Aspects of Branes and Orbifolds in String Theory

by

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Abstract

The main theme of this thesis is branes and orbifolds in string theory, with some digression towards other directions in the final chapters. After reviewing background material on D-branes and orbifolds, along with some mathematical fundamentals, I describe the following results.

First, a classification of discrete torsion for a large class of orbifolds and a technique for extracting the matter content of D-brane probe gauge theories on orbifolds with discrete torsion.

Subsequently, I discuss an algorithm, called stepwise projection, that illuminates the structure of the so-called exceptional quivers, and hints towards the brane realization of the associated gauge theories.

Next, I present the computation of the partition function of the coset conformal field theory describing the two-dimensional black hole. This computation confirms earlier results concerning the spectrum of the black hole and it enable us to identify the physical Hilbert space.

This theory appears in the exact string theory description of configurations of Neveu-Schwarz 5-branes and in the conformal field theory description of certain orbifolds in singular limits; thus, it is only a mild digression from the main theme of the thesis.

This is not so, however, for the last topic discussed here. We will change gears completely and discuss our extension of Witten's construction of boundary string field theory to the superstring. As in the bosonic case, the main tool we use is the Batalin-Vilkovisky formalism. Our construction proves a recent conjecture regarding the spacetime action of the supersymmetric theory and a related conjecture concerning quantum field theories on two-dimensional spaces with boundaries.

Thesis Supervisor: Amihay Hanany
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This thesis is based on research I did between the February of 2000 and the February of 2002, under the supervision of Amihay Hanany.

I would like to thank Amihay wholeheartedly for giving me the opportunity to work on string theory with him and his group, despite the fact that I was already in my 4th year of graduate studies. It is reassuring to know that in the competitive arena of theoretical physics, and especially of string theory, there are people like Amihay that manage to be good humans and good scientists at the same time. Needless to say, the kindness and genuine interest Amihay showed to me were absolutely crucial in my attempt to repair damages due to bad choices and lack of maturity on my side, and due to bad advising, misguidance, irresponsibility and neglect from a variety of other agents.

I am deeply grateful to Amihay for being a real teacher to me in many ways; sometimes explaining basic things that I should probably know, or answering to some really naive questions. His advice on other aspects of scientific life, besides science per se, was also immensely helpful and I feel particularly privileged that he shared his rich experience and wisdom on these issues with me.

I would like to thank Jeffrey Goldstone and Barton Zwiebach, who, as members of my thesis committee, provided constructive comments and suggestions concerning my thesis. In particular, Barton pointed out several misleading statements in my discussion on off-shell strings and string field theory in the last chapter, and I wish to thank him for his help.

I would like to express my gratitude to Bo Feng and Yang-Hui He for collaboration [1, 2, 3], to Vasilis Niarchos for discussions and collaboration [4], to Jan Troost for discussions and collaboration [5], and to Pavlos Kazakopoulos for discussions. I am also indebted to Pantelis Manousselis for many discussions throughout the years.

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Chapter 1

Introduction

During the last few years we witnessed a tremendous development of the branch of theoretical physics known as string theory. At the same time, it is quite surprising that despite all the progress and the huge effort devoted to this field, it still remains unclear if it is has anything to do with the world around us [1]. Thus, instead of looking at string theory only as a candidate for providing a unified quantum description of all elementary particles and their interactions, one needs to adopt a more relaxed attitude; string theory can be viewed also as a tool, which we can use, for example, to gain insight on the dynamics of gauge theories, or to understand the nature of spacetime geometry at very small distances. Needless to say, we can expect that the application of string theory in this fashion will eventually teach us a lot about string theory per se, and hopefully it may be even able to clarify if strings and other extended objects are really the unified framework we are looking for.

The bulk of the developments during the so-called "second superstring revolution" was in this spirit. In particular, the importance of D-branes can hardly be overestimated. Besides opening a widow to the non-perturbative aspects of string theory in general, the concept of D-branes as defects whose world-volume supports certain

[1]Some advocates of string theory would claim that there is actually some experimental evidence in favor of strings, namely, the fact that it "predicts" gravity. In particular, one can use string theory - at least in principle - to compute amplitudes in quantum gravity. However, the lack of a manifestly non-perturbative and background-independent formulation of string theory, narrows significantly the scope of such claims. And, of course, string theory predicts 10-dimensional gravity!

quantum field theories as consistent truncations of the full string theory spectrum, proved to be a very powerful tool in uncovering a rich variety of field theory results. The basic idea is that one can decouple the theory living on the brane from the degrees of freedom living in the bulk of spacetime and from the massive states living on the brane. This way one is left with a field theory, usually of Yang-Mills type, describing the interaction of massless modes localized on the brane's world-volume. Since the variables specifying the original string theory background appear as coupling constants or other field theory parameters for the D-brane theory, one obtains the means of understanding highly non-trivial aspects of the field theory by translating them to substantially less difficult problems in string theory.

There are several ways of engineering gauge theories by using D-branes. One involves string theory backgrounds that contain singularities in the classical geometry, usually of orbifold type. It is well-known (see chapter 2 for further discussion) that perturbative string theory makes perfect sense on such singular geometries, in contrast to plain quantum field theory which is ill-defined in such situations. The reason is the extended nature of the string, which allows for the so-called twisted sectors. The states in these sectors correspond to spacetime fields which resolve the singularities, i.e. provide extra degrees of freedom that make string theory non-singular. Orbifolds were extensively used in the context of perturbative string theory as tractable examples of compact manifolds on which the 10-dimensional superstring theory could be compactified and produce a 4-dimensional UV completion of the standard model incorporating gravity.

One can engineer a rich variety of gauge theories by placing D-branes on orbifold singularities. The nature of the theory, for example the amount of supersymmetry, the matter content, and the interactions, are determined by the geometrical characteristics of the orbifold and the presence of background spacetime fields. In this context we usually say that the D-branes are probing the orbifold. This is due to a fundamental property of D-branes: they perceive spacetime geometry through the moduli space of supersymmetric vacua of their world-volume gauge theory. As we will discuss in the next chapter, in some cases the analysis of the moduli space provides a

physical realization of abstract mathematical methods of describing the corresponding orbifold geometries. Moreover, the precise ways in which the singularities are resolved in the eyes of the D-brane observer and the eyes of the fundamental string observer are usually complementary.

It is of obvious significance to understand the range of theories obtainable by the above technique. One can associate at least one gauge theory to a given orbifold geometry. However, the D-brane probe gauge theory is also sensitive to changes in the background fields besides the geometry; hence, one can increase the scope of these techniques by turning on suitable fluxes consistent with the equations of motion. A simple way to achieve that is by giving vacuum expectations values to spacetime fields corresponding to flat gauge potentials, i.e. non-trivial field configurations with zero energy. However, this is expected to correspond to motion along flat directions of the same theory and not to a completely different theory.

The next candidate are fluxes corresponding to non-trivial field strengths in spacetime. This time, however, the field configurations carry energy and therefore one expects that the orbifold geometry should also be modified, so that the full background still satisfies the spacetime equations of motion. The only way to avoid that is by localizing the fluxes on the orbifold fixed points, which are in any case points of infinite curvature. This is exactly what discrete torsion stands for. In general, turning on this type of fluxes changes considerably the theory on the D-brane probe, affecting both the spectrum and the superpotential. In particular, in many cases some of the blow-up modes, i.e. degrees of freedom associated to the resolution of the singularities, are absent. This is in accordance with similar expectations from the perturbative treatment of orbifolds with discrete torsion.

In order to acquire a better handle on the "space" of all possible D-brane probe gauge theories, we need a classification of discrete torsion for each orbifold of interest. This is our first result: a complete classification of discrete torsion for all orbifolds of the form $\mathbb{C}^3/\Gamma$, where Γ is a finite subgroup of the symmetry group $SO(6)$ of $\mathbb{C}^3$. It is well-known that the possibilities of discrete torsion are classified by the second group cohomology of Γ. Thus, our classification amounts to explicitly computing these

cohomologies for all finite subgroups of $SO(6)$. Since finite subgroups that belong only in an $SU(2)$ part of $SO(6)$ produce theories with 8 supercharges, and the latter are known to be very rigid, we verify that there can be no non-trivial discrete torsion for these groups. Subgroups in $SU(3) \subset SO(6)$ are known to yield theories with 4 supercharges, whose superpotential is not determined by supersymmetry. We indeed find that many such groups admit discrete torsions. Moreover, since most discussions in literature focus on abelian orbifold groups, we study in some detail a non-abelian example, hoping to uncover further aspects of discrete torsion and its effects.

The basic ingredient in the study of D-brane probe gauge theories is the knowledge of the spectrum and the interactions. In general, these are determined by the orbifold group and actually they can be summarized in a quiver diagram. For a given group (without discrete torsion), this diagram is unique and it encodes the ring structure of the set of irreducible representations of the group. A natural question is if the techniques that exist in obtaining this information are extendable to the case where discrete torsion has been turned on. Our next result indicates that this is indeed the case. In particular, we find that the covering group of the orbifold group contains all information we need to determine the spectrum of the D-brane probe for all choices of discrete torsion. We should emphasize also that our approach is computationally more convenient than other methods that have been proposed in the literature to tackle the same problem.

A very intriguing finding is that the quiver of the covering group splits into disconnected components, each of them summarizing the D-brane gauge theory for a given choice of discrete torsion. It still remains to be clarified if the covering group and the aforementioned quiver splitting is just a mathematical convenience or has a deeper physical meaning.

Our next work is related to a different method of engineering gauge theories from branes and its connection with the "brane on orbifolds" approach. This method, invented by Hanany and Witten, is based on the fact that configurations of intersecting branes, including not only D-branes but NS5-branes as well, can be manufactured so that they realize theories in diverse dimensions and with a wide variety of gauge

groups. The advantage of this method, known as "brane setups", is that it provides in many cases a very intuitive framework for analyzing gauge theory dynamics. In particular, there are situations where just by inspecting the brane setup one can arrive to highly non-trivial results about the corresponding quantum field theories.

This powerful feature of the above approach is balanced by the difficulty one faces when he tries to engineer theories with more involved matter content. In it well-known however that there is actually a correspondence between a class of Hanany-Witten setups and the D-brane probe theories obtained from orbifolds associated to the A- and D-type subgroups of $SU(2)$. This correspondence, which is an application of T-duality between the respective backgrounds, illuminates some aspects of the theory on the orbifold side in a very heuristic manner. However, for the last type of $SU(2)$ orbifolds, known as the exceptionals or E-type, the corresponding brane setups are not known.

Our work, called stepwise projection, provides some information for the dual brane configuration of E-type orbifold theories. This is achieved by analyzing some striking relations between the quivers and the geometry of the associated brane setup. More concretely, we orbifold the theory with one generator at a time and we compare its action on the quiver versus its action on the brane setup.

Applying our method to the case of D-type quivers results in a novel way of understanding the nature of the T-dual brane configurations. Thus, by extending our analysis to the E-type quivers we conclude that the brane setup should involve a new object, so far unknown in string theory, which is defined through a $\mathbb{Z}^3$ action on the world-sheet. This is to be contrasted with the D-type brane setups which include a variant of the orientifold plane; the last is defined by gauging world-sheet parity, which is a $\mathbb{Z}^2$ action. We also point out some interesting similarities of the stepwise pattern we uncovered with the A-D-E classification of modular invariant partition functions for $SU(2)$ WZW theories.

In constructing the Hanany-Witten models we mentioned that they include a different kind of brane, known as NS5-brane. This object is by far less tractable to study, as it does not admit a simple perturbative description. Recall that the

dynamics of D-branes are determined perturbatively by the open strings ending on them. As every brane however, the NS5-brane has a simple description as a solitonic solution of low-energy effective supergravity theory. It is well-known that in string theory the near horizon region of this solution, called the "throat" because of its shape, admits an exact conformal field theory description in terms of an $SU(2)$ WZW model and a linear dilaton background.

The linear dilaton renders the description problematic as it implies large string coupling as we approach the NS5-brane. In particular, on the NS5-brane the string coupling is infinite. Therefore, the direct analysis of this CFT has little to offer in our understanding of the NS5-brane and its dynamics.

A different route is to consider the spacetime description of the NS5-brane. It has been argued that in the limit where the string coupling approaches zero, the degrees of freedom on the NS5-brane decouple from the bulk modes and one is left with an interacting theory of closed strings propagating on the NS5-brane world-volume. This mysterious 6-dimensional string theory, which does not contain gravity, is known as little string theory.

This exotic phase of string theory is quite intriguing. Its analysis is hindered by the fact that the theory on the NS5-branes is strongly coupled and hence it does not allow for a world-sheet approach. Understanding little string theory better is very significant as it may shed some light on aspects of string theory without the extra complications due to gravity. Note that this idea of taking decoupling limits and defining theories, whose microscopic formulation may not be explicitly known in some cases, proved to be one of the most exciting new developments in string theory. For example, taking appropriate decoupling limits on D-branes with constant B-fields on their world-volume gives rise to non-commutative gauge theories. Furthermore, by turning on electric backgrounds one finds non-commutative theories of open strings that are non-gravitational (recall that gravity exists as a quantum effect in all standard open string theories).

The interplay between string theory and field theory, which resulted in a large number of new results, is based exactly on the philosophy of decoupling limits. This

philosophy has close ties with the notion of holography. The latter is a general principle suggesting that theories with gravity are equivalent in some cases to theories without gravity in one dimension less.

Holography appeared in string theory through the AdS/CFT correspondence and the Matrix formulation of M-theory. The AdS/CFT conjecture asserts that the decoupled theory on a D3-brane, which is a 4-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory, is equivalent holographically to type IIB string theory on the near horizon geometry of the D3-brane. This conjecture has been used to uncover properties of strongly coupled SYM theories by studying supergravity. A large number of results, some of them unexpected, was obtained this way.

In a similar vein, it has been conjectured that the two descriptions of NS5-branes we discussed earlier are holographically dual. More precisely, the spectrum and interactions of little string theory should be encoded in the conformal field theory of the throat.

The strong coupling singularity however still prevents us from fully exploiting this duality. One of the few ways to resolve it, while maintaining an exact CFT description, is by separating the NS5-branes on a circle. It can be shown that the conformal field theory associated to this configuration involves the $SL(2)/U(1)$ coset CFT.

The last one is also known as the 2-dimensional black hole, since the $SL(2)/U(1)$ coset CFT corresponds to a sigma model describing string propagation on a black hole solution of 2-dimensional gravity coupled to a dilaton.

Our next result is the computation of the partition function of this coset CFT using path-integral techniques. The partition function is used to derive the spectrum of the black hole, previously known only through algebraic CFT analysis. In particular, we show how this spectrum is organized in representations of the affine Lie algebra of $SL(2)$ by decomposing the partition function into the corresponding characters. We also find the precise bounds on the spin of the physically allowed discrete representations, and we determine the density of the continuous representations. Our upper bound for the spin proves a related conjecture that was based on the holographic

description of little string theories.

Our study is also motivated by the recent improvement in our understanding of the non-compact WZW theory based on the $SL(2)$ group manifold. More precisely, the identification of the spectrum of the WZW model and its completeness was firmly established. Further motivation for our computation is presented in the beginning of chapter 5.

The last work presented in this thesis is in a different line of developments, for which however the concept of D-branes played also a substantial role. The difference lays in the fact that we will be interested in an off-shell formulation of string theory. This type of formulation, known as string field theory, is in contrast to the first-quantized on-shell description of strings.

The recent developments in this field demonstrated that open string field theory has an impressive predictive power which extends beyond the perturbative spectrum. In this context, some aspects of the condensation of open strings tachyons were clarified.

Our work is more formal, as it is related to the precise formulation of boundary superstring field theory. This is an off-shell theory of open strings, which was originally constructed by Witten only for bosonic strings. In our work we extend Witten's theory to the NS sector of the superstring. In particular, we will find that the superstring version is considerably simpler than the bosonic one, in contrast to the situation with other open string field theories. Our analysis proves also a recent conjecture regarding the spacetime action of this theory. More details on the motivation for studying this problem can be found at the beginning of chapter 6.

Chapter 2

D-branes, Singularities, and Gauge Theories: A Selective Review

In this chapter we discuss some aspects of the physics and mathematics of D-brane probes that will provide the platform on which the next two chapters will be based. Thus, there are no new results here but only a very selective presentation of background material.

We start by reviewing D-branes and their dynamics following the traditional route of extracting the massless spectrum and writing the corresponding low-energy effective field theory.

We continue with a synopsis of orbifold compactifications of perturbative string theory and we discuss in detail a simple example, namely $T^4/\mathbb{Z}_2$, in order to set the scene for the appearance of D-branes. Subsequently, we collect for convenience some information on Calabi-Yau manifolds, singularities, ALE and ALF spaces, quivers etc. These will provide the mathematical basis for the discussion of D-branes probing generic orbifolds, which will come next. There we explain in detail the Douglas-Moore construction of the D-brane probe gauge theory as a projection of the original $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory [1]. We also formulate the projection algorithm in the more abstract language of Lawrence, Nekrasov, and Vafa, which we will employ in

[1] We will use calligraphic $\mathcal{N}$ to denote 4-dimensional supersymmetry, while supersymmetry in D spacetime dimensions will be denoted by N_D.

chapter 3.

Finally, we elaborate further on some aspects of the D-brane probe theories with $\mathcal{N} = 2$ supersymmetry. This case is particularly interesting since the construction of the Higgs branch of the moduli space is a physical realization of Kronheimer's HyperKähler quotient approach to ALE gravitational instantons.

2.1 Review of D-brane dynamics

D-branes where found in [10]; their importance however was fully appreciated only when Polchinski showed in [11] that they are charged under the Ramond-Ramond (R-R) $U(1)$ gauge fields. This was an interesting observation for many reasons; first, the perturbative string spectrum is neutral under the R-R fields and it is natural to wonder what would be the stringy origin of the appropriate sources. Moreover, Polchinski's work essentially provided a microscopic description of the so-called (extremal) black p-branes, which were well-known solitonic solutions of the low-energy effective supergravities charged under the R-R fields, as hyperplanes on which open strings can end. Thus their dynamics should be fully determined by open string theory. The interplay of their open string theory description with their closed string theory one (as solitonic black branes), motivated many of the recent developments in string theory (the most important example being the AdS/CFT correspondence perhaps).

2.1.1 Generalities

D-branes are dynamical hyperplanes on which the endpoints of open strings can end. Recall than the usual boundary conditions of open strings are of Neumann type, i.e. if $\sigma \in [0, \pi]$ is the spatial coordinate on the string world-sheet, we have for example

$$\partial_\sigma X(\sigma, \tau)|_{\sigma=0} = 0, \tag{2.1.1}$$

where $X(\sigma,\tau)$ is a spacetime coordinate and the boundary condition is imposed at the endpoint $\sigma = 0$. As usual τ denotes world-sheet time.

Dirichlet boundary conditions break Poincare invariance in the D-dimensional ambient spacetime as they fix the string endpoint at a particular spacetime submanifold. Thus, ones needs to impose Neumann boundary conditions in all D spacetime coordinates in order to obtain an open string theory with the maximal possible spacetime symmetry.

However, there are situations, in field theory for example, where extended objects that break Poincare invariance (e.g. like domain walls), exhibit interesting dynamical properties and are often the starting point for analyzing non-perturbative effects. Similarly, one would hope that the study of analogous objects in string theory would be quite useful. Even more important is however the fact that if one wants to extend the T-duality symmetry of closed strings to open string theories, is obliged to consider open strings with Dirichlet boundary conditions [10]. Thus, the incorporation of D-branes is essentially a consistency requirement.

Concretely, a Dp-brane is defined as a (p+1)-dimensional hyperplane in spacetime extended in p spatial directions (so that along with the time direction its world-volume is (p+1)-dimensional) with the property that open strings can end on its world-volume. This is a non-trivial requirement since the open strings can be thought of as flux tubes and accordingly they can only terminate in regions where an appropriate gauge field exists to absorb their flux. For example, the flux of the usual type I open strings is absorbed by the 1-form gauge field A^M that exists in the D=10, N=1 SYM multiplet. In this case one can imagine that all of 10-dimensional spacetime is the world-volume of a number of D9-branes. The endpoint of the open string is an electrically charged particle for the 1-form gauge field. Interesting discussions of this interpretation of the string endpoints along with generalizations to other branes ending on branes can be found in [12, 13].

Thus one is led to suspect that on generic Dp-branes there should be gauge fields confined on them. In the presence of a Dp-brane extended along $X^0, X^1, \ldots, X^p$ in 10-dimensional spacetime, the endpoint, say at $\sigma = 0$, of an open string ending on

the D-brane satisfies:

$$\begin{aligned} \partial_\sigma X^i &= 0, \quad i = 0, 1, \ldots, p, \\ \partial_\tau X^m &= 0, \quad m = p+1, \ldots, 9. \end{aligned} \tag{2.1.2}$$

The string's endpoint is confined on the hyperplane spanned by $X^0, X^1, \ldots, X^p$ since we have Neumann boundary conditions in these directions and thus it is free to move along those, while it has a fixed position in the transverse space spanned by $X^{p+1}, \ldots, X^9$ since the value of these coordinates does not depend on the world-volume time τ. The values of $X^{p+1}, \ldots, X^9$ can be thought of as the coordinates of the D-brane in the transverse $(9-p)$-dimensional space.

The D-brane is really a dynamical object carrying tension (besides R-R charge) since it should balance the force exerted by the strings ending on it. However, in order for a D-brane configuration to be stable it should preserve some supersymmetry, in other words it should be a BPS object. BPS means that it saturates the Bogomol'nyi-Prasad-Sommerfeld inequality and in a supersymmetric theory such states are annihilated by a subset of the supercharges, i.e. they leave some supersymmetry unbroken. The stability of BPS objects is due to topological reasons as they minimize the energy of field configurations subject to certain topological constraints (for example, having the same winding number).

In general, putting Dp-branes in flat spacetime so that they are parallel to each other leaves half of the original supersymmetry unbroken. For example, in type II theories, backgrounds with D-branes have 16 supersymmetries which are realized on the D-brane world-volume. We have seen that the theory living on configurations of parallel D-branes should involve gauge fields and now we learn that it should actually have $\mathcal{N} = 4$ supersymmetry. There is essentially a unique theory with the above features: the 10D, N=1 SYM. By dimensional reduction with can obtain various theories in $d < 10$ with the same amount of supersymmetry; one well-known example is the 4D, $\mathcal{N} = 4$ SYM which is superconformal and exhibits S-duality.

2.1.2 Massless spectrum and the effective action

It was argued by Witten [14] that at low-energies the effective field theory living on a collection of parallel and coincident Dp-branes should be the dimensional reduction of the 10D, N=1 SYM to (p+1)-dimensions with $U(N)$ gauge group. For a single D-brane it is easy to see how one acquires a $U(1)$ gauge field on the world-volume. Recall that after the GSO projection, one has a massless state in the NS sector of the superstring $\psi^M_{-1/2}|k\rangle_{NS}$ which is interpreted as a gauge boson in spacetime, and also its supersymmetric partner which is just the ground state of the R sector $|\alpha; k\rangle_R$. Here $\alpha = 1, \cdots, 16$ is a spinor index, since in 10D this state is a Majorana-Weyl spinor with 16 real components. When we study an open string ending on a Dp-brane we have to impose the boundary conditions (2.1.2). We see that in this case the momentum k has only $p+1$ components since the string is fixed in the $9-p$ transverse directions. Note also that the presence of the D-brane breaks the spacetime Lorentz symmetry $SO(9,1) \to SO(p,1) \times SO(9-p)$, where the first factor is realized as the Lorentz symmetry of the D-brane's world-volume while the second corresponds to the fact that the D-brane is invariant under rotations in its transverse space (in which it is point-like). We conclude that the 10-dimensional gauge boson state will decompose into a (p+1)-dimensional gauge boson $\psi^i_{-1/2}|k\rangle_{NS}$ and $9-p$ scalar fields $\psi^m_{-1/2}|k\rangle_{NS}$. Both of them are confined on the D-brane since they cannot have momentum transverse to it. If we decompose similarly the fermionic ground state, we realize that the resulting (massless) spectrum falls exactly into the dimensional reduction of the 10D, N=1 SYM multiplet to (p+1)-dimensions.

In this light, the 6 mysterious scalars present in the 4D, $\mathcal{N}=4$ SYM theory are just the 6 transverse positions of the D3-brane we use to engineer this particular SYM from string theory. More generally, the scalars on a D-brane's world-volume parameterize its position in the transverse space and they can also be interpreted as Goldstone bosons corresponding to the spontaneous breaking of translational invariance due to the presence of the D-brane. Note also that these scalars depend on the world-volume coordinates of the D-brane and thus they describe the embedding of the D-brane into

the transverse space.

Of course one expects that the SYM description is only valid for low-energies and weak fields. One can study the massless modes in more detail by integrating out all other massive open string excitations and the resulting theory is the so-called Born-Infeld theory, originally introduced by Born and Infeld to cure the divergences of classical electromagnetism [15]. As was shown in [16], the action for a single Dp-brane is given by

$$S = -T_p \int d^{p+1}\sigma e^{-\phi} \sqrt{-\det(G_{ij} + B_{ij} + 2\pi\alpha' F_{ij})} \tag{2.1.3}$$

where $T_p \sim \frac{1}{g_s \alpha'^{(p+1)/2}}$ is the Dp-brane tension, $e^{-\phi}$ is the way the dilaton couples to the D-brane and it has that form since its vacuum expectation value (vev), which has been absorbed in T_p, is the inverse of the open string coupling g_s since this action results from a tree level (disk) computation in open string theory, G_{ij} is the metric induced on the world-volume from the spacetime metric, similarly B_{ij} is the pull-back of the NS-NS 2-form, and F_{ij} is the field strength of the $U(1)$ gauge field living on the D-brane. Note that this is actually the action for a bosonic D-brane; the dynamics of a single D-brane in a superstring theory are governed by the supersymmetric generalization of the Born-Infeld theory written above [17, 18, 19, 20] along with appropriate Chern-Simons type couplings to the bulk R-R fields [21, 22].

Assuming that the D-brane is nearly flat, that the field strength is weak, and that the antisymmetric tensor is vanishing, one can easily show that the Born-Infeld action (2.1.3) reproduces the bosonic piece of the action of the 10D, N=1 $U(1)$ SYM dimensionally reduced to (p+1) dimensions with Yang-Mills coupling $g_{YM}^2 \sim g_s \alpha'^{(p-3)/2}$.

When more than one D-brane are present, the computation of the exact action for the massless modes becomes very involved and actually the general action, i.e. the non-abelian Born-Infeld, is still unknown (terms up to order F^5 are known precisely and there are some partial results for order F^6; see [23] for an early discussion and [24, 25] for the current state of affairs).

Under the simplifying assumptions we stated above for the single D-brane Born-

Infeld theory, one can argue that for N Dp-branes we should get the same theory but with different gauge symmetry, i.e. the (p+1)-dimensional reduction of the 10D, N=1 SYM with $U(N)$ gauge group.

The fact that the gauge symmetry is bigger than one would expect from a combination of independent $U(1)$ theories from each D-brane, is called gauge symmetry enhancement and it is due to strings that stretch between different D-branes. When the D-branes are separated, these strings have a mass proportional to the distance between the supporting branes. When the D-branes coincide, the mass of the strings stretching between them becomes zero and we obtain extra massless states in addition to the massless open string states due to strings starting and ending on the same D-brane. Since we are dealing with oriented strings, we have a total of N^2 massless states filling the adjoint of $U(N)$. Note that different D-branes should be thought of as providing Chan-Paton (CP) factors [26] to the string endpoints. Thus, one would expect that the type I theory with SO(32) gauge group should somehow contain 32 D9-branes (since the gauge field lives in 10D). We will see that this is indeed true when we discuss orientifolds in chapter 4.

2.1.3 The moduli space

The bosonic piece of the action for N Dp-branes is

$$S = \frac{1}{4g_{YM}^2} \int d^{p+1}\sigma \, \mathrm{Tr}\Big(- F_{ij}F^{ij} - 2(D_i X^m)^2 + [X^m, X^n]^2 \Big) \tag{2.1.4}$$

where X^m are hermitian matrices that transform in the adjoint of $U(N)$. If we had abelian gauge symmetry, the vacuum expectation values of $2\pi\alpha' X^m$ [2] would correspond to the position of the Dp-brane in the (9-p)-dimensional transverse space. In the non-abelian case the interpretation of the matrices X^m as coordinates is more involved.

What one has to do is to look for classical solutions of the potential $\mathrm{Tr}[X^m, X^n]^2$,

[2] We have assumed that the matrices X^m have mass dimension one and thus we need to multiply them we α' in order to get quantities with length dimensions.

in order to find the possible vevs of X^m that parameterize the classical moduli space of the SYM. It is easy to see that the moduli space is spanned by diagonal matrices

$$\langle X^m \rangle = \begin{pmatrix} x_1^m & 0 & 0 & \ddots \\ 0 & x_2^m & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \ddots & 0 & 0 & x_N^m \end{pmatrix} \tag{2.1.5}$$

since in this case the potential is vanishing and being positive definite it obtains its minimum value. The entries of the above matrices are constant and thus the kinetic terms in the action are also vanishing. The gauge fields and fermion vevs are of course vanishing since we want to preserve Lorentz invariance on the D-branes.

The eigenvalues (i.e. diagonal entries) of the X^m matrices can be interpreted as the coordinates of the N Dp-branes in the transverse space. The moduli space is thus $\mathbb{R}^{N(9-p)}/S_N$ where S_N is the group of permutations of N objects. This is the Weyl group of $U(N)$ and is the remnant of the $U(N)$ gauge symmetry which at a generic point in the moduli space is broken to $U(1)^N$. The D-brane interpretation of the above facts is that we have separated the D-branes in the transverse space and the only piece of gauge symmetry left is the symmetry under interchange of the D-branes since they are indistinguishable. The gauge group is the product of the $U(1)$ gauge group on each of the D-branes and when some of them are coincident we have gauge symmetry enhancement. The maximum possible gauge symmetry is achieved when all D-branes lie on top of each other, in other words when we are at the origin of the moduli space (which is actually a singular point since it is fixed under the action of the Weyl group S_N. This ties nicely with the general fact that extra massless states appear at singularities of moduli spaces and that they lead to gauge symmetry enhancement at these points).

Using the standard terminology of supersymmetric gauge field theories, we can say that separating the D-branes corresponds to moving into the Coulomb branch of the world-volume gauge theory. This is one of the many instances where the spacetime characteristics of D-brane configurations are associated with properties of the low-

energy effective field theory living on the D-branes. The brane setups we will discuss at chapter 4 are motivated by this basic idea.

Note than for a generic (not vacuum) configuration of D-branes the matrices X^m may not commute and so they may not be simultaneously diagonalizable. Then the spacetime interpretation of these matrices is not straightforward and it implies that the geometry seen by the D-branes is in a sense non-commutative. A non-zero off-diagonal term X^m_{ab} corresponds to a string that stretches between the a-th and b-th D-brane and extends along the m-th transverse direction.

The dynamics of supersymmetric gauge theories with 16 supercharges is quite well understood because the large amount of supersymmetry imposes severe constraints on the interactions. In particular, there are no quantum corrections to the classical moduli space. Thus one needs to extend the above considerations so that the theories realized on the D-branes have less supersymmetry and, accordingly, richer dynamics. There are three ways to accomplish that and two of them will be discussed extensively in this thesis. The first is by orbifolding the space transverse to the D-branes so that the supersymmetry in the bulk is reduced and, subsequently, the field theory on the D-branes has less that 16 supersymmetries. We will elaborate on this type of theories in this chapter.

The second way is by considering brane configurations involving several D-branes of various dimensionalities, perhaps intersecting, so that the bulk supersymmetry is again reduced. Aspects of these constructions will be discussed in detail in chapter 4. This method is actually related to the first one, i.e. D-branes probing orbifolds, by T-duality symmetries of the underlying string backgrounds. Some of the results we will present in chapter 4 concern exactly this correspondence.

Finally, the third method is known as geometric engineering and it is based on the fact that gauge symmetry enhancement can also occur when D-branes wrap vanishing cycles in singular geometries. By selecting appropriate geometries one can engineer a large number of field theories and, as in the previous two cases, many of their properties can be determined from the string background into which the D-branes are embedded. This method is also related to the previous two, see for example [27].

2.2 Orbifolds in perturbative string theory

We turn now our attention to a short discussion of orbifold compactifications of perturbative string theory. Orbifolds are a generalization of manifolds which allow for a class of well-behaved singularities, in the sense that the singularities are fixed points of the quotient of a smooth space with a discrete symmetry group.

Orbifolds made their appearance in string theory in [28, 29]. The original motivation was that they provided examples of compact spaces that break some of the supersymmetry of the original 10-dimensional space but at the same time, as they differ only slightly from flat spaces, the analysis of string theory on them is quite straightforward. The subsequent development of orbifold compactifications with phenomenological interest was enormous (see [30] for a review).

Currently, model-building based on orbifolds is not as popular as it was used to be for many reasons. First, there is a better understanding of more complicated geometries which may provide more viable phenomenology. Moreover, the recent developments showed that studying perturbative string theory and its compactifications is not the whole story and that in order to make contact with 4-dimensional physics, a different starting point which also captures non-perturbative effects may be necessary. However, in order to motivate the study of D-branes on orbifold geometries, we will present some rudiments of perturbative string theory on simple orbifolds.

We will discuss the well-known example [31, 32, 33] of type II strings propagating on the orbifold $T^4/\mathbb{Z}_2$. In the conventions of [7] the world-sheet action for superstrings is

$$S = \frac{1}{4\pi}\int d^2z\Big(\frac{2}{\alpha'}\partial X^M\bar{\partial}X^M + \psi^M\bar{\partial}\psi_M + \tilde{\psi}^M\partial\tilde{\psi}_M\Big) \tag{2.2.6}$$

where $\psi^M(z)$ and $\tilde{\psi}^M(\bar{z})$ are holomorphic and antiholomorphic Majorana-Weyl fermions with $M = 0, 1, \ldots, 9$.

Assume now that the directions $X^m := X^6, X^7, X^8, X^9$ are compact with the same radius R and orbifold them as $\mathcal{I}_4 : X^m \to -X^m$. Due to world-sheet superconformal invariance there is a similar action on the fermions, i.e. $\psi^m, \tilde{\psi}^m \to -\psi^m, -\tilde{\psi}^m$. We

will use Greek indices $\mu, \nu, \ldots$ for the six unorbifolded directions $(0, 1, \ldots, 5)$ and Latin indices $i, j, \ldots$ for the same directions after eliminating X^0, X^1 by going to light-cone gauge.

We focus for convenience on the holomorphic part. The discussion of the antiholomorphic part will be similar and the full spectrum will be found as usual by combining the two parts. The prescription [3] for orbifolding a conformal field theory tells us to consider first the untwisted sector, where the bosons X^μ, X^m are periodic and the fermions have the usual periodic (Ramond, "R") or anti-periodic (Neveu-Schwarz,"NS") boundary conditions and keep states invariant under the orbifold projection. We have moreover to compute the states in the so-called twisted sectors where the bosons that correspond to the orbifolded directions X^m are anti-periodic and the boundary conditions of the corresponding world-sheet fermions are similarly reversed. Finally we project again to states invariant under the orbifold projection. Recall that the twisted sector states are demanded from modular invariance. Finally, we have to make the Gliozzi-Scherk-Olive (GSO) projection which gets rid of the NS sector tachyon and results in a supersymmetric spectrum in spacetime. This amounts in keeping only states with even world-sheet fermion number F_s in the NS sector, i.e. $(-1)^{F_s} = 1$, and states with either even or odd F_s in the R sector. The last choice is essentially the choice of chirality in spacetime of the corresponding states.

The (light-cone) quantization of (2.2.6) results in the usual bosonic tower of excitations from the fields X^i, X^m and the NS and R sector of states from the fields ψ^i, ψ^m. As usual we are mostly interested in massless states. From the untwisted NS sector we get a vector $\psi^i_{-1/2}|k\rangle_{NS}$ of the $SO(4)$ little group of the $SO(5,1)$ Lorentz symmetry of the unorbifolded directions and four scalars $\psi^m_{-1/2}|k\rangle_{NS}$. Employing the usual (j_1, j_2) notation for representations of $SO(4) \cong SU(2) \times SU(2)$ with j_1, j_2 the respective $SU(2)$ spins, we see that we get one $(\frac{1}{2}, \frac{1}{2})$ and four $(0,0)$ states. Note that the orbifold action is $\psi^i, \psi^m \to \psi^i, -\psi^m$ and thus the vector is even while the scalars are odd under the projection. The NS sector ground state $|k\rangle_{NS}$ is projected out by the GSO because it has fermion number one due to superghost contributions.

[3]More details on general orbifoldings can be found in chapter 3.

For the R sector states we will use the standard labelling of spinor components by their eigenvalues s_1, s_2, s_3, s_4 under the commuting set of generators $\Sigma^{23}, \Sigma^{45}, \Sigma^{67}, \Sigma^{89}$ in the spinor representation of the Lorentz group. These correspond to rotations in the mutually orthogonal planes $X^2 - X^3, X^4 - X^5, X^6 - X^7, X^8 - X^9$ and their eigenvalues are $\pm 1/2$. The spin in the plane $X^0 - X^1$ is $s_0 = 1/2$ because we have restricted ourselves to the physical spectrum by going to light-cone gauge [7]. The GSO projected states have either an even number of $-1/2$ spins or an odd one. We will call the corresponding R sectors as R_+ or R_- respectively.

We need now to find out the transformation properties of these states under the orbifold projection. Since the projection acts only on the last four directions, the properties of the R sector physical ground states $|s_1, s_2, s_3, s_4\rangle_R$ will depend only on s_3 and s_4. Observing that the orbifold projection $X^m \to -X^m$ is a rotation by π in the planes $X^6 - X^7$ and $X^8 - X^9$, we see that the orbifold action on the spinors is $\exp(i\pi\Sigma^{67} + i\pi\Sigma^{89})$ and thus a state will be invariant if $s_3 + s_4 = 0$.

We also have to GSO project to a definite chirality in spacetime, which we choose to be the negative one R_-. That means that we need an odd number of $-\frac{1}{2}$ s in $|s_1, s_2, s_3, s_4\rangle_R$ and thus for the invariant state with $s_3 = -s_4$ we want $s_1 = s_2$. We see that we get two states in the $(\frac{1}{2}, 0)$ of $SO(4)$. Similarly the odd states with $s_3 = s_4$ should have $s_1 = -s_2$ and hence we have two states in the $(0, \frac{1}{2})$ representation. Had we chosen the opposite GSO projection R_+ in the R sector, the even states would be now in the $(0, \frac{1}{2})$ representation while the even ones would be in the $(\frac{1}{2}, 0)$.

We summarize all of the above in the following table:

sector	states	orbifold	$SU(2) \times SU(2)$
NS_{GSO}	$\psi^i_{-1/2}$	+	$(\frac{1}{2}, \frac{1}{2})$
	$\psi^m_{-1/2}$	-	$4(0, 0)$
R_-	$\lvert s_1, s_2, s_3, s_4\rangle, s_1 = s_2, s_3 = -s_4$	+	$2(\frac{1}{2}, 0)$
	$\lvert s_1, s_2, s_3, s_4\rangle, s_1 = -s_2, s_3 = s_4$	-	$2(0, \frac{1}{2})$
R_+	$\lvert s_1, s_2, s_3, s_4\rangle, s_1 = -s_2, s_3 = -s_4$	+	$2(0, \frac{1}{2})$
	$\lvert s_1, s_2, s_3, s_4\rangle, s_1 = s_2, s_3 = s_4$	-	$2(\frac{1}{2}, 0)$

Note that we have put only the GSO projected NS sector states but we present the

states in the R sectors for both chiralities.

We move on now to the twisted states. The X^m are anti-periodic and the corresponding oscillators have half-integral modding. The periodic boundary conditions for the world-sheet fermions ψ^m are opposite of what they used to be in the untwisted sector. Thus we have four fermion zero-modes ψ_0^m in the NS sector while in the R sector the eight zero-modes of the untwisted case are reduced to four, namely ψ_0^i, since the fields ψ_0^m are now anti-periodic.

One can easily find that in both the NS and R twisted sectors the zero-point energies are zero and thus we obtain massless ground states falling into representations of the corresponding fermionic zero-modes. Explicitly, in the NS sector we will get $SO(4)$ scalars $|s_3, s_4\rangle$ which will be even or odd under the orbifold projection if $s_3 = -s_4$ or $s_3 = s_4$ respectively. The GSO projection is $\exp(i\pi s_3 + i\pi s_4) = 1$ and thus only the even states with $s_3 = -s_4$ will survive the GSO. The R sector ground states will be invariant under the orbifold projection and they will be labelled by $|s_1, s_2\rangle$. After the GSO projection we will get either a $(\frac{1}{2}, 0)$ corresponding to $s_1 = s_2$ or a $(0, \frac{1}{2})$ representation of $SO(4)$ corresponding to $s_1 = -s_2$. From these we keep only the representation that corresponds to the same GSO projection in the untwisted R sector. For example, since above we chose negative chirality (odd number of $-1/2$s), we have to keep the $(0, \frac{1}{2})$ states in the twisted R sector.

The following table synopsizes the twisted states:

sector	states	orbifold	$SU(2) \times SU(2)$
NS_{GSO}	$\|s_3, s_4\rangle, s_3 = -s_4$	$+$	$2(0,0)$
R_-	$\|s_1, s_2\rangle, s_1 = -s_2$	$+$	$(0, \frac{1}{2})$
R_+	$\|s_1, s_2\rangle, s_1 = s_2$	$+$	$(\frac{1}{2}, 0)$

We have kept only the GSO projected NS sector states but we show the twisted R sectors states for both chiralities.

The last step is to tensor the states we found above for the holomorphic part with the similar ones for the antiholomorphic. In type IIA we have to make different GSO projections in the R sectors, i.e. we have to combine R_+ with R_-. We will only consider the bosonic part of the spacetime spectrum which will arise from the

NS-NS and R-R sectors. Of course we keep only states invariant under the orbifold projection.

In the untwisted NS-NS sector we will get $(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},\frac{1}{2}) = (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0)$ and $16(0,0)$ states. Similarly the untwisted R-R sector contains states $4(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = 4(\frac{1}{2},\frac{1}{2})$ and $4(0,\frac{1}{2})\otimes(\frac{1}{2},0) = 4(\frac{1}{2},\frac{1}{2})$. In other words, from the NS-NS sector we obtain the graviton $G_{\mu\nu}$, antisymmetric tensor field $B_{\mu\nu}$ and dilaton Φ along with 16 scalars and from the R-R sector we obtain 8 vectors.

The twisted states of the type IIA theory are as follows. From the NS-NS sector we get $4(0,0)$ states and from the R-R sector we get a $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ state, i.e. an $SO(4)$ vector.

To find the overall spacetime spectrum we have to take into account that on $T^4/\mathbb{Z}_2$ we have 16 twisted sectors since there is a total of 16 fixed points. We conclude that type IIA string theory on $T^4/\mathbb{Z}_2$ has, besides the standard gravity multiplet fields $G_{\mu\nu}, B_{\mu\nu}, \Phi$, $80 = 16 + 16\times 4$ scalars and $24 = 8 + 16$ vectors in $5+1$-dimensions. It is also straightforward to work out the fermionic spectrum. One would then conclude that the resulting 6D supergravity theory has $N_6 = (1,1)$ supersymmetry. We will see shortly how this result about the surviving supersymmetry can be obtained for a generic orbifold.

In type IIB we make the same GSO projection in the R sectors. The NS-NS spectrum is of course the same as that of type IIA in both the untwisted and the twisted sectors. The untwisted R-R sector contains $2\,(\frac{1}{2},0)\otimes 2(\frac{1}{2},0) = 4(1,0)\oplus 4(0,0)$ and $2\,(0,\frac{1}{2})\otimes 2(0,\frac{1}{2}) = 4(0,1)\oplus 4(0,0)$ states, while each of the 16 twisted sectors contributes a $(0,\frac{1}{2})\otimes(0,\frac{1}{2}) = (0,1)\oplus(0,0)$ state. All in all we get the gravity multiplet fields, 19 self-dual antisymmetric tensors $(0,1)$, 3 anti-self-dual antisymmetric tensors $(1,0)$, one antisymmetric tensor $(1,0)\oplus(0,1)$, and 104 scalars.

It is quite remarkable that precisely the same spectrum is obtained from compactification of type II theories on the manifold known as K3, which we will discuss in more detail in the next section. Our purpose here was to demonstrate a simple and interesting example of an orbifold compactification of string theory.

2.3 From Calabi-Yau manifolds to quivers

We present now some background material from algebraic and differential geometry with a string-theoretic flavor, which will be sine qua non for our explorations in the following chapters.

2.3.1 Calabi-Yau spaces and singularities

Calabi-Yau spaces are fundamental in string theory as they provide one way of linking 10-dimensional with 4-dimensional physics. Very simply put, a Calabi-Yau space M is a compact $2n$-dimensional manifold with $SU(n)$ holonomy. Recall that the holonomy group characterizes the possible rotations a vector can be subject to under parallel translations along closed loops, and that for a generic oriented Riemannian manifold of dimension $2n$ the holonomy is $SO(2n)$. Note than $SU(n)$ holonomy implies that M is Kähler (and thus complex) and that it can be endowed with a Ricci-flat metric. Yau's theorem [34] actually assures us that if M is Kähler and has vanishing first Chern class $c_1(M)$, we can always find a Ricci-flat metric for a given Kähler class J.

In order to motivate the need for $SU(n)$ holonomy in string theory compactifications, we will analyze the supersymmetry preserved after orbifolding a part of flat spacetime. The unbroken supersymmetry is easily determined by examining the way the spinors in 10D decompose into spinors of the remaining dimensions. For any of the five perturbative string theories, the original spinors of $SO(9,1)$ are Majorana-Weyl with 16 real components. In type I and heterotic theories we have one such spinor, i.e. the original supersymmetry is $N_{10}(1,0)$, while in type II theories we have two, having opposite chirality in type IIA, that is $N_{10}(1,1)$, and the same chirality in type IIB, $N_{10}(2,0)$.

