2.2.1.** *The restriction map*

$$r^* : H_T^*(M) \to H_T^*(M^T) \tag{2.5}$$

*induced by inclusion $r : M^T \hookrightarrow M$ is an injection.*

*Proof.* By a localization theorem of Borel (see [Bo] or [GS2]), the kernel of (2.5) is the torsion submodule of $H_T^*(M)$. However, if $M$ is equivariantly formal, then $H_T^*(M)$ is free as an $S(\mathfrak{t}^*)$-module, so the kernel has to be zero. □

Thus, as in the symplectic case, $H_T^*(M)$ imbeds as a subring of the ring

$$H_T^*(M^T) = \bigoplus_{x \in M^T} S(\mathfrak{t}^*)\,. \tag{2.6}$$

We will give an explicit description of this subring in Section 2.2.3.

### The Euler characteristic

It follows from (2.4) that, if $M$ is a homogeneous space of the form $G/K$, with $T \subseteq K$, then the Euler characteristic of $M$ is equal to

$$\chi(M) = \sum_i \dim H^{2i}(M).$$

In particular, the Euler characteristic is non-zero. It is easy to see that the converse is true as well.

**Proposition 2.2.2.** *If $M = G/K$ and the rank of $K$ is strictly less than the rank of $G$, then the Euler characteristic of $G/K$ is zero.*

*Proof.* Let $h$ be an element of $T$ with the property that

$$\{h^N\ ; -\infty < N < \infty\}$$

is dense in $T$. Suppose that the action of $h$ on $G/K$ fixes a coset $g_0K$. Then $g_0^{-1}hg_0 \in K$, i.e. $h$ is conjugate to an element of $K$ and hence conjugate to an element $h_1$ of the Cartan subgroup $T_1$ of $K$. However, if the iterates of $h$ are dense in $T$, so must be the iterates of $h_1$ and hence $T_1 = T$. Suppose now that $h = \exp\xi, \xi \in \mathfrak{t}$. If $h$ has no fixed points, then the vector field $\xi_M$ can have no zeroes and hence the Euler characteristic of $M$ has to be zero. □

**The fixed points**

We prove in this section that the action of $T$ on $M$ is a GKM action. Hence, we must show that

1. $M^T$ is finite; and
2. For every codimension one subgroup $H$ of $T$, $\dim M^H \leq 2$.

We will show that if $M$ is of the form $G/K$, with $T \subseteq K$, then it has the two properties above, and we will also show that it has the following third property:

3. For every subtorus $H$ of $T$ and every connected component $X$ of $M^H$, $X^T \neq \emptyset$.

It is well known that these properties hold for the homogeneous space $\mathcal{O} = G/T$. The first two properties can be checked directly (see [GZ1], and the third property holds because $\mathcal{O}$ is a compact symplectic manifold, the action of $T$ is Hamiltonian, and every Hamiltonian action has a fixed point. Therefore, to prove that $M$ satisfies properties 1-3, it suffices to prove the following theorem.

**Theorem 2.2.3.** *For every subtorus $H$ of $T$, the map*

$$\mathcal{O} = G/T \to G/K = M \tag{2.7}$$

*sends $\mathcal{O}^H$ onto $M^H$.*

*Proof.* Let $p_0$ be the identity coset in $M$ and $q_0$ the identity coset in $\mathcal{O}$. Let $h$ be an element of $H$ with the property that

$$\{h^N\ ;\ -\infty < N < \infty\}$$

is dense in $H$. If $p = gp_0 \in M^H$, then $g^{-1}hg \in K$; so $g^{-1}hg = ata^{-1}$, with $a \in K$ and $t \in T$. Thus $hga = gat$ and hence $hq = q$, where $q = gaq_0$. But under the map (2.7), $q_0$ is sent to $p_0$, so $q$ is sent to $gap_0 = gp_0 = p$. □

In particular, Theorem 2.2.3 tells us that the map $\mathcal{O}^T \to M^T$ is surjective. However,

$$\mathcal{O}^T = N_G(T)/T = W_G \,,$$

so $M^T$ is the image of $W_G = N_G(T)/T$ in $G/K$. But $N_G(T) \cap K = N_K(T)$, the normalizer of $T$ in $K$, so

$$(N_G(T) \cap K)/T = W_K \,,$$

and hence we proved:

**Proposition 2.2.4.** *There is a bijection*

$$M^T \simeq W_G/W_K \,;$$

*in particular, $W_G = N_G(T)/T$ acts transitively on $M^T$.*

### The one-skeleton

Next we compute the connected components of the sets $M^H$, where $H$ is a codimension one subgroup of $T$. Let $X$ be one of these components. Then $X^T \neq \emptyset$, since the action of $T/H$ on $\mathcal{O}^H$ is Hamiltonian and $\mathcal{O}^H \to M^H$ is surjective. Moreover, since $M$ is simply connected, it is orientable, and hence every connected component of $M^H$ is orientable. So, if $X$ is not an isolated point of $M^H$, then it has to be either a circle, a 2-torus, or a 2-sphere, and the first two possibilities are ruled out by the condition $X^T \neq \emptyset$. We conclude:

**Theorem 2.2.5.** *Let $H$ be a codimension one subgroup of $T$ and let $X$ be a connected component of $M^H$. Then $X$ is either a point or a 2-sphere.*

**Remark 2.2.6.** *By the Korn-Lichtenstein theorem, every faithful action of $S^1$ on the 2-sphere is diffeomorphic to the standard action of "rotation about the z-axis". Therefore the action of the circle $S^1 = T/H$ on the 2-sphere $X$ in the theorem above has to be diffeomorphic to the standard action. In particular, $\#X^T = 2$.*

We now explicitly determine what these 2-spheres are. By Theorem 2.2.3, each of these 2-spheres is the conjugate by an element of $N_G(T)$ of a 2-sphere containing the identity coset $p_0 \in M = G/K$; so we begin by determining the 2-spheres containing $p_0$.

### The space $\mathfrak{g}/\mathfrak{k}$

The tangent space $T_{p_0}M$ can be identified with $\mathfrak{g}/\mathfrak{k}$, and the isotropy representation of $T$ on this space decomposes into a direct sum of two-dimensional $T$-invariant subspaces

$$T_{p_0}M = \oplus V_{[\alpha]} \,, \tag{2.8}$$

labelled by the roots modulo $\pm 1$,

$$\alpha \in \Delta_{G,K}/\pm 1 \,. \tag{2.9}$$

One can also regard this as a labelling by the *positive* roots in $\Delta_{G,K}$; however, since this set of positive roots is not fixed by the natural action of $W_K$ on $\Delta_{G,K}$, this is not an *intrinsic* labelling. (This fact is of importance in Section 2.3, when we discuss the existence of $G$-invariant almost complex structures on $M$.) Now let $H$ be a codimension one subgroup of $T$, let $\mathfrak{h} \subset \mathfrak{t}$ be the Lie algebra of $H$, and let $M^H$ be the set of $H$-fixed points. Then

$$T_{p_0} M^H = (T_{p_0} M)^H \,.$$

Hence, if $X$ is the connected component of $M^H$ containing $p_0$, and if $X$ is not an isolated point, then $(T_{p_0} M)^H$ has to be one of the $V_{[\alpha]}$'s in the sum (2.8). Hence the adjoint action of $H$ on $\mathfrak{g}/\mathfrak{k}$ has to leave $V_{[\alpha]}$ pointwise fixed. However, an element $g = \exp t$ of $T$ acts on $V_{[\alpha]}$ by the rotation

$$\chi_\alpha(g) = \begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix} \,, \tag{2.10}$$

so the stabilizer group of $V_{[\alpha]}$ is the group

$$H_\alpha = \{ g \in T \,;\, \chi_\alpha(g) = 1 \} \,. \tag{2.11}$$

Let $C(H_\alpha)$ be the centralizer of $H_\alpha$ in $G$ and let $G_\alpha$ be the semisimple component of $C(H_\alpha)$. Then $G_\alpha$ is either $SU(2)$ or $SO(3)$, and since $G_\alpha$ is contained in $C(H_\alpha)$, $G_\alpha p_0$ is fixed pointwise by the action of $H$. Moreover, since $G_\alpha \not\subseteq K$, the orbit $G_\alpha p_0$ cannot consist of the point $p_0$ itself; hence

$$G_\alpha p_0 = X \,. \tag{2.12}$$

The Weyl group of $G_\alpha$ is contained in the Weyl group of $G$ and consists of two elements: the identity and a reflection, $\sigma = \sigma_\alpha$, which leaves fixed the hyperplane $\ker \alpha \subset \mathfrak{t}$, and maps $\alpha$ to $-\alpha$. Therefore, since $\alpha \notin \Delta_K$, $\sigma_\alpha p_0 \neq p_0$, and hence $p_0$ and $\sigma_\alpha p_0$ are the two $T$-fixed points on the 2-sphere (2.12).

Now let $p = w p_0$ be another fixed point of $T$, with $[w] \in W_G/W_K$. Let $a$ be a representative for $w$ in $N_G(T)$ and let $L_a : G \to G$ be the left action of $a$ on $G$. If $X$ is the 2-sphere (2.12), then the 2-sphere $L_a(X)$ intersects $M^T$ in the two fixed points $w p_0$ and $w\sigma_\alpha p_0$, and its stabilizer group in $T$ is the group

$$a H_\alpha a^{-1} = w H_\alpha w^{-1} = H_{w\alpha} \,, \tag{2.13}$$

where $H_\alpha$ is the group (2.11).

### The GKM graph of $M$

This concludes our classification of the set of 2-spheres in the one-skeleton of $M$. Now note that if $X$ is such a two-sphere and $H$ is the subgroup of $T$ stabilizing it, then the orbit space $X/T$ consists of two $T$-fixed points and a connected one dimensional set of orbits having the orbitype of $T/H$. Thus these $X$'s are in one-to-one correspondence with the edges of the GKM graph of $M$. Denoting this graph by $\Gamma$, we summarize the graph-theoretical content of what we have proved so far.

**Theorem 2.2.7.** *The GKM data associated to the action of $T$ on the homogeneous space $M = G/K$ consists of a graph $\Gamma$ with the following additional structure.*

*(1) The vertices of $\Gamma$ are in one-to-one correspondence with the elements of $W_G/W_K$;*

*(2) Two vertices $[w]$ and $[w']$ are on a common edge of $\Gamma$ if and only if $[w'] = [w\sigma_\alpha]$ for some $\alpha \in \Delta_{G,K}$;*

*(3) The edges of $\Gamma$ containing the vertex $[w]$ are in one-to-one correspondence with the roots, modulo $\pm 1$, in the set $\Delta_{G,K}$;*

*(4) If $\alpha$ is such a root, then the stabilizer group* (2.1) *labelling the edge corresponding to this root is the group* (2.13).

In particular, the labelling (2.1) of the graph $\Gamma$ can be viewed as a labelling by elements $[\alpha]$ of $\Delta_G/\pm 1$. We call this labelling a *pre-axial function*.

### The connection on $\Gamma$

One last structural component of the graph $\Gamma$ remains to be described: Given any graph, $\Gamma$, and vertex, $p$, of $\Gamma$, let $E_p$ be the set of *oriented* edges of $\Gamma$ with initial vertex $p$. A *connection* on $\Gamma$ is a function which assigns to each oriented edge, $e$, a bijective map

$$\nabla_e : E_p \to E_q \ ,$$

where $p$ is the initial vertex of $e$ and $q$ is the terminal vertex. Every GKM graph has a natural connection. For the graph $\Gamma$ described in Theorem 2.2.7 this connection is the following. Let $e$ be the oriented edge of $\Gamma$ joining $[w]$ to $[w\sigma_\alpha]$. If $e' \in E_{[w]}$ is the oriented edge joining $[w]$ to $[w\sigma_\delta]$, then $\nabla_e(e') = e''$, where $e''$ is the edge joining $[w\sigma_\alpha]$ and $[w\sigma_\alpha\sigma_\delta]$. This connection is compatible with the pre-axial function (2.1) in the sense that, for every vertex $p$, and every pair of oriented edges, $e, e' \in E_p$, the roots labelling $e, e'$, and $e'' = \nabla_e(e')$ are coplanar in $\mathfrak{t}^*$.

**Simplicity**

A graph is said to be *simple* if every pair of vertices is joined by at most one edge. Most of the graphs above do not have this property. There is however an important class of subgroups, $K$, for which the graph associated with $G/K$ does have this property.

**Theorem 2.2.8.** *If $K$ is the stabilizer group of an element of $\mathfrak{t}$, then the graph $\Gamma$ is simple.*

*Proof.* A root $\alpha \in \Delta_G$ is in $\Delta_K$ if and only if the restriction of $\alpha$ to the subspace $\mathfrak{t}^{W_K}$ of $\mathfrak{t}$ is zero. Let $\alpha, \delta \in \Delta_{G,K}$ such that $\alpha \neq \pm\delta$, and let $\sigma_\alpha, \sigma_\delta$ be the reflections of $\mathfrak{t}$ defined by $\alpha$ and $\delta$. Then $\sigma_\alpha \neq \sigma_\delta$ and the subspace of $\mathfrak{t}$ fixed by $\sigma_\alpha\sigma_\delta$ is the codimension 2 subspace on which both $\alpha$ and $\delta$ vanish. If $\sigma_\alpha\sigma_\delta \in W_K$, then this subspace contains $\mathfrak{t}^{W_K}$, so $\alpha$ and $\delta$ are both vanishing on $\mathfrak{t}^{W_K}$, contradicting our assumption that $\alpha, \delta \notin \Delta_K$. □

Another way to prove Theorem 2.2.8 is to observe that $M = G/K$ is a coadjoint orbit of the group $G$. In particular, it is a Hamiltonian $T$-space and $\Gamma$ is the one-skeleton of its moment polytope.

### 2.2.3 The GKM definition of the cohomology ring

We recall how the data encoded in the GKM graph determines the equivariant cohomology ring $H_T^*(M)$. The inclusion $r : M^T \to M$ induces a map in cohomology

$$r^* : H_T^*(M) \to H_T^*(M^T) = \text{Maps}(M^T, S(\mathfrak{t}^*)) = \text{Maps}(W_G/W_K, S(\mathfrak{t}^*)) ,$$

and the fact that $M$ is equivariantly formal implies that $r^*$ is injective. Let $H^*(\Gamma, \alpha)$ be the set of maps

$$f : W_G/W_K \to S(\mathfrak{t}^*) \tag{2.14}$$

that satisfy the compatibility condition:

$$f([w\sigma_\alpha]) - f([w]) \in (w\alpha)S(\mathfrak{t}^*) . \tag{2.15}$$

for every edge $([w], [w\sigma_\alpha])$ of $\Gamma$.

The Goresky, Kottwitz and MacPherson theorem [GKM], Theorem 1.7.3 asserts that

$$H_T^*(M) \simeq r^*(H_T^*(M)) = H^*(\Gamma, \alpha) .$$

In the next section we construct a direct isomorphism between this ring $H_T^*(M)$ and the Borel ring given in (2.3).

### 2.2.4 Equivalence between the Borel picture and the GKM picture

From the inclusion, $r$, of $M^T$ into $M$, one gets a restriction map

$$i^* : H_T^*(M) \to H_T^*(M^T) \; ; \tag{2.16}$$

and, since $M$ is equivariantly formal, $i^*$ maps $H_T^*(M)$ bijectively onto the subring $H_T^*\Gamma$ of $H_T^*(M^T)$. However, as we pointed out in Section 2.2.1,

$$H_T^*(M) \simeq S(\mathfrak{t}^*)^{W_K} \otimes_{S(\mathfrak{t}^*)^{W_G}} S(\mathfrak{t}^*) \; ;$$

so, by combining (2.16) and (2.3), we get an isomorphism

$$\mathcal{K} : S(\mathfrak{t}^*)^{W_K} \otimes_{S(\mathfrak{t}^*)^{W_G}} S(\mathfrak{t}^*) \to H_T^*(\Gamma) \; . \tag{2.17}$$

The purpose of this section is to give an explicit formula for this map. Note that since $M^T$ is a finite set,

$$H_T^*(M^T) = \bigoplus_{p \in M^T} H_T^*(p) = \bigoplus_{p \in M^T} S(\mathfrak{t}^*) = \mathrm{Maps}(M^T, S(\mathfrak{t}^*)) \; .$$

**Theorem 2.2.9.** *On decomposable elements, $f_1 \otimes f_2$, of the product* (2.3),

$$\mathcal{K}(f_1 \otimes f_2) = g \in \mathit{Maps}(M^T, S(\mathfrak{t}^*)) \; , \tag{2.18}$$

*where, for $w \in W_G$ and $p = wp_0 \in M^T$,*

$$g(wp_0) = (wf_1)f_2 \; . \tag{2.19}$$

*Proof.* We first show that (2.18) and (2.19) do define a ring homomorphism of the ring (2.3) into $H^*(\Gamma, \alpha)$. To show that (2.19) doesn't depend on the representative $w$ chosen, we note that if $wp_0 = w'p_0$, then $\sigma = w(w')^{-1} \in W_K$. Thus

$$g(w'p_0) = (w'f_1)f_2 = (w\sigma f_1)f_2 = (wf_1)f_2 = g(wp_0) \; ,$$

since $f_1 \in S(\mathfrak{t}^*)^{W_K}$. Next, we note that if $f \in S(\mathfrak{t}^*)^{W_G}$, then

$$\mathcal{K}(f_1 f \otimes f_2) = \mathcal{K}(f_1 \otimes f f_2) \; ,$$

since

$$w(f_1 f)f_2 = (wf_1)(wf)f_2 = (wf_1)f f_2 \; .$$

Thus, by the universality property of tensor products, $\mathcal{K}$ does extend to a mapping of the

ring (2.3) into the ring $\mathrm{Maps}(M^T, S(\mathfrak{t}^*))$. Next, let $\alpha$ be a root and let $\sigma \in W_G$ be the reflection that interchanges $\alpha$ and $-\alpha$ and that is the identity on the hyperplane

$$\mathfrak{h} = \{\xi \in \mathfrak{t} \,;\, \alpha(\xi) = 0\} \,.$$

Suppose that $p$ and $p'$ are two adjacent vertices of $\Gamma$ with $p' = \sigma p$. To show that $g = \mathcal{K}(f_1 \otimes f_2)$ is in $H^*(\Gamma, \alpha)$, we must show that the quotient

$$\frac{g(p') - g(p)}{\alpha}$$

is in $S(\mathfrak{t}^*)$. However, if $p = wp_0$, then

$$g(p') - g(p) = (\sigma w f_1 - w f_1) f_2 \,,$$

and since $\sigma$ is the identity on $\mathfrak{h}$, the restriction of the polynomial $w\sigma f_1$ to $\mathfrak{h}$ is equal to the restriction of the polynomial $wf_1$ to $\mathfrak{h}$; hence

$$\frac{g(p') - g(p)}{\alpha} \in S(\mathfrak{t}^*) \,.$$

Finally, we show that the map $\mathcal{K}$ defined by (2.18) and (2.19) has the same equivariance properties with respect to the action of the Weyl group $W_G$ as does the map (2.17). Note that under the identification (2.3), the action of $W_G$ on $H_T^*(M)$ becomes the action

$$w(f_1 \otimes f_2) = f_1 \otimes w f_2 \,,$$

since in the right hand side of (2.3), the first factor is $H_G^*(M)$, so $W_G$ acts trivially on it. In particular,the ring of $W_G$-invariants in $H_T^*(M)$ is

$$S(\mathfrak{t}^*)^{W_K} \otimes_{S(\mathfrak{t}^*)^{W_G}} S(\mathfrak{t}^*)^{W_G} = S(\mathfrak{t}^*)^{W_K} \,,$$

which is consistent with the identifications

$$H_G^*(M) = S(\mathfrak{k}^*) = S(\mathfrak{t}^*)^{W_K} = H_T^*(M)^{W_G} \,. \tag{2.20}$$

On the other hand, the action of $W_G$ on the space

$$H_T^*(M^T) = \mathrm{Maps}(M^T, S(\mathfrak{t}^*))$$

is just the action

$$(wg)(p) = w(g(w^{-1}p)) \,;$$

so to check that the map $\mathcal{K}$ defined by (2.18) and (2.19) is $W_G$-equivariant, we must show that if

$$g = \mathcal{K}(f_1 \otimes f_2) \quad \text{and } g^w = \mathcal{K}(f_1 \otimes wf_2) \ ,$$

then for all points $p = \sigma p_0$,

$$g^w(p) = (wg)(p) \ .$$

However,

$$g^w(p) = (\sigma f_1)(wf_2) = w((w^{-1}\sigma f_1)f_2) = wg(w^{-1}p) = (wg)(p) \ .$$

Let us now prove that the map $\mathcal{K}$ coincides with the map (2.17). We first note that $\mathcal{K}$ is a morphism of $S(\mathfrak{t}^*)$-modules. For $f \in S(\mathfrak{t}^*)$,

$$\mathcal{K}(f_1 \otimes f_2 f) = \mathcal{K}(f_1 \otimes f_2)f \ .$$

Thus, it suffices to verify that $\mathcal{K}$ agrees with the map (2.17) on elements of the form $f_1 \otimes 1$. That is, in view of the identification (2.20), it suffices to show that $\mathcal{K}$, restricted to $S(\mathfrak{t}^*)^{W_K} \otimes 1$, agrees with the map (2.17), restricted to $H_T^*(M)^{W_G}$. However, if $f \in H_T^*(M)^{W_G}$, then $r^*f \in H_T^*(M^T)^{W_G}$, so it suffices to show that $r^*f$ and $\mathcal{K}(f \otimes 1)$ coincide at $p_0$, the identity coset of $M = G/K$. This is equivalent to showing that in the diagram below

$$\begin{array}{ccccc} H_G^*(M) & \longrightarrow & H_K^*(M) & \longrightarrow & H_K^*(p_0) \\ \downarrow & & & & \downarrow \\ S(\mathfrak{k}^*)^{W_K} & & \longrightarrow & & S(\mathfrak{k}^*)^{W_K} \end{array}$$

the bottom arrow is the identity map. However, the bottom arrow is clearly the identity on $S^0(\mathfrak{k}^*)^K = \mathbb{C}$ and the two maps on the top line are $S(\mathfrak{k}^*)^K$-module morphisms. □

## 2.3 Almost complex structures and axial functions

### 2.3.1 Axial functions

A $G$-invariant almost complex structure on $M = G/K$ is determined by an almost complex structure on the tangent space $T_{p_0}M$,

$$J_{p_0} : T_{p_0}M \simeq \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{k} \ .$$

For an arbitrary point $gp_0 \in M$, the almost complex structure on

$$T_{gp_0}M = (dL_g)_{p_0}(T_{p_0}M)$$

is given by

$$J_{gp_0}((dL_g)_{p_0}(X)) = (dL_g)_{p_0}(J_{p_0}(X)) \ ,$$

for all $X \in \mathfrak{g}/\mathfrak{k}$. This definition is independent on the representative $g$ chosen if and only if $J_{p_0}$ is $K$-invariant. Therefore $G$-invariant almost complex structures on $G/K$ are in one to one correspondence with $K$-invariant almost complex structures on $\mathfrak{g}/\mathfrak{k}$.

