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How to construct a gravitating quantum electron star

Andrea Allais[1] and John McGreevy[2,*]
[1]*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Department of Physics, University of California at San Diego, La Jolla, California 92093, USA*

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Motivated by the holographic study of Fermi surfaces, we develop methods to solve Einstein gravity coupled to fermions and gauge fields, with anti–de Sitter boundary conditions and a chemical potential.

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I. INTRODUCTION

In this paper we address the following question: what asymptotically anti–de Sitter (AdS) spacetimes result from a finite density of gravitating, charged fermions? While this question is a natural one in the context of the study of gravity solutions with the covariant infrared cutoff provided by AdS, there is also a second set of motivations coming from condensed matter physics.

There is considerable experimental evidence for the existence of materials whose electronic structure cannot be described by Fermi liquid theory, nor by any other known effective theory. A signature of these exotic materials is the presence of a well-defined Fermi surface, without long-lived quasiparticles. More precisely, as in ordinary Fermi liquids, the electron operator Green's function has a zero-frequency singularity located over an entire surface in momentum space (the Fermi surface). However, contrary to Fermi liquid behavior, the relative frequency width of this resonance does not vanish as k approaches the Fermi surface.

The short life of the quasiparticles indicates that strong interactions are at play, a fact which stymies conventional theoretical investigation. The holographic duality [1–3], on the other hand, provides a different starting point, far from weak coupling. Hence, the gravitational system we study in this paper, seen through the eye of the duality, could be a useful toy model displaying non-Fermi liquid phenomenology. In the present scarcity of viable approaches, such a model would be very valuable even if it has exotic short-distance physics.

A. Statement of the problem

Our target field theory is a relativistic conformal field theory (CFT) with a gravity dual, a global U(1) symmetry (our proxy for fermion number), and a fermion operator charged under the symmetry (our proxy for bare electrons). As a gravity dual, we are led to study asymptotically AdS boundary conditions on gravity coupled to quantum electrodynamics. We want to study the CFT at nonzero U(1) charge density, so we turn on a chemical potential μ for the U(1) symmetry; this is encoded in the boundary conditions $A_t|_{\mathrm{bound}} = \mu$.

Wilsonian naturalness suggests that we use the following action in the bulk (for a review of the duality from this point of view, see e.g. Refs. [4,5]):

$$Z = \int [\mathcal{D}(g, A, \psi)] \exp\left[\mathrm{i} \int \mathrm{d}^4 x \sqrt{g} \left(\frac{R - 2\Lambda}{\kappa^2} - \frac{F^2}{4q^2} + \bar{\psi}(\mathrm{i}\not{D} - m)\psi \right) \right], \tag{1}$$

where D is a derivative covariant under coordinate transformations and U(1) gauge transformations. We normalize the gauge field so that the fermion field has unit charge.

This model has been has been investigated in Refs. [6–17] under various (drastic) simplifying assumptions, which we briefly summarize.

The large-N limit in the CFT implies $\kappa^2 \Lambda \ll 1$ and $q \ll 1$, and suppresses the fluctuations of the metric and of the gauge field. References [6–9] worked in the limit $\kappa \to 0$, $q \to 0$ with finite κ/q, where the backreaction of the fermions on the background can be ignored. This results in the Reissner-Nördstrom extremal black hole groundstate [18,19], with its AdS_2 near-horizon region and associated zero-temperature entropy.

In this approximation, one finds [9] fermion Green's functions of non-Fermi liquid character, of the form

$$G \sim \frac{1}{\omega^{2\nu} + |k| - k_F}. \tag{2}$$

However, the fact that the backreaction can be ignored means that the charge and energy density carried by the fermions is negligible compared to the charge and mass of the black hole. From the dual point of view, this describes a system in which the fermions that form a non-Fermi liquid are only a small fraction of all the degrees of freedom. The large bath which makes up most of the system has some issues: the finite zero-temperature entropy and associated instabilities encourage us to lift the too-strong assumption that bulk matter fields do not affect the dynamics.

In Ref. [10], the bulk fermions were treated as a charged, gravitating fluid, in a Thomas-Fermi approximation. According to Ref. [10], this approximation is valid in the limit of large fermion mass $m^2/\Lambda \gg 1$, and results in a

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[*] On leave from Department of Physics, MIT, Cambridge, Massachusetts 02139, USA.

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much less exotic IR geometry. However, a fermion with a large mass is dual to an operator with a large anomalous dimension, so this large-mass limit leads to a somewhat unphysical class of field theories. Moreover, the associated large quantum numbers imply parametrically many Fermi surfaces in the fermion Green's function [13].

Reference [16] studied the limit of a large external magnetic field, where the bulk fermions are effectively one-dimensional and may be treated using bosonization.

B. This paper

In this work, we will retain the full quantum nature of the fermionic field, while treating the path integrals over the metric and the gauge field in the saddle-point approximation,[1] as is natural at large N. It is reasonable to hope that bulk fermions with $m^2/\Lambda \sim 1$, which must be treated quantum mechanically, realize a happy medium between the too-exotic AdS$_2$ solution, which results from no fermions, and the classical electron star, which results from heavy bulk fermions.

The leading contribution to the partition function comes from the on-shell action, evaluated on the field configuration that solves the equations of motion

$$\begin{cases} D_\mu F^{\mu\nu} = q^2 J^\nu, \\ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 [T^{(\psi)}_{\mu\nu} + T^{(A)}_{\mu\nu}], \end{cases} \tag{3}$$

where

$$J^\mu = \langle \bar\psi \gamma^\mu \psi \rangle, \tag{4}$$

$$T^{(\psi)}_{\mu\nu} = \langle \bar\psi \gamma_{(\mu} \mathrm{i} D_{\nu)} \psi \rangle, \tag{5}$$

$$T^{(A)}_{\mu\nu} = \frac{1}{q^2}\left[F_{\mu\rho} F^{\rho}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\beta\alpha} \right]. \tag{6}$$

The expectation values are computed with respect to the fermionic field path integral,

$$\langle \cdots \rangle = \frac{1}{Z_\psi} \int [\mathcal{D}\psi] \cdots \exp\left[\mathrm{i} \int \mathrm{d}^4 x \sqrt{g}\, \bar\psi (\mathrm{i} \not{D} - m) \psi \right]. \tag{7}$$

The solution of the saddle-point equations constitutes a rather difficult problem, because the fermion current and stress tensor are *nonlocal* functionals of the background fields g and A, for which there is no hope of finding an explicit closed-form expression. In this sense the system (3) is more similar to an integro-differential system of equations than to a system of differential equations.

Like for other integro-differential systems, a solution can be found numerically via an iterative approach. Starting from some configuration of g and A, one computes numerically the corresponding fermion current and stress tensor. Then one uses these to construct a new configuration of g and A via Eq. (3), and repeats the process. Eventually, one hopes, the process converges to a fixed point, which is a solution of the system of equations.

Once a solution to the field equations is found, small perturbations about it can deliver all the correlators of the dual field theory. This approach was used successfully in the frozen-geometry approximation (valid when $\kappa \to 0$) in Refs. [14,15].

We emphasize that this iterative scheme for solving the saddle-point equations has two separate parts:

(1) Given a fixed set of currents[2] $\langle j^\mu \rangle$, $\langle T^\mu{}_\nu \rangle$, solve Eq. (3) to find the gauge field A and geometry g. This is a relatively standard problem, which we treat in Sec. III.

(2) Given the background A, g, evaluate the currents. This problem is more difficult and less familiar. We will lavish a great deal of attention on it, in Sec. II.

In Sec. IV we present our results: the first gravitating quantum electron stars without a large magnetic field. The eager reader may safely read this last section first, as reference to Secs. II and III is kept to a minimum.

Further details can be found in the appendices. Of particular note is Appendix B, where we describe the many alluring ways in which one should *not* approach the construction of a gravitating quantum electron star.

II. COMPUTATION OF THE FERMIONIC CURRENTS

A. Regularization and renormalization

In the continuum, the currents are divergent quantities. They must be regulated and then renormalized while preserving the symmetries of the low-energy theory, that is, gauge invariance and general covariance [20]. The simplest way of doing so, at least conceptually, is to use a covariant regulator. For example, one could set up the problem in Euclidean space, and use a heat-kernel regulator to define the bare currents,

$$J^\mu_0 = \langle \bar\psi \gamma^\mu e^{-s^2 \not{D}^2} \psi \rangle, \tag{8}$$

$$T^{\mu\nu}_0 = \langle \bar\psi \gamma^{(\mu} \mathrm{i} D^{\nu)} e^{-s^2 \not{D}^2} \psi \rangle. \tag{9}$$

With this choice of regulator, the bare currents have the following small-s expansion:

$$J^\mu_0 = c_1 \log \frac{s}{L_{\mathrm{IR}}} D_\nu F^{\mu\nu} + J^\mu_R + \mathcal{O}(s^2), \tag{10}$$

---

[1] The neglection of metric and gauge fluctuations, in this context, is equivalent to the Hartree-Fock approximation.

[2] For brevity, we refer collectively to the charge current and the stress tensor as currents.

$$
\begin{aligned}
T_0^{\mu\nu} = {} & \frac{c_2}{s^4} g^{\mu\nu} + \frac{1}{s^2}(c_3 G^{\mu\nu} + c_4 m^2 g^{\mu\nu}) \\
& + \log\frac{s}{L_{\mathrm{IR}}}(c_5 H^{(1)\mu\nu} + c_6 H^{(2)\mu\nu} + c_7 m^2 G^{\mu\nu} \\
& + c_8 m^4 g^{\mu\nu} + c_9 q^2 T^{(A)\mu\nu}) + T_R^{\mu\nu} + \mathcal{O}(s^2), \qquad (11)
\end{aligned}
$$

where the coefficients c_i are (known) rational multiples of $1/\pi^2$, $H^{(1)}$ and $H^{(2)}$ are tensors involving four derivatives of the metric, and $T^{(A)}$ is the Maxwell stress tensor. L_{IR} is some infrared renormalization scale of choice: changing it amounts to a finite renormalization of coupling constants, as described below.

The divergent terms in the series are local functionals of the background. This is because they come from high-energy, short-wavelength modes, which are sensitive only to the local physics. They are geometric objects with all the necessary symmetries: they are covariantly conserved tensors, with the right dimension, and transform appropriately under charge conjugation. This is because the regulator is gauge and diffeomorphism invariant.

By looking at Eq. (3), it is clear that the divergent terms can be absorbed in the renormalization of the charge q, the cosmological constant Λ and Newton's constant κ^2, with the exception of $H^{(1)}$ and $H^{(2)}$. These renormalize two higher-derivative corrections to the Einstein-Hilbert action. In fact,

$$
H^{(1)\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{g} R^2, \qquad (12)
$$

$$
H^{(2)\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \sqrt{g} R^{\alpha\beta} R_{\alpha\beta}. \qquad (13)
$$

In the spirit of retaining only the very low-energy physics of the theory, we will set the renormalized coefficient of these higher-derivative terms to zero.

After an appropriate renormalization of the couplings, we are left with the renormalized, finite currents J_R and T_R, which is what we take to stand on the right-hand side of Eq. (3). These quantities receive contributions from the whole spectrum of modes of the fermionic field, they are sensitive to the infrared physics, and therefore they are nonlocal, nongeometric functionals of the background.

Although the regularization and renormalization prescription described above is conceptually very simple, it proved unfeasible to follow in practice. For various technical reasons, the heat-kernel regularization, or any other covariant method like dimensional, zeta function or Pauli-Villars regularization, turns out not to be well suited for the numerical computation we require (see Appendix B 1).

Instead, we resorted to point-splitting regularization. Starting from a point x, we shoot out a geodesic, in a direction specified by a unit vector t, and we take a point x' along it, at a geodesic distance s from x. We then define the regularized current and stress tensor at x as

$$
J_0^{\mu}(x) = \langle \bar{\psi}(x') \gamma^{\mu} \psi(x) \rangle, \qquad (14)
$$

$$
T_0^{\mu\nu}(x) = \langle \bar{\psi}(x') \gamma^{(\mu} \mathrm{i} D^{\nu)} \psi(x) \rangle. \qquad (15)
$$

A small-s expansion has been worked out for these quantities also. However, because the regulator breaks gauge and diffeomorphism invariance, it involves contractions of local geometric tensors with the vector t, and the terms are not covariantly conserved. In the massless case $m = 0$ it has the form[3]

$$
\begin{aligned}
T_0^{\mu\nu} = {} & -\frac{1}{\pi^2 s^4}(g^{\mu\nu} - 4t^{\mu}t^{\nu}) - \frac{1}{16\pi^2 s^2}\Big[\frac{2}{3}(G^{\mu\nu} + R t^{\mu} t^{\nu}) \\
& - 4R^{(\mu}{}_{\lambda} t^{\nu)} t^{\lambda} + \frac{4}{3}(g^{\mu\nu} R_{\lambda\xi} - R^{\mu}{}_{\lambda}{}^{\nu}{}_{\xi}) t^{\lambda} t^{\xi}\Big] \\
& - \frac{1}{160\pi^2} \log\frac{s}{L_{\mathrm{IR}}}\Big(H^{(2)\mu\nu} - \frac{1}{3}H^{(1)\mu\nu}\Big) \\
& + T_{\mathrm{finite}}^{\mu\nu}(t) + \mathcal{O}(s^2). \qquad (16)
\end{aligned}
$$

It is very important to stress that T_{finite}, besides being a nonlocal, nongeometric object, still depends on the vector t, i.e. on the regularization scheme, and hence cannot be interpreted as a renormalized quantity. To obtain a well-defined renormalized stress tensor, it is necessary to proceed with what is called adiabatic renormalization [20].

Adiabatic renormalization consists in subtracting from T_{finite} an additional, finite counterterm that precisely compensates for all the symmetry-breaking effects of the regulator. To determine what this counterterm is, it is necessary to compute the bare stress tensor within a derivative, or adiabatic, expansion: the background is assumed to change on length scales much larger than the correlation length of the fermion field, set by the mass m.