We are interested in orbifolds of the form $\mathbb{C}^3/\Gamma$ where Γ is a finite subgroup of the rotational symmetry group $SO(6) \cong SU(4)$ of $\mathbb{C}^3 \cong \mathbb{R}^6$. The discussion is actually the same for compact orbifolds where instead of flat space we orbifold a flat torus, as we did in the previous section.

If we assume that Γ is in one of the $SU(2)$ subgroups of $SU(2)\times SU(2) \cong SO(4) \subset$

$SU(4)$ so that it acts only on a $\mathbb{C}^2$ subspace of $\mathbb{C}^3$, the decomposition we are looking for is

$$SO(9,1) \to SO(5,1) \times SO(4) \cong SO(5,1) \times SU(2) \times SU(2) \tag{2.3.7}$$

and the Majorana-Weyl spinor breaks as

$$\mathbf{16} \to (\mathbf{4},\mathbf{2}) \oplus (\mathbf{4}',\mathbf{2}') \to (\mathbf{4},(\mathbf{2},\mathbf{1})) \oplus (\mathbf{4}',(\mathbf{1},\mathbf{2})) \tag{2.3.8}$$

and since Γ lies in one of the $SU(2)$ groups, say the first, we are left with a $(\mathbf{4}',(\mathbf{1},\mathbf{2}))$ representation of $SO(5,1) \times SO(4)$. The $\mathbf{2}$ of $SU(2)$ is pseudo-real with two real components while the $\mathbf{4}$ of $SO(5,1)$ is the so-called symplectic Majorana representation which combined with the $\mathbf{2}$ results in the complex $\mathbf{4}$ of $SO(5,1)$. This has 8 real components and thus is the equivalent of $\mathcal{N}=2$ in 4D. In general, orbifolds of this type will preserve half of the supersymmetry of the original 10-dimensional theory.

If Γ lies in $SU(3)$ we have

$$SO(9,1) \to SO(3,1) \times SO(6) \to SO(3,1) \times SU(3) \tag{2.3.9}$$

and the Majorana-Weyl spinor breaks as

$$\mathbf{16} \to (\mathbf{2},\mathbf{4}) \oplus (\bar{\mathbf{2}},\bar{\mathbf{4}}) \to (\mathbf{2},\mathbf{3}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{2}},\bar{\mathbf{3}}) \oplus (\bar{\mathbf{2}},\mathbf{1}) \tag{2.3.10}$$

and thus for each Majorana-Weyl spinor we get a $\mathbf{2} \oplus \bar{\mathbf{2}}$ spinor of $SO(3,1)$ since it is a singlet under the $SU(3)$ subgroup of $SO(9,1)$ and hence invariant under the projection by Γ. Note that the spinor we get has 4 real components and not 8 since we had a Majorana condition on the $\mathbf{16}$ we started from and so we get $\mathbf{2} \oplus \bar{\mathbf{2}}$ which is real.

The final result is that a quarter of the supersymmetry in 10D survives the projection and we will get either $\mathcal{N}=1$ or $\mathcal{N}=2$ starting with type I and heterotic or type II strings respectively. Of course the same results are valid for the compact case

where we orbifold the torus T^6.

We observe now that for an orbifold of a flat space $\mathbb{R}^n$ with a finite group of symmetries Γ, the holonomy group is $\mathrm{hol}(\mathbb{R}^n/\Gamma) = \Gamma$. Non-trivial holonomy implies obviously non-zero curvature; for example, in the orbifold spaces it is well-known that the fixed points under the orbifold projection are singularities where the curvature is infinite.

We can now understand easily how the above analysis of unbroken supersymmetry generalizes to generic manifolds; the rule is that compactification on a manifold with holonomy in $SU(3)$ preserves $1/4$ of the original supersymmetry while holonomy in $SU(2)$ preserves one-half of the supersymmetry. For a flat space such as $\mathbb{R}^n$ or a d-dimensional torus T^d, the holonomy group is trivial and no supersymmetry is broken at all.

One more intuitive way to see the role of holonomy is the following. Suppose we compactify a D-dimensional theory on a $(D-d)$-dimensional compact space, so that at low energies physics is d-dimensional. To understand the supersymmetry of the d-dimensional theory, we need to find how the spinors of the original $SO(D-1,1)$ Lorentz symmetry decompose under the maximal subalgebra $SO(d-1,1) \times SO(D-d) \subset SO(D-1,1)$. The holonomy of M will be in general a subgroup of $SO(D-d)$ and it will act on the corresponding representations. Only spinors of $SO(D-1,1)$ whose decomposition is invariant under the action of the holonomy group will give rise to well-defined supersymmetries in d-dimensions. In other words, the supersymmetries we observe in d-dimensions correspond to spinors that are in singlets of the holonomy group under the decomposition $SO(d-1,1) \times SO(D-d) \subset SO(D-1,1)$. This is of course just a rigorous way to say that well-defined spinors in d-dimensions are only those corresponding to spinors of the original theory that stay the same when we travel in a loop in the compact space.

It should also be evident now why we are interested in Calabi-Yau manifolds, especially those with 3 complex dimensions. As they preserve only one quarter of supersymmetry, compactifying heterotic strings on them results in 4-dimensional theories with $\mathcal{N}=1$ supersymmetry, which is a well-motivated theoretical requirement. More

precisely we obtain a $\mathcal{N}=1$ supergravity theory and since we started with $\mathcal{N}=4$ supersymmetry and we are still left with a quarter of it, we see that the Calabi-Yau compactification is actually a stable solution of 10-dimensional supergravity theory., i.e. it is BPS. Or, from another point of view, Calabi-Yau manifolds solve automatically the Einstein field equations since they are Ricci-flat.

There are many ways to construct manifolds with $SU(3)$ holonomy ("CY 3-folds") and actually it is believed that there are billions of them. However, so far there are no convincing phenomenological models based on CY 3-folds despite many years of theoretical efforts. Moreover, even if a single CY 3-fold could be found that reproduces the minimally supersymmetric Standard Model, one would have in principle to explain how and why string theory selects that particular one. The last problem is actually not so pressing as it was used to be, since with the advent of duality symmetries it was realized that many of these seemingly different Calabi-Yau manifolds result in the same theories after compactification.

The moduli space of parameters of all Calabi-Yau spaces contains special points where submanifolds of the Calabi-Yau shrink to zero size. These singularities are particularly interesting; for example, they can give rise to enhanced gauge symmetries through non-perturbative effects. Furthermore, in many cases these singularities are orbifold-like and thus one can study them in more detail using the techniques we sketched in the previous section for the perturbative regime and the method of D-branes on orbifolds we are about to discuss for the non-perturbative regime.

A concrete example of such a singular point in the moduli space of Calabi-Yau's is hinted by the observation made at the end of the previous section. The spectrum for both type IIA and type IIB string theory on the orbifold $T^4/\mathbb{Z}_2$ is the same as the spectrum of these string theories on the unique manifold with $SU(2)$ holonomy, known as the K3 (see [35] for a comprehensive review). Unique means that all such manifolds with $SU(2)$ holonomy are diffeomorphic to each other. The possible metrics on them are parametrized by 58 moduli and for some special values of these moduli the manifold degenerates to the $T^4/\mathbb{Z}_2$ space. Physically, the moduli that control the singular nature of the K3 manifold belong to the twisted NS-NS sector states.

Condensation of these fields to generic values results in a desingularization of the orbifold, which is also known as "blowing-up the singularities". This shows that string theory has an intrinsic mechanism for smoothing out geometries that classically look singular [36]. The reason for all these stringy "miracles" is of course the extended nature of the string.

2.3.2 K3 and the Eguchi-Hanson space

We discuss now in more detail the K3 and related geometries. The precise definition of K3 is a a compact, Kähler manifold S with complex dimension 2 and having $h^{1,0}(S) = 0$ and vanishing canonical class, i.e. $K(S) = 0$. As we discussed in the previous subsection, the orbifold $T^4/\mathbb{Z}_2$ is actually a singular limit in the moduli space of K3 metrics and this can be easily proved by explicitly checking that it satisfies the defining properties of the K3 given above.

The K3 space is the unique Calabi-Yau manifold of complex dimension 2 [4] and its Euler characteristic is $\chi(S) = 24$. One can construct an example of a K3 space as a hypersurface in the 3-complex-dimensional projective space $\mathbb{C}P^3$, given by the vanishing locus of

$$f(x_1, x_2, x_3, x_4) = x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0 \tag{2.3.11}$$

where $x_i, i = 1, 2, 3, 4$ are homogeneous coordinates in $\mathbb{C}P^3$. This is also known as the quartic surface, i.e. hypersurface of degree 4 in $\mathbb{C}P^3$.

Another way to construct a non-singular K3 manifold, known as Kummer's surface, is by replacing the 16 fixed point singularities of the orbifold with non-singular spaces so that the resulting manifold is still a K3. Each of these singularities is of the form $\mathbb{C}^2/\mathbb{Z}_2$. Remarkably, there is the so-called Eguchi-Hanson space [37, 38] which has $\mathbb{C}^2/\mathbb{Z}_2$ as a singular limit.

[4] We assume that a Calabi-Yau manifold has precisely $SU(n)$ holonomy and not a subgroup thereof. Otherwise, the 4-torus, which has trivial holonomy, is the only other 2-complex-dimensional Calabi-Yau manifold besides the K3.

The Eguchi-Hanson space is actually a gravitational instanton, i.e. a 4-dimensional Euclidean space with self-dual curvature tensor so that it satisfies Einstein's equations automatically (and hence it is precisely the gravitational analogue of ordinary Yang-Mills instantons). The metric of this space is

$$ds^2_{EH} = \frac{dr^2}{\left(1-(\frac{a}{r})^4\right)} + r^2\left(1-(\frac{a}{r})^4\right)(d\psi + \cos\theta d\phi)^2 + r^2(d\theta^2 + \sin^2\theta d\phi)^2 \quad (2.3.12)$$

with $\theta \in [0,\pi), \phi \in [0,2\pi), \psi \in [0,4\pi)$ being Euler angles on S^3.

The first observation about this space is that asymptotically, at $r \to \infty$, approaches $\mathbb{R}^4$. However, it is actually only locally Euclidean because the coordinate singularity at $r = a$ requires $\psi \equiv \psi + 2\pi$ and thus a $\mathbb{Z}_2$ identification at the $\mathbb{R}^4$ at infinity. The apparent singularity at $r = a$ is called a bolt and it has the topology of $\mathbb{R}^2 \times S^2$ where the 2-sphere has radius a. In the limit $a \to 0$ the 2-sphere collapses to a point and we have a real (i.e. not coordinate) singularity. Moreover, the rest of the space is flat as $a \to 0$ modulo the $\mathbb{Z}_2$ identification. Thus in this limit the Eguchi-Hanson geometry becomes exactly the orbifold $\mathbb{C}^2/\mathbb{Z}_2$.

One can imagine now that the 16 fixed points in $T^4/\mathbb{Z}_2$ are replaced by smooth Eguchi-Hanson spaces, in other words one excises a neighborhood of the singularity and glues an Eguchi-Hanson space at the open slot. Since both T^4 and the Eguchi-Hanson space are Ricci-flat, it is not surprising that the resulting space is also Ricci-flat. The parameter a of each Eguchi-Hanson space is one of the three moduli that control the size and shape of each of the 16 singularities of the orbifold. For finite a the points of infinite curvature in $T^4/\mathbb{Z}_2$ are replaced by finite-sized 2-spheres (known as exceptional divisors in algebraic geometry). This is an example of the procedure known as blowing-up in algebraic geometry, where singularities are resolved by replacing them with 2-spheres. We have $3 \times 16 = 48$ parameters that control the blow-ups of the 16 fixed points and along with the 10 parameters that specify a T^4, we get the 58 metric parameters of the generic K3.

Note that we obtain 16 2-cycles from each of the 16 fixed points of the orbifolds and 6 more 2-cycles from the independent 2-cycles of the T^4. We thus have 22 2-

cycles in total and it is straightforward to see that reducing the spacetime fields of the type II supergravities on these 2-cycles results exactly in the massless spectrum of the corresponding type II superstring theories on the orbifold $T^4/\mathbb{Z}_2$.

Starting with the NS-NS fields in 10-dimensions, direct dimensional reduction to 6-dimensions gives the corresponding supergravity multiplet. Recall that, in the orbifold limit $T^4/\mathbb{Z}_2$, we have in addition 16 scalars from the untwisted NS-NS sector and 4 scalars from each of the 16 twisted NS-NS sectors. One from the four scalars in the 16 twisted NS-NS sectors results from the reduction of the 10-dimensional antisymmetric tensor field on the 16 2-cycles that appear from the blow-up of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. The other three of the four scalars appear likewise from the reduction of the 10-dimensional metric tensor on the 16 2-cycles. From the reduction of the antisymmetric tensor on the 6 T^4 2-cycles we get 6 of the untwisted NS-NS scalars and finally the rest 10 of those scalars come from the reduction of the metric tensor on T^4, since the generic metric of a 4-torus is parametrized by 10 numbers.

In type IIA the 10-dimensional R-R sector contains a 1-form, a 3-form and a 5-form. The reduction of the 1-form results in a 6-dimensional 1-form while the 3-form produces 6 1-forms from the reduction on the 2-cycles of T^4 and 16 1-forms from the reduction on the 16 blown-up 2-cycles. These last 16 1-forms are the 16 vectors we found in the twisted R-R sector of the orbifold. We also have one extra 1-form from the reduction of the 10D 5-form on the K3 space. In the orbifold limit, the 8 1-forms belong in the untwisted R-R sector.

In type IIB the original 10D R-R sector consists a 0-form, a 2-form and a 4-form with self-dual field strength. The reduction of the 2-form on the 16 blown-up 2-cycles gives 16 scalars which are identified with the scalars from the 16 twisted R-R sectors. Similarly, reducing the 4-form produces 16 self-dual 2-forms which also belong in the twisted R-R sector. Moreover, reduction of the 2-form on the 6 2-cycles of the T^4 along with the direct reduction of the 10D 0-form and the reduction of the 4-form on the K3, yields the 8 scalars of the untwisted R-R sector. Finally, reducing the self-dual 4-form on these 2-cycles - three of which are self-dual and three anti-self-dual - results in 3 self-dual and 3 anti-self-dual 2-forms which along with the direct

reduction of the 2-form produce the 4 self-duals and 4 anti-self-dual 2-forms of the untwisted R-R sector.

We notice that the fields found as twisted states from the string theory point of view, are now just the supergravity fields reduced on the blown-up 2-cycles. We see that string theory on the orbifold makes sense - while, for example, the supergravity theory does not - and through the twisted sector states it actually knows about the hidden 2-cycles. Away from the orbifold point there are no twisted-states but one can use the standard compactification prescription for the supergravity fields, as we just demonstrated. Note that the states in the untwisted sectors in the string theory approach correspond to either the direct reduction of the supergravity fields to 6-dimensions, or to reduction either on the 2-cycles of the T^4, which have finite size, or on the full 4-dimensional compact space.

The Eguchi-Hanson space is actually the first and simplest of a family of Asymptotically Locally Euclidean (ALE) spaces. One of these families, the multi-Eguchi-Hanson spaces, was found by Gibbons and Hawking [39]. Their construction is motivated by the fact that the singular limit of the Eguchi-Hanson space is just the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ and one can start wondering what the resulting geometry will be if $\mathbb{Z}_2$ is replaced by some other group.

Let us parameterize $\mathbb{C}^2$ with $z_1, z_2 \in \mathbb{C}$. The $SO(4) \cong SU(2)_L \times SU(2)_R$ symmetry of $\mathbb{C}^2$ acts on z_1, z_2 as

$$\begin{pmatrix} z_1 & i\bar{z}_2 \\ i\bar{z}_2 & \bar{z}_1 \end{pmatrix} \to g_L \begin{pmatrix} z_1 & i\bar{z}_2 \\ i\bar{z}_2 & \bar{z}_1 \end{pmatrix} g_R \tag{2.3.13}$$

with $g_L \in SU(2)_L, g_R \in SU(2)_R$. We can now define the orbifold $\mathbb{C}^2/\Gamma$ by identifying points in $\mathbb{C}^2$ under the action of a finite subgroup Γ of $SU(2)_L$. In other words Γ acts on $\mathbb{C}^2$ in the fundamental 2-dimensional representation, i.e. for every element $g \in \Gamma$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{2.3.14}$$

The origin $z_1 = z_2 = 0$ is obviously a fixed point of the Γ action and thus we have a curvature singularity there. Since the asymptotic geometry of these orbifolds is locally Euclidean, we can obtain smooth ALE spaces by resolving the singularity at the origin. More precisely, Kronheimer has shown [40, 41] that we obtain smooth ALE spaces with self-dual metrics, i.e. gravitational instantons, for Γ being a finite subgroup of $SU(2)_L$. These spaces are characterized by their boundary at infinity which is S^3/Γ. However, before discussing further their geometry, we need to review the classification of finite subgroups of $SU(2)$ and some related topics.

2.3.3 Finite groups, quivers, and the McKay correspondence

The classification of finite subgroups of $SU(2)$ is closely related to the classification of regular polygons, dihedra and polyhedra, also known as Platonic solids. This of course is due to that locally $SO(3) \cong SU(2)$ and for every finite subgroup Γ of $SO(3)$ one can ask for the corresponding solid object that has Γ as its group of symmetries. The classification of finite subgroups of $SU(2)$ was done by Klein [43] and these subgroups are also known as Kleinians.

The finite subgroups of $SU(2)$ have an A-D-E classification. The A_{k-1} series consists of cyclic groups with order k, i.e. $\mathbb{Z}_k$, with $k \geq 2$. In the defining 2-dimensional representation, the generator is

$$\beta_k := \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{pmatrix}, \qquad \text{with} \quad \omega_n := e^{\frac{2\pi i}{n}}. \tag{2.3.15}$$

The $D_{k+2}, k \geq 2$ series groups are binary extensions of the dihedral groups and are denoted as $\hat{\mathcal{D}}_k$. The hat stands for the binary extension which is due to the fact that we are interested in the double cover of $SO(3)$, that is $SU(2)$. Thus, finite subgroups of $SO(3)$ like the dihedral - which is the symmetry group of the dihedron - receive a $\mathbb{Z}_2$ extension when are lifted to $SU(2)$. The generators of $\hat{\mathcal{D}}_k$ are

$$\beta_{2k} = \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix}, \qquad \gamma := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.3.16}$$

Note that since $\gamma^2 = \beta_{2k}^k$ the order of $\hat{\mathcal{D}}_k$ is $4k$. There is a normal subgroup generated by β_{2k} so that $\hat{\mathcal{D}}_k/\mathbb{Z}_{2k} = \mathbb{Z}_2$.

The three exceptional subgroups E_6, E_7, E_8 correspond to the binary extensions of the tetrahedral $\hat{\mathcal{T}}$, the octahedral $\hat{\mathcal{O}}$ and the icosahedral $\hat{\mathcal{I}}$ group. The binary tetrahedral group $\hat{\mathcal{T}}$ has order 24 and it is generated by the elements of $\hat{\mathcal{D}}_2$ combing them with

$$\delta := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \omega_8^7 & \omega_8^6 \\ \omega_8^5 & \omega_8 \end{pmatrix}. \tag{2.3.17}$$

Note that $\hat{\mathcal{T}}/\hat{\mathcal{D}}_2 = \mathbb{Z}_3$.

The binary octahedral group $\hat{\mathcal{O}}$ has order 48 and it is generated by the elements of $\hat{\mathcal{T}}$ with the addition of

$$\epsilon := \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} = \begin{pmatrix} \omega_8 & 0 \\ 0 & \omega_8^7 \end{pmatrix}. \tag{2.3.18}$$

Finally, the binary icosahedral group $\hat{\mathcal{I}}$ is generated by

$$\zeta := -\begin{pmatrix} \omega_5^3 & 0 \\ 0 & \omega_5^2 \end{pmatrix}, \qquad \eta := \frac{1}{\omega_5^2 - \omega_5^3}\begin{pmatrix} \omega_5 + \omega_5^4 & 1 \\ 1 & -\omega_5 - \omega_5^4 \end{pmatrix} \tag{2.3.19}$$

and it has order 120.

The representation theory of these groups is particularly interesting. For the A_{k-1} groups we have k 1-dimensional irreducible representations (irreps) $r_i, i = 1, \ldots, k$ since the groups are abelian. Recall that the set of representations is actually a ring, i.e. we can consider tensor products and sums of representations. A large amount of information about this ring is contained in the matrix $a_{ij}^{\mathbf{2}}$ defined as

$$\mathbf{2} \otimes r_i = \bigoplus_j a_{ij}^{\mathbf{2}}\, r_j \tag{2.3.20}$$

where $\mathbf{2}$ is the 2-dimensional representation of the finite group induced by the defining 2-dimensional representation of $SU(2)$.

There is a very interesting way of depicting the information encoded in a^2_{ij}. For each irrep draw a node labelled with the dimension of the irrep and connect the nodes i and j with a^2_{ij} lines. The graph obtained this way is known as a quiver and for the case of the A_{k-1} groups it is not difficult to see that it is actually the extended (affine) Dynkin diagram of the A-type root systems with the labels identified with the so-called marks or Kac labels in Lie algebra theory. Recall that the extended Dynking diagram is obtained from the usual one by adding one more node corresponding to the negative of the highest root [5]. The label associated to a given node is half the sum of the labels associated to adjacent nodes. This correspondence extends to the D and E-type finite subgroups of $SU(2)$ and it is known as the McKay correspondence. The general statement is that the matrices a^2_{ij} for each A-D-E subgroup of $SU(2)$ are equal to $2\delta_{ij} - c_{ij}$ where c_{ij} the (extended) Cartan matrix of the corresponding affine Lie algebra [44]. In other words, a^2_{ij} is just the adjacency matrix of the affine Dynkin diagram. The quivers corresponding to finite subgroups of $SU(2)$ are presented in figure 2-1.

The classification of finite subgroups of $SU(3)$ and $SU(4)$ is also known [45, 46] (see also [47] for a review); so far however there hasn't been found an analogue of the McKay correspondence. One can of course construct the quivers associated to a given finite subgroup and as we will see in the next section, these quivers are still very useful as a way of depicting the matter content of D-brane probe gauge theories.

Since in chapter 3 we will classify discrete torsion in $SU(3)$ orbifolds, we briefly review now the corresponding subgroups. Due to $SU(2) \subset SU(3)$, all A-D-E subgroups of $SU(2)$ are inherited to $SU(3)$ of course. The pure $SU(3)$ subgroups include a new abelian group $\mathbb{Z}_n \times \mathbb{Z}_m$ besides the abelian $\mathbb{Z}_k \subset SU(2)$. The generators of

[5]If $\{a_i\}$ is the set of simple roots, the highest root θ is the unique root whose expansion as $\theta = \sum_i m_i a_i$ maximizes the sum $\sum_i m_i$. By repeated subtraction of simple roots from θ one can obtain all possible roots. The numbers $\{m_i\}$ are the so-called marks and are exactly the labels of the quiver nodes.

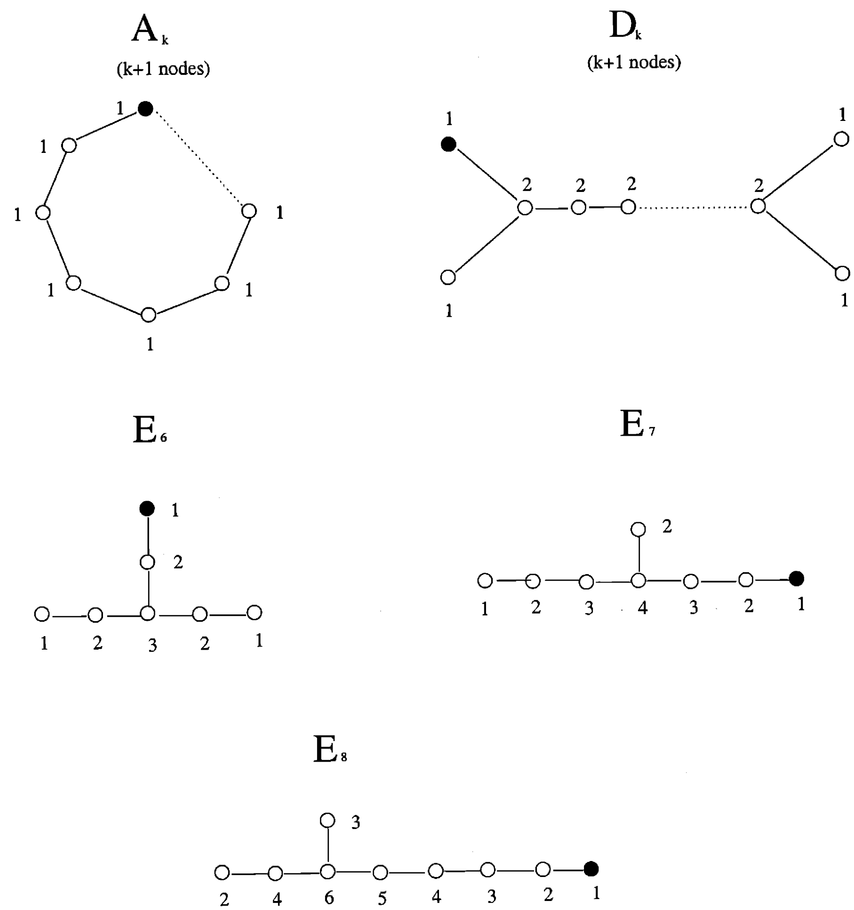

Figure 2-1: The quivers associated to finite subgroups of $SU(2)$. Each node corresponds to an irreducible representation, with the trivial representation being the filled node. The links connect a given node to nodes corresponding to irreps that appear in the decomposition of the tensor product of the irrep associated to the original node with the 2-dimensional representation induced by the defining 2-dim. representation of $SU(2)$. These graphs are the same as the Dynkin diagrams of the affine simply-laced Lie algebras, where each node corresponds to a simple root and the filled node is associated to the negative of the highest root.

$\mathbb{Z}_n \times \mathbb{Z}_m$ are

$$\begin{pmatrix} \omega_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega_n^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_m & 0 \\ 0 & 0 & \omega_m^{-1} \end{pmatrix}. \tag{2.3.21}$$

In addition we have two infinite series: the Δ_{3n^2} with order $3n^2$ for $n \in \mathbb{Z}^+$ and the Δ_{6n^2} with order $6n^2$ for $n \in 2\mathbb{Z}^+$. These can be thought of as the analogues of the series of binary dihedral subgroups of $SU(2)$.

Finally, there is a finite number of "exceptional" subgroups of $SU(3)$ known as $\Sigma_{3\cdot 36}, \Sigma_{3\cdot 60}, \Sigma_{3\cdot 168}, \Sigma_{3\cdot 216}, \Sigma_{3\cdot 360}$ where the subscript is the corresponding order. In addition, if one considers $SU(3)$ modded out by its $\mathbb{Z}_3$ center, there are the following extra subgroups: $\Sigma_{36}, \Sigma_{60} \cong A_5 \cong \mathcal{I}, \Sigma_{72}, \Sigma_{168}, \Sigma_{216}, \Sigma_{360} \cong A_6$, where by A_k we denote the alternating symmetric group of k elements. These are similar to the ordinary tetrahedral, octahedral, and icosahedral groups which are subgroups of $SU(2)/\mathbb{Z}_2 \cong SO(3)$, i.e. $SU(2)$ modded out by its center $\mathbb{Z}_2$, but they are not subgroups of the full $SU(2)$.

After this brief intermezzo on finite groups, we are ready to return to our discussion of generalizations of the Eguchi-Hanson space.

2.3.4 ALE, ALF spaces and $SU(2)$ orbifolds

According to Kronheimer, the 4-dimensional ALE spaces have an A-D-E classification induced by the corresponding classification of the finite subgroups of $SU(2)$ (a brief review of Kronheimer's construction is given in the next subsection). The boundary at infinity is the same as the boundary of the orbifold $\mathbb{C}^2/\Gamma$ with $\Gamma \subset SU(2)$, that is the Lens space S^3/Γ. The associated metrics are known explicitly only for the A-type cases and these were the metrics found by Gibbons and Hawking [39] as generalizations of the Eguchi-Hanson metric.

The A_{k-1} type metric is

$$ds^2_{A_{k-1}} = V d\vec{x}^2 + V^{-1}(dx^4 - \vec{\omega} \cdot d\vec{x})^2 \tag{2.3.22}$$

where $\vec{x}$ coordinates in $\mathbb{R}^3$, x^4 is periodic and

$$V = \sum_{i=1}^{k} \frac{1}{|\vec{x} - \vec{x}_i|}, \quad \vec{\nabla} V = \vec{\nabla} \times \vec{\omega}. \tag{2.3.23}$$

It is an open problem to find the explicit metrics for the D and E-type ALE gravitational instantons.

This class of metrics actually arises as a limit of another type of gravitational instantons known as multi-Taub-NUT spaces [39] (see also [42, 38] for further discussions). The metric of these spaces is of the form (2.3.22) but with

$$V = 1 + \sum_{i=1}^{k} \frac{1}{|\vec{x} - \vec{x}_i|}. \tag{2.3.24}$$

Whereas the asymptotic boundary of the A-type ALE is $S^3/\mathbb{Z}_k$, the asymptotic geometry of the multi-Taub-NUT is not locally Euclidean but locally flat and thus these spaces are known also as Asymptotically Locally Flat (ALF). The boundary at infinity is a non-trivial S^1 bundle, parameterized by the periodic x^4, over the 2-spheres of constant $|\vec{x}|$.

When the k centers $\vec{x}_i, i = 1, \ldots, k$ of the multi-Taub-NUT are close to each other and, in addition, we are only interested in the geometry very close to them, i.e. $|\vec{x} - \vec{x}_i| << 1$ for every i, we can drop the 1 in the expression for the V and we obtain the A-type ALE metric.

The multi-Taub-NUT and the corresponding A_{k-1} ALE spaces have $k-1$ linearly independent 2-cycles which provide a basis for the second homology group of those spaces. These 2-cycles can be found as follows [48]. For every pair of centers $\vec{x}_i$ and $\vec{x}_i$ in $\mathbb{R}^3$, the S^1 fibration due to x^4 produces a cycle $S_{ij}^{(k-1)}$ with spherical topology since the radius of the x^4 fiber approaches zero near the centers. It can be easily seen that the minimal area cycle results from the fibration of x^4 along the straight line joining the two centers. An independent set of such 2-cycles is $S_{i,i+1}^{(k-1)}$ for $i = 1, \ldots, k-1$.

The intersection matrix of these 2-cycles is minus the Cartan matrix of the A_{k-1} Lie algebra. For $k = 2$ the space we obtain is the Eguchi-Hanson space and the

2-cycle $S^{(1)}_{12}$ is just the S^2 we found previously arising near the bolt singularity. In the general case, as the distances $|\vec{x}_i - \vec{x}_j|$ approach zero, the 2-cycles $S^{(k-1)}_{ij}$ shrink and we approach the orbifold space $\mathbb{C}^2/\mathbb{Z}_k$. We expect that string theory on these orbifolds should provide the necessary blow-up modes for the collapsed 2-cycles in terms of twisted states; that is indeed the case [49].

Notice that when we are very close to a given shrinking 2-cycle, say $S^{(k-1)}_{ij}$, in the generic A_{k-1} space, we can drop the 1 and the terms involving $|\vec{x} - \vec{x}_k|, k \neq i, j$ from (2.3.22) and we get an Eguchi-Hanson metric. Hence, the local geometry near every 2-cycle is A_1.

Even though the explicit metrics for the D and E-type ALE spaces are not known, the topology of these spaces has a structure similar to that discussed above for the A-type ALE space. The set of independent 2-cycles is in 1-1 correspondence with the set of non-trivial irreps (set of non-trivial conjugacy classes) of the relevant finite subgroup of $SU(2)$ and their intersection matrix is minus the corresponding Cartan matrix. This is of course just another manifestation of the McKay correspondence we discussed in the previous subsection. It relates the topological data (intersection matrix) of the resolved orbifold $\mathbb{C}^2/\Gamma$ where $\Gamma \subset SU(2)$ with the algebraic data ($a^{\mathcal{R}}_{ij}$ matrix) of Γ encoded in the associated quiver. In particular, the independent 2-cycles can be identified with the nodes of the quiver and the pattern of their intersection is described by the links between the nodes. Note that the extra node assigned to the trivial irrep, and which corresponds to the negative of the highest root in the Lie algebra context, is not associated to a 2-cycle. In the orbifold limit where the 2-cycles collapse to zero size, the ALE spaces with asymptotic boundary S^3/Γ approach the corresponding orbifolds $\mathbb{C}^2/\Gamma$. Because Γ is subject to an A-D-E classification, the orbifolds $\mathbb{C}^2/\Gamma$ are also known as A-D-E singularities.

There is a nice algebraic description of the orbifolds $\mathbb{C}^2/\Gamma$ and their possible resolutions as affine [6] varieties, i.e. solutions of polynomials equations in $\mathbb{C}^3$. The idea is to define new variables in terms of the z_1, z_2 coordinates of $\mathbb{C}^2$, which are

[6]This is in contrast to projective varieties which are the locus of solutions of polynomials in projective spaces and thus are suitable for describing compact complex manifolds.

invariant under the action of the orbifold group on z_1, z_2. The constraints arising between the new variables are precisely the defining equations of the variety that corresponds to the orbifold. We will denote the polynomial corresponding to $\mathbb{C}^2/\Gamma$ as $\mathcal{W}_\Gamma$.

For example, for the A_{k-1} series the action of the orbifold is $z_1, z_2 \to \omega_k z_1, \omega_k^{-1} z_2$ and we can define the invariant combinations $x = z_1 z_2, y = z_1^k, z = z_2^k$. We immediately obtain

$$\mathcal{W}_{A_{k-1}} = xy - z^k = 0. \tag{2.3.25}$$

Similarly, for the D and E-type one can show that

$$\begin{aligned} \mathcal{W}_{D_{k+2}} &= x^2 + y^2 z + z^{k+1} = 0 && (2.3.26) \\ \mathcal{W}_{E_6} &= x^2 + y^3 + z^4 = 0 && (2.3.27) \\ \mathcal{W}_{E_7} &= x^2 + y^3 + yz^3 = 0 && (2.3.28) \\ \mathcal{W}_{E_7} &= x^2 + y^3 + z^5 = 0 && (2.3.29) \end{aligned}$$

where x, y, z are defined appropriately. The singularity is associated to the point $x = y = z = 0$ where $\mathcal{W}_\Gamma = \partial_x \mathcal{W}_\Gamma = \partial_y \mathcal{W}_\Gamma = \partial_z \mathcal{W}_\Gamma = 0$.

The blowing-up of the collapsed 2-cycles has a simple algebraic description in the above formalism [50, 51]. One defines the chiral ring associated to a given singularity $\mathcal{W}_\Gamma(x, y, z) = 0$ as the quotient

$$\mathcal{Q}_\Gamma := \frac{\mathbb{C}[x, y, z]}{\partial \mathcal{W}_\Gamma}, \tag{2.3.30}$$

i.e. as the quotient of the ring of polynomials in the variables x, y, z with complex coefficients, with respect to the polynomials $\partial_x \mathcal{W}_\Gamma(x, y, z), \partial_y \mathcal{W}_\Gamma(x, y, z), \partial_z \mathcal{W}_\Gamma(x, y, z)$. The dimension of this ring equals the number of non-trivial irreps $r(\Gamma)$ of the group Γ and a basis will be denoted by $\mathcal{P}_i(x, y, z), i = 1, \ldots, r(\Gamma)$. For example, $\mathcal{Q}_{A_{k-1}}$ is $(k-1)$-dimensional since we exclude the trivial representation.

The resolved orbifold is described now as the vanishing locus of deformed polynomials $\mathcal{W}_\Gamma(x, y, z; t_1, t_2, \ldots, t_{r(\Gamma)})$ which are given by

$$\mathcal{W}_\Gamma(x, y, z; t_1, t_2, \ldots, t_{r(\Gamma)}) = \mathcal{W}_\Gamma(x, y, z) + \sum_{i=1}^{r(\Gamma)} t_i \mathcal{P}_i(x, y, z) \tag{2.3.31}$$

with $t_i \in \mathbb{C}$. Note that the reason we exclude deformations proportional to the derivatives of $\mathcal{W}(x, y, z)$ is that these are trivial, i.e. they can be transformed away by shifting the variables x, y, z, and thus the singularity remains, except that now is located at a different point. These deformed polynomials correspond to complex structure deformations of the original singular variety since by changing the defining equation we explicitly change the complex structure of the associated affine variety. The coefficients t_i are coordinates on the moduli space of complex structures of the orbifold. Non-zero t_i correspond to blown-up 2-cycles, i.e. exceptional divisors. Note that in general there are two ways to resolve algebraic singularities; one is the blowing-up which corresponds to a change in the Kähler form and the other is the deformation of the complex structure. It is a special feature of the ALE spaces that these two types of resolution match, as we saw above [35].

For the A_{k-1} series the t_i moduli are associated to the $k-1$ independent positions of the centers $\vec{x}_i, i = 2, \ldots, k$ in $\mathbb{R}^3$ where we have fixed $\vec{x}_1 = 0$. These moduli appear as twisted fields in the conformal field theory describing string propagation on the orbifold $\mathbb{C}^2/\Gamma$ [49].

We should mention here that besides the $T^4/\mathbb{Z}_2$ there are other singular limits of K3 spaces. For example $T^4/\mathbb{Z}_k, k = 3, 4, 6$ are also possible (see [31] for an analysis of heterotic strings on such orbifolds). More generally, the singularities in K3 are of A-D-E type, i.e. they can be approximated by orbifolds $\mathbb{C}^2/\Gamma$ with $\Gamma \subset SU(2)$ and the smooth K3 manifold is obtained by blowing-up the collapsed 2-cycles. Note also that since the holonomy of ALE and ALF spaces with A-D-E classification is a subgroup of $SU(2)$, we can consider them as non-compact Calabi-Yau manifolds with 2 complex dimensions. Thus, if we are interested in understanding the effects due to singularities in a compact K3 without the complications of gravity, we can study the

non-compact ALE with the same singularities.

The ALE and ALF spaces also appear in string theory in a variety of situations where the geometry of a brane is dualized; two particularly useful cases are the M-theory lift of a collection of D6-branes which is described purely in geometric terms in 11-dimensions as an ALF space which degenerates to an A-type singularity when the D6-branes are coincident, and the T-dual geometry of NS5-branes. The last case will be discussed in more detail in chapter 4.

2.3.5 Holonomy and HyperKähler geometries

We mentioned previously that a generic oriented n-dimensional Riemannian manifold M has $SO(n)$ holonomy. Moreover, in string theory compactifications the holonomy of the compact part of spacetime is an indicator of the surviving supersymmetry. Hence, it will be useful to know what subgroups of $SO(n)$ can appear as holonomy groups and how this choice is reflected on the geometry of of the corresponding manifolds. This classification was undertaken by Berger [54] and we will briefly review it here following [55].

We will denote the holonomy group of M by $\mathrm{hol}(M)$. The possibilities are:

- $\mathrm{hol}(M) \subseteq U(n/2) \Longleftrightarrow M$ is Kähler;

  $\mathrm{hol}(M) \subseteq SU(n/2) \Longleftrightarrow M$ is Kähler and Ricci-flat (i.e. Calabi-Yau);

- $\mathrm{hol}(M) \subseteq Sp(n/4) \Longleftrightarrow M$ is HyperKähler (and hence Ricci-flat);

- $\mathrm{hol}(M) \subseteq Sp(1) \times Sp(n/4) \Longleftrightarrow M$ is quaternionic Kähler (it is Einstein but neither Ricci-flat nor Kähler);

- $\mathrm{hol}(M) \subseteq G_2 \Longleftrightarrow M$ is 7-dimensional and Ricci-flat;

- $\mathrm{hol}(M) \subseteq Spin(7) \Longleftrightarrow M$ is 8-dimensional and Ricci-flat;

- $\mathrm{hol}(M) = H \Longleftrightarrow M$ is a symmetric space G/H where G, H are Lie groups.

We should mention now that the K3 and the ALE/ALF gravitational instantons are actually HyperKähler since they are 4-dimensional and their holonomy is

$SU(2) \cong Sp(1)$. In general, a 4-dimensional simply-connected Riemannian manifold is HyperKähler when its Riemann curvature 2-form is either self-dual or anti-self-dual. Thus all 4-dimensional HyperKähler spaces are gravitational instantons; in particular, they are Ricci-flat.

A general method of constructing HyperKähler manifolds starting from simpler higher-dimensional HyperKähler metrics is via the so-called HyperKähler Quotients [52]. The construction of ALE instantons by Kronheimer [40, 41] is an interesting application of the HyperKähler quotient technique, especially due to the fact that it is physically realized by D-brane probes on the corresponding orbifold limits of ALE.

We review now some aspects of HyperKähler quotients in order to establish terminology for our later discussion in the context of D-brane probes. Our presentation follows reviews of the same material in a string theory context as appeared in [49, 53]. Recall that an independent definition of a HyperKähler space M is as a Riemannian manifold equipped with three covariant constant complex structures [7] I, J, K which satisfy the quaternionic algebra $I^2 = J^2 = K^2 = -\mathbb{I}; IJ = -K, JK = -I, KI = -J$. Note that given I, J, K we actually have an S^2 of possible complex structures $Q^2 = -\mathbb{I}$ parametrized as $Q = a + bI + cJ + dK$ with $a, b, c, d \in \mathbb{R}$ with $b^2 + c^2 + d^2 = 1$. All HyperKähler spaces have dimension $n = 0 \bmod 4$.

Accordingly, we can define three closed 2-forms as $\omega_I(v_1, v_2) = g(v_1, Iv_2)$, $\omega_J(v_1, v_2) = g(v_1, Jv_2)$, $\omega_K(v_1, v_2) = g(v_1, Kv_2)$, where g the metric tensor. Each one of those is a Kähler form on M.

The method of HyperKähler quotients starts from a $4n$-dimensional HyperKähler space M and produces a $4(n-k)$-dimensional HyperKähler space $\tilde{M}$, where $\tilde{M}$ is a quotient of M modulo a subgroup of its isometry group G. Recall that isometries are generated by independent Killing vectors fields V so that $\mathcal{L}_V g = 0$, where $\mathcal{L}_V$ is the Lie derivative along V.

A Killing vector field V is called triholomorphic if $\mathcal{L}_V \omega_{I,J,K} = 0$ for all three Kähler forms. The term triholomorphic stands for the fact that V is holomorphic with

[7] In general, a complex structure is an automorphism of the tangent bundle that squares to minus the identity.

respect to the three complex structures I, J, K of M. Since $\mathcal{L}_V \omega_{I,J,K} = i_V d\omega_{I,J,K} + d(i_V \omega_{I,J,K}) = 0$ and $\omega_{I,J,K}$ are closed, we have $d(i_V \omega_{I,J,K}) = 0$. Locally integrating the last formula results in three Killing potentials $\mu^V_{I,J,K}$, i.e.

$$d\mu^V_{I,J,K} = i_V \omega_{I,J,K}. \tag{2.3.32}$$

We see that for each Killing vector field we obtain a set of three functions. Generalizing this correspondence, we are led to the definition of the moment map

$$\mu : x \in M \to \mu(x) \in \mathbb{R}^3 \times g^*, \tag{2.3.33}$$

that is, for each point on M we get three elements of g^*, the dual of the Lie algebra of G, and thus for each Killing vector field we can obtain a set of three functions, which are exactly the Killing potentials defined earlier. The name moment map originates from a similar device in symplectic geometry which is associated either to momentum or to angular momentum or, more generally, to a conserved quantity under a group of symmetries of phase space.

Assume now that M is endowed with a compact group G of k freely acting triholomorphic isometies. We can consider the so-called level set of the k moment maps, which is

$$P(\vec{\zeta}) = \{x \in M \mid \vec{\mu}(x) = \vec{\zeta}\} \tag{2.3.34}$$

with $\vec{\zeta} \in Z(g)^*$, which is the the dual to the center of g. Note that we use a vector notation for the three independent components of the moment map and of the parameter ζ.

Note than $P(\vec{\zeta})$ is invariant under the action of G since we chose $\vec{\zeta}$ in the dual of the center of the Lie algebra of G. It can be proved now that the quotient $\tilde{M} := P(\vec{\zeta})/G$ is a HyperKähler space with dimension $\text{dim}\tilde{M} = \text{dim}P(\vec{\zeta}) - \text{dim}G = (\text{dim}M - 3\text{dimG}) - \text{dim}G = 4(n-k)$.

We will now discuss the construction of generic 4-dimensional ALE instantons

using HyperKähler quotients [40, 41]. The starting point is the HyperKähler space $M_\Gamma = (\mathcal{R} \otimes \mathrm{End}(V_\Gamma))_\Gamma$ where $\mathcal{R}$ the defining 2-dimensional representation $\mathcal{R}$ [8] of $\Gamma \subset SU(2)$ and $\mathrm{End}(V_\Gamma)$ is the set of endomorphisms of the $|\Gamma|$-dimensional vector space V_Γ) on which Γ acts by its regular representation (see the footnote in the next page for some mathematical definitions). The set M_Γ consists of Γ invariant pairs of endomorphisms of $\mathrm{End}(V_\Gamma)$. Note that as is common, we will use interchangeably the same symbol, for example $\mathcal{R}$, to denote both an abstract representation and the vector space on which this representation acts.

Representing the elements of M_Γ as doublets

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{2.3.35}$$

with $z_1, z_2 \in \mathrm{End}(V_\Gamma)$ being $|\Gamma| \times |\Gamma|$ complex matrices, the Γ-invariance condition takes the form:

$$\mathcal{R}(g) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \gamma_{reg}(g) z_1 \gamma_{reg}^{-1}(g) \\ \gamma_{reg}(g) z_2 \gamma_{reg}^{-1}(g) \end{pmatrix}, \quad \forall g \in \Gamma. \tag{2.3.36}$$

We use γ_{reg} to denote the matrices that furnish the regular representation of Γ.