If $M = G/K$ has a $G$-invariant almost complex structure, then the isotropy representations of $T$ on $T_{p_0}M$ is a complex representation, and therefore its weights are well-defined (not just well-defined up to sign). Let

$$T_{p_0}M = \mathfrak{g}/\mathfrak{k} = \bigoplus_{[\delta]} V_{[\delta]}$$

be the root space decomposition of $\mathfrak{g}/\mathfrak{k}$. Then $V_{[\delta]}$ is a one-dimensional complex representation of $T$; let $\widetilde{\delta} \in \{\pm\delta\}$ be the weight of this complex representation:

$$\exp t \cdot X_{\widetilde{\delta}} = e^{i\widetilde{\delta}(t)} X_{\widetilde{\delta}} \quad , \text{ for all } t \in \mathfrak{t} \ .$$

Thus, the map

$$s : \Delta_{G,K}/\pm 1 \to \Delta_{G,K} \ , \ s([\delta]) = \widetilde{\delta} \ , \tag{2.21}$$

is a $W_K$-equivariant right inverse of the projection $\Delta_{G,K} \to \Delta_{G,K}/\{\pm 1\}$. Let $\Delta_0 \subset \Delta_{G,K}$ be the image of $s$.

The existence of a map (2.21) is equivalent to the condition

$$w\alpha \neq -\alpha \quad , \text{ for all } w \in W_K \ , \ \alpha \in \Delta_{G,K} = \Delta_G - \Delta_K \ , \tag{2.22}$$

hence (2.22) is a necessary condition for the existence of a $G$-invariant almost complex structure on $M$. We will see in the next section that this condition is also sufficient.

We can now define a labelling of the *oriented* edges, $E_\Gamma$, of the GKM graph $\Gamma$, as follows. Let $[w] \in W_G/W_K$ be a vertex of the graph and let $e = ([w], [w\sigma_\delta])$ be an oriented edge of the graph, with $\delta \in \Delta_0$. This edge corresponds to the subspace $V_{[w\delta]}$ (see (2.13)) in the decomposition

$$T_{[w]}M = \bigoplus_{\delta \in \Delta_0} V_{[w\delta]} \ ,$$

and the $G$-invariance of the almost complex structure implies that $T$ acts on $V_{[w\delta]}$ with weight $w\delta$. We define $\alpha : E_\Gamma \to \mathfrak{t}^*$ by

$$\alpha([w], [w\sigma_\delta]) = w\delta \ , \text{ for all } \delta \in \Delta_0, w \in W_G \ . \tag{2.23}$$

**Theorem 2.3.1.** *The map $\alpha : E_\Gamma \to \mathfrak{t}^*$ has the following properties:*

1. *If $e_1$ and $e_2$ are two oriented edges with the same initial vertex, then $\alpha(e_1)$ and $\alpha(e_2)$ are linearly independent;*

2. *If $e$ is an oriented edge and $e^{-1}$ is the same edge, with the opposite orientation, then $\alpha(e^{-1}) = -\alpha(e)$;*

3. *If $e$ and $e'$ are oriented edge with the same initial vertex, and if $e'' = \nabla_e(e')$, then $\alpha(e'') - \alpha(e')$ is a multiple of $\alpha(e)$.*

*Proof.* The first assertion is a consequence of the fact that the only multiples of a root $\alpha$ that are roots are $\pm\alpha$.

If $e$ is the oriented edge that joins $[w]$ to $[w\sigma_\delta]$ and that is labelled by $w\delta \in w\Delta_0$, then

$$\alpha(e^{-1}) = (w\sigma_\delta)(\delta) = -w\delta = -\alpha(e^{-1}) .$$

Finally, if $e$ joins $[w]$ to $[w\sigma_\delta]$ and if $e'$ joins $[w]$ to $[w\sigma_\gamma]$ (with $\delta, \gamma \in \Delta_0$), then $e''$ joins $[w\sigma_\delta]$ to $[w\sigma_\delta\sigma_\gamma]$, and

$$\alpha(e'') - \alpha(e) = w\sigma_\delta\gamma - w\gamma = w(\sigma_\delta\gamma - \gamma) = -\langle\gamma, \delta\rangle w\delta = -\langle\gamma, \delta\rangle\alpha(e) .$$

□

Equivalently, Theorem 2.3.1 says that $\alpha : E_\Gamma \to \mathfrak{t}^*$ is an *axial function* compatible with the connection $\nabla$.

### 2.3.2 Invariant almost complex structures

As we have seen in Section 2.3.1, (2.22) is a necessary condition for the existence of a $G$-invariant almost complex structure on $M = G/K$; in this section we show that it is also a sufficient condition.

**Theorem 2.3.2.** *If the condition*

$$w\alpha \neq -\alpha \quad , \text{ for all } w \in W_K \ , \ \alpha \in \Delta_{G,K} = \Delta_G - \Delta_K \ ,$$

*is satisfied, then $M$ admits a $G$-invariant almost complex structure.*

*Proof.* Consider the complex representation of $K$ on $(\mathfrak{g}/\mathfrak{k})_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}$ and let

$$(\mathfrak{g}/\mathfrak{k})_{\mathbb{C}} = \bigoplus_j V_j$$

be the decomposition into irreducible representations; $(\mathfrak{g}/\mathfrak{k})_{\mathbb{C}}$ is self dual, hence

$$\bigoplus_j V_j = (\mathfrak{g}/\mathfrak{k})_{\mathbb{C}} = (\mathfrak{g}/\mathfrak{k})^*_{\mathbb{C}} = \bigoplus_j V_j^* = \bigoplus_j \overline{V_j}$$

Therefore $\overline{V_j} = V_\ell$ for some $\ell$. If $\alpha$ is a highest weight of $V_j$, then condition (2.22) implies that $-\alpha$ is not a weight of $V_j$; however, $-\alpha$ is a weight of $\overline{V_j}$, hence $\overline{V_j} \neq V_j$. Therefore

$$(\mathfrak{g}/\mathfrak{k})_{\mathbb{C}} = \bigoplus_j (V_j \oplus \overline{V_j}) = U \oplus \overline{U}$$

as complex $K$-representations, and this induces a $K$-invariant almost complex structure

$$J : \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{k}$$

as follows: If $x \in \mathfrak{g}/\mathfrak{k}$, then there exists a unique $y \in \mathfrak{g}/\mathfrak{k}$ such that $x + iy \in U$, and we define $J(x) = y$. As we have shown before, this is equivalent to the existence of a $G$-invariant almost complex structure on $M$. □

An alternative way of proving Theorem 2.3.2 is to observe that the condition (2.22) is equivalent to the existence of a $W_K$-equivariant section $s : \Delta_{G,K}/\pm 1 \to \Delta_{G,K}$. Let $s$ be such a section and let $\Delta_0 \subset \Delta_G - \Delta_K$ be the image of $s$. Then (see (2.8))

$$\mathfrak{g}/\mathfrak{k} = \bigoplus_{\alpha \in \Delta_0} V_{[\alpha]}$$

and one can define a $K$-invariant almost complex structure $J$ by requiring that for each $\alpha \in \Delta_0$, $J$ acts on $V_{[\alpha]}$ by

$$J \begin{pmatrix} X_\alpha \\ X_{-\alpha} \end{pmatrix} = \begin{pmatrix} X_{-\alpha} \\ -X_\alpha \end{pmatrix} . \tag{2.24}$$

## 2.4 Morse theory on the GKM graph

### 2.4.1 Betti numbers

Henceforth we assume that $M$ admits a $G$-invariant almost complex structure, determined (see (2.24)) by the image $\Delta_0 \subset \Delta_{G,K}$ of a section $s : \Delta_{G,K}/\pm 1 \to \Delta_{G,K}$. Let $\Gamma$ be the GKM graph of $M$ and let

$$\alpha : E_\Gamma \to \mathfrak{t}^*$$

be the axial function (2.23). Then the edges whose initial vertex is the identity coset in $W_G/W_K$ are labelled by vectors in $\Delta_0$.

Let $\xi \in \mathfrak{t}$ be a regular element of $\mathfrak{t}$, *i.e.*

$$\delta(\xi) \neq 0 \quad , \quad \text{for all } \delta \in \Delta_G \subset \mathfrak{t}^* \, .$$

For a vertex $[w] \in W_G/W_K$, define the *index* of $[w]$ to be

$$ind_{[w]}(\xi) = \#\{e \in E_{[w]} \; ; \; \alpha(e)(\xi) < 0\} \, ,$$

and for each $k \geq 0$, let the $k$th *Betti number* of $\Gamma$ be defined by

$$\beta_k(\Gamma) = \#\{[w] \in W_G/W_K \; ; \; ind_{[w]} = k\} \, .$$

The index of a vertex obviously depends on $\xi$, but the Betti numbers do not. This is shown in [GZ1], and we will prove this abstractly for graphs in Chapter 3.

In general these Betti numbers are not equal to the Betti numbers

$$\beta_{2k}(M) = \dim H^{2k}(M)$$

of $M = G/K$; however, we show in the next section that there is a large class of homogeneous spaces for which they are equal. One should note that $\beta_{2k}(M)$ is the dimension of the *ordinary* cohomology of $M$ as a *vector space*, while $\beta_k(\Gamma)$ counts the number of generators of degree $2k$ of the *equivariant* cohomology ring of $M$, as a *free module* over the symmetric algebra $S(\mathfrak{t}^*)$.

### 2.4.2 Morse functions

Let $\xi \in \mathfrak{t}$ be a regular element.

**Definition 2.4.1.** *A function $f : W_G/W_K \to \mathbb{R}$ is called a* Morse function *compatible with $\xi$ if for every oriented edge $e = ([w], [w'])$ of the GKM graph, the condition $f([w']) > f([w])$ is satisfied whenever $\alpha(e)(\xi) > 0$.*

Morse functions do not always exist; however, there is a simple necessary and sufficient condition for the existence of a Morse function. Every regular element $\xi \in \mathfrak{t}$ determines an orientation $o_\xi$ of the edges of $\Gamma$: an edge $e \in E_\Gamma$ points upward (with respect to $\xi$) if $\alpha_e(\xi) > 0$, and points downward if $\alpha_e(\xi) < 0$. The associated directed graph $(\Gamma, o_\xi)$ is the graph with all upward-pointing edges.

**Proposition 2.4.2.** *There exists a Morse function compatible with $\xi$ if and only if the directed graph $(\Gamma, o_\xi)$ has no cycles.*

### 2.4.3 Invariant complex structures

In this section we show that the existence of Morse functions on the GKM graph, which is a combinatorial condition, has geometric implications for the space $M = G/K$.

**Theorem 2.4.3.** *The GKM graph $(\Gamma, \alpha)$ admits a Morse function compatible with a regular $\xi \in \mathfrak{t}$ if and only if the almost complex structure determined by $\alpha$ is a $K$-invariant complex structure on $M$. Moreover, if this is the case, then the combinatorial Betti numbers agree with the topological Betti numbers. That is,*

$$\beta_k(\Gamma) = \beta_{2k}(M) \quad .$$

*Proof.* Let $f : W_G/W_K \to \mathbb{R}$ be a Morse function compatible with $\xi$, and let $[w]$ be a vertex of the GKM graph where $f$ attains its minimum. If we replace $\xi$ by $w^{-1}(\xi)$ and $f$ by $(w^{-1})^* f$, then the minimum of this new function is $p_0$. Thus, without loss of generality, we may assume that the minimum vertex $[w]$ is the identity coset in $W_G/W_K$. Then

$$\Delta_0 = \{\delta \in \Delta_{G,K} \; ; \; \delta(\xi) > 0\} \; ,$$

hence $\Delta_0$ is the intersection of $\Delta_{G,K}$ with the positive Weyl chamber determined by $\xi$. Let

$$\mathfrak{p} = \mathfrak{k}_{\mathbb{C}} \oplus \left( \bigoplus_{\delta \in \Delta_0} \mathfrak{g}_\delta \right) .$$

Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, hence the almost complex structure determined by $\alpha$ is actually a complex structure.

If $G_{\mathbb{C}}$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and if $P$ is the Lie subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{p}$, then

$$M = G/K = G_{\mathbb{C}}/P \; ,$$

hence $M$ is a coadjoint orbit of $G$. Then $M$ is a Hamiltonian $T$-space and the GKM graph of $M$ is the 1-skeleton of the moment polytope, and therefore the combinatorial Betti numbers agree with the topological Betti numbers.

On the other hand, if the almost complex structure is integrable then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $M = G/K \subset \mathfrak{g}^*$ is a coadjoint orbit of $G$. Let $\Phi : G/K \to \mathfrak{g}^*$ be the moment map, that is inclusion as coadjoint orbit. For a generic direction $\xi \in \mathfrak{t} \subset \mathfrak{g}$, the map $f : W_G/W_K \to \mathbb{R}$ given by

$$f([w]) = \langle \Phi([w]), \xi \rangle$$

(with $W_G/W_K \to G/K \to \mathfrak{g}^*$) is a Morse function on the GKM graph compatible with $\xi$. □

## 2.5 Examples

### 2.5.1 Non-existence of almost complex structures

Let $G$ be a compact Lie group such that $\mathfrak{g}_{\mathbb{C}}$ is the simple Lie algebra of type $B_2$. Let $\alpha_1, \alpha_1 +$

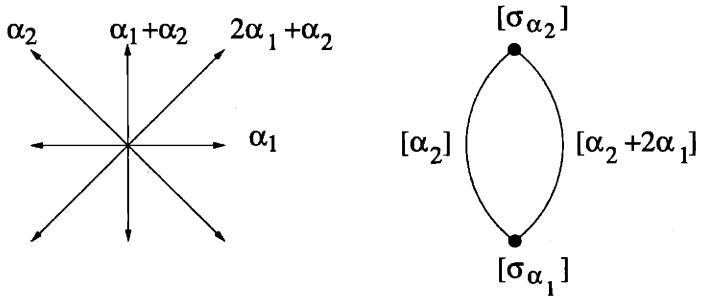



Figure 2-1: The weights of $SO(5)$ and graph for the homogeneous space $SO(5)/(SU(2) \times SU(2))$.

$\alpha_2$ be the short positive roots and let $\alpha_2, \alpha_2 + 2\alpha_1$ be the long positive roots. Let $K$ be the subgroup of $G$ corresponding to the root system consisting of the short roots. Then $\mathfrak{k}_{\mathbb{C}} = D_2 = A_1 \times A_1$ and $K \simeq SU(2) \times SU(2)$. The quotient $W_G/W_K$ has two classes: the class of $\sigma_{\alpha_1} \in W_K$ and the class of $\sigma_{\alpha_2} \in W_G - W_K$.

The GKM graph $\Gamma$ has two vertices, joined by two edges, and the edges are labelled by $[\alpha_2], [\alpha_2 + 2\alpha_1] \in \Delta_{G,K}/\pm 1$. If $w = \sigma_{\alpha_1+\alpha_2}\sigma_{\alpha_1} \in W_K$, then $w\alpha_2 = -\alpha_2$ and $\alpha_2 \in \Delta_{G,K}$, hence one cannot define an axial function on $\Gamma$. In this example, $G/K = S^4$, which does not admit an almost complex structure.

### 2.5.2 Non-existence of Morse functions

Let $G$ be a compact Lie group such that $\mathfrak{g}_{\mathbb{C}}$ is the simple Lie algebra of type $G_2$. Let $\alpha_1, \alpha_1 + \alpha_2$, and $2\alpha_1 + \alpha_2$ be the short positive roots and let $\alpha_2, 2\alpha_2 + 3\alpha_1, \alpha_2 + 3\alpha_1$ be the long positive roots. Let $K$ be the subgroup of $G$ corresponding to the root system consisting of the short roots. Then $\mathfrak{k}_{\mathbb{C}} = A_2$ and $K \simeq SU(3)$. The quotient $W_G/W_K$ has two classes: the class of $\sigma_{\alpha_1} \in W_K$ and the class of $\sigma_{\alpha_2} \in W_G - W_K$.

The GKM graph $\Gamma$ has two vertices, joined by three edges, and the edges are labelled by $[\alpha_2], [2\alpha_2 + 3\alpha_1], [\alpha_2 + 3\alpha_1] \in \Delta_{G,K}/\pm 1$. There are two $W_K$-equivariant sections of the projection $\Delta_{G,K} \to \Delta_{G,K}/\pm 1$, corresponding to $\{\alpha_2, \alpha_2+3\alpha_1, -2\alpha_2-3\alpha_1\}$ and $\{-\alpha_2, -\alpha_2-3\alpha_1, 2\alpha_2 + 3\alpha_1\}$. If

$$\Delta_0 = \{\alpha_2, \alpha_2 + 3\alpha_1, -2\alpha_2 - 3\alpha_1\} ,$$

then the axial function is shown in Figure 2-2 and there is no Morse function on $\Gamma$: the corresponding almost complex structure is not integrable. In this example, $G/K = S^6$,

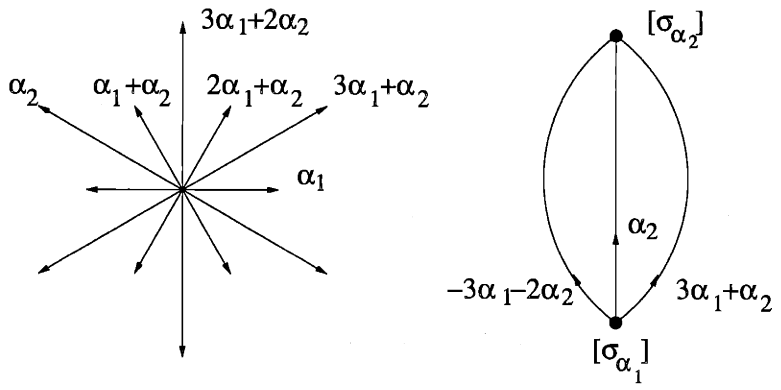



Figure 2-2: The weights of $G_2$ and graph for $G_2/SU(3)$.

which admits an almost complex structure, but no invariant complex structure.

### 2.5.3 The existence of several almost complex structures

Let $G = SU(3)$ and $K = T$. Then the homogeneous space $G/K$ is the manifold of complete flags in $\mathbb{C}^3$. The root system of $G$ is $A_2$, with positive roots $\alpha_1, \alpha_2$, and $\alpha_1 + \alpha_2$ of equal length. The Weyl group of $G$ is $W_G = S_3$, the group of permutations of $\{1, 2, 3\}$, and $W_K = 1$, hence $W_G/W_K = W_G = S_3$.