Within this approximation, it is possible to compute all the orders of the expansion (16)—including T_{finite}—explicitly as functionals of the background. They all turn out to be local terms, made of contractions of geometric tensors with the vector t. This means that the adiabatic expansion completely misses the nonlocal, infrared physics. However, it retains all the symmetry-breaking effects of the regulator, which affects only local, UV physics. Therefore, by subtracting from the bare stress tensor T_0 its adiabatic expansion, up to and including the finite term, and taking the limit $s \to 0$, a well-defined renormalized stress tensor T_R is obtained. When other covariant regularization procedures are feasible, it has been shown that this approach yields the same renormalized stress tensor T_R, up to a finite renormalization of Λ, κ^2 and the other couplings. The current can be regularized and renormalized according to the same procedure.

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[3] The precise form of the expansion depends on the details of the point-splitting prescription. This formula is taken from Ref. [21], and is derived according to the definitions there.

B. The conformal anomaly and covariant conservation

An important check of the regularization and renormalization prescription we are adopting is the ability to produce a covariantly conserved stress tensor. In a static geometry the bare stress tensor, defined by

$$T_0^{\mu\nu}(x) = \langle \bar{\psi}(x')\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle, \tag{17}$$

is covariantly conserved if $x - x' = (\Delta t, 0, 0, 0)$ with constant Δt. In fact, we have

$$\begin{aligned} D_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\rho}T^{\rho\nu} + \Gamma^\nu{}_{\mu\rho}T^{\mu\rho} \\ &= \langle \bar{\psi}(x')\vec{\partial}_\mu\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle \\ &\quad + \langle \bar{\psi}(x')\overleftarrow{\partial}_\mu\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle + \Gamma^\mu{}_{\mu\rho}T^{\rho\nu} \\ &\quad + \Gamma^\nu{}_{\mu\rho}T^{\mu\rho}, \end{aligned} \tag{18}$$

and, because the sections do not depend on time, the derivatives can be promoted to covariant derivatives,

$$\begin{aligned} D_\mu T^{\mu\nu} &= \langle \bar{\psi}(x')\vec{D}_\mu\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle \\ &\quad + \langle \bar{\psi}(x')\overleftarrow{D}_\mu\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle. \end{aligned} \tag{19}$$

Using the field equations of motion, it is easy to show that this quantity is zero. It is also possible to show directly that the stress tensor computed with the adiabatic expansion is covariantly conserved, and so is the renormalized stress tensor.

If another point-splitting prescription is taken—for example, one in which Δt depends on position—then neither the bare stress tensor nor the adiabatic stress tensor are conserved. However, the difference of the two in the limit $s \to 0$ is conserved.

Another important check is the manifestation of the conformal anomaly. In fact, it is well known that, for a massless field, the trace of the stress tensor on curved spacetime is not zero, but is proportional to a local geometric functional of the background. Using the equations of motion, it is easy to show that

$$T^\mu{}_\mu(x, x') = m\langle \bar{\psi}(x)\psi(x)\rangle + \text{contact terms}, \tag{20}$$

and it would be natural to conclude that, in the massless limit, the trace is zero. This is certainly true for the bare stress tensor. On the other hand, the adiabatic expansion assumes that the correlation length $1/m$ is much smaller than the other length scales, and hence it breaks down in the massless limit. This becomes manifest as a $1/m$ divergence in $\langle \bar{\psi}(x)\psi(x)\rangle$, which cancels the factor of m, and gives a finite contribution in the massless limit. Since the renormalized stress tensor is the difference between the bare and the adiabatic quantity, it also acquires a finite trace in the massless limit. The trace of the stress tensor obtained in this way correctly reproduces the conformal anomaly,

$$\begin{aligned} T^\lambda{}_\lambda = \frac{1}{2880\pi^2}\bigg(&-\frac{7}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} \\ &+ \frac{5}{4}R^2 - 3\Box R\bigg). \end{aligned} \tag{21}$$

C. A choice of background

It is not possible to carry out the computation of the bare currents in a completely arbitrary background, even with time-translation invariance and spatial-rotation invariance. Even restricting to an asymptotically anti–de Sitter space is not enough: some knowledge of the interior is needed. Therefore, we choose to target a class of metrics,

$$\mathrm{d}s^2 = -e_t^2(r)\mathrm{d}t^2 + e_r^2(r)\mathrm{d}r^2 + e_s^2(r)\mathrm{d}\Omega_2^2, \tag{22}$$

that are smoothly connected to global anti–de Sitter space

$$\mathrm{d}s^2 = \frac{L^2}{\frac{4}{\pi^2}\cos^2\frac{\pi r}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}r^2 + \frac{4}{\pi^2}\sin^2\frac{\pi r}{2}\,\mathrm{d}\Omega_2^2\right). \tag{23}$$

That is, we take $r \in (0, 1)$, with e_s vanishing linearly at $r = 0$ and all the sections e_μ diverging like $(1 - r)^{-1}$ at $r = 1$. It is useful to think of the spatial sections of this class of metrics as 3-balls, with the center at $r = 0$, and the edge at $r = 1$, where the conformal factor diverges.

The spherical spatial sections should be regarded as a (covariant and natural) IR regulator, which replaces the artificial hard wall z_m employed in Ref. [15], following Ref. [14]. Such a regulator has been used to good effect in the analogous problem of charged scalar fields in AdS [22].[4] A noncovariant regulator such as a hard wall is an obstruction to building a covariant bulk stress tensor. This regulator has the further virtue (in contrast to the hard wall) of uniquely specifying the IR boundary conditions on the bulk spinor fields, simply by regularity.

A further practical reason for this choice is that this class of spaces is compact from the point of view of the Dirac Hamiltonian, which therefore has a discrete spectrum and normalizable eigenfunctions. This is a big advantage for a numerical computation, which is lost, for example, in spaces with a horizon in the interior.

We will take the gauge field to have the form

$$A = \Phi(r)\mathrm{d}t, \tag{24}$$

and the chemical potential μ sets the boundary condition for Φ,

$$\Phi(1) = \mu. \tag{25}$$

From the dual point of view, this choice of background is analogous to defining the field theory on a sphere of radius $R = e_s(1)L$, instead of flat space. The sphere is just an infrared regulator, whose effect becomes negligible in the limit $\mu \gg \frac{1}{R}$.

---

[4] We thank Simon Gentle for useful discussions of this point.

The fermion field is a free field, so all the information is contained in the Green's function

$$S(x,x') \equiv \langle \psi(x)\bar{\psi}(x')\rangle, \tag{26}$$

which satisfies

$$\sqrt{g}(\mathrm{i}\gamma\cdot D - m)S(x,x') = \mathrm{i}\delta(x-x'). \tag{27}$$

The currents, in terms of the Green's function, are given by

$$j^{\mu}(x) = -\mathrm{Tr}[\gamma^{\mu}S(x,x')], \tag{28}$$

$$T^{\mu\nu}(x) = -\mathrm{Tr}[\gamma^{(\mu}\mathrm{i}D^{\nu)}S(x,x')]. \tag{29}$$

Specifying to our background, we introduce the Hamiltonian Dirac matrices,

$$\alpha^t = \gamma^t, \quad \alpha^i = \gamma^t\gamma^i, \quad (\alpha^{\mu})^{\dagger} = \alpha^{\mu}, \quad \{\alpha^{\mu},\alpha^{\nu}\} = \delta^{\mu\nu}, \tag{30}$$

and the covariant derivatives are

$$D_t = \partial_t + \mathrm{i}\Phi + \frac{e_t'}{2e_r}\alpha^r, \tag{31}$$

$$D_r = \partial_r, \tag{32}$$

$$D_{\theta} = \partial_{\theta} - \frac{e_s'}{2e_r}\alpha^r\alpha^{\theta}, \tag{33}$$

$$D_{\phi} = \partial_{\phi} - \sin\theta\frac{e_s'}{2e_r}\alpha^r\alpha^{\phi} - \frac{1}{2}\cos\theta\alpha^{\theta}\alpha^{\phi}. \tag{34}$$

Since our background is static, it is advantageous to move to a Hamiltonian picture by defining

$$S(x,x') = [g^{-\frac{1}{4}}\sqrt{e_t}]_x G(x,x')\alpha^t[g^{-\frac{1}{4}}\sqrt{e_t}]_{x'}, \tag{35}$$

so that G satisfies the equation

$$(\mathrm{i}\partial_t - H)G(x,x') = \mathrm{i}\delta(x-x'), \tag{36}$$

with

$$\begin{aligned} H = {} & -\mathrm{i}\frac{e_t}{e_r}\alpha^r\left[\partial_r + \frac{e_t'}{2e_t} - \frac{e_r'}{2e_r}\right] \\ & - \mathrm{i}\frac{e_t}{e_s}\left[\alpha^{\theta}\partial_{\theta} + \alpha^{\phi}\frac{1}{\sin\theta}\partial_{\phi}\right] + \Phi + me_t\alpha^t. \end{aligned} \tag{37}$$

The Hamiltonian H is self-adjoint with respect to the scalar product

$$(\psi_1,\psi_2) = \int \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\,\psi_1^{\dagger}(r,\theta,\phi)\psi_2(r,\theta,\phi). \tag{38}$$

In our background, only the time component of the current and the diagonal components of the stress tensor have a nonzero expectation value. In terms of the Hamiltonian Green's function G these are given by

$$J^t = -f_{x,x'}\,\mathrm{Tr}[G(x,x')], \tag{39}$$

$$T^t{}_t = -\mathrm{i}f_{x,x'}\,\mathrm{Tr}\left[\left(\partial_t + \mathrm{i}\Phi + \frac{e_t'}{2e_r}\alpha^r\right)G(x,x')\right], \tag{40}$$

$$T^r{}_r = -\mathrm{i}f_{x,x'}\frac{e_t}{e_r}\,\mathrm{Tr}\left[\left(\alpha^r\partial_r - \alpha^r\frac{e_r'}{2e_r} - \alpha^r\frac{e_s'}{e_s}\right)G(x,x')\right], \tag{41}$$

$$T^{\theta}{}_{\theta} = -\mathrm{i}f_{x,x'}\frac{e_t}{e_s}\mathrm{Tr}\left[\left(\alpha^{\theta}\partial_{\theta} - \alpha^{\theta}\frac{\cos\theta}{2\sin\theta} + \alpha^r\frac{e_s'}{2e_r}\right)G(x,x')\right], \tag{42}$$

$$\begin{aligned} T^{\phi}{}_{\phi} = -\mathrm{i}f_{x,x'}\frac{e_t}{e_s}\,\mathrm{Tr}\Bigg[\Bigg(&\alpha^{\phi}\frac{\partial_{\phi}}{\sin\theta} + \alpha^{\theta}\frac{\cos\theta}{2\sin\theta} \\ & + \alpha^r\frac{e_s'}{2e_r}\Bigg)G(x,x')\Bigg], \end{aligned} \tag{43}$$

where

$$f_{x,x'} = g^{-\frac{1}{4}}(x)g^{-\frac{1}{4}}(x')\frac{e_t(x')}{e_t(x)}, \tag{44}$$

and all other sections are evaluated at x.

D. Adiabatic expansion

We now show how the Green's function, and hence the bare currents, can be computed within a small derivative—or adiabatic—expansion [23].

To outline the idea behind the computation, it is useful to consider the simpler problem

$$[-\nabla^2 + m^2 + V(x)]G(x,x') = \delta(x-x'). \tag{45}$$

We take V to be varying slowly compared to the correlation length $1/m$, and we expand $V(x)$ about x'. Then, Eq. (45) can be written symbolically as

$$[G_0^{-1} + A]G = \mathbb{1} \tag{46}$$

with

$$G_0^{-1} = -\nabla^2 + m^2, \tag{47}$$

$$A = V_{,i}(x')(x-x')_i + \frac{1}{2}V_{,ij}(x')(x-x')_i(x-x')_j + \cdots, \tag{48}$$

and it is formally solved by a series in A,

$$G = G_0\sum_{n=0}^{\infty}(-AG_0)^n. \tag{49}$$

The matrix products in this series actually stand for convolutions, so it is useful to go to momentum space,

$$G(x;x') = \int đ^d k e^{\mathrm{i}k\cdot(x-x')}G(k;x'), \tag{50}$$

where $đk \equiv \frac{\mathrm{d}k}{2\pi}$. So,

$$G_0^{-1} = k^2 + m^2, \tag{51}$$

$$A = V_{,i}(x')\mathrm{i}\partial_{k_i} - \frac{1}{2}V_{,ij}(x')\partial_{k_i}\partial_{k_j} + \cdots. \tag{52}$$

Using the identity

$$\partial_{k_i}G_0 = -2k_iG_0^2, \tag{53}$$

we have

$$\begin{aligned} G(k;x') = G_0 &+ (2\mathrm{i}V_{,i}k_i - V_{,ii})G_0^3 + (4V_{,ij}k_ik_j \\ &+ 2V_{,i}V_{,j})G_0^4 - 12V_{,i}V_{,j}k_ik_jG_0^5 + \cdots, \end{aligned} \tag{54}$$

where we have retained terms involving up to two derivatives of the potential. Now we revert to position space. We have

$$\begin{aligned} \int đ^dke^{\mathrm{i}k\cdot s}G_0(k) &= \frac{1}{(2\pi)^{\frac{d}{2}}}F_{d-2}(s,m), \\ F_n(s,m) &= \left(\frac{m}{s}\right)^{\frac{n}{2}}K_{\frac{n}{2}}(ms), \end{aligned} \tag{55}$$

where $s_i = x_i - x'_i$ and $s = \sqrt{s_is_i}$, and we use the identities

$$\int đ^dke^{\mathrm{i}k\cdot s}G_0^n(k) = -\frac{1}{2m(n-1)}\frac{\partial}{\partial m}\int đ^dke^{\mathrm{i}k\cdot s}G_0^{n-1}(k), \tag{56}$$

$$\int đ^dkk_ie^{\mathrm{i}k\cdot s}f(k) = -\mathrm{i}\partial_{s_i}\int đ^dke^{\mathrm{i}k\cdot s}f(k) \tag{57}$$

to carry out the Fourier transform. The final result is

$$\begin{aligned} (2\pi)^{\frac{d}{2}}G(x,x') = F_{d-2} &- \left(\frac{1}{4}V_{,i}s_i + \frac{1}{12}V_{,ij}s_is_j\right)F_{d-4} \\ &- \left(\frac{1}{24}V_{,ii} - \frac{1}{32}V_{,i}V_{,j}s_is_j\right)F_{d-6} \\ &+ \frac{1}{96}V_{,i}V_{,i}F_{d-8} + \cdots. \end{aligned} \tag{58}$$

Then we can expand in series for small s, and we have, for $d = 4$,

$$\begin{aligned} (2\pi)^2G(x,x') = \frac{1}{s^2} &+ \frac{m^2}{4}L - \frac{m^2}{4} - \frac{1}{24}V_{,ii}m^2 + \frac{1}{48}\frac{V_{,i}V_{,i}}{m^4} \\ &+ \frac{1}{8}V_{,i}s_iL + \frac{m^4}{32}s^2L - \frac{1}{96}V_{,ii}s^2L \\ &+ \frac{1}{24}V_{,ij}s_is_jL + \mathcal{O}(s^2), \end{aligned} \tag{59}$$

where $L = \log m^2s^2/4 + \gamma_E$. A structure similar to that of Eq. (16) starts to be apparent.