It will be convenient to use also a quaternionic notation of points on M_Γ as

$$z = \begin{pmatrix} z_1 & -z_2^\dagger \\ z_2 & z_1^\dagger \end{pmatrix} \tag{2.3.37}$$

where now the action of $\mathcal{R}$ is taken on the left.

It is well-known that the regular representation can be decomposed in terms of the irreps r_i of Γ as

$$V_\Gamma = \bigoplus_i V_i \otimes \mathbb{C}^{\dim r_i} \tag{2.3.38}$$

where V_i the vector space acted by r_i. The condition of Γ-invariance can be written

[8] This was called **2** in previous subsections.

as

$$
\begin{aligned}
&M_\Gamma = (\mathcal{R} \otimes \mathrm{End}(V_\Gamma))_\Gamma = \\
&(\mathcal{R} \otimes \mathrm{Hom}(\bigoplus_i V_i \otimes \mathbb{C}^{\dim r_i}, \bigoplus_j V_j \otimes \mathbb{C}^{\dim r_j}))_\Gamma = \\
&(\mathcal{R} \otimes (\bigoplus_{ij} \mathrm{Hom}(V_i, V_j)_\Gamma \otimes \mathrm{Hom}(\mathbb{C}^{\dim r_i}, \mathbb{C}^{\dim r_j}))) = \\
&\bigoplus_{ij} a_{ij}^2 \mathrm{Hom}(\mathbb{C}^{\dim r_i}, \mathbb{C}^{\dim r_j}),
\end{aligned}
\tag{2.3.39}
$$

where we used the Schur lemma in the form $\mathrm{Hom}(r_i, r_j)_\Gamma = \delta_{ij}$ [9] and the decomposition (2.3.20). It can be easily seen that $\dim M_\Gamma = 4|\Gamma|$.

The three HyperKähler forms can be combined into a real (1,1)-form and a com-

[9] Let us elaborate on this mathematical abstractions a bit. First a few definitions: a homomorphism from a group G into a group H is a map $T : G \to H$ such that for every pair $x, y \in G$ we have $T(x *_G y) = T(x) *_H T(y)$. Here we use $*_G$ and $*_H$ to denote the group multiplication in G and H respectively. The set of all homomorphisms from G into H will be called $\mathrm{Hom}(G, H)$. If G and H are vector spaces, i.e. abelian groups with respect to addition, over the same field $\mathbb{F}$, a homomorphism is a linear map. More concretely, if $\dim G = n$ and $\dim H = m$, $\mathrm{Hom}(G, H)$ is the set of all $m \times n$ matrices with elements in $\mathbb{F}$. One can also think of $\mathrm{Hom}(G, H)$ as an element of $G^* \times H$ where G^* the dual of G, i.e. the set of all linear maps from G to $\mathbb{F}$. An endomorphism on G is a homomorphism from G into G and the set of all endomorphisms of G will be called $\mathrm{End}(G) \equiv \mathrm{Hom}(G, G)$. An isomorphism is an 1-1 homomorphism. An automorphism is an isomorphism which is also an endomorphism, i.e. 1-1 homomorphism from G into G. The set of all automorphisms $\mathrm{Aut}(V)$ of a vector space V over a field $\mathbb{F}$ with $\dim V = n$ is denoted usually as $GL(V)$ and it is isomorphic (as a vector space) to the set $GL(n, \mathbb{F})$ of $n \times n$ invertible matrices with elements in $\mathbb{F}$. Note that the set $\mathrm{End}(V)$ is bigger than $\mathrm{Aut}(V)$ since the first includes non-invertible matrices. In deriving (2.3.39) we also used the fact that for two vector spaces V, W the following identity is true: $V \otimes \mathrm{End}(W) = \mathrm{Hom}(W, V \otimes W)$, as it can be easily checked.

Suppose now we are given two vector spaces V_1 and V_2 and an element of $\mathrm{Hom}(V_1, V_2)$, that is a linear map $E : V_1 \to V_2$ that takes an element $v_1 \in V_1$ to an element $Ev_1 \in V_2$. Assume moreover that V_1 and V_2 carry two representations, γ_1 and γ_2 respectively, of the group Γ. That means that for every element $g \in \Gamma$, $\gamma_i(g), i = 1, 2$ is a linear map from $V_i, i = 1, 2$ to itself, i.e. $\gamma_i(g) : V_i \to V_i, i = 1, 2$. Now, the action of Γ on the vector spaces $V_i, i = 1, 2$ induces an action on $\mathrm{Hom}(V_1, V_2)$ in the following way: suppose an element E of $\mathrm{Hom}(V_1, V_2)$ takes $v_1 \in V_1$ to $v_2 = Ev_1 \in V_2$. We define the action of Γ on E as $g \cdot E, g \in \Gamma$ so that $(g \cdot E)\gamma_1(g)v_1 = \gamma_2(g)v_2$ with $v_2 = Ev_1$. We conclude that $g \cdot E = \gamma_2(g) E \gamma_1^{-1}(g)$. We say that E is Γ-invariant (or Γ-equivariant) if $g \cdot E = E \ \forall g \in \Gamma$. In that case, the standard version of Schur's lemma from group representation theory tell us that if γ_1 and γ_2 are inequivalent representations of Γ, the only Γ-invariant element of $\mathrm{Hom}(V_1, V_2)$ is $E = 0$. If however γ_1 and γ_2 are equivalent representations, and thus they have the same dimension, say n, the set of Γ-invariant E is $E = \lambda \mathbb{I}_n$ where $\lambda \in \mathbb{C}$. Thus we can synopsize this by $\mathrm{Hom}(V_i, V_j)_\Gamma = \delta_{ij}$ assuming that V_i and V_j correspond to non-equivalent irreps when $i \neq j$.

plex (2,0)-form which can be written as

$$\begin{aligned} \omega_{\mathbb{R}} &= \mathrm{Tr}(dz_1 \wedge dz_1^\dagger) + \mathrm{Tr}(dz_2 \wedge dz_2^\dagger) \\ \omega_{\mathbb{C}} &= \mathrm{Tr}(dz_1 \wedge dz_2). \end{aligned} \tag{2.3.40}$$

Finally, we need to identify the isometry group of M_Γ. It is straightforward to check that a $U(|\Gamma|)$ transformation U, acting on M_Γ as

$$z_1 \to U z_1 U^\dagger, \qquad z_2 \to U z_2 U^\dagger \tag{2.3.41}$$

preserves both the three HyperKähler forms and the flat metric on M_Γ and thus corresponds to a triholomorphic isometry. However, we have to make sure that the action of $U(|\Gamma|)$ is consistent with the definition of M_Γ, i.e. that acting with U on Γ-invariant z_1 and z_2 (cf. eq. (2.3.36)) leaves them Γ-invariant. Since U can be thought of as an element of $\mathrm{Hom}(V_\Gamma, V_\Gamma)$, the requirement of Γ-invariance tell us that we should only consider elements of $\mathrm{Hom}(V_\Gamma, V_\Gamma)_\Gamma$. In more concrete terms, we should keep only the Us that satisfy $\gamma_{reg}(g)U = U\gamma_{reg}(g)$ for all $g \in \Gamma$.

An analysis of $\mathrm{Hom}(V_\Gamma, V_\Gamma)_\Gamma$ along the lines of (2.3.39) shows that the surviving group of freely acting triholomorphic isometries of M_Γ is $G = \otimes_i U(\mathrm{dim} r_i)$ with the $U(1)$ factor corresponding to the trivial irrep removed since it acts trivially on M_Γ.

The moment maps can be found from the explicit expressions for the HyperKähler forms given above. We obtain

$$\begin{aligned} \mu_{\mathbb{R}}(z) &= [z_1, z_1^\dagger] + [z_2, z_2^\dagger] \\ \mu_{\mathbb{C}}(z) &= [z_1, z_2]. \end{aligned} \tag{2.3.42}$$

The dual of the center of the Lie algebra of G is spanned by elements of the form $\oplus_i \zeta_i \mathbb{I}_{\mathrm{dim} r_i}$ where $\zeta_i \in \mathbb{R}$ satisfying $\sum_i \zeta_i = 0$. Introducing three such parameters $\vec{\zeta} = (\zeta^1, \zeta^2, \zeta^3)$ and using the notation $\zeta_{\mathbb{R}} = \zeta^3, \zeta_{\mathbb{C}} = \zeta^1 + i\zeta^2$ we can define the level

set $P_\Gamma(\vec{\zeta})$ of $\vec{\zeta}$ as the locus of points on M_Γ for which

$$\begin{aligned} [z_1, z_1^\dagger] + [z_2, z_2^\dagger] &= \zeta_{\mathbb{R}} \\ [z_1, z_2] &= \zeta_{\mathbb{C}}. \end{aligned} \tag{2.3.43}$$

The last step is to quotient the level set $P_\Gamma(\vec{\zeta})$ with the group G. The resulting space has real dimension 4 and it is HyperKähler by construction. As was shown by Kronheimer, varying $\vec{\zeta}$ results in all possible 4-dimensional ALE spaces whose boundary at infinity is S^3/Γ. Furthermore, the choice of Γ among all finite subgroups of $SU(2)$ exhausts all possibilities for locally Euclidean geometry at infinity. Hence, we conclude that the 4-dimensional ALE instantons fall into an A-D-E classification.

A final note is in order about the role of the parameters $\vec{\zeta}$. Recall that we have three of them for each irreducible representation of Γ (satisfying one constraint, so that only those that correspond to non-trivial irreps should be considered independent) and, according to the McKay correspondence, each irrep is associated to a 2-cycle in the ALE geometry. Thus we have three parameters for every independent 2-cycle and it should be evident that they correspond to the three blow-up moduli for those cycles; in particular, for the A-type multi-centered ALE metric they are just the positions of the centers in $\mathbb{R}^3$. More generally, setting $\vec{\zeta} = 0$ corresponds to the singular orbifold limit $P_\Gamma(0) = \mathbb{C}^2/\Gamma$. Moreover, we know that string theory provides an explicit realization of these parameters as twisted field condensates. Thus we expect that the way they appear in the level set defining conditions (2.3.43) should be somehow implied by string theory; this is indeed the case, however it is not in the context of perturbative string theory but rather in the non-perturbative framework of D-brane probes of the geometry. Thus, we will now change gears and delve into the subject of D-branes on orbifolds, where the mathematical material we reviewed in this section will be put to full use.

2.4 D-branes on orbifolds

As we have seen in the previous sections, the study of singular geometries such as orbifolds in string theory is particularly interesting for two reasons: first, one obtains a spectrum that resembles in some cases the spectrum of supersymmetric Grand Unified Theories (GUTs). Thus, this type of geometries is a promising starting point for string phenomenology models. The second reason is more abstract; it is related to the fact that string propagation on singular manifolds makes perfect sense since the stringy degrees of freedom essentially "resolve" the singularities. This is an amazing property of string theory and shows clearly that strings see spacetime differently than point-like objects. It is thus very natural to wonder about the way D-branes perceive spacetime and, in particular, singular geometries. The final reason is that orbifolds provide a tractable example of compactification on non-trivial manifolds. As we have seen, Calabi-Yau manifolds can develop orbifold singularities at certain points in their moduli space. At these points and near them, the study of D-branes on the Calabi-Yau manifold - a difficult and so far unsolved problem - is immensely simpified by using the machinery of D-branes on orbifolds.

2.4.1 The generic D-brane probe gauge theory

The study of D-branes on orbifolds was initiated by Douglas and Moore in [56]. Subsequently several papers generalized their construction and elucidated the projection algorithm that gives the gauge theory on the D-brane probes. A partial list of references is [57, 58, 59, 60, 61].

The starting point is to define precisely what we mean by a D-brane on a singular manifold. Since the case of generic singularities seems to be still open, we will restrict to the simplest and most well-studied example which is that of orbifold geometries. As we will see in the ensuing, this class of singular geometries is already very rich and provides additional insight into the way string theory resolves spacetime singularities.

We start with N parallel and coincident D3-branes which lie at the origin of the 6-dimensional transverse space $\mathbb{C}^3$. The theory on the D-branes is the $\mathcal{N}=4$ SYM with

$U(N)$ gauge group and $SU(4) \cong SO(6)$ R-symmetry. The latter reflects the $SO(6)$ symmetry of the transverse space which leaves our D-brane configuration invariant. There is a gauge field A^μ in the adjoint representation of $U(N)$ and uncharged under $SU(4)$, four adjoint Weyl fermions Ψ^a in the the fundamental $\mathbf{4}$ of $SU(4)$ and six adjoint scalars X^m in the antisymmetric $\mathbf{6}$ of $SU(4)$. Altogether we have the usual vector multiplet of the $\mathcal{N} = 4$ theory in 4D in the adjoint representation of the gauge group.

Recall that the geometry of spacetime is a derived concept from the point of view of D-branes and it arises as their moduli space, i.e. the space of all degenerate supersymmetric vacua of their world-volume gauge theories. This is something we explicitly derived in the first section for the case of flat transverse geometry. It is natural to ask what is the geometry as perceived by the D-branes for more general transverse spaces. The simplest example after flat space are of course the quotients of flat space with discrete symmetries thereof, i.e. orbifolds.

Suppose we orbifold the transverse space into $\mathbb{C}^3/\Gamma$ where Γ is a finite subgroup of $SO(6)$ which is the group of symmetries of $\mathbb{C}^3 \cong \mathbb{R}^6$. For an element g of the orbifold group we first need to specify a 6-dimensional representation $\mathcal{R}(g)$ that describes the way the orbifold group acts on the transverse $\mathbb{R}^6$. It should be evident that the orbifold projection actds also on the D-brane spectrum since the world-volume fields fall in representations of the R-symmetry group which is identified with the symmetry of the transverse space.

The most natural way to understand the orbifold action on the world-volume degrees of freedom is by appaealing to the consistency of the original D-brane configuration under the orbifold projection. In other words, if we have a D-brane in the quotient space we should have all of its mirror images corresponding to all elements of the orbifold group in the original flat transverse space. This means that the orbifold acts on the CP factors associated to the configuration of N D-branes.

Let i denote a CP label. The action $\gamma(g)$ on the CP indices should be correlated with the geometric action on the D-branes, i.e. if $X^m[i]$ are the coordinates of the i-th D-brane we should have $\mathcal{R}^n_m(g)X^m[i] = X^n[\gamma(g) \cdot i]$.

We proceed now to find the action on the world-volume fields. The NS ground state of an open string stretching from the i-th to the j-th D-brane can be symbolized as $|k^\mu; ij\rangle_{NS}$ with k^μ the momentum along the world-volume (its momentum in the transverse directions is of course zero since the string has Dirichlet boundary conditions along them). A generic state of the scalar fields X^m_{ij} on the D-branes would be $X^m_{ij}\psi^m_{-1/2}|k^\mu; ij\rangle_{NS}$ and the action of the orbifold group element g would be

$$g : X^m_{ij}\psi^m_{-1/2}|k^\mu; ij\rangle_{NS} \to X^m_{ij}\mathcal{R}^n_m(g)\psi^n_{-1/2}\,\gamma(g)_{ik}|k^\mu; kl\rangle_{NS}\gamma(g)^{-1}_{lj}. \tag{2.4.44}$$

The form of the action on the CP indices is required by the invariance of traces of products of wavefunctions [62]. This is of course necessary since the orbifold group has to be a symmetry of the world-sheet conformal field theory. The representation γ has to be unitary; this is actually not a very restrictive requirement since every representation of a finite group is equivalent to a unitary one.

We find that the action on the world-volume scalars is [56]

$$g : X^m_{ij} \to \mathcal{R}^m_n(g)\gamma(g)_{ik}X^n_{kl}\gamma(g)^{-1}_{lj} \tag{2.4.45}$$

and similarly, for the gauge fields we obtain

$$g : A^\mu_{ij} \to \gamma(g)_{ik}A^\mu_{kl}\gamma(g)^{-1}_{lj}. \tag{2.4.46}$$

Finally, the action on the fermions is similar to the action on the scalars but with $\mathcal{R}^m_n(g)$ replaced by the appropriate 4-dimensional representation of $SU(4)$.

The theory of the D-branes on the orbifold is the original $\mathcal{N} = 4, U(N)$ gauge theory truncated to fields that are invariant under the projections, i.e. satisfy

$$\begin{aligned} X^m_{ij} &= \mathcal{R}^m_n(g)\gamma(g)_{ik}X^n_{kl}\gamma(g)^{-1}_{lj} \\ A^\mu_{ij} &= \gamma(g)_{ik}A^\mu_{kl}\gamma(g)^{-1}_{lj} \end{aligned} \tag{2.4.47}$$

and similarly for the fermions, for every element g of the orbifold group. Furthermore,

the classical world-volume action can be found from the $\mathcal{N} = 4$ action with the conditions (2.4.47) imposed on the original fields.

One can proceed now and solve (2.4.47) directly, as it is done for example in [61]. Instead, we will follow the more abstract treatment of [59]. The approach of the last reference is to consider the projection of the original D-brane gauge theory in a purely field theoretic context, i.e. as a projection of the $\mathcal{N} = 4$, 4D SYM under the action of a subgroup Γ of the $SU(4)$ R-symmetry. It is required that Γ acts on the gauge group; this is of course just the action on the CP factors in the D-brane picture. This approach is computationally convenient because it requires only knowledge of the representation ring of Γ and the associated Clebsch-Gordan coefficients but not of the actual form of the representation γ in terms of matrices.

Any finite dimensional representation γ of Γ can be written as a tensor sum of irreps r_i of Γ. Recall that the number of irreps of a finite group is equal to the number of conjugacy classes. Decomposing γ in irreps gives

$$\gamma = \bigoplus_i \mathbb{C}^{n_i} \otimes r_i \tag{2.4.48}$$

where n_i the number of times r_i appears in γ. Since γ is required to act on the gauge group as in (2.4.46), we need to have $\sum_i n_i \mathrm{dim} r_i = N$.

The gauge field A^{μ}_{ij} can be thought of as a homomorphism from $\mathbb{C}^n$ to itself, i.e. $A^{\mu}_{ij} \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. In other words it is an endomorphism of $\mathbb{C}^n$. Note that it is not an automorphism since the only condition on A^{μ}_{ij} is that it is hermitian. It will also be useful to denote this as $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ where $(\mathbb{C}^n)^*$ is the vector space dual to $\mathbb{C}^n$ (that is the space of all linear maps from $\mathbb{C}^n$ to $\mathbb{C}$). This algebraic machinery will immensely facilitate the solution of the projection equations. Indeed, from (2.4.47) we see that the surviving gauge field is just the Γ-invariant part of A^{μ}_{ij}, which we will denote by $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)_{\Gamma}$. In other words we are instructed to keep only the trivial (singlet) representation of Γ when we decompose the gauge field in irreps of Γ. More

explicitly:

$$\begin{aligned}&\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)_\Gamma = \bigoplus_{i,j}\mathrm{Hom}(\mathbb{C}^{n_i}\otimes r_i,\mathbb{C}^{n_j}\otimes r_j)_\Gamma = \\ &\bigoplus_{i,j}(\mathbb{C}^{n_i}\otimes(\mathbb{C}^{n_j})^*\otimes r_i\otimes r_j^*)_\Gamma = \bigoplus_{i,j}\mathbb{C}^{n_i}\otimes(\mathbb{C}^{n_j})^*\end{aligned} \tag{2.4.49}$$

where we have used the Schur lemma, i.e. $\mathrm{Hom}(r_i, r_j)_\Gamma = \delta_{ij}$.

We see that the gauge field (and in general all fields that are singlets under the orbifold group) break into adjoint representations $\mathrm{Hom}(\mathbb{C}^{n_i},\mathbb{C}^{n_i})$. Hence, the unbroken gauge group is

$$\mathrm{G}_{proj} = \prod_i U(n_i) \tag{2.4.50}$$

and usually one ignores the $U(1)$ factors since they result in IR free theories and writes

$$\mathrm{G}_{proj} = \prod_i SU(n_i). \tag{2.4.51}$$

Note that in terms of D-branes, we started with a $U(N)$ gauge theory with $N = \sum_i n_i \mathrm{dim} r_i$ in the covering space and the physical D-branes that live in the orbifold have a $\prod_i SU(n_i)$ gauge symmetry, i.e. the D-branes have been arranged in groups of n_i each. We will later discuss a nice geometrical interpretation of these D-branes.

The projection in the matter sector works similarly. Recall that we have six scalars in the $\mathbf{6}$ of $SU(4)$ and four Weyl fermions in the $\mathbf{4}$. We will denote either one of these representations as $\mathcal{R}$; in (2.4.47) for example $\mathcal{R}$ stands for the $\mathbf{6}$. Now, the fields transforming under the $\mathcal{R}$ of $SU(4)$ and under the γ of the orbifold group Γ - where γ as specified in the above - are elements of $\mathcal{R}\otimes\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)$. Abusing the notation as usual, we use $\mathcal{R}$ to denote either the representation or the vector space on which this representation acts.

The Γ-invariant fields are

$$
\begin{aligned}
&(\mathcal{R} \otimes \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)_\Gamma = \bigoplus_{i,j} (\mathcal{R} \otimes \mathbb{C}^{n_i} \otimes (\mathbb{C}^{n_j})^* \otimes r_i \otimes r_j^*)_\Gamma = \\
&\bigoplus_{i,j,k} (a_{ik}^{\mathcal{R}}\, r_k \otimes \mathbb{C}^{n_i} \otimes (\mathbb{C}^{n_j})^* \otimes r_j^*)_\Gamma = \bigoplus_{i,j} a_{ij}^{\mathcal{R}}\, \mathbb{C}^{n_i} \otimes (\mathbb{C}^{n_j})^*
\end{aligned} \tag{2.4.52}
$$

were we have introduced the coefficients $a_{ij}^{\mathcal{R}}$ defines from

$$
\mathcal{R} \otimes r_i = \bigoplus_j a_{ij}^{\mathcal{R}}\, r_j. \tag{2.4.53}
$$

We conclude that the spectrum consists of $a_{ij}^{\mathbf{4}}$ Weyl fermions and $a_{ij}^{\mathbf{6}}$ real scalars in the $(n_i, \bar{n}_j)$ representation of the gauge group $\prod_i SU(n_i)$ for each pair i, j. The bar denotes the conjugate of the fundamental representation of $SU(n_j)$. Notice that there is a striking similarity of the above formalism with the HyperKähler quotient construction of ALE instantons; this is not accidental. In fact, when $\Gamma \subset SU(2)$ they are exactly the same; the D-brane probe gauge theory puts Kronheimer's abstract mathematical construction in a physical context. We will elaborate more on this correspondence in the next subsection.

One can now plug the projected fields in the action of the original $\mathcal{N} = 4$ theory and find the superpotential of the projected theory. Since we won't discuss superpotentials in this thesis we refer the reader to [59, 61] for details. We would like however to mention that since the theories we obtain after the orbifolding have less supersymmetry that the original theory, the superpotential can contain in principle any term consistent with supersymmetry; for example we can have Fayet-Iliopoulos (FI) terms which of course are absent from the original $\mathcal{N} = 4$ theory.

We elaborate now further on the case where the orbifold is chosen to act according to the regular representation γ_{reg} on the CP factors. Each irrep r_i appear $\mathrm{dim} r_i$ times in the decomposition of γ_{reg} and it is well-known that $\mathrm{dim}\gamma_{reg} = \sum_i \mathrm{dim}^2 r_i = |\Gamma|$. Thus the number of D-branes N on the covering space should satisfy $N = 0 \bmod \Gamma$ and the number of physical D-branes on the orbifold is $n := N/\Gamma$.

According to the previous analysis, the gauge group we obtain is

$$\mathrm{G}_{proj} = \prod_i SU(n\mathrm{dim}r_i), \tag{2.4.54}$$

and it is straightforward to show that the gauge coupling for each factor is $\tau_i = \frac{\mathrm{dim}r_i}{|\Gamma|}\tau$, with τ the original $\mathcal{N} =$ complex coupling.

There is a very elegant way to summarize the matter content and interactions in terms of a quiver diagram by assigning a node to each factor $SU(n\mathrm{dim}r_i)$ of the gauge group (2.4.54) and bosonic/fermionic arrows between nodes according to way the 6-dimensional/4-dimensional representations of Γ (inherited from $SU(4)$) act on each irrep. The number of bosonic/fermionic arrows between the i-th and j-th node is $a_{ij}^{\mathbf{6}}/a_{ij}^{\mathbf{4}}$ and physically each arrow corresponds to a bifundamental $(n\mathrm{dim}r_i, \overline{n\mathrm{dim}r_j})$ scalar/Weyl fermion under the gauge groups $SU(n\mathrm{dim}r_i)$ and $SU(n\mathrm{dim}r_j)$. In particular, when $i = j$ we have a scalar/Weyl-fermion in the adjoint of the gauge group $SU(n\mathrm{dim}r_i)$ associated to the i-th node.

The quiver also encodes the interactions inherited from the superpotential $\mathrm{Tr}\epsilon_{abc}\Phi^a\Phi^b\Phi^c$ of the original $\mathcal{N} = 4$ theory. Here we use Φ to denote a chiral multiplet of $\mathcal{N} = 1$ supersymmetry and we have three of them $\Phi^a, a = 1, 2, 3$, each having a complex scalar that corresponds to two of the six scalars of the $\mathcal{N} = 4$ vector multiplet.

For each triangle in the quiver consisting of two fermionic and one bosonic arrows we obtain a Yukawa interaction, while each square made by bosonic arrows corresponds to a quartic scalar interaction. The coefficients of each interaction can be determined by writing the original $\mathcal{N} = 4$ Lagrangian in terms of the fields that satisfy the orbifold projection, i.e. equations (2.4.47). The term quiver gauge theory is used in general to mean a gauge theory whose matter content and interactions are encoded in a quiver in the way we described above.

2.4.2 Aspects of D3-branes on ALE spaces

We discuss now in more detail the theories on D-branes probing orbifolds of the form $\mathbb{C}^2/\Gamma$ where $\Gamma \subset SU(2)$. As we have shown, these orbifolds break 1/2 of the bulk

supersymmetry and the D-branes break a further 1/2, so that for type II theories we are left with $\mathcal{N} = 2$, that is 8 supercharges. Our discussion here is influenced from [57, 60].

The spectrum

The analysis of the previous subsection applies to every finite subgroup $\Gamma \subset SU(4)$. When Γ lies in $SU(2)$ however, there are considerable simplifications and, in particular, the quivers encoding the matter content can be constructed in terms of the quivers we associated to finite subgroups of $SU(2)$ in the previous section. This is due to the fact that for $\Gamma \subset SU(2) \subset SU(4)$, the 6-dimensional and 4-dimensional representations of $SU(4)$ which act on the 6 real scalars and the 4 Weyl fermions respectively, are decomposed as $\mathbf{6} \rightarrow \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{2}$ and $\mathbf{4} \rightarrow \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{2}$, where $\mathbf{1}$ is the trivial irrep of Γ and $\mathbf{2}$ is the irrep inherited from the 2-dimensional defining representation of $SU(2)$. From (2.4.53) we conclude that $a^{\mathbf{6}}_{ij} = 2\delta_{ij} + 2a^{\mathbf{2}}_{ij}$ and $a^{\mathbf{4}}_{ij} = 2\delta_{ij} + a^{\mathbf{2}}_{ij}$, where $a^{\mathbf{2}}_{ij}$ was defined in the previous section. Recall now that the $SU(2)$ quivers where a graphic way of representing the data encoded in $a^{\mathbf{2}}_{ij}$; we had one node for each irrep, as we also have for the quivers we defined above for the D-brane theory, and the i-th and j-th node being connected with $a^{\mathbf{2}}_{ij}$ links.

When $\Gamma \subset SU(2)$ the D-brane probe gauge theory has $\mathcal{N} = 2$ supersymmetry and thus we can organize its spectrum in terms of the corresponding multiplets. In addition there is an $SU(2)$ R-symmetry, which survives the projection since $SU(4) \supset SO(4) \cong SU(2) \times SU(2)$ and the orbifold group $\Gamma \subset SU(2)$ is taken along one of the $SU(2)$ factors and hence the other is left invariant. The vector multiplet contains the vector field, a complex scalar and two Weyl fermions. The first two are R-symmetry singlets while the two Weyl fermions form an R-symmetry doublet. The other multiplet of interest is the hypermultiplet; this consists of two complex scalars which are in the doublet of the $SU(2)$ R-symmetry and two Weyl fermions which are R-symmetry singlets.

Recalling now that $a^{\mathbf{6}}_{ij}/a^{\mathbf{4}}_{ij}$ count real scalars/Weyl fermions between nodes, we see from their explicit form in terms of $a^{\mathbf{2}}_{ij}$ that the spectrum consists of *(i)* an $\mathcal{N} = 1$

chiral multiplet in the adjoint of each gauge group, which comes from the δ_{ij} part $a^{\mathbf{6}}_{ij}/a^{\mathbf{4}}_{ij}$ and when combined with the $\mathcal{N}=1$ vector multiplet results in a $\mathcal{N}=2$ vector multiplet and *(ii)* $a^{\mathbf{2}}_{ij}$ chiral multiplets in the bifundamental $(n\dim r_i, \overline{n\dim r_j})$ and $a^{\mathbf{2}}_{ji}=a^{\mathbf{2}}_{ij}$ chiral multiplets in the conjugate representation $(n\dim r_j, \overline{n\dim r_i})$, so that their combination produces $a^{\mathbf{2}}_{ji}$ hypermultiplets in the $(n\dim r_i, \overline{n\dim r_j})$. We have used here the fact that there are $a^{\mathbf{2}}_{ij}$ Weyl fermions between the i-th and j-th node and $2a^{\mathbf{2}}_{ij}$ real scalars, which can of course be organized in terms of a chiral multiplet and similarly for the j-th and i-th node. Since $a^{\mathbf{2}}_{ji}=a^{\mathbf{2}}_{ij}$, we eventually conclude that the links between nodes in the $SU(2)$ quivers counts hypermultiplets. Note that we don't need to use arrows since a link corresponds to two chiral multiplets, one in the $(n\dim r_i, \overline{n\dim r_j})$ representation and thus pointing from i to j and the other chiral multiplet in the conjugate representations and thus pointing the opposite way.

To summarize, the quivers associated to finite subgroups Γ of $SU(2)$, which are shown in figure 2-1, encode the spectrum of D-branes probing orbifolds of the form $\mathbb{C}^2/\Gamma$. Nodes correspond to vector multiplets and links to hypermultiplets of $\mathcal{N}=2$ supersymmetry. Note that the symmetry of the quiver coefficients $a^{\mathbf{2}}_{ji}=a^{\mathbf{2}}_{ij}$ means that we obtain a non-chiral spectrum, as we should since $\mathcal{N}=2$ supersymmetry is not chiral. For $\Gamma\subset SU(3)$ the quiver coefficients are not symmetric and in general the corresponding spectrum of the D-brane probe is chiral.

The moduli space

We have mentioned quite a few times that strings have the property to feel a smooth geometry, even when they are put on singular spaces like orbifolds. In the perturbative string theory description, i.e. when fundamental strings are used as probes of the geometry, this phenomenon is due to the presence of twisted sectors; one can ask now if a similar mechanism exists for D-brane probes. The $\mathcal{N}=2$ theories studied above are a relatively simple framework where this question can be investigated in detail. The reason is that the resolved geometry of the orbifold $\mathbb{C}^2/\Gamma, \Gamma\subset SU(2)$, i.e. the ALE space, is well-known and thus we know what to look for.

The main idea is that D-branes perceive spacetime through the moduli space of

vacua of the low-energy field theory that lives on them. When enough supersymmetry is present, one can expect that an analysis of the classical moduli space is very close to the true geometry as observed by the D-branes. Since turning on twisted sector vevs correspond to resolving the singularities of the geometry, we expect that the couplings of the twisted fields with the D0-brane world-volume degrees of freedom will have a similar effect on the moduli space. In particular, for the theories we are going to discuss the construction of the moduli space is entirely the same as Kronheimer's HyperKähler quotient construction of ALE instantons. Thus, the machinery of D-branes on orbifolds is a framework where this abstract mathematical technique acquires a physical interpretation.

In order to compute the moduli space of the D-brane probe theory, we need to know the superpotential, which, in $\mathcal{N}=2$ theories is fully determined by supersymmetry. As we have mentioned above, the superpotential of the orbifold theory can be found by putting in the original $\mathcal{N}=4$ superpotential the projected fields.

On can actually start directly with the scalar potential of the $\mathcal{N}=4$ theory which reads

$$V=-\sum_{m,n}\mathrm{Tr}[X^m,X^n]^2, \tag{2.4.55}$$

where X^m are the 6 hermitian non-abelian scalars that parametrize the transverse positions of the D3-branes. Since the $\mathcal{N}=2$ multiplets contain complex scalars, we define $Y^1=X^1+iX^2, Y^2=X^3+iX^4, Y^3=X^5+iX^6$. Recalling that under $\Gamma\subset SU(2)\subset SU(4)$ the 6-dimensional representation of $SU(4)$ breaks as $\mathbf{6}\to\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{2}$, we chose the complex structure so that Y^1 is a Γ singlet and Y^2,Y^3 are Γ doublets. Hence, Y^1 is the complex scalar of the $\mathcal{N}=2$ vector multiplet and Y^2,Y^3 are the two complex scalars in the hypermultiplet. If the D3-branes are extended along the x^1,x^2,x^3 directions and the orbifold is along the x^6,x^7,x^8,x^9 directions, the field $Y^1=X^1+iX^2$ parametrize the D3-brane position in x^4,x^5 while Y^2,Y^3 parametrize its position along the orbifolded directions.

The $\mathcal{N}=4$ scalar potential can be written in terms of the $Y^a, a=1,2,3$ as (see

for istance [61])

$$V = \frac{1}{2}\Big(\mathrm{Tr}\sum_{a,b=1}^{3}|[Y^a, Y^b]|^2 + \frac{1}{2}\mathrm{Tr}\sum_{a=1}^{3}[Y^a, Y^{a\dagger}]^2\Big) \tag{2.4.56}$$

where the first is the F-term and the second the D-term. The second is the trace of the square of the moment map $\rho = \sum_{a=1}^{3}[Y^a, Y^{a\dagger}]$; comparing with the moment maps that appeared in the HyperKähler quotient construction, we notice that ρ is essentially $\mu_{\mathbb{R}}$. This is the first hint that solving for the moduli space of the D-brane prove gauge theory we repeat Kronheimer's construction in a physical context.

In order to make transparent the different physical interpretation of the field Y^1 with the fields Y^2, Y^3, we re-write the above potential in terms of the complex scalars Y^2, Y^3 and the two real scalars X^1, X^2 that correspond to Y^1. This gives

$$V \sim \sum_{a,b=2}^{3}\mathrm{Tr}|[Y^a, Y^b]|^2 + 2\sum_{a,i}\mathrm{Tr}|[Y^a, X^i]|^2 - \sum_{i,j=1}^{2}\mathrm{Tr}[X^i, X^j]^2. \tag{2.4.57}$$

The moduli space of supersymmetric vacua of (2.4.57) is the locus of points where $V = 0$. When the scalars Y^2, Y^3 in the hypermultiplets are zero and the scalars in the vector multiplet X^1, X^2 have non-zero values in the Cartan subalgebra of the gauge group, the unbroken gauyge symmetry is generically a product of $U(1)$s and we are at the Coulomb branch. It is well-known that the metric on this branch receives 1-loop and non-perturbative corrections; it was actually calculated in the seminal work of Seiberg and Witten [63, 64].

Since these scalars parametrize the D3-brane positions in the orbifold fixed plane $x^4 - x^5$, we see that the Coulomb branch corresponds to motion of the D3-branes as independent objects inside the fixed plane.

When $X^1 = X^2 = 0$ but Y^2, Y^3 take non-zero values so that $V = 0$, we are in the Higgs branch. It is a generic feature of $\mathcal{N} = 2$ theories that this branch is exact, i.e. it does not receive quantum corrections and thus its metric can be readily computed from the classical action. It can be seen that the gauge symmetry breaks completely except for a $U(1)$ part.

Geometrically, the vevs of Y^2, Y^3 correspond to the positions of the D3-branes inside the orbifold. Due to the orbifold projection, the original D3-branes we put on the covering space have to move so that they respect the orbifolding. In the physical space, i.e. the orbifold, we just see a single D3-brane away from the fixed point where the gauge symmetry on its world-volume is exactly the $U(1)$ we mentioned above. Note that if we had started with $n|\Gamma|$ D3-branes on the covering space $\mathbb{C}^2$, we would have left with n physical D3-branes on the orbifold and the gauge symmetry of the Higgs branch would be $U(n)$.

The connection with Kronheimer's construction of ALE instantons comes from the fact that twisted sector moduli in the bulk, which are known to correspond to blow-up modes of the orbifold singularity, couple as Fayet-Iliopoulos terms on the D3-brane world-volume [56]. As it was shown in [56] for the A-type subgroups of $SU(2)$ and in [57] for the D- and E-type, these couplings change the equations describing the Higgs branch and the resulting moduli space is the resolved ALE. This is based on the fact that the conditions for minimizing V with non-zero Y^2, Y^3 in the presense of FI terms is exactly the same as the equations (2.3.43) describing the level set in the HyperKähler quotient construction of the ALEs. Note that the starting point of this construction, i.e. the space M_Γ defined explicitly in (2.3.36), corresponds precisely to the space of the four matric scalars X^3, X^4, X^5, X^6 subject to the orbifold projection (2.4.47). This is easy to see by replacing $z_1 \to Y^2 = X^3+iX^4$ and $z_2 \to Y^3 = X^5+iX^6$ in (2.3.36).

The other ingredient of the HyperKähler quotient construction is the quotienting. In the language of D-branes this is just the quotient one has to perform on the space of solutions of $V = 0$ with respect to the gauge group action on the scalars. This is of course necessary in order to describe the moduli space in terms of gauge-invariant variables. Note that in the HyperKähler quotient construction, the original $U(|\Gamma|$ symmetry is projected to its Γ-invariant part; in the D-brane picture this is just the analogue of the projection on the gauge fields in (2.4.47).

We see that indeed all elements of the abstract mathematical construction of Kronheimer find their physical analogues when the gauge theory of a D-brane on

an $SU(2)$ orbifold is examined. This shows the power of D-brane techniques and it emphasizes the complementarity of D-brane methods with perturbative approaches in analyzing stringy geometry. For example, the ways fundamental strings and D-branes perceive the resolved geometry, though equivalent, are completely different.

Fractional branes

We mentioned above that in the Coulomb branch the original D3-branes are free to move inside the fixed plane. One can imagine putting all of them except one at infinity. Since we don't have anymore the images of this D-brane under the orbifold action, we see that we cannot go to the Higgs branch and obtain a single physical brane probing the orbifold. Instead, the single copy is stuck at the fixed plane. This type of D-brane localized at fixed points of orbifolds is known as fractional brane since it corresponds to a fraction of what would be perceived as a single D-brane on the orbifold [65, 66].

The existence of fractional branes is necessary due to the fact that the choice of the orbifold group action on the CP factors is not constrained by any consistency conditions. Thus, if we choose a single irrep instead of the regular representation as we have been doing so far, it is straightforward to see that the theory we obtain corresponds to a fractional brane. More precisely, every irrep is associated to a different fractional brane. We should note here that these considerations are general and not only for $SU(2)$ orbifolds.

For an irrep r_i the tension of the corresponding fractional brane is $T_i = \frac{\mathrm{dim} r_i}{|\Gamma|} T$, where T the tension of the physical D-brane. When we choose the regular representation we get $\mathrm{dim} r_i$ fractional branes with tension T_i and the total tension is T as it should.

There is a nice geometrical interpretation of fractional branes. One can show that the fractional brane corresponding to the irrep r_i is charged under the corresponding twisted R-R field. Recall that twisted fields are in 1-1 correspondence with conjugacy classes of the orbifold group and hence to irreps. Moreover, when the shrunken 2-cycles are blown-up, the twisted R-R fields arise from dimensional reduction of the

ordinary bulk R-R fields. This motivates the identification of fractional Dp-branes with D(p+2)-branes that wrap the vanishing 2-cycles in the orbifold and provides a natural way to understand the coupling of the fractional D-branes to the twisted R-R fields.

Chapter 3

Discrete Torsion, Schur Multipliers, and Quivers

In this chapter we discuss some aspects of D-branes on orbifolds with discrete torsion, focusing on the classification of possible discrete torsions and on determining the quivers of the corresponding D-brane probe gauge theories. We first present a short review of discrete torsion in perturbative string theory and consequently we recall some facts about D-brane probes on orbifolds with discrete torsion.

Next, we discuss our work [1] on the classification of discrete torsion for orbifolds of the form $\mathbb{C}^n/G$, where G is a finite discrete subgroup of $SU(n)$ with $n = 2, 3, 4$. This is heavily based on mathematical tools related to Group Cohomology and, in particular, to the so-called Schur multiplier, which is a compact way to encode the possible discrete torsions.

We continue by analyzing our work [2] on quivers associated to orbifolds with discrete torsion. The main mathematical tool there is the covering group. As we will find out, it is a powerful tool in the construction of D-brane probe theories since it dispenses the need of hard to find mathematical information like explicit values of cocycles and projective character tables. In addition, the use of the covering group will provide us with an elegant way of unifying quivers that correspond to different choices of discrete torsion.

Nomenclature

Throughout this chapter, unless otherwise specified, we shall adhere to the following conventions for notation [1] :

ω_n	n-th root of unity;
G	finite group of order $\|G\|$;
Γ	generic representation of G;
$\mathbb{F}$	(algebraically closed) number field;
$\mathbb{F}^*$	multiplicative subgroup of $\mathbb{F}$;
$\langle x_i \| y_j \rangle$	the group generated by elements $\{x_i\}$ with relations y_j;
$< G_1, G_2, \ldots, G_n >$	group generated by the generators of groups $G_1, G_2, \ldots, G_n$;
$\gcd(m,n)$	the greatest common divisor of m and n;
$\mathcal{D}_n$, $\hat{\mathcal{D}}_n$	the ordinary (order $2n$) and the binary (order $4n$) dihedral group;
$\mathcal{T}, \mathcal{O}, \mathcal{I}$	the tetrahedral, octahedral and icosahedral group;
$E_{6,7,8}$	the binary tetrahedral, octahedral and icosahedral group;
A_n and S_n	alternating and symmetric groups on n elements;
$H \lhd G$	H is a normal subgroup of G;
$A \rtimes B$	semi-direct product of A and B;
$Z(G)$	center of G;
$N_G(H)$	the normalizer of $H \subset G$;
$G' := [G,G]$	the derived (commutator) group of G;
$[x,y]$	$:= xyx^{-1}y^{-1}$, the group commutator of x, y;
$\exp(G)$	exponent of group G;
G^*	the covering group of G;
$A = M(G)$	the Schur multiplier of G;
$\mathrm{char}(G)$	the ordinary (linear) character table of G, given as an $(r+1)\times r$ matrix with r the number of conjugacy classes and the extra row for the class numbers;
$Q_\alpha(G, \mathcal{R})$	the α-projective quiver for G associated to the representation $\mathcal{R}$.

[1]Note some minor differences from the notation used for finite subgroups of $SU(2)$ in the previous chapter.

3.1 Discrete torsion in perturbative orbifolds

3.1.1 Generalities on orbifolds

Before presenting the way discrete torsion was found by Vafa [67], let us review briefly the generic form of an orbifold partition function [28, 29]. Suppose we consider closed strings propagating on the orbifold $\mathcal{M}/G$, where G a discrete subgroup of symmetries of the space $\mathcal{M}$. The Hilbert space of the orbifolded theory should contain states invariant under the orbifold projection, i.e $g|\psi\rangle = |\psi\rangle$ for every element $g \in G$. This condition is enforced in the partition function by putting the projector $\mathcal{P} = \frac{1}{|G|}\sum_{g\in G} g$ inside the trace.

This would be the end of the story for point particles. For strings however, the possibility of having twisted boundary conditions can also be considered, and actually it is necessary for modular invariance as we will discuss momentarily. If $X(\sigma_1, \sigma_2)$ is a closed string toroidal world-sheet field, with $\sigma_1, \sigma_2 \in [0, 2\pi)$, the twisted boundary condition is

$$X(\sigma_1 + 2\pi, \sigma_2) = hX(\sigma_1, \sigma_2) \tag{3.1.1}$$

where h a generic element of the orbifold group G. σ_1 is the spatial coordinate on the closed string while σ_2 is the Euclidean world-sheet time. A torus with complex structure modulus τ, can be thought of as the complex plane parametrized by $z = \sigma_1 + \tau\sigma_2$ modulo the identifications $z \cong z + 2\pi \cong z + 2\pi\tau$.

Quantizing the theory in a sector twisted by h will in general result in a spectrum different from the spectrum of the untwisted sector, i.e. the sector with the usual periodic boundary conditions for world-sheet bosons and periodic (R) or anti-periodic (NS) boundary conditions for world-sheet fermions. The reason is that changing the boundary conditions will in general change both the moddings of the oscillator expansion of the world-sheet fields and the associated zero-point energy. Even if, however, these remain unchanged, the states obtained are still different from those in the untwisted sector since the vacuum in the latter is the original vacuum acted by

the so-called twist field. We will use $\mathcal{H}_h$ to denote the Hilbert space of states in the h-twisted sector.