The GKM graph is the bi-partite graph $K_{3,3}$ : it has 6 vertices and each vertex has 3 edges incident to it, labelled by $[\alpha_1]$, $[\alpha_2]$, and $[\alpha_1 + \alpha_2]$. There are $2^3$ possible $W_K$-invariant sections, hence eight $G$-invariant almost complex structures on $G/K$. If

$$\Delta_0 = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} ,$$

then the corresponding almost complex structure is integrable and there is a Morse function on $\Gamma$ compatible with $\xi \in \mathfrak{t}$ such that both $\alpha_1(\xi)$, and $\alpha_2(\xi)$ are positive. This Morse function is given by $f(w) = \ell(w)$ where $\ell(w)$ is the length of $w$. In this case, this is the number of inversions in $w$. However, if

$$\Delta_0 = \{\alpha_1, \alpha_2, -\alpha_1 - \alpha_2\} ,$$

then the corresponding almost complex structure is not integrable and there is no Morse function on $(\Gamma, \alpha)$ : for every vertex $w$ of $\Gamma$, there exist three edges $e_1, e_2$, and $e_3$, going out of $w$, such that

$$\alpha_{e_1} + \alpha_{e_2} + \alpha_{e_3} = 0 ,$$

hence there is no vertex of $\Gamma$ on which a Morse function compatible with some $\xi \in \mathfrak{t}$ can achieve its minimum.

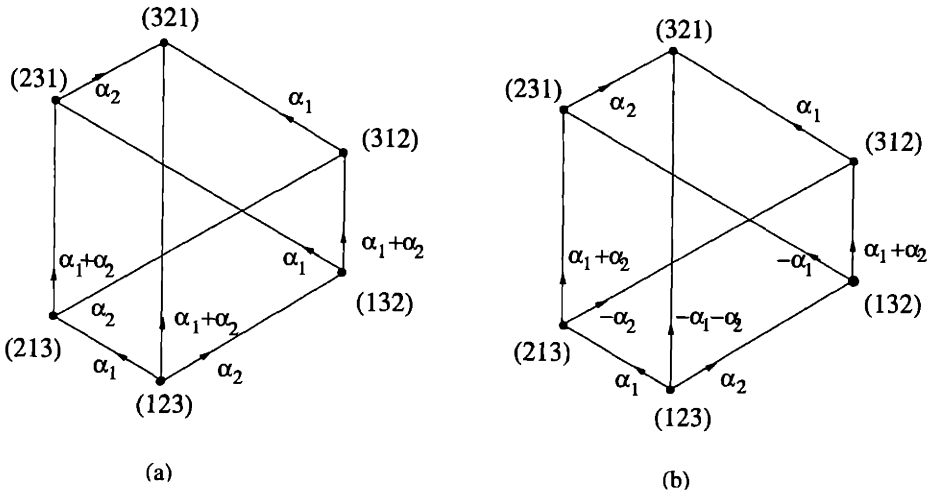



Figure 2-3: Two choices of almost complex structure for $SU(3)/T$.

# Chapter 3

# Graphs and equivariant cohomology

In Chapter 2, we explored the ramifications of the GKM computation of equivariant cohomology for homogeneous spaces $G/K$. In this chapter, we combinatorialize the geometric concepts discussed in Chapters 1 and 2. We will be particularly interested in Cayley graphs, as they are the combinatorial analogue of homogeneous spaces.

## 3.1 Preliminaries

In this section, we summarize basic definitions and results about graphs from [BoGH]. In that paper, the goal is to give a combinatorial interpretation to the Betti numbers defined below. We will examine these definitions in greater detail for homogeneous graphs.

When a graph $\Gamma$ comes from a GKM manifold, there is one additional structure on the graph that is of fundamental importance. This is the *axial function*. When we try to strip the geometry from this picture, and try to make purely combinatorial definitions, we describe the structure of an axial function in two pieces: a *connection* and an *axial function*.

### 3.1.1 Connections and geodesic subgraphs

Let $\Gamma = (V, E)$ be a graph with finite vertex set $V$ and edge set $E$. We will assume that $\Gamma$ has no multiple edges and no loops. In the previous chapter, there were examples of non-simple graphs arising in geometry. However, for convenience, we will restrict our attention here to simple graphs. We count each edge twice, once with each of its two possible orientations. When $x$ and $y$ are adjacent vertices we write $e = (x, y)$ for the edge from $x$ to $y$ and $e^{-1} = (y, x)$ for the edge from $y$ to $x$. Given an oriented edge $e = (x, y)$, we write $x = \iota(e)$ for the initial vertex and $y = \tau(e)$ for the terminal vertex.

**Definition 3.1.1.** *The* star *of a vertex $x$, written $St(x)$, is the set of edges leaving $x$,*

$$St(x) = \{e \mid \iota(e) = x\}.$$

The star of a vertex is the combinatorial analogue of the tangent space to a manifold at a point. In the manifold setting, the tangent space breaks up into weight spaces, each corresponding to one of the edges $e \in St(x)$.

**Definition 3.1.2.** *A* connection *on a graph $\Gamma$ is a set of functions $\nabla_{(x,y)}$ or $\nabla_e$, one for each oriented edge $e = (x, y)$ of $\Gamma$, such that*

1. $\nabla_{(x,y)} : St(x) \to St(y)$,
2. $\nabla_{(x,y)}(x, y) = (y, x)$, *and*
3. $\nabla_{(y,x)} = (\nabla_{(x,y)})^{-1}$.

It follows that each $\nabla_{(w,y)}$ is bijective, so each connected component of $\Gamma$ is regular: all vertices have the same valence. Every regular graph has at least one connection, and often many. Henceforth we will assume $\Gamma$ comes equipped with a specified connection $\nabla$. In the geometric picture, the connection is built into the definition of axial function. From the combinatorial point of view, it is interesting to study the connection itself. The connection will be of particular use in Section 3.2.2.

**Definition 3.1.3.** *A* 3-geodesic *is a sequence of four vertices $(x, y, z, w)$ with edges $\{x, y\}$, $\{y, z\}$, and $\{z, w\}$ for which $\nabla_{(y,z)}(y, x) = (z, w)$. We inductively define a $k$-geodesic as a sequence of $k + 1$ vertices in the natural way. We may identify a geodesic by specifying either its edges or its vertices, and we will refer to* edge geodesics *or* vertex geodesics *as appropriate. The three consecutive edges $(d, e, f)$ of a 3-geodesic will be called an* edge chain.

**Definition 3.1.4.** *A* closed geodesic *is a sequence of edges $e_1, \ldots, e_n$ such that each consecutive triple $(e_i, e_{i+1}, e_{i+2})$ is an edge chain for each $1 \leq i \leq n$, modulo $n$.*

A little care is required to understand when a geodesic is closed, since it may in fact use some edges in $St(x)$ multiple times. It is not closed until it returns to the same pair of edges in the same order. That is analogous to the fact that a periodic geodesic in a manifold is an immersed submanifold, not an embedded submanifold. The period completes only when it returns to a point with the same velocity (tangent vector).

**Remark 3.1.5.** *Because there is a unique closed geodesic through each pair of edges in the star of a vertex, the set of all closed geodesics completely determines the connection on $\Gamma$. We will sometimes use this fact to describe a connection.*

We define totally geodesic subgraphs of a graph by analogy to totally geodesic submanifolds of a manifold.

**Definition 3.1.6.** *Given a graph $\Gamma$ with a connection $\nabla$, we say that a subgraph $(V_0, E_0) = \Gamma_0 \subseteq \Gamma$ is* totally geodesic *if all geodesics starting in $E_0$ stay within $E_0$.*

This definition is equivalent to saying that a totally geodesic subgraph $\Gamma_0$ is one in which, for every two adjacent vertices $x$ and $y$ in $\Gamma_0$,

$$\nabla_{(x,y)}(St(x) \cap E_0) \subseteq E_0.$$

Suppose now that $P = \{e_1, \ldots, e_n\}$ is any cycle in $\Gamma$: $\tau(e_i) = \iota(e_{i+1})$ modulo $n$. Then following the connection around $P$ leads to a permutation

$$\nabla_P = \nabla_{e_n} \circ \cdots \circ \nabla_{e_1} \circ \nabla_{e_0}$$

of $St(x)$.

**Definition 3.1.7.** *The* holonomy group $Hol(\Gamma_x)$ *at vertex $x$ of $\Gamma$ is the subgroup of the permutation group of $St(x)$ generated by the permutations $\nabla_P$ for all cycles $P$ that pass through $x$.*

It is easy to see that the holonomy groups $Hol(\Gamma_x)$ for the vertices $x$ in each connected component of $\Gamma$ are isomorphic. When $\Gamma$ is connected and $d$-regular we call that group the holonomy group of $\Gamma$ and think of it as a subgroup of $S_d$.

### 3.1.2 Axial functions

We described in Section 1.6 how a graph arising from a GKM manifold has associated to it an *axial function*, namely an assignment of a vector to each oriented edge $e$. We will not repeat this definition here, but refer the reader to Defintion 1.6.6. It follows immediately from the definition that the images under $\alpha$ of all geodesics of $\Gamma$ are planar. What matters about the axial function is the *direction* of $\alpha(e)$ in $\mathbb{R}^n \setminus \{0\}$, not its actual value. We consider two axial functions $\alpha$ and $\alpha'$ to be equivalent if

$$\frac{\alpha(e)}{||\alpha(e)||} = \frac{\alpha'(e)}{||\alpha'(e)||}$$

for all edges $e$. Notice that $\alpha$ is *not* equivalent to $-\alpha$.

If $e = (x, y)$ is an edge, we will denote $\alpha(e)$ by $\alpha(x, y)$, rather than using two sets of parentheses. We picture an edge chain as a succession of vectors joined head to tail in their plane, as shown in the figure below. A picture of an equivalent axial function will show vectors with the same orientations, but different lengths.

**Definition 3.1.8.** *An* immersion *of $(\Gamma, \alpha)$ is a map $F : V \to \mathbb{R}^n$ such that*

$$\alpha(x, y) = F(y) - F(x).$$

Our picture of an immersed vertex chain $(x, y, z, w)$ is shown below. Here, the endpoints of the vectors do make sense, as the vertices are points in $\mathbb{R}^n$.

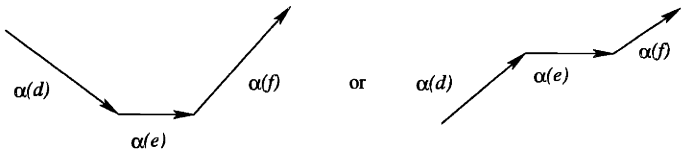



Figure 3-1: This shows how we picture the axial function on an edge chain.

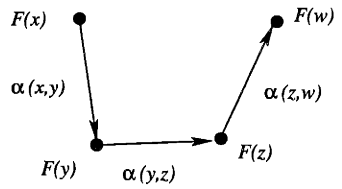



Figure 3-2: This shows how we picture the axial function on an immersed vertex chain.

For an axial function $\alpha$, the edges $\alpha(d)$ and $\alpha(f)$ must lie on the same side of $\alpha(e)$ in the plane in which they lie.

**Definition 3.1.9.** *An axial function $\alpha$ is $k$-independent if for every $x \in V$ and every $k$ edges $e_1, \ldots, e_k \in St(x)$, the vectors $\alpha(e_1), \ldots, \alpha(e_k)$ are linearly independent. By assumption, $\alpha$ is 2-independent.*

**Theorem 3.1.10.** *If the axial function $\alpha$ is 3-independent, then it determines the connection.*

*Proof.* Let $d$ and $e$ be edges with $\tau(d) = \iota(e)$. Then 3-independence implies that there is only *one* edge $f$ with $\tau(e) = \iota(f)$ and and $\alpha(f)$ in the plane determined by $\alpha(d)$ and $\alpha(e)$, □

**Definition 3.1.11.** *The product of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is the graph*

$$\Gamma = \Gamma_1 \times \Gamma_2 = (V, E),$$

*with vertex set $V = V_1 \times V_2$. Two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent if and only if*

1. *$x_1 = x_2$ and $\{y_1, y_2\} \in E_2$; or*

(2) *$y_1 = y_2$ and $\{x_1, x_2\} \in E_1$.*

Suppose now that each $\Gamma_i$ is equipped with a connection $\nabla_i$ and axial function $\alpha_i : E_i \to \mathbb{R}^{n_i}$. Then we can define a connection $\nabla$ on $\Gamma$ in a natural way by specifying the

closed geodesics as the closed geodesics in each component and some closed geodesics of length 4 which go between $\Gamma_1$ and $\Gamma_2$.

The figure below shows one each of the two kinds of geodesics for the example in which $\Gamma_1$ is a 3-cycle and $\Gamma_2$ is an edge.

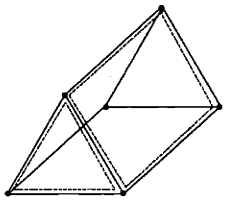


Figure 3-3: This shows the product of two graphs, showing one geodesic of each type.

We define an axial function $\alpha : E \to \mathbb{R}^{n_1+n_2}$ by

$$\alpha((x_1, y_1), (x_2, y_2)) = (\alpha_1(x_1, x_2), \alpha_2(y_1, y_2)),$$

where (by definition) $\alpha_i((x, x)) = 0$. We leave it to the reader to check that $\nabla$ is a well-defined connection, and that $\alpha$ is indeed an axial function compatible with $\nabla$. Note that this generalizes the example of the hypercube, which is an $n$-fold product of an edge.

### 3.1.3 Betti numbers

Suppose $\Gamma$ is a graph with a connection $\nabla$ and an axial function $\alpha$ mapping edges to $\mathbb{R}^n$. The images under $\alpha$ of the chains in $\Gamma$ are planar; we will study how those chains wind in their planes. To that end choose an arbitrary orientation for each such plane $P$. Then whenever $\alpha(e) \in P$ the direction $\alpha(e)^\perp$ is a well defined direction in $P$. (If $\alpha$ is immersible then $\alpha(e)^\perp$ is a well defined vector in $p$.)

Throughout this section we will assume $\alpha$ is 2-independent. That is, no two edges in the star of a vertex of $\Gamma$ are mapped by $\alpha$ into the same line in $\mathbb{R}^n$. Thus any two edges at a vertex determine a unique plane, which we have assumed is oriented.

Recall that a vector $\xi \in \mathbb{R}^n \setminus \{0\}$ is a *regular value* if for all $e \in E$, $\xi \not\perp \alpha(e)$. In this case, we will call $\xi$ *generic*. We can define the index of a vertex and Betti numbers of a graph in an identical fashion to our definitions for homogeneous spaces in Chapter 2.

**Definition 3.1.12.** *The* index *of a vertex $x \in V$ with respect to a generic direction $\xi$ is the number of edges $e \in St(x)$ such that*

$$\alpha(e) \cdot \xi < 0.$$

*We call those the* down *edges. Let $\beta_i(\xi)$ be the number of vertices $x \in V$ such that the index of $x$*

*is exactly $i$.*

**Theorem 3.1.13.** *If $\Gamma$ is a graph with connection $\nabla$ and an axial function $\alpha$, then the Betti numbers $\beta_i$ do not depend on the choice of direction $\xi$.*

*Proof.* Imagine the direction $\xi$ varying continuously in $\mathbb{R}^n$. It is clear from the definitions above that the indices of vertices can change only when $\xi$ crosses one of the hyperplanes $\alpha(x,y)^\perp$. Let us suppose that $(x,y)$ is the only edge of $\Gamma$ at which the value of the axial function is a multiple of $\alpha(x,y)$. Then at such a crossing only the indices of the vertices $x$ and $y$ can change. Suppose $\xi$ is near $\alpha(x,y)^\perp$. Since $\alpha$ is an axial function, the connection mapping $St(x)$ to $St(y)$ preserves down edges, with the single exception of edge $(x,y)$ itself. That edge is down for one of $x$ and $y$ and up for the other. Thus the vertices $x$ and $y$ have indices $i$ and $i+1$ for $\xi$ on one side of $\alpha(x,y)^\perp$ and indices $i+1$ and $i$ on the other. Thus the number of vertices with index $i$ does not change as $\xi$ crosses $\alpha(x,y)^\perp$. If there are several edges of $\Gamma$ at which the axial function is a multiple of $\alpha(x,y)$, the same argument works, since by the 2-independence of $\alpha$, none of those edges can share a common vertex. □

Henceforth we will assume $\alpha$ is inflection free. The motivation for the following definitions comes from Morse theory.

**Definition 3.1.14.** *When the $\beta_i(\xi)$ are independent of $\xi$, we call them the* Betti numbers *of $\Gamma$ (or, more precisely, the Betti numbers of the pair $(\Gamma,\alpha)$ ).*

The following proposition is the combinatorial version of Poincaré duality.

**Proposition 3.1.15.** *When the Betti numbers of a graph are independent of the choice of $\xi$, then $\beta_i(\Gamma) = \beta_{d-i}(\Gamma)$ for $i = 0,\ldots,d$.*

*Proof.* Choose some $\xi$ with which to compute the Betti numbers of $\Gamma$. Then simply replace $\xi$ by $-\xi$, and a vertex of index $i$ becomes a vertex of index $d-i$. □

**Definition 3.1.16.** *Given a generic $\xi$, a* Morse function compatible with $\xi$ *on a graph with an axial function $\alpha$ is a map $f : V \to \mathbb{R}$ such that if $(x,y)$ is an edge, $f(x) > f(y)$ whenever $\alpha(x,y)\cdot\xi > 0$.*

There is a simple necessary and sufficient condition for the existence of a Morse function compatible with $\xi$. Recall that Proposition 2.4.2 says that a Morse function compatible with $\xi$ exists if and only if there exists no closed cycle $(e_1,\ldots,e_n)$ with $e_1 = e_n$, in $\Gamma$ for which all the edges $e_i$ are "up" edges. We prove this here.

*Proof of Proposition 2.4.2.* The necessity of this condition is obvious since $f$ has to be strictly increasing along such a path. To prove sufficiency, for every vertex $p$, define $f(p)$ to be the length $N$ of the longest path $(e_1,\ldots,e_N)$ in $\Gamma$ of up edges $\tau(e_N) = p$ The hypothesis that

there is no cycle of up edges guarantees that this function is well-defined, and it is easy to check that it is a Morse function. □

**Remark 3.1.17.** *One can easily arrange for $f$ in the above proof to be an* injective *map of $V$ into $\mathbb{R}$ by perturbing it slightly.*

When $\alpha$ is immersible, so that $\alpha(x,y) = f(y) - f(x)$, then we can define a Morse function on $\Gamma$ by setting $m(x) = f(x) \cdot \xi$ for any generic direction $\xi$. Then $m(x)$ increases along each up edge. The vertices with index $i$ resemble critical points of Morse index $i$ in the Morse theory of a manifold. We call the $\beta_i$ Betti numbers because when a graph we are studying is the GKM graph of a manifold, the $\beta_i$ indeed correspond to the Betti numbers of the manifold, and they are the dimensions of the cohomology groups of the manifold.

**Remark 3.1.18.** *When an inflection-free 2-independent axial function is projected generically into a plane, it retains those properties, so the Betti numbers of $\Gamma$ can be computed using a generic direction in a generic plane projection. In most of our examples $\alpha$ is immersible. In these cases we are of course drawing a planar embedding of $\Gamma$. Thus the figures in this paper are more than mere suggestions of some high dimensional truth. They actually capture all the interesting information about $\Gamma$.*

**Definition 3.1.19.** *The generating function $\beta$ for the Betti numbers of $\Gamma$ is the polynomial*

$$\beta(z) = \sum_{i=0}^{n} \beta_i z^i.$$

**Remark 3.1.20.** *When $\Gamma$ is $d$-regular, $\beta$ is of degree $d$. The sum of the Betti numbers, $\beta(1)$, is just the number of vertices of $\Gamma$*

**Remark 3.1.21.** *It is clear that $\beta_0 > 0$ if a Morse function exists, because the vertex at which the Morse function assumes its minimum value has no down edges.*

We can relate the Betti numbers of the product of two graphs to the Betti numbers of the two multiplicands as follows. The proof is left to the reader.

**Proposition 3.1.22.** *Let $\Gamma$ and $\Delta$ be graphs with Betti numbers generated by $\beta_\Gamma(z)$ and $\beta_\Delta(z)$ respectively. Then the generating function for the Betti numbers of the product graph $\Gamma \times \Delta$ is the polynomial product*

$$\beta_\Gamma(z) \cdot \beta_\Delta(z).$$

In [BoGH], the following theorem relates the Betti numbers and the cohomology ring $H^*(\Gamma, \alpha)$.