A computation along the same lines can be carried out for the fermionic Green's function

$$\begin{aligned} &\bigg(\mathrm{i}\partial_t + \mathrm{i}\frac{e_t}{e_r}\alpha^r\left[\partial_r + \frac{e'_t}{2e_t} - \frac{e'_r}{2e_r}\right] + \mathrm{i}\frac{e_t}{e_s}\bigg[\alpha^\theta\partial_\theta \\ &\quad + \alpha^\phi\frac{1}{\sin\theta}\partial_\phi\bigg] - \Phi - me_t\alpha^t\bigg)G = \mathrm{i}\delta. \end{aligned} \tag{60}$$

It involves the same steps, including the integrals (after Wick rotation), but is algebraically much messier. In fact, it is necessary use a computer algebra system, and we found it advantageous to specify the direction of point splitting from the beginning. Once the Green's function is found, the currents can be computed by taking the opportune derivatives. The result is reported in Appendix A.

As an alternative to the method described, there are also covariant ways to carry out the adiabatic expansion [21], which, however, are more difficult to implement on the computer.

E. Computation of the bare currents

We compute the bare currents by expanding the Green's function on a basis of eigenfunctions of the Dirac Hamiltonian H. The problem can be reduced to one dimension by exploiting translational invariance in the time direction, and the spherical symmetry of the spatial sections.

In order to exploit the spherical symmetry, we must introduce spinor spherical harmonics. For the two-sphere, they are solutions of the eigenvalue equation

$$\left[\sigma^2(-\mathrm{i}\partial_\theta) + \sigma^1\frac{1}{\sin\theta}(-\mathrm{i}\partial_\phi)\right]Y_{\ell m}(\theta,\phi) = \ell Y_{\ell m}(\theta,\phi). \tag{61}$$

The spectrum is quantized, with $\ell \in \{+1,-1,+2,-2,\ldots\}$, and m labels the degeneracy $2|\ell|$ of each eigenspace. The spinor harmonics are orthonormal and complete,

$$\int \mathrm{d}\theta\mathrm{d}\phi Y^\dagger_{\ell m}(\theta,\phi)Y_{\ell'm'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}, \tag{62}$$

$$\sum_{\ell m}Y_{\ell m}(\theta,\phi)Y^\dagger_{\ell m}(\theta',\phi') = \delta(\theta-\theta')\delta(\phi-\phi'), \tag{63}$$

and they also satisfy (note that the sum runs only over the degeneracy index m)

$$\sum_m Y^\dagger_{\ell m}(\theta,\phi)Y_{\ell m}(\theta,\phi) = \frac{\ell}{2\pi}\sin(\theta), \tag{64}$$

$$\sum_m Y^\dagger_{\ell m}(\theta,\phi)\sigma^2(-\mathrm{i}\partial_\theta)Y_{\ell m}(\theta,\phi) = \mathrm{sign}\ell\frac{\ell^2}{4\pi}\sin(\theta), \tag{65}$$

$$\sum_m Y^\dagger_{\ell m}(\theta,\phi)\sigma^1\frac{-\mathrm{i}\partial_\phi}{\sin\theta}Y_{\ell m}(\theta,\phi) = \mathrm{sign}\ell\frac{\ell^2}{4\pi}\sin(\theta). \tag{66}$$

We organize the Dirac matrices as

$$\alpha^r = \mathbb{1} \otimes \sigma^2, \qquad \alpha^\theta = \sigma^2 \otimes \sigma^3,$$
$$\alpha^\phi = \sigma^1 \otimes \sigma^3, \qquad \alpha^t = \mathbb{1} \otimes \sigma^1, \tag{67}$$

and we exploit this direct product structure to write

$$G(x, x') = \int \text{đ}\omega\, e^{-\mathrm{i}\omega(t-t')} \sum_{\ell m} [Y_{\ell m}(\theta, \phi) Y^\dagger_{\ell m}(\theta', \phi')] \otimes G_{\omega\ell}(r, r'), \tag{68}$$

where $G_{\omega\ell}$ is the Green's function of a simple one-dimensional differential operator,

$$(\omega - H_\ell) G_{\omega\ell}(r, r') = \mathrm{i}\delta(r - r'), \tag{69}$$

with

$$H_\ell = -\mathrm{i}\sigma^2 \frac{e_t}{e_r}\left[\partial_r + \frac{e_t'}{2e_t} - \frac{e_r'}{2e_r}\right] + \ell \frac{e_t}{e_s}\sigma^3 + \Phi + m e_t \sigma^1. \tag{70}$$

H_ℓ is a self-adjoint operator, with an orthonormal and complete set of real eigenfunctions. In the class of backgrounds we are considering, the spectrum is discrete, and we label it with an index n,

$$H_\ell \psi_{n\ell}(r) = \omega_{n\ell} \psi_{n\ell}(r). \tag{71}$$

Then we have

$$G(x, x') = \int \text{đ}\omega \frac{\mathrm{i}e^{-\mathrm{i}\omega(t-t')}}{\omega - \omega_{n\ell}} \sum_{n\ell m} [Y_{\ell m}(\theta, \phi) Y^\dagger_{\ell m}(\theta', \phi')] \otimes [\psi_{n\ell}(r) \psi^\dagger_{n\ell}(r')]. \tag{72}$$

We perform a Wick rotation, and we take the point splitting to be along the imaginary time direction, i.e. $t - t' = \pm \mathrm{i}s$. Then we symmetrize with respect to the sign of s and we have

$$G(x, x') = \sum_{n\ell m} \Theta_{n\ell}(s) [Y_{\ell m}(\theta, \phi) Y^\dagger_{\ell m}(\theta', \phi')] \otimes [\psi_{n\ell}(r) \psi^\dagger_{n\ell}(r')], \tag{73}$$

where

$$\Theta_{n\ell}(s) = \frac{1}{2} \operatorname{sign}\omega_{n\ell} e^{-|s\omega_{n\ell}|}. \tag{74}$$

Now we substitute in the expression for the currents. To keep things as easy as possible, we let $r = r'$, $\theta = \theta'$, $\phi = \phi'$, that is, we do not point split in the other directions. Using the spinor harmonics identities, and the reality of wave functions, we have

$$J^t = \frac{1}{e_t e_r e_s^2} \sum_{n\ell} \frac{|\ell|}{2\pi} \Theta_{n\ell}(s) \psi^\dagger_{n\ell}(r) \psi_{n\ell}(r), \tag{75}$$

$$T^t{}_t = \frac{1}{e_t e_r e_s^2} \sum_{n\ell} \frac{|\ell|}{2\pi} \Theta_{n\ell}(s)(-\omega_{n\ell} + \Phi) \psi^\dagger_{n\ell}(r) \psi_{n\ell}(r), \tag{76}$$

$$T^r{}_r = \frac{1}{e_t e_r e_s^2} \sum_{n\ell} \frac{|\ell|}{2\pi} \Theta_{n\ell}(s) \frac{e_t}{e_r} \psi^\dagger_{n\ell}(r)(-\mathrm{i}\sigma^2) \psi'_{n\ell}(r), \tag{77}$$

$$T^\theta{}_\theta = T^\phi{}_\phi = \frac{1}{e_t e_r e_s^2} \sum_{n\ell} \frac{|\ell|}{2\pi} \Theta_{n\ell}(s) \frac{\ell}{2} \frac{e_t}{e_s} \psi^\dagger_{n\ell}(r) \sigma^3 \psi_{n\ell}(r). \tag{78}$$

With these manipulations, the computation of the bare currents has been reduced to the problem of diagonalizing the one-dimensional Hamiltonian H_ℓ and carrying out the mode sums above. Both tasks can be carried out numerically, and the second is feasible especially thanks to the exponential suppression of the high-energy modes, due to the factor $\Theta_{n\ell}(s)$.

F. Diagonalization of the Dirac Hamiltonian

Let us now show how the Hamiltonian H_ℓ can be diagonalized numerically. The spectrum and the eigenfunctions must be computed with very high accuracy, and efficiently. Using a finite-differences discretization of the Hamiltonian is not sufficient for the purpose. It is necessary to resort to spectral methods [24], which consist in approximating the eigenfunctions with polynomials of high degree, instead of a set of values on a uniformly spaced grid. This allows a better representation of the derivative operator, and yields a spectrum and eigenfunctions accurate to order e^{-n}, where n is both the degree of the approximating polynomial and the rank of the matrix to be numerically diagonalized. This should be compared with the accuracy of finite-differences methods, which is only polynomial in n.

In order to use spectral methods, the metric must be further specified. In fact, the background (22) possesses residual reparametrization invariance, which we use to impose the constraint $e_t(r) = e_r(r)$, so that the metric takes the form

$$\mathrm{d}s^2 = \frac{1}{\beta^2(r)}(-\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2(r)\mathrm{d}\Omega_2^2), \qquad A = \Phi(r)\mathrm{d}t. \tag{79}$$

Demanding a space with the same asymptotics as global AdS, we take

$$\alpha \sim r, \qquad \beta \sim b_0 r^0, \quad \text{for } r \to 0, \tag{80}$$

$$\alpha \sim a_0(1 - r)^0, \qquad \beta \sim \frac{1 - r}{L}, \quad \text{for } r \to 1, \tag{81}$$

$$\alpha(-r) = -\alpha(r), \qquad \beta(-r) = \beta(r), \qquad \Phi(-r) = \Phi(r). \tag{82}$$

The reparametrization invariance could be used to impose a different constraint instead of $e_r = e_t$, leading to different asymptotics of the sections. However, this choice has the big advantage that all the terms in the Hamiltonian

$$H_\ell = -\mathrm{i}\sigma^2 \partial_r + \frac{\ell}{\alpha(r)}\sigma^1 + \Phi(r) + \frac{m}{\beta(r)}\sigma^3 \tag{83}$$

are analytic for $r \in [0, 1]$. This is crucial for the possibility of using spectral methods.

The eigenfunctions, however, are not analytic at $r = 1$. To have a good polynomial approximation, the nonanalytic behavior at the boundaries must be determined and factored out. For $r \to 0$, retaining only the leading terms, we have

$$H \sim -\mathrm{i}\sigma^2 \partial_r + \frac{\ell}{r}\sigma^1, \qquad \psi(r) \sim \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} r^\lambda, \tag{84}$$

and there are two solutions:

$$a_1 = 0, \qquad \lambda = \ell \quad \text{or} \quad a_2 = 0, \qquad \lambda = -\ell. \tag{85}$$

The first solution is normalizable for $\ell > 0$, the second for $\ell < 0$. For $r \to 1$ we have

$$H \sim -\mathrm{i}\sigma^2 \partial_r + \frac{mL}{1-r}\sigma^3, \qquad \psi(r) \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} (1-r)^\nu. \tag{86}$$

There are two solutions:

$$b_1 = -b_2, \qquad \nu = mL \quad \text{or} \quad b_1 = b_2, \qquad \nu = -mL. \tag{87}$$

The first solution is normalizable for $mL > -1/2$, the second for $mL < 1/2$.

Finally, we notice that the Hamiltonian has parity symmetry,

$$\sigma^3 H(-r) \sigma^3 = H(r), \tag{88}$$

and hence the eigenfunctions can be taken to have definite parity,

$$\psi(-r) = \pm\sigma^3 \psi(r). \tag{89}$$

We can collect all this information by writing the normalizable[5] eigenfunctions as

$$\psi(r) = r^{|\ell|}(1-r^2)^{mL}\phi(r), \tag{90}$$

where $\phi(r)$ is analytic at $r = 0$ and at $r = 1$ and it satisfies

$$\phi(-r) = -\mathrm{sign}(\ell)\sigma^3 \phi(r), \qquad \phi_1(1) + \phi_2(1) = 0. \tag{91}$$

Therefore, ϕ can be well approximated by a polynomial, and we can construct a complete basis of spinors with polynomial components that satisfies the boundary conditions and parity constraints:

$$\begin{aligned} \phi_a(r) &= \begin{pmatrix} +Q_{a-1}(1)Q_a(r) \\ -Q_a(1)Q_{a-1}(r) \end{pmatrix} \quad \begin{matrix} \text{odd } a, \\ \ell > 0, \end{matrix} \\ \phi_a(r) &= \begin{pmatrix} -Q_a(1)Q_{a-1}(r) \\ +Q_{a-1}(1)Q_a(r) \end{pmatrix} \quad \begin{matrix} \text{even } a, \\ \ell > 0, \end{matrix} \end{aligned} \tag{92}$$

$$\begin{aligned} \phi_a(r) &= \begin{pmatrix} -Q_a(1)Q_{a-1}(r) \\ +Q_{a-1}(1)Q_a(r) \end{pmatrix} \quad \begin{matrix} \text{odd } a, \\ \ell < 0, \end{matrix} \\ \phi_a(r) &= \begin{pmatrix} +Q_{a-1}(1)Q_a(r) \\ -Q_a(1)Q_{a-1}(r) \end{pmatrix} \quad \begin{matrix} \text{even } a, \\ \ell < 0, \end{matrix} \end{aligned} \tag{93}$$

where the Q_a are polynomials, such that Q_a has degree $a \in \{0, 1, 2, \ldots\}$ and the same parity as a. Since H_ℓ is self-adjoint with respect to the scalar product