The physical spectrum of every twisted sector has also to be invariant under the orbifold projection. The torus partition function in the h-twisted sector will thus take the form

$$Z_h(q,\bar{q}) = \mathrm{Tr}_{\mathcal{H}_h}(\mathcal{P} q^{L_0} \bar{q}^{\bar{L}_0}). \tag{3.1.2}$$

Recall now that the projection by $\mathcal{P}$ can be thought of as twisting in the σ_2 direction, i.e. along the periodic world-sheet (Euclidean) time. Thus, we can think of the projection $\mathcal{P}$ as introducing twisted sectors with respect to the τ direction. Let us symbolize by $Z_{g,h}(p,\bar{q})$ the partition sum in the sector twisted by g in the τ direction and by h in the sigma direction, that is the contribution of fields satisfying

$$\begin{aligned} X(\sigma_1 + 2\pi, \sigma_2) &= hX(\sigma_1, \sigma_2) & (3.1.3) \\ X(\sigma_1, \sigma_2 + 2\pi) &= gX(\sigma_1, \sigma_2). & (3.1.4) \end{aligned}$$

Note that consistency requires that $gh = hg$, i.e. only commutative pairs of twistings contribute. Then $Z_h(q,\bar{q}) = \frac{1}{|G|}\sum_{g\in G} Z_{g,h}(p,\bar{q})$ and the total partition function $Z(q,\bar{q})$ of the theory will be the sum over all sectors twisted in the σ direction (we think of the untwisted sector as twisted with the identity element of the orbifold group). Hence, we can write

$$Z(q,\bar{q}) = \frac{1}{|G|} \sum_{g,h\in G; gh=hg} Z_{g,h}(p,\bar{q}) \tag{3.1.5}$$

where we sum over commuting pairs of G elements.

The necessity of considering the twisted sectors stems from the requirement of modular invariance of the toroidal partition function. The generic modular transformation is $(\sigma_1,\sigma_2) \to (\sigma_1', \sigma_2') = (a\sigma_1 - b\sigma_2, -c\sigma_1 + d\sigma_2)$ where $a,b,c,d \in \mathbb{Z}$ satisfy $ad - bc = 1$. In terms of the torus modulus τ this is the familiar $SL(2,\mathbb{Z})$ transformation $\tau \to (a\tau + b)/(c\tau + d)$. One can then show [9] that the boundary conditions

(3.1.4) take the form

$$X(\sigma_1' + 2\pi, \sigma_2') = g^b h^a X(\sigma_1', \sigma_2') \tag{3.1.6}$$

$$X(\sigma_1', \sigma_2' + 2\pi) = g^d h^c X(\sigma_1', \sigma_2') \tag{3.1.7}$$

in terms of the modular transformed coordinates (σ_1', σ_2'). Thus the original pair of twistings is mapped under a modular transformation as $(g, h) \to (g', h') = (g^d h^c, g^b h^a)$. Note that if $gh = hg$ we have $g'h' = h'g'$ and the new twistings are also valid.

We see that even if we had started only with the projected [2] untwisted Hilbert space $h = 1$, we would have to include the twisted ones with $h' = g^{-b}$ to ensure modular invariance. Our choice of summation over all commuting pairs g, h in the general partition function defined above is obviously modular invariant [3] since the action of the modular group permutes the pairs.

Finally, consider all sectors twisted by elements fhf^{-1} in the conjugacy class of a given element h, i.e. $X(\sigma_1+2\pi, \sigma_2) = fhf^{-1}X(\sigma_1, \sigma_2)$ for every $f \in G$. The projection into G invariant states is enforced by the projector $\mathcal{P} = \frac{1}{|G|}\sum_{g\in G} g$. Note now that the spatial twist of the field gX is $gfh(gf)^{-1}$ and thus it still lies in the conjugacy class of h. In particular, the action of g changes the spatial twist $g : h \to ghg^{-1}$. We conclude that the twisted states that are invariant under the orbifold projection are in the same conjugacy class of spatial twists and accordingly the number of independent twisted sectors equals the number of conjugacy classes. The trivial conjugacy class of the identity corresponds to the untwisted sector. For abelian groups each element has its own conjugacy class and thus the number of independent twisted sectors plus the untwisted one is equal to the order of the group.

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[2]The original theory, i.e. unprojected and untwisted, is modular invariant since $(g = 1, h = 1)$ is fixed under the action of the modular group.

[3]In principle one can consider more complicated situations where only a subset of all possible pairs is considered; in most applications however ones uses the simple prescription of summing over all commuting pairs.

3.1.2 Discrete torsion in perturbative orbifolds

It is natural to ask if the construction of the partition function $Z(q,\bar{q})$ in the previous subsection is the most general that can be envisaged. For a given orbifold background $\mathcal{M}/G$, Vafa showed [67] that a more general prescription for constructing a modular invariant partition function using $Z_{g,h}(p,\bar{q})$ as building blocks, is

$$Z(q,\bar{q}) = \frac{1}{|G|} \sum_{g,h\in G; gh=hg} \epsilon(g,h) Z_{g,h}(p,\bar{q}) \qquad (3.1.8)$$

with $\epsilon(g,h)$ being the $U(1)$-valued discrete torsion coefficients. Alternatively, this modification is tantamount to different projectors $\mathcal{P}_h = \frac{1}{|G|}\sum_{g\in G}\epsilon(g,h)g$ in each twisted sector Hilbert space $\mathcal{H}_h$.

The requirement of modular invariance of (3.1.8) imposes the following condition on the discrete torsion coefficients:

$$\epsilon(g,h) = \epsilon(g^d h^c, g^b h^a) \qquad (3.1.9)$$

for the modular transformation $\tau \to (a\tau+b)/(c\tau+d)$. Furthermore, modular invariance at higher genus along with the factorization property of loop amplitudes imply that for abelian G, $\epsilon(g,h)$ forms a 1-dimensional representation of G with respect to the world-sheet time twisting g, i.e.

$$\epsilon(g_1 g_2, h) = \epsilon(g_1, h)\ \epsilon(g_2, h). \qquad (3.1.10)$$

For non-abelian G the above condition is generalized to [68]

$$\epsilon(g_1 g_2, h) = \epsilon(g_1, g_2 h g_2^{-1})\ \epsilon(g_2, h). \qquad (3.1.11)$$

It can be shown [67] that the possible inequivalent choices of the set of discrete torsion coefficients are classified by the second $U(1)$-valued group cohomology of G, denoted by $H^2(G,U(1))$. We will elaborate further on the relevant mathematics in the next section. Here, let us only mention that for each element $\alpha(g,h)$ of $H^2(G,U(1))$,

which is known as a cocyle, the corresponding discrete torsion coefficient is given by [67]

$$\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(h,g)} \tag{3.1.12}$$

for abelian orbifolds. For non-abelian orbifolds it was proposed in [68] that this relation is modified to

$$\epsilon(g,h) = \frac{\alpha(g,ghg^{-1})}{\alpha(h,g)}, \tag{3.1.13}$$

so that (3.1.11) is satisfied. From this we conclude that the relations

$$\epsilon(g,g) = 1 \tag{3.1.14}$$

$$\epsilon(g,h) = 1/\epsilon(h,g), \tag{3.1.15}$$

which along with the conditions (3.1.9) and (3.1.11), can be taken as the defining properties of the discrete torsion coefficients.

The incorporation of discrete torsion in an orbifold background can be thought of as selecting a different representation of the action of G on the twisted sectors $\mathcal{H}_h$ [69]. Originally, the action of G is $g \to \hat{g}$, where by $\hat{g}$ we denote the orbifold action on the space of states. For generic discrete torsion coefficients this action is modified to $g \to \epsilon(g,h)\hat{g}$ for the h-twisted sector. The last map is still a representation of G due to the property (3.1.11) of $\epsilon(g,h)$ and the fact that the action of g_2 on states twisted by h changes the twisting to $g_2hg_2^{-1}$, as we discussed in the previous subsection.

Some recent studies of conformal field theories corresponding to orbifolds with discrete torsion can be found in [70, 71, 72].

3.2 D-branes on orbifolds with discrete torsion

The inclusion of discrete torsion in the open string sector was first considered by Douglas [73], who studied D-brane probes on a $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete

torsion turned on. Subsequently, [74] extended this work to more general $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds.

The basic idea is that the projection rules (2.4.47) for the D-brane probe gauge theory remain unchanged; what needs modification however is the way the orbifold group acts on the CP factors. Instead of using ordinary representations of the orbifold group, one has to consider projective ones [73]. This modification immediately incorporates the effects of non-trivial discrete torsion.

In other words, for the invariant matter fields which survive the orbifold, Φ such that $\gamma^{-1}(g)\Phi\gamma(g) = r(g)\Phi, \quad \forall\ g \in G$, we now need the representation

$$\begin{array}{ll}\gamma(g)\gamma(h) = \alpha(g,h)\gamma(gh), \quad g,h \in G \ \text{ with} & \\ \alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \qquad \alpha(x,\mathbb{I}_G) = 1 = \alpha(\mathbb{I}_G,x) & \forall x,y,z \in G,\end{array} \tag{3.2.16}$$

where $\alpha(g,h)$ is known as a cocycle. These cocycles constitute, up to the equivalence

$$\alpha(g,h) \sim \frac{c(g)c(h)}{c(gh)}\alpha(g,h), \tag{3.2.17}$$

the so-called second cohomology group $H^2(G, U(1))$ of G, where c is any function (not necessarily a homomorphism) mapping G to $U(1)$. We shall formalize all these definitions in the next section.

In fact, one can show [67] that the choice

$$\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(h,g)},$$

for α obeying (3.2.16) actually satisfies (3.1.10), whereby linking the concepts of discrete torsion in the closed and open string sectors.

The main result of [73, 74] is that in the case of non-trivial discrete torsion, there are not enough twisted closed string moduli to fully resolve the singularities. In particular, for the specific case of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ there are $n-1$ conifold singularities that cannot be resolved. This analysis substantiated similar results obtained earlier in the closed string sector [69].

Along the line of [73, 74] a series of papers [75, 76, 77, 78] presented a new approach to study the moduli space of the D-brane probe theory, which demonstrates that the corresponding moduli spaces are actually examples of non-commutative spaces. Since non-trivial discrete torsion is related to the existence of B-fields localized at the orbifold fixed points [69], the observations of the aforementioned papers are complementary to the well-known fact that turning on a B-field along a D-brane world-volume changes the corresponding gauge theory into a non-commutative one. In the case of non-trivial discrete torsion, the orbifold geometry - as probed by D-branes - becomes non-commutative, while the world-volume of the D-branes remains commutative.

Other aspects of discrete torsion in connection with D-branes have been studied in [82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100].

The focus of all of the above works however has mainly been on abelian orbifolds [4]. One of our intentions in this chapter is to initiate a study of non-abelian orbifolds, which may shed some light into the nature of the corresponding geometries when non-trivial B-fields are turned on.

[4]Some non-abelian examples have been discussed in the context of orbifold conformal field theories in [70].

3.3 Classifying discrete torsion: Schur multipliers

This section is based on our work on the classification of possible discrete torsions for orbifolds of the form $\mathbb{C}^n/G$ where G is a finite subgroup of $Su(n), n = 2, 3, 4..$ The main mathematical tool we use is the so-called Schur multiplier. Most of the material in this and the next section is based on [1].

3.3.1 Projective representations, group cohomology, and Schur multipliers

In this subsection we explain in some detail the mathematics relevant in the classification of discrete torsion. Our main sources of information are [101, 102, 103].

Projective representations

We begin by first formalizing (3.2.16), the group representation of our interest:

Definition 3.3.1 *A projective representation of G over a field $\mathbb{F}$ (throughout we let $\mathbb{F}$ be an algebraically closed field with characteristic $p \geq 0$) is a mapping $\rho : G \to GL(V)$ such that*

$$(A) \quad \rho(x)\rho(y) = \alpha(x,y)\rho(xy) \ \ \forall \ \ x,y \in G; \qquad (B) \quad \rho(\mathbb{I}_G) = \mathbb{I}_V.$$

Here $\alpha : G \times G \to \mathbb{F}^*$ is a mapping, known as the factor system of ρ, whose meaning we shall clarify later. Of course we see that if $\alpha = 1$ trivially, then we have our familiar ordinary representation of G to which we shall refer as linear. Indeed, the mapping ρ into $GL(V)$ defined above is naturally equivalent to a homomorphism into the projective linear group $PGL(V) \cong GL(V)/\mathbb{F}^*\mathbb{I}_V$, and hence the name "projective". In particular we shall be concerned with projective *matrix* representations of G where we take $GL(V)$ to be $GL(n, \mathbb{F})$. The term α-projective representation will stand for a projective representation with factor system α.

The function α can not be arbitrary. Two immediate restrictions can be placed thereupon purely from the structure of the group:

$$\begin{array}{lll} (a) & \text{Group Associativity} & \Rightarrow \quad \alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \quad \forall x,y,z \in G \\ (b) & \text{Group Identity} & \Rightarrow \quad \alpha(x,\mathbb{I}_G) = 1 = \alpha(\mathbb{I}_G,x), \quad \forall x \in G. \end{array} \tag{3.3.18}$$

Group cohomology and the Schur multiplier

The study of such functions on a group satisfying (3.3.18) is precisely the subject of the theory of Group Cohomology. In general we let α to take values in A, an abelian coefficient group ($\mathbb{F}^*$ is certainly a simple example of such an A) and call them cocycles. The set of all cocycles, which we will name $Z^2(G,A)$ is an abelian group. We subsequently define a set of functions satisfying

$$B^2(G,A) := \{(\delta g)(x,y) := g(x)g(y)g(xy)^{-1}\} \quad \text{for any } g : G \to A \text{ such that } g(\mathbb{I}_G) = 1, \tag{3.3.19}$$

and call them coboundaries. It is then obvious that $B^2(G,A)$ is a (normal) subgroup of $Z^2(G,A)$ and in fact constitutes an equivalence relation in the following sense:

$$\alpha(x,y) \sim \frac{g(x)g(y)}{g(xy)}\alpha(x,y). \tag{3.3.20}$$

Thus it becomes a routine exercise in cohomology to define

$$H^2(G,A) := Z^2(G,A)/B^2(G,A),$$

the second (group) cohomology group of G.

Cohomologous cocycles correspond to projectively equivalent representations. That means that if $\rho_i : G \to GL(V_i), i = 1,2$ are two projective representations with cocycles $\alpha_1(x,y) = \delta g(x,y)\alpha_2(x,y)$, there is a map $\rho_2(x) = g(x)U\rho_1(x)U^{-1}$, where $U : V_1 \to V_2$ is a vector space isomorphism. When $g(x) = 1, \forall x \in G$ we have $a_1 = a_2$ and the representations ρ_1 and ρ_2 are called linearly equivalent. This is of course the

standard equivalence we encounter in linear representation theory.

We infer that the projective representations of G are classified by its second cohomology $H^2(G, \mathbb{F}^*)$. To facilitate the computation thereof, we shall come to an important concept:

Definition 3.3.2 *The Schur multiplier $M(G)$ of the group G is the second cohomology group with respect to the trivial action of G on $\mathbb{C}^*$:*

$$M(G) := H^2(G, \mathbb{C}^*).$$

Since we shall be mostly concerned with the field $\mathbb{F} = \mathbb{C}$, the Schur multiplier is exactly what we need. However, the properties thereof are more general. In fact, for any algebraically closed field $\mathbb{F}$ of zero characteristic, $M(G) \cong H^2(G, \mathbb{F}^*)$. In our case of $\mathbb{F} = \mathbb{C}$, it can be shown that

$$H^2(G, \mathbb{C}^*) \cong H^2(G, U(1)).$$

This terminology is the more frequently encountered one in the physics literature.

The calculation of the Schur multiplier $M(G)$ of a given group G will indicate possibilities of projective representations of the said group, or in a physical language, the possibilities of turning on discrete torsion in string theory on the orbifold space $\mathcal{M}/G$. In particular, if $M(G) \cong \mathbb{I}$, then the second cohomology of G is trivial and no non-trivial discrete torsion is allowed.

A first look at the covering group

The study of the actual projective representation of G is very involved and what is usually done in fact is to "lift to an ordinary representation." What this means is that for a central extension[5] A of G to G^*, we say a projective representation ρ of G lifts to a linear representation ρ^* of G^* if (i) $\rho^*(a \in A)$ is proportional to $\mathbb{I}$ and (ii)

[5] i.e., A in the center $Z(G^*)$ and $G^*/A \cong G$ according to the exact sequence $1 \to A \to G^* \to G \to 1$.

there is a section[6] $\mu : G \to G^*$ such that $\rho(g) = \rho^*(\mu(g)), \quad \forall g \in G$. Likewise it lifts projectively if $\rho(g) = t(g)\rho^*(\mu(g))$ for a map $t : G \to \mathbb{F}^*$. Now we are ready to give the following:

Definition 3.3.3 *We call G^* a covering group[7] of G over $\mathbb{F}$ if the following are satisfied:*
(i) $\exists$ a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G lifts projectively to an ordinary representation of G^*;
(ii) $|A| = |H^2(G, \mathbb{F}^*)|$.

The following theorem, initially due to Schur, characterizes covering groups:

Theorem 3.3.1 ([101] p143) *$G^\star$ is a covering group of G over $\mathbb{F}$ if and only if the following conditions hold:*
(i) $G^\star$ has a finite subgroup A with $A \subseteq Z(G^\star) \cap [G^\star, G^\star]$;
(ii) $G \cong G^\star/A$;
(iii) $|A| = |H^2(G, F^\star)|$
where $[G^\star, G^\star]$ is the derived group[8] $G^{\star'}$ of G^*.

3.3.2 Schur multipliers and string theory orbifolds

After presenting the relevant mathematical tools, we embark in our classification programme. As we have discussed in chapter 2, the orbifolds of interest are of the form $\mathbb{C}^n/G$ where G a finite subgroup of $SU(n), n = 2, 3, 4$. We need only to compute $M(G)$ for the all possible groups G, to know the discrete torsion afforded by the said orbifold theories.

[6] i.e., for the projection $f : G^* \to G$, $\mu \circ f = \mathbb{I}_G$.

[7] *Sometimes is also known as representation group.*

[8] *For a group G, $G' := [G, G]$ is the group generated by elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.*

The Schur multiplier of the discrete subgroups of $SU(2)$

Recall that the finite subgroups of $SU(2)$ have an ADE classification. The presentations [9] of these groups are given in the following table:

G	Name	Order	Presentation
A_n	Cyclic, $\cong \mathbb{Z}_{n+1}$	n	$\langle a \mid a^n = \mathbb{I}\rangle$
$\hat{\mathcal{D}}_n$	Binary Dihedral	$4n$	$\langle a, b \mid b^2 = a^n, abab^{-1} = \mathbb{I}\rangle$
E_6	Binary Tetrahedral	24	$\langle a, b \mid a^3 = b^3 = (ab)^3\rangle$
E_7	Binary Octahedral	48	$\langle a, b \mid a^4 = b^3 = (ab)^3\rangle$
E_8	Binary Icosahedral	120	$\langle a, b \mid a^5 = b^3 = (ab)^3\rangle$

(3.3.21)

We present now a powerful result due to Schur (1907) (q.v. col. 2.5, chap. 11 of [102]) which aids us to explicitly compute large classes of Schur multipliers for finite groups:

Theorem 3.3.2 ([101] p383) *Let G be generated by n elements with (minimally) r defining relations and let the Schur multiplier $M(G)$ have a minimum of s generators. Then*

$$r \geq n + s.$$

In particular, $r = n$ implies that $M(G)$ is trivial and $r = n + 1$, that $M(G)$ is cyclic.

theorem 3.3.2 could be immediately applied to $G \in SU(2)$.

Let us proceed with the computation case-wise. The A_n series has 1 generator with 1 relation, thus $r = n = 1$ and $M(A_n)$ is trivial. Now for the $\hat{\mathcal{D}}_n$ series, we note briefly that the usual presentation is $\hat{\mathcal{D}}_n := \langle a, b \mid a^{2n} = \mathbb{I}, b^2 = a^n, bab^{-1} = a^{-1}\rangle$; however, we can see easily that the last two relations imply the first, or explicitly: $a^{-n} := (bab^{-1})^n = ba^nb^{-1} = a^n$, (q.v. [102] example 3.1, chap. 11), whence making $r = n = 2$, i.e., 2 generators and 2 relations, and further making $M(\hat{\mathcal{D}}_n)$ trivial. Thus too are the cases of the 3 exceptional groups, each having 2 generators with

[9] A presentation of a finite group is its specification as the quotient of a free group produced by a set of generators, modulo a a set of relations between the generators.

2 relations. In summary then we have the following corollary of theorem 3.3.2, the well-known [97] result that

Corollary 3.3.1 *All discrete finite subgroups of $SU(2)$ have second cohomology $H^2(G, \mathbb{C}^*) = \mathbb{I}$, and hence afford no non-trivial discrete torsion.*

The above result can in fact be hinted from physical considerations without recourse to heavy mathematical machinery. The orbifold theory for $G \subset SU(2)$ preserves $\mathcal{N} = 2$ supersymmetry on the world-volume of the D3-brane probe. Moreover, non-trivial discrete torsion does not break any supersymmetry. Since the superpotential of $\mathcal{N} = 2$ theories is determined by supersymmetry and thus there are no allowed deformations thereof, while deformations of the superpotential are a generic feature of turning on discrete torsion, we conclude that there can be no non-trivial discrete torsion variants of $\mathcal{N} = 2$ theories. Accordingly, the finite subgroups of $SU(2)$ should not admit discrete torsion, exactly as we expect from the triviality of the corresponding Schur multipliers.

To address more complicated groups we need a methodology to compute the Schur multiplier. We quote one method below, a result originally due to Schur:

Theorem 3.3.3 ([103] p54) *Let $G = F/R$ be the defining finite presentation of G with F the free group of rank n and R is (the normal closure of) the set of relations. Suppose $R/[F, R]$ has the presentation $\langle x_1, \ldots, x_m; y_1, \ldots, y_n \rangle$ with all x_i of finite order, then*

$$M(G) \cong \langle x_1, \ldots, x_n \rangle.$$

Two more theorems of great usage are the following:

Theorem 3.3.4 ([103] p17) *Let the exponent*[10] *of $M(G)$ be $\exp(M(G))$, then*

$$\exp(M(G))^2 \textit{ divides } |G|.$$

And for direct products, another fact due to Schur,

[10] *i.e., the lowest common multiple of the orders of the elements.*

Theorem 3.3.5 ([103] p37)

$$M(G_1 \times G_2) \cong M(G_1) \times M(G_2) \times (G_1 \otimes G_2),$$

where $G_1 \otimes G_2$ *is defined to be* $\mathrm{Hom}_{\mathbb{Z}}(G_1/G_1', G_2/G_2')$.

Armed with the above theorems and with the aid of the Computer Algebra package GAP [104] using the algorithm developed for the p-Sylow subgroups of Schur Multiplier [105], we can attack the formidable task of giving the explicit Schur Multiplier of the list of groups of our interest. We will leave most of the details of the analytic computations in Appendix A. Without further ado then, we now proceed with the list of Schur multipliers for the discrete subgroups of $SU(n)$ for $n = 3, 4$, i.e., the $\mathcal{N} = 1, 0$ orbifold theories.

The Schur multiplier of the discrete subgroups of $SU(3)$

The classification of the discrete finite groups of $SU(3)$ is well-known (see e.g. [45], and [60, 149] for a discussion thereof in the context of string theory). It was pointed out in [106] that the usual classification of these groups does not include the so-called intransitive groups (see for example [47] for definitions) which are of less mathematical interest.From a physical standpoint however, they all give well-defined orbifolds. More specifically [60], all the ordinary polyhedral subgroups of $SO(3)$, namely the ordinary dihedral group D_{2n} and the ordinary tetrahedral $\mathcal{T} \cong A_4 \cong \Delta_{3\cdot 2^2}$, octahedral $\mathcal{O} \cong S_4 \cong \Delta_{6\cdot 2^2}$ and icosahedral $\mathcal{I} \cong \Sigma_{60}$, due to the embedding $SO(3) \hookrightarrow SU(3)$, are obviously (intransitive) subgroups of $SU(3)$ and hence we shall include these as well in what follows. We discuss some tricky aspects of the intransitives in Appendix B. Note that the Δ_{6n^2} series does not actually include the cases for n odd [149]; therefore n shall be restricted to be even.

The Schur multipliers of the $SU(3)$ discrete subgroups are presented in the fol-

lowing table:

	G	Order	Schur Multiplier $M(G)$
Intransitives	$\mathbb{Z}_n \times \mathbb{Z}_m$	$n \times m$	$\mathbb{Z}_{\gcd(n,m)}$
	$< \mathbb{Z}_n, \hat{\mathcal{D}}_{2m} >$	$\begin{cases} n \times 4m & n \text{ odd} \\ \frac{n}{2} \times 4m & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{I} & n \bmod 4 \neq 1 \\ \mathbb{Z}_2 & n \bmod 4 = 0, m \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & n \bmod 4 = 0, m \text{ even} \end{cases}$
	$< \mathbb{Z}_n, E_6 >$	$\begin{cases} n \times 24 & n \text{ odd} \\ \frac{n}{2} \times 24 & n \text{ even} \end{cases}$	$\mathbb{Z}_{\gcd(n,3)}$
	$< \mathbb{Z}_n, E_7 >$	$\begin{cases} n \times 48 & n \text{ odd} \\ \frac{n}{2} \times 48 & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{I} & n \bmod 4 \neq 0 \\ \mathbb{Z}_2 & n \bmod 4 = 0 \end{cases}$
	$< \mathbb{Z}_n, E_8 >$	$\begin{cases} n \times 120 & n \text{ odd} \\ \frac{n}{2} \times 120 & n \text{ even} \end{cases}$	$\mathbb{I}$
	Ordinary Dihedral $\mathcal{D}_n$	$2n$	$\mathbb{Z}_{\gcd(n,2)}$
	$< \mathbb{Z}_n, \mathcal{D}_m >$	$\begin{cases} n \times 2m & m \text{ odd} \\ n \times 2m & m \text{ even, n odd} \\ \frac{n}{2} \times 2m & m \text{ even, n even} \end{cases}$	$\begin{cases} \mathbb{Z}_{\gcd(n,2)} & m \text{ odd} \\ \mathbb{Z}_2 & m \text{ even, n} \bmod 4 = 1,2,3 \\ \mathbb{Z}_2 & m \bmod 4 \neq 0, n \bmod 4 = 0 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & m \bmod 4 = 0, n \bmod 4 = 0 \end{cases}$
Transitives	Δ_{3n^2}	$3n^2$	$\begin{cases} \mathbb{Z}_n \times \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{Z}_n, & \gcd(n,3) = 1 \end{cases}$
	Δ_{6n^2} (n even)	$6n^2$	$\mathbb{Z}_2$
	$\Sigma_{60} \cong A_5$	60	$\mathbb{Z}_2$
	Σ_{168}	168	$\mathbb{Z}_2$
	Σ_{108}	36×3	$\mathbb{I}$
	Σ_{216}	72×3	$\mathbb{I}$
	Σ_{648}	216×3	$\mathbb{I}$
	Σ_{1080}	360×3	$\mathbb{Z}_2$

(3.3.22)

The question of whether any discrete subgroup of $SU(3)$ admits non-cyclic discrete torsion was posed in [97]. From the results in table (3.3.22) we see that indeed, various intransitives give rise to non-cyclic Schur Multipliers as well as the transitive $\Delta(3n^2)$ series for n a multiple of 3. Some details of the computations of $M(\Delta_{3n^2})$ and $M(\Delta_{6n^2})$ are given in Appendix A.

The Schur multiplier of the discrete subgroups of $SU(4)$

The finite subgroups of $SU(4)$, which give rise to non-supersymmetric orbifold theories, are presented in modern notation in [47]. In the notation of the last reference, the Schur multipliers of the exceptional finite subgroups of $SU(4)$ read:

G	Order	Schur Multiplier $M(G)$
I$*$	60×4	$\mathbb{I}$
II$* \cong \Sigma_{60}$	60	$\mathbb{Z}_2$
III$*$	360×4	$\mathbb{Z}_3$
IV$*$	$\frac{1}{2}7! \times 2$	$\mathbb{Z}_3$
VI$*$	$2^6 3^4 5 \times 2$	$\mathbb{I}$
VII$*$	120×4	$\mathbb{Z}_2$
VIII$*$	120×4	$\mathbb{Z}_2$
IX$*$	720×4	$\mathbb{Z}_2$
X$*$	144×2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XI$*$	288×2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XII$*$	288×2	$\mathbb{Z}_2$
XIII$*$	720×2	$\mathbb{Z}_2$
XIV$*$	576×2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XV$*$	1440×2	$\mathbb{Z}_2$

G	Order	Schur Multiplier $M(G)$
XVI$*$	3600×2	$\mathbb{Z}_2$
XVII$*$	576×4	$\mathbb{Z}_2$
XVIII$*$	576×4	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XIX$*$	288×4	$\mathbb{I}$
XX$*$	7200×4	$\mathbb{I}$
XXI$*$	1152×4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXII$*$	$5 \times 16 \times 4$	$\mathbb{Z}_2$
XXIII$*$	$10 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXIV$*$	$20 \times 16 \times 4$	$\mathbb{Z}_2$
XXV$*$	$60 \times 16 \times 4$	$\mathbb{Z}_2$
XXVI$*$	$60 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
XXVII$*$	$120 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXVIII$*$	$120 \times 16 \times 4$	$\mathbb{Z}_2$
XXIX$*$	$360 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XXX$*$	$720 \times 16 \times 4$	$\mathbb{Z}_2$

(3.3.23)

3.4 $\mathcal{D}_n$ orbifolds: discrete torsion for a non-abelian example

We investigate now in depth the discrete torsion for a non-Abelian orbifold. The ordinary dihedral group $\mathcal{D}_n \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ of order $2n$, has the presentation

$$\mathcal{D}_n = \langle a, b | a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

As tabulated in (3.3.22), the Schur Multiplier is $M(\mathcal{D}_n) = \mathbb{I}$ for n odd and $\mathbb{Z}_2$ for n even [101]. Therefore the n odd cases are no different from the ordinary linear representations since they have trivial Schur multiplier and hence trivial discrete torsion. On the other hand, for the n even case, we will demonstrate the following result:

Proposition 3.4.1 *The binary dihedral group $\hat{\mathcal{D}}_n$ of the D-series of the discrete subgroups of $SU(2)$ (otherwise called the generalized quaternion group) is the covering group of $\mathcal{D}_n$ when n is even.*

Proof: The definition of the binary dihedral group $\hat{\mathcal{D}}_n$, of order $4n$, is

$$\hat{\mathcal{D}}_n = \langle a, b | a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1} \rangle,$$

as we saw in subsection 3.1. Let us check against the conditions of theorem 3.3.1. It is well-known that $\hat{\mathcal{D}}_n$ is the double cover of $\mathcal{D}_n$ and whence an $\mathbb{Z}_2$ central extension. First we can see that $A = Z(\hat{\mathcal{D}}_n) = \{1, a^n\} \cong \mathbb{Z}_2$ and condition (ii) is satisfied. Second we find that the commutators are $[a^x, a^y] := (a^x)^{-1}(a^y)^{-1}a^x a^y = 1$, $[a^x b, a^y b] = a^{2(x-y)}$ and $[a^x b, a^y] = a^{2y}$. From these we see that the derived group $[\hat{\mathcal{D}}_n, \hat{\mathcal{D}}_n]$ is generated by a^2 and is thus equal to $\mathbb{Z}_n$ (since a is of order $2n$). An important point is that only when n is even does A belong to $Z(\hat{\mathcal{D}}_n) \cap [\hat{\mathcal{D}}_n, \hat{\mathcal{D}}_n]$. This result is consistent with the fact that for odd n, $\mathcal{D}_n$ has trivial Schur multiplier. Finally of course, $|A| = |H^2(G, \mathbb{F}^*)| = 2$. Thus conditions (i) and (iii) are also satisfied. We therefore conclude that for even n, $\hat{\mathcal{D}}_n$ is the covering group of $\mathcal{D}_n$.

3.4.1 The irreducible representations

With the above proposition, we know by the very definition of the covering group, that the projective representations of $\mathcal{D}_n$ should be encoded in the well-known linear representations of $\hat{\mathcal{D}}_n$ (see for instance [106]). The latter has four 1-dimensional and $n-1$ 2-dimensional irreps. The matrix representations of these 2-dimensionals for the generic elements a^p, ba^p $(p = 0, ..., 2n-1)$ are given below:

$$a^p = \begin{pmatrix} \omega_{2n}^{lp} & 0 \\ 0 & \omega_{2n}^{-lp} \end{pmatrix} \qquad ba^p = \begin{pmatrix} 0 & i^l \omega_{2n}^{-lp} \\ i^l \omega_{2n}^{lp} & 0 \end{pmatrix}, \tag{3.4.24}$$

with $l = 1, ..., n-1$; these are denoted as χ_2^l.

On the other hand, the four 1-dimensionals are

	$n = $ even				$n = $ odd			
	a^{even}	$a(a^{\text{odd}})$	$b(ba^{\text{even}})$	$ba(ba^{\text{odd}})$	a^{even}	$a(a^{\text{odd}})$	$b(ba^{\text{even}})$	$ba(ba^{\text{odd}})$
χ_1^1	1	1	1	1	1	1	1	1
χ_1^2	1	-1	1	-1	1	-1	ω_4	$-\omega_4$
χ_1^3	1	1	-1	-1	1	1	-1	-1
χ_1^4	1	-1	-1	1	1	-1	$-\omega_4$	ω_4

(3.4.25)

We can subsequently obtain all irreducible projective representations of $\mathcal{D}_n$ from the above (henceforth n will be even).

Recalling that $\hat{\mathcal{D}}_n/\{1, a^n\} \cong \mathcal{D}_n$ from property (ii) of theorem 3.3.1, we can choose one element of each of the transversals of $\hat{\mathcal{D}}_n$ with respect to the $\mathbb{Z}_2$ to be mapped to $\mathcal{D}_n$. For convenience we choose $b^x a^y$ with $x = 0, 1$ and $y = 0, 1, ..., n-1$, a total of $4n/2 = 2n$ elements. Thus we are effectively expressing $\mathcal{D}_n$ in terms of $\hat{\mathcal{D}}_n$ elements.

For the matrix representation of $a^n \in \hat{\mathcal{D}}_n$, there are two cases. In the first, we have $a^n = 1 \times I_{d \times d}$ where d is the dimension of the representation. This case includes all four 1-dimensional representations and $(n/2 - 1)$ 2-dimensional representations in (3.4.24) for $l = 2, 4, ..., n-2$. Because a^n has the same matrix form as $\mathbb{I}$, we see that the elements $b^x a^y$ and $b^x a^{y+n}$ also have the same matrix form. Consequently, when we map them to $\mathcal{D}_n$, they automatically give the irreducible linear representations of

$\mathcal{D}_n$.

In the other case, we have $a^n = -1 \times I_{d\times d}$ and this happens when $l = 1, 3, ..., n-1$. It is precisely these cases[11] which give the irreducible projective representations of $\mathcal{D}_n$. Now, because a^n has a different matrix form from $\mathbb{I}$, the matrices for $b^x a^y$ and $b^x a^{y+n}$ differ. Therefore, when we map $\hat{\mathcal{D}}_n$ to $\mathcal{D}_n$, there is an ambiguity as to which of the matrix forms, $b^x a^y$ or $b^x a^{y+n}$, to choose as those of $\mathcal{D}_n$.

This ambiguity is exactly a feature of projective representations. Preserving the notations of theorem 3.3.1, we let $G^* = \bigcup\limits_{g_i \in G} Ag_i$ be the decomposition into transversals of G for the normal subgroup A. Then choosing one element in every transversal, say $A_q g_i$ for some fixed q, we have the ordinary (linear) representation thereof being precisely the projective representation of g_i. Of course different choices of A_q give different but projectively equivalent (projective) representations of G.

By this above method, we can construct all irreducible projective representations of $\mathcal{D}_n$ from (3.4.24). We can verify this by matching dimensions: we end up with $n/2$ 2-dimensional representations inherited from $\hat{\mathcal{D}}_n$ and $2^2 \times (n/2) = 2n$, which of course is the order of $\mathcal{D}_n$ as it should.

3.4.2 The quiver diagram and the matter content

The projection for the matter content Φ is well-known (see e.g., [59, 60, 61]):

$$\gamma^{-1}(g)\Phi\gamma(g) = \mathcal{R}(g)\Phi, \tag{3.4.26}$$

for $g \in G$ with $\mathcal{R}, \gamma$ appropriate linear and linear/projective representations respectively. For the group $\mathcal{D}_n$ we choose the generators (with action on $\mathbb{C}^3$) as [106]

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-1} \end{pmatrix} \qquad b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.4.27}$$

[11] Sometimes also called negative representations in such cases.

Now we can use these explicit forms to work out the matter content (the quiver diagram) and superpotential. For the regular representation, we choose $\gamma(g)$ as block-diagonal in which every 2-dimensional irreducible representation repeats twice with labels $l = 1, 1, 3, 3, .., n-1, n-1$ (as we have shown in the previous section that the even labels correspond to the linear representation of $\mathcal{D}_n$). With this $\gamma(g)$, we calculate the matter content below.

For simplicity, in the actual calculation we would not use (3.4.26) but rather the standard method given by Lawrence, Nekrasov and Vafa [59], generalized appropriately to the projective case by [97]. We can do so because we are armed with definition 3.5.5 and results from the previous subsection, and directly use the linear representation of the covering group: we lift the action of $\mathcal{D}_n$ into the action of its covering group $\hat{\mathcal{D}}_n$. It is easy to see that we get the same matter content either by using the projective representations of the former or the linear representations of the latter.

From the point of view of the covering group, the representation $\mathcal{R}(g)$ in (3.4.26) is given by

$$\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2 \tag{3.4.28}$$

and the representation $\gamma(g)$ is given by $\gamma \longrightarrow \sum\limits_{l=0}^{n/2-1} 2\chi_2^{2l+1}$. We remind ourselves that the $\mathbf{3}$ must in fact be always a linear representation of $\mathcal{D}_n$ while $\gamma(g)$ is the one that has to be projective when we include discrete torsion [73].

For the purpose of tensor decompositions we recall the result for the binary dihedral group [106] :

	$n = \text{even}$	$n = \text{odd}$
$\mathbf{1} \otimes \mathbf{1}'$	$\chi_1^2\chi_1^2 = \chi_1^1 \quad \chi_1^3\chi_1^3 = \chi_1^1 \quad \chi_1^4\chi_1^4 = \chi_1^1$ $\chi_1^2\chi_1^3 = \chi_1^4 \quad \chi_1^2\chi_1^4 = \chi_1^3 \quad \chi_1^3\chi_1^4 = \chi_1^2$	$\chi_1^2\chi_1^2 = \chi_1^3 \quad \chi_1^3\chi_1^3 = \chi_1^1 \quad \chi_1^4\chi_1^4 = \chi_1^3$ $\chi_1^2\chi_1^3 = \chi_1^4 \quad \chi_1^2\chi_1^4 = \chi_1^1 \quad \chi_1^3\chi_1^4 = \chi_1^2$
$\mathbf{1} \otimes \mathbf{2}$	$\chi_1^h\chi_2^l = \begin{cases} \chi_2^l & h = 1,3 \\ \chi_2^{n-l} & h = 2,4 \end{cases}$	
$\mathbf{2} \otimes \mathbf{2}'$	$\chi_2^{l_1}\chi_2^{l_2} = \chi_2^{(l_1+l_2)} + \chi_2^{(l_1-l_2)}$ where	$\chi_2^{(l_1+l_2)} = \begin{cases} \chi_2^{(l_1+l_2)} & \text{if} \quad l_1 + l_2 < n, \\ \chi_2^{2n-(l_1+l_2)} & \text{if} \quad l_1 + l_2 > n, \\ \chi_1^2 + \chi_1^4 & \text{if} \quad l_1 + l_2 = n. \end{cases}$ $\chi_2^{(l_1-l_2)} = \begin{cases} \chi_2^{(l_1-l_2)} & \text{if} \quad l_1 > l_2, \\ \chi_2^{(l_2-l_1)} & \text{if} \quad l_1 < l_2, \\ \chi_1^1 + \chi_1^3 & \text{if} \quad l_1 = l_2. \end{cases}$

(3.4.29)

From these relations we immediately obtain the matter content. Firstly, there are $n/2$ $U(2)$ gauge groups ($n/2$ nodes in the quiver). Secondly, because $\chi_1^3\chi_2^l = \chi_2^l$ we have one adjoint scalar for every gauge group. Thirdly, since $\chi_2^2\chi_2^{2l+1} = \chi_2^{2l-1} + \chi_2^{2l+3}$ (where for $l = 0$, χ_2^{2l-1} is understood to be χ_2^1 and for $l = n/2 - 1$, χ_2^{2l+3} is understood to be χ_2^{n-1}), we have two bi-fundamental chiral supermultiplets. We summarize these results in figure 3-1.

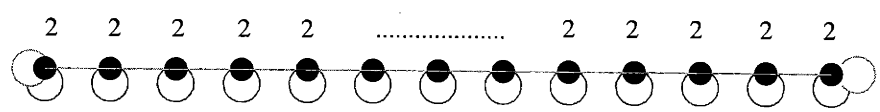

Figure 3-1: The quiver diagram of the ordinary dihedral group $\mathcal{D}_n$ with non-trivial projective representation. In this case of discrete torsion being turned on, we have a product of $n/2$ $U(2)$ gauge groups (nodes). The line connecting two nodes without arrows means that there is one chiral multiplet in each direction. Therefore we have a non-chiral theory.

We want to emphasize that by lifting to the covering group, in general we not only find the matter content (quiver diagram) as we have done above, but also the superpotential as well. The relevant formula was found in [59] and it is reviewed in the previous chapter. It can be applied here without any modification (of course, one

can use the matrix form of the group elements to obtain the superpotential directly as it is done for example in [73, 74, 75, 76, 77, 86, 87, 88], but using the Clebsch-Gordan coefficients is in general more convenient).

Knowing the above quiver (cf. figure 3-1) of the ordinary dihedral group $\mathcal{D}_n$ with discrete torsion, we wish to question ourselves as to the relationships between this quiver and that of its covering group, the binary dihedral group $\hat{\mathcal{D}}_n$ without discrete torsion (as well as that of $\mathcal{D}_n$ without discrete torsion). The usual quiver of $\hat{\mathcal{D}}_n$ is well-known [57]; we give an example for $n = 4$ in part (a) of figure 3-2. The quiver is obtained by choosing the decomposition of $\mathbf{3} \longrightarrow \chi_1^1 + \chi_2^1$ (as opposed to (3.4.28) because this is the linear representation of $\hat{\mathcal{D}}_n$); also $\gamma(g)$ is in the regular representation of dimension $4n$. A total of $(n-1)+4 = n+3$ nodes results. We recall that when getting the quiver of $\mathcal{D}_n$ with discrete torsion in the above, we chose the decomposition of $\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2$ in (3.4.28) which provided a linear representation of $\mathcal{D}_n$. Had we made this same choice for $\hat{\mathcal{D}}_n$, our familiar quiver of $\hat{\mathcal{D}}_n$ would have split into two parts: one being precisely the quiver of $\mathcal{D}_n$ without discrete torsion (see [106] for a discussion) and the other, that of $\mathcal{D}_n$ with discrete torsion as presented in figure 3-1. These are given respectively in parts (b) and (c) of figure 3-2.

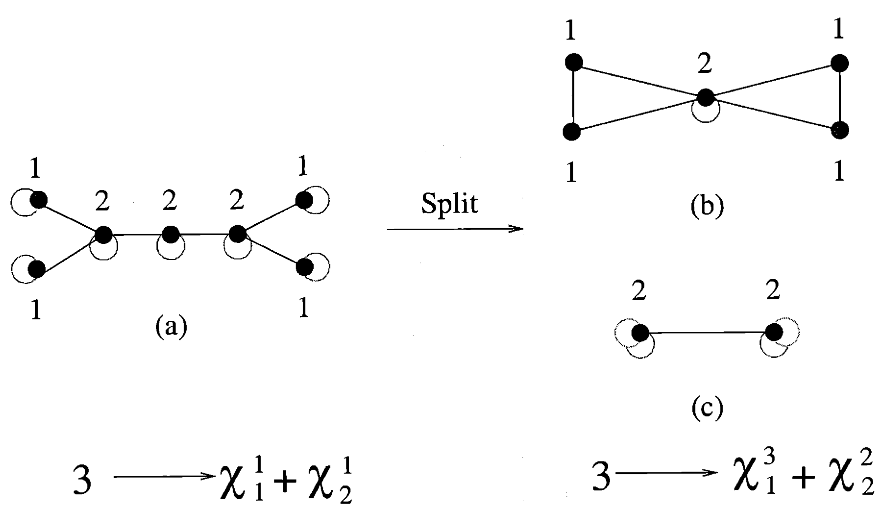

Figure 3-2: (a) The quiver diagram of the binary dihedral group $\hat{\mathcal{D}}_4$ without discrete torsion; (b) the quiver of the ordinary dihedral group $\mathcal{D}_4$ without discrete torsion; (c) the quiver of the ordinary dihedral group $\mathcal{D}_4$ with discrete torsion.

From this discussion, we see that in some sense discrete torsion is connected with different choices of decomposition in the usual orbifold projection. We should emphasize at this point that the example of $\mathcal{D}_n$ is very special because its covering group $\hat{\mathcal{D}}_{2n}$ belongs to $SU(2)$. In general, the covering group does not belong to $SU(3)$ and the meaning of the usual orbifold projection of the covering group in string theory seems vague.

3.5 Quivers and covering groups

After the classification of discrete torsion presented in the previous section, we move on towards the formulation of the gauge theory of the D-brane probe. The gauge theory is determined by the matter content, which can be succinctly represented using quivers, and the superpotential. In this section we will discuss a powerful method of obtaining the quivers that correspond to orbifolds with discrete torsion. The advantages of our method is that it dispenses of the need of mathematical information such as the explicit form of the cocycles or the projective character tables, which in general is difficult to obtain, and instead it requires only the character table of the covering group. The last of course is easily constructed using well-known techniques. Another strong point of our method is the fact that it determines the quivers for all choices of discrete torsion (including the case of no discrete torsion at all) at the same time and, in fact, it demonstrates that all of these independent quivers are components of a single disconnected quiver associated to the covering group. This is an intriguing property of the covering group and its precise interpretation in string theory remains to be clarified.

Most of the results in this section are taken from [2]. Another method of obtaining quivers for orbifolds with discrete torsion has been published in [152].