**Theorem 3.1.23.** *Suppose $\Gamma$ is a graph equipped with a connection $\nabla$, a 3-independent axial function $\alpha : E \to \mathbb{R}^n$, and a $\xi$-compatible Morse function $f$. Suppose further that each two-face of $\Gamma$ has zeroth Betti number equal to 1. Then the dimension of $H^r(\Gamma, \alpha)$ is given by the formula*

$$\sum_{\ell=0}^{r} \binom{r-\ell+n-1}{n-1} \beta_\ell. \tag{3.1}$$

## 3.2 Graphs and equivariant cohomology

### 3.2.1 Equivariant classes

Let $S$ be the polynomial ring in the variables $x = (x_1, \ldots, x_n)$ and $S^k$ the $k^{\text{th}}$ graded component $S$: the space of homogeneous polynomials of degree $k$. $S^k$ has dimension $\binom{n+k-1}{n-1}$.

Let $\Gamma = (V, E)$ be a regular $d$-valent graph with connection $\nabla$ and axial function $\alpha : E \to \mathbb{R}^n$. Then for every edge $e$, of $\Gamma$ we will identify the vector, $\alpha(e) \in \mathbb{R}^n$, with the linear function $\alpha_e(x) = \alpha(e) \cdot x$ so we can think of $\alpha_e$ as an element of $S^1$. Finally, for $g$ and $h \in S$ we will say that

$$g \equiv h \mod \alpha$$

when $g - h$ vanishes on the hyperplane, $\alpha(x) = 0$.

In Section 1.6, we defined the equivariant cohomology $H^*(\Gamma, \alpha)$ of a graph with an axial function.

**Definition 3.2.1.** *Let $m$ be the numbers of vertices of $\Gamma$. An $m$-tuple of polynomials*

$$g_p \in S, \quad p \in V$$

*is an* equivariant class *if for every $e = (p, q) \in E$*

$$g_p \equiv g_q \mod \alpha_e. \tag{3.2}$$

*Henceforth we will write $< g >$ for such an $m$-tuple. We say that this class has* degree $k$ *if for all $p$, $g_p \in S^k$, and we note that the set of all equivariant classes of degree $k$ is $H^k(\Gamma, \alpha)$.*

It's clear that every equivariant class is in the space

$$H^*(\Gamma, \alpha) = \bigoplus_{k=0}^{\infty} H^k(\Gamma, \alpha).$$

Moreover this space is clearly a graded module over the ring $S$. That is, if $< g_p >$ satisfies (3.2) then for every $h \in S$, so does $< hg_p >$. More generally, if $< g_p >$ and $< h_p >$ satisfy (3.2) then so does $< g_p \cdot h_p >$. So $H^*(\Gamma, \alpha)$ is not just a module, but in fact a *graded ring*,

and $S$ sits in this ring as the subring of constant classes

$$g_p = g \text{ for all } p.$$

In this section we describe some methods for constructing solutions of the compatibility conditions (3.2). These methods will rely heavily on the ideas that we introduced in Sections 3.1.1 and 3.1.2.

Let $F : V \to \mathbb{R}^n$ be an immersion of $\Gamma$. If we identify vector $F(p)$ with the monomial

$$f_p(x) = F(p) \cdot x$$

then (3.2) is just a rephrasing of the identity (1.1), so $< f_p >$ is an equivariant class of degree 1. More generally if

$$\mathfrak{p}(z) = \sum_{i=0}^{k} \mathfrak{p}_i(x) z^i$$

is a polynomial in $z$ whose coefficients are polynomials in $x = (x_1, \ldots, x_n)$ then the $m$-tuple of polynomials

$$< \sum_{i=0}^{k} \mathfrak{p}_i(x) f_p^i > \tag{3.3}$$

is an equivariant class. The class $< f_p >$ itself corresponds to the case $k = 1, \mathfrak{p}_0 = 0, \mathfrak{p}_1 = 1$.

### 3.2.2 The complete graph

In one important case this construction gives *all* solutions of (3.2). Namely let $K_{n+1} = (V, E)$ be the complete graph on $n + 1$ vertices with the natural connection. Then every immersion $F : V \to \mathbb{R}^n$ defines an axial function compatible with that connection by setting

$$\alpha(p, q) = F(q) - F(p) \tag{3.4}$$

for every oriented edge $e = (p, q)$. We will prove

**Theorem 3.2.2.** *If the axial function (3.4) is two-independent, every equivariant class can be written* uniquely *in the form*

$$< g_p > = < \sum_{i=0}^{k} \mathfrak{p}_i(x) f_p^i > \tag{3.5}$$

*for some polynomials* $\mathfrak{p}_i$.

*Proof.* By induction on $n$. Let $\{p_1, \ldots, p_{n+1}\}$ be the vertices of $V$, and let $< g_p >$ be an

equivariant class. By induction there exists a polynomial

$$\mathfrak{p}(z) = \sum_{i=0}^{k} \mathfrak{p}_i z^i$$

with coefficients in $S$ such that $\mathfrak{p}(g_{p_i}) = f_{p_i}$ for $i = 1, \ldots, n$. Hence the $n+1$-tuple of polynomials

$$< f_p - p(g_p) > n$$

is an equivariant class vanishing on $p_1, \ldots, p_n$. Therefore, by (3.2) and the two-independence of the axial function (3.3)

$$f_{p_n} - \mathfrak{p}(g_{p_n}) = h \prod_{i<n} (g_{p_n} - g_{p_i})$$

for some polynomial $h \in S$. Let

$$\mathfrak{q}(z) = h \prod_{i \leq n} (z - g_{p_i}) .$$

Then

$$\mathfrak{q}(g_{p_{n+1}}) = f_{p_{n+1}} - \mathfrak{p}(g_{p_{n+1}})$$

and

$$\mathfrak{q}(g_{p_i}) = 0$$

for $i < n$. Thus the theorem is true for $\#V = n+1$ with $\mathfrak{p}$ replaced by $\mathfrak{p} + \mathfrak{q}$.

The uniqueness of $\mathfrak{p}$ follows from the Vandermonde identity

$$\det \begin{pmatrix} 1 & g_{p_1} & \cdots & g_{p_1}^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & g_{p_n} & \cdots & g_{p_n}^{n-1} \end{pmatrix} = \prod_{i>j} g_{p_i} - g_{p_j} ,$$

the right-hand side of which is non-zero by the two-independence of the axial function (3.3).

□

### 3.2.3 Holonomy and equvariant classes

The complete graph is the only example we know of for which the methods of the previous section give *all* the equivariant classes. In this section we describe an alternative method which is effective in examples in which one has information about the holonomy group of the graph $\Gamma$. To simplify the exposition below we will confine ourselves to the case in

which the axial function $\alpha$ is exact.

Let $p_0$ be a vertex of $\Gamma$. The holonomy group, $\mathrm{Hol}(\Gamma_{p_0})$ is by definition a subgroup of the group of permutations of the elements of $St(p_0)$, so if we enumerate its elements in some order

$$e_i^0 \in St(p_0) \quad i = 1, \dots d$$

we can regard $\mathrm{Hol}(\Gamma_{p_0})$ as a subgroup of the permutation group $S_d$ on $\{1, \dots, d\}$. Let $\mathfrak{q}(z_1, \dots, z_d)$ be a polynomial in $d$ variables with scalar coefficients. We will say that $\mathfrak{q}$ is $\mathrm{Hol}(\Gamma_{p_0})$ *invariant* if for every $\sigma \in \mathrm{Hol}(\Gamma_{p_0})$

$$\mathfrak{q}(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = \mathfrak{q}(z_1, \dots, z_n)\,.$$

Now fix such a $\mathfrak{q}$ and construct a polynomial assignment $< g_p >$ as follows. Given a path, $\gamma$ in $\Gamma$ joining $p_0$ to $p$ the connection gives us a holonomy map

$$\nabla_\gamma : St(p_0) \to St(p)$$

mapping $e_1^0, \dots, e_d^0$ to $e_1, \dots, e_d$. Set

$$g_p = \mathfrak{q}(\alpha_{e_1}(x), \dots, \alpha_{e_d}(x))\,. \tag{3.6}$$

The invariance of $\mathfrak{q}$ guarantees that this definition is independent of the choice of $\gamma$. Let us show that $< g_p >$ satisfies the compatibility conditions (3.2). Let $e = (p, q)$. The map $\Gamma$

$$\nabla_p : St(p) \to St(q)$$

maps $e_1, \dots, e_d$ to $e_1', \dots, e_d'$ and by the exactness of $\alpha$

$$\alpha_{e_i'} \equiv \alpha_{e_i} \mod \alpha_e\,.$$

Hence

$$\mathfrak{q}(\alpha_{e_1'}, \dots, \alpha_{e_d'}) \equiv \mathfrak{q}(\alpha_{e_1}, \dots, \alpha_{e_d}) \mod \alpha_e\,.$$

If the holonomy group is small this construction provides many solutions of (3.2). Even if $\mathrm{Hol}(\Gamma_{p_0})$ is large this method yields some interesting solutions. For instance if $\mathfrak{q}$ is a symmetric polynomial in $z_1, \dots, z_d$, (3.6) is a solution of (3.2).

### 3.2.4 Totally geodesic subgraphs and equivariant classes

A third method for constructing solutions of (3.2) makes use of totally geodesic subgraphs. Whenever $\Gamma_0 = (V_0, E_0)$ is a totally geodesic subgraph of degree $j$ then for every $p \in V_0$, $St(p)$ is a disjoint union of $St(p, \Gamma_0)$ and its complement, which we can regard as the *tangent*

and *normal* spaces to $\Gamma_0$ at $p$. Let

$$g_p = \prod_{e \perp \Gamma_0} \alpha(e) \,, \tag{3.7}$$

a homogeneous polynomial of degree $d - j$. By the results of Section 3.2.3, the assignment $< g_p >$ sending $p \to g_p$, is an equivariant on $V_0$, and we can extend this class to $V$ by setting

$$g_p = 0 \tag{3.8}$$

for $p \in V - V_0$. Then (3.7) and (3.8) *do* define an equivariant class on $V$. Clearly the compatibility conditions (3.2) are satisfied if $e = (p, q)$ is either an edge of $V_0$ or if $p$ and $q$ are both in $V - V_0$. If $p \in V_0$ and $q \in V - V_0$ then $\alpha(p, q)$ is one of the factors in the product (3.7); so in this case the condition (3.2) is also satisfied.

It is hence of great importance to compute the holonomy and totally geodesic subgraphs in our examples, which we do in the following section. An interesting question is when the above constructions give complete sets of generators for $H^*(\Gamma, \alpha)$.

## 3.3 Examples

### 3.3.1 The complete graph

Let $M = \mathbb{C}P^{n-1}$ be complex projective $n-1$ space, the set of all (complex) lines in $\mathbb{C}^n$. Then $T^{n-1}$ acts naturally on $M$, and $M$ is a GKM space. The associated graph is the complete graph $K_n$ on $n$ vertices.

Our standard view of $K_n$ embeds with vertices the standard basis vectors in $\mathbb{R}^n$. That embedding is a regular simplex in the $n-1$-dimensional subspace $\Sigma x_i = 1$. The exact axial function is determined by assigning to each vertex the difference between its end points. The following figure shows a part of the connection determined by that axial function for $K_4$: it moves edges across the triangular faces.

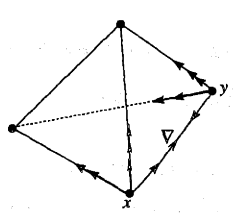



Figure 3-4: This shows the connection we defined above on the graph $K_4$.

When we think of $K_4$ just as an abstract 4-regular graph we find that it has 10 different connections (up to graph automorphism). But in each of these connections other than the standard one there is at least one geodesic of length at least 4, so none of those connections has a 3-independent immersion. So we will study only the standard view.

**Proposition 3.3.1.** *The geodesics of $K_n$ are the triangles. The connected totally geodesic subgraphs are the complete subgraphs.*

*Proof.* It's clear that the geodesics are the triangles. Let $\Gamma_0$ be a connected totally geodesic subgraph and $p$ and $q$ two vertices of $\Gamma_0$. Then transporting edge $e = (p, q)$ along a path in $\Gamma_0$ from $p$ to $q$ we eventually reach a triangle containing $q$. At that point the image of $e$ transports to an edge of $\Gamma_0$ so $e$ must have been part of $\Gamma_0$ to begin with. □

It's easy to compute the holonomy of $K_n$.

**Proposition 3.3.2.** $Hol(K_n) \cong S_{n-1}$.

*Proof.* If you follow the connection along triangle $(p, q, r)$ from $p$ back to itself you interchange $(p, q)$ and $(p, r)$. Thus the holonomy group acting on $St(p)$ contains all the transpositions. □

**Proposition 3.3.3.** *The Betti numbers of $K_n$ are invariant of choice of direction $\xi$ and are* $(1, 1, \ldots, 1)$.

*Proof.* The geodesics are triangles, hence convex. hence inflection free, so the Betti numbers are well defined. Let $\xi = (1, 2, \ldots, n)$. Then the number of down edges at the vertex corresponding to the $i^{th}$ coordinate vector is the number of $j$'s less than $i$. □

### 3.3.2 The Johnson graph

Let $M$ be the $k$-Grassmannian $\mathcal{G}r(k, n)$, the set of all (complex) $k$-dimensional subspaces of $\mathbb{C}^n$. The $n-1$-dimensional torus $T$ acts on $M$, and this is a GKM action. The associated graph $J(k, n)$ is the Johnson graph.

**Definition 3.3.4.** *Given a set $A$ and an integer $k \leq \#A$, we define the* Johnson graph $J(k, A)$ *to be the graph with vertices corresponding to $k$-element subsets of $A$, with two vertices $S_1, S_2 \in V$ adjacent if $\#(S_1 \cap S_2) = k - 1$.*

If $A_1$ and $A_2$ have the same cardinality, then $J(k, A_1)$ is isomorphic to $J(k, A_2)$. We will denote $J(k, \{1, \ldots, n\})$ by $J(k, n)$. Notice that $J(k, n)$ is an $k \cdot (n-k)$-regular graph.

The easiest way to describe the natural connection on $J(2, 4)$ is to describe its geodesics. They are the triangles $Q \cup \{a\}$, $Q \cup \{b\}$, $Q \cup \{c\}$ for $k-1$ element sets $Q$ and distinct $a, b, c$ and the planar squares $Q \cup \{a, b\}$, $Q \cup \{b, c\}$, $Q \cup \{c, d\}$, $Q \cup \{d, a\}$, for $k-2$ element sets $Q$ and distinct $a, b, c, d$.

The triangles are actual faces of the polytope. The squares are more like equators, as in the picture of the octahedron $J(2, 4)$ below.

We can also define the connection itself on $J(k, n)$. Let $S_1$ and $S_2$ be two adjacent vertices in $J(k, n)$. We think of the edge pointing from $S_1$ to $S_2$ as an ordered pair $(i, j)$,

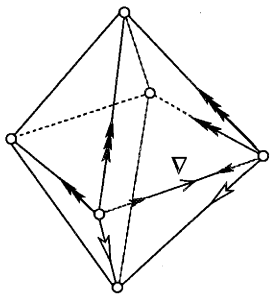
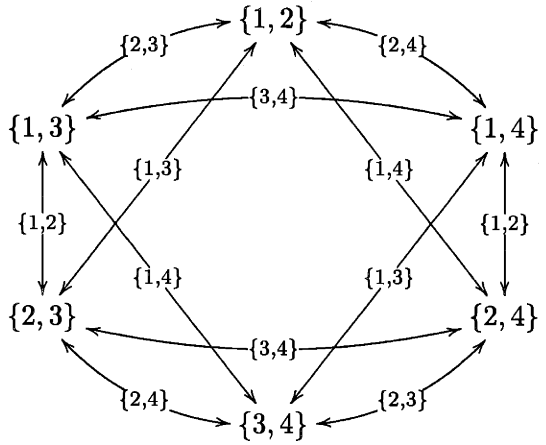



Figure 3-5: This shows the connection we defined on the Johnson graph $J(2,4)$.

where $i \in S_1 \setminus S_2$ and $j \in S_2 \setminus S_1$. Thus the edge $(i,j)$ corresponds to removing $i$ from $S_1$ and adding $j$ to get $S_2$. Here we show the Johnson graph $J(2,4)$.




Suppose $S_1$ and $S_2$ are adjacent vertices in the Johnson graph $J(k,n)$, via the pair $(i,j)$. Then the natural connection on $J(k,n)$ is defined as follows.

$$\nabla_{S_1,(i,j)}(a,b) = \begin{cases} (a,i) & a \in S_2,\ b \in S_2, \\ (a,b) & a \in S_2,\ b \notin S_2, \\ (j,a) & a \notin S_2,\ b \in S_2, \\ (j,b) & a \notin S_2,\ b \notin S_2. \end{cases}$$

Using the connection of $J(k,n)$ given above, we can determine all of the totally geodesic subgraphs of $J(k,n)$.

**Proposition 3.3.5.** *If $\Gamma_0$ is a totally geodesic subgraph of $\Gamma = J(k,n)$, then*

$$\Gamma_0 \cong J(\ell_1, A_1) \times \cdots \times J(\ell_r, A_r,),$$

*where the $A_i$ are subsets of $\{1,\ldots,n\}$ of size $a_i \geq \ell_i$, and $\{1,\ldots,n\}$ is the disjoint union of the $A_i$.*

*Proof.* Let $p_0 = S_0 \subseteq \{1, \ldots, n\}$ be a vertex of $\Gamma_0$. Then the edges in $E_{p_0}|_{\Gamma_0}$ form a subgraph of the complete bipartite graph with partite classes $S_0$ and $S_0^c = \{1, \ldots, n\} \setminus S_0$. Consider the connected components of this subgraph, and label them $\Gamma_1^{p_0}, \ldots, \Gamma_r^{p_0}$. Let $A_i$ be the vertex set of $\Gamma_i^{p_0}$. Then for any vertex $p = S \subseteq \{1, \ldots, n\}$, we claim that $\#S \cap A_i = \ell_i$ is independent of $S$. Since $\Gamma_0$ is connected, there is a path from $S_0$ to $S$. Furthermore, because $\Gamma_0$ is totally geodesic, there is a path which is minimal. That is, if $(i_1, j_1), \ldots, (i_\ell, j_\ell)$ is the path from $S_0$ to $S$, then $\{i_1, \ldots, i_\ell\} \cap \{j_1, \ldots, j_\ell\}$ is empty. To prove this, we need to consider two cases. First, consider the path in $\Gamma_0$

$$S_1 \xrightarrow{(a,b)} S_2 \xrightarrow{(b,c)} S_3.$$

Then $\nabla_{S_2,(b,a)}(b, c) = (a, c)$ is an edge from $S_1$ to $S_3$, in $\Gamma_0$ because it is totally geodesic. Thus, we can avoid adding $b$ and then removing it. Next, consider the path in $\Gamma_0$

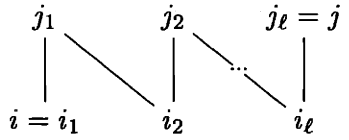
$$S_1 \xrightarrow{(a,b)} S_2 \xrightarrow{(c,a)} S_3.$$

Then $\nabla_{S_2,(b,a)}(c, a) = (c, b)$ is an edge from $S_1$ to $S_3$, in $\Gamma_0$ because it is totally geodesic. Thus, we can avoid removing $a$ and adding it back again. Hence there is a path from $S_0$ to $S$ which is minimal. That is, $S = (S_0 \setminus \{i_1, \ldots, i_\ell\}) \cup \{j_1, \ldots, j_\ell\}$, where a minimal path from $S_0$ to $S$ is

$$S_0 \xrightarrow{(i_1,j_1)} S_1 \xrightarrow{(i_2,j_2)} \cdots \xrightarrow{(i_\ell,j_\ell)} S_\ell = S.$$

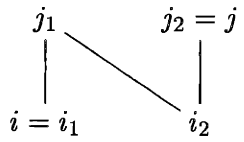
By assumption, $\{i_1, \ldots, i_\ell\} \cap \{j_1, \ldots, j_\ell\}$ is empty. But then using the connection, we can push any edge $(i_a, j_a)$ back to the very same edge $(i_a, j_a)$ going out of $S_0$, and so this edge is in $\Gamma_0$ because it is totally geodesic. Thus, in our bipartite graph, we have $i_a$ connected to $j_a$ for all $a = 1, \ldots, \ell$. Thus, $\#S \cap A_i$ isthe same as $\#S_0 \cap A_i$, for all $i$. And so $\#S \cap A_i$ is independent of our choice of $S$ in $\Gamma_0$.