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle &= \int_{-1}^{1} \mathrm{d}r \psi_1^\dagger(r) \psi_2(r) \\ &= \int_{-1}^{1} \mathrm{d}r r^{2|\ell|}(1-r^2)^{2mL} \phi_1^\dagger(r)\phi_2(r) \\ &\equiv (\phi_1, \phi_2), \end{aligned} \tag{94}$$

we take the polynomials Q_a to be orthogonal with respect to the same scalar product,

$$(Q_a, Q_b) = h_a \delta_{ab}. \tag{95}$$

These may be constructed from the Jacobi polynomials as follows:

$$Q_{2a}(r) = P_a^{(2mL, |\ell| - \frac{1}{2})}(2r^2 - 1), \tag{96}$$

$$Q_{2a+1}(r) = r P_a^{(2mL, |\ell| + \frac{1}{2})}(2r^2 - 1). \tag{97}$$

Now we cast the differential operator H_ℓ to a rank-n matrix H_{ab}, by projecting it to the Hilbert space spanned by the first n elements of the basis $\psi_a(r) = r^{|\ell|}(1-r^2)^{mL}\phi_a(r)$. We have

$$\begin{aligned} H_{ab} &= \langle \psi_a | H_\ell | \psi_b \rangle \\ &= \int_0^1 \mathrm{d}r r^{2|\ell|}(1-r^2)^{2mL} \Big[\frac{1}{2}(\phi_{a2}\phi'_{b1} - \phi_{a1}\phi'_{b2} \\ &\quad - \phi'_{a2}\phi_{b1} + \phi'_{a1}\phi_{b2}) + \frac{\ell}{\alpha(r)}(\phi_{a2}\phi_{b1} + \phi_{a1}\phi_{b2}) \\ &\quad + \frac{m}{\beta(r)}(\phi_{a1}\phi_{b1} - \phi_{a2}\phi_{b2}) \\ &\quad + \Phi(r)(\phi_{a1}\phi_{b1} + \phi_{a2}\phi_{b2}) \Big]. \end{aligned} \tag{98}$$

Even when using orthogonal polynomials, the basis ψ_a turns out to be not orthogonal because it involves linear combinations of polynomials of different degree. Therefore, it has a nontrivial overlap matrix

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[5] Here we consider the case $mL > -\frac{1}{2}$. The other case $mL < +\frac{1}{2}$ is equivalent.

$$G_{ab} = \langle \psi_a | \psi_b \rangle = \int_0^1 \mathrm{d}r r^{2|\lambda|} z^{2m} (\phi_{a1}\phi_{b1} + \phi_{a2}\phi_{b2}). \tag{99}$$

Given the matrices H, G, the approximate eigenfunctions of H_ℓ are obtained by solving the generalized eigenvalue problem

$$Hv = \omega G v. \tag{100}$$

Since H is Hermitian and G is Hermitian and positive definite, the eigenvalues ω are real and the vectors v form a basis. The matrix U that has the eigenvectors v for columns satisfies[6]

$$U^\star_{ia} G_{ab} U_{jb} = \delta_{ij}, \qquad U^\star_{ia} H_{ab} U_{jb} = \omega_i \delta_{ij}. \tag{101}$$

Therefore, ω_i are the approximate eigenvalues of the Dirac Hamiltonian H_ℓ, and the corresponding approximate eigenfunctions are given by

$$\psi_i(r) = r^{|\lambda|}(1-r^2)^m U_{ia}\phi_a(r). \tag{102}$$

The question remains of how to compute the matrix elements G_{ab} and H_{ab}. By using the recursion relation for the polynomials Q_a, it is possible compute analytically the quantities

$$h_a = (Q_a, Q_a), \qquad K_{ab} = (Q_a, Q_b') - (Q_a', Q_b), \tag{103}$$

with which it is then possible to directly compute G_{ab} and the matrix elements of the kinetic term of the Hamiltonian. The terms involving α, β and Φ instead can be reduced to the form $(Q_a, f(r)Q_b)$. We compute them [25] by expanding Q_a, Q_b over approximate eigenfunctions of the position operator r, also called cardinal functions. These are polynomials C_i such that

$$(C_i, C_j) = \delta_{ij}, \qquad (C_i, rC_j) = r_i \delta_{ij}. \tag{104}$$

They can be obtained as linear combinations of the polynomials Q_a by diagonalizing the matrix

$$R_{ab} = \frac{(Q_a, rQ_b)}{\sqrt{h_a h_b}}, \tag{105}$$

which can be computed analytically, again using the recursion relation. Let V be the orthogonal matrix that diagonalizes R: $V^\dagger R V = r$. Then

$$Q_a(r) = \sqrt{h_a} V_{ai} C_i(r), \tag{106}$$

and we have

$$(Q_a, fQ_b) = \sqrt{h_a} U_{ai} (C_i, f(r) C_j) U_{bj} \sqrt{h_b}. \tag{107}$$

Now the operator r inside f is acting against an approximate eigenfunction, and we have

$$(Q_a, fQ_b) \simeq \sqrt{h_a} U_{ai} f(r_i) U_{bi} \sqrt{h_b}. \tag{108}$$

This approximation becomes an equality if the function f is a polynomial and the total degree of $Q_a Q_b f$ is less than $2n$. Otherwise it allows for an error, which is exponentially small in n, provided f is analytic over the interval $[-1, 1]$. Using this approximate integration, the matrix elements of H_{ab} can be computed.

G. Near-boundary singularity

It is well known from the literature on Casimir energy [26] that quantum fields in spaces with a boundary have peculiar behavior. This issue is very relevant to the problem at hand, because AdS is a space with a boundary. It turns out that the boundary at $r = 1$ causes the currents to approach their $s = 0$ profile in a nonuniform way, more or less like

$$f(r, s) = r^{\frac{1}{s}} \quad \text{for } 0 < r < 1 \tag{109}$$

approaches its limit $f(r, 0) = 0$.

The top panel in Fig. 1 exemplifies this phenomenon. It shows the charge density $\rho \equiv \beta^4 J^t$ in pure AdS geometry, with an electric potential[7]

$$\Phi = \frac{V}{16}(15r^2 - 5r^4 + r^6). \tag{110}$$

The charge density is plotted at nonzero s, but after the adiabatic expansion has been subtracted, so it has a finite limit as $s \to 0$. On the left, ρ is shown as a function of r, for several values of s (the bluer the smaller). On the right ρ is shown as a function of s, for several values of r (the bluer the closer to the boundary). The r-profiles approach nonuniformly a limiting flat curve. We find that this kind of behavior is present whenever $\Phi'(1) \neq 0$. Intuitively, it can be explained as a layer of charge at $r = 1$ induced by the electric field at the boundary through vacuum polarization.[8]

Although the $s \to 0$ limit is well defined and finite, the fact that it is approached nonuniformly makes it extremely hard to reach in practice. In fact, the computational cost grows exponentially as s decreases, because more and more wave functions must be retained to compute the bare currents. This would put beyond reach the computation of the currents near the boundary.

Fortunately, the contribution to the current that comes from the boundary conditions, and that vanishes nonuniformly at $s = 0$, can be computed analytically and subtracted. The plot at the bottom of Fig. 1 shows ρ after the subtraction of

$$\Delta\rho = \frac{\Phi'(1)}{6\pi^2 s}\left[3\left(1 + 4\frac{z^2}{s^2}\right)\tan^{-1}\left(\frac{s}{2z}\right) - \frac{s}{z}\left(1 + 6\frac{z^2}{s^2}\right)\right],$$
$$z = 1 - r: \tag{111}$$

---

[6] Note that U is not unitary.

[7] This potential has $\Phi'(1) \neq 0$, $\Phi''(1) = \Phi^{(3)}(1) = 0$.

[8] This layer is a fictitious, finite-cutoff effect that disappears as $s \to 0$, but was mistaken for a real effect in Ref. [15].

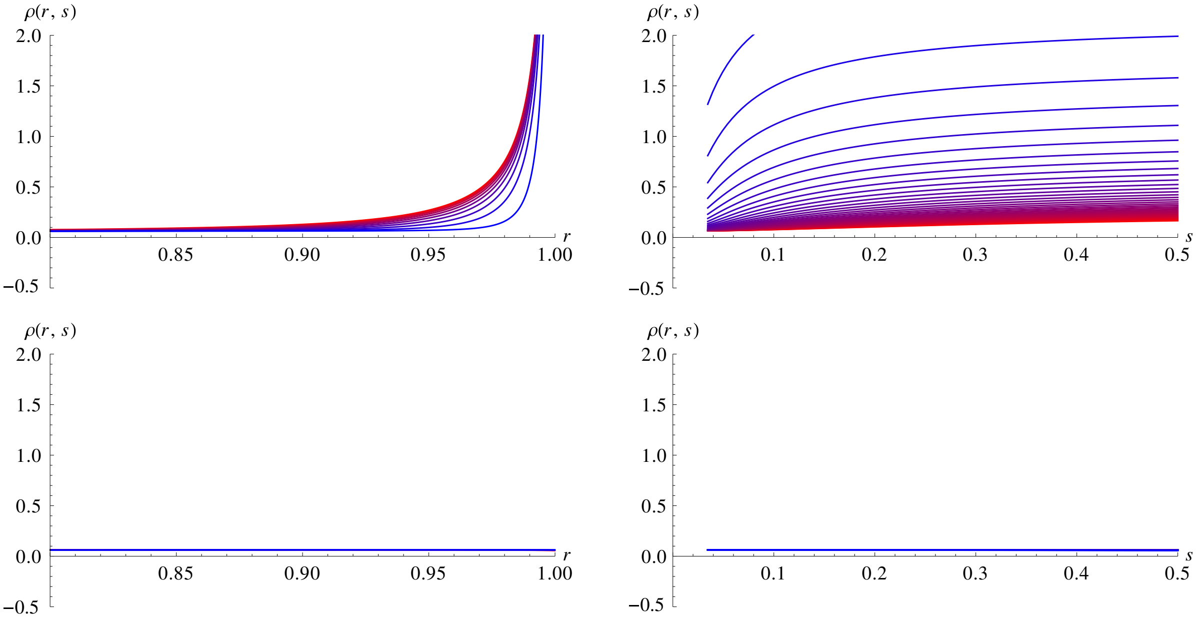

FIG. 1 (color online). Near-boundary behavior of the charge density, before (top) and after (bottom) the subtraction of a boundary counterterm. On the left the radial (r) dependence is plotted, and the regulator (s) dependence is encoded in the color; vice versa on the right. After the subtraction the convergence in s is completely uniform in r and the extrapolation to $s = 0$ is robust.

the nonuniform singularity has been removed. This counterterm correctly accounts for the finite-s effect of the boundary conditions, without altering the $s \to 0$ limit, because it vanishes at $s = 0$ for any $r \in [0, 1)$. After this subtraction, the limit $s \to 0$ is easily and safely taken by extrapolation.

There are also nonuniform singularities in the stress tensor, when $\alpha^{(3)}(1) \neq 0$ or, for $m \neq 0$, singularities proportional to $\beta'(1)$ and $\beta^{(3)}(1)$. For each of these singularities, a boundary counterterm like Eq. (111) must be and has been derived.

These counterterms can be obtained using an approach similar to the adiabatic expansion, assuming that the length scale over which the background varies is much larger than both the distance $z = 1 - r$ from the boundary and the point-splitting separation s. Using the simple example

$$[-\partial_z^2 - \nabla^2 + V(z, x)]G(z, x; z', x') = \delta(z - z')\delta(x - x'), \tag{112}$$

we can expand for small z and small $x - x'$, and write the previous equation as

$$[G_0^{-1} + A]G = \mathbb{1}, \tag{113}$$

with

$$G_0^{-1} = -\partial_z^2 - \nabla^2, \tag{114}$$

$$A = V(0, x') + V_{,z}(0, x')z + V_{,i}(0, x')(x - x')_i + \cdots. \tag{115}$$

The Green's function is given by

$$G = G_0 \sum_{n=0}^{\infty} (-AG_0)^n, \tag{116}$$

but, in this case, G_0 must account for the boundary conditions on the fields at $z = 0$. For example, for Dirichlet boundary conditions,

$$G_0(z, x; z', x') = 2\int \text{đ}q\,\text{đ}^{d-1}p\,\frac{\sin(qz)\sin(qz')e^{ip(x-x')}}{q^2 + p^2}. \tag{117}$$

Because there is no translational invariance, it is better to stay in position space in the z direction, so we write

$$G_0(z, x; z', x') = \int \text{đ}^{d-1}p\,G_p(z, z')e^{ip(x-x')}, \tag{118}$$

where

$$G_p(z, z') = \begin{cases} -\frac{1}{p}\sinh(pz)e^{-pz'} & \text{for } z < z', \\ -\frac{1}{p}\sinh(pz')e^{-pz} & \text{for } z > z'. \end{cases} \tag{119}$$

Let us consider the contribution of the term $V_{,z}(0, x')$ to the Green's function G. We have

$$\Delta G(z, x; z', x') = -\int \text{đ}^{d-1}p\,e^{ip(x-x')} \times \int_0^{\infty} \mathrm{d}\zeta\, G_p(z, \zeta)V_{,z}\zeta G_p(\zeta, z'). \tag{120}$$

For simplicity, we let $z' = z$ and we have

$$\begin{aligned}\Delta G(z, x; z, x') &= -V_{,z}\int đ^{d-1}p e^{ip(x-x')}\frac{z}{4p^3}[1-(1+pz)e^{-2pz}] \\ &= \frac{V_{,z}z}{4\pi^2}\left[\log\left(\frac{4z^2}{s^2}+1\right)+2\frac{z}{s}\tan^{-1}\frac{s}{2z}\right], \qquad (121)\end{aligned}$$

where the last result is specific to $d = 4$, and where $s = |x - x'|$.

If we expand this expression at small s, we find the divergent term

$$\Delta G(z, x; z, x') \sim -\frac{V_{,z}z}{4\pi^2}\log\frac{s^2}{4z^2}. \qquad (122)$$

This term was already obtained from the adiabatic expansion. It is the second term in Eq. (59), with m^2 replaced by $V_{,z}z$. In fact, $V(z)$ is locally a mass term $m^2(z)$, and the current expansion is reproducing the divergence $m^2(z)\log s$ through its series expansion about $z = 1$.