3.5.1 Covering groups and projective characters

More on the covering group

As we have discussed in detail in the previous sections, a projective representation of G over $\mathbb{C}$ is a mapping $\rho : G \to GL(V)$ such that $\rho(\mathbb{I}_G) = \mathbb{I}_V$ and $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$ for any elements $x, y \in G$. The function α, known as the cocycle, is a map $G \times G \to \mathbb{C}^*$ which is classified by $H^2(G, \mathbb{C}^*)$, the second $\mathbb{C}^*$-valued cohomology of G.

The study of the projective representations of a given group G is greatly facilitated by introducing an auxiliary object G^*, the covering group of G, which "lifts projective representations to linear ones." Suppose that there is a central extension according to the exact sequence $1 \to A \to G^* \to G \to 1$ such that A is in the center of G^*. Thus we have $G^*/A \cong G$. Now we say

Definition 3.5.4 *A projective representation ρ of G lifts to a linear representation ρ^* of G^* if*
(i) $\rho^*(a \in A)$ is proportional to $\mathbb{I}$ and
(ii) there is a section[12] $\mu : G \to G^*$ such that $\rho(g) = \rho^*(\mu(g))$, $\forall g \in G$.

Likewise it lifts projectively if $\rho(g) = t(g)\rho^*(\mu(g))$ for a map (not necessarily a homomorphism) $t : G \to \mathbb{C}^*$.

Definition 3.5.5 *G^* is called a covering group (also known as representation group) of G over $\mathbb{C}$ if the following are satisfied:*

(i) *$\exists$ a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G lifts projectively to an ordinary representation of G^*;*

(ii) $|A| = |M(G)| = |H^2(G, \mathbb{C}^*)|$.

The covering group will play a central role in this section; as we will show, the matter content of an orbifold theory with group G having discrete torsion switched-on is encoded in the quiver diagram of G^*.

We recall now theorem (3.3.1) from section 1.3:

[12] *i.e., for the projection $f : G^* \to G$, $\mu \circ f = \mathbb{I}_G$.*

Theorem 3.5.6 ([102] p143) *$G^\star$ is a covering group of G over $\mathbb{C}$ if and only if the following conditions hold:*
(i) $G^\star$ has a finite subgroup A with $A \subseteq Z(G^\star) \cap [G^\star, G^\star]$;
(ii) $G \cong G^\star/A$;
(iii) $|A| = |M(G)|$.

In the above, $[G^\star, G^\star]$ is the derived group $G^{*'}$ of G^*. For a group H, $H' := [H, H]$ is the group generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$ for $x, y \in H$. We stress that conditions (ii) and (iii) are easily satisfied while (i) is the more stringent imposition.

The solution of the problem of finding covering groups is certainly not unique: G in general may have more than one covering groups (e.g., the quaternion and the dihedral group of order 8 are both covering groups of $\mathbb{Z}_2 \times \mathbb{Z}_2$). To characterize the relation between different covering groups of the same group we start with the following definition:

Definition 3.5.6 *Two groups G and H are said to be isoclinic if there exist two isomorphisms*

$$\alpha : G/Z(G) \overset{\cong}{\to} H/Z(H) \quad \text{and} \quad \beta : G' \overset{\cong}{\to} H'$$

such that $\alpha(x_1 Z(G)) = x_2 Z(H)$ and $\alpha(y_1 Z(G)) = y_2 Z(H) \Rightarrow \beta([x_1, y_1]) = [x_2, y_2]$,

where we have used the standard notation that $xZ(G)$ is a coset representative in $G/Z(G)$. We note in passing that every abelian group is obviously isoclinic to the trivial group $\langle \mathbb{I} \rangle$.

To be able to characterize pairs of isoclinic groups we need one additional concept:

Definition 3.5.7 *If G and H are groups and $f : G \to H$ a surjective homomorphism, we call f isoclinic if*

$$\text{Ker}(f) \cap G' = 1.$$

Then we have the following useful lemma:

Lemma 3.5.1 ([102] p439) *Two groups G and H are isoclinic if there is an isoclinic surjective homomorphism $f : G \to H$.*

We introduce the concept of isoclinism because of the following important theorem of Hall:

Theorem 3.5.7 ([102] p441) *Any two covering groups of a given finite group G are isoclinic.*

Knowing that the covering groups of G are not isomorphic to each other, but isoclinic, a natural question to ask is how many non-isomorphic covering groups can one have. The following theorem of Schur gives a partial answer:

Theorem 3.5.8 ([102] p149) *For a finite group G, let*

$$G/G' = \mathbb{Z}_{e_1} \times ... \times \mathbb{Z}_{e_r}$$

and

$$M(G) = \mathbb{Z}_{f_1} \times ... \times \mathbb{Z}_{f_s}$$

be decompositions of these Abelian groups into cyclic factors. Then the number of non-isomorphic covering groups of G is less than or equal to

$$\prod_{1 \le i \le r, 1 \le j \le s} \gcd(e_i, f_j).$$

Projective characters

By virtue of the construction of the covering group G^* of G , we have the following 1-1 correspondence which will enable us to compute α-projective representations of G in terms of the linear representations of G^*:

Theorem 3.5.9 ([102] p139; [108] p8) *Let G^* be the covering group of G and $\lambda : A \to \mathbb{C}^*$ a homomorphism. Then*

(i) *For every linear representation $L : G^* \to GL(V)$ of G^* such that $L(a) = \lambda(a)\mathbb{I}_V \; \forall a \in A$, there is an induced projective representation P on G defined by*

$$P(g) := L(r(g)), \; \forall \; g \in G,$$

with $r : G \to G^*$ the map that associates to each coset $g \in G \cong G^*/A$ a representative element[13] *in G^*; and vice versa,*

(ii) *Every α-projective representation for $\alpha \in M(G)$ lifts to an ordinary representation of G^*.*

An immediate consequence of the above is the fact that knowing the linear characters of G^* suffices to establish the projective characters of G for all α [107]. The final conclusion is that one does not need to know a priori the specific values of the cocycles $\alpha(x,y)$ for all $x, y \in G$ (a stupendous task indeed) in order to construct the α-projective character table of G. As we will see in the ensuing, this result will facilitate enormously the computation of the quivers assocated to orbifolds with discrete torsion.

3.5.2 Explicit calculation of covering groups

We have prepared ourselves in the previous section the rudiments of the theory of covering groups; in the present section we will demonstrate these covers for the discrete finite subgroups of $SU(3)$. First we shall illustrate our techniques with the case of $\mathcal{D}_n$, the ordinary dihedral group, before tabulating the complete results.

The Covering Group of The Ordinary Dihedral Group

The standard presentation of the ordinary dihedral group of order $2n$ is:

$$\mathcal{D}_n = \langle \tilde{\alpha}, \tilde{\beta} | \tilde{\alpha}^n = 1, \tilde{\beta}^2 = 1, \tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = \tilde{\alpha}^{-1} \rangle.$$

We recall from the section 1.3 that the Schur multiplier for $G = \mathcal{D}_n$ is $\mathbb{Z}_2$ when n is even and trivial otherwise, thus we restrict ourselves only to the case of n even. We let $M(\mathcal{D}_n)$ be $A = \mathbb{Z}_2$ generated by $\{a | a^2 = \mathbb{I}\}$ and that the covering group is $G^* = \langle \alpha, \beta, a \rangle$.

[13] *i.e., $r(g)A \to g$ is the isomorphism $G^*/A \stackrel{\cong}{\to} G$.*

Having defined the generators we proceed to constrain relations there-among. Of course, $A \subset Z(G^*)$ immediately implies that $\alpha a = a\alpha$ and $\beta a = a\beta$. Moreover, α, β must map to $\tilde{\alpha}, \tilde{\beta}$ when we identify $G^\star/A \cong \mathcal{D}_n$ (by part (ii) of theorem 3.3.1). This means that $\mathbb{I}_G$ must have a preimage in $A \subset G^*$, giving us: $\alpha^n \in A, \beta^2 \in A$ and $\beta\alpha\beta^{-1}\alpha \in A$ by virtue of the presentation of G. And hence we have 8 possibilities, each being a central extension of $\mathcal{D}_n$ by A:

$$\begin{array}{l}
G_1^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_2^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_3^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_4^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_5^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_6^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_7^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_8^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle
\end{array} \tag{3.5.30}$$

Therefore, conditions (ii) and (iii) of theorem 3.3.1 are satisfied. One must check (i) to distinguish the covering group among these 8 central extensions in (3.5.30). Now since A is actually the center, it suffices to check whether $A \subset G_i^{*'} = [G_i^*, G_i^*]$.

We observe G_1^* to be precisely $\mathcal{D}_n \times \mathbb{Z}_2$, from which we have $G_1^{*'} = \mathbb{Z}_{n/2}$, generated by α^2. Because $A = \{\mathbb{I}, a\}$ clearly is not included in this $\mathbb{Z}_{n/2}$ we conclude that G_1^* is not the covering group. For G_2^*, we have $G_2^{*'} = \langle \alpha^2 a \rangle$, which means that when $n/2 =$ odd (recall that $n =$ even), $G_2^{*'}$ can contain a and hence $A \subset G_2^{*'}$, whereby making G_2^* a covering group. By the same token we find that $G_3^{*'} = \langle \alpha^2 \rangle$, $G_4^{*'} = \langle \alpha^2 a \rangle$, $G_5^{*'} = \langle \alpha^2 \rangle$, $G_6^{*'} = \langle \alpha^2 a \rangle$, and $G_7^{*'} = \langle \alpha^2 \rangle$.

We summarize these results in the following table:

Group	$G^{*'}$	$Z(G^*)$	$G^*/Z(G^*)$	Covering Group?
G_1^*	$\mathbb{Z}_{n/2} = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
$G_2^*(n = 4k+2)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathcal{D}_n$	yes
$G_2^*(n = 4k)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
G_3^*	$\mathbb{Z}_{n/2} = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
$G_4^*(n = 4k+2)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathcal{D}_n$	yes
$G_4^*(n = 4k)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2} \rangle$	D_n	no
G_5^*	$\mathbb{Z}_n = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathcal{D}_n$	yes
$G_6^*(n = 4k+2)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_4 = \langle \alpha^{n/2} \rangle$	D_n	no
$G_6^*(n = 4k)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathcal{D}_n$	yes
G_7^*	$\mathbb{Z}_n = \langle \alpha^2 \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathcal{D}_n$	yes
$G_8^*(n = 4k+2)$	$\mathbb{Z}_{n/2} = \langle \alpha^2 a \rangle$	$\mathbb{Z}_4 = \langle \alpha^{n/2} \rangle$	D_n	no
$G_8^*(n = 4k)$	$\mathbb{Z}_n = \langle \alpha^2 a \rangle$	$\mathbb{Z}_2 = \langle a \rangle$	$\mathcal{D}_n$	yes

Whence we see that G_1^* and G_3^* are not covering groups, while for $n/2 = $ odd $G_{2,4}^*$ are covers, for $n/2 = $ even $G_{6,8}^*$ are covers as well and finally $G_{5,7}^*$ are always covers. Incidentally, G_7^* is actually the binary dihedral group and we have already argued in section 1.3 that this is indeed the covering group of the dihedral group.

In accordance with theorem 3.5.7, these different covers must be isoclinic to each other. Checking against definition 3.5.6, we see that for G^* being $G_{2,4}^*$ with $n = 4k+2$, $G_{6,8}^*$ with $n = 4k$ and $G_{5,7}^*$ for all even n, $G^{*'} \cong \mathbb{Z}_n$ and $G^*/Z(G^*) \cong \mathcal{D}_n$; furthermore the isomorphisms α and β in the definition are easily seen to satisfy the prescribed conditions. Therefore all these groups are indeed isoclinic. We make one further remark, for both the cases of $n = 4k$ and $n = 4k+2$, we have found 4 non-isomorphic covering groups. Recall theorem 3.5.8, here we have $f_1 = 2$ and $G/G' = \mathbb{Z}_2 \times \mathbb{Z}_2$ (note that n is even) and so $e_1 = e_2 = 2$, whence the upper limit is exactly $2 \times 2 = 4$ which is saturated here. This demonstrates that our method is general enough to find all possible covering groups.

Covering groups for the discrete finite subgroups of $SU(3)$

By methods entirely analogous to the one presented in the above subsection, we can arrive at the covering groups for the discrete finite groups of $SU(3)$ as tabulated in the next few pages. As mentioned earlier however, the covering group is not unique. The particular ones we have chosen in the following table are the same as generated by the computer package GAP using the Holt algorithm [104].

Intransitives

- $G = \ \mathbb{Z}_m \times \mathbb{Z}_n = \langle\tilde{\alpha},\tilde{\beta}|\tilde{\alpha}^n = 1, \tilde{\beta}^m = 1, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}\rangle;$

$$M(G) = \mathbb{Z}_{p=\gcd(m,n)} = \langle a|a^p = \mathbb{I}\rangle;$$

$$G^* = \langle\alpha,\beta,a|\alpha a = a\alpha, \beta a = a\beta, a^p = 1, \alpha^n = 1, \beta^m = 1, \alpha\beta = \beta\alpha a\rangle. \tag{3.5.31}$$

- $G = \ \langle\mathbb{Z}_{n=4k},\hat{\mathcal{D}}_m\rangle = \langle\tilde{\alpha},\tilde{\beta},\tilde{\gamma}|\tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^m, \tilde{\beta}^{2m} = 1, \tilde{\beta}^m = \tilde{\gamma}^2, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1}\rangle;$

$$\begin{cases} m \text{ even} & M(G) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a,b|a^2 = 1 = b^2, ab = ba\rangle; \\ & G^* = \ \langle\alpha,\beta,\gamma,a,b|ab = ba, \alpha a = a\alpha, \alpha b = b\alpha, \beta a = a\beta, \beta b = b\beta, \\ & \qquad \gamma a = a\gamma, \gamma b = b\gamma, a^2 = 1 = b^2, \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha b, \\ & \qquad \alpha^{n/2} = \beta^m, \beta^{2m} = 1, \beta^m = \gamma^2, \gamma\beta\gamma^{-1} = \beta^{-1}\rangle. \\ m \text{ odd,} & M(G) = \mathbb{Z}_2 = \langle a|a^2 = 1\rangle; \\ & G^* = \ \langle\alpha,\beta,\gamma,a|a^2 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha\beta = \beta\alpha, \\ & \qquad \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^m, \beta^{2m} = 1, \beta^m = \gamma^2, \gamma\beta\gamma^{-1} = \beta^{-1}\rangle. \end{cases} \tag{3.5.32}$$

- $G = \ \langle\mathbb{Z}_{n=3k},E_6\rangle$

k odd $\quad G \cong \mathbb{Z}_n \times E_6 = \langle\tilde{\alpha},\tilde{\beta},\tilde{\gamma}|\tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^n = 1, \tilde{\beta}^3 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2\rangle;$

$$M(G) = \mathbb{Z}_3 = \langle a|a^3 = \mathbb{I}\rangle;$$

$$\begin{aligned} G^* = \ & \langle\alpha,\beta,\gamma,a|a^3 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^n = 1, \\ & \alpha\beta = \beta\alpha a^{-1}, \alpha\gamma = \gamma\alpha a, \beta^3 = \gamma^3 = (\beta\gamma)^2\rangle. \end{aligned}$$

$k = 2(2p+1) \quad G \cong \mathbb{Z}_{n/2} \times E_6$

$k = 4p \quad G \cong (\mathbb{Z}_n \times E_6)/\mathbb{Z}_2 = \langle\tilde{\alpha},\tilde{\beta},\tilde{\gamma}|\tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^3, \tilde{\beta}^3 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2\rangle;$

$$M(G) = \mathbb{Z}_3 = \langle a|a^3 = \mathbb{I}\rangle;$$

$$\begin{aligned} G^* = \ & \langle\alpha,\beta,\gamma,a|a^3 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^{n/2} = \beta^3, \\ & \alpha\beta = \beta\alpha a^{-1}, \alpha\gamma = \gamma\alpha a, \beta^3 = \gamma^3 = (\beta\gamma)^2\rangle. \end{aligned} \tag{3.5.33}$$

- $G = \ \langle\mathbb{Z}_{n=4k},E_7\rangle = \langle\tilde{\alpha},\tilde{\beta},\tilde{\gamma}|\tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^4, \tilde{\beta}^4 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2\rangle;$

$$M(G) = \mathbb{Z}_2 = \langle a|a^2 = \mathbb{I}\rangle;$$

$$\begin{aligned} G^* = \ & \langle\alpha,\beta,\gamma,a|a^2 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^{n/2} = \beta^4, \\ & \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha, \beta^4 = \gamma^3 = (\beta\gamma)^2\rangle. \end{aligned} \tag{3.5.34}$$

- $G = \langle \mathbb{Z}_n, D_{2m} \rangle$

n odd, m even
$$G = \mathbb{Z}_n \times D_{2m} = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}^n = 1, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\beta}^m = 1, \tilde{\gamma}^2 = 1, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle;$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha(\beta/\gamma) = (\beta/\gamma)\alpha, \alpha^n = 1, \beta^m = a, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

n even, m odd
$$G = \mathbb{Z}_n \times D_{2m}$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha a, \alpha^n = 1, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

m even, n = 2(2l + 1)
$$G = \mathbb{Z}_{n/2} \times D_{2m}$$

$n = 4k, m = 2(2l+1)$
$$G = (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}^{n/2} = \beta^{m/2}, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\beta}^m = 1, \tilde{\gamma}^2 = 1, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle;$$
$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^{m/2}, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle$$

$n = 4k, m = 4l$
$$G = (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2$$
$$M(G) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = 1, b^2 = 1, ab = ba \rangle;$$
$$G^* = \langle \alpha, \beta, \gamma, a, b | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha b, \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^{m/2}, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \tag{3.5.35}$$

where we have used the shorthand notation $(x/y/\ldots/z)$ to mean the relation to be applied to each of the elements $x, y, \ldots, z$.

Transitives

We first have the two infinite series:

- $$G = \Delta_{3n^2} = \langle \alpha, \beta, \gamma | \alpha^n = \beta^n = \gamma^3 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle;$$

$$\begin{cases} \gcd(n,3) = 1, n \text{ even} & M(G) = \mathbb{Z}_n = \langle a | a^n = 1 \rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a | (\alpha/\beta/\gamma)a = a(\alpha/\beta/\gamma), \\ & \quad a^n = \alpha^n a^{n/2} = \beta^n a^{n/2} = \gamma^3 = 1, \\ & \quad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle; \\ \gcd(n,3) = 1, n \text{ odd} & M(G) = \mathbb{Z}_n; \\ & G^* = \langle \alpha, \beta, \gamma, a | (\alpha/\beta/\gamma)a = a(\alpha/\beta/\gamma), \\ & \quad a^n = \alpha^n = \beta^n = \gamma^3 = 1, \\ & \quad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle; \\ \gcd(n,3) \neq 1, n \text{ even} & M(G) = \mathbb{Z}_n \times \mathbb{Z}_3 = \langle a, b | a^n = 1, b^3 = 1 \rangle; \\ & G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), \\ & \quad ab = ba, a^n = b^3 = \gamma^3 = \alpha^n a^{n/2} b = 1, \\ & \quad \beta^n a^{n/2} = b, \alpha\beta = \beta\alpha a b, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle; \\ \gcd(n,3) \neq 1, n \text{ odd} & M(G) = \mathbb{Z}_n \times \mathbb{Z}_3; \\ & G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), \\ & \quad a^n = b^3 = \gamma^3 = \alpha^n b = \beta^n b^{-1} = 1, \\ & \quad ab = ba, \alpha\beta = \beta\alpha a b, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle; \end{cases} \tag{3.5.36}$$

- $$G = \Delta_{6n^2} = \langle \alpha, \beta, \gamma, \delta | \alpha^n = \beta^n = \gamma^3 = \delta^2 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma, \\ \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta \rangle;$$

$$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle;$$

$$\begin{cases} \gcd(n,4) = 4 & G^* = \langle \alpha, \beta, \gamma, \delta, a | \alpha^n = \beta^n = \gamma^3 = \delta^2 = a^2 = 1, \\ & \quad (\alpha/\beta/\gamma/\delta)a = a(\alpha/\beta/\gamma/\delta), \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma, \\ & \quad \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta \rangle; \\ \gcd(n,4) = 2 & G^* = \langle \alpha, \beta, \gamma, \delta, a | \alpha^n a = \beta^n a = \gamma^3 = \delta^2 = a^2 = 1, \\ & \quad (\alpha/\beta/\gamma/\delta)a = a(\alpha/\beta/\gamma/\delta), \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma, \\ & \quad \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta \rangle; \end{cases} \tag{3.5.37}$$

Next we present the three exceptionals that admit discrete torsion:

- $$G = \quad \Sigma_{60} \cong A_5 = \langle \alpha, \beta | \alpha^5 = \beta^3 = (\alpha\beta^{-1})^3 = (\alpha^2\beta)^2 = 1$$
$$\alpha\beta\alpha\beta\alpha\beta = \alpha\gamma\alpha^{-1}\beta\alpha^2\beta\alpha^{-2}\beta = 1\rangle;$$
$$M(G) = \mathbb{Z}_2; \tag{3.5.38}$$
$$G^* = \langle \alpha, \beta, a | \alpha^5 = a, \beta^3 = a^2 = 1, (\alpha/\beta)a = a(\alpha/\beta)$$
$$(\alpha\beta^{-1})^3 = 1, (\alpha^2\beta)^2 = a\rangle;$$

- $$G = \quad \Sigma_{168} = \langle \alpha, \beta, \gamma | \gamma^2 = \beta^3 = \beta\gamma\beta\gamma = (\alpha\gamma)^4 = 1, \alpha^2\beta = \beta\alpha, \alpha^3\gamma\alpha^{-1}\beta = \gamma\alpha\gamma\rangle;$$
$$M(G) = \mathbb{Z}_2;$$
$$G^* = \langle \alpha, \beta, \gamma, \delta | \delta^2 = \gamma^2\delta = \beta^3\delta = (\beta\alpha)^3 = (\alpha\gamma)^3 = 1,$$
$$\beta\gamma\beta = \gamma, \alpha\delta = \delta\alpha, \beta^2\alpha^2\beta = \alpha, \beta^{-1}\alpha^{-1}\beta\gamma\alpha^{-1}\gamma = \gamma\alpha\beta\rangle;$$
(3.5.39)

- $$G = \quad \Sigma_{1080} = \langle \alpha, \beta, \gamma, \delta | \alpha^5 = \beta^2 = \gamma^2 = \delta^2 = (\alpha\beta)^2(\beta\gamma)^2 = (\beta\delta)^2 = 1,$$
$$(\alpha\gamma)^3 = (\alpha\delta)^3 = 1, \gamma\beta = \delta\gamma\delta, \alpha^2\gamma\beta\alpha^2 = \gamma\alpha^2\gamma\rangle;$$
$$M(G) = \mathbb{Z}_2;$$
$$G^* = \langle \alpha, \beta, \gamma, \delta, \epsilon | \alpha^5 = \epsilon^2 = \gamma^2\epsilon^{-1} = \beta^2\epsilon^{-1} = \delta^2\epsilon^{-1} = (\alpha\delta)^3 = 1,$$
$$\alpha^{-1}\epsilon\alpha = \beta^{-1}\epsilon\beta = \gamma^{-1}\epsilon\gamma = \delta^{-1}\epsilon\delta = \epsilon,$$
$$(\alpha\beta)^2 = (\beta\gamma)^2 = (\beta\delta)^2 = \gamma\beta\delta\gamma\delta = (\alpha\gamma)^3 = \epsilon,$$
$$\alpha^2\gamma\beta\alpha^2\gamma\alpha^{-2}\gamma = 1\rangle;$$
(3.5.40)

3.5.3 Covering groups, discrete torsion, and quiver diagrams

The strategy

The introduction of the host of the above concepts is not without a cause. In this section we shall provide an algorithm which permits the construction of the quiver $Q_\alpha(G,\mathcal{R})$ of an orbifold theory with group G having discrete torsion α turned-on, and with a linear representation $\mathcal{R}$ of G acting on the transverse space.

Our method dispenses of the need of the knowledge of the cocycles $\alpha(x,y)$, which in general is a formidable task from the viewpoint of cohomology, but which the literature may lead one to believe to be required for finding the projective representations. We shall demonstrate that the problem of finding these α-representations is reducible to the far more manageable duty of finding the covering group, constructing its character table (which is of course straightforward) and then applying the usual prodecure of extracting the quiver therefrom. One advantage of this method is that we immediately obtain the quiver for all cocycles (including the trivial cocycle which corresponds to having no discrete torsion at all) and in fact the values of $\alpha(x,y)$ (which we shall address in the next section) in a unified framework.

All the data we require are

(i) G and its (ordinary) character table char(G);

(ii) The covering group G^* of G and its (ordinary) character table char(G^*).

We first recall from [73] that turning on discrete torsion α in an orbifold projection amounts to using an α-projective representation Γ_α of $g \in G$

$$\Gamma_\alpha(g) A \Gamma_\alpha^{-1}(g) = A, \qquad \Gamma_\alpha(g) \Phi \Gamma_\alpha^{-1}(g) = \mathcal{R}(g) \cdot \Phi \qquad (3.5.41)$$

on the gauge field A and matter fields Φ.

The above equations have been phrased in a more axiomatic setting (in the language of [59]), by virtue of the fact that much of ordinary representation theory of finite group extends in direct analogy to the projective case, in [97]. However, we emphasize that with the aid of the linear representation of the covering group, one

can perform the orbifold projection with discrete torsion entirely in the setting of [59] without usage of the formulas in [97] generalized to twisted group algebras and modules. In other words, if we use the matrix of the linear representation of G^* instead of that of the corresponding projective representation of G, we will arrive at the same gauge group and matter contents in the orbifold theory. This can be alternatively shown as follows.

When we lift the projective matrix representation of G into the linear one of G^*, every matrix $\rho(g)$ will map to $\rho(ga_i)$ for every $a_i \in A$. The crucial fact is that $\rho(ga_i) = \lambda_i \rho(g)$ where λ_i is simply a phase factor. Now in (4.4.6) $\Gamma_\alpha(g)$ and $\Gamma_\alpha^{-1}(g)$ always appear in pairs, when we replace them by $\Gamma(ga_i)$ and $\Gamma^{-1}(ga_i)$, the phase factor λ_i will cancel out and leave the result invariant. This shows that the two results, the one given by projective matrix representations of G and the other by linear matrix representations of G^*, will give identical answers in orbifold projections.

An illustrative example: $\Delta_{3\cdot 3^2}$

Without much further ado, an illustrative example of the group $\Delta_{3\cdot 3^2} \in SU(3)$ shall serve to enlighten the reader. We recall from (3.5.36) that this group of order 27 has presentation $\langle \alpha, \beta, \gamma | \alpha^3 = \beta^3 = \gamma^3 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle$ and its covering group of order 243 (since the Schur multiplier is $\mathbb{Z}_3 \times \mathbb{Z}_3$) is $G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), a^3 = b^3 = \gamma^3 = \alpha^3 b = \beta^3 b^{-1} = 1, ab = ba, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle$.

Next we require the two (ordinary) character tables. Recall that the character tables are given as $(r+1) \times r$ matrices with r being the number of conjugacy classes (and equivalently the number of irreps), and the first row giving the conjugacy class numbers.

$$\mathrm{char}(\Delta_{3\cdot 3^2}) = \qquad (3.5.42)$$

1	1	1	3	3	3	3	3	3	3	3
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3
1	1	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$
1	1	1	ω_3	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	1	$\bar{\omega}_3$	1	ω_3
1	1	1	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	1	ω_3	ω_3	$\bar{\omega}_3$	1
1	1	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3
1	1	1	$\bar{\omega}_3$	ω_3	ω_3	1	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	1
1	1	1	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	ω_3	1	ω_3	1	$\bar{\omega}_3$
3	$3\bar{\omega}_3$	$3\omega_3$	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0

$$\mathrm{char}(\Delta^*_{3\cdot 3^2}) =$$

1	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	1	1	1	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	1	1	1
1	1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	1	1	1
1	1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	1	1	1	ω_3	ω_3	ω_3	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
3	3	3	3	3	3	3	3	3	$3\bar{\omega}_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	$3\omega_3$	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	0	0	0	X	Y	Z	0	0	0	0	0	0	0	0	0	P	M	N
3	$3\omega_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	0	0	0	Z	X	Y	0	0	0	0	0	0	0	0	0	M	N	P
3	$3\omega_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	0	0	0	Y	Z	X	0	0	0	0	0	0	0	0	0	N	P	M
3	$3\bar{\omega}_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	M	P	N	0	0	0	0	0	0	0	0	0	Z	Y	X
3	$3\bar{\omega}_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	N	M	P	0	0	0	0	0	0	0	0	0	Y	X	Z
3	$3\bar{\omega}_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	P	N	M	0	0	0	0	0	0	0	0	0	X	Z	Y
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	0	0	A	$-B$	C	$-A$	$-C$	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	0	0	$-B$	C	A	$-C$	B	$-A$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	0	0	C	A	$-B$	B	$-A$	$-C$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	$3\omega_3$	$3\bar{\omega}_3$	3	$3\bar{\omega}_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	$-C$	$-A$	B	0	0	0	0	0	0	$-B$	C	A	0	0	0	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	$3\omega_3$	$3\bar{\omega}_3$	3	$3\bar{\omega}_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	$-A$	B	$-C$	0	0	0	0	0	0	A	$-B$	C	0	0	0	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	$3\omega_3$	$3\bar{\omega}_3$	3	$3\bar{\omega}_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	B	$-C$	$-A$	0	0	0	0	0	0	C	A	$-B$	0	0	0	0	0	0
3	$3\bar{\omega}_3$	$3\omega_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	N	P	0	0	0	Z	X	Y	0	0	0
3	$3\bar{\omega}_3$	$3\omega_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	P	M	N	0	0	0	X	Y	Z	0	0	0
3	$3\bar{\omega}_3$	$3\omega_3$	$3\omega_3$	3	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	N	P	M	0	0	0	Y	Z	X	0	0	0
3	3	3	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	$-A$	$-C$	B	A	$-B$	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	$-C$	B	$-A$	$-B$	C	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	B	$-A$	$-C$	C	A	$-B$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\omega_3$	$3\bar{\omega}_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	X	Z	Y	0	0	0	P	N	M	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\omega_3$	$3\bar{\omega}_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Y	X	Z	0	0	0	N	M	P	0	0	0
3	$3\omega_3$	$3\bar{\omega}_3$	$3\bar{\omega}_3$	3	$3\omega_3$	$3\omega_3$	$3\bar{\omega}_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Z	Y	X	0	0	0	M	P	N	0	0	0
3	$3\bar{\omega}_3$	$3\omega_3$	$3\bar{\omega}_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	$-B$	A	C	0	0	0	0	0	0	$-C$	B	$-A$	0	0	0	0	0	0
3	$3\bar{\omega}_3$	$3\omega_3$	$3\bar{\omega}_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	A	C	$-B$	0	0	0	0	0	0	$-A$	$-C$	B	0	0	0	0	0	0
3	$3\bar{\omega}_3$	$3\omega_3$	$3\bar{\omega}_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar{\omega}_3$	0	0	0	0	0	0	0	0	C	$-B$	A	0	0	0	0	0	0	B	$-A$	$-C$	0	0	0	0	0	0

(3.5.43)

with $A := -\omega_3+\bar{\omega}_3, B := \omega_3+2\bar{\omega}_3, C := 2\omega_3+\bar{\omega}_3; M := -\omega_9^2-2\bar{\omega}_9^4, N := \omega_9^2+\bar{\omega}_9^4, P := -\omega_9^2 + \bar{\omega}_9^4; X := \omega_9^4 - \bar{\omega}_9^2, Y := \omega_9^4 + 2\bar{\omega}_9^2, Z := -2\omega_9^4 - \bar{\omega}_9^2$.

A comparative study of these two tables shall suffice to demonstrate the method.

We have taken extreme pains to re-arrange the columns and rows of char(G^*) for the sake of perspicuity; whence we immediately observe that char(G) and char(G^*) are unrelated but that the latter is organized in terms of "cohorts" [107] of the former. What this means is as follows: columns 1 through 9 of char(G^*) have their first 11 rows (not counting the row of class numbers) identical to the first column of char(G), so too is column 10 of char(G^*) with column 2 of char(G), et cetera, with $\{11\} \to \{3\}$, $\{12, 13, 14\} \to \{4\}$, $\{15, 16, 17\} \to \{5\}$, $\{18, 19, 20\} \to \{6\}$, $\{21, 22, 23\} \to \{7\}$, $\{24, 25, 26\} \to \{8\}$, $\{27, 28, 29\} \to \{9\}$, $\{30, 31, 32\} \to \{10\}$, and $\{33, 34, 35\} \to \{11\}$; using the notation that $\{X\} \to \{Y\}$ for the first 11 rows of columns $\{X\} \subset$ char(G^*) are mapped to column $\{Y\} \subset$ char(G). These are the so-called "splitting conjugacy classes" in G^* which give the (linear) char(G) [108]. In other words, (though the conjugacy class numbers may differ), up to repetition char(G) $\subset$ char(G^*). This of course is in the spirit of the technique of Frøbenius Induction of finding the character table of a group from that of its subgroup; an application of this technique in the context of string theory orbifolds will be given in the next chapter. Thus the first 11 rows of char(G^*) corresponds exactly to the linear irreps of G. The rest of the rows we shall shortly observe to correspond to the projective representations.

To understand these above remarks, let $A := \mathbb{Z}_3 \times \mathbb{Z}_3$ so that $G^*/A \cong G$. Now $A \subseteq Z(G^*)$, hence the matrix forms of all of its elements must be $\lambda \mathbb{I}_{d\times d}$, where d is the dimension of the irreducible representation and λ some phase factor. Indeed the first 9 columns of char(G^*) have conjugacy class number 1 and hence correspond to elements of this centre. Bearing this in mind, if we only tabulated the phases λ (by suppressing the factor $d = 1$ or 3 coming from $\mathbb{I}_{d\times d}$) of these first 9 columns, we arrive

at the following table (removing the first row of conjugacy class numbers):

rows	$\mathbb{I}$	a	a^2	b	ab	a^2b	b^2	ab^2	a^2b^2
$2-12$	1	1	1	1	1	1	1	1	1
$13-15$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$
$16-18$	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3
$19-21$	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
$22-24$	1	ω_3	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	1	$\bar{\omega}_3$	1	ω_3
$25-27$	1	$\bar{\omega}_3$	ω_3	ω_3	1	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	1
$28-30$	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3
$31-33$	1	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	1	ω_3	ω_3	$\bar{\omega}_3$	1
$34-36$	1	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	ω_3	1	ω_3	1	$\bar{\omega}_3$

This of course can be recognized as the character table of $\mathbb{Z}_3 \times \mathbb{Z}_3 = A$ (and with foresight we have labelled the elements of the group in the above table). This certainly is to be expected: G^* can be written as cosets gA for $g \in G$, whence lifting the (projective) matrix representation $M(g)$ of g simply gives $\lambda M(g)$ for λ a phase factor corresponding to the representation (or character as A is always Abelian) of A.

What is happening should be clear: all of this is merely part (i) of theorem 3.3.1 at work. The phases λ are precisely as described in the theorem. The trivial phase 1 gives rows $2-12$, or simply the ordinary representation of G while the remaining 8 non-trivial phases give, in groups of 3 rows from char(G^*), the projective representations of G. And to determine to which cocycle the projective representation belongs, we need and only need to determine the the 1-dimensional irreps of A. We shall show in section 5 how to read out the actual cocycle values $\alpha(g,h)$ for $g,h \in G$ directly with the knowledge of A and G^* without char(G^*).

Having understood the structure of the character table of the covering group in connection with the character group of the original group, we can at last analyze the quiver diagrams. Recall first that it is the group action on the Chan-Paton bundle that we choose to be projective; the space-time action inherited from $\mathcal{N} = 4$ R-symmetry

remains ordinary, i.e. $\mathcal{R}$ from (4.4.6) must still be a linear representation.

Now we evoke an obvious result: the tensor product of an α-projective representation with that of a β-representation gives an $\alpha\beta$-projective representation (cf. [102] p119), i.e.,

$$\Gamma_\alpha(g) \otimes \Gamma_\beta(g) = \Gamma_{\alpha\beta}(g). \tag{3.5.44}$$

We recall that from (4.4.6) and in the language of [59, 97] the bi-fundamental matter content $a_{ij}^{\mathcal{R}}$ is given in terms of the irreducible representations r_i of G as

$$\mathcal{R} \otimes r_i = \bigoplus_j a_{ij}^{\mathcal{R}} r_j, \tag{3.5.45}$$

(with of course $\mathcal{R}$ linear and r_i projective representations). Because $\mathcal{R}$ is an $\alpha = 1$ (linear) representation, (3.5.44) dictates that if r_i in (3.5.45) is a β-representation, then the right-hand thereof must be written entirely in terms of β-representations r_j. In other words, the various projective representations corresponding to the different cocycles should not mix under (3.5.45). What this signifies for the matter matrix is that $a_{ij}^{\mathcal{R}}$ is block-diagonal and the quiver diagram $Q(G^*, \mathcal{R})$ for G^* splits into precisely $|A|$ pieces, one of which is the ordinary (linear) quiver for G and the rest, the various quivers each corresponding to a different value of the cocycle.

Thus motivated, let us present the quiver diagram for $\Delta^*_{3 \cdot 3^2}$ in figure 3-3. The splitting does indeed occur as desired, into precisely $|\mathbb{Z}_3 \times \mathbb{Z}_3| = 9$ pieces, with (i) being the usual $\Delta_{3 \cdot 3^2}$ quiver (cf. [60, 149]) and the rest, the quivers corresponding to the 8 non-trivial projective representations.

The general method

Having expounded upon the detailed example of $\Delta_{3 \cdot 3^2}$ and witnessed the subtleties, we now present, in an algorithmic manner, the general method of computing the quiver diagram for an orbifold G with discrete torsion turned on:

1. Compute the character table char(G) of G;

2. Compute a covering group G^* of G and its character table char(G^*);

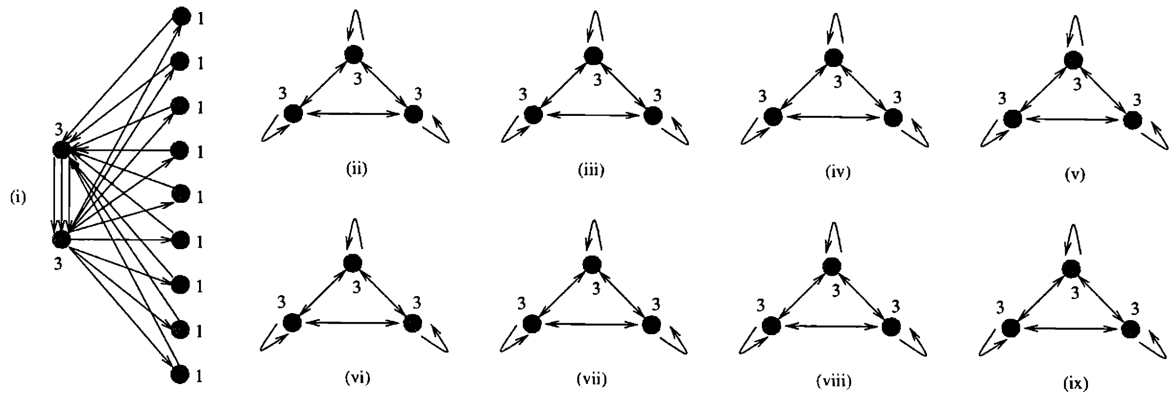

Figure 3-3: The Quiver Diagram for $\Delta^*_{3\cdot 3^2}$ (the Space Invaders Quiver!): piece (i) corresponds to the usual quiver for $\Delta_{3\cdot 3^2}$ while the remaining 8 pieces (ii) to (ix) are for the cases of the 8 non-trivial discrete torsions (out of the $\mathbb{Z}_3 \times \mathbb{Z}_3$) turned on.

3. Judiciously re-order the rows and columns of char(G^*):

   - Columns must be arranged into cohorts of char(G), i.e., group the columns which contain a corresponding column in char(G) together;
   - Rows must be arranged so that modulo the dimension of the irreps, the columns with conjugacy class number 1 must contain the character table of the Schur multiplier $A = M(G)$ (recall that $G^*/A \cong G$);
   - Thus char(G) is a sub-matrix (up to repetition) of char(G^*);

4. Compute the (ordinary) matter matrix $a_{ij}^{\mathcal{R}}$ and hence the quiver $Q(G^*, \mathcal{R})$ for a representation $\mathcal{R}$ which corresponds to a linear representation of G.

Now we have our final result:

Theorem 3.5.10 *$Q(G^*, \mathcal{R})$ has $|M(G)|$ disconnected components (sub-quivers) in 1-1 correspondence with the quivers $Q_\alpha(G, \mathcal{R})$ of G for all possible cocycles (discrete torsions) $\alpha \in A = M(G)$. Symbolically,*

$$Q(G^*, \mathcal{R}) = \bigsqcup_{\alpha \in A} Q_\alpha(G, \mathcal{R}).$$

In particular, $Q(G^*,\mathcal{R})$ contains a piece for the trivial $\alpha = 1$ which is precisely the case without discrete torsion, viz., $Q(G,\mathcal{R})$.

This algorithm facilitates enormously the investigation of the matter spectrum of orbifold gauge theories with discrete torsion as the associated quivers can be found without any recourse to explicit evaluation of the cocycles and projective character tables. Another fine feature of this new understanding is that, not only the matter content, but also the superpotential can be directly calculated by the explicit formulae in [59] using the ordinary Clebsch-Gordan coefficients of G^*.

As we have mentioned in the beginning of this section, the covering group G^* in general is not unique. How could we guarantee that the quivers obtained at the end of the day will be independent of the choice of the covering group? We can appeal directly to our previous remark, that using the explicit form of (4.4.6) we see that the phase factor λ (being a $\mathbb{C}$-number) always cancels out. In other words, the linear representation of whichever G^* we use, when applied to orbifold projections (4.4.6) shall result in the same matrix form for the projective representations of G. Whence we conclude that the quiver $Q(G^*,\mathcal{R})$ obtained at the end will ipso facto be independent of the choice of the covering group G^*.

Some examples

With the method at hand, we move on to the host of other subgroups of $SU(3)$ as tabulated in the previous section. The character tables char(G) and char(G^*) for the examples we have worked out are given in Appendix C. We present the cases of $\Sigma_{60,168,1080}$, the exceptionals which admit nontrivial discrete torsion and some first members of the Delta series in figure 3-4 to figure 3-10.

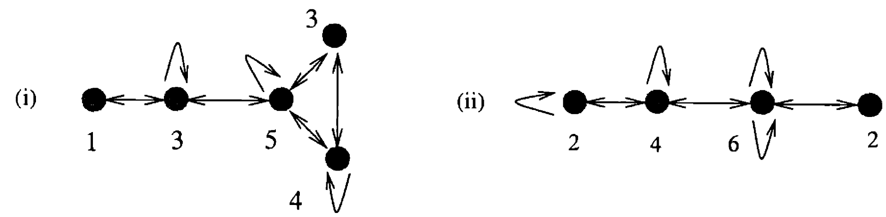

Figure 3-4: The quiver diagram of Σ_{60}^*: piece (i) is the ordinary quiver of Σ_{60} and piece (ii) has discrete torsion turned on.

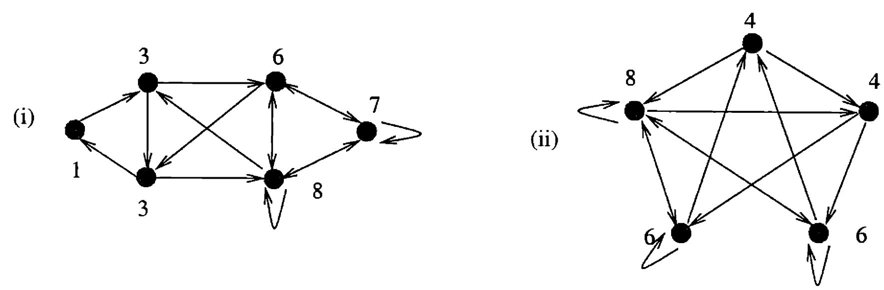

Figure 3-5: The quiver diagram of Σ_{168}: piece (i) is the ordinary quiver of Σ_{168} and piece (ii) has discrete torsion turned on.

3.5.4 Finding the cocycle values

A useful by-product of the method is that we can actually find the values of the 2-cocycles from the covering group. Here we require even less information: only G^* is needed.

Recall that the Schur multiplier is $A \subset Z(G^*)$, so every element therein has its own conjugacy class in G^*. Hence for all linear representations of G^*, the character of $a_k \in A$ will have the form $d\chi_i(a_k)$ where d is the dimension of that particular irrep of G^* and $\chi_i(a_k)$ is the character of a_k in A in its $i-th$ 1-dimensional irrep (A is always Abelian and thus has only 1-dimensional irreps). This property has a very important consequence: merely reading out the factor $\chi_i(a_k)$ from char(G^*), we can determine which linear representations will give which projective representations of G. Indeed, two projective representations of G belong to the same cocycle when and only when the factor $\chi_i(a_k)$ is the same for every $a_k \in A$.

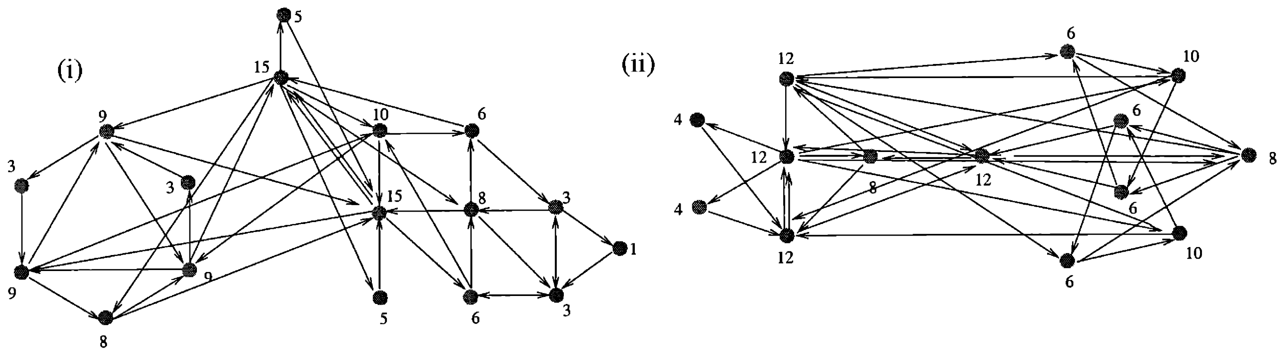

Figure 3-6: The quiver diagram of Σ_{1080}: piece (i) is the ordinary quiver of Σ_{1080} and piece (ii) has discrete torsion turned on.