Thus, we need only consider the case that the bipartite graph given by edges at $S_0$ in $\Gamma_0$ is connected. Call this bipartite graph $\Gamma^{p_0}$. To prove the proposition, it is sufficient to show that this is the complete bipartite graph with partite classes $S_0$ and $S_0^c$. Suppose $i \in S_0$ and $j \in S_0^c$. We want to prove that there is an edge $(i, j)$ in $\Gamma^{p_0}$. Because the bipartite graph is connected, we have a path

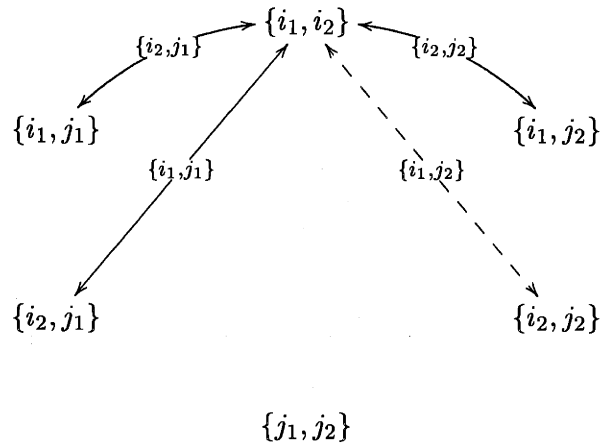



where $i_a \in S_0$ and $j_a \in S_0^c$. We proced by induction on $\ell$. Suppose $\ell = 2$. Then we have the

path




and we want to show that $(i, j)$ is an edge. We can now think of a piece of $\Gamma_0$ as isomorphic to a subgraph of $J(\{i_1, i_2, j_1, j_2\}, 2)$. Using the path, we have the following solid edges in $J(2,4)_0$. and we want to show that the dashed edge is also in $J(2,4)_0$.

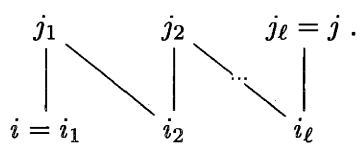



But now using our connection and the fact that $\Gamma_0$ is totally geodesic, we have the following sequence of edges in $\Gamma_0$.

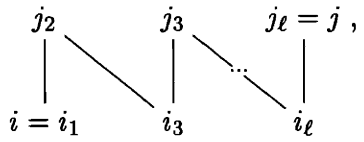
1. $\nabla_{\{i_1,i_2\},(i_1,j_1)}(i_2, j_1) = (i_2, i_1) \in E_{\{i_2,j_1\}}$.
2. $\nabla_{\{i_1,i_2\},(i_1,j_1)}(i_2, j_2) = (i_2, j_2) \in E_{\{i_2,j_1\}}$.
3. $\nabla_{\{i_2,j_1\},(i_2,j_2)}(i_2, i_1) = (j_2, i_1) \in E_{\{j_1,j_2\}}$.
4. $\nabla_{\{j_1,j_2\},(j_2,i_1)}(j_2, i_1) = (i_1, j_2) \in E_{\{i_1,j_1\}}$.
5. $\nabla_{\{i_1,j_1\},(j_1,i_2)}(i_1, j_2) = (i_1, j_2) \in E_{\{i_1,i_2\}}$.

But this last equality tells us precisely that the dashed edge in the diagram above is indeed an edge in the subgraph $\Gamma_0$.

Now suppose that $\ell$ is greater than 2. So we have the path



But we proved above that we actually have an edge from $i_1$ to $j_2$. So we have the path




which has length $\ell - 1$. Thus, by induction, we now have an edge from $i$ to $j$. This completes the proof of the proposition. □

We can also compute the holonomy of $J(n,k)$. We do not include the proof here.

**Proposition 3.3.6.** $Hol(J(n,k)) \cong S_k \times S_{n-k}$.

### 3.3.3 The dihedral group $\mathcal{D}_n$

Let $\mathcal{D} = \mathcal{D}$ be the group of symmetries of the regular $n$-gon: the dihedral group with $2n$ elements. Then $\mathcal{D}$ is a reflection group of type $I_2(n)$, following the notational conventions of Humphreys [Hu]. It is generated by two reflections, and contains $n$ reflections and $n$ rotations. If we let $\Delta$ be the set of reflections in $\mathcal{D}$, then the Cayley graph $\Gamma = (\mathcal{D}, \Delta)$ has vertices corresponding to elements of $\mathcal{D}$. $\sigma \in \mathcal{D}$ is connected to $\tau\sigma$ for every $\tau \in \Delta$. Just half the vertices of $\Gamma$ correspond to symmetries that preserve the orientation of the $n$-gon, and $\sigma$ preserves orientation if and only if $\tau\sigma$ reverses it. Thus the graph is bipartite. The only $n$-regular bipartite graph on $2n$ vertices is $K_{n,n}$.

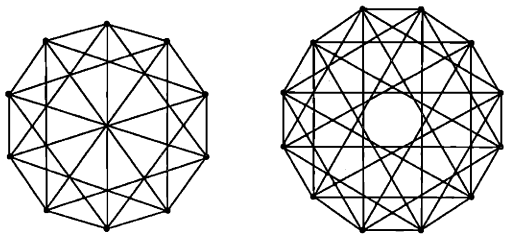
$D_n$ has a natural holonomy free connection defined just as for the permutahedron, using the reflection generating one vertex from another as the label for the corresponding edge. The natural embedding of $D_n$ as the vertices of a regular $2n$-gon produces an exact axial function with inflection free geodesics for that connection.

$\mathcal{D}_3$ is $K_{3,3}$ and also the permutahedron $S_3$ discussed above. The figure below shows two more examples.

This class of graphs is particularly interesting because $\mathcal{D}_n = K_{n,n}$ is the graph associated with a minifold only when $n = 1, 2, 3, 4, 6$. This is precisely when $\mathcal{D}_n$ is a Weyl group. Thus, these provide examples where combinatorics may go further than differential geometry.

We will leave as an exercise the following Betti number count.

**Proposition 3.3.7.** *The Betti numbers of $K_{n,n}$ are invariant of choice of direction $\xi$ and are* $(1, 2, \ldots, 2, 1)$.



(a) (b)

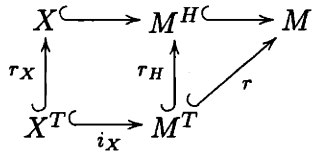
Figure 3-6: This shows the Cayley graphs for (a) $\mathcal{D}_5$ and (b) $\mathcal{D}_6$.

# Chapter 4

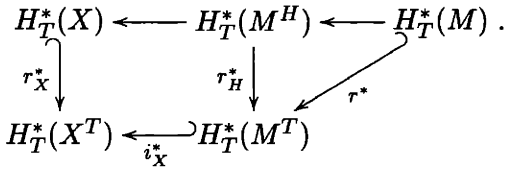
# $S^1$ actions and equivariant cohomology

## 4.1 Preliminaries

As in the introduction, we will use the following notation. Suppose $M$ is a compact, connected symplectic manifold with a Hamiltonian torus action of $T = T^n$. Let $H$ be a codimension one subtorus of $T$ and let $X$ be a connected component of $M^H$. Then there are natural inclusion maps




inducing maps in equivariant cohomology




The GKM theorem is concerned with the case when each component $X$ has dimension at most two. We will be concerned with the case when the dimension of $X$ is at most four.

## 4.2 Reduction to the study of circle actions

It has long been a "folk theorem" that, for a Hamiltonian torus action on a symplectic manifold, the associated equivariant cohomology is determined by $S^1$ actions on certain submanifolds. Recently, Tolman and Weitsman [TW2] used equivariant Morse theory to

prove that the cohomology is determined by that of the *one-skeleton*, the subspace given by the closure of all points whose orbit under the torus action is one-dimensional. Here we use the Chang-Skjelbred theorem, Theorem 1.5.1, to make this folk theorem precise.

**Theorem 4.2.1.** *A class $f \in H_T^*(M^T)$ is in the image of $r^*$ if and only if*

$$i_X^*(f) \in r_X^*(H_T^*(X))$$

*for all codimension-1 subtori $H \subset T$ and all connected components $X$ of $M^H$, where $i_X^*$ restricts a class to the fixed points of $X$ and $r_X^*$ is restriction to the fixed points for each component $X$.*

*Proof.* By Theorem 1.5.1, $f \in im(r^*)$ if and only if $f$ is in the intersection over all codimension-1 subtori $H$ of $r_H^*(H_T^*(M^H))$. Equivalently,

$$f \in \bigcap_H r_H^*(\bigoplus_X H_T^*(X)),$$

where the direct sum is taken over all connected components $X$ of $M^H$. Let $k_X : H_T^*(X) \to H_T^*(M^H)$ be the map which extends any class on $X$ to 0 on other components of $M^H$. Let $k_{X^T} : H_T^*(X^T) \to H_T^*(M^T)$ be the same map on the fixed point sets. Then

$$r_H^*(\bigoplus_X H_T^*(X)) = \bigoplus_X r_X^* \circ k_X(H_T^*(X)).$$

As $k_{X^T} \circ r_X^* = r_H^* \circ k_X$, we have that $f$ is in $im(r^*)$ if and only if

$$f \in \bigoplus_X k_{X^T} \circ r_X^*(H_T^*(X)), \tag{4.1}$$

for all $H$. Now note that $i_X^* \circ k_{X^T} = id$. Because the connected components $X$ are disjoint, we can now apply $i_X^*$ to (4.1) to get

$$i_X^*(f) \in r_X^*(H_T^*(X)), \tag{4.2}$$

for every $H$ and $X$. However, since $\bigoplus_X i_X^*$ is an injection, we can apply $\bigoplus_X k_{X^T}$ to (4.2) to get (4.1). Thus, (4.2) and (4.1) are equivalent. This completes the proof. □

This result provides another proof for Theorem 1.5.4 of Tolman and Weitsman [TW2].

**Definition 4.2.2.** *Let $N \subset M$ be the set of points whose orbits under the $T$ action are 1-dimensional. The one-skeleton $M^{(1)}$ of $M$ is the closure $\overline{N}$.*

Tolman and Weitsman show that the image of $r^* : H_T^*(M) \to H_T^*(M^T)$ is equal to the image of the cohomology of the one-skeleton. This assertion is Theorem 1.5.4 of Section 1.5. We will give its proof now.

*Proof of Theorem 1.5.4.* Because $T$ acts effectively, $N$ consists of points fixed by some codimension-1 torus $H \subset T$ but not by all of $T$, i.e.

$$N = \bigcup_H M^H \backslash M^T$$

where the union is taken over all codimension-1 tori $H \subset T$. As noted above, this is a finite union over all codimension-1 $H$ which appear as isotropy subgroups of points in $M$. Then $\overline{N} = \bigcup_H M^H$, and the inclusion $\gamma_H : M^H \hookrightarrow M$ factors through the inclusion $\gamma : \overline{N} \hookrightarrow M$ for each codimension-1 torus $H$ in $T$. It follows that the induced maps in cohomology also factor. Furthermore, there is an inclusion

$$H_T^*(\overline{N}) \hookrightarrow \bigoplus_{i=1}^{k} H_T^*(M^{H_i}),$$

where $H_i, i = 1, \ldots, k$ are the codimension-1 tori which appear as isotropy subgroups of $T$. Theorem 4.2.1 implies that the map $r^* : H_T^*(M) \to H_T^*(M^T)$ factors through the map

$$\bigoplus_{i=1}^{k} r^*_{M^{H_i}} : \bigoplus_{i=1}^{k} H_T^*(M^{H_i}) \longrightarrow H_T^*(M^T)$$

But then $r^*$ must factor through $j^* : H_T^*(\overline{N}) \to H_T^*(M^T)$. □

Now suppose that $M^T$ consists of isolated fixed points. Then

$$H_T^*(M^T) = \bigoplus_{p \in M^T} S(\mathfrak{t}^*)$$

and any $f \in H_T^*(M^T)$ is a map $f : M^T \to S(\mathfrak{t}^*)$. Furthermore, as $X$ and $X^T$ have trivial $H$ actions, we can rewrite Theorem 4.2.1 in the following way.

**Theorem 4.2.3.** *Under the above hypotheses, the image of $r^*$ is the set of $f : M^T \to S(\mathfrak{t}^*)$ such that*

$$i_X^*(f) : X^T \to S(\mathfrak{t}^*)$$

*is in the image of*

$$r_X^* : H_T^*(X) \to H_T^*(X^T) = \bigoplus_{p \in X^T} S(\mathfrak{t}^*),$$

*where $i_X^*$ restricts a class to the fixed points of $X$ and $r_X^*$ is restriction to the fixed points for each component $X$.*

## 4.3 An extension of a theorem of GKM

When the one-skeleton has pieces of dimensions 2 and 4, we can still have a GKM-like theorem.

First, let $X$ be a compact, connected symplectic four-manifold with an effective Hamiltonian $G = S^1$ action with isolated fixed points $X^{S^1} = \{p_1, \ldots, p_d\}$. Then the equivariant cohomology can be computed as follows.

**Proposition 4.3.1.** *Let $X$ be a compact symplectic 4-manifold with an $S^1$ action as above. The map $r^* : H^*_{S^1}(X) \to H^*_{S^1}(X^{S^1})$ induced by inclusion is an injection with image*

$$\{(f_1, \ldots, f_d) \in \bigoplus_{i=1}^{d} S(\mathfrak{s}^*) \mid f_i - f_j \in x \cdot \mathbb{C}[x],\ \sum_{i=1}^{d} \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(\mathfrak{s}^*)\}, \tag{4.3}$$

*where $\alpha_1^i$ and $\alpha_2^i$ are the (linearly dependent) weights of the $S = S^1$ isotropy action on $T_{p_i}X$.*

*Proof.* The map $r^*$ is injective because $M$ is equivariantly formal. We know that the $f_i$ must satisfy the first condition because the functions constant on all the vertices are the only equivariant classes in degree 0, as $\dim H^0_{S^1}(M) = 1$. The second condition is necessary as a direct result of the ABBV localization formula. Notice that this condition gives us one relation in degree 2 cohomology. A dimension count shows us that these conditions are sufficient. As an $S(\mathfrak{s}^*)$-module, $H^*_S(X) \cong H^*(X) \otimes H^*_S(pt)$. Thus, the equivariant Poincaré polynomial is

$$\begin{aligned} P^S_t(X) &= (1 + (d-2)t^2 + t^4) \cdot (1 + t^2 + t^4 + \ldots) \\ &= 1 + (d-1)t^2 + dt^4 + \cdots + dt^{2n} + \cdots . \end{aligned}$$

As $H^*_S(X)$ is generated in degree 2, the $d-1$ degree 2 classes given by the $(f_1, \ldots, f_d)$ subject to the ABBV condition generate the entire cohomology ring. Thus we have found all the conditions. □

We now prove a slightly more general proposition. Let $\pi_H$ be the map $S(\mathfrak{g}^*) \to S(\mathfrak{h}^*)$.

**Proposition 4.3.2.** *Let $X$ be a compact symplectic 4-manifold with a Hamiltonian $G$ action with $d$ isolated fixed points. Suppose further that there is a codimension-1 subtorus $H$ which acts trivially. The map $r^* : H^*_G(X) \to H^*_G(X^G)$ induced by inclusion is an injection with image*

$$\{(f_1, \ldots, f_d) \in \bigoplus_{i=1}^{d} S(\mathfrak{g}^*) \mid f_i - f_j \in \ker(\pi_H),\ \sum_{i=1}^{d} \frac{f_i}{\alpha_1^i \alpha_2^i} \in S(\mathfrak{g}^*)\}, \tag{4.4}$$

*where $\alpha_1^i$ and $\alpha_2^i$ are the (linearly dependent) weights of the $G$ isotropy action on $T_{p_i}X$.*

*Proof.* As in the case where $X \cong S^2$,

$$H_G^*(X) = H_{G/H}^*(X) \otimes S(\mathfrak{h}^*).$$

Again, choose a complement $L$ to $H$, and write $S(\mathfrak{l}^*) \cong \mathbb{C}[x]$. Then $H_{G/H}^*(X^G)$ can be identified with $\bigoplus_{p \in X^G} = C[x]$. By Proposition 4.3.1, we have $f \in H_G^*(X^G)$ is in the image of $r^* H_G^*(X) \to H_G^*(X^G)$ if and only if the first component of $f$ in $\bigoplus_{p \in X^G} = C[x]$ satisfies the conditions (4.3). But then $f$ must satisfy the conditions (4.4). □

We now discuss the more general case, which extends the result due to [GKM] (Corollary 1.7.2). Suppose that $M$ is a compact, connected symplectic manifold with an effective Hamiltonian $G$ action. Suppose further that this $G$ action has only isolated fixed points $M^G = \{p_1, \ldots, p_d\}$ and that $\dim X_H \leq 4$ for all $H \subset G$ and $X_H$ a connected component of $M^H$, as above. As before, let $f_i \in H_G^*(pt)$ denote the restriction of $f \in H_G^*(M)$ to the fixed point $p_i$. The equivariant cohomology of $M$ can be computed as follows.

**Theorem 4.3.3.** *The image of the injection $r^* : H_G^*(M) \to H_G^*(M^G)$ is the subalgebra of functions $(f_{p_1}, \ldots, f_{p_d}) \in \bigoplus_{i=1}^d S(\mathfrak{g}^*)$ which satisfy*

$$\begin{cases} \pi_H(f_{p_{i_j}}) = \pi_H(f_{p_{i_k}}) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = X_H^G \\ \sum_{j=1}^{l} \frac{f_{i_j}}{\alpha_1^{i_j} \alpha_2^{i_j}} \in S(\mathfrak{g}^*) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = X_H^G \text{ and } \dim X_H = 4 \end{cases}$$

*for all $H \subset G$ codimension-1 tori, where $\alpha_1^{ij}$ and $\alpha_2^{ij}$ are the (linearly dependent) weights of the $G$ action on $T_{p_{i_j}} X_H$.*

*Proof.* By Theorem 4.2.1, $im(r^*)$ consists of $(f_1, \ldots, f_d)$ which have certain properties restricted to each $X_H$. Proposition 4.3.2 lists these restrictions for each $X_H$ of dimension 4. The conditions for $X_H$ of dimension 2 are stated in the GKM theorem. A quick check shows that these are exactly the conditions listed above. □

## 4.4 Hypergraphs and equivariant cohomology

The goal of this section is to survey some results relating equivariant cohomology and hypergraphs. The Chang-Skjelbred theorem says that in the symplectic setting, in order to understand equivariant cohomology of Hamiltonian $T$-spaces, we need only understand the equivariant cohomology of hypergraphs. The GKM theorem states that under certain dimension restrictions, we need only consider graphs. The generalization given in Theorem 4.3.3 allows us to begin extending our understanding of equivariant cohomology of graphs to hypergraphs.

We first restrict our attention to the case when $M$ has isolated fixed points. In Theorem 4.3.3, we relaxed the condition that the one-skeleton be two-dimensional, and we were still able to compute the equivariant cohomology in this case. There is another approach to this situation, given by the work of Y. Karshon [Ka]. Karshon proves that every Hamiltonian $S^1$ action on a 4-manifold with isolated fixed points necessarily extends to a $T^2$ action.

**Theorem 4.4.1 (Karshon).** *Every four-dimensional, compact Hamiltonian $S^1$-space with isolated fixed points comes from a* Kähler toric variety *by restricting the action to a sub-circle.*

This idea of extension will be key in studying hypergraphs. When we can extend a Hamiltonian $S^1$-action to a $T^2$-action on $M$, we are able to find the embedded spheres inside $M$, which allows us to define the $S^1$ equivariant cohomology of $M$ as the projection of the $T^2$ cohomology, by restricting to the sub-circle isomorphic to the original $S^1$. We must show some amount of care, however. S. Tolman has shown that there is one sense in which a higher dimensional analogue of Theorem 4.4.1 is not true.

**Theorem 4.4.2 (Tolman).** *There exists a compact 6-dimensional symplectic non-Kähler manifold $M$ with a Hamiltonian $T^2$-action with isolated fixed points. In particular, the $T^2$-action is not the restriction of a $T^3$-action on $M$, and $M$ is not a toric variety.*

However, consider the case when there is a Hamiltonian $S^1$ action on a compact symplectic 6-manifold, with isolated fixed points. In this case, if we could extend the $S^1$-action to a $T^2$-action, the one-skeleton of the $T^2$ action would necessarily be at most 4-dimensional, and so we could compute its equivariant cohomology. Ultimately, we are left with the question, when does a Hamiltonian $S^1$-action with isolated fixed points extend to a $T^2$-action? The answer is unknown both for six-dimensional manifolds and in general. Accordingly, we will analyze the combinatorics of this situation.

Suppose $\Gamma = (V, E)$ is a hypergraph. That is, $V$ is a set of vertices and $E \subseteq \mathcal{P}(V)$ is a set of hyperedges, which are subsets of $V$. If $p \in e$, then we say $p$ is *incident* to $e$. Our hypergraphs have the additional property that for every $e \in E$, $\#e \geq 2$ and for every pair of hyperedges $e$ and $f$,

$$e \cap f = \begin{cases} \{p\} & \text{for some } p \in V, \text{ or} \\ \emptyset \end{cases}$$

Notice that a graph is a hypergraph where all the hyperedges have size exactly 2. The hypergraphs with which we are concerned are all regular in the sense that the tangent space to a fixed point has a fixed dimension $2n$, and so there are exactly $n$ isotropy weights at each vertex. To make this precise, we define multiplicity and valency.