If we subtract both this logarithmic divergence and the $\mathcal{O}(s^0)$ term, we obtain a quantity that vanishes as $s \to 0$,

$$\Delta G(z, x; z, x') = \frac{V_{,z}z}{4\pi^2}\left[\log\left(1+\frac{s^2}{4z^2}\right)+2\frac{z}{s}\tan^{-1}\frac{s}{2z}-1\right], \qquad (123)$$

and which would be a boundary counterterm for the coincidence limit of the Green's function. Unfortunately, this example is limited in that this expression vanishes uniformly, so this counterterm is not necessary. However, by the same means but more complicated algebra, it is possible to compute the counterterm (111), which instead is of crucial importance.

III. SOLUTION OF EINSTEIN'S EQUATIONS

Given an electromagnetic current and a stress tensor with enough symmetries ("cohomogeneity one"), the solution of Einstein's and Maxwell's equations is a relatively standard problem. Let us briefly describe the method we used.

With the ansatz

$$\mathrm{d}s^2 = \frac{1}{\beta^2(r)}(-\mathrm{d}t^2+\mathrm{d}r^2+\alpha^2(r)\mathrm{d}\Omega_2^2), \qquad A = \Phi(r)\mathrm{d}t, \qquad (124)$$

the equations

$$\begin{cases} D_\mu F^{\mu\nu} = q^2 J^\nu, \\ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \end{cases} \qquad (125)$$

become

$$\begin{cases} \beta^4\left(\Phi'' + \frac{2\alpha'}{\alpha}\Phi'\right) & = q^2 J^t, \\ \beta^2\left(\frac{2\alpha''}{\alpha} - \frac{4\alpha'\beta'}{\alpha\beta} + \frac{\alpha'^2-1}{\alpha^2} - 2\frac{\beta''}{\beta} + 3\frac{\beta'^2}{\beta^2}\right) - \frac{3}{L^2} & = \kappa^2 T^t{}_t, \\ \beta^2\left(-\frac{4\alpha'\beta'}{\alpha\beta} + \frac{\alpha'^2-1}{\alpha^2} + 3\frac{\beta'^2}{\beta^2}\right) - \frac{3}{L^2} & = \kappa^2 T^r{}_r, \\ \beta^2\left(\frac{\alpha''}{\alpha} - \frac{2\alpha'\beta'}{\alpha\beta} - 2\frac{\beta''}{\beta} + 3\frac{\beta'^2}{\beta^2}\right) - \frac{3}{L^2} & = \kappa^2 T^s{}_s, \end{cases} \qquad (126)$$

where L is the radius of the asymptotic AdS geometry, that is, $\Lambda = -3/L^2$. Because of spherical symmetry, we have $T^\theta{}_\theta = T^\phi{}_\phi$, and we defined $T^s{}_s \equiv T^\theta{}_\theta = T^\phi{}_\phi$.

The first equation is simply Gauss' law; it is a linear equation, and does not require further discussion. The three Einstein's equations are not independent, because the Einstein tensor is covariantly conserved, that is, $D_\mu G^\mu{}_\nu = 0$ identically. This fact constrains the stress tensor to be convariantly conserved too,

$$T^r{}_{r,r} + \left(2\frac{\alpha'}{\alpha} - 3\frac{\beta'}{\beta}\right)T^r{}_r + \frac{\beta'}{\beta}T^t{}_t + 2\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha}\right)T^s{}_s = 0, \qquad (127)$$

and this reduces the independent components from three to two.

We demand that $\alpha(0) = 0$ (regularity in IR) and $\beta(1) = 0$ (asymptotically AdS in UV). This sets the coordinate location of the center of the space ($r = 0$) and the boundary ($r = 1$). It is useful to expand the equations near these two points to understand the asymptotic behavior of the sections. In order to do so, some knowledge of the behavior of the stress tensor is needed, which can be inferred by computing them explicitly in a few sample backgrounds. Based on this, we can assume that the stress tensor is analytic at the boundary, and that $T^t{}_t(1) = T^r{}_r(1) = T^s{}_s(1)$. This is due to the symmetry of AdS space, which forces the stress tensor at the boundary to be simply a correction to the cosmological constant. This correction comes from the high-energy modes, and hence is independent of Φ, so we absorb it directly into Λ. The next three derivatives vanish, so we have

$$\kappa^2 T^t{}_t = t_4(1-r)^4 + \mathcal{O}(1-r)^5, \qquad (128)$$

$$\kappa^2 T^s{}_s = s_4(1-r)^4 + \mathcal{O}(1-r)^5, \qquad (129)$$

and

$$\begin{aligned}\alpha(r) = a_0 - \frac{(1-r)^2}{2a_0} + a_3(1-r)^3 + \bigg[&\frac{a_0L^2}{4}(t_4-s_4) \\ &+ \frac{1}{24a_0^3}\bigg](1-r)^4 + \mathcal{O}(1-r)^5, \qquad (130)\end{aligned}$$

$$\beta(r) = \frac{1-r}{L} - \frac{(1-r)^3}{6a_0^2 L} + b_4(1-r)^4 + \left[\frac{L}{10}(t_4 - 2s_4) + \frac{1}{120a_0^4 L}\right](1-r)^5 + \mathcal{O}(1-r)^6. \tag{131}$$

At the center we have

$$\alpha(r) = r - \frac{1}{6b_0^2}\left(\frac{1}{L^2} + \kappa^2 T^s{}_s(0) - \frac{2}{3}\kappa^2 T^t{}_t(0)\right)r^3 + \mathcal{O}(r^5), \tag{132}$$

$$\beta(r) = b_0 - \frac{1}{2b_0}\left(\frac{1}{L^2} + \frac{1}{2}\kappa^2 T^s{}_s(0) - \frac{1}{6}\kappa^2 T^t{}_t(0)\right) + \mathcal{O}(r^4). \tag{133}$$

Moreover, since the equations are symmetric under $r \rightarrow -r$, we can take α to be odd and β and Φ to be even, provided that the currents are also even. Since the currents are even when the sections have definite parity [see Eq. (89)], this assumption is self-consistent.

The constants a_0, a_3, b_0 and b_4 are not fixed by the series expansion. They are four integration constants, which take a precise value in the unique solution that matches the two expansions at the edges. The constant a_0 is related to the radius of the sphere of the boundary theory, whereas a_3 and b_4 are related to the expectation value of the boundary stress tensor [27,28], as described in Sec. IV [see Eq. (137)].

On a more practical level, we solve the equations using spectral methods. We represent the sections α, β and Φ as polynomials of moderate degree. There is some freedom in choosing what basis to use for the space of polynomials. We use Chebyshev polynomials of the appropriate parity as a starting point, and we find it important to take linear combinations, so that each basis element satisfies

$$\alpha'(1) = 0, \qquad \beta(1) = 0, \qquad \beta''(1) = 0, \qquad \Phi(1) = 0. \tag{134}$$

The condition $\beta''(1) = 0$ is particularly important for the stability of Newton's method.

IV. RESULTS AND DISCUSSION

Let us briefly recapitulate the setup. We are considering a quantum fermionic field in interaction with classical gravity and a classical U(1) gauge field, in asymptotically anti–de Sitter spacetime. The class of backgrounds we are considering is described by the following ansatz[9]:

$$\mathrm{d}s^2 = \frac{1}{\beta^2(z)}[-\mathrm{d}t^2 + \mathrm{d}z^2 + \alpha^2(z)\mathrm{d}\Omega_2^2], \qquad A = \Phi(z)\mathrm{d}t. \tag{135}$$

The AdS boundary is at $z = 0$, where the conformal factor $\beta(z) \sim z/L$, and L is the radius of the asymptotic AdS geometry. We assume that spacetime ends smoothly at $z = z_m$, and the spatial sections can be visualized as 3-balls with center at $z = z_m$ and edge at $z = 0$. Global AdS is a metric of this class, with $\beta = \sin\frac{z}{L}$, $\alpha = L\cos\frac{z}{L}$, $z_m = \pi\frac{L}{2}$.

From the dual point of view, we are considering a 2 + 1-dimensional conformal field theory, defined on a sphere of radius $R = \alpha(0)$. This CFT has a global U(1) symmetry, for which we turn on a chemical potential $\mu = \Phi(0)$, and a fermionic operator charged under the U(1) symmetry, whose correlation functions we wish to study.

The model depends on four dimensionless parameters: q, κ/L, mL, μR. The U(1) coupling q, the gravity coupling κ/L and the fermion mass mL should be thought of as parametric labels (like the number of species of fields) specifying the dual CFT. Then, for a given CFT, dimensionless quantities depend on μ and R only through the combination μR.

The duality allows the computation of several CFT quantities, particularly the U(1) charge density ρ_b, the energy density ϵ_b, the pressure p_b, and the fermion spectral function. The thermodynamic responses are given by[10]

$$\rho_b = -\Phi'(0), \tag{136}$$

$$\epsilon_b = \frac{1}{R}\alpha^{(3)}(0) - \frac{L}{3}\beta^{(4)}(0), \tag{137}$$

$$p_b = -\frac{1}{2R}\alpha^{(3)}(0) + \frac{L}{3}\beta^{(4)}(0). \tag{138}$$

For what concerns the spectral function, it is important to notice that the CFT is defined on a sphere, and hence the spectrum of the many-body Hamiltonian is discrete, and single-particle states are labeled by the partial wave number ℓ. Consequently, the fermion spectral function

$$A(\ell, \omega) = \sum_\alpha [|\langle\alpha|c_\ell^\dagger|\mathrm{gd}\rangle|^2\delta(\omega - E_\alpha) + |\langle\alpha|c_\ell|\mathrm{gd}\rangle|^2\delta(\omega + E_\alpha)] \tag{139}$$

is composed of a discrete set of delta functions that track the many-body eigenvalues. However, for $\mu R \gg 1$ the effect of the infrared regulator R becomes negligible, and the flat-space spectral function is recovered. In fact, in this regime, one can identify ℓ/R with a continuous momentum

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[9] These coordinates are related to the coordinates in Eq. (22) by a rigid rescaling $t \rightarrow z_m t$, $z \rightarrow z_m(1 - r)$. The discussion of the results is slightly more transparent in these coordinates.

[10] The result for ϵ_b and p_b requires holographic renormalization as described in Ref. [28].

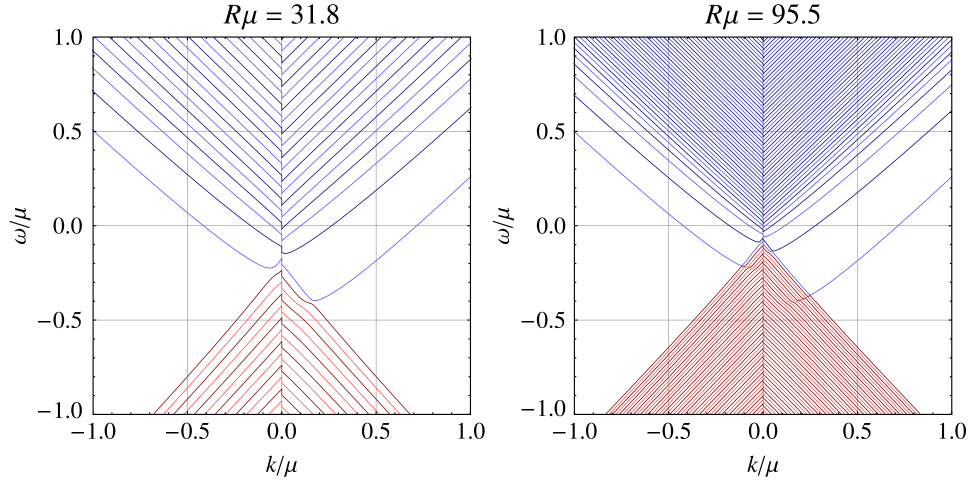

FIG. 2 (color online). Example of the spectral function: $\kappa^2/L^2 = 0.0$, $q^2 = 4.0$, $mL = 0.0$. Compare with Fig. 3 of Ref. [15]. Only discrete values of $k = \ell/R$ carry spectral weight, but here lines are shown as if k were a continuous variable. The difference is negligible as $\mu R \to \infty$.

label k, and the delta functions merge into a continuum. Hence, the regime $\mu R \gg 1$ is of the greatest interest. Holographically, the location of the delta functions in the k-ω plane coincides with the spectrum of the bulk Dirac Hamiltonian.

As an example of how the continuum is approached, consider Fig. 2. The plots refer to a frozen global AdS geometry, with a self-consistently determined gauge field, and display the location of the delta-function peaks of the spectral function. As μR increases, a continuum emerges in the light cone $\omega^2 > k^2$, outside of which a number of isolated bands remain. These results agree with our previous findings in Ref. [15], but here they have been derived with far greater care for all the regularization and renormalization issues, therefore giving an important check of the correctness of our previous work. For an interpretation see Ref. [15].

Moving to the more interesting case of a dynamic metric, Fig. 3 shows the profiles for $\alpha(z)$, $\beta(z)$ and $\Phi(z)$ in a set of self-consistent solutions. Without loss of generality, we set $\mu L = 1$. Then, by dialing z_m, we are able to vary the radius R of the boundary sphere. If $R < \frac{3}{2}L$, the fermions do not contribute any charge or energy density, and hence the background is given by global AdS geometry with constant electric potential,

$$\begin{aligned} ds^2 &= \frac{L^2}{R^2 \sin^2 \frac{z}{R}}\left[-dt^2 + dz^2 + R^2 \cos^2 \frac{z}{R}\, d\Omega_2^2\right], \\ A &= \frac{1}{L}\, dt. \end{aligned} \tag{140}$$

This is because the spectrum of the Dirac Hamiltonian in global AdS is discrete and gapped, the lowest positive-energy state being $\omega_0 = \frac{3}{2R}$. If the electric potential is smaller than this threshold, no charge is induced in the bulk. From the dual point of view, the infrared regulator opens a gap of order $1/R$ in the spectrum of charged excitation, and hence the system is incompressible for sufficiently small R. The critical solution $R = 3/2$ is shown with a dashed line in Fig. 3.

As R goes beyond the critical value, the fermions start contributing nonzero charge and energy density, which then backreacts on the gauge field and the geometry.