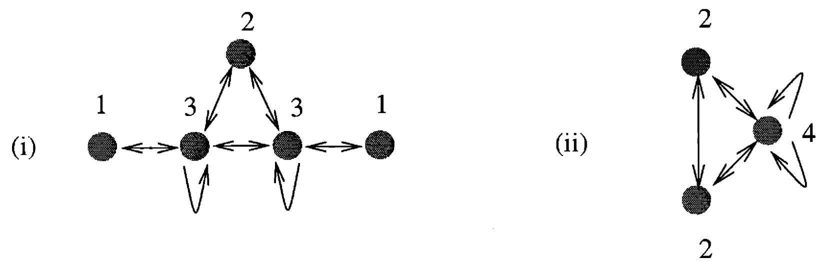

Figure 3-7: The quiver diagram of $\Delta_{6\cdot 2^2}$: piece (i) is the ordinary quiver of $\Delta_{6\cdot 2^2}$ and piece (ii) has discrete torsion turned on.

Next we recall how to construct the matrix forms of projective representations of G. $G^\star/A \equiv G$ implies that G^* can be decomposed into cosets $\bigcup\limits_{g\in G} gA$. Let $ga_i \in G^*$ correspond canonically to $\tilde{g} \in G$ for some fixed $a_i \in A$; then the matrix form of $\tilde{g}$ can be set to that of ga_i and furnishes the projective representation of $\tilde{g}$. Different choices of a_i will give different but projectively equivalent projective representations of G.

Note that if we have $\tilde{g}_i\tilde{g}_j = \tilde{g}_k$ in G, then in G^*, $g_ig_j = g_ka_{ij}^k$, or $(g_ia_i)(g_ja_j) = g_ka_k(a_{ij}^ka_ia_ja_k^{-1})$, but since (g_ia_i) is the projective matrix form for $\tilde{g}_i \in G$, this is exactly the definition of the cocyle from which we read:

$$\alpha(\tilde{g}_i, \tilde{g}_j) = \chi_p(a_{ij}^ka_ia_ja_k^{-1}), \tag{3.5.46}$$

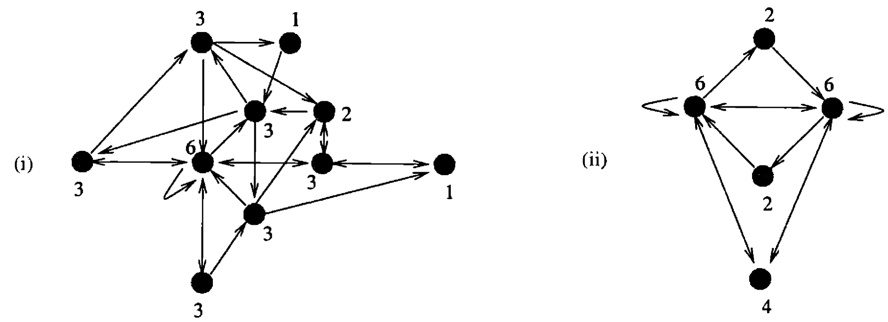

Figure 3-8: The quiver diagram of $\Delta_{6\cdot 4^2}$: piece (i) is the ordinary quiver of $\Delta_{(6\cdot 4^2}$ and piece (ii) has discrete torsion turned on.

where $\chi_p(a)$ is the p-th character of the linear representation of $a \in A$ defined above.

We can prove that (3.5.46) satisfies the 2-cocycle axioms (i) and (ii). Firstly notice that if $\tilde{g}_i = \mathbb{I} \in G$, we have $g_i = \mathbb{I} \in G^\star$; whence $a_{ij}^k = \delta_j^k \; \forall \; i$ and

$$(i) \quad \alpha(\mathbb{I}, \tilde{g}_j) = \chi_p(\delta_j^k a_j a_k^{-1}) = \chi_p(\mathbb{I}) = 1.$$

Secondly if we assume that $\tilde{g}_i\tilde{g}_j = \tilde{g}_q$, $\tilde{g}_q\tilde{g}_k = \tilde{g}_h$ and $\tilde{g}_j\tilde{g}_k = \tilde{g}_l$, we have $\alpha(\tilde{g}_i, \tilde{g}_j)\alpha(\tilde{g}_i\tilde{g}_j, \tilde{g}_k) = \chi_p(a_{ij}^q a_i a_j a_q^{-1})\chi_p(a_{qk}^h a_q a_k a_h^{-1}) = \chi_p(a_{ij}^k a_{qk}^h a_i a_j a_k a_h^{-1})$ and $\alpha(\tilde{g}_i, \tilde{g}_j\tilde{g}_k)\alpha(\tilde{g}_j, \tilde{g}_k) = \chi_p(a_{jk}^l a_j a_k a_l^{-1})\chi_p(a_{il}^h a_i a_l a_h^{-1}) = \chi_p(a_{il}^h a_{jk}^l a_i a_j a_k a_h^{-1})$. However, because $(g_i g_j)g_k = g_q a_{ij}^q g_k = g_h a_{ij}^q a_{qk}^h = g_i(g_j g_k) = g_i g_l a_{jk}^l = g_h a_{il}^h a_{jk}^l$ we have $a_{ij}^k a_{qk}^h = a_{il}^h a_{jk}^l$, and so

$$(ii) \quad \alpha(\tilde{g}_i, \tilde{g}_j)\alpha(\tilde{g}_i\tilde{g}_j, \tilde{g}_k) = \alpha(\tilde{g}_i, \tilde{g}_j\tilde{g}_k)\alpha(\tilde{g}_j, \tilde{g}_k).$$

Let us summarize the result. To read out the cocycle according to (3.5.46) we need only two pieces of information: the choices of the representative element in G^* (i.e., $a_i \in A$), and the definitions of G^* which allows us to calculate the $a_{ij}^k \in A$. We do not even need to calculate the character table of G^* to obtain the cocycle.

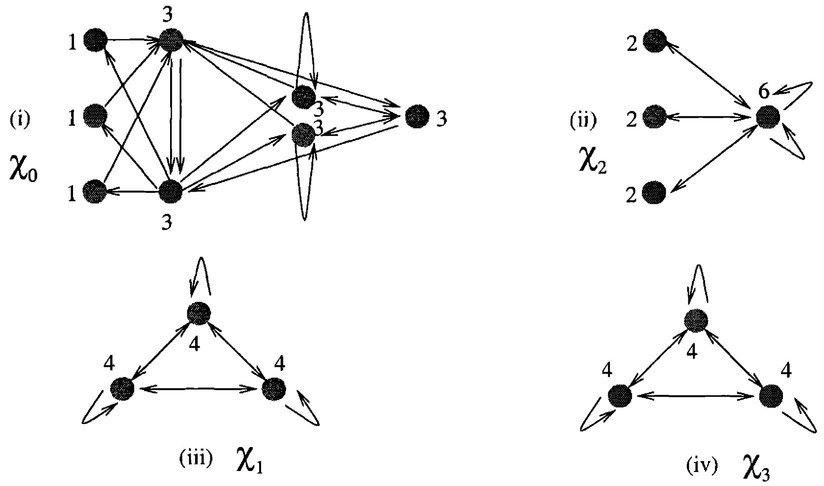

Figure 3-9: The quiver diagram of $\Delta_{3\cdot 4^2}$: piece (i) is the ordinary quiver of $\Delta_{3\cdot 4^2}$ and pieces (ii-iv) have discrete torsion turned on. We recall that the Schur Multiplier is $\mathbb{Z}_4$.

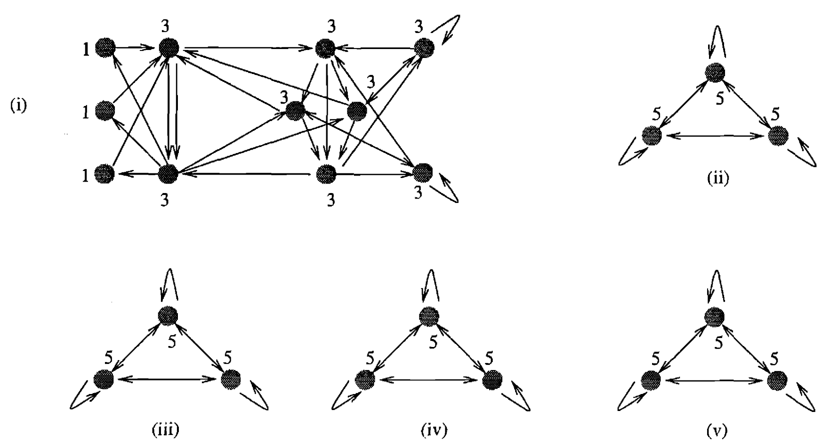

Figure 3-10: The quiver diagram of $\Delta_{3\cdot 5^2}$: piece (i) is the ordinary quiver of $\Delta_{3\cdot 4^2}$ and pieces (ii-v) have discrete torsion turned on. We recall that the Schur Multiplier is $\mathbb{Z}_5$.

Chapter 4

Brane Configurations and Stepwise Orbifold Projections

The focal point of this chapter is the relation between theories living on brane configurations with theories living on D-brane probes. The latter were discussed in chapter two.

We start by reviewing orientifolds; they are a necessary ingredient in constructing theories with orthogonal or symplectic gauge groups from D-branes. Subsequently, we synopsize the Hanany-Witten type construction and its variant known as the elliptic brane model. Unfortunately, we won't be able to make justice to the huge related literature but we will only concentrate on aspects of brane setups directly related to our work. A comprehensive review of brane setups and related field theory results can be found in [111].

We present then the state of affairs about the connection of brane setup theories with D-brane probe theories, which is essentially an application of T-duality. Thus, we examine in some detail the duality between configurations of NS5-branes and certain orbifold geometries. For the exceptional quiver gauge theories this connection is not clear; in particular, the T-dual brane setup is not known. One small step towards this goal is our method, called stepwise projection, which provides some hints about the corresponding the brane setup [3].

The main idea is that the brane probe theory is characterized by symmetries,

apparent already in the respective quivers, which are also reflected in their brane setup realization. We first demonstrate in detail the potential of this method for the case of D-type quiver gauge theories where the corresponding brane setup realization involving the so-called ON^0-planes is well-known. We apply then our techniques to the E_6 singularity where a brane realization is lacking. Even though we won't solve the problem, we find that a crucial ingredient may be a generalization of orientifolds involving world-sheet $\mathbb{Z}_3$ symmetries.

Nomenclature

Throughout this chapter, unless otherwise specified, we shall adhere to the following conventions for notation :

ω_n	n-th root of unity;
Γ	finite group of order $\|\Gamma\|$;
$\langle x_i \| y_j \rangle$	the group generated by elements $\{x_i\}$ with relations y_j;
$\langle G_1, G_2, \ldots, G_n \rangle$	group generated by the generators of groups $G_1, G_2, \ldots, G_n$;
$\hat{\mathcal{D}}_n$	the binary (order $4n$) dihedral group;
$E_{6,7,8}$	the binary tetrahedral, octahedral and icosahedral group;
$R^{\bullet}_{G(n)}(x)$	a representation of the element $x \in G$ of dimension n with $\bullet$ denoting properties such as regularity, irreducibility, etc., and/or simply a label;
S^T	the transpose of the matrix S;
$A \otimes B$	the tensor product of matrices A and B with block matrix elements $A_{ij}B$.

4.1 Review of orientifolds

In this section we provide a selective review on certain aspects of orientifold planes [10, 112, 113, 114, 115] which we will use in the ensuing. More thorough presentations can be found in [116, 117].

In general, an orientifold plane is the fixed plane of a spacetime symmetry in conjunction with the world-sheet parity transformation Ω which exchanges left-movers with right-movers. The combined transformation has of course to be a symmetry of the underlying conformal field theory; the theory after the projection is found using the general rules of orbifold projections in CFT which we sketched in the previous

chapter.

The most notable example of an orientifold is the type I theory. The orientation reversal operation Ω on a closed string with spatial coordinate $\sigma \in [0, 2\pi)$ amounts to $\Omega : \sigma \to 2\pi - \sigma$ or, in terms of variables on the complex plane, $\Omega : z \leftrightarrow \bar{z}$. Type IIB theory in the Green-Schwarz formalism includes, besides the usual bosonic terms which are Ω invariant, one left-moving and one right-moving world-sheet field corresponding to spinors of the same chirality in spacetime. Hence, Ω is a symmetry of the world-sheet theory and gauging it [1] results in a theory of unoriented open and closed strings with half of the original type IIB supersymmetry. In addition, the spectrum contains a 10-dimensional $SO(32)$ gauge field.

One can compactify type I strings and then by T-duality a wealth of new objects, known as orientifold planes, can be obtained. One direct way to define an orientifold p-plane (Op-plane) is as the background

$$\mathbb{R}^{1,p} \times \left(\mathbb{R}^{9-p} \,/\, \mathcal{I}_{9-p}\Omega\mathcal{J}\right) \tag{4.1.1}$$

where

$$\mathcal{J} = \begin{cases} \mathbb{I}, & p = 0,1 \mod 4 \\ (-1)^{F_L}, & p = 2,3 \mod 4 \end{cases} . \tag{4.1.2}$$

The orientifold is the fixed plane of the spacetime parity projection $\mathcal{I}_{9-p}$. p is odd in type IIB and even in type IIA theories.

In the case where D-branes are present, the orientifold projection has an action on their CP factors, much like the action we encountered in the discussion of D-brane on orbifolds in chapter two. The unitary gauge symmetry realized on the world-volume of parallel and coincident Dp-branes reduces to either an orthogonal or a symplectic subgroup when an Op-plane is also put on the stack. The distinction between

[1] We recall that gauging a discrete symmetry of the world-sheet theory just means orbifolding the world-sheet theory with respect to this symmetry. The term gauging reflects the fact that by imposing the invariance under the orbifold projection on the states in both the twisted and untwisted sectors, we essentially demand that the symmetry is local on the world-sheet.

the two types of projections on the CP factors is reflected on the R-R charge of the corresponding orientifold. We should mention here that unlike D-branes, orientifolds are not dynamical since they are just the fixed locus of a spacetime symmetry. Nevertheless it can be shown that they have tension and carry R-R charges.

It is easy to verify that the generic orientifold projection leaves half of the original 32 supersymmetries unbroken and hence the resulting object is BPS. There can actually be supersymmetric configurations involving orientifolds and D-branes. The simplest of course is just parallel and coincident D-branes on top of the orientifold fixed plane; the resulting theory has still 16 supercharges.

The coupling of orientifolds to R-R fields is responsible for the gauge symmetry of the type I theory. Recall that we obtained type I strings by projecting type IIB with the world-sheet parity Ω. Since there is no spacetime action involved, we obtain a O9-plane. Now, a straightforward world-sheet computation shows that this orientifold carries -32 units of D9-brane charge [116]. The corresponding R-R field is the 10-form potential of type IIB string theory which survives the orientifold projection. Since the O9-plane fills all of space-time, the 10-form flux would have nowhere to go and a tadpole would appear making the theory inconsistent [2] . Thus, the only way to render the theory consistent is by canceling the flux; this can be achieved by placing 32 D9-branes and on their 10-dimensional world-volume an $SO(32)$ gauge theory will be realized, which is exactly the gauge symmetry of type I strings (whose discovery was originally based on anomaly cancellation conditions [118]).

The orientifold which results in orthogonal gauge symmetry is called Op^- and its R-R charge is -2^{p-4}. This refers to charge under the R-R $(p+1)$-form and its unit of R-R charge corresponds to a single Dp-brane, i.e. without its mirror image due to the spacetime parity projection [3] . The orientifold that results in symplectic gauge

[2]This can be seen also by the fact that the transverse space to the O9-plane is a point and hence, being compact, should not contain any flux. This is a generalization of the familiar statement that an electric charge on a single periodic spatial direction is inconsistent since the flux would wind around the circle an infinite number of times resulting in infinite energy.

[3]These D-branes are frequently called half D-branes since they are stuck on the orientifold fixed plane and what would be seen as a physical D-brane in the projected half of spacetime would be a pair of them moving symmetrically with respect to the fixed plane in the covering space. They are of course an example of the fractional branes we discussed in chapter 2 for generic spacetime

groups is denoted by Op^+ and it has 2^{p-4} units of R-R charge. To summarize, placing n physical Dp-branes on an Op^--plane (Op^+-plane) leads to a $SO(2n)$ ($Sp(n)$) gauge group.

As was first found in [119] and further elucidated in [120], the Op^+ variety arises from the Op^- one when an NS-NS discrete torsion is turned on. Moreover, there is a further variant that corresponds to the possibility of turning on a R-R discrete torsion. The reason is that the transverse space of an orientifold p-plane is $\mathbb{RP}^{8-p}$ and it can be shown that for $p \leq 5$ it supports discrete torsion cohomologies for the 3-form field strength $H = dB$ corresponding to the NS-NS antisymmetric tensor field and for the $(6-p)$-form R-R field strength. Both of these cohomologies are $\mathbb{Z}_2$ and thus we have four possibilities [4] We should mention that the situation is slightly different for either $p \leq 1$ or $p \geq 6$. For the first case there are more than two non-trivial cohomologies, see [121] for a discussion; for the second case one expects only the variants due to NS-NS discrete torsion but there are further subtleties that depend on the specific value of p, some of which have been studied in [122]

Turning on the R-R discrete torsion results in the $\widetilde{Op}^-$ and $\widetilde{Op}^+$ variants. Placing n physical Dp-branes on top of the first one yields a $SO(2n+1)$ gauge symmetry. Therefore we can view this orientifold as a bound state of an Op^- with a half Dp-brane. As expected, its R-R charge is $1 - 2^{p-4}$.

The last variant is more mysterious: it realizes the same gauge symmetry as Op^+, namely $Sp(n)$ groups, and it also has the same R-R charge. In some cases it differs from the Op^+ in the spectrum of monopoles and the theta angle of the corresponding gauge theory. More on the $\widetilde{Op}^+$ can be found in [120].

A natural question concerns the behavior of orientifolds at strong coupling. In type IIA theory we expect that as $g_s \to \infty$ the perturbative orientifolds are transmuted to non-perturbative M-theory analogues. Since a microscopic definition of M-theory is so far lacking, the analogue of world-sheet parity is obscure. The approach so far has

orbifolds.

[4] As we will discuss, these discrete torsions can be also understood in terms of crossings of the orientifold with certain branes.

been to define the M-theory lifts of orientifolds in a spacetime fashion based on the fact that M-theory on a circle is equivalent to type IIA strings. In other words, one can define the orientifolds of M-theory by specifying their action on the spacetime fields of 11-dimensional supergravity in a way consistent with the action of world-sheet parity reversal on their 10-dimensional reductions. These objects are called OMp-planes and it can be shown that M-theory contains OM1-, OM2-, OM5-, OM6- and an OM9-plane. Their properties have been studied in [123, 120].

The strong coupling behavior of type IIB orientifolds can be extracted from the rules of S-duality. In particular, it is well-known [116] that Ω is exchanged with the projection to even spacetime fermion number in the left-moving sector of the closed string. The last projection is symbolized usually by $(-1)^{F_L}$ and hence we can write succinctly: $S\Omega S^{-1} = (-1)^{F_L}$, with S denoting the S-duality transformation. It is easy to see that $(-1)^{F_L}$ is a symmetry of the world-sheet theory of both the type IIA and type IIB strings. A direct way of checking the above S-duality transformation is by comparing the action of Ω and $(-1)^{F_L}$ on the massless supergravity fields; we find that Ω leaves invariant the metric, the dilaton and the antisymmetric R-R 2-form while the axion, the self-dual 4-form and the NS-NS 2-form are odd; S-duality exchanges the two antisymmetric tensors and indeed the NS-NS 2-form is invariant under $(-1)^{F_L}$ while the R-R 2-form is odd.

Orbifolding type IIB strings with respect to the $(-1)^{F_L}$ world-sheet symmetry results in type IIA strings (a short proof of that can be found in [116]). Note that one should be cautious when applies such orbifoldings and dualities at the same time [124]; the Ω projection is S-dual to the $(-1)^{F_L}$ one, but the theories obtained after orbifolding type II with them, namely type I and type IIA respectively, are not S-dual. Instead, type I is S-dual to the heterotic theory with $SO(32)$ gauge group while at strong coupling type IIA becomes M-theory.

Due to the nature of the S-duality transformation, we can see from definition (4.1.2) that the O3-, O7-, and O9-planes will stay the same under S-duality. More precisely, since the specific type of each orientifold depends also on the choice of discrete torsions, one of which is in the NS-NS sector and the other in the R-R, the

type of the orientifold may change; the perturbative definition however will remain the same. This is no longer true for the O1- and the O5-planes; instead, we will get objects corresponding to the fixed planes of a spacetime parity combined with the action of $(-1)^{F_L}$. For more information on these objects see [120, 121].

We focus now on the case of the O5-plane, since this is directly related to the subject of this chapter. Recall first that the $O5^-$-plane carries -1 D5-brane charge (we count physical charge corresponding to two D5-branes in the covering space). Hence, the S-dual object, which is defined as type IIB modded out by reflection in four coordinates and $(-1)^{F_L}$, has -1 NS5-brane charge. We will denote this object ON^-.

It seems that this way we have found a perturbative definition for an object with NS5-brane charge. In reality, if we compute the spectrum of IIB/$\mathcal{I}_4(-1)^{F_L}$ we will find that there is twisted sector localized on the fixed 5-plane of the spacetime projection. The twisted sector is a vector multiplet of 6-dimensional $(1,1)$ supersymmetry in the adjoint of $SO(2)$. Since before S-dualizing we just had an $O5^-$-plane, which cannot support any states on its world-volume without adding D-branes, we see that a natural explanation of this puzzle is that IIB/$\mathcal{I}_4(-1)^{F_L}$ actually describes the S-dual of an $O5^-$-plane with a D5-brane on it. Hence, the resulting object should have no NS5-brane charge and that is consistent with the fact that we have a perturbative CFT definition thereof. More comments on the varieties of O5-planes can be found in [120].

The object defined by IIB/$\mathcal{I}_4(-1)^{F_L}$ will be called ON^0-plane and it was analyzed in detail in [124, 125, 126]. It was shown there that the twisted sector of its type IIA analogue is a tensor multiplet of 6-dimensional $(0,2)$ supersymmetry. Note that the supersymmetry on the fixed plane is chiral for type IIA and non-chiral for type IIB; this is a consequence of the $(-1)^{F_L}$ projection which - as we mentioned previously - takes type IIA to type IIB and vice versa in a flat background. It is intriguing that the spectrum of these objects is exactly the same with that of the corresponding NS5-brane and thus, one may be tempted to think that these orbifolds provide perturbative description of NS-branes. Unfortunately this is too optimistic; the correct way to view

these fixed planes is as the analogue of orientifolds for NS5-branes. Now, the pure orientifold of this type would have -1 unit of NS5-brane charge and thus would be invisible in all perturbative considerations. The $\mathcal{I}_4(-1)^{F_L}$ orbifolding assumes validity of the perturbative description of strings with no NS-NS fluxed and - as we argued above - it automatically contains a "hidden" NS5-brane which cancels its negative charge. Thus, the theory on the fixed plane will be the theory on the corresponding NS5-brane, i.e. either the $N_6 = (0,2)$ in type IIA or the $N_6 = (1,1)$ in type IIB, with $SO(2)$ gauge symmetry. Recall that placing a physical D-brane on an Op-plane also results in an $SO(2)$ symmetry (instead of the $U(2)$ symmetry one would get if there was one physical D-brane, i.e. two D-branes on the covering space, without the orientifold).

Since $SO(2) \cong U(1)$, which is the gauge symmetry of a single NS5-brane, one may erroneously conclude that the $\mathcal{I}_4(-1)^{F_L}$ orbifold provides a perturbative description of the NS5-brane!

In general, placing n physical NS5-branes on an ON^0-plane results in a 6-dimensional theory with $SO(2n+2)$ gauge symmetry. In type IIB this is an ordinary Yang-Mills theory with $N_6 = (1,1)$ supersymmetry while in type IIA the symmetry is realized as a non-abelian theory of self-dual antisymmetric 2-forms acting on the tensionless strings of $N_6 = (0,2)$ theory.

The most important feature of the ON^0-plane is the twisted sector. Since its field content is the same as that of a type IIB NS5-brane, all D-branes that can end on the NS5-brane can also end on the ON^0-plane. Note that this is in contrast with what happens for ordinary Op-planes; as there is no twisted sector for the Ω projection, there are no fields localized on their world-volume and accordingly they cannot support any D-branes. A D-brane ending on an ON^0-plane can be either electrically or magnetically charged with respect to the twisted $SO(2)$ gauge field. In particular, a D-string on the ON^0-plane is a point-like electric source for the gauge field whereas a D3-brane is a magnetic 2-brane [5] .

[5]Indeed, the magnetic dual of a point-like electric charge in six dimensions is a 2-dimensional object.

A detailed analysis of the possible D-branes ending on the orbifold plane was presented in [126], where the machinery of boundary states was employed. It was shown that there are two types of $SO(2)$ charge, negative and positive, and a configuration involving D-branes of both kinds is still supersymmetric. In the case of D3-branes this can be easily seen by switching to an S-dual configuration [127].

Furthermore, it was shown in [128] that if there are n_+ and n_- D3-branes of positive and negative $SO(2)$ charge respectively on the ON^0-plane, the gauge symmetry realized on the D3-branes is $U(n_+) \times U(n_-)$. This is due to the fact that the vector multiplet coming from open strings stretching between D3-branes of different charge is removed by the orbifold projection. A heuristic way to see this is by considering the Weyl subgroup of the gauge group, which corresponds to the indistinguishability of the D-branes. Since branes with different charge are clearly distinguishable, the Weyl group is not $S_{n_+ + n_-}$ but rather $S_{n_+} \times S_{n_-}$.

Note that the world-volume theory of the D3-brane is $\mathcal{N} = 2$. The open string states decompose into the vector multiplets we mentioned above and hypermultiplets. The latter are projected out when they arise from strings that connect D3-branes of the same charge while they survive when they arise from open strings connecting mixed D3-branes. In fact this is true when the D3-brane are half-infinite, i.e. have one end supported on the ON^0-plane while the other end extends to infinity. When the other end is also supported on an NS5-branes or another ON^0-plane the situation is different. For the case of an ON^0-plane at the other end, a variety of possibilities exists corresponding to the freedom of choosing the D3-brane charges on this orbifold plane. Here we are only interested in the case where the other end is supported on an NS5-brane; in this case there are no hypermultiplets at all (more details on these issues can be found in [127, 128].).

4.2 Gauge theories from brane configurations: an ultra-brief review

As we have reviewed in chapter 2, on a stack of parallel and coincident Dp-branes the massless modes of open strings are described by the dimensional reduction of the $\mathcal{N}_{10} = 1$ SYM in 10-dimensions to (p+1)-dimensions. This class of theories however has a large amount of supersymmetry, namely 16 supercharges, and thus one needs to conceive ways to break supersymmetry so that the dynamics are richer and the theories more phenomenologically appealing. In addition, stacks of parallel branes result to simple unitary groups and it is of interest to extend the possibilities so that symplectic and orthogonal as well as product of simple groups can be considered.

We have seen that a way to engineer semi-simple gauge groups is by placing the stack of D-branes on an orbifold singularity. Then, there is a partial or total breaking of supersymmetry and one can obtain gauge groups which are products of unitary groups. The precise form of the theory we get is determined by algebraic properties of the orbifold group, like its representation ring and if it is embedded in $SU(2), SU(3)$ or $SU(4)$.

Unfortunately, these constructions, though powerful, tend to obscure the intuitive connections one hopes to get from D-branes concerning their spacetime characteristics and properties of the effective field theory thereof. For example, we have seen that for a stack of coincident and parallel D3-branes, the Coulomb branch is just the 6-dimensional transverse space and separating the branes corresponds to picking a particular point in the Coulomb branch. Thus, we would like a framework where such a heuristic connection is preserved even though the theories are more complicated than $\mathcal{N} = 4$ SYM with $U(N)$ gauge symmetry.

Such a framework was provided in the pioneering work of Hanany and Witten [129]. Subsequently, there was a tremendous activity in generalizing the original setup of [129] for various situations, involving theories with different amounts of supersymmetry in diverse dimensions and with a wide variety of gauge groups. A large number of results concerning all these theories was obtained with relative easiness and

without complicated calculations. There actually lies the power of this method; by mapping properties of the effective D-brane field theories to geometric characteristics of the corresponding brane setup, results that would be obtain directly, i.e. using the Lagrangian description of the theories, after a substantial amount of work, can be found easily; sometimes just by inspecting the geometry of the brane setup.

4.2.1 Hanany-Witten brane setups

The basic configuration of [129] consists of a number k of parallel NS5-branes in type IIB string theory, which extend along directions X^1, X^2, X^3, X^4, X^5 and may have different positions $X^6_i, i = 1, \ldots, k$, while for simplicity we first assume that they have the same position in the rest three directions. It is well-known that D3-branes can end of NS5-branes (such statements are easily shown by starting with the case of a fundamental string ending on a Dp-brane and using appropriate combinations of dualities). Thus we can imagine that between consecutive NS5-branes, say the i-th and the $i+1$-th, where $i = 1, \ldots, k-1$, we stretch n_i D3-branes so that they extend along X^1, X^2 and X^6. The numbers n_i may be different. Note that the D3-branes and the NS5-brane have two spatial directions in common.

This configuration preserves $1/4$ of the original supersymmetry; the NS5-branes are BPS objects and break $1/2$ of the bulk supersymmetry while the D2-branes breaks this further to $1/4$ of the original supersymmetry. We can also put D5-branes in the above background without extra reduction of supersymmetry. This can be done by placing the D5-branes so that they are point-like on the X^3, X^4, X^5, X^6 directions. Notice that they share two directions with both the D2-branes and the NS5-branes. In general, we can place m_i D5-branes with X^6 coordinate between those of the i-th and $i+1$-th NS5-brane.

The massless fluctuations on the D3-branes are described by a theory with $32/4 = 8$ supercharges. Moreover, for D3-branes which extend indefinitely in all directions this would be a $3+1$-dimensional theory. Since however in the above setup the X^6 direction on the D3-branes is finite, we expect that at low-energies this direction will be invisible and that the effective field theory will be only $2+1$-dimensional. Thinking

of this as compactification, we see that the gauge coupling of the D3-branes stretched between the i-th and $i+1$-th NS5-brane will be

$$\frac{1}{g_i^2} = \frac{|X_i^6 - X_{i+1}^6|}{g_s}, \tag{4.2.3}$$

since the gauge coupling of an infinite D3-brane is equal to the string coupling g_s.

The parameters specifying the positions of the 5-branes correspond to massless fields on the 5-branes. Hence, in principle, their fluctuations should be taken into account when one studies the the D3-brane effective world-volume. It should be evident however that since the 5-branes are infinite in two directions not shared by the D3-branes, they can be thought of as much "heavier" objects whose dynamics can be neglected compared with the D3-brane dynamics. The way to make this rigorous is by taking a decoupling limit, which in this case should decouple the bulk (i.e. gravity) modes, the massive string excitations and the massless excitations on the 5-branes. Then, the only fluctuating degrees of freedom will be those on the lowest-dimensional brane, i.e. the D3-brane, while the fields on the 5-branes will correspond to frozen classical backgrounds and coupling constants for the D3-brane theory.

The precise decoupling limits are as follows: we take M_{pl} and M_s to infinity so that gravity (whose coupling in 10-dimensions is $G \sim k^2 \sim M_{pl}^{-8}$ and hence it vanishes at the IR where the Planck mass becomes big [6]) and the massive string modes (whose scale is set by $M_s \sim a'^{-1/2}$) are decoupled. These limits should be taken so that the coupling on the D3-brane is kept fixed and we still have a non-trivial theory on the D3-branes. However, we want the effective field theory on the D3-branes to be 3-dimensional and thus we need to make the mass of the Kaluza-Klein type excitations along the finite intervals between the NS5-branes large enough so that there are no signatures of an extra finite dimension to the low-energy 3-dimensional observer. Since the Kaluza-Klein modes have masses that go like $1/L$, where L is the distance in the X^6 coordinate of the NS5-branes that support the D3-brane we study, while the mass scale in the 3D-theory is set by the Yang-Mills coupling constant [7] $g^2 = g_s/L$,

[6]Equivalently, gravity is an irrelevant interaction and thus it is free at low energies.

we conclude that we need weak string coupling $g_s << 1$ in order for the Kaluza-Klein modes to decouple. Finally, the gauge couplings on the NS5-branes and the D5-branes are $1/M_s^2$ and g_s/M_s^2 respectively and they both vanish in the decoupling limit we have taken. This demonstrates that our assumptions about neglecting the fluctuations on the 5-brane are well-based and hence the IR behavior of this brane configuration is indeed governed by the effective field theory of the massless modes on the D3-branes.

We thus have a 3-dimensional theory with $N_3 = 4$ supersymmetry. Since the $\mathcal{N} = 2$ theory in 4-dimensions has the same supersymmetry, we can use the corresponding multiplets, which we discussed in chapter 2, to organize the spectrum of the theory on the D3-branes. Evidently, we have vector multiplets coming from strings connecting D3-branes. More precisely, we obtain a gauge group factor $U(n_i)$ from the n_i D3-branes that are stretched between the i-th and $i+1$-th NS and hence the total gauge groups is $\prod_{i=1}^{k-1} SU(n_i)$. Note that a $N_3 = 4$ vector multiplet has a gauge field and 3 scalars in its bosonic content. The scalars parameterize the positions of the D3-branes on the NS5-brane, i.e. along X^3, X^4, X^5.

In addition there are two types of massless hypermultiplets. One arises when D3-branes at opposite sides of the same NS5-brane are brought very close. Intuitively one expects that this is due to a fundamental string that stretches between the D3-branes and passes through the NS5-brane. This string gives rise to a BPS state that becomes massless when the D3-brane meet. Since however the string is close to an NS5-brane, which involves a strongly coupled region exactly at the NS5-brane location, this argument - based on semi-classical considerations - may not be valid. In [129] the existence of this hypermultiplet was conjectured and further arguments, based mostly on consistency with expectations from field theory, were presented in its support. The validity of this conjecture was firmly established in [130] with a careful analysis of D-branes in the background of NS5-branes, where the exact CFT

[7]Recall that a gauge theory in 3-dimensions is super-renormalizable and it has a dimensionfull gauge coupling g^2 with mass dimension 1 where the corresponding mass is the UV cutoff of the theory.

description of the latter was employed.

Since there would be one such hypermultiplet for each D3-brane pair between a given NS5-brane, we expect that the full theory on the D3-branes will contain a hypermutiplet in the $(n_1, \overline{n_2}) \oplus (n_2, \overline{n_3}) \oplus \ldots \oplus (n_{k-2}, \overline{n_{k-1}})$ representation of the gauge group.

The second type of hypermultiplets arises from fundamental strings that stretch between the D3-branes and the D5-branes. The existence of such states, which become massless when the separation of the D3-branes with the D5-branes in the X^3, X^4, X^5 directions vanish, is straightforward to establish in the context of perturbative string theory. A single D5-brane between the i-th and $i+1$-th NS5-brane will give rise to a hypermultiplet in the fundamental representation of the corresponding gauge group, i.e. $SU(n_i)$. If there are f_i D5-branes, the $SU(f_i)$ gauge symmetry of the D5-branes will be a flavor symmetry from the D3-brane point of view and thus we will obtain f_i hypermultiplets (all of them in the fundamental of the gauge symmetry group of the D3-branes) in the fundamental of the flavor group $SU(f_i)$. The fact that the gauge symmetry on the D5-branes is perceived as a global symmetry on the D3-branes is due to the fact that in the decoupling limit the gauge bosons on the D5-branes do not interact and thus the gauge symmetry becomes global.

This is the basic brane setup, also known as the Hanany-Witten brane configuration. Application of T-dualities yields a family of p-dimensional gauge theories with 8 supercharges. These theories arise from brane setups with Dp-branes and D(p+2)-branes replacing the D3-brane and D5-brane of the original Hanany-Witten model. Note than $p = 1, 2, 3, 4, 5$; however the case of $p = 5$ is more involved and it actually leads to the so-called brane web models [131].

There are numerous applications of this type of engineering gauge theories from brane setups (see [111] for a extensive review). Since lack of space does not permit even a synopsis of these results, we will restrict ourselves in giving just the flavor of the subject. For example, it is straightforward to identify the Coulomb branch of the D3-brane gauge theory, which is the branch where the scalars in the vector multiplet have non-zero vevs, with the positions of the D3-branes on the NS5-branes. This is of

course a fancy way of saying that the scalars in the vector multiplets parameterize the points where the D3-branes are suspended on the NS5-branes along the X^3, X^4, X^5 directions. The very first application of this brane model was that since the D3-branes appear also as magnetic monopoles on the NS5-branes, i.e. their positions parameterize the monopole moduli space of the 6-dimensional field theory on the NS5-branes, there should be a correspondence between the Coulomb branch of the D3-brane theory and the monopole moduli space of the NS5-brane theory. Indeed, a very simple example [132] of such correspondence along with some generalizations [133] were already known from purely field theoretic analysis; the power of brane setups is that they can be used to explain or motivate such non-trivial correspondences without the need of doing any computation at all. In the above case, the correspondence we mentioned can be found just be inspecting the brane setup! Using this logic [129] presented further generalizations of this correspondence whose proof within the Lagrangian description of quantum field theory may be highly non-trivial.

This example clearly demonstrates the great potential of brane setups in studying aspects of certain quantum field theories whose analytic examination may be prohibitive. Hence, it would be very important to know if other classes of theories that can be engineered in string theory by different means, have also a realization in terms of brane setups. In the following section we will review what is known about the brane realization of the theories on D-branes probing orbifolds of the form $\mathbb{C}^2/\Gamma$ with $\Gamma \subset SU(2)$.

4.2.2 Inclusion of orientifolds

It is straightforward to include orientifolds in Hanany-Witten type configurations and thus obtain products of orthogonal and symplectic groups. What is new in this case is that when an orientifold crosses an NS5-brane its NS-NS $\mathbb{Z}_2$ discrete torsion is switched. This was originally found in [134] as a requirement imposed by field theory expectations. Moreover, by applying a sequence of S-dualities one can show that all discrete torsions that characterize the orientifold type can be obtained from the Op^- plane by crossing it with appropriate D-branes and NS5-branes [120].

Using the conformal field theory description of NS5-branes and the formalism of crosscap states [8], the change of the orientifold type when an NS5-brane is crossed was proved in [135] from the world-sheet perspective.

4.3 Branes on orbifolds and brane setups

The connection between brane setups and branes on orbifolds is an application of T-duality. The very basic idea is that T-duality can exchange backgrounds with purely geometric interpretation, like orbifolds, with backgrounds corresponding to certain branes. In combination with the well-known transformation properties of D-branes under T-duality we have a set of rules which can map physical quantities and parameters of the brane setup theory to those of the brane theory on the orbifold.

The most well-known example of the above connection is between ALF spaces and NS5-branes. This is well-established in general; nevertheless, there are still some aspects of this correspondence that are still puzzling. We will this discuss this case in some detail.

4.3.1 NS5-branes and T-duality

The fact that NS5-branes on a transverse circle are T-dual to the ALF geometry, with the dual circle being the x^4 coordinate in the ALF metric (2.3.22), was discovered in [136]. More precisely, it was shown that the CFT describing string propagation on an blown-up orbifold of the form $\mathbb{C}^2/\Gamma, \Gamma \subset SU(2)$ is T-dual to the CFT describing string propagation in the near horizon geometry of the NS5-brane. The last one is a WZW model on the $SU(2) \cong S^3$ group manifold at level $k = N-2$ with N being the number of NS5-branes along with a linear dilaton with background charge $Q = \sqrt{2/N}$ and four free fermions.

Before elaborating further on the T-duality, let us mention that there are some subtleties which have been investigated in [137, 138]. They are related to the fact

[8]The crosscap states represent orientifolds in a given CFT, i.e. they are the corresponding "boundary states".

that naively T-dualizing the ALF background with respect to the isometry in the x^4 direction, which defines a topologically trivial 1-cycle since the size of the circle vanishes at the centers of the ALF metric and hence string winding is not conserved, results in a configuration representing a "smeared" NS5-brane on the T-dual circle, which apparently conserves compact momentum. The precise way the NS5-branes are localized on the transverse circle, thus breaking translational invariance so that momentum is not conserved, is still not entirely clear. Actually, this seems to be a particular case of applying the Buscher T-duality rules [139, 140] to isometries along topologically trivial circles on which string winding is not conserved and hence the T-dual background should not be translationally invariant in the circle. It would be interesting to investigate the general situation further. In any case, we will accept the T-duality rules as they appear in the literature and leave these subtleties for future study.

Starting from the k-centered multi-Taub-NUT (ALF) metric given by (2.3.22) and (2.3.24), we can T-dualize with respect to the isometry along the x^4 coordinate. Note that since ALF spaces are Ricci-flat it is straightforward to embed them in string theory. We can assume that the ALF space spans the directions x^1, x^2, x^3, x^4 (in (2.3.22) we use the notation $\vec{x} = x^1, x^2, x^3$), i.e. they are 5-branes. We will use for convenience the symbol x^4 for the coordinate of the T-dual circle as well; of course, as a world-sheet field this is the Hodge-dual scalar of the original x^4 coordinate [141].

The T-dual configuration consists of k NS5-branes which are point-like in the directions x^1, x^2, x^3, x^4 and their position in x^1, x^2, x^3 is given by the centers $\vec{x}_i, i = 1, \ldots, k$ of the ALF metric. When two centers merge and the associated 2-cycle collapses to zero size, the corresponding NS5-branes have also the same position in the transverse space x^1, x^2, x^3. In particular, when all centers are identified and thus locally we obtain an ALE singularity, all of the T-dual NS5-branes are localized on the T-dual circle. However, their positions on this circle may be different and, contrary to naive expectations, the NS5-branes are not coincident in general. What accounts for these moduli in the T-dual geometric description ?

To answer this, let us first denote by d_i the distance between the i-th and the

$i+1$-th NS5-brane on the T-dual circle. Obviously we have $\sum_{i=1}^{k} d_i = \text{const.}$.

A non-trivial property of this T-duality is that the distances d_i correspond to the fluxes of the B-field through the 2-cycles $S_{i,i+1}^{(k-1)}$ for $i = 1, \ldots, k$, i.e. [9]

$$d_i = \int_{S_{i,i+1}^{(k-1)}} B. \tag{4.3.4}$$

This is of course expected from the fact that homologically only the first $k-1$ 2-cycles are linearly independent. In fact, when the circle on which the NS5-brane are placed shrinks to zero size, we have $\sum_{i=1}^{k} d_i = 0$ and we can write $S_{k,1}^{(k-1)} = -\oplus_{i=1}^{k-1} S_{i,i+1}^{(k-1)}$, where by $\oplus$ we mean homological summation. In the T-dual geometry this limit corresponds to making the radius of the x^4 coordinate infinite. In this limit the ALF space transmutes into the ALE one and the relation between the 2-cycles written above is well-known; in the light of the McKay correspondence, it is actually the topological analogue of the formula for the negative of the highest root, i.e. $-\theta = -\sum_{i=1}^{k-1} a_i$ where $-\theta$ the extra node of the affine Dynkin diagram that is to be associated with the linearly dependent 2-cycle $S_{k,1}^{(k-1)}$ and a_i are the simple roots associated to linearly independent 2-cycles $S_{i,i+1}^{(k-1)}$.

Note that the flux is non-zero even though the size of the 2-cycle is vanishing. We should mention here that these are fluxes due to flat B-fields, i.e. $dB = 0$, and hence they depend only on the homology class of the 2-cycle. Hence, they are similar to Wilson lines or, more correctly, Wilson surfaces. In particular, they don't contribute to the energy of the background. This is of course consistent with the fact that the configuration of NS5-branes on the circle is BPS and so moving the NS5-branes is a marginal deformation that doesn't cost energy. Since the B-fields on the 2-cycles are due to twisted sector NS-NS moduli in the orbifold description of the background, we see that these geometric deformations, i.e. moving the NS5-branes on the circle, correspond to changing the vevs of the corresponding twisted fields in the T-dual CFT.

The correspondence of the distances between the NS5-branes with the B-field

[9] We use the convention e $S_{k,k+1}^{(k-1)} \cong S_{k,1}^{(k-1)}$ for convenience.

fluxes is very important. As pointed out in [142], the ordinary orbifold compactifications of string theory, for example on orbifold limits of K3, correspond to situations where the B-field fluxes are non-zero. The corresponding conformal field theories are of course well-behaved. On the other side, it is known that type IIA theory on a K3 exhibits gauge symmetry enhancement at singularities of the moduli space [143, 144]. The enhancement is due to extra massless states that arise from D2-branes wrapping the vanishing 2-cycles in the K3; thus, they correspond to non-perturbative states out of reach of the conformal field theory description. These theories are known to be T-dual to world-volume theories of coincident NS5-brane in type IIB. [10]

We see that when all NS5-branes are coincident and hence the associated T-dual moduli, i.e. the B-field fluxes, are zero, we have a situation where the conformal field theory description breaks down as it is unable to predict the extra massless states. Thus, the statement that string theory makes sense on singular manifolds needs refining; the singularities associated to purely geometric characteristics of the manifold are not singular from the string point of view because string propagation is determined not only by metric properties of the underlying spacetime manifolds, but also by other moduli which are not taken into account when why specify a background solely in geometrical terms. When these moduli are vanishing we obtain intrinsically stringy singularities whose conformal field theory description is expected to be problematic, probably due to strong coupling effects [11] . Note that this idea is consistent with the heuristic argument that it is due to the extended nature of strings that their propagation on orbifolds is non-singular; the string feels that B-field fluxes exactly because it is a 1-dimensional extended object and thus it bears a natural coupling with this field.