**Definition 4.4.3.** *Given a hyperedge $e \in E$, we associate to $e$ a positive integer*

$$mult(e) \leq \#e - 1,$$

*its* multiplicity.

**Remark 4.4.4.** *In the manifold setting, the multiplicity is half the dimension of the corresponding submanifold in the one-skeleton.*

We require that $\text{mult}(e) \geq 2$ if $\#e \geq 3$. Now we are ready to define regularity for hypergraphs.

**Definition 4.4.5.** *Given a vertex $v$ in $V$, its* valency *is*

$$val(v) = \sum_{e \ni v} mult(e).$$

*A hypergraph $\Gamma$ is* regular *if $val(v) = val(w)$ for every $v, w \in V$.*

**Remark 4.4.6.** *Geometrically, the valency of a regular hypergraph is half the dimension of the corresponding $T$-manifold.*

We will be particularly interested in regular hypergraphs which have multiplicities at most than 2. In this case, the one-skeleton has dimension at most 4, and so using Theorem 4.3.3, we have a recipe for its equivariant cohomology. Additionally, when a hypergraph comes from a manifold, each hyperedge is labeled with a codimension one subgroup $T_e \subseteq T$ which fixes the corresponding submanifold in the one-skeleton.

As a result of the Chang-Skjelbred Theorem, we need only be able to compute the cohomology of a hyperedge. In other words, to iterate a principle of Sue Tolman's, the computation of equivariant cohomology for torus actions always reduces to the computation of equivariant cohomology for circle actions. At this point, then, we will restrict our attention to the combinatorics of hyperedges, since these are precisely the manifolds with circle actions that we must understand. A hypergraph that is a hyperedge has a vertex set $V$ and one hyperedge which is the set of all the vertices. Accordingly, let $\Xi = e = (V, \{V\})$ be a hyperedge with multiplicity $m$. Geometrically, each vertex will have associated to it $m$ weights corresponding to the $T/T_e$ action. Thus, these weights will sit naturally in $\mathfrak{t}_e^{\perp}$. Combinatorially, suppose we have an assignment of $m$ non-zero vectors in $\mathfrak{t}^*$ to each vertex $p$ in $V$, for some vector space $\mathfrak{t}$. For each $p \in V$, we will denote the $m$ vectors assigned to $p$ by $v_1, \ldots, v_m$. The $v_i$'s need not be distinct. Let

$$A_e = \{(p, v_i) \mid p \in V,\ i = 1, \ldots, m\}.$$

In this case, we make the following definition.

**Definition 4.4.7.** *Given a hyperedge $e = (V, \{V\})$, and vector assignment as above, a* weight pairing *is an involution $\Psi : A_e \to A_e$ such that for every $p \in V$, $\Psi((p, v_j)) = (q_j, -v_j) \in A_e$ for some $p \neq q_j \in V$, and $\#\{q_j \mid j = 1, \ldots, m\} = m$. The* associated graph *to this weight pairing*

*$\Psi$ is the graph with vertex set $V$ and edge set $(p,q) \in E$ whenever $\Psi((p,v_i)) = (q,v_j)$ for some $i$ and $j$.*

We are trying to mimic the geometry of the following. Suppose $S \cong S^1$ acts on $M$ with corresponding hypergraph $\Xi$. Geometrically, we are looking for a torus $T$ containing $S$ as a closed subgroup which acts on $M$ in a GKM fashion. In this case, suppose $\pi : \mathfrak{t}^* \to \mathfrak{s}^*$ is the natural projection induced by inclusion $\mathfrak{s} \subseteq \mathfrak{t}$. Then we can compute

$$H_S^*(M) = \pi(H_T^*(M)),$$

since we understand $H_T^*(M)$ via the GKM theorem.

**Definition 4.4.8.** *Suppose $\Xi$ is a hyperedge with a vector assignment $\alpha$ of vectors in a vector space $\mathfrak{s}^*$ and weight pairing $\Psi$. A* graph extension $\Gamma_\Xi$ *of $\Xi = e = (V, \{V\})$ is the graph with vertex set $\tilde{V}_e = V$ and edge set $\tilde{E}_e = \{\{p,q\} \mid \Psi(\alpha^i_{p,e}) = \alpha^j_{q,e}\}$.*

*Moreover, suppose that $\alpha_e$ extends to an axial function $\tilde{\alpha}_e$ on $\Gamma_\Xi$ with $\pi_e : \mathfrak{t}^* \to \mathfrak{s}^*$. Then the map $\pi_e$ extends to a map $\pi_e : S(\mathfrak{t}^*) \to S(\mathfrak{s}^*)$. We define the* hypergraph cohomology *of $e$ to be*

$$H^*(e, \alpha_e) = \left\{ \pi_e(f) \;\middle|\; f \in H^*(\Gamma_\Xi, \tilde{\alpha}_e) \right\}.$$

*The ring $H^*(\Gamma_\Xi, \tilde{\alpha}_e)$ is well-defined, since $\Gamma_\Xi$ is a graph with axial function $\tilde{\alpha}_e$. Note that $H^*(e, \alpha_e)$ depends on our choice of extension.*

We give $H^*(e,\alpha)$ a ring structure by point-wise multiplication. A product of two maps will still satisfy the above conditions since $\pi_e$ is a ring morphism. Also, notice that $H^*(e,\alpha)$ contains $S(\mathfrak{s}^*)$ as a subring, the ring of constant maps from $V$ to $S(\mathfrak{s}^*)$. Thus, $H^*(e,\alpha)$ is a module over $S(\mathfrak{s}^*)$.

The goal of defining this abstract hyperedge cohomology ring is that when our hypergraph corresponds to some symplectic manifold with a Hamiltonian torus action, this hypergraph cohomology ring should coincide with the equivariant cohomology ring of the manifold. If there is a submanifold $X$ in the one-skeleton, and the $S^1$ action extends to a $T^k$ action which is a GKM action, then there is an extension of the hyperedge corresponding to $X$ in the fashion described combinatorially above. In fact, if $X$ is a four-dimensional manifold, and the extension is the one given in Theorem 4.4.1, then this definition agrees with the recipe given in Theorem 4.3.3. The remaining open question is when a Hamiltonian $S^1$ action on $M^{2n}$ with isolated fixed points extend to a $T^2$ action.

Thus far in this Chapter, we have relaxed the second of the GKM hypotheses, concerning the dimension of the one-skeleton. It is also possible to relax the first of the conditions, concerning the fixed point sets. H. Li has some results classifying semi-free Hamiltonian $S^1$ actions on symplectic 6-manifolds [L].

## 4.5 Examples

Here we demonstrate the use of Theorem 4.3.3 in computing equivariant cohomology. In the first example, we compute the $S^1$-equivariant cohomology of $\mathbb{C}P^2$ with a Hamiltonian circle action. In the second, we calculate the $T^2$-equivariant cohomology of $\mathbb{C}P^3$. Finally, we find the equivariant cohomology of a two manifolds obtained by symplectic reduction.

### 4.5.1 $S^1$ action on $\mathbb{C}P^2$.

Consider $\mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Let $T = S^1$ act on $\mathbb{C}P^2$ by

$$e^{i\theta} \cdot [z_0 : z_1 : z_2] = [e^{-i\theta} z_0 : z_1 : e^{i\theta} z_2].$$

This action has three fixed points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.

The weights at these fixed points are

| Fixed point | Weights |
|---|---|
| $p_1 = [1 : 0 : 0]$ | $x, 2x,$ |
| $p_2 = [0 : 1 : 0]$ | $-x, x,$ |
| $p_3 = [0 : 0 : 1]$ | $-2x, -x,$ |

where we have identified $\mathfrak{t}^*$ with degree one polynomials in $\mathbb{C}[x]$. As cohomology elements, these are assigned degree two. The image of the equivariant cohomology $H^*_{S^1}(\mathbb{C}P^2)$ in $H^*_{S^1}(\{p_1, p_2, p_3\}) \cong \bigoplus_{i=1}^3 \mathbb{C}[x]$ is the subalgebra generated by the triples of functions $(f_1, f_2, f_3)$ such that

$$f_i - f_j \in x \cdot \mathbb{C}[x] \text{ for every } i \text{ and } j, \text{ and}$$

$$\frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} \in \mathbb{C}[x].$$

### 4.5.2 $T^2$ action on $\mathbb{C}P^3$.

We use the cohomology computed above to compute the $T^2$-equivariant cohomology of $\mathbb{C}P^3$.

The second example we consider is a $T^2$ action on $\mathbb{C}P^3$. Consider $\mathbb{C}P^3$ with homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$. Let $T^2$ act on $\mathbb{C}P^3$ by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot [z_0 : z_1 : z_2 : z_3] = [e^{-i\theta_1} z_0 : z_1 : e^{i\theta_1} z_2 : e^{i\theta_2} z_3].$$

This action has four fixed points, $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$ and $[0 : 0 : 0 : 1]$. The image of the moment map for this action is show in the figure below.

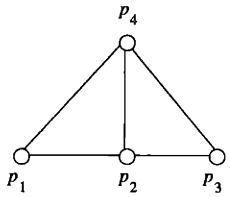



Figure 4-1: This shows the image of the moment map for $T^2$ acting on $M = \mathbb{C}P^3$, as described above.

The weights at these fixed points are

| Fixed point | Weights |
|---|---|
| $p_1 = [1:0:0:0]$ | $x, 2x, x+y,$ |
| $p_2 = [0:1:0:0]$ | $-x, x, y,$ |
| $p_3 = [0:0:1:0]$ | $-2x, -x, y-x,$ |
| $p_4 = [0:0:0:1]$ | $-x-y, -y, x-y.$ |

Theorem 4.3.3 tells us that the image of the equivariant cohomology $H^*_{T^2}(\mathbb{C}P^3)$ in

$$H^*_{T^2}(\{p_1, p_2, p_3, p_4\}) \cong \bigoplus_{i=1}^{4} \mathbb{C}[x, y]$$

is the ring of functions $(f_1, f_2, f_3, f_4)$ such that

$$f_i - f_j \in (x) \cdot \mathbb{C}[x, y] \quad \text{for every } i, j \in \{1, 2, 3\},$$

$$\begin{aligned} \frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} &\in \mathbb{C}[x, y], \\ f_1 - f_4 &\in (y+x) \cdot \mathbb{C}[x, y], \\ f_2 - f_4 &\in (y) \cdot \mathbb{C}[x, y], \\ f_3 - f_4 &\in (y-x) \cdot \mathbb{C}[x, y]. \end{aligned}$$

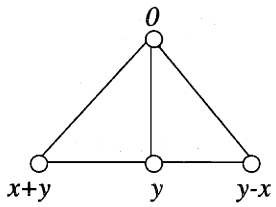



Figure 4-2: This shows an equivariant class of $H^*_{T^2}(\mathbb{C}P^3)$, shown as an element of the equivariant cohomology of the fixed points.

### 4.5.3 $S^1$ action on an $S^1$-reduction of $SU(3)/T$.

Let $\mathcal{O}_\lambda$ be the coadjoint orbit of $SU(3)$ through the generic point $\lambda \in \mathfrak{t}^*$, the dual of the Lie algebra $\mathfrak{t}$ of the maximal 2-torus $T$ in $SU(3)$. Recall that $T$ acts on $\mathcal{O}_\lambda$ in a Hamiltonian fashion, and (one choice of) the moment map

$$\Phi_T : \mathcal{O}_\lambda \longrightarrow \mathfrak{t}^*$$

takes each matrix to its diagonal entries. Equivalently, $\Phi_T$ is the composition of the inclusion of $\mathcal{O}_\lambda$ into $\mathfrak{su}(3)^*$ and projection of $\mathfrak{su}(3)^*$ onto $\mathfrak{t}^*$.

We compute the equivariant cohomology of $M = \mathcal{O}_\lambda//H$, the *symplectic reduction* of $\mathcal{O}_\lambda$ by a circle $H$ chosen such that the reduced space is a manifold. Let $H \subset T$ be any copy of $S^1$ which fixes a two-sphere in $\mathcal{O}_\lambda$. Then the moment map $\Phi_H : \mathcal{O}_\lambda \to \mathfrak{h}^*$ for the $H$ action is the map $\Phi_T$ followed by the projection $\pi_H : \mathfrak{t}^* \to \mathfrak{h}^*$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{t}$. The symplectic reduction at $\mu$ by $H$ is by definition

$$M = \mathcal{O}_\lambda//H := \Phi_H^{-1}(\mu)/H,$$

where $\mu$ is a regular value for $\Phi_H$. Note that there is a residual $T/H \cong S^1$ action on $M$. We use Theorem 4.2.1 to calculate the the corresponding equivariant cohomology of $M$.

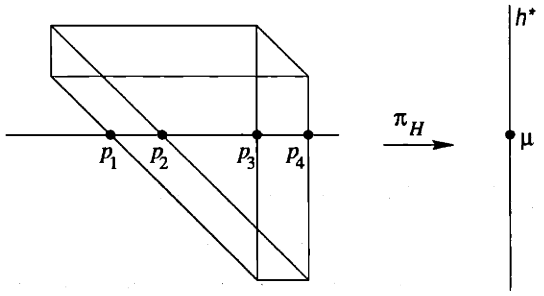



Figure 4-3: On the left is the image of the moment map for $T$ acting on $\mathcal{O}_\lambda$. The cut through the moment polytope for $\mathcal{O}_\lambda$ corresponds to the symplectic reduction of $\mathcal{O}_\lambda$ by $H$ at $\mu$, for some choice of $\mathfrak{h}^\perp$.

One can easily see that there are four fixed points of this action, which we denote by $p_i$ for $i = 1, \ldots, 4$. For each $p_i$, $\Phi_T^{-1}(p_i)$ lies on a two-sphere in $\mathcal{O}_\lambda$, denoted $S_i^2$, which is fixed by a a subgroup $H_i \cong S^1$ of $T$. Note that $H_i$ is complementary to $H$ in $T$.

The weights of the $T/H$ action on the tangent space $T_{p_i}M$ are determined by the $T$ action on $S_i^2$. Let $n_i$ and $s_i$ be fixed points of the $T$ action on $S_i^2$. Note that the condition that $\mu$ be a regular value of $\Phi_H$ ensures that $\Phi_T^{-1}(p_i) \neq n_i, s_i$. Furthermore, by assumption the set $\Phi_T^{-1}(p_i)$ is point-wise fixed by $H_i$. Thus in the reduction, the $T/H$ action on $T_{p_i}M$ is isomorphic to the $H_i$ action on this space.

Denote the weights of the $T$ action on $T_{n_i}\mathcal{O}_\lambda$ by $\pm\alpha_1, \pm\alpha_2$, and $\pm\alpha_3 = \pm(\alpha_1+\alpha_2)$, where

the signs depend on $i$. The weights of the $H$ action on the reduction $M$ are determined by projecting the $\alpha_i$ to $\mathfrak{h}_1{}^*$.

At $p_1$, the weight $\alpha_3$ projects to 0 and the other two weights both project to the generator $x$ of $S(\mathfrak{h}_1{}^*) \cong \mathbb{C}[x]$. Similarly, at $p_2$ the weights are $x$ and $-x$, at $p_3$ they are $x$ and $-x$ and at $p_4$ they are both $-x$. The image of the moment map $\Phi_H : M \to \mathfrak{h}^*$, with weights, is shown in Figure 4.5.3.

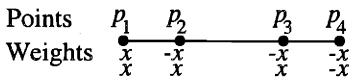



Figure 4-4: The image of the moment map for the $T/H$ action on $M = \mathcal{O}_\lambda//H$, with the weights for the isotropy action on the tangent space of the fixed points.

Finally, this tells us that the equivariant cohomology of $M$ is

$$H^*_{S^1}(M) \cong \left\{ f : V \to \mathbb{C}[x] \;\middle|\; f_i - f_j \in x \cdot \mathbb{C}[x], \; \frac{f_1}{x^2} - \frac{f_2}{x^2} - \frac{f_3}{x^2} + \frac{f_4}{x^2} \in \mathbb{C}[x] \right\}.$$

Notice that this computation leads us to the $T/S^1$-equivariant cohomology of $M \cong \mathcal{O}_\lambda//S^1$ for a coadjoint orbit of $SU(n)$, as the submanifolds that appear are identical to those shown the above $SU(3)$ case.

### 4.5.4 $T^2$ action on an $S^1$-redution of $SU(4)/T$

Let $M = \mathcal{O}_\lambda$ be a 12-dimesional coadjoint orbit for $SU(4)$. Then the moment polytope for the $T^3$ action on $M$ is a truncated octahedron. Let $H \subset T$ be any copy of $S^1$ which fixes a two-sphere in $\mathcal{O}_\lambda$. Then the moment map $\Phi_H : \mathcal{O}_\lambda \to \mathfrak{h}^*$ for the $H$ action is the map $\Phi_T$ followed by the projection $\pi_H : \mathfrak{t}^* \to \mathfrak{h}^*$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{t}$. Let $\mu$ be a regular value for $\Phi_H$. Then the symplectic reduction at $\mu$ by $H$ is

$$M = \mathcal{O}_\lambda//H := \Phi_H^{-1}(\mu)/H.$$

Note that there is a residual $T/H \cong T^2$ action on $M$. Moreover, the moment polytope for this $T^2$ action is simply $\pi_H^{-1}(\mu)$. In the first figure below, we show the moment map image for $M$, with $\pi_H^{-1}(\mu)$ shaded. Next, we show the moment polytope for $M//S^1$ for some choice of $\mu$ and $\lambda$ in the second figure below.

This reduction satisfies the dimension restrictions of Theorem 4.3.3, since each component of the 2-skeleton of $M$ is 6-dimensional, so each component of the 1-skeleton of the reduction $M//S^1$ is 4-dimensional. Thus, we can use Theorem 4.3.3 to calculate the the corresponding equivariant cohomology of $M//S^1$.

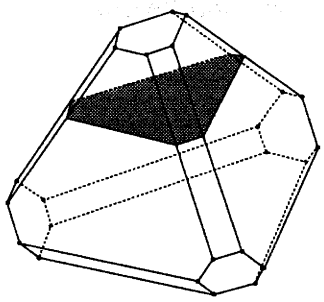


Figure 4-5: This shows a cut corresponding to an $S^1$ reduction of the $T^3$ action on $SU(4)/T$.

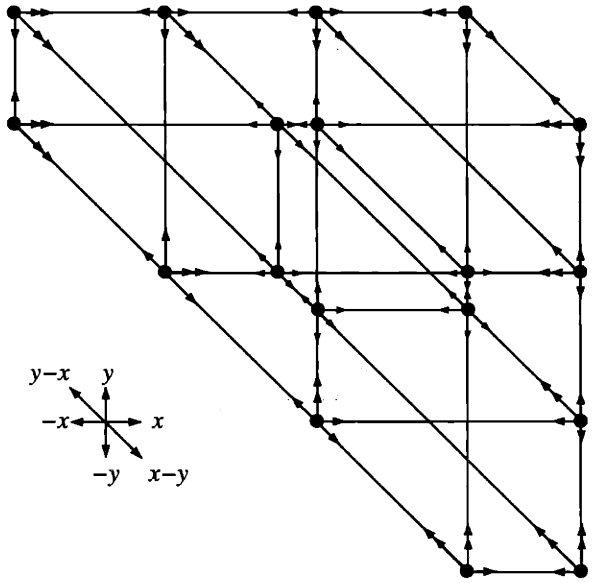



Figure 4-6: This shows the hypergraph associated to an $S^1$ reduction of the $T^3$ action on $SU(4)/T$, with isotropy weights.

In general, if $M$ is a GKM manifold, and if the two-skeleton of a $T^n$ manifold is at most 6-dimensional, then we can apply Theorem 4.3.3 to an $S^1$-rection of $M$.

# Chapter 5

# Real Loci

## 5.1 Preliminaries

Atiyah observed in [A] that if $M$ is a compact symplectic manifold and $\tau$ a Hamiltonian action of an $n$-dimensional torus $T$ on $M$, then the cohomology groups of $M$ can be computed from the cohomology groups of the fixed point set $M^T$ of $\tau$. Explicitly

$$H^*(M;\mathbb{R}) \quad = \quad \sum_{i=1}^{N} H^{*-d_i}(F_i;\mathbb{R}), \tag{5.1}$$

where the $F_i$ are the connected components of $M^T$ and $d_i$ is the Bott-Morse index of $F_i$. This result is also true in equivariant cohomology:

$$H_T^*(M;\mathbb{R}) = \sum_{i=1}^{N} H_T^{*-d_i}(F_i;\mathbb{R}) = \sum_{i=1}^{N} H^{*-d_i}(F_i \times BT;\mathbb{R}). \tag{5.2}$$

This is a consequence of Atiyah's result and equivariant formality for Hamiltonian $T$-manifolds, as discussed in Chapter 1.