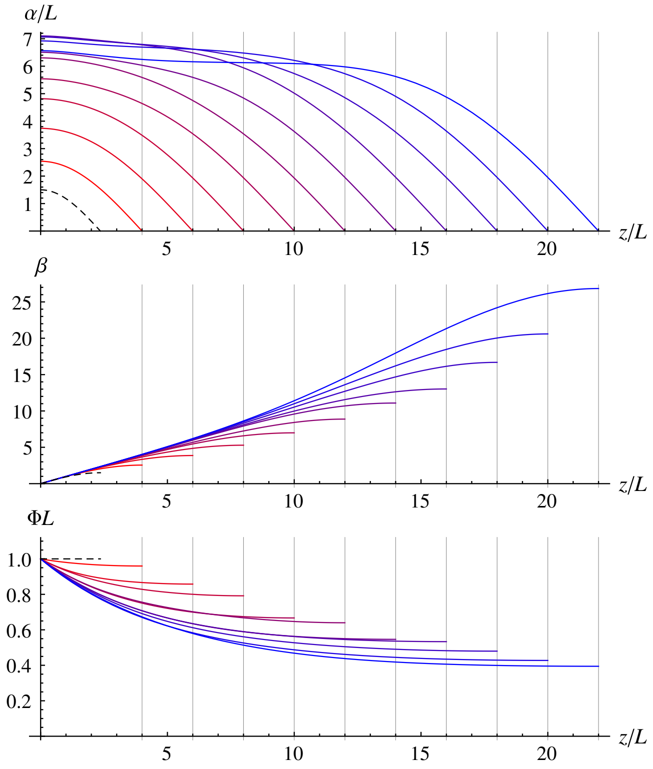

FIG. 3 (color online). Self-consistent profiles for $\kappa^2/L^2 = 0.1$, $q^2 = 1.0$, $mL = 0.0$.

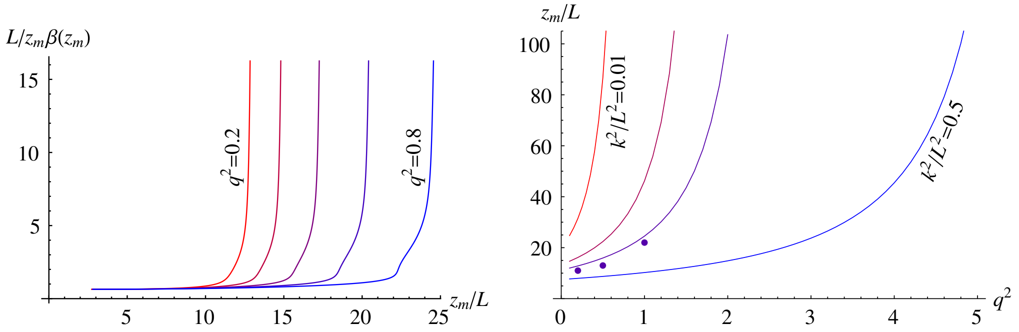

FIG. 4 (color online). Location of the phase transition in the Thomas-Fermi approximation. Left: $\beta(z_m)$ against z_m. From red to blue, $q^2 = 0.2, 0.4, 0.6, 0.8$, $\kappa^2/L^2 = 0.1$, $mL = 0.0$. Right: The critical z_m as a function of q^2. The points show the approximate location of the numerical instability. From red to blue, $\kappa^2/L^2 = 0.01, 0.05, 0.1, 0.5$, $mL = 0.0$.

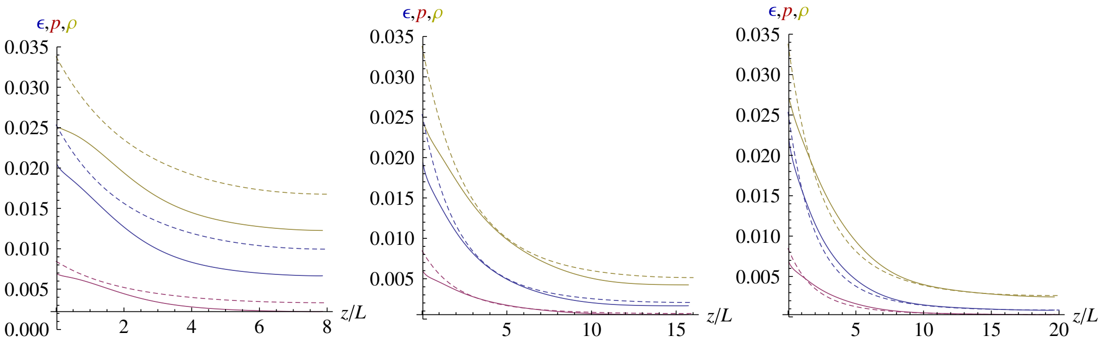

FIG. 5 (color online). Comparison of the Thomas-Fermi approximation (dashed line) against the exact answer (solid line). $q^2 = 1.0$, $\kappa^2/L^2 = 0.1$, $mL = 0.0$. From left to right, $z_m/L = 8, 16, 20$.

The expectation is that, at large R, the background would approach some asymptotic, R-independent behavior, at least for $z \ll z_m$. What we discover instead is that rather soon R stops growing as z_m is increased and, more or less at the same point, the iterative algorithm becomes unstable and fails to converge. A notable feature of this instability is that, iteration after iteration, the value $\beta(z_m)$ grows beyond bounds, suggesting that the system would like to develop a horizon in the interior.

Close to the instability, the present method of solution is not reliable enough to determine whether there is an actual singularity or merely an algorithmic problem, but it gives some indications that the first option is the correct one. Therefore, to better investigate the issue, we solve the problem in the same setup, but within the Thomas-Fermi approximation (previously used with spherical spatial geometry in Refs. [29,30]). The results are shown in Figs. 4 and 5.[11] It is manifest that $\beta(z_m)$ diverges at a finite value of z_m, over the whole range of q^2 and κ^2/L^2 that we have explored, indicating the presence of an actual singularity. The points in the right plot display the approximate location of the onset of the instability of the iterative algorithm, and agree quite well with the phase boundary determined by the Thomas-Fermi approximation.

Based on these results, we conclude that our model develops a physical singularity as z_m increases, such that $\beta(z_m)$ diverges at a finite value of z_m. While this phase transition may be interesting in itself, it precludes the possibility of studying the large-μR limit with the current method of solution. In fact, the geometry at $\mu R \gg 1$ is likely to have a zero-temperature horizon or some other singularity in the interior, and this would cause the Dirac operator to be noncompact. While it may be possible to generalize the current method of solution to a Hamiltonian with a continuous spectrum, it is beyond the scope of the present article.

The accuracy with which the Thomas-Fermi approximation is able to predict the location of the numerical instability raises questions on its regime of validity. According to Ref. [10], the Thomas-Fermi approximation is justified in the regime $mL \gg 1$, but we find it to be a good description even at $mL = 0$. In Fig. 5 we compare the currents computed within the Thomas-Fermi approximation against the exact ones, in a self-consistent background.

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[11] Note that the IR geometry of these Thomas-Fermi solutions still satisfies the boundary conditions (132) and (133); it is not a Lifshitz geometry as in the Poincaré case.

It is apparent that, as z_m grows, the Thomas-Fermi approximation becomes better. On the other hand, the approximation should be consistent if locally

$$\frac{|\nabla k_F(x)|}{k_F^2(x)} \ll 1, \tag{141}$$

which in the current setup ($mL = 0$) translates to

$$\frac{1}{\beta\Phi^2}\frac{d}{dz}(\beta\Phi) \ll 1. \tag{142}$$

This condition breaks down if β diverges, as we have seen happens at a finite critical value of z_m. Near this critical value, it is not possible to reliably compute the exact answer, so we have no way of verifying this breakdown. We may summarize our present understanding of the Thomas-Fermi approximation by saying that it is inadequate at small z_m, it improves at larger values, and probably breaks down near the critical z_m.

We end this paper with a discussion of possible future directions.

(i) The phase transition that we observe in the Thomas-Fermi approximation certainly deserves further investigation, and currently lacks a clear holographic interpretation. We emphasize that it is a zero-temperature phase transition different than the one described in Ref. [13]. It appears to be a confinement-deconfinement transition [3,31] driven by the fermion density.

(ii) It would be very important to develop a better understanding of the regime of validity of the Thomas-Fermi approximation.

(iii) In order to explore the large-μR regime, which is of the greatest physical interest, it may be necessary to generalize the methods presented in this paper to situations in which the Hamiltonian is not a compact operator. On the other hand, it may be sufficient to replace the radius of the sphere with some other infrared regulator that is better behaved.

(iv) To gain complete control over the approximations made in solving the model, it would be important to estimate the amplitude of the fluctuations of the metric and the gauge field, which have been neglected (in the Hartree-Fock approximation). This amounts to computing the current-current correlators. The same information could also be used to substitute Newton's method to the current naive iteration algorithm, allowing for a faster and more stable solver. This would probably allow us to explore values of z_m closer to the critical point.

(v) In holographic duality, we are accustomed to interpreting the radial dependence of (bosonic) bulk fields as encoding running couplings in the dual quantum field theory (QFT) (along with some information about the quantum state). The holographic interpretation of the quantum state of the bulk fermion fields poses an interesting question of principle for holographic duality. It appears to provide a concrete example of the "quantum renormalization group" described in Refs. [32–34].

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APPENDIX A: ADIABATIC EXPANSION OF THE CURRENTS

Here we write the explicit adiabatic expansion for the currents, in the background

$$\mathrm{d}s^2 = \frac{1}{\beta^2(r)}[-\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)], \tag{A1}$$

where the currents are defined by

$$j^\mu(x) = -\mathrm{Tr}[\gamma^\mu S(x, x')], \tag{A2}$$

$$T^{\mu\nu}(x) = -\mathrm{Tr}[\gamma^{(\mu}\mathrm{i}D^{\nu)} S(x, x')], \tag{A3}$$

with

$$\sqrt{g}(\mathrm{i}\gamma \cdot D - m)S(x, x') = \mathrm{i}\delta(x - x'), \tag{A4}$$

and $x = (t, r, \theta, \phi)$, $x' = (t \pm \mathrm{i}s, r, \theta, \phi)$. Symmetrized over the sign of s,

$$\begin{aligned}\frac{\pi^2}{\beta^4}J^t = &-\frac{2\Phi}{s^2} + L\left(\frac{\alpha'\Phi'}{3\alpha} + \frac{\Phi''}{6}\right) - \frac{\Phi\alpha''}{6\alpha} - \frac{\alpha'\Phi'}{6\alpha} - \frac{\Phi\alpha'^2}{12\alpha^2}\\ &+ \frac{\Phi}{12\alpha^2} - \frac{\beta'\Phi'}{6\beta} - \frac{\Phi''}{12} - \frac{1}{3}\Phi^3 + \mathcal{O}(s^2),\end{aligned} \tag{A5}$$

$$\begin{aligned}\frac{\pi^2}{\beta^4}T^t{}_t ={}& \frac{6}{s^4} + \frac{1}{s^2}\left(\frac{\alpha''}{6\alpha} + \frac{\alpha'^2}{12\alpha^2} - \frac{1}{12\alpha^2} + 3\Phi^2\right) + L\left(\frac{\alpha^{(4)}}{120\alpha} - \frac{\alpha''^2}{240\alpha^2} + \frac{\alpha'^4}{240\alpha^4} + \frac{\alpha^{(3)}\alpha'}{120\alpha^2} - \frac{\alpha'^2\alpha''}{60\alpha^3} - \frac{1}{240\alpha^4} + \frac{1}{12}\Phi'^2\right)\\ &+ \frac{\alpha^{(4)}}{120\alpha} - \frac{\alpha^{(3)}\beta'}{60\alpha\beta} - \frac{\alpha''\beta''}{40\alpha\beta} + \frac{17\alpha''\beta'^2}{720\alpha\beta^2} + \frac{\Phi^2\alpha''}{12\alpha} - \frac{\alpha''^2}{240\alpha^2} - \frac{\beta^{(3)}\alpha'}{60\alpha\beta} - \frac{\alpha'^2\beta''}{144\alpha^2\beta} - \frac{11\alpha'\beta'^3}{360\alpha\beta^3} + \frac{3\alpha'^2\beta'^2}{160\alpha^2\beta^2}\\ &+ \frac{17\alpha'\beta'\beta''}{360\alpha\beta^2} + \frac{\Phi\alpha'\Phi'}{3\alpha} + \frac{\Phi^2\alpha'^2}{24\alpha^2} + \frac{\alpha'^4}{240\alpha^4} + \frac{\alpha^{(3)}\alpha'}{120\alpha^2} - \frac{\alpha'\alpha''\beta'}{72\alpha^2\beta} - \frac{\alpha'^2\alpha''}{60\alpha^3} + \frac{\beta''}{144\alpha^2\beta} - \frac{\beta'^2}{288\alpha^2\beta^2} - \frac{\Phi^2}{24\alpha^2}\\ &- \frac{1}{240\alpha^4} - \frac{\beta^{(4)}}{240\beta} + \frac{7\beta''^2}{480\beta^2} + \frac{11\beta'^4}{960\beta^4} + \frac{\beta^{(3)}\beta'}{80\beta^2} - \frac{23\beta'^2\beta''}{720\beta^3} + \frac{1}{6}\Phi\Phi'' + \frac{1}{24}\Phi'^2 + \frac{\Phi^4}{4} + \mathcal{O}(s^2),\end{aligned} \tag{A6}$$

$$\begin{aligned}\frac{\pi^2}{\beta^4}T^r{}_r ={}& -\frac{2}{s^4} + \frac{1}{s^2}\left(-\frac{\alpha'^2}{12\alpha^2} + \frac{1}{12\alpha^2} - \Phi^2\right) + L\left(\frac{\alpha''^2}{240\alpha^2} + \frac{\alpha'^4}{240\alpha^4} - \frac{\alpha^{(3)}\alpha'}{120\alpha^2} - \frac{1}{240\alpha^4} + \frac{1}{12}\Phi'^2\right) - \frac{\alpha^{(3)}\beta'}{120\alpha\beta} + \frac{\alpha''\beta''}{120\alpha\beta}\\ &- \frac{\beta^{(3)}\alpha'}{120\alpha\beta} - \frac{\alpha'^2\beta''}{60\alpha^2\beta} - \frac{\alpha'\beta'^3}{72\alpha\beta^3} - \frac{7\alpha'^2\beta'^2}{480\alpha^2\beta^2} + \frac{\alpha'^3\beta'}{60\alpha^3\beta} + \frac{\alpha'\beta'\beta''}{24\alpha\beta^2} - \frac{\Phi\alpha'\Phi'}{6\alpha} - \frac{\Phi^2\alpha'^2}{24\alpha^2} + \frac{\alpha'\alpha''\beta'}{120\alpha^2\beta} - \frac{\beta'^2}{288\alpha^2\beta^2}\\ &+ \frac{\Phi^2}{24\alpha^2} - \frac{\beta''^2}{160\beta^2} + \frac{11\beta'^4}{960\beta^4} + \frac{\beta^{(3)}\beta'}{80\beta^2} - \frac{\beta'^2\beta''}{40\beta^3} - \frac{1}{24}\Phi'^2 - \frac{1}{12}\Phi^4 + \mathcal{O}(s^2),\end{aligned} \tag{A7}$$