[10]There is also a similar story with type IIB on K3 which gives rise to self-dual tensionless strings from D3-branes wrapping the vanishing 2-cycles and which appear also on type IIA NS5-branes that are put on top of each other; the tensionless string in this picture is due to D2-branes that end on the NS5-branes and give rise to strings on the NS5-brane world-volume.

[11]This is due to the appearance of a linear dilaton background when one examines the region very close to the singularity [136] or the NS5-branes [188, 189] .

4.3.2 Elliptic models and D-branes on A-type singularities

We apply now the T-duality rule of the previous subsection to the so-called elliptic model. This is just the original Hanany-Witten brane setup with the X^6 coordinate been compactified. Let us ignore the possibility of putting D5-branes and concentrate only on the D3-branes stretched between the NS5-branes along the X^6 direction. We also assume that the same number of D3-branes are between each pair of NS5-branes and we go to the origin of the Coulomb branch where the D3-branes are connected and hence they can leave the NS5-branes and move in the X^7, X^8, X^9 directions. This motion corresponds to the Higgs branch of the D3-brane world-volume theory.

In the case where the NS5-branes are localized on the circle - as we implicitly assume throughout - the 2-cycles of the T-dual geometry are vanishing and thus we are in the orbifold limit of A_{k-1} ALE, that is $\mathbb{C}^2/\mathbb{Z}_k$. Furthermore, the D3-branes became D2-branes which are point-like in the X^6, X^7, X^8, X^9 directions. These are also the directions into which the orbifold extends. Hence, the branes probe the orbifold geometry and the theory is in the Higgs branch, as we discussed in chapter 2 and in line with the brane setup picture above. When the theory on the NS5-branes is in the Coulomb branch, i.e. the D3-branes are disconnected and stretch between the NS5-branes, T-duality maps the D3-branes to fractional D2-branes. We see that the Hanany-Witten setup is a powerful tool that enables us in a sense to look inside the orbifold fixed point (note that we keep the NS5-branes at the same point in the transverse $\mathbb{R}^3$ and so the 2-cycles of the T-dual ALE are not resolved).

It is a matter of inspection to verify that the gauge theories are actually the same. In the orbifold picture we get a unitary group for each irrep and a hypermultiplet in the bifundamental of irreps that correspond to adjacent nodes on the associated quiver; in the brane picture, each set of D3-branes between adjacent NS5-branes give a unitary group and the bifundamental hypermultiplet comes from strings connecting these D3-branes with the ones lying at the other side of the NS5-branes. Further details on this correspondence can be found in [146, 147].

It is natural to ask if there are brane setups that realize the theories on D-brane

probes of $\mathbb{C}^2/\Gamma$ where Γ the other finite subgroups of $SU(2)$. This question is equivalent of course to asking what are the objects (analogues of the NS5-branes) whose spacetime description is T-dual to such orbifolds.

Note that the $\mathbb{Z}_k$ quiver has a very intriguing similarity with the spacetime arrangement of the NS5-branes in the brane setup dual. This observation was employed in [148] which discussed brane setups for non-abelian subgroups of $SU(3)$. This chapter is based on an application of a similar idea for the exceptional subgroups of $SU(2)$. Before explaining our results however, we have to review the situation for the D-type subgroups of $SU(2)$ whose brane setups are also known. This will provide further evidence for the validity of the correspondence we hinted above relating the quiver of the finite group with the configurations of branes in the dual description.

4.3.3 ON-planes and D-branes on D-type singularities

The brane configuration that reproduces the D-type quiver gauge theory was found in [128]. It relies on the properties of the ON^0-plane which we have discussed in the previous section.

Let us consider the brane setup shown in part IV of figure 4-2. There are two parallel ON^0-planes at the endpoints of an interval and k parallel NS5-branes in between. Note that this interval is just the circle of the elliptic model modded out by the reflection due to the ON^0-planes; the endpoints on which the ON^0-planes are placed are the fixed points of the reflection. In addition, there are n D3-branes of positive $SO(2)$ charge and the same number of negatively charged D3-branes between the first ON^0-plane and the adjacent NS5-branes and similarly for the last slot. Finally, we place $2n$ D3-branes between each adjacent pair of NS5-branes. Using the rules of Hanany-Witten brane configurations in conjunction with the properties of D-branes ending on ON^0-planes, we see that the theory we obtain has an $U(n)^4 \times \prod_{i=1}^{k-1} U(2n)$ gauge symmetry and bifundamental hypermultiplets corresponding to D3-branes supported at the two sides of each NS5-brane.

Comparing with the spectrum of $4kn$ D2-branes probing the orbifold $\mathbb{C}^2/\hat{\mathcal{D}}_k$ as given by the corresponding quiver in figure 2-1, we observe that they are the same! It

is evident that the property of ON^0-planes to support two different types of D-branes leading to a semi-simple gauge group is very crucial in arriving at this correspondence.

It is straightforward to extend the discussion of the previous subsection for this case. The D3-branes suspended between the ON^0-planes and the NS5-branes, are fractional branes that in the orbifold picture result from D4-branes wrapping the four vanishing 2-cycles of $\mathbb{C}^2/\hat{\mathcal{D}}_k$ corresponding to the four 1-dimensional irreps of $\hat{\mathcal{D}}_k$. The rest $k-1$ irreps of $\hat{\mathcal{D}}_k$ are 2-dimensional and the corresponding 2-cycles give rise to the fractional branes that map to D3-branes supported between adjacent NS5-branes in the brane setup. The discussion of the Higgs and Coulomb branches is also similar.

This correspondence is a consequence of the T-duality that connects the two backgrounds, i.e. the orbifold $\mathbb{C}^2/\hat{\mathcal{D}}_k$ and the configuration of parallel ON^0-planes and NS5-branes. Since the metric of the D-type ALF is not explicitly known, this T-duality has been advocated by indirect arguments, the agreement between the field theory on D-branes probes and the brane setup being one of them. Another argument based on S-duality is given in [127].

For the orbifolds of the form $\mathbb{C}^2/E_6, \mathbb{C}^2/E_7, \mathbb{C}^2/E_8$ the T-dual geometry is not known. The main objective of the rest of this chapter is is to analyze the corresponding quiver gauge theories with the hope that some hints will be obtained regarding the T-dual geometry.

4.4 Stepwise orbifold projections

In this section we discuss the method of stepwise projection which is a first step towards the construction of brane setups realizing the exceptional quiver gauge theories. Our work was motivated by [149, 148, 150] where an attempt was made to establish the brane setup which corresponds to the three-dimensional non-Abelian orbifolds $\mathbb{C}^3/\{\Gamma \subset SU(3)\}$ with $\Gamma = \Delta_{3n^2}$ and Δ_{6n^2}. The key idea was to arrive at these theories by judiciously quotienting the well-known orbifold $\mathbb{C}^3/\{\mathbb{Z}_k \times \mathbb{Z}_l \subset SU(3)\}$ whose brane configuration is the brane box model. In the process of this quotienting, a non-trivial $\mathbb{Z}_3$ action on the brane box is required.

Our objective is first to systematize the aforementioned quotienting idea and secondly to apply it for the particular cases of E-type singularities.

The material in this section is based on [3]. We should mention that independent and variant forms of this method have been in germination (see e.g. [151]). In particular, a very similar idea has been presented in [152].

4.4.1 The orbifold projection in detail

The general methodology of how the finite group structure of the orbifold projects the gauge theory has been discussed in chapter 2. In our forth-coming discussion we shall not use the abstract formulation of [59] but the more explicit one in terms of representation matrices. Throughout we shall focus on two dimensional orbifolds $\mathbb{C}^2/\{\Gamma \subset SU(2)\}$. Recall that the parent theory has an $SU(4) \cong Spin(6)$ R-symmetry from the $\mathcal{N} = 4$ SUSY, where the $U(n)$ gauge bosons A^{μ}_{IJ} with $I, J = 1, ..., n$ are R-singlets, the Weyl fermions $\Psi^{i=1,2,3,4}_{IJ}$ are in the fundamental $\mathbf{4}$ of $SU(4)$ and the scalars $\Phi^{i=1,..,6}_{IJ}$ are in the antisymmetric $\mathbf{6}$.

The orbifold imposes a projection condition upon these fields due to the finite group Γ. Let $R^{reg}_{\Gamma}(g)$ be the regular representation of $g \in \Gamma$:

$$R^{reg}_{\Gamma}(g) := \bigoplus_i \Gamma_i(g) \otimes \mathbb{I}_{\dim(\Gamma_i)}$$

where $\{\Gamma_i\}$ are the irreducible representations of Γ. In matrix form, $R^{reg}_{\Gamma}(g)$ is composed of blocks of irreps, with each of dimension j repeated j times. Therefore it is a matrix of size $\sum\limits_i \dim(\Gamma_i)^2 = |\Gamma|$. Let the set of all irreps be $\text{Irreps}(\Gamma) = \{\Gamma^{(1)}_1, \ldots, \Gamma^{(1)}_{m_1}; \Gamma^{(2)}_1, \ldots, \Gamma^{(2)}_{m_2}; \ldots\ldots; \Gamma^{(n)}_1, \ldots, \Gamma^{(n)}_{m_n}\}$, consisting of m_j irreps of dimen-

sion j, then

$$R_\Gamma^{reg} := \begin{pmatrix} \Gamma_1^{(1)} & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & \Gamma_{m_1}^{(1)} & & & & & & & & \\ & & & \begin{pmatrix} \Gamma_1^{(2)} & \\ & \Gamma_1^{(2)} \end{pmatrix} & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & \begin{pmatrix} \Gamma_{m_2}^{(2)} & \\ & \Gamma_{m_2}^{(2)} \end{pmatrix} & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & \begin{pmatrix} \Gamma_1^{(n)} & & \\ & \ddots & \\ & & \Gamma_1^{(n)} \end{pmatrix}_{n\times n} & & & \\ & & & & & & & & \ddots & & \\ & & & & & & & & & \begin{pmatrix} \Gamma_{m_n}^{(n)} & & \\ & \ddots & \\ & & \Gamma_{m_n}^{(n)} \end{pmatrix}_{n\times n} \end{pmatrix}. \tag{4.4.5}$$

Of the parent fields A^μ, Ψ, Φ, only those invariant under the group action will remain in the orbifolded theory; this imposition is what we mean by *surviving the projection*:

$$\begin{aligned} A^\mu &= R_\Gamma^{reg}(g)^{-1} A^\mu R_\Gamma^{reg}(g) \\ \Psi^i &= \rho(g)^i_j \; R_\Gamma^{reg}(g)^{-1} \Psi^j R_\Gamma^{reg}(g) \\ \Phi^i &= \rho'(g)^i_j \; R_\Gamma^{reg}(g)^{-1} \Phi^j R_\Gamma^{reg}(g) \qquad \forall \;\; g \in \Gamma, \end{aligned} \tag{4.4.6}$$

where ρ and ρ' are induced actions because the matter fields carry R-charge (while the gauge bosons are R-singlets). Clearly if $\Gamma = \langle x_1, ..., x_n \rangle$, it suffices to impose (4.4.6) for the generators $\{x_i\}$ in order to find the matter content of the orbifold gauge theory; this observation we shall liberally use henceforth.

Letting $n = N|\Gamma|$ for some large N and $n_i = \dim(\Gamma_i)$, the subsequent gauge group becomes $\prod\limits_i U(n_i N)$ with a^4_{ij} Weyl fermions as bifundamentals $\left(\mathbf{n_i N}, \overline{\mathbf{n_j N}}\right)$ as well as a^6_{ij} scalar bifundamentals, where we have defined a^4_{ij} and a^6_{ij} in chapter 2. These bifundamentals are pictorially summarized in quiver diagrams whose adjacency matrices are the a_{ij}'s.

Since we shall henceforth be dealing primarily with $\mathbb{C}^2$ orbifolds, we have $\mathcal{N} = 2$ gauge theory in four dimensions [56, 59]. In particular we choose the induced group action on the R-symmetry to be $\mathbf{4} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}$ and $\mathbf{6} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}^2$. For this reason

[59, 60] we can specify the final fermion and scalar matter matrices by a single quiver characterized by the **2** of $SU(2)$ as the trivial **1**'s give diagonal 1's.

4.4.2 The method

The philosophy of the method is straightforward: say we are given a group $\Gamma_1 = \langle x_1, ..., x_n \rangle$ with quiver diagram Q_1 and $\Gamma_2 = \langle x_1, ..., x_{n+1} \rangle \supset \Gamma_1$ with quiver Q_2, we wish to determine Q_2 from Q_1 by the projection (4.4.6) by $\{x_1, ..., x_n\}$ followed by another projection by x_{n+1}. We now proceed to analyze the well-known examples of the cyclic and binary dihedral quivers in this approach.

$\hat{\mathcal{D}}_k$ quivers from A_k quivers

We shall concern ourselves with orbifold theories of $\mathbb{C}^2/\mathbb{Z}_k$ and $\mathbb{C}^2/\hat{\mathcal{D}}_k$. Let us first recall that the cyclic group $A_{k-1} \cong \mathbb{Z}_k$ has a single generator

$$\beta_k := \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{pmatrix}, \qquad \text{with} \quad \omega_n := e^{\frac{2\pi i}{n}}$$

and that the generators for the binary dihedral group D_k are

$$\beta_{2k} = \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix}, \qquad \gamma := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We further recall from chapter 2 that $\hat{\mathcal{D}}_k/\mathbb{Z}_{2k} \cong \mathbb{Z}_2$.

Now all irreps for $\mathbb{Z}_k$ are 1-dimensional (the k^{th} roots of unity), and (4.4.5) for the generator reads

$$R^{reg}_{\mathbb{Z}_k}(\beta_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_k & 0 & 0 & 0 \\ 0 & 0 & \omega_k^2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \omega_k^{k-1} \end{pmatrix}.$$

On the other hand, D_k has 1 and 2-dimensional irreps and (4.4.5) for the two generators become

$$R^{reg}_{\hat{\mathcal{D}}_k}(\beta_{2k}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} \omega_{2k}^{k-1} & 0 \\ 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} \omega_{2k}^{k-1} & 0 \\ 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} \end{pmatrix}$$

and

$$R^{reg}_{\hat{\mathcal{D}}_k}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i^{k \bmod 2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} -1 & 0 \\ 0 & -i^{k \bmod 2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & i^{k-1} \\ i^{k-1} & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & i^{k-1} \\ i^{k-1} & 0 \end{pmatrix} \end{pmatrix}.$$

In order to see the structural similarities between the regular representation of β_{2k} in $\Gamma_1 = \mathbb{Z}_{2k}$ and $\Gamma_2 = \hat{\mathcal{D}}_k$, we need to perform a change of basis. We do so such that each pair (say the j^{th}) of the 2-dimensional irreps of $\hat{\mathcal{D}}_2$ becomes as follows:

$$\Gamma^{(2)}(\beta_{2k}) = \begin{pmatrix} \begin{pmatrix} \omega_{2k}^{j} & 0 \\ 0 & \omega_{2k}^{-j} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \omega_{2k}^{j} & 0 \\ 0 & \omega_{2k}^{-j} \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \omega_{2k}^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \omega_{2k}^{-j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

where $j = 1, 2, \ldots, k-1$. In this basis, the 2-dimensionals of γ become

$$\Gamma^{(2)}(\gamma) = \begin{pmatrix} \begin{pmatrix} 0 & i^{j} \\ i^{j} & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & i^{j} \\ i^{j} & 0 \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} 0 & i^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{pmatrix}.$$

Now for the 1-dimensionals, we also permute the basis:

$$\Gamma^{(1)}(\beta_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Gamma^{(1)}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i^{k \bmod 2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i^{k \bmod 2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i^{k \bmod 2} & 0 \\ 0 & 0 & 0 & -i^{k \bmod 2} \end{pmatrix}.$$

Therefore, we have

$$R^{reg}_{\hat{\mathcal{D}}_k}(\beta_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & -1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \omega_{2k} & 0 & & 0 & 0 \\ 0 & 0 & 0 & \omega_{2k}^{-1} & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & \omega_{2k}^{k-1} & 0 \\ 0 & 0 & 0 & 0 & & 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which by now has a great resemblance to the regular representation of $\beta_{2k} \in \mathbb{Z}_{2k}$; indeed, after one final change of basis, by ordering the powers of ω_{2k} in an ascending fashion while writing $\omega_{2k}^{-j} = \omega_{2k}^{2k-j}$ to ensure only positive exponents, we arrive at

$$\begin{aligned} R^{reg}_{\hat{\mathcal{D}}_k}(\beta_{2k}) &= \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & \omega_{2k} & 0 & & 0 \\ 0 & 0 & \omega_{2k}^2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \omega_{2k}^{2k-1} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= R^{reg}_{\mathbb{Z}_{2k}}(\beta_{2k}) \otimes \mathbb{I}_2, \end{aligned} \tag{4.4.7}$$

the key relation which we need.

Under this final change of basis,

$$
R^{reg}_{\hat{\mathcal{D}}_k}(\gamma) = \begin{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \begin{pmatrix} i^{k-3} & 0 \\ 0 & i^{k-3} \end{pmatrix} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} i^{k-2} & 0 \\ 0 & i^{k-2} \end{pmatrix} & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & \begin{pmatrix} i^{k-1} & 0 \\ 0 & i^{k-1} \end{pmatrix} \\
\vdots & & & & \begin{pmatrix} i^{k \bmod 2} & 0 \\ 0 & -i^{k \bmod 2} \end{pmatrix} & & & \vdots \\
0 & \begin{pmatrix} i^{k-3} & 0 \\ 0 & i^{k-3} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \begin{pmatrix} i^{k-2} & 0 \\ 0 & i^{k-2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \begin{pmatrix} i^{k-1} & 0 \\ 0 & i^{k-1} \end{pmatrix} & 0 & 0 & 0 & 0
\end{pmatrix}. \qquad (4.4.8)
$$

Our strategy is now obvious. We shall first project according to (4.4.6), using (4.4.7), which is equivalent to a projection by $\mathbb{Z}_{2k}$, except with two identical copies (physically, this simply means we place twice as many D3-brane probes). Thereafter we shall project once again using (4.4.8) and the resulting theory should be that of the $\hat{\mathcal{D}}_k$ orbifold.

An illustrative example

Let us turn to a concrete example, namely $\mathbb{Z}_4 \to \hat{\mathcal{D}}_2$. The key points to note are that $\hat{\mathcal{D}}_2 := \langle \beta_4, \gamma \rangle$ and $\mathbb{Z}_4 \cong \langle \beta_4 \rangle$. We shall therefore perform stepwise projection by β_4 followed by γ.

Equation (4.4.7) now reads

$$
R^{reg}_{\hat{\mathcal{D}}_2}(\beta_4) = R^{reg}_{\mathbb{Z}_4}(\beta_4) \otimes \mathbb{I}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix} \otimes \mathbb{I}_2. \qquad (4.4.9)
$$

We have the following matter content in the parent (pre-orbifold) theory: gauge field A^{μ}, fermions $\Psi^{1,2,3,4}$ and scalars $\Phi^{1,2,3,4,5,6}$ (suppressing gauge indices IJ). Projection by $R^{reg}_{\hat{\mathcal{D}}_2}(\beta_4)$ in (4.4.9) according to (4.4.6) gives a $\mathbb{Z}_4$ orbifold theory, which restricts

the form of the fields to be as follows:

$$A^\mu, \Psi^{1,2}, \Phi^{1,2} = \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \square \end{pmatrix} ; \quad \Psi^3, \Phi^{3,5} = \begin{pmatrix} & \square & & \\ & & \square & \\ & & & \square \\ \square & & & \end{pmatrix} ; \quad \Psi^4, \Phi^{4,6} = \begin{pmatrix} & & & \square \\ \square & & & \\ & \square & & \\ & & \square & \end{pmatrix} \tag{4.4.10}$$

where $\square$ are 2×2 blocks. We recall from the previous section that we have chosen the R-symmetry decomposition as $\mathbf{4} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}$ and $\mathbf{6} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}^2$. The fields in (4.4.10) are defined in accordance thereto: the fermions $\Psi^{1,2}$ and scalars $\Phi^{1,2}$ are respectively in the two trivial $\mathbf{1}$'s of the $\mathbf{4}$ and $\mathbf{6}$; (Ψ^3, Ψ^4), (Φ^3, Φ^4) and (Φ^5, Φ^6) are in the doublet $\mathbf{2}$ of Γ inherited from $SU(2)$.

Indeed, the $R^{reg}_{\mathbb{Z}_4}(\beta_4)$ projection would force $\square$ to be numbers and not matrices as we do not have the extra $\mathbb{I}_2$ tensored to the group action, in which case (4.4.10) would be 4×4 matrices prescribing the adjacency matrices of the $\mathbb{Z}_4$ quiver. For this reason, the quiver diagram for the $\mathbb{Z}_4$ theory as drawn in part (I) of figure 4-1 has the nodes labelled 2's instead of the usual Dynkin labels of 1's for the A-series. In physical terms we have placed twice as many image D-brane probes. The key point is that because $\square$ are now matrices (and (4.4.10) are 8×8), further projection internal thereto may change the number and structure of the product gauge groups and matter fields.

Having done the first step by the β_4 projection, next we project with the regular representation of γ:

$$R^{reg}_{\hat{\mathcal{D}}_2}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & 0 & 0 \end{pmatrix} := \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{I}_2 & 0 & 0 \end{pmatrix}. \tag{4.4.11}$$

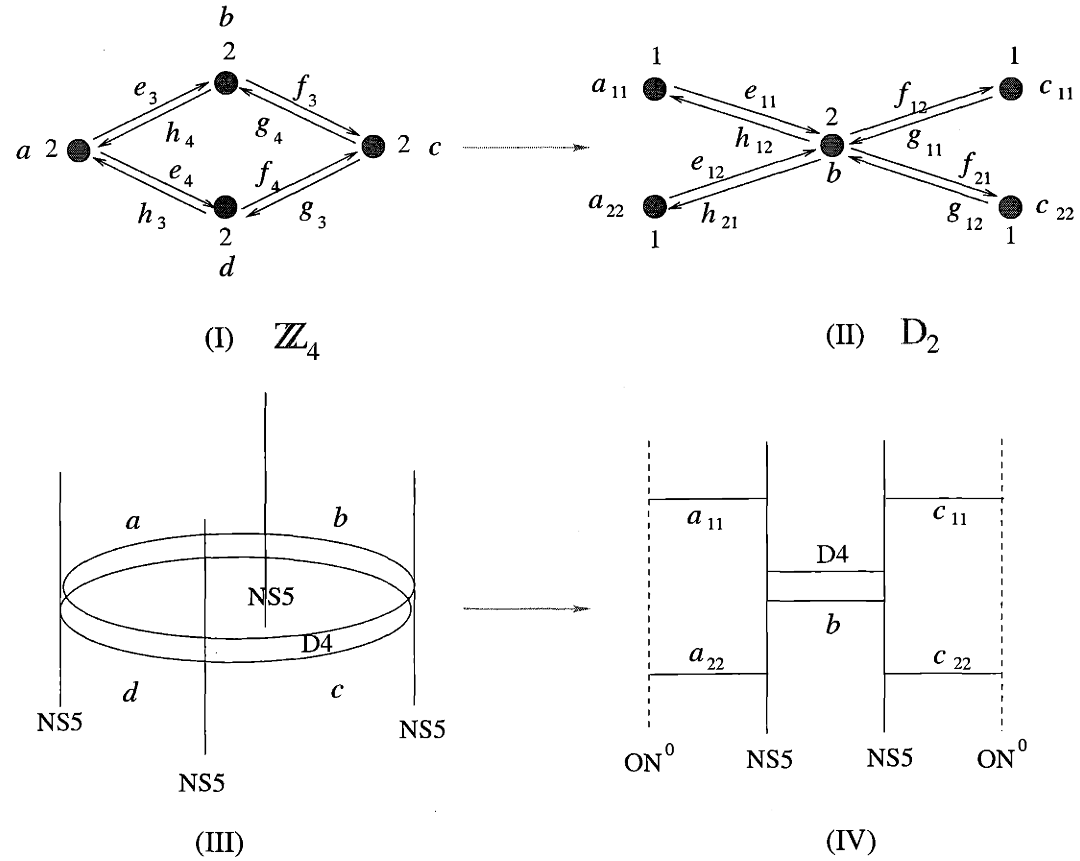

Figure 4-1: From the fact that $\hat{\mathcal{D}}_2 := \langle \beta_4, \gamma \rangle$ is generated by $\mathbb{Z}_4 = \beta_4$ together with γ, our stepwise projection, first by β_4, and then by γ, gives 2 copies of the $\mathbb{Z}_4$ quiver in Part (I) and then the $\hat{\mathcal{D}}_2$ quiver in Part (II) by appropriate joining/splitting of the nodes and arrows. The brane configurations for these theories are given in Parts (III) and (IV).

In accordance with (4.4.10), let the gauge field be

$$A^\mu := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

with a, b, c, d denoting the 2×2 blocks $\square$, (4.4.6) for (4.4.11) now reads

$$A^\mu = R^{reg}_{\hat{\mathcal{D}}_2}(\gamma)^{-1} \cdot A^\mu \cdot R^{reg}_{\hat{\mathcal{D}}_2}(\gamma) \Rightarrow$$

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{I}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{I}_2 & 0 & 0 \end{pmatrix},$$

giving us a set of constraining equations for the blocks:

$$\sigma_3 \cdot a \cdot \sigma_3 = a; \qquad d = b; \qquad \sigma_3 \cdot c \cdot \sigma_3 = c. \tag{4.4.12}$$

Similarly, for the fermions in the **2**, viz.,

$$\Psi^3 = \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi^4 = \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix},$$

the projection (4.4.6) is

$$\gamma \cdot \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix} = R^{reg}_{\hat{\mathcal{D}}_2}(\gamma)^{-1} \cdot \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix} \cdot R^{reg}_{\hat{\mathcal{D}}_2}(\gamma).$$

We have used the fact that the induced action $\rho(\gamma)$, having to act upon a doublet, is simply the 2×2 matrix γ herself. Therefore, writing it out explicitly, we have

$$i\begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{I}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{I}_2 & 0 & 0 \end{pmatrix}$$

and

$$i\begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{I}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{I}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{I}_2 & 0 & 0 \end{pmatrix},$$

which gives the constraints

$$f_4 = -h_3 \cdot \sigma_3; \qquad g_4 = \sigma_3 \cdot g_3; \qquad h_4 = -f_3 \cdot \sigma_3; \qquad e_4 = \sigma_3 \cdot e_3. \qquad (4.4.13)$$

The doublet scalars $(\Phi^{3,5}, \Phi^{4,6})$ of course give the same results, as should be expected from supersymmetry.

In summary then, the final fields which survive both β_4 and γ projections (and thus the entire group $\hat{\mathcal{D}}_2$) are

$$A^\mu = \begin{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} & & & \\ & b & & \\ & & \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix} & \\ & & & b \end{pmatrix}; \quad \begin{cases} e_3 = \begin{pmatrix} e_{11} & e_{12} \\ 0 & 0 \end{pmatrix}, & f_3 = \begin{pmatrix} 0 & f_{12} \\ 0 & f_{22} \end{pmatrix}, \\ g_3 = \begin{pmatrix} g_{11} & g_{12} \\ 0 & 0 \end{pmatrix}, & h_3 = \begin{pmatrix} 0 & h_{12} \\ 0 & h_{22} \end{pmatrix}, \end{cases}$$

$$\Psi^3 = \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi^4 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3 \cdot e_3 \\ -h_3 \cdot \sigma_3 & 0 & 0 & 0 \\ 0 & \sigma_3 \cdot g_3 & 0 & 0 \\ 0 & 0 & -f_3 \cdot \sigma_3 & 0 \end{pmatrix}. \qquad (4.4.14)$$

The key features to be noticed are now apparent in the structure of these matrices in (4.4.14). We see that the 4 blocks of A^μ in (4.4.10), which give the four nodes of the $\mathbb{Z}_4$ quiver, now undergo a metamorphosis: we have written out the components of a, c explicitly and have used (4.4.12) to restrict both to diagonal matrices, while b and d are identified, but still remain blocks without internal structure of interest. Thus we have a total of 5 non-trivial constituents $a_{11}, a_{22}, c_{11}, c_{22}$ and b, precisely the 5 nodes of the $\hat{\mathcal{D}}_2$ quiver (see parts (I) and (II) of figure 4-1). Thus nodes of the quiver merge and split as we impose further projections, as we mentioned a few paragraphs ago.

As for the bifundamentals, i.e. the arrows of the quiver, (4.4.10) prescribes the blocks $e_{3,4}, f_{3,4}, g_{3,4}$ and $h_{3,4}$ as the 8 arrows of Part (I) of figure 4-1. After the projection by γ, and imposing the constraint (4.4.13) as well as the fact that all entries

of matter matrices must be non-negative, we are left with the 8 fields $e_{11,12}$, $f_{12,22}$, $g_{11,12}$ and $h_{12,22}$, precisely the 8 arrows in the $\hat{\mathcal{D}}_2$ quiver (see Part (II) of figure 4-1).

The general case

The generic situation of obtaining the $\hat{\mathcal{D}}_k$ quiver from that of $\mathbb{Z}_{2k}$ is completely analogous. We would always have two end nodes of the $\mathbb{Z}_{2k}$ quiver each splitting into two while the middle ones coalesce pair-wise, as is shown in figure 4-2.

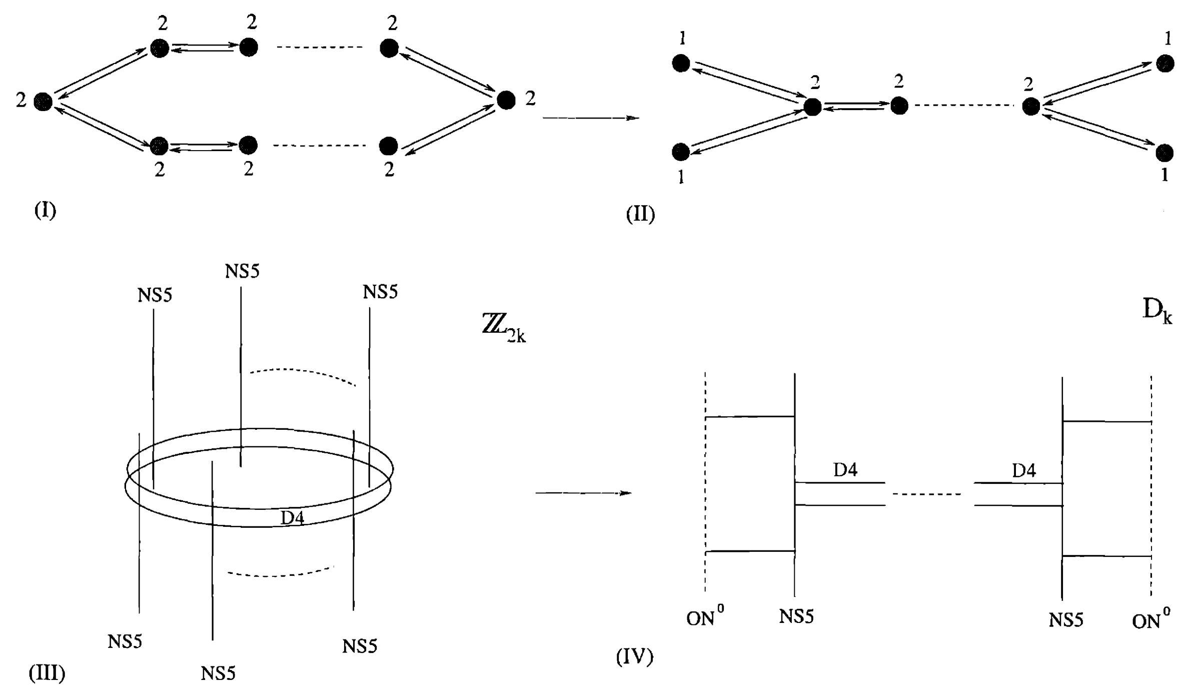

Figure 4-2: Obtaining the $\hat{\mathcal{D}}_k$ quiver (II) from the $\mathbb{Z}_{2k}$ quiver (I) by the stepwise projection algorithm. The brane setups are given respectively in (IV) and (III).

4.4.3 The E_6 Quiver from $\hat{\mathcal{D}}_2$

We now move on to tackle the binary tetrahedral group E_6 (with the relation that $E_6/\hat{\mathcal{D}}_2 \cong \mathbb{Z}_3$), whose generators are

$$\beta_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \delta := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}.$$

We observe therefore that it has yet one more generator δ than $\hat{\mathcal{D}}_2$, hence we need to continue our stepwise projection from the previous subsection, with the exception that we should begin with more copies of $\mathbb{Z}_4$. To see this let us first present the irreducible matrix representations of the three generators of E_6:

	β_4	γ	δ
$\Gamma_1^{(1)}$	1	1	1
$\Gamma_2^{(1)}$	1	1	ω_3
$\Gamma_3^{(1)}$	1	1	ω_3^2
$\Gamma_4^{(2)}$	β_4	γ	δ
$\Gamma_5^{(2)}$	β_4	γ	$\omega_3\delta$
$\Gamma_6^{(2)}$	β_4	γ	$\omega_3^2\delta$
$\Gamma_7^{(3)}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{i}{2} & \frac{i}{\sqrt{2}} & -\frac{i}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{2} & -\frac{i}{\sqrt{2}} & \frac{i}{2} \end{pmatrix}$

The regular representation for these generators is therefore a matrix of size $3 \cdot 1^2 + 3 \cdot 2^2 + 3^3 = 24$, in accordance with (4.4.5).

Our first step is as with the case of $\hat{\mathcal{D}}_2$, namely to change to a convenient basis wherein β_4 becomes diagonal:

$$R_{E_6}^{reg}(\beta_4) = R_{\mathbb{Z}_4}^{reg}(\beta_4) \otimes \mathbb{I}_6. \tag{4.4.15}$$

The only difference between the above and (4.4.9) is that we have the tensor product with $\mathbb{I}_6$ instead of $\mathbb{I}_2$, therefore at this stage we have a $\mathbb{Z}_4$ quiver with the nodes labeled 6 as opposed to 2 as in Part (I) of figure 4-1. In other words we have 6 times the usual number of D-brane probes.

Under the basis of (4.4.15),

$$R^{reg}_{E_6}(\gamma) = \begin{pmatrix} \Sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{I}_6 \\ 0 & 0 & \Sigma_3 & 0 \\ 0 & i\mathbb{I}_6 & 0 & 0 \end{pmatrix} \quad \text{where} \quad \Sigma_3 := \sigma_3 \otimes \mathbb{I}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.4.16}$$

Subsequent projection gives a $\hat{\mathcal{D}}_2$ quiver as in part (II) of figure 4-1, but with the nodes labeled as $3, 3, 6, 3, 3$, three times the usual. Note incidentally that (4.4.15) and (4.4.16) can be re-written in terms of regular representations of $\hat{\mathcal{D}}_2$ directly: $R^{reg}_{E_6}(\beta_4) = R^{reg}_{\hat{\mathcal{D}}_2}(\beta_4) \otimes \mathbb{I}_3$ and $R^{reg}_{E_6}(\gamma) = R^{reg}_{\hat{\mathcal{D}}_2}(\gamma) \otimes \mathbb{I}_3$. To this fact we shall later turn.

To arrive at E_6, we proceed with one more projection, by the last generator δ, the regular representation of which, observing the table above, has the form (in the basis of (4.4.15))

$$R^{reg}_{E_6}(\delta) = \begin{pmatrix} S_1 & 0 & S_2 & 0 \\ 0 & \omega_8^{-1}P & 0 & \omega_8^{-1}P \\ S_3 & 0 & S_4 & 0 \\ 0 & -\omega^8 P & 0 & \omega_8 P \end{pmatrix} \tag{4.4.17}$$

where

$$S_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes R^{reg}_{\mathbb{Z}_3}(\beta_3), \quad S_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_3 := -i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_4 := i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I}_3$$

and

$$P := R^{reg}_{\mathbb{Z}_3}(\beta_3) \otimes \frac{1}{\sqrt{2}}\mathbb{I}_2; \quad \text{recalling that} \quad R^{reg}_{\mathbb{Z}_3}(\beta_3) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}.$$

The inverse of (4.4.17) is readily determined to be

$$R^{reg}_{E_6}(\delta)^{-1} = \begin{pmatrix} \tilde{S}_1 & 0 & -S_3 & 0 \\ 0 & \frac{1}{2}\omega_8 P^{-1} & 0 & -\frac{1}{2}\omega_8^{-1}P^{-1} \\ S_2^T & 0 & -S_4^T & 0 \\ 0 & \frac{1}{2}\omega_8 P^{-1} & 0 & \frac{1}{2}\omega_8^{-1}P^{-1} \end{pmatrix}, \qquad \tilde{S}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes R^{reg}_{\mathbb{Z}_3}(\beta_3)^{-1}.$$

Thus equipped, we must use (4.4.6) with (4.4.17) on the matrix forms obtained in (4.4.14) (other fields can of course be checked to have the same projection), with of course each number therein now being 3×3 matrices. The final matrix for A^μ is as in (4.4.14), but with

$$a_{11} = \begin{pmatrix} a_{11(1)} & 0 & 0 \\ 0 & a_{11(2)} & 0 \\ 0 & 0 & a_{11(3)} \end{pmatrix}_{3\times 3}; \quad c_{11} = c_{22} = a_{22}; \quad b = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}_{6\times 6}$$

where a_{22}, c_{ii} are 3×3 while b_{ii} are 2×2 blocks. We observe therefore, that there are 7 distinct gauge group factors of interest, namely $a_{11(1)}, a_{11(2)}, a_{11(3)}, a_{22}, b_{11}, b_{22}$ and b_{33}, with Dynkin labels $1, 1, 1, 3, 2, 2, 2$ respectively. What we have now is the E_6 quiver and the bifundamentals split and join accordingly; the reader is referred to Part (I) of figure 4-3.

4.4.4 The E_6 quiver from $\mathbb{Z}_6$

Let us make use of an interesting fact, that actually $E_6 = \langle \beta_4, \gamma, \delta \rangle = \langle \beta_4, \delta \rangle = \langle \gamma, \delta \rangle$. Therefore, alternative to the previous subsection wherein we exploited the sequence $\mathbb{Z}_4 = \langle \beta_4 \rangle \xrightarrow{+\gamma} \hat{\mathcal{D}}_2 \xrightarrow{+\delta} E_6$, we could equivalently apply our stepwise projection on $\mathbb{Z}_6 = \langle \delta \rangle \xrightarrow{+\beta_4} E_6$.

Let us first project with δ, an element of order 6 and the regular representation of which, after appropriate rotation is

$$R^{reg}_{E_6}(\delta) = R^{reg}_{\mathbb{Z}_6}(\delta) \otimes \mathbb{I}_4. \tag{4.4.18}$$

Therefore at this stage we have a $\mathbb{Z}_6$ quiver with labels of six 4's due to the $\mathbb{I}_4$; this is drawn in Part (II) of figure 4-3. The gauge group we shall denote as $A^\mu := \mathrm{Diag}(a,b,c,d,e,f)_{24\times 24}$, with $a, b, \cdots, f$ being 4×4 blocks.

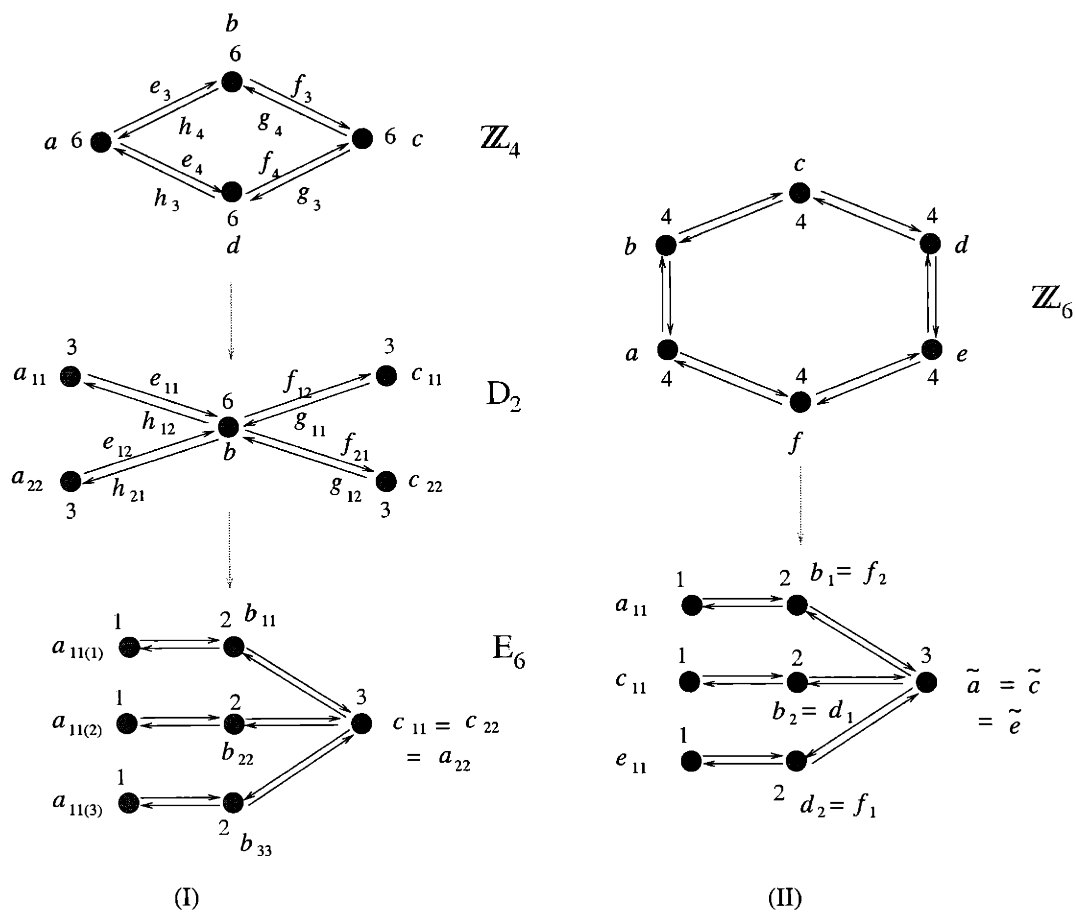

Figure 4-3: Obtaining the quiver diagram for the binary tetrahedral group E_6. We compare the two alternative stepwise projections: (I) $\mathbb{Z}_4 = \langle\beta_4\rangle \to \hat{\mathcal{D}}_2 = \langle\beta_4,\gamma\rangle \to E_6 = \langle\beta_4,\gamma,\delta\rangle$ and (II) $\mathbb{Z}_6 = \langle\delta\rangle \to E_6 = \langle\delta,\beta_4\rangle$.

Next we perform projection by $R^{reg}_{E_6}(\beta_4)$ in the rotated basis, splitting and joining

the gauge groups (nodes) as follows

$$A^\mu = \begin{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & \tilde{a} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} c_{11} & 0 \\ 0 & \tilde{c} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} e_{11} & 0 \\ 0 & \tilde{e} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \end{pmatrix} ; \; s.t. \quad \begin{array}{l} \tilde{a} = \tilde{c} = \tilde{e}, \\ b_2 = d_1, \\ d_2 = f_1, \\ f_2 = b_1, \end{array}$$

which upon substitution of the relations, gives us 7 independent factors: a_{11}, c_{11} and e_{11} are numbers, giving 1 as Dynkin labels in the quiver; b_1, b_2 and d_2 are 2×2 blocks, giving the 2 labels; while $\tilde{a}$ is 3×3, giving the 3. We refer the reader to Part (II) of figure 4-3 for the diagrammatical representation.

4.5 Towards brane setups for generic orbifolds

Our procedure outlined above is originally inspired by a series of papers [149, 148, 150] where the quivers for the Δ series of $\Gamma \subset SU(3)$ were observed to be obtainable from the $\mathbb{Z}_n \times \mathbb{Z}_n$ series after an appropriate identification. In particular, it was noted that

$$\Delta_{3n^2} = \langle \left\{ \mathbb{Z}_n \times \mathbb{Z}_n := \begin{pmatrix} \omega_n^i & 0 & 0 \\ 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_n^{-i-j} \end{pmatrix}_{i,j=0,\cdots,n-1} \right\}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle \text{ and}$$

subsequently the quiver for Δ_{3n^2} is that of $\mathbb{Z}_n \times \mathbb{Z}_n$ modded out by a certain $\mathbb{Z}_3$ quotient. Similarly, the quiver for

$$\Delta_{6n^2} = \langle \mathbb{Z}_n \times \mathbb{Z}_n, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \rangle$$

is that of $\mathbb{Z}_n \times \mathbb{Z}_n$ modded out by a certain S_3 quotient. In [150], it was further commented that the Σ series could be likewise treated.

The motivation for those studies was to realize a brane-setup for the non-Abelian $SU(3)$ orbifolds as geometrical quotients of the well-known Abelian case of $\mathbb{Z}_m \times \mathbb{Z}_n$, viz., the brane box models. The key idea was to recognize that the irreducible

representations of these groups could be labelled by a double index $(l_1, l_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ up to identifications.

Our purpose here is to establish an algorithmic treatment along similar lines, which would be generalizable to arbitrary finite groups. Indeed, since any finite group Γ is finitely generated, starting from the cyclic subgroup (with one single generator), our stepwise projection would give the quiver for Γ as appropriate splitting and joining of nodes, i.e., as a certain geometrical action, of the $\mathbb{Z}_n$ quiver.

4.5.1 Stepwise projection and induced representations

To see why stepwise projection works on a more axiomatic level, we need to turn to a brief review of the theory of induced representations.

It was a fundamental observation of Frøbenius that the representations of a group could be constructed from those of an arbitrary subgroup. The aforementioned chain of groups, where we tried to relate the regular representations, is precisely in this vein. We shall now briefly review this theory in the spirit of the above discussions, largely following the nomenclature of [153].