In [Du], Duistermaat proved a "real form" version of (5.1). Let $\sigma : M \to M$ be an anti-symplectic involution with the property that

$$\sigma \circ \tau_g \quad = \quad \tau_{g^{-1}} \circ \sigma. \tag{5.3}$$

**Definition 5.1.1.** *Let $X = M^\sigma$ be the fixed point set of $\sigma$. We call $X$ the* real locus *of $M$.*

The motivating example of this setup is a complex manifold $M$ with a complex conjugation $\sigma$. Duistermaat proved that components of the moment map are Morse functions not only for $M$, but also for $X$, and used this to compute the ordinary cohomology of $X$.

**Theorem 5.1.2 (Duistermaat).** *Suppose $M$ is a symplectic manifold with a Hamiltonian torus*

*action $\tau$ and an anti-symplectic involution $\sigma$. Let $X = M^\sigma$ denote the real locus of $M$. Then*

$$H^*(X;\mathbb{Z}_2) = \sum_{i=1}^{N} H^{*-\frac{d_i}{2}}(F_i^\sigma;\mathbb{Z}_2), \tag{5.4}$$

*where $F_i^\sigma$ are the real loci of the fixed point sets of $M$, and the $d_i$ are the indices of the fixed point sets $F_i$.*

The $\mathbb{Z}_2$ coefficients are essential here; the theorem does not hold with real coefficients.

The first result in this chapter is an equivariant analogue of (5.4) similar to the equivariant analogue (5.2) of Atiyah's result (5.1). By (5.3), the group

$$T_{\mathbb{R}} = \{g \in T \mid g^2 = id\} \cong (\mathbb{Z}_2)^n \tag{5.5}$$

acts on $X$ and we will prove

$$H^*_{T_{\mathbb{R}}}(X;\mathbb{Z}_2) \;=\; \sum_{i=1}^{N} H^{*-\frac{d_i}{2}}_{T_{\mathbb{R}}}(F_i^\sigma;\mathbb{Z}_2). \tag{5.6}$$

The idea of the proof will be to derive (5.6) from (5.4) by a simple trick.

The isomorphisms (5.1), (5.2), (5.4), and (5.6) are all isomorphisms in *additive* cohomology. We also consider below the ring structure of $H^*_{T_{\mathbb{R}}}(X;\mathbb{Z}_2)$. We first note that our results thus far, combined with a theorem of Allday and Puppe, suffice to prove that $X$ is equivariantly formal. The second main theorem of this chapter will be a $\mathbb{Z}_2$ version of the GKM theorem for the manifold $X$. We define the one-skeleton of the real locus to be the set

$$X^{(1)} = \{x \in X \mid \#(T_{\mathbb{R}} \cdot x) \leq 2\}. \tag{5.7}$$

Assume in addition to the above that $M^T = X^{T_{\mathbb{R}}}$ and the real locus of the one-skeleton is the same as the one-skeleton of the real locus. We will call a manifold with these properties a mod 2 *GKM manifold*. Let $r : X^{T_{\mathbb{R}}} \to X$ be the natural inclusion map. As a result of equivariant formality and localization, the map

$$r^* : H^*_{T_{\mathbb{R}}}(X;\mathbb{Z}_2) \;\to\; H^*_{T_{\mathbb{R}}}(X^{T_{\mathbb{R}}};\mathbb{Z}_2) \tag{5.8}$$

is injective, and by factoring through the one-skeleton, we achieve the desired analogue of the GKM theorem. This completely determines the ring structure of $H^*_{T_{\mathbb{R}}}(X;\mathbb{Z}_2)$. This theorem was proved independently by Schmid [S] using different techniques from ours.

In Chapter 4, we generalized the GKM result to the case where the one-skeleton has dimension at most 4. Assume in addition to the dimension hypothesis that $M^T = X^{T_{\mathbb{R}}}$ and the real locus of the one-skeleton is the same as the one-skeleton of the real locus. We

will call a manifold with these properties a mod 2 *GH manifold*. The third main result in this chapter is a $\mathbb{Z}_2$ version of Theorem 4.3.3 for the real locus $X$.

We reëmphasize that Duistermaat's techniques only apply to additive cohomology. Since we are able to obtain results concerning the ring structure of the equivariant cohomology and its relationship to ordinary cohomology, we also obtain statements about the ring structure of the ordinary cohomology as well. Indeed, in many cases, Duistermaat's isomorphism (5.4) turns out to give a ring isomorphism. (See Corollaries 5.5.7 and 5.5.8 to Theorem 5.5.6 and Corollaries 5.6.6 and 5.6.7 to Theorem 5.6.5.) When describing these ring isomorphisms, we will make use of the following notation. The symbol

$$H^{2*}(M;\mathbb{Z}_2)$$

will denote the subring

$$\bigoplus_i H^{2i}(M;\mathbb{Z}_2) \subseteq H^*(M;\mathbb{Z}_2),$$

endowed with a new grading wherein a class in $H^{2i}(M;\mathbb{Z}_2)$ is given degree $i$ (and similarly for equivariant cohomology). Then under suitable hypotheses, the additive isomorphism of Duistermaat becomes an isomorphism of graded rings.

In Section 5.7.2, we discuss an application of the results of this chapter to string theory. The $\mathbb{Z}_2$-equivariant cohomology ring of $T^n$ with $\mathbb{Z}_2$ coefficients classifies all possible orientifold configurations of Type II string theories, compactified on $T^n$. We explain how to compute this cohomology ring.

## 5.2 Additive equivariant cohomology

We will first prove the equivariant analogue of Theorem 5.1.2, computing the additive structure of the equivariant cohomology of $X$.

**Theorem 5.2.1.** *Suppose $M$ is a symplectic manifold with a Hamiltonian action $\tau$ of a torus $T^n = T$ and an anti-symplectic involution $\sigma$. Let $X = M^\sigma$ denote the real locus of $M$. Then the group $T_\mathbb{R}$ acts on $X$, and the $T_\mathbb{R}$-equivariant cohomology of $X$ with $\mathbb{Z}_2$ coefficients is*

$$H^*_{T_\mathbb{R}}(X;\mathbb{Z}_2) \;=\; \sum_{i=1}^{N} H^{*-\frac{d_i}{2}}_{T_\mathbb{R}}(F_i^\sigma;\mathbb{Z}_2). \tag{5.9}$$

*Proof.* Consider the product action of $T^n$ on

$$M \times \underbrace{(\mathbb{C}^d \times \cdots \times \mathbb{C}^d)}_{n},$$

in which each $S^1$ factor acts by multiplication on the corresponding factor of $\mathbb{C}^d$. This is a

Hamiltonian action. If $(\phi_1, \ldots, \phi_n) = \Phi : M \to \mathbb{R}^n$ is the moment map associated with $\tau$, then the moment map of this product action is $\Psi = (\psi_1, \ldots \psi_n)$, with

$$\psi_i(m, z_{1,1}, \ldots, z_{1,d}, \ldots, z_{d,d}) = \phi_i(m) + \sum_{j=1}^{d} |z_{i,j}|^2.$$

Let $a = (a_1, \ldots a_n) \in \mathbb{R}^n$. If $a_i > \sup(\phi_i)$ for every $i$, then $\Psi^{-1}(a)$ and $M \times S^{2d-1} \times \cdots \times S^{2d-1}$ are equivariantly diffeomorphic, so the reduced space

$$M_{\text{red}} = M//_a T^n = \psi^{-1}(a)/T^n$$

is diffeomorphic to $M \times_{T^n} (S^{2d-1} \times \cdots \times S^{2d-1})$. Moreover, there is another action of $T^n$ on $M \times \mathbb{C}^d \times \cdots \times \mathbb{C}^d$, namely $\tau$ coupled with the trivial action on $(\mathbb{C}^d)^n$. Since this commutes with the product action, it induces a Hamiltonian action of $T^n$ on $M_{\text{red}}$. In addition, one gets from $\sigma$ an involution

$$(m, z_1, \ldots, z_d) \mapsto (\sigma(m), \overline{z_1}, \ldots, \overline{z_d})$$

of $M \times \mathbb{C}^d \times \cdots \times \mathbb{C}^d$. This induces an anti-symplectic involution $\tilde{\sigma}$ on $M_{\text{red}}$. Thus, one can apply Duistermaat's theorem to $M_{\text{red}}$ to get a formula for the cohomology of the space

$$M_{\text{red}}^{\tilde{\sigma}} = X \times_{T_{\mathbb{R}}} (S^{d-1} \times \cdots \times S^{d-1})$$

in terms of the cohomology of the spaces

$$Z_i^d := F_i^\sigma \times_{T_{\mathbb{R}}} (S^{d-1} \times \cdots \times S^{d-1}) = F_i^\sigma \times (\mathbb{R}P^{d-1} \times \cdots \times \mathbb{R}P^{d-1}).$$

Now $F_i^\sigma \times BT_{\mathbb{R}}$ is obtained from $Z_i^d$ by attaching cells of dimension $d$ and higher. So, for fixed $k$, the sequence $H^k(Z_i^d; \mathbb{Z}_2)$ stabilizes as $d$ grows large, and moreover is equal to the equivariant cohomology of $X$. Thus one obtains from (5.1.2) the following real analogue:

$$H^*_{T_{\mathbb{R}}}(X; \mathbb{Z}_2) = \sum H^{*-\frac{d_i}{2}}_{T_{\mathbb{R}}}(F_i^\sigma; \mathbb{Z}_2),$$

where $T_{\mathbb{R}} = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. □

A similar result is obtained by Schmid under slightly different hypotheses in [S], using techniques from equivariant Morse theory.

## 5.3 Equivariant formality

As a result of Theorem 5.2.1, we have the inequality

$$\dim_{\mathbb{Z}_2} H^*(X;\mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H^*(X^{T_{\mathbb{R}}};\mathbb{Z}_2).$$

But Theorem 3.4.10 of [AP] claims the opposite inequality, with equality if and only if $X$ is equivariantly formal. Thus, we conclude the following.

**Theorem 5.3.1.** *The equivariant cohomology $H^*_{T_{\mathbb{R}}}(X,\mathbb{Z}_2)$ is a free module over $H^*_{T_{\mathbb{R}}}$ generated in dimension zero. Moreover, as an $H^*_{T_{\mathbb{R}}}$ module, $H^*_{T_{\mathbb{R}}}(X,\mathbb{Z}_2)$ is isomorphic to*

$$H^*_{T_{\mathbb{R}}} \otimes_{\mathbb{Z}_2} H^*(X;\mathbb{Z}_2). \tag{5.10}$$

This is proved in [BiGH] by a direct computation of the appropriate spectral sequence, but in the interest of brevity, we do not include this computation here.

## 5.4 The Chang-Skjelbred theorem in the $\mathbb{Z}_2$ setting

As a result of the mod 2 localization theorem, the kernel and cokernel of the map

$$r^* : H^*_{T_{\mathbb{R}}}(X;\mathbb{Z}_2) \to H^*_{T_{\mathbb{R}}}(X^{T_{\mathbb{R}}};\mathbb{Z}_2)$$

are torsion submodules. As a result of the collapse of the spectral sequence proved in the previous section, then, $r^*$ is an injection. In the case of the original manifold $M$, the Chang-Skjelbred theorem [CS] identifies the image of this map. They note that an analogous result holds for a 2-torus action with $\mathbb{Z}_2$ coefficients, and indeed, since localization holds over $\mathbb{Z}_2$, this is straight forward.

**Theorem 5.4.1.** *Suppose that $H^*_{T_{\mathbb{R}}}(M,\mathbb{Z}_2)$ is a free $H^*_{T_{\mathbb{R}}}$-module. For a subgroup $H_{\mathbb{R}} < T_{\mathbb{R}}$, let $r_{H_{\mathbb{R}}} : M^{(T_{\mathbb{R}})^*} \hookrightarrow M^{H_{\mathbb{R}}}$ denote the inclusion. Then we have*

$$r^* H^*_{T_{\mathbb{R}}}(M,\mathbb{Z}_2) = \bigcap_{\substack{H_{\mathbb{R}}<T_{\mathbb{R}} \\ |H_{\mathbb{R}}|=2^{n-1}}} r^*_{H_{\mathbb{R}}} H^*_{T_{\mathbb{R}}}(M^{H_{\mathbb{R}}};\mathbb{Z}_2).$$

We omit the proof here. The reader may find a proof in [BiGH].

Now suppose that $Z_{H_{\mathbb{R}}}$ is a connected component of $M^{H_{\mathbb{R}}}$ for some subgroup $H_{\mathbb{R}}$ of $T_{\mathbb{R}}$ of order $|H_{\mathbb{R}}| = 2^{n-1}$. Let $r_{Z_{H_{\mathbb{R}}}}$ be the inclusion

$$r_{Z_{H_{\mathbb{R}}}} : Z^{T_{\mathbb{R}}}_{H_{\mathbb{R}}} \to Z_{H_{\mathbb{R}}}$$

of the fixed points of $Z_{H_{\mathbb{R}}}$ into $Z_{H_{\mathbb{R}}}$. Let $r_{Z_{H_{\mathbb{R}}}}$ be the inclusion

$$r_{Z_{H_{\mathbb{R}}}} : Z_{H_{\mathbb{R}}}^{T_{\mathbb{R}}} \to M^{T_{\mathbb{R}}}$$

of the fixed points of $Z_{H_{\mathbb{R}}}$ into all of the fixed points. Then we have the following corollary of Theorem 5.4.1.

**Corollary 5.4.2.** *Suppose that $H^*_{T_{\mathbb{R}}}(M, \mathbb{Z}_2)$ is a free $H^*_{T_{\mathbb{R}}}$-module. A class*

$$f \in H^*_{T_{\mathbb{R}}}(M^{T_{\mathbb{R}}}; \mathbb{Z}_2)$$

*is in the image of $r^*$ if and only if*

$$r^*_{Z_{H_{\mathbb{R}}}}(f) \in r^*_{Z_{H_{\mathbb{R}}}}(H^*_{T_{\mathbb{R}}}(Z_{H_{\mathbb{R}}}; \mathbb{Z}_2))$$

*for every subgroup $H_{\mathbb{R}}$ of $T_{\mathbb{R}}$ of order $|H_{\mathbb{R}}| = 2^{n-1}$ and every connected component $Z_{H_{\mathbb{R}}}$ of $M^{H_{\mathbb{R}}}$.*

*Proof.* The proof is identical to the proof of Theorem 4.2.1 in Chapter 4. It follows directly from Theorem 5.4.1. □

**Remark 5.4.3.** *In his thesis [S], C. Schmid proves the injectivity of $r^*$ directly, using equivariant Morse-Kirwan theory. Thus, one is left to wonder if a* mod 2 *version of Kirwan's surjectivity (Theorem 1.4.2) holds. This is addressed in [GH3].*

## 5.5 A real locus version of the GKM theorem

The goal of this section is to prove an analogue of the GKM theorem (Corollary 1.7.2) for the real locus $X$ of $M$. The proof will require two hypotheses on $X$, namely

$$X^{T_{\mathbb{R}}} = M^T \tag{5.11}$$

and

$$X^{(1)} = X \cap M^{(1)}, \tag{5.12}$$

where $M^{(1)}$ is the one-skeleton of $M$ and $X^{(1)}$ the one-skeleton of $X$. We will begin by analyzing these conditions and their implications, much in the way that we analyzed the implications of the GKM conditions in Section 1.6. Let $\mathbb{Z}^*_T$ be the weight lattice of $T$. By the mod 2 reduction of a weight $\tilde{\alpha} \in \mathbb{Z}^*_T$, we mean its image $\alpha$ in $\mathbb{Z}^*_T/2\mathbb{Z}^*_T$. We will prove a real analogue of Theorem 1.6.5.

**Theorem 5.5.1.** *Suppose $M$ satisfies the hypotheses of Theorem 1.6.5. Then the conditions $X^{T_{\mathbb{R}}} = M^T$ and $X^{(1)} = X \cap M^{(1)}$ are satisfied if and only if, for every $p \in M^T$, the* mod 2 *reduced weights, $\alpha_{i,p}$, are all distinct and non-zero.*

*Proof.* Let $Y$ be a connected component of $M^{T_{\mathbb{R}}}$. Then $Y$ is a $T$-invariant symplectic submanifold of $M$, and the action of $T$ on it is Hamiltonian, so it contains at least one $T$-fixed point $p$. However, the hypotheses above imply that the linear isotropy action of $T_{\mathbb{R}}$ on $T_pM$ has no fixed points other than the origin. Hence, $\dim(Y) = 0$ and $Y = \{p\}$. This argument applies to all the connected components of $M^{T_{\mathbb{R}}}$, hence the connected components are just the fixed points of $T$, and thus $X^{T_{\mathbb{R}}} = M^T$.

The proof that $X^{(1)} = X \cap M^{(1)}$ is similar. Let $H_{\mathbb{R}}$ be a subgroup of $T_{\mathbb{R}}$ of index 2, and let $Y$ be a connected component of $M^{H_{\mathbb{R}}}$. Then $Y$ is a $T$-invariant submanifold of $M$, and because $\sigma \circ \tau_g = \tau_{g^{-1}} \circ \sigma$, it is also $\sigma$-invariant. Let $p \in Y$ be a $T$-fixed point, and let

$$T_pM = V_1 \oplus \cdots \oplus V_d$$

be the decomposition of $T_pM$ into the 2-dimensional weight spaces corresponding to the $\tilde{\alpha}_{i,p}$. By the hypotheses on the reduced weights $\alpha_{i,p}$, either

$$(T_pM)^{H_{\mathbb{R}}} = \{0\},$$

in which case $Y = \{p\}$, or

$$(T_pM)^{H_{\mathbb{R}}} = V_i = T_pY \tag{5.13}$$

for some $i$. Let $\chi_i$ be the character of $T$ associated with the representation of $T$ on $V_i$ and let $H = \ker(\chi_i)$. Then $H_{\mathbb{R}} \subset H$ and

$$(T_pM)^H = V_i.$$

Thus, by (5.13), $Y$ is the connected component of $M^H$ containing $p$, and in particular, $Y$ is contained in $M^{(1)}$. Thus,

$$Y^\sigma \subseteq X \cap M^{(1)}.$$

Applying this argument to all index 2 subgroups $H_{\mathbb{R}}$ of $T_{\mathbb{R}}$ and all connected components of the fixed point sets of these groups, one obtains the inclusion

$$X^{(1)} \subseteq X \cap M^{(1)}.$$

The reverse inclusion is obvious. This completes the proof. □

The theorem above motivates the following definition.

**Definition 5.5.2.** *If $M$ is a GKM manifold, and if for every $p \in M^T$, the* mod 2 *reduced weights, $\alpha_{i,p}$, are all distinct and non-zero, we will say that $M$ is a* mod 2 GKM manifold.

This definition imposes some rather severe restrictions on the manifold $M$. For instance, the cardinality of the set of mod 2 reduced weights, $\mathbb{Z}_T^*/2\mathbb{Z}_T^*$, is $2^n$. Therefore,

since the reduced weights $\alpha_{i,p}$ are distinct and non-zero for $i = 1, \ldots, d$, we must have that $d \leq 2^n - 1$. Hence,

$$\dim(M) = 2d \leq 2^{n+1} - 2. \tag{5.14}$$

For example, if $n = 2$, then $\dim(M) \leq 6$.

There are also relatively few compact *homogeneous* symplectic manifolds (i.e. coadjoint orbits) are mod 2 GKM manifolds. Consider coadjoint orbits of the classical compact simple Lie groups associated with the Dynkin diagrams $A_n$, $B_n$, $C_n$ and $D_n$. Let $\varepsilon_i$, for $i = 1, \ldots, n$, be the standard basis vectors of $\mathbb{R}^n$. The positive roots associated to the Dynkin diagram $A_n$ consist of

$$\varepsilon_i - \varepsilon_j, \;\; i < j;$$

so their mod 2 reductions are distinct and non-zero. However, for $B_n$, $C_n$, and $D_n$, this list of positive roots contains

$$\varepsilon_i - \varepsilon_j \text{ and } \varepsilon_i + \varepsilon_j, \;\; i < j,$$

so we conclude

**Theorem 5.5.3.** *Each coadjoint orbit of $SU(n)$ is a* mod 2 *GKM space. However, for other compact simple Lie groups,* no *maximal coadjoint orbit can be a* mod 2 *GKM space.*

On the other hand, on a more positive note, one has

**Theorem 5.5.4.** *If $M$ is a non-singular toric variety, it is a* mod 2 *GKM space.*

*Proof.* If $M$ is a non-singular toric variety, the weights $\tilde{\alpha}_{i,p}$, $i = 1, \ldots, n$, are a $\mathbb{Z}$-basis for $\mathbb{Z}_T^*$, so their images in $\mathbb{Z}_T^*/2\mathbb{Z}_T^*$ are a $\mathbb{Z}_2$ basis of $\mathbb{Z}_T^*/2\mathbb{Z}_T^*$. □

We will now prove a real locus version of the GKM theorem with $\mathbb{Z}_2$ coefficients. Recall from Chapter 1 that Corollary 1.7.2 of GKM characterizes the image of $r^* : H_T^*(M; \mathbb{C}) \to H_T^*(M^T; \mathbb{C})$ in terms of the weights of the isotropy representations of $T$ on the tangent spaces at the fixed points.