$$\begin{aligned}\frac{\pi^2}{\beta^4}T^\theta{}_\theta ={}& \frac{\pi^2}{\beta^4}T^\phi{}_\phi\\ ={}& -\frac{2}{s^4} + \frac{1}{s^2}\left(-\frac{\alpha''}{12\alpha} - \Phi^2\right) + L\left(-\frac{\alpha'^4}{240\alpha^4} + \frac{\alpha'^2\alpha''}{120\alpha^3} + \frac{1}{240\alpha^4} - \frac{1}{12}\Phi'^2\right) + \frac{\alpha^{(3)}\beta'}{120\alpha\beta} - \frac{\alpha''\beta''}{120\alpha\beta} - \frac{7\alpha''\beta'^2}{1440\alpha\beta^2}\\ &- \frac{\Phi^2\alpha''}{24\alpha} - \frac{\beta^{(3)}\alpha'}{80\alpha\beta} + \frac{\alpha'^2\beta''}{120\alpha^2\beta} + \frac{\alpha'\beta'^3}{720\alpha\beta^3} - \frac{\alpha'^2\beta'^2}{120\alpha^2\beta^2} - \frac{\alpha'^3\beta'}{120\alpha^3\beta} + \frac{\alpha'\beta'\beta''}{90\alpha\beta^2} - \frac{\Phi\alpha'\Phi'}{12\alpha} + \frac{\alpha'\alpha''\beta'}{60\alpha^2\beta} - \frac{\beta^{(4)}}{240\beta}\\ &+ \frac{7\beta''^2}{480\beta^2} + \frac{11\beta'^4}{960\beta^4} + \frac{\beta^{(3)}\beta'}{80\beta^2} - \frac{23\beta'^2\beta''}{720\beta^3} - \frac{1}{12}\Phi\Phi'' - \frac{1}{24}\Phi'^2 - \frac{1}{12}\Phi^4 + \mathcal{O}(s^2).\end{aligned} \tag{A8}$$

APPENDIX B: HOW NOT TO CONSTRUCT A GRAVITATING QUANTUM ELECTRON STAR

In the course of developing the method of solution we described, we encountered many approaches that seemed natural choices, but revealed themselves to be complete blunders. We briefly describe them here as a warning to any person that would get involved in this kind of problem in the future.

1. Other regulators

We began our investigations by looking at the problem with a frozen metric [15]. In this case, only the charge density is needed. The charge density is a mildly divergent quantity, and hence not very sensitive to the regularization and renormalization procedure.

In our preliminary work, we discretized the Dirac Hamiltonian using finite differences, and we considered a planar instead of spherical boundary. To avoid dealing with a continuous spectrum, we terminated the geometry with an artificial hard wall, following Ref. [14]. In this case the partial wave number ℓ is replaced by the transverse momentum k, and the radius R of the sphere is replaced by the distance of the wall from the boundary.

In this setup, the lattice spacing a provides a natural cutoff on the high-frequency modes. The contribution of each k mode to the charge density is finite in the limit $a \to 0$, because positive-frequency and negative-frequency modes make contributions with the opposite sign. The sum over the k modes is logarithmically divergent, but it can easily be regulated with a hard cutoff on the momentum k. A change in this cutoff is equivalent to charge renormalization.

For the charge density this regularization and renormalization scheme works just fine, but it is not recommendable to use it when the geometry becomes dynamical. First of all, a hard-wall termination of the geometry makes little sense when Einstein's equations are involved, so a geometric infrared regulator is needed. This is why we introduced the spherical geometry.

Second, the contribution of each k mode to the energy density is not finite in the limit $a \to 0$, because positive-frequency and negative-frequency modes make contributions with the same sign. One needs to carry out some kind of subtraction to get rid of this infinity. But it is not obvious how to determine the counterterm. Since the infinity is strongly tied to the lattice physics, there is no procedure analogous to the adiabatic expansion that can give analytic information about the divergences. Moreover, even if one were able to obtain a finite subtracted quantity, it is not clear whether it would be a meaningful quantity, i.e. whether the subtraction procedure succeeded in restoring general covariance.

Understanding the importance of general covariance as a guidance for the regularization and renormalization procedure, it is tempting to use a covariant regulator. For example, one can try the heat-kernel regulator

$$J_0^\mu = \langle \bar{\psi} \gamma^\mu e^{-s^2 \not{D}^2} \psi \rangle, \tag{B1}$$

$$T_0^{\mu\nu} = \langle \bar{\psi} \gamma^{(\mu} \mathrm{i} D^{\nu)} e^{-s^2 \not{D}^2} \psi \rangle. \tag{B2}$$

When using this regulator, all divergences are proportional to local geometric objects. The renormalization procedure consists in simply subtracting them, and general covariance is preserved throughout the process. In practice, one would expand over eigenfunctions of $\not{D}$,

$$\not{D} \psi_n(x) = \lambda_n \psi_n(x), \tag{B3}$$

and write

$$J_0^\mu = \sum_n \bar{\psi}_n \gamma^\mu \psi_n e^{-s^2 \lambda^2}, \tag{B4}$$

$$T_0^{\mu\nu} = \sum_n \bar{\psi}_n \gamma^{(\mu} \mathrm{i} D^{\nu)} \psi_n e^{-s^2 \lambda^2}. \tag{B5}$$

Unfortunately, this approach has several problems. First of all, the operator $\not{D}$ is not self-adjoint when the real time component Φ of the gauge field is nonzero. Consequently, the numerical diagonalization of $\not{D}$ is problematic. Second, the label n stands for the momenta in the time, radial and transverse directions. It is not possible to carry out the sum over any of these momenta analytically, even though there is time-translational invariance and translational or spherical symmetry along the transverse directions. Therefore, the sum over n truly is at best a double sum, with each term involving the diagonalization of a non-Hermitian matrix, and a summation over the eigenfunctions. Moreover, the momentum in the radial direction is continuous, because the operator $\not{D}$ is noncompact at the boundary of AdS, so it is necessary to introduce a hard-wall infrared regulator near the boundary. It is apparent that this is not quite the way to go.

A more promising approach is to use a Pauli-Villars regulator. One introduces a number of additional fictitious spinor fields, with appropriately chosen masses M_i and statistics σ_i (bosonic spinor fields may be needed), so that their contribution to the currents exactly cancels the contribution of the physical field at large energy. Explicitly,

$$j_0^\mu(x) = \mathrm{Tr}[\gamma^\mu S_m(x,x)] + \sum_i \sigma_i \mathrm{Tr}[\gamma^\mu S_{M_i}(x,x)], \tag{B6}$$

$$\begin{aligned} T_0^{\mu\nu}(x) &= \mathrm{Tr}[\gamma^{(\mu} \mathrm{i} D^{\nu)} S_m(x,x)] \\ &\quad + \sum_i \sigma_i \mathrm{Tr}[\gamma^{(\mu} \mathrm{i} D^{\nu)} S_{M_i}(x,x)]. \end{aligned} \tag{B7}$$

The masses and statistics can be found by studying the problem in flat space. In this case one has for the stress tensor

$$T_0^{\mu\nu}(x) = \int \mathrm{d}^d p\, p^\mu p^\nu \left(\frac{1}{p^2 + m^2} + \sum_i \sigma_i \frac{1}{p^2 + M_i^2} \right). \tag{B8}$$

With an appropriate choice of σ_i and M_i, the integral can be made convergent. For example, in two dimensions one can take

$$\begin{aligned} &\sigma_1 = \sigma_2 = -1, \qquad \sigma_3 = 1, \\ &M_1 = M_2 = M, \qquad M_3 = \sqrt{2M^2 - m^2}. \end{aligned} \tag{B9}$$

Clearly the currents diverge in the large-M limit, but the coefficients of the divergent terms in a series expansion are local geometric objects because this regulator is manifestly covariant. These terms can be subtracted, yielding well-defined renormalized currents.

A Pauli-Villars regulator makes it possible to express the currents in terms of the Dirac Hamiltonian. Introducing a set of eigenfunctions of the Dirac Hamiltonian

$$H_{\ell, M_i} \psi_{n\ell i} = \omega_{n\ell i} \psi_{n\ell i}, \tag{B10}$$

we have, for example,

$$T_0^{tt}(x) = \frac{1}{2} \sum_{n,\ell} \sum_i \sigma_i \psi_{n\ell i}^\dagger(x) [\mathrm{abs}\,\omega_{n\ell i} - \Phi(x) \mathrm{sign}\,\omega_{n\ell i}] \psi_{n\ell i}(x), \tag{B11}$$

where we have included m in the list of the masses M_i. The sum is convergent by construction, so the contribution of the higher-frequency modes is less and less important. The problem has been reduced to a single sum, each term of which involves the diagonalization of a handful of Hermitian matrices, and a summation over their eigenfunctions. This is a marked improvement over the heat-kernel regulator.

Unfortunately, the suppression of high-frequency modes is only polynomial. This makes it necessary to compute a great number of terms in the ℓ and n sum, and hence to diagonalize matrices of large size. Eventually, because of this reason, we choose to resort to point-splitting regulation, which yields exponential suppression of the high-energy modes.

2. Parallel transport

The point-separated expressions

$$J_0^\mu(x) = \langle \bar{\psi}(x') \gamma^\mu \psi(x) \rangle, \tag{B12}$$

$$T_0^{\mu\nu}(x) = \langle \bar{\psi}(x')\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle \tag{B13}$$

may look awkward to the careful reader, because the spinors $\psi(x)$ and $\bar{\psi}(x')$ do not transform in a complementary way under gauge transformations and diffeomorphisms, and hence the bare currents are not tensors. One may be tempted to introduce a more covariant expression,

$$J_0^{\mu}(x) = \langle \bar{\psi}(x')P(x',x)\gamma^{\mu}\psi(x)\rangle, \tag{B14}$$

$$T_0^{\mu\nu}(x) = \langle \bar{\psi}(x')P(x',x)\gamma^{(\mu}\mathrm{i}D^{\nu)}\psi(x)\rangle, \tag{B15}$$

where P is the spinor parallel transport, satisfying

$$\begin{cases} D_{\mu}P(x,x') = 0, \\ P(x,x) = 1. \end{cases} \tag{B16}$$

While there is certainly nothing wrong in doing so, it is not necessary, the reason being that the subtraction of the adiabatic expansion cancels all the covariance-breaking effects of the regulator. To show this explicitly, let us assume the covariant definitions (B14), and show that the parallel transport has no effect after subtraction of the adiabatic expansion. Let us consider the U(1) current. We have

$$J_0^{\mu}(x) = -\mathrm{Tr}[\gamma^{\mu}S(x,x')P(x',x)]. \tag{B17}$$

The propagator $S(x,x')$ diverges as x' approaches x, and we have[12]

$$S(x,x') = \frac{1}{s^3}S_3(x,x') + \frac{1}{s^2}S_2(x,x') + \cdots + S_0(x,x'), \tag{B18}$$

where $s^2 = (x-x')^{\mu}(x-x')_{\mu}$ and the S_i have a finite limit as $s \longrightarrow 0$. On the other hand, $P(x,x) = 1$, so

$$P(x,x') = 1 + sP_1(x,x') + s^2P_2(x,x') + \cdots. \tag{B19}$$

Therefore, the portion of the current that depends on P and that does not vanish as $s \longrightarrow 0$ is

$$\begin{aligned} J_0^{\mu}(x)|_P = &-\frac{1}{s^2}\mathrm{Tr}[S_3P_1] - \frac{1}{s}(\mathrm{Tr}[S_3P_2] + \mathrm{Tr}[S_2P_1]) \\ &- (\mathrm{Tr}[S_3P_3] + \mathrm{Tr}[S_2P_2] + \mathrm{Tr}[S_1P_1]). \end{aligned} \tag{B20}$$

This expression depends only on the divergent terms of S, which are captured in full by the adiabatic expansion. Therefore, after subtraction of the adiabatic expansion and the limit $s \longrightarrow 0$, there is no dependence of P left, and the result is the same as if it had not been included from the beginning.

Besides complicating the algebra unnecessarily, the inclusion of the parallel transport has another undesirable consequence. As shown in Sec. II B, if the point splitting is in the time direction, with constant coordinate separation, the bare stress tensor is covariantly conserved. This property is lost if the parallel transport is included. Obviously it is restored by the regularization and renormalization procedure, but there is some advantage in having it throughout the process.

3. WKB instead of adiabatic expansion

It is tempting to try to use the WKB approximation to determine the high-energy behavior of the eigenfunctions of the Dirac Hamiltonian. If that were possible, one could subtract the high-energy behavior directly in the mode sum, for example,

$$J^t = \frac{1}{e_te_re_s^2}\sum_{n\ell}\frac{|\ell|}{2\pi}[\psi_{n\ell}^{\dagger}(r)\psi_{n\ell}(r) - \psi_{n\ell}^{\mathrm{WKB}\dagger}(r)\psi_{n\ell}^{\mathrm{WKB}}(r)], \tag{B21}$$

and compute the renormalized currents directly as an altogether finite sum, without the need for any other subtraction or limit, provided that the subtraction can be shown to preserve general covariance. This program works when the background is spatially uniform, and depends on time, but it fails when there is nontrivial spatial dependence.