Let $\Gamma_1 = \langle x_1, ..., x_n \rangle$ and $\Gamma_2 = \langle x_1, ..., x_{n+1} \rangle$. We see thus that $\Gamma_1 \subset \Gamma_2$. Now let $R_{\Gamma_1}(x)$ be a representation (not necessarily irreducible) of the element $x \in \Gamma_1$. Extending it to Γ_2 gives

$$R_{\Gamma_2}(y) = \begin{cases} R_{\Gamma_1}(x) & \text{if} \quad y = x \in \Gamma_1 \\ 0 & \text{if} \quad y \notin \Gamma_1 \end{cases}$$

It follows then that if we decompose Γ_2 as (right) cosets of Γ_1,

$$\Gamma_2 = \Gamma_1 t_1 \cup \Gamma_1 t_2 \cup \cdots \cup \Gamma_1 t_m$$

we have an induced representation of Γ_2 as

$$R_{\Gamma_2}(y) = R_{\Gamma_1}(t_i y t_j^{-1}) = \begin{pmatrix} R_{\Gamma_1}(t_1 y t_1^{-1}) & R_{\Gamma_1}(t_1 y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_1 y t_m^{-1}) \\ R_{\Gamma_1}(t_2 y t_1^{-1}) & R_{\Gamma_1}(t_2 y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_2 y t_m^{-1}) \\ \vdots & \vdots & & \vdots \\ R_{\Gamma_1}(t_m y t_1^{-1}) & R_{\Gamma_1}(t_m y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_m y t_m^{-1}) \end{pmatrix}. \tag{4.5.19}$$

A important property of (4.5.19) is that it has only one member of each row or column non-zero and whereby it is essentially a generalized permutation (see e.g., 3.1 of [153]) matrix acting on the Γ_1-stable submodules of the Γ_2-module.

Now, for the case at hand the coset decomposition is simple due to the addition of a single new generator: the (right) transversals $t_1, \cdots, t_m$ are simply powers of the extra generator x_{n+1} and m is simply the index of $\Gamma_1 \subset \Gamma_2$, namely $|\Gamma_2|/|\Gamma_1|$, i.e.,

$$t_i = x_{n+1}^{i-1} \qquad i = 1, 2, \cdots, m; \quad m = \frac{|\Gamma_2|}{|\Gamma_1|}. \tag{4.5.20}$$

Now let us define an important concept for an element $x \in \Gamma_2$

Definition 4.5.8 *We call a representation $R_{\Gamma_2}(x)$ factorisable if it can be written, up to possible change of bases, as a tensor product $R_{\Gamma_2}(x) = R_{\Gamma_1}(x) \otimes \mathbb{I}_k$ for some integer k.*

Factorisability of the element, in the physical sense, corresponds to the ability to initialize our stepwise projection algorithm, by which we mean that the orbifold projection by this element is performed on k copies as in the usual sense, i.e., a stack of k copies of the quiver. Subsequently we could continue with the stepwise algorithm to demonstrate how the nodes of these copies merge or split. In the corresponding D-brane picture this simply means that we should consider k copies of each image D-brane probe in the covering space.

The natural question to ask is of course why our examples in the previous section permitted factorisable generators so as to in turn permit the performance of the stepwise projection. The following claim shall be of great assurance to us:

Proposition 4.5.2 *Let H be a subgroup of G, then the representation $R_G(x)$ for an element $x \in H \subset G$ induced from $R_H(x)$ according to (4.5.19) is factorisable and k is equal to $|G|/|H|$, the index of H in G.*

Proof: Take $R_H(x \in H)$, and tensor it with $\mathbb{I}_{k=|G|/|H|}$; this remains of course a representation for $x \in H$. It then remains to find the representations of $x \not\in H$, which we supplement by the permutation actions of these elements on the H-cosets. At the end of the day we arrive at a representation $R'_G(x)$ of dimension k, such that it is factorisable for $x \in H$ and a general permutation for $x \not\in H$. However by the uniqueness theorem of induced representations (q.v. e.g. [154], theorem 11) such a linear representation $R'_G(x)$ must in fact be isomorphic to $R_G(x)$. Thus by explicit construction we have shown that $R_G(x \in H) = R_H(x) \otimes \mathbb{I}_k$. □

We can be more specific and apply proposition 4.1 to our case of the two groups the second of which is generated by the first with one additional generator. Using the elegant property that the induction of a regular representation remains regular (q.v. e.g., 3.3 of [154]), we have:

Corollary 4.5.2 *Let Γ_1 and Γ_2 be as defined above, then*

$$R^{reg}_{\Gamma_2}(x_i) = R^{reg}_{\Gamma_1}(x_i) \otimes \mathbb{I}_{|\Gamma_2|/|\Gamma_1|} \qquad \textit{for common generators} \qquad i = 1, 2, \ldots, n.$$

In particular, since any $G = \langle x_1, \ldots, x_n \rangle$ contains a cyclic subgroup generated by, say x_1 of order m, i.e., $\mathbb{Z}_m = \langle x_1 \rangle$, we conclude that

Corollary 4.5.3 $R^{reg}_G(x_1) = R^{reg}_{\mathbb{Z}_m}(x_1) \otimes \mathbb{I}_{|G|/m}$, *and hence the quiver for G can always be obtained by starting with the $\mathbb{Z}_m$ quiver using the stepwise projection.*

Let us revisit the examples in the previous section equipped with the above knowledge. For the case of $\Gamma_1 = \mathbb{Z}_4 = \langle \beta_4 \rangle$ and $\Gamma_2 = \hat{\mathcal{D}}_2$ with the extra generator γ, (4.5.20) becomes $t_1 = \mathbb{I}$ and $t_2 = \gamma$ as the index of $\mathbb{Z}_4$ in $\hat{\mathcal{D}}_2$ is $\frac{|\hat{\mathcal{D}}_2|=8}{|\mathbb{Z}_4|=4} = 2$. The induced representation of β_4 according to (4.5.19) reads

$$R_{\hat{\mathcal{D}}_2}(\beta_4) = \begin{pmatrix} R^{reg}_{\mathbb{Z}_4}(\mathbb{I}\beta_4\mathbb{I}^{-1}) & R^{reg}_{\mathbb{Z}_4}(\mathbb{I}\beta_4\gamma^{-1}) \\ R^{reg}_{\mathbb{Z}_4}(\gamma\beta_4\mathbb{I}^{-1}) & R^{reg}_{\mathbb{Z}_4}(\gamma\beta_4\gamma^{-1}) \end{pmatrix} = \begin{pmatrix} R^{reg}_{\mathbb{Z}_4}(\beta_4) & 0 \\ 0 & R^{reg}_{\mathbb{Z}_4}(\beta_4^{-1}) \end{pmatrix}$$

using the fact that $\gamma\beta_k\gamma^{-1} = \beta_k^{-1}$ in $\hat{\mathcal{D}}_k$ for the last entry. Recalling that $R^{reg}_{\mathbb{Z}_4}(\beta_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix}$, this is subsequently equal to $R^{reg}_{\mathbb{Z}_4} \otimes \mathbb{I}_2$ after appropriate permutation of basis. Thus corollary 4.1 manifests its validity as we see that the $R_{\hat{\mathcal{D}}_2}$ obtained by Frøbenius induction of $R^{reg}_{\mathbb{Z}_4}$ is indeed regular and moreover factorisable, as (4.4.9) dictates.

Similarly with the case of $\mathbb{Z}_6 \to E_6$, we see that corollary 4.1 demands that for the common generator δ, $R^{reg}_{E_6}(\delta)$ should be factorisable, as is indeed indicated by (4.4.18). So too is it with $\mathbb{Z}_4 \to E_6$, where $R^{reg}_{E_6}(\beta_4)$ should factorize, precisely as shown by (4.4.15).

The above have actually been special cases of corollary 4.2, where we started with a cyclic subgroup; in fact we have also presented an example demonstrating the general validity of proposition 4.1. In the case of $\hat{\mathcal{D}}_2 \to E_6$, we mentioned earlier that $R^{reg}_{E_6}(\beta_4) = R^{reg}_{\hat{\mathcal{D}}_2}(\beta_4) \otimes \mathbb{I}_3$ and $R^{reg}_{E_6}(\gamma) = R^{reg}_{\hat{\mathcal{D}}_2}(\gamma) \otimes \mathbb{I}_3$ for the common generators as was seen from (4.4.15) and (4.4.16); this is exactly as expected by proposition (4.5.2).

4.5.2 Stepwise projection on brane setups

It should be clear by now what happens at a mathematical level. However, this is only half of the story; we expect T-duality to take D-branes at generic orbifold singularities to brane setups. As we have discussed in the previous section, the brane setups for the A and D-type orbifolds $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^2/D_n$ have been realized (see [146, 147] and [128] respectively). Other works have also attempted to establish such setups for more complicated singularities [106, 155, 148, 150].

In particular, the problem of finding a consistent brane-setup for the remaining case of the exceptional groups $E_{6,7,8}$ of the $A-D-E$ orbifold singularities of $\mathbb{C}^2$ (and indeed analogues thereof for $SU(3)$ and $SU(4)$ subgroups) so far has been proven to be stubbornly intractable. An original motivation for our work was to formulate an algorithmic outlook wherein such a problem, with the insight of the algebraic structure of an appropriate chain of certain relevant groups, may be addressed systematically.

The $\mathbb{Z}_2$ action on the brane setup

Let us attempt to recast our previous mathematical discussion in a physical language. First we try to interpret the action by $R^{reg}_{\hat{\mathcal{D}}_k}(\gamma)$ in (4.4.8) on the $\mathbb{Z}_{2k}$ quiver as a string-theoretic action on brane setups to get the corresponding brane setup of $\hat{\mathcal{D}}_k$ from that of $\mathbb{Z}_{2k}$.

Now the brane configuration for the $\mathbb{Z}_{2k}$ orbifold is the well-known elliptic model consisting of $2k$ NS5-branes arranged in a circle with D4-branes stretched in between as shown in Part (III) of figure 4-1. After stepwise projection by γ, the quiver in Part (I) becomes that in Part(II) (see figure 4-2 also). There is an obvious $\mathbb{Z}_2$ quotienting involved, where the nodes i and $2k-i$ for $i = 1, 2, ..., k-1$ are identified while each of the nodes 0 and k splits into two parts. Of course, this symmetry is not immediately apparent from the properties of γ, which is a group element of order 4. This phenomenon is true in general: the order of the generator used in the stepwise projection does not necessarily determine what symmetry the parent quiver undergoes to arrive at the resulting quiver; instead we must observe a posteriori the shapes of the respective quivers.

Let us digress a moment to formulate the above results in the language used in [148, 150]. We adopt their idea of labelling the irreducible representations of Δ by $\mathbb{Z}_n \times \mathbb{Z}_n$ up to appropriate identifications, which in our terminology is simply the stepwise projection of the parent $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver. As a comparison, we apply this idea to the case of $\mathbb{Z}_{2k} \to \hat{\mathcal{D}}_k$. Therefore we need to label the irreps of $\hat{\mathcal{D}}_k$ or appropriate tensor sums thereof, in terms of certain (reducible) 2-dimensional representations of $\mathbb{Z}_{2k}$. Motivated by the factorization property (4.4.9), we chose these representations to be

$$R^{l}_{\mathbb{Z}_{2k}(2)} := R^{l,irrep}_{\mathbb{Z}_{2k}(1)} \oplus R^{l,irrep}_{\mathbb{Z}_{2k}(1)} \tag{4.5.21}$$

where $l \in \mathbb{Z}_{2k}$, and amounts to precisely a $\mathbb{Z}_{2k}$-valued index on the representations of $\hat{\mathcal{D}}_k$ (since $\mathbb{Z}_{2k}$ is Abelian), which with foresight, we shall later use on $\hat{\mathcal{D}}_k$. We observe

that such a labelling scheme has a symmetry

$$R^{l}_{\mathbb{Z}_{2k}(2)} \cong R^{-l}_{\mathbb{Z}_{2k}(2)},$$

which is obviously a $\mathbb{Z}_2$ action. Note that $l = 0$ and $l = k$ are fixed points of this $\mathbb{Z}_2$.

We can now associate the 2-dimensional irreps of $\hat{\mathcal{D}}_k$ with the non-trivial equivalence classes of the $\mathbb{Z}_{2k}$ representations (4.5.21), i.e., for $l = 1, 2, \ldots, k-1$ we have

$$R^{l}_{\mathbb{Z}_{2k}(2)} \cong R^{-l}_{\mathbb{Z}_{2k}(2)} \rightarrow R^{l,irrep}_{D_k(2)}. \tag{4.5.22}$$

These identifications correspond to the merging nodes in the associated quiver diagram. As for the fixed points, we need to map

$$\begin{array}{l} R^{0}_{\mathbb{Z}_{2k}(2)} \rightarrow R^{1,irrep}_{D_k(1)} \oplus R^{2,irrep}_{D_k(1)} \\ R^{k}_{\mathbb{Z}_{2k}(2)} \rightarrow R^{3,irrep}_{D_k(1)} \oplus R^{4,irrep}_{D_k(1)}. \end{array} \tag{4.5.23}$$

These fixed points are associated precisely with the nodes that split.

This construction shows clearly how, in the labelling scheme of [148, 150], our stepwise algorithm derives the $\hat{\mathcal{D}}_k$ quiver as a $\mathbb{Z}_2$ projection of the $\mathbb{Z}_{2k}$ quiver. The consistency of this description is verified by substituting the representations $R^{l}_{\mathbb{Z}_{2k}(2)}$ in the $\mathbb{Z}_{2k}$ quiver relations $\mathcal{R} \otimes R^{l}_{\mathbb{Z}_{2k}(2)} = \bigoplus_{\bar{l}} a^{\mathbb{Z}_{2k}(\mathcal{R})}_{l\bar{l}} R^{\bar{l}}_{\mathbb{Z}_{2k}(2)}$ using (4.5.22) and (4.5.23), which results exactly in the $\hat{\mathcal{D}}_k$ quiver relations. We can of course apply the stepwise projection for the case of $\mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \Delta$, and would arrive at the results in [148, 150].

In the brane setup picture, the identification of the nodes i and $2k - i$ for $i = 1, 2, ..., k-1$ corresponds to the identification of these intervals of NS5-branes as well as the D4-branes in between in the X^{6789} directions (with direction-6 compact). Thus the $\mathbb{Z}_2$ action on the $\mathbb{Z}_{2k}$ quiver should include a space-time action which identifies $X^{6789} = -X^{6789}$. Similarly, the splitting of gauge fields in intervals 0 and k hints the existence of a $\mathbb{Z}_2$ action on the string world-sheet. Thus the overall $\mathbb{Z}_2$ action should include two parts: a space-time symmetry which identifies and a world-sheet symmetry which splits respective gauge groups.

What then is this action physically? What object in string theory performs the tasks in the above paragraph? Fortunately, the space-time parity and string world-sheet $(-1)^{F_L}$ actions [124, 126, 125] are precisely the aforementioned symmetries. In other words, the ON^0-plane is that which we seek. This of course was expected; the brane setup for $\hat{\mathcal{D}}_k$ theories, as given in [128], is indeed a configuration which uses the ON^0-plane to project out or identify fields in a manner consistent with our discussions.

The general action on the brane setup ?

We have proven above that our stepwise projection algorithm is a constructive method of arriving at any orbifold quiver by appropriate quotient of the $\mathbb{Z}_n$ quiver. It is natural to ask if we can apply this technique to find the appropriate object in string theory which would perform such a quotient, much in the spirit of the orientifold prescribing $\mathbb{Z}_2$ in the above example, on the well-known $\mathbb{Z}_n$ brane setup.

Let us consider the E_6 example. The action by δ in the case of $\hat{\mathcal{D}}_2 \to E_6$ and that of β_4 in the case of $\mathbb{Z}_6 \to E_6$, can be visualized in Parts (I) and (II) of figure 4-3 to be an $\mathbb{Z}_3$ action on the respective parent quivers. In particular, the identifications $c_{11} \sim c_{22} \sim a_{22}$ and $\tilde{a} \sim \tilde{c} \sim \tilde{e}; b_1 \sim f_2, b_2 \sim d_1, d_2 \sim f_1$ respectively for Parts (I) and (II) are suggestive of a $\mathbb{Z}_3$ action on X^{6789}. The tripartite splittings for b, a_{11} and a, b, d respectively also hint at a $\mathbb{Z}_3$ action on the string world-sheet.

Again let us phrase the above results in the scheme of [148, 150] and manifestly show how the E_6 quiver results from a $\mathbb{Z}_3$ projection of the $\hat{\mathcal{D}}_2$ quiver. We define the following representations of $\hat{\mathcal{D}}_2$: $R^0_{D_2(6)} = R^{irrep}_{D_2(2)} \oplus R^{irrep}_{D_2(2)} \oplus R^{irrep}_{D_2(2)}$ and $R^l_{D_2(3)} = R^{l,irrep}_{D_2(1)} \oplus R^{l,irrep}_{D_2(1)} \oplus R^{l,irrep}_{D_2(1)}$ where $l \in \mathbb{Z}_4$ labels the four 1-dimensional irreducible representations of $\hat{\mathcal{D}}_2$. There is an identification

$$R^l_{D_2} \cong R^{f(l)}_{D_2}$$

where

$$f(0) = 0, \quad f(1) = 2, \quad f(2) = 3, \quad f(3) = 1.$$

Clearly this is a $\mathbb{Z}_3$ action on the index l. Note that we have two representations labelled with $l = 0$ which are fixed points of this action. In the quiver diagram of $\hat{\mathcal{D}}_2$ these correspond to the middle node and another one arbitrarily selected from the remaining four, both of which split into three. The remaining three nodes are consequently merged into a single one (see figure 4-3). To derive the E_6 quiver we need to map the nodes of the parent $\hat{\mathcal{D}}_2$ quiver as

$$\begin{aligned} R^0_{D_2(6)} &\to R^{1,irrep}_{E_6(2)} \oplus R^{2,irrep}_{E_6(2)} \oplus R^{3,irrep}_{E_6(2)} \\ R^0_{D_2(3)} &\to R^{1,irrep}_{E_6(1)} \oplus R^{2,irrep}_{E_6(1)} \oplus R^{3,irrep}_{E_6(1)} \\ R^l_{D_2(3)} \cong R^{f(l)}_{D_2(3)} &\to R^{irrep}_{E_6(3)}, \qquad l \in \mathbb{Z}_4 - \{0\}. \end{aligned}$$

Consistency requires that if we replace R_{D_2} in the $\hat{\mathcal{D}}_2$ quiver defining relations and then use the above mappings, we get the E_6 quiver relations for $R^{irrep}_{E_6}$.

The origin of this $\mathbb{Z}_3$ analogue of the orientifold $\mathbb{Z}_2$-projection is thus far unknown. If an object with this property is to exist, then the brane setup for the E_6 theory could be implemented; on the other hand if it does not, then we would be suggested at why the attempt for E_6 has been prohibitively difficult.

The $\mathbb{Z}_3$ action has been noted to arise in [148] in the context of quotienting the $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver to arrive at the quiver for the Δ-series. Indeed from our comparative study we see that in general, labelling the irreps by a multi-index is precisely our stepwise algorithm in disguise, as applied to a product Abelian group: the $\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ orbifold. Therefore in a sense we have explained why the labelling scheme of [148, 150] should work.

We expect a similar story to hold for E_7 and E_8: we could perform stepwise projection thereupon and mathematically obtain their quivers as appropriate quotients of the $\mathbb{Z}_n$ quiver by the symmetry S of the identification and splitting of nodes. To find a physical brane setup, we would then need to find an object in string theory which has an S action on space-time and the string world-sheet. Note that the above are cases of the $\mathbb{C}^2$ orbifolds; for the $\mathbb{C}^k$-orbifold we should initialize our algorithm with, and perform stepwise projection on the quiver of $\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ ($k-1$ times),

i.e., the brane box [147] and cube [156] ($k = 2, 3$).

In practice, the cases of E_7 and E_8 are much more complicated than E_6, and in light of the fact that the $\mathbb{Z}_3$ action corresponding to E_6 is unknown, it would be of little value to perform in detail the stepwise projection for these groups at this point.

4.6 A connection with modular invariants of $SU(2)$ WZW models

We should also mention here that a similar stepwise pattern appears to exist in the classification of modular invariant partition functions for $SU(2)$ Wess-Zumino-Witten models. It is well-known that this classification if of A-D-E type [157, 158]. A natural way to obtain the D-type modular invariants is by orbifolding the original $SU(2) \cong S^3$ WZW equipped with the A-type diagonal modular invariant, with respect to parity on S^3. This is a $\mathbb{Z}^2$ orbifold which effectively transforms the target space into $SO(3) \cong SU(2)/\mathbb{Z}_2$.

As we have discussed in chapter 3, the partition function of an orbifold theory has to incorporate twisted sectors in order to be modular invariant. For the particular WZW model, it was shown in [159] that the partition function of the orbifolded theory is exactly what we would now call the D-type modular invariant of $SU(2)$.

Hence, a $\mathbb{Z}_2$ projection of the A-type modular invariants leads to the D-type ones. In the light of our previous discussions, this should not be very surprising. Notice in particular that our $\mathbb{Z}_2$ projection that takes us from the A-type quiver gauge theory to the D-type one involves a spacetime parity! However, the role of the $(-1)^{F_L}$ is a bit obscure.

From the point of view of WZW partition functions, note that in the superstring case one has to consider also the fermionic partners of the bosonic WZW fields. It is well-known that a supersymmetric WZW on a group manifold G at level k can be written as a bosonic WZW model on the same group manifold with level $k-2$ along with four free fermions. This is achieved after a chiral rotation that decouples the

fermions (see for instance [160]).

The spacetime parity acts also on the world-sheet fermions since they belong to the same representation of the Lorentz group as the bosonic spacetime coordinates. Hence, their partition function will also change accordingly. It was shown in [161] that projecting in addition with $(-1)^{F_L}$, undoes the parity projection on the fermions and results in the D-type partition function for the bosonic $SU(2)$ WZW model along with the standard free fermion partition sum. Note that this model should describe the near-horizon region of an ON^0-plane along with $k+2$ parallel and coincident physical NS5-branes.

It would be interesting to push this correspondence one step further and attempt to understand the nature of the $\mathbb{Z}_3$ projection we found above for E_6 using the well-known form of the corresponding $SU(2)$ modular invariants. In fact, a method similar to our stepwise projection was employed in a systematic search for modular invariant partition functions of WZW theories in [162]. In particular, the same $\mathbb{Z}_3$ was shown to be required in order to derive the E_6 type $SU(2)$ partition function starting from simpler ones. It would be interesting to redo the computation of [161] for this example and thus find explicitly the $\mathbb{Z}_3$ world-sheet action we need in order to realize the configuration of branes that are T-dual to the $\mathbb{C}^2/E_6$ orbifold.

The fact however that the exceptional modular invariants exist only for specific values of the level seems to suggest that a semi-classical limit where the object we are looking for has a macroscopic spacetime description may be absent. This may be the reason for the failure of all attempts to realize the corresponding quiver gauge theories using branes.

It is natural to suspect that the stepwise projection pattern we exploited in this chapter in the context of branes and quivers, persists in all A-D-E classifications. It would be interesting to understand the general principles behind this pattern, especially for the exceptional cases.

Chapter 5

The Partition Function and the Spectrum of the Two-Dimensional Black Hole

The main topic of this chapter is the computation of the partition function of the two-dimensional Euclidean black hole conformal field theory, i.e. the coset $SL(2,\mathbb{R})/U(1)$. This computation will enable us to determine the spectrum of the black hole and verify the original algebraic results of Dijkgraaf, Verlinde and Verlinde (DVV) [170]. In particular, we find confirmation for the bound on the spin of the discrete representations and we determine the density of the continuous representations.

Before presenting our computation in detail, we motivate the study of the two-dimensional back hole from a variety of directions. These include the holographic description of decoupled theories on NS5-branes or singularities and the necessity of obtaining a better understanding of exact string propagation on curved backgrounds. In addition, the study of this particular conformal field theory is also interesting on its own right since it presents several novel subtleties related to the non-compactness of the target space.

The material presented in this chapter is based on [5].

5.1 Two-dimensional black hole: motivation

The two-dimensional back hole conformal field theory, based on the coset $SL(2)/U(1)$, was first found and analyzed in [163, 164, 165, 166, 167, 168, 170]. In particular, it was shown in [169] that it corresponds to a black hole solution of 2-dimensional gravity coupled to a dilaton. There also exists a construction of this model as a gauged Wess-Zumino-Witten (WZW) based on the $SL(2, \mathbb{R})$ group manifold, which is equivalent to its coset CFT description (see [171, 172, 173, 174] for the connection between generic coset CFTs [175] and gauged WZW theories).

The spectrum of the conformal field theory was first determined in [170], along with a precise analysis of the target space geometry including α' corrections. To first order in α' the geometry of $SL(2, \mathbb{R})/U(1)$ is that of a semi-infinite cigar, with metric

$$ds^2 = dr^2 + 4\tanh^2\frac{r}{2}d\theta^2. \tag{5.1.1}$$

Here $r \in [0, \infty)$ is the non-compact direction along the cigar, while $\theta \in [0, 2\pi)$ parameterizes the compact direction around the cigar. Note that asymptotically the radius of the compact dimension becomes constant.

The analysis of [170] indicated that the corrected sigma-model metric actually is

$$ds^2 = dr^2 + \frac{4\tanh^2\frac{r}{2}}{1 - \frac{2}{k}\tanh^2\frac{r}{2}}d\theta^2. \tag{5.1.2}$$

Another derivation of this result, based on the quantum effective action of the gauged WZW theory, was presented in [176].

Since exact non-compact curved gravitational backgrounds in string theory are not at all abundant, the supersymmetric version of the 2D black hole provides an interesting candidate and it was extensively used in this context (see for instance [177, 178]). An interesting property of the supersymmetric coset is that, in conjunction with the parafermion system $SU(2)/U(1)$, it realizes an $N_2 = 4$ superconformal algebra [179]. Sigma-models with this type of world-sheet symmetry are not subject to either perturbative or non-perturbative corrections in α'.

for earlier work).

This series of papers has answered important questions in non-compact Wess-Zumino-Witten theories and opened the road to a more extensive study of similar models. It is natural then to re-examine some aspects of the $SL(2,\mathbb{R})/U(1)$ black hole in the light of our improved understanding of the parent $SL(2,\mathbb{R})$ WZW theory. In particular, we can re-analyze [223] the toroidal partition function for the $SL(2,\mathbb{R})/U(1)$ coset conformal field theory.

As we have argued, this particular coset appears in a variety of interesting situations in string theory. In order to write down the toroidal partition function for string theory on these backgrounds, and thus determine the corresponding string spectrum, the most crucial ingredient is the black hole background partition sum. In the rest of this chapter we will address the computation of the partition function from a path-integral point of view. We should mention that our computation is technically close to the analysis of the free energy of string theory on AdS_3 in [198].

5.2 The $SL(2,\mathbb{R})/U(1)$ coset toroidal partition function

The general treatment of gauged Wess-Zumino-Witten models is well-known [171, 172, 173, 174]. For a general group manifold G the Wess-Zumino-Witten action is:

$$S[g] = \frac{k}{2\pi}\int_{\mathrm{WS}} d^2z \, \mathrm{Tr}(\partial g^{-1}\bar{\partial} g) + \frac{ik}{12\pi}\int_{\mathrm{B}} \mathrm{Tr}(g^{-1}dg)^3 \tag{5.2.3}$$

where $g(z,\bar{z})$ is a group element, the level of the WZW-model is $k \in \mathbb{R}$, and B is a three-dimensional manifold with the world-sheet as a boundary. Our world-sheet is a two-torus T^2.

For compact group manifolds, the level k is in general quantized but for $SL(2,\mathbb{R})$ we have $H^3(SL(2,\mathbb{R}),\mathbb{R}) = 0$ so that the action is independent of our choice of manifold B for any real k. The Wess-Zumino-Witten model has an affine symmetry $G(z)\times G(\bar{z})$. We will gauge an axial abelian subgroup of the symmetry group $g \to hgh$,

The $SL(2,\mathbb{R})/U(1)$ coset makes its appearance in a variety of places in string theory. It is known that conformal field theories near ALE singularities include such a factor [136]. It also arises as part of the CFT that describes the near horizon of NS5-branes separated on a circle [180, 181, 182, 183]. In this context, it is conjectured [182, 183] that the $SL(2,\mathbb{R})/U(1) \times SU(2)/U(1)$ CFT provides a holographic formulation of the decoupled theory on the NS5-branes. This is related to a similar conjecture [184] which asserts that the holographic dual of the decoupled theory on a stack of parallel and coincident NS5-branes, which is known as little string theory [185], is the conformal field theory of the throat region, i.e. the $SU(2)$ WZW model along with a linear dilaton CFT [186, 187, 188, 189]. Note that this last conjecture is similar in spirit to AdS/CFT [190]; it relates holographically the decoupled non-gravitational theory living on a brane with the theory of closed strings on the background created by the same brane.

The first conjecture is based on the fact that the linear dilaton theory with a tachyon condensate is conjectured to be T-dual to the cigar coset theory [191] (recent work [192, 193] provided evidence for the validity of the supersymmetric version of this T-duality, which is between an $N_2 = 2$ Liouville theory [194, 195] and the Kazama-Suzuki $SL(2,\mathbb{R})/U(1)$ model [196]).

Our main motivation for studying the $SL(2,\mathbb{R})/U(1)$ is the recent progress in understanding non-compact Wess-Zumino-Witten models. In particular, the spectrum and correlation functions of string theory on AdS_3, the covering space of $SL(2,\mathbb{R})$, were analyzed in detail in [197, 198, 199]. The spectrum for the non-compact WZW model was determined in [197], using intuition for long strings obtained from [200, 201] and the technical tool of spectral flow [202], thereby solving the long-standing problem of determining the correct Hilbert space for the $SL(2,\mathbb{R})$ WZW-model [203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214]. Next, in [198], the computation of the free energy for string theory on AdS_3[1] by path-integral methods gave additional support to the spectrum proposed in [197]. Finally, in [199] the completeness of the Hilbert space was checked by computing various correlators (see also e.g. [218, 219, 220, 221, 222]

[1]See also e.g. [215, 216, 217].

yielding the action:

$$S_{gauged}[g;A] = S[g] - \frac{k}{\pi}\int_{\rm WS} d^2z\, {\rm Tr}(\bar{A}\bar{\partial}gg^{-1} + Ag^{-1}\partial g + g^{-1}\bar{A}gA + A\bar{A}) \qquad (5.2.4)$$

with the one-form gauge field $A_{(1)}$ defined as $A_{(1)} = A\,dz + \bar{A}\,d\bar{z}$. Our gauged theory is anomaly free (see e.g. [224]).

Next we concentrate on the Lorentzian $SL(2,\mathbb{R})$ group manifold. A standard parametrisation of the group elements g is in terms of Euler angles [170]:

$$g \;=\; e^{\frac{i}{2}\theta_L\sigma_2}e^{\frac{r}{2}\sigma_1}e^{\frac{i}{2}\theta_R\sigma_2} \qquad (5.2.5)$$

where $0 \leq r \leq \infty, 0 \leq \theta_L < 2\pi, -2\pi \leq \theta_R < 2\pi$. We gauge the axial U(1) symmetry $g \to hgh$ where $h = \exp(\frac{i}{2}\lambda\sigma_2)$. The coordinates shift under gauge transformations as $\theta_{L,R} \to \theta_{L,R} + \lambda$ and the gauge field transforms as

$$A \longrightarrow A + d\lambda. \qquad (5.2.6)$$

The gauged WZW action (5.2.4) becomes:

$$\begin{aligned} S[r,\theta_R,\theta_L;A] = \tfrac{k}{2\pi}\textstyle\int d^2z\Big(&\tfrac{1}{2}(\partial r\bar{\partial}r - \partial\theta_L\bar{\partial}\theta_L - \partial\theta_R\bar{\partial}\theta_R - 2\cosh r\bar{\partial}\theta_L\partial\theta_R) + \\ &A(\bar{\partial}\theta_R + \cosh r\bar{\partial}\theta_L) + \bar{A}(\partial\theta_L + \cosh r\partial\theta_R) - A\bar{A}(\cosh r + 1)\Big). \end{aligned} \qquad (5.2.7)$$

In [169] it was shown, by integrating out the gauge field classically, that the coset has a cigar geometry that can be interpreted as a Euclidean black hole. Gauging a non-compact abelian subgroup would have resulted in the Lorentzian two-dimensional black hole.

The gauged theory can be re-written in terms of the sum of an $SL(2,\mathbb{R})$ model and a $U(1)$ model. We thereto introduce the coordinates $\theta = \frac{1}{2}(\theta_L - \theta_R)$ and $\tilde{\theta} = \frac{1}{2}(\theta_L + \theta_R)$. In terms of these coordinates the action can be written in the manifestly gauge invariant form (since shifts $\tilde{\theta} \to \tilde{\theta} + \lambda$ are compensated by shifts in the gauge

field $A \to A + \partial\lambda$ and $\bar{A} \to \bar{A} + \bar{\partial}\lambda$):

$$S[r,\theta,\tilde{\theta};A,\bar{A}] = \tfrac{k}{2\pi}\int d^2z\Big[\tfrac{1}{2}\partial r\bar{\partial}r + (\cosh r - 1)\Big(\partial\theta\bar{\partial}\theta + (A - \partial\tilde{\theta})\bar{\partial}\theta - (\bar{A} - \bar{\partial}\tilde{\theta})\partial\theta\Big) \\ -(\cosh r + 1)(A - \partial\tilde{\theta})(\bar{A} - \bar{\partial}\tilde{\theta})\Big]. \qquad (5.2.8)$$

To re-write the action in product form, we first Hodge-decompose the gauge field on the torus as:

$$A = \partial\rho_L + \frac{i}{2\tau_2}(u_1\bar{\tau} - u_2) \qquad (5.2.9)$$

$$\bar{A} = \bar{\partial}\rho_R - \frac{i}{2\tau_2}(u_1\tau - u_2) \qquad (5.2.10)$$

where $\rho_L^* = \rho_R$ is well-defined on the torus and the holonomies u_1, u_2, parameterize the Wilson lines on the toroidal world-sheet with modular parameter τ. Ignoring the holonomies for now, we can follow the treatment on the sphere (as in e.g. [176]). We introduce the new variables $\rho = \frac{1}{2}(\rho_L - \rho_R)$, $\tilde{\rho} = \frac{1}{2}(\rho_L + \rho_R)$ and $\kappa = \theta + \rho$, $\tilde{\kappa} = \tilde{\theta} - \tilde{\rho}$, in terms of which the action becomes:

$$S[r,\kappa,\tilde{\kappa};\rho,\tilde{\rho}] = \tfrac{k}{2\pi}\int d^2z\Big(\tfrac{1}{2}\partial r\bar{\partial}r + (\cosh r - 1)\partial\kappa\bar{\partial}\kappa - (\cosh r + 1)\partial\tilde{\kappa}\bar{\partial}\tilde{\kappa} + \\ (\cosh r - 1)(\partial\kappa\bar{\partial}\tilde{\kappa} - \bar{\partial}\kappa\partial\tilde{\kappa})\Big) + \tfrac{k}{\pi}\int d^2z\, \partial\rho\bar{\partial}\rho. \qquad (5.2.11)$$

Since under a gauge transformation $\rho_{L,R} \to \rho_{L,R} + \lambda$ the fields $\kappa, \tilde{\kappa}$ and ρ do not transform, the above action is gauge invariant.

We can read (5.2.11) as the action for the $SL(2,\mathbb{R}) \times U(1)$ model. Of course, on a toroidal world-sheet we need to take care in following the holonomies in the gauge field through the coordinate redefinitions. Note however that already for the spherical topology, subtleties arise that have hitherto been successfully ignored. Indeed, the quantity κ is a linear combination of a real and imaginary field, but will nevertheless be treated as a real field [170, 176]. We will briefly return to the subtle issues of analytic continuation in the following, although we will not resolve all of them unambiguously. The above decomposition was put to good use in [170] to argue for the spectrum of

the CFT and to calculate the exact black hole background, and in [176] to compute the effective action and re-derive the exact metric in a path-integral approach. We note that the holonomies of the gauge field have been transformed, via the field redefinitions, into non-trivial windings (over the two 1-cycles of the torus) for the matter fields $\kappa, \tilde{\kappa}$ and ρ. They will be crucial in the following.[2]

Before delving into the main part of the computation of the partition function we gauge-fix the action by choosing $\tilde{\rho} = 0$. We then need to include the ghost action

$$S_{ghosts}[b, c] = \frac{1}{\pi} \int d^2z (b\bar{\partial}c + \bar{b}\partial\bar{c}). \tag{5.2.12}$$

5.2.1 Computing the partition function

This subsection contains the core of the computation of the toroidal partition function. We will discuss the various techniques needed for the computation in some detail. Our computation owes a lot to the analysis in [198, 223]. Of course, since [198] computes the free energy of AdS_3 string theory while we are interested in the partition function on the Euclidean black hole background, we need to adapt their computational techniques creatively.[3]

Our previous treatment of the model was in accord with standard conventions on Euler angles, but to make the computation of the partition function feasible, it is very useful to parameterize the $SL(2,\mathbb{R})$ part of the model in terms of the coordinates introduced in [223]. After continuing the path integral to Euclidean signature to make it well defined (effectively transforming the model into the $SL(2,C)/SU(2)$ coset model – for discussions see [223] and [199]), the coordinate transformation becomes:

$$\begin{aligned} v &= \sinh\frac{r}{2}e^{i\kappa} && (5.2.13) \\ \bar{v} &= \sinh\frac{r}{2}e^{-i\kappa} && (5.2.14) \end{aligned}$$

[2] Our toroidal treatment of the holonomies will naturally turn out to be equivalent to the BRST analysis of the gauge invariant states in [170].

[3] To avoid confusion, note that the temperature introduced in [198] is the temperature of AdS_3. The Euclidean black hole is an analytically continued version of the Lorentzian black hole with a different time direction.

$$\phi = i\tilde{\kappa} - \log\cosh\frac{r}{2}. \tag{5.2.15}$$

Writing the total action in terms of these variables results in:

$$\begin{aligned} S[\phi, v, \bar{v}; \rho; b, c] &= \frac{k}{\pi}\int d^2z\Big(\partial\phi\bar{\partial}\phi + (\partial\bar{v} + \bar{v}\partial\phi)(\bar{\partial}v + v\bar{\partial}\phi)\Big) + \\ &\quad \frac{k}{\pi}\int d^2z\, \partial\rho\bar{\partial}\rho + \int d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}). \end{aligned} \tag{5.2.16}$$

Note that the fields $\phi, v, \bar{v}$ and ρ have non-trivial holonomies. In order to perform the path integral we will decompose them in a periodic part and a holonomy part:

$$\begin{aligned} \phi &= \hat{\phi} + \frac{1}{4\tau_2}\Big((u_1\bar{\tau} - u_2)z + (u_1\tau - u_2)\bar{z}\Big) \\ v &= \hat{v}\exp\Big(-\frac{1}{4\tau_2}((u_1\bar{\tau} - u_2)z - (u_1\tau - u_2)\bar{z})\Big) \\ \bar{v} &= \hat{\bar{v}}\exp\Big(+\frac{1}{4\tau_2}((u_1\bar{\tau} - u_2)z - (u_1\tau - u_2)\bar{z})\Big) \\ \rho &= \hat{\rho} + \frac{1}{4\tau_2}\Big((u_1\bar{\tau} - u_2)z + (u_1\tau - u_2)\bar{z}\Big), \end{aligned} \tag{5.2.17}$$

where the hatted fields are periodic.[4]

The coset partition function then reads

$$Z_{cs}(\tau) = \int \mathcal{D}\hat{\phi}\mathcal{D}\hat{v}\mathcal{D}\hat{\bar{v}}\mathcal{D}\hat{\rho}\mathcal{D}b\mathcal{D}c \int_{-\infty}^{+\infty} du_1 du_2 e^{-S[\phi,v,\bar{v};\rho;b,c]}. \tag{5.2.18}$$

Ray-Singer torsion

The core of the computation uses the Ray-Singer analytic torsion [225], which arises from the path integral over $\hat{v}, \hat{\bar{v}}$. The relevant piece of the action, after substituting (5.2.17), is

$$S_{v,\bar{v}} = \Big(\partial + \partial\hat{\phi} + \frac{1}{2\tau_2}(u_1\bar{\tau} - u_2)\Big)\,\hat{\bar{v}}\,\Big(\bar{\partial} + \bar{\partial}\hat{\phi} + \frac{1}{2\tau_2}(u_1\tau - u_2)\Big)\,\hat{v} \tag{5.2.19}$$

[4] Trying to follow the holonomies of the gauge field through the field redefinitions we gave before gives rise to the difficulties we mentioned related to analytic continuation and reality of the fields. We chose the holonomies to be consistent with complex conjugation for v, reality for ϕ, etc. We believe the resulting spectrum gives sufficient justification for this choice of analytic continuation.

Note that the action is quadratic in $\hat{v}, \hat{\bar{v}}$. Following [223], we observe that we can disentangle the $\hat{\phi}$-dependence by a chiral rotation. The integral over $v, \bar{v}$ then becomes the regularised determinant of the Laplacian on a space of functions that have non-trivial holonomies around the cycles of the two-torus. Precisely this determinant was defined in [225] by using ζ-function regularisation. The regularised determinant is called the analytic torsion[5]:

$$\det\left|\partial + \tfrac{1}{2\tau_2}(u_1\bar{\tau} - u_2)\right|^{-2} = \frac{(q\bar{q})^{-2/24}}{|\sin(\pi(u_1\tau - u_2))|^2} \frac{e^{\frac{2\pi}{\tau_2}(\mathrm{Im}(u_1\tau - u_2))^2}}{|\prod_{r=1}^{\infty}(1-e^{2\pi i r\tau - 2\pi i(u_1\tau - u_2)})(1-e^{2\pi i r\tau + 2\pi i(u_1\tau - u_2)})|^2} \ . \tag{5.2.20}$$

We introduced the usual notation $q = \exp(2\pi i\tau)$. The analytic torsion is periodic in the holonomies u_1 and u_2, as we would expect from gauge invariance.[6] If needed (for instance in order to check modular properties [225]), the analytic torsion can be re-written in terms of the θ_1-function.

Free contributions

We treat the other contributions to the partition function which are basically the familiar free contributions, but some factors need to be treated with care. First of all note that there is a shift $k \to k-2$ in the kinetic term of $\hat{\phi}$ because of the contribution of the chiral rotation that we performed to disentangle ϕ and $v, \bar{v}$. The path integration over $\hat{\phi}$ and $\hat{\rho}$ will each give the usual periodic boson partition sum $\tau_2^{-1/2}|\eta(\tau)|^{-2}$ with overall factor $2\sqrt{k(k-2)}$. Moreover, the holonomy contributes an overall exponential factor. Finally, the contribution from the ghosts b, c that we introduced to gauge fix the $U(1)$ symmetry is $\tau_2|\eta(\tau)|^4$. It is natural that the net effect of the gauge field is to cancel the free boson contribution to the $SL(2,\mathbb{R})$ partition function.

[5]Note that the computation of the analytic torsion on the torus (cf. [225], p. 165-169), naturally resembles the usual computation of the partition function for a compact boson.

[6]It is also evident from the mathematical definition of analytic torsion in terms of a complex line bundle with non-trivial character $\chi(m\tau + n) = e^{2\pi i(mu_1 + nu_2)}$. Note that the authors of [198] appropriately use an analytically continued version of the analytic torsion that is not periodic.

Holonomies

It is convenient at this point to break the holonomy parameters u_1 and u_2 in integer and fractional parts, i.e. $u_1 = s_1 + w,\ u_2 = s_2 + m$ with $s_1, s_2 \in [0,1)$ and $w, m \in \mathbb{Z}$ running over the integers. Since the Ray-Singer torsion is periodic, it is only the overall exponential factor that depends on the integers w and m that parameterize the non-trivial windings for the compact bosons.

5.2.2 Combining ingredients

Combining all of the above we obtain for the modular invariant partition function:

$$\begin{aligned} Z_{cs}(\tau) &= 2(k(k-2))^{1/2} \int_0^1 ds_1 ds_2 \\ &\sum_{w,m=-\infty}^{+\infty} \frac{(q\bar{q})^{-2/24}}{|\sin(\pi(s_1\tau - s_2))|^2} \\ &\frac{e^{-\frac{k\pi}{\tau_2}|(s_1+w)\tau-(s_2+m)|^2+\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau-s_2))^2}}{|\prod_{r=1}^{\infty}(1-e^{2\pi i r\tau-2\pi i(s_1\tau-s_2)})(1-e^{2\pi i r\bar{\tau}+2\pi i(s_1\bar{\tau}-s_2)})|^2}. \end{aligned} \tag{5.2.21}$$

If we are interested in incorporating the coset theory as a factor in a string theory background $SL(2,\mathbb{R})/U(1) \times \mathcal{M}$, we combine it with the modular invariant partition function $Z_{\mathcal{M}}$ for strings propagating on $\mathcal{M}$ and the reparametrization ghosts partition function Z_{ghosts}. We then integrate the modular parameter τ over the fundamental domain F_0 of the usual $SL(2,Z)$ action on the complex τ-plane to obtain:

$$\mathcal{Z} = \int_{F_0} \frac{d\tau d\bar{\tau}}{\tau_2} Z_{\mathcal{M}}(\tau) Z_{cs}(\tau) Z_{ghosts}(\tau). \tag{5.2.22}$$

The general form of the partition function corresponding to the background $\mathcal{M}$ is

$$Z_{\mathcal{M}}(\tau) = (q\bar{q})^{-c_{\mathcal{M}}/24} \sum_i q^{h_i} \bar{q}^{\bar{h}_i} \tag{5.2.23}$$

where i labels all states of the CFT on $\mathcal{M}$ and $h_i, \bar{h}_i$ are the left-moving and right-moving conformal