To prove an analogue of this for the real locus of a symplectic manifold, we must first compute the $\mathbb{Z}_2$-equivariant cohomology with $\mathbb{Z}_2$ coefficients of $\mathbb{R}P^1$. Recall that $S^1$ acts on $\mathbb{C}P^1$ by $\theta \cdot [z_0 : z_1] = [z_o : e^{i\theta} z_1]$. This is a Hamiltonian action, with respect to the Fubini-Study symplectic form on $\mathbb{C}P^1$. Furthermore, complex conjugation is an anti-symplectic involution on $\mathbb{C}P^1$, with fixed point set $\mathbb{R}P^1$. There is a residual action of $\mathbb{Z}_2$ on $\mathbb{R}P^1 \cong S^1$ which reflects $S^1$ about the $y$-axis.

**Lemma 5.5.5.** *Let $N$ and $S$ denote the fixed points of the $\mathbb{Z}_2$ action on $\mathbb{R}P^1$. Then the image of the map*

$$r^* : H_{\mathbb{Z}_2}^*(\mathbb{R}P^1; \mathbb{Z}_2) \to H_{\mathbb{Z}_2}^*(N; \mathbb{Z}_2) \oplus H_{\mathbb{Z}_2}^*(S; \mathbb{Z}_2)$$

*is the set of pairs* $(f_N, f_S)$ *such that*

$$f_N + f_S \in x \cdot \mathbb{Z}_2[x].$$

*Proof.* It is clear that the constant functions are equivariant classes in

$$H^0_{\mathbb{Z}_2}(\mathbb{R}P^1; \mathbb{Z}_2).$$

Furthermore, we know that $\dim H^0_{\mathbb{Z}_2}(\mathbb{R}P^1; \mathbb{Z}_2) = 1$, and so these are the only equivariant classes. Finally, $\dim H^i_{\mathbb{Z}_2}(\mathbb{R}P^1; \mathbb{Z}_2) = 2$ for $i > 0$, and so indeed, the condition stated is the only condition of pairs $(f_N, f_S) \in H^*_{\mathbb{Z}_2}(N; \mathbb{Z}_2) \oplus H^*_{\mathbb{Z}_2}(S; \mathbb{Z}_2)$. □

We now state and prove a mod 2 version of the GKM theorem.

**Theorem 5.5.6.** *Suppose* $M$ *is a* mod 2 *GKM manifold. An element*

$$f \in H^*_{T_{\mathbb{R}}}(X^{T_{\mathbb{R}}}; \mathbb{Z}_2)$$

*can be thought of as a map* $f : V_\Gamma \to \mathbb{Z}_2[x_1, \ldots, x_n]$*, and such a map* $f$ *is in the image of* $r^*$ *if and only if, for each edge* $e = \{p, q\}$ *of* $\Gamma$

$$f_p - f_q \in \alpha_e \cdot \mathbb{Z}_2[x_1, \ldots, x_n],$$

*where* $\alpha_e \in \mathbb{Z}_2[x_1, \ldots, x_n]$ *is the image of the weight* $\tilde{\alpha}_e$.

*Proof.* The result follows immediately from Corollary 5.4.2 and Lemma 5.5.5. □

The results of this section and the previous section have been proved independently by Schmid [S]. Schmid uses an equivariant Morse theoretic approach, and consequently the proofs are quite different.

As a result of equivariant formality, we get two corollaries of Theorem 5.5.6 concerning the relation between the ring structure of the cohomology of $M$ and the cohomology of $X$.

**Corollary 5.5.7.** *Suppose that* $M$ *is a GKM manifold and a* mod 2 *GKM manifold. Then there is a graded ring isomorphism*

$$H^{2*}_T(M; \mathbb{Z}_2) \cong H^*_{T_{\mathbb{R}}}(X; \mathbb{Z}_2).$$

**Corollary 5.5.8.** *Suppose that* $M$ *is a GKM manifold and a* mod 2 *GKM manifold. Then there is a graded ring isomorphism*

$$H^{2*}(M; \mathbb{Z}_2) \cong H^*(X; \mathbb{Z}_2).$$

Note that this last corollary strengthens Duistermaat's original result from an isomorphism of vector spaces to an isomorphism of rings.

**Remark 5.5.9.** *Some of the results of this section, including Theorem 5.5.6, are valid not only for the real locus $X$ of a Hamiltonian $T$-manifold, but more generally for any compact $T_{\mathbb{R}}$-manifold $X$ which satisfies the following properties:*

1. *$X$ is equivariantly formal;*
2. *$X^{T_{\mathbb{R}}}$ is finite; and*
3. *the weights of $X$ satisfy the properties of a* mod 2 *GKM manifold.*

## 5.6 Extending the real locus version of the GKM theorem

In Chapter 4, we generalized Corollary 1.7.2 to the case where the one-skeleton has dimension at most 4. The goal of this section is to prove a real version of that theorem with $\mathbb{Z}_2$ coefficients. Again, we require the hypotheses that the $(\mathbb{Z}_2)^n$-fixed points of the real locus are the same as the $T$-fixed points of $M$ as in (5.11); and that the real locus of the one-skeleton is the same as the one-skeleton of the real locus, as in (5.12). Finally, we require

$$\#M^T < \infty,$$

and

$$\dim(M^{(1)}) \leq 4.$$

If a manifold satisfies these last two hypotheses, we will say that it is a *GH manifold*. These hypotheses have a nice interpretation in terms of the isotropy representations of $T$ at the fixed points of $M$.

**Theorem 5.6.1.** *The conditions $\#M^T < \infty$ and $\dim(M^{(1)}) \leq 4$ are satisfied if and only if the weights $\alpha_{i,p}$ of the isotropy representation of $T$ on $T_pM$ have the property that every three span a vector subspace of dimension at least two.*

These hypotheses on $M$ have real analogues, namely that $\#X < \infty$ and the one-skeleton $X^{(1)}$ of $X$ is at most 2-dimensional. We will state without proof the following real analogue of Theorem 5.6.1.

**Theorem 5.6.2.** *Suppose that $M$ satisfies the hypotheses of Theorem 5.6.1. If the conditions $X^{T_{\mathbb{R}}} = M^T$ and $X^{(1)} = M^{(1)} \cap X$ are satisfied, then for every $p \in M^T$, the* mod 2 *reduced weights $\alpha_{i,p}^{\#}$ are all non-zero, and each element of $S((T_{\mathbb{R}})^*) = \mathbb{Z}_2[x_1, \ldots, x_n]$ appears no more than twice.*

The proof of this theorem is nearly identical to that of Theorem 5.5.1. The hypotheses of this theorem, although weaker than those of Theorem 5.5.1, still impose restrictions on the manifold $M$. The cardinality of the set of mod 2 reduced weights is $2^n$. Since the weights

are non-zero, and each weight can appear at most twice,

$$d \leq 2 \cdot (2^n - 1),$$

and so

$$\dim(M) = 2d \leq 2 \cdot (2 \cdot (2^n - 1)) = 2^{n+2} - 4.$$

For instance, if $n = 2$, $\dim(M) \leq 12$. We will now show an example where the condition that the reduced weights be non-zero is *not* satisfied.

EXAMPLE 1. Consider $\mathbb{C}P^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$. Let $T = S^1$ act on $\mathbb{C}P^2$ by

$$e^{i\theta} \cdot [z_0 : z_1 : z_2] = [e^{-i\theta} z_0 : z_1 : e^{i\theta} z_2].$$

This action has three fixed points: $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$.

The weights at these fixed points are

| Fixed point | Weights |
|---|---|
| $p_1 = [1 : 0 : 0]$ | $x, 2x$ |
| $p_2 = [0 : 1 : 0]$ | $-x, x$ |
| $p_3 = [0 : 0 : 1]$ | $-2x, -x$ |

where we have identified $\mathfrak{t}^*$ with degree one polynomials in $\mathbb{C}[x]$. As cohomology elements, these are assigned degree two. Using Theorem 4.3.3, we can compute the $S^1$ equivariant cohomology of $\mathbb{C}P^2$ as follows. The image of the equivariant cohomology $H^*_{S^1}(\mathbb{C}P^2)$ in

$$H^*_{S^1}(\{p_1, p_2, p_3\}) \cong \bigoplus_{i=1}^{3} \mathbb{C}[x]$$

is the subalgebra generated by the triples of functions $(f_1, f_2, f_3)$ such that

$$f_i - f_j \in x \cdot \mathbb{C}[x] \text{ for every } i \text{ and } j, \text{ and}$$

$$\frac{f_1}{2x^2} - \frac{f_2}{x^2} + \frac{f_3}{2x^2} \in \mathbb{C}[x].$$

However, when we try to compute the $\mathbb{Z}_2$ equivariant cohomology of $\mathbb{R}P^2$, the real locus of $\mathbb{C}P^2$, we run into a problem. The mod 2 reduced weights are given in the table below.

| Fixed point | Weights |
|---|---|
| $p_1 = [1 : 0 : 0]$ | $x, 0,$ |
| $p_2 = [0 : 1 : 0]$ | $x, x,$ |
| $p_3 = [0 : 0 : 1]$ | $0, x.$ |

The problem with this $\mathbb{Z}_2$ action on $\mathbb{R}P^2$ is that it no longer has isolated fixed points. There is an entire $\mathbb{R}P^1$ which is fixed by this $\mathbb{Z}_2$ action. Thus, we cannot compute the $\mathbb{Z}_2$ equivariant cohomology of $\mathbb{R}P^2$ using these methods. $\diamondsuit$

We make the following definition, analogous to the definition of mod 2 GKM manifolds given in Section 5.5.

**Definition 5.6.3.** *Suppose that $M$ is a GH manifold, and furthermore that $X^{T_{\mathbb{R}}} = M^T$ and $X^{(1)} = M^{(1)} \cap X$. In this case, we will say that $M$ is a* mod 2 GH space.

We can use results of Chapter 4 to compute the $(\mathbb{Z}_2)^n$ equivariant cohomology of a mod 2 GH manifold. We now prove mod 2 analogues of Proposition 4.3.2 and Theorem 4.3.3.

**Lemma 5.6.4.** *Let $M$ be a compact, connected symplectic 4-manifold with an effective Hamiltonian $S^1$ action with isolated fixed points $M^{S^1} = \{p_1, \ldots, p_d\}$. Suppose further that $M$ is a* mod 2 *GH manifold with real locus $X$. The map*

$$r^* : H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \to H^*_{\mathbb{Z}_2}(X^{\mathbb{Z}_2}; \mathbb{Z}_2)$$

*induced by inclusion is an injection with image*

$$\left\{ (f_1, \ldots, f_d) \in \bigoplus_{i=1}^{d} \mathbb{Z}_2[x] \;\middle|\; \begin{array}{l} f_i - f_j \in x \cdot \mathbb{Z}_2[x], \\ \sum_{i=1}^{d} \frac{f_i}{\alpha_1^i \alpha_2^i} \in \mathbb{Z}_2[x] \end{array} \right\}, \tag{5.15}$$

*where $\alpha_1^i$ and $\alpha_2^i$ are the linearly dependent weights of the $\mathbb{Z}_2$ isotropy representation on $T_{p_i}X$. (In this case, $\alpha_1^i = \alpha_2^i = x$.)*

*Proof.* The map $r^*$ is injective because $X$ is equivariantly formal. We know that the $f_i$ must satisfy the first condition because the functions constant on all the vertices are the only equivariant classes in degree 0, as $\dim H^0_{\mathbb{Z}_2}(X; \mathbb{Z}_2) = 1$. The second condition is necessary as a direct result of the $\mathbb{Z}_2$ version of the localization theorem proved in Section 5.4. Notice that this condition gives us one relation in degree 1 cohomology. A dimension count shows us that these conditions are sufficient. As an $S((\mathbb{Z}_2)^*)$-module, $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2) \cong H^*(X; \mathbb{Z}_2) \otimes H^*_{\mathbb{Z}_2}(pt; \mathbb{Z}_2)$. Thus, the equivariant Poincaré polynomial is

$$\begin{aligned} P_t^{\mathbb{Z}_2}(X) &= (1 + (d-2)t + t^2) \cdot (1 + t + t^2 + \ldots) \\ &= 1 + (d-1)t + dt^2 + \cdots + dt^n + \cdots . \end{aligned}$$

As $H^*_{\mathbb{Z}_2}(X; \mathbb{Z}_2)$ is generated in degree 1, the $d-1$ degree 1 classes given by the $(f_1, \ldots, f_d)$ subject to the localization condition generate the entire cohomology ring. Thus we have found all the conditions. $\square$

We now prove the mod 2 analogue of Theorem 4.3.3.

**Theorem 5.6.5.** *Suppose that $M$ is a* mod 2 *GH manifold with $T$ fixed points fixed points $M^T = \{p_1, \ldots, p_d\}$. Let $f_i \in H^*_{T_\mathbb{R}}$ denote the restriction of $f \in H^*_{T_\mathbb{R}}(X)$ to the fixed point $p_i$. The image of the injection $r^* : H^*_{T_\mathbb{R}}(X) \to H^*_{T_\mathbb{R}}(X^{T_\mathbb{R}})$ is the subalgebra of functions $(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d H^*_{T_\mathbb{R}}$ which satisfy*

$$\begin{cases} \pi^*_{H_\mathbb{R}}(f_{i_j}) = \pi^*_{H_\mathbb{R}}(f_{i_k}) & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = Z^{T_\mathbb{R}}_{H_\mathbb{R}} \\ \sum_{j=1}^{l} \frac{f_{i_j}}{\alpha_1^{i_j} \alpha_2^{i_j}} \in H^*_{T_\mathbb{R}} & \text{if } \{p_{i_1}, \ldots, p_{i_l}\} = Z^{T}_{H_\mathbb{R}} \text{ and } \dim Z_{H_\mathbb{R}} = 4 \end{cases}$$

*for all subgroups $H_\mathbb{R}$ of $T_\mathbb{R}$ of order $|H_\mathbb{R}| = 2^{n-1}$ and all connected components $Z_{H_\mathbb{R}}$ of $X^{H_\mathbb{R}}$, where $\alpha_1^{i_j}$ and $\alpha_2^{i_j}$ are the (linearly dependent) weights of the $T_\mathbb{R}$ action on $T_{p_{i_j}} Z_{H_\mathbb{R}}$.*

*Proof.* This follows immediately from Corollary 5.4.2 and Lemma 5.6.4. □

There are two immediate corollaries in this setting, analogous to Corollaries 5.5.7 and 5.5.8.

**Corollary 5.6.6.** *Suppose that $M$ is a GH manifold, and that $M^T = X^{T_\mathbb{R}}$ and $M^{(1)} \cap X = X^{(1)}$. Then there is a graded ring isomorphism*

$$H^{2*}_T(M; \mathbb{Z}_2) \cong H^*_{T_\mathbb{R}}(X; \mathbb{Z}_2).$$

**Corollary 5.6.7.** *Suppose that $M$ is a GH manifold, and that $M^T = X^{T_\mathbb{R}}$ and $M^{(1)} \cap X = X^{(1)}$. Then there is a graded ring isomorphism*

$$H^{2*}(M; \mathbb{Z}_2) \cong H^*(X; \mathbb{Z}_2).$$

## 5.7 Examples

### 5.7.1 Toric varieties

The equivariant cohomology of a Kähler toric variety can be computed in two different ways. On the one hand, we can use the GKM theory discussed above to compute the equivariant cohomology in terms of the polytope $\Delta$ associated to the variety. On the other hand, following Danilov [D], one can compute the equivariant cohomology ring directly, as a polynomial ring over Chern classes of normal bundles associated to certain codimension-one subvarieties, modulo a certain ideal.

Similarly, the equivariant cohomology of real toric varieties can be computed in two ways. Since the real points of a Kähler toric variety are a real GKM space, we can compute the corresponding equivariant cohomology in terms of the graph underlying $\Delta$. On the

other hand, this is also a polynomial ring in the Stiefel-Whitney classes for the real toric variety. This alternate computation has been discussed in [DJ].

### 5.7.2 An application to string theory

Consider the $\mathbb{Z}_2$ action on $T^n$, which reflects each copy of $S^1$. Then the equivariant cohomology ring

$$H^*_{\mathbb{Z}_2}(T^n; \mathbb{Z}_2)$$

classifies all possible orientifold configurations of Type II string theories, compactified on $T^n$. See Section 3 and Appendix C of [dB] for more details. Yang-Hui He pointed this example out to us. Using the results of Section 5.5, we can now compute this equivariant cohomology.

First, we recognize $T^n$ as the real locus of $M = \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 = (\mathbb{C}P^1)^n$. This space $M$ has a natural $T^n$ action, where the $i$th copy of $S^1$ acts in the standard fashion on the $i$th copy of $\mathbb{C}P^1$. We can compute the $(\mathbb{Z}_2)^n$-equivariant cohomology of this space quite easily. The GKM graph associated to $(\mathbb{C}P^1)^n$ with the $T^n$ action described above is the $n$-dimensional hypercube. The vertices correspond to the binary words of length $n$. Two binary words are connected by an edge if they differ in exactly one bit. Suppose $v$ and $w$ differ in exactly the $i$th bit. Then the weight associated to the edge $(v, w)$ is $x_i$. Thus, when $n = 3$, the GKM graph and weights are shown in the figure below.

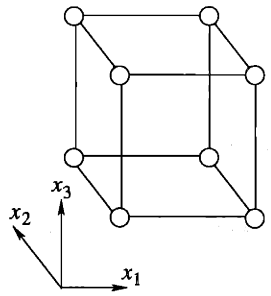



Figure 5-1: This shows the GKM graph and the weights for $(\mathbb{C}P^1)^3$.

Notice that the reduced weights are all non-zero and are distinct in $\mathbb{Z}_T/2\mathbb{Z}_T$. Thus, we can apply Theorem 5.5.6 to compute

$$H_{(\mathbb{Z}_2)^n}(T^n; \mathbb{Z}_2).$$

That is, the equivariant cohomology is the set of functions $f : V \to \mathbb{Z}_2[x_1, \ldots, x_n]$ such that for every edge $(v, w) \in E$, we have

$$f(v) + f(w) \in x_i \cdot \mathbb{Z}_2[x_1, \ldots, x_n].$$

We can now consider the copy of $\mathbb{Z}_2$ sitting diagonally inside $(\mathbb{Z}_2)^n$. This copy of $\mathbb{Z}_2$ acts on $T^n$, and this is the action that originally interested physicists. We can now compute the $\mathbb{Z}_2$-equivariant cohomology simply by projecting

$$\pi : S(((\mathbb{Z}_2)^n)^*) = \mathbb{Z}_2[x_1, \ldots, x_n] \to \mathbb{Z}_2[x] = S((\mathbb{Z}_2)^*)$$

where $x_i$ gets sent to $x$. Then

$$H^*_{\mathbb{Z}_2}(T^n; \mathbb{Z}_2) = \pi(H_{(\mathbb{Z}_2)^n}(T^n; \mathbb{Z}_2)).$$

### 5.7.3 $T^2$ on $SO(5)/T$

Let $M = SO(5)/T$. Then $T^2$ acts on $M$ by right multiplication. This is a Hamiltonian action, with moment polytope shown below. We can use the GKM recipe to compute

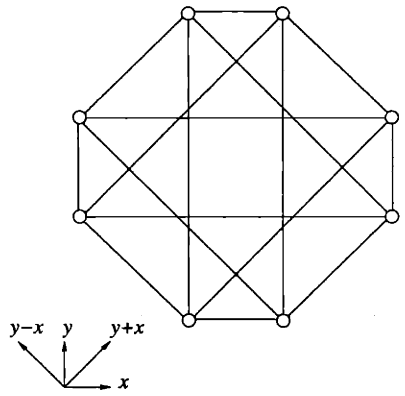



Figure 5-2: This shows the GKM graph and the weights for the $T^2$ action on $SO(5)/T$.

the equivariant cohomology. Moreover, there is a natural involution on $M$, induced by complex conjugation. However, the real locus $X = M^\sigma$ does not satisfy the mod 2 GKM conditions, since at each vertex, it has weights $x+y$ and $x-y$, which are equivalent modulo 2. It does satisfy the weight restrictions of a mod 2 GH space, but it does not satisfy the restriction on one-skeletons. Thus, we also cannot use the results of Section 5.6 to compute the equivariant cohomology of the real locus.

This is a simple enough example that one can compute the equivariant cohomology directly. However, there are more complex examples for which we still do not know how to compute the equivariant cohomology.

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