The issue can be demonstrated (by replacing the Dirac equation) with the more familiar Schrödinger equation, in more than one dimension. It arises even if the Schrödinger operator (i.e. the potential) depends only on one variable r. So we consider

$$-\nabla^2\psi(r,y) + (V(r) - E)\psi(r,y) = 0. \tag{B22}$$

Translation invariance in y—the proxy for the QFT spatial slices, of which there can be more than one for the present discussion—and the linearity of Eq. (B22) allow us to Fourier decompose,

$$\psi(r,y) = e^{\mathrm{i}ky}\psi(r), \tag{B23}$$

so that

$$-\psi''(r) + (V(r) + k^2 - E)\psi(r) = 0. \tag{B24}$$

For $E \gg V(x) + k^2$ we may use a WKB ansatz,

$$\psi(r) = \frac{\mathcal{N}}{\sqrt{q(x)}}\exp\left(\mathrm{i}\int_{-\infty}^{r}q(r')\mathrm{d}r'\right), \tag{B25}$$

and $q(r)$ must satisfy

$$q^2(r) - (E - k^2 - V(r)) - \epsilon\left(\frac{3}{4}\frac{q'(r)^2}{q(r)^2} - \frac{q''(r)}{2q(r)}\right), \tag{B26}$$

where $\epsilon = 1$ is a bookkeeping parameter. The WKB approximation treats the last two terms as an approximation by writing

$$q(r) = q_0(r) + \epsilon q_1(r) + \mathcal{O}(\epsilon). \tag{B27}$$

Solving order-by-order in ϵ gives

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[12] There are also logarithmic divergences, which do not matter for the following.

$$q_0(r) = \sqrt{E - k^2 - V(r)},$$
$$q_1(r) = \frac{1}{8}\frac{V''(r)}{q_0(r)^3} + \frac{1}{32}\frac{V'(r)^2}{q_0(r)^5}. \tag{B28}$$

In this approximation, the contribution to the particle density from a given mode is

$$|\psi(r,y)|^2 = \frac{\mathcal{N}}{q(r)} = \frac{\mathcal{N}}{q_0(r)} - \epsilon\frac{\mathcal{N}q_1(r)}{q_0^2}\Bigg|_{\epsilon=1}$$
$$= \mathcal{N}\left(\frac{1}{q_0(r)} - \frac{1}{8}\frac{V''(r)}{q_0(r)^5} + \frac{1}{32}\frac{V'(r)^2}{q_0(r)^7}\right). \tag{B29}$$

Now assume for argument that V and V' vanish at $r \longrightarrow \pm\infty$, so that the states at large r are plane waves, which we can take to be incoming,

$$p = q(-\infty) = \sqrt{E - k^2}. \tag{B30}$$

One can verify by putting the system in a box that the correct integration measure (for adding up the contributions of the modes to the total density or total energy) is

$$\int \text{đ}k - \text{đ} - p \equiv \int \frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}p}{2\pi}, \tag{B31}$$

so we can label the states by (E, k) rather than (p, k). So

$$q_0(r) = \sqrt{p^2 - V(r)} \tag{B32}$$

and, for example, the (heat-kernel) regulated energy density is

$$\varepsilon(r,y) = \int \text{đ}k - \text{đ} - pE(k,p,r)|\psi(r,y)|^2 e^{-sE(k,p,r)}, \tag{B33}$$

with

$$E(k,p,r) \equiv \sqrt{k^2 + p^2 + V(r)} \tag{B34}$$

and

$$|\psi(r,y)|^2 = \mathcal{N}\left(\frac{1}{\sqrt{p^2 - V(r)}} - \frac{1}{8}\frac{V''(r)}{(p^2 - V(r))^{5/2}} + \frac{1}{32}\frac{V'(r)^2}{(p^2 - V(r))^{5/2}}\right). \tag{B35}$$

This expression has a nonintegrable singularity when $p^2 \longrightarrow V(r)$. This is to be expected, since the WKB approximation is valid in the limit $p^2 \gg V(r)$.

The problem for our purposes is that we need to integrate over p. We could consider restricting the integral to large $|p| > M$. But this excludes a slice of the integration region *that includes very high-energy modes*, and hence modifies the s dependence of the integral. The problem is that the singularity from the breakdown of WKB is present not only for small $k^2 + p^2$, and one cannot exclude it without excluding high-energy modes.

The point then is that no matter how large E is, there are still modes of large enough k to cause the WKB approximation to break down.

One can attempt a higher-dimensional generalization of WKB. Unfortunately, this does not work because WKB in higher dimensions (at least as we understand it) is nonlocal. Suppose we want to solve

$$-\nabla^2\psi(r,y) + (V(r,y) - E)\psi(r,y) = 0 \tag{B36}$$

and substitute

$$\psi(r,y) = A(r,y)\exp\left(\mathrm{i}\int_\infty^{(r,y)} \vec{q}(\vec{r})\cdot \mathrm{d}\vec{r}\right). \tag{B37}$$

We arrive at the system of equations

$$\begin{cases} \frac{1}{A}\nabla^2 A + (E - V + \vec{q}^2) = 0, \\ \vec{\nabla}\cdot(A^2\vec{q}) = 0, \\ \partial_r q_y - \partial_y q_r = 0, \end{cases} \tag{B38}$$

where the last equation ensures that the phase in Eq. (B37) does not depend on the path. The amplitude $A(r,y)$, however, depends on the potential $V(r', y')$ at arbitrarily distant points.

We note in passing that this problem seems to present an obstruction to a systematic implementation of the expansion studied in Ref. [17].

4. Localization of eigenstates of the lattice Dirac Hamiltonian

In this section we explain why it is necessary to use the point-splitting regulator s in addition to the lattice regularization of the Dirac operator. One might consider simply using a lattice regularization of the Dirac Hamiltonian, with near-neighbor derivatives, or some improvement thereof, for example using spectral methods. However, only a fraction of the eigenstates of the resulting lattice operator have anything at all to do with the continuum limit. This fact is visible in a plot of the eigenvalues versus mode number (Fig. 6, top left) and of the successive differences between eigenvalues (Fig. 6, top right), for a 1 + 1-dimensional case. In the middle of the spectrum, the dispersion is linear and agrees with the expected slope in the continuum (straight line). Further, the eigenstates come in pairs consisting of a left-mover and a right-mover, as one expects from plane waves in the continuum. Away from the middle of the spectrum, we find only one-dimensional representations of the parity operator. We observe that there is a sharp boundary where the degeneracy ends.

That this is a sharp "mobility edge" in the spectrum, separating extended and localized states, can be seen as follows. A measure of localization is the inverse participation ratio (IPR), defined as

$$\mathrm{IPR}_k[\psi] = \int dx|\psi(x)|^{2k}$$

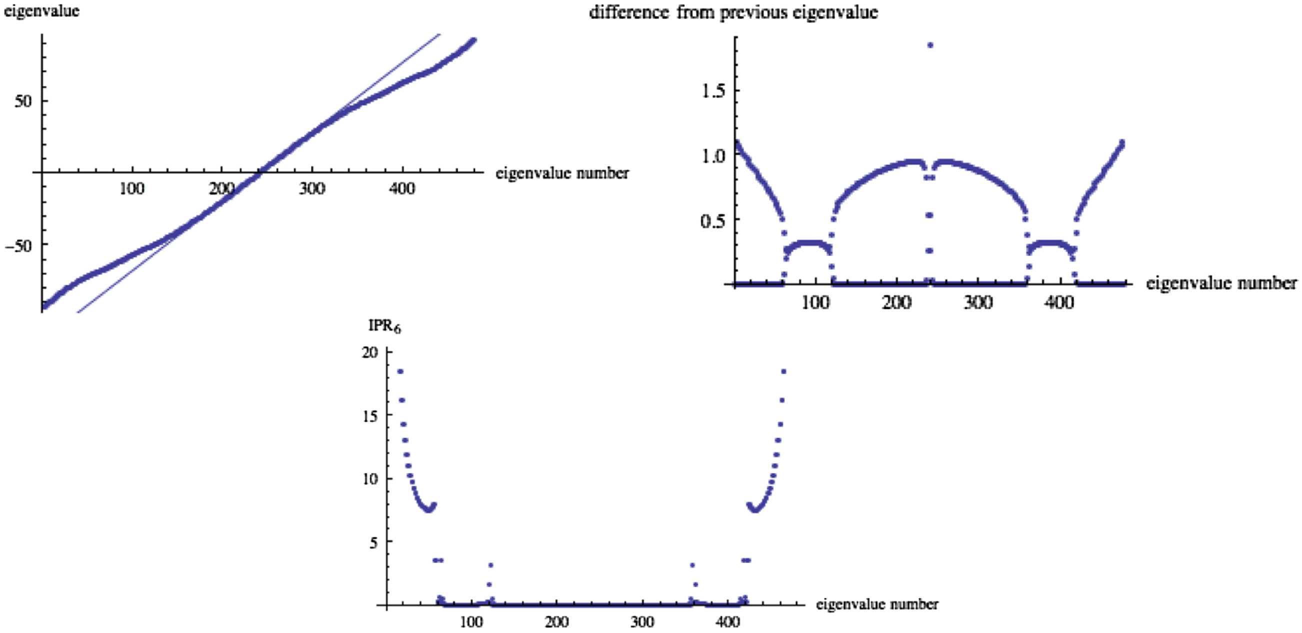

FIG. 6 (color online). In this example, $f_t = 1 + .3\cos z + .2\cos 2z$, $f_x = 1 + .4\cos z - .2\cos 2z$. $n = 249$ sites. In the first plot, the line represents the continuum dispersion. The second plot shows successive differences of eigenvalues; the zeros in this plot indicate eigenvalues that come in pairs related by parity, as is true of all modes in the continuum. Note the presence of several bands of parity-paired states.

for a normalized wave function $\int dx|\psi(x)|^2 = 1$. The results can be seen in Fig. 6. A sample localized wavefunction is shown in Fig. 7, along with the curvature well in which it is localized.

Leaving out the localized modes would make it impossible to resolve the identity, and one cannot construct smooth sources without them. The problem of central interest to us is the four-dimensional bulk Dirac operator, with translation invariance in the QFT spatial directions. Each momentum mode satisfies a $1 + 1$ Dirac equation (with a complex k-dependent mass). Therefore, this same problem persists for us in higher dimensions.

More explicitly, we consider a massless Dirac field in the geometry

$$ds^2 = f_t(z)dt^2 + f_z(z)dz^2 + f_x(z)d\vec{x}^2$$

with electrostatic potential Φ. It will be useful to write vielbeins $e_\mu^2 \equiv f_\mu$. (Below we will take $d = 0$, $z \simeq z + 2\pi$ is a circle and f_t, f_z are chosen to be periodic, and $\Phi = 0$.) In terms of

$$\Psi(z, \vec{x}, t) = e^{-i\omega t + i\vec{k}\cdot\vec{x}} \sqrt{\frac{e_t}{\prod_{\mu=t,z,\vec{x}} e_\mu}}\,\psi(z)$$

the Dirac equation $(\not{D} + m)\Psi = 0$ can be written as $H\psi = \omega\psi$. In this basis, the Dirac Hamiltonian (at fixed k) is self-adjoint with respect to the usual inner product,

$$(\psi_1, \psi_2) = \int dz\,\psi_1^\star(z)\psi_2(z).$$

It is convenient to study a different basis, where derivatives of the metric do not appear,

$$\Psi(z, \vec{x}, t) = e^{-i\omega t + i\vec{k}\cdot\vec{x}} \sqrt{\frac{e_z}{\prod_{\mu=t,z,\vec{x}} e_\mu}}\,\tilde{\psi}(z),$$

in which the Dirac equation $(\not{D} + m)\Psi = 0$ can be written as $\tilde{H}\tilde{\psi} = \omega\tilde{\psi}$, where $\tilde{H}$ has no derivatives of e_μ. We have

$$\tilde{\psi} = \sqrt{\frac{e_t}{e_z}}\psi,$$

and therefore

$$\tilde{H}\sqrt{\frac{e_t}{e_z}}\psi = \omega\sqrt{\frac{e_t}{e_z}}\psi,$$

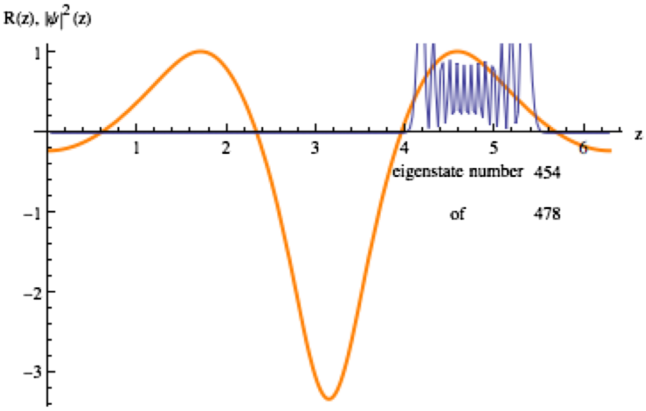

FIG. 7 (color online). A plot of the curvature (orange, thick) in the example above, along with one of the visibly localized eigenfunctions (blue, thin).

which means

$$H = \sqrt{\frac{e_z}{e_t}}\tilde{H}\sqrt{\frac{e_t}{e_z}} = \underbrace{\sqrt{\frac{e_t}{e_z}}}_{W}\underbrace{\frac{e_z}{e_t}\tilde{H}}_{\hat{H}}\underbrace{\sqrt{\frac{e_t}{e_z}}}_{W}. \tag{B39}$$

Here W is diagonal in position space, and $\hat{H}$ is a Hermitian matrix, free from derivatives of e_μ, with which H is isospectral. Explicitly, with $k = 0$,

$$H = \begin{pmatrix} -m\sqrt{f_t} + \Phi & -\left(\frac{f_t}{f_z}\right)^{1/4}\partial_z\left(\frac{f_t}{f_z}\right)^{1/4} \\ \left(\frac{f_t}{f_z}\right)^{1/4}\partial_z\left(\frac{f_t}{f_z}\right)^{1/4} & m\sqrt{f_t} + \Phi \end{pmatrix} = W\begin{pmatrix} -m\sqrt{f_z} + \Phi\sqrt{\frac{f_z}{f_t}} & -\partial_z \\ \partial_z & m\sqrt{f_z} + \Phi\sqrt{\frac{f_z}{f_t}} \end{pmatrix}.$$

A straightforward way to put this operator on the lattice is simply to replace $\psi(z)$ with a column vector of values at equidistant points $\psi(z_i)$, and to replace

$$\begin{pmatrix} . & -\partial_z \\ \partial_z & . \end{pmatrix} \mapsto \begin{pmatrix} . & B \\ B^t & . \end{pmatrix}$$

with

$$B = \frac{1}{\Delta z}\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{B40}$$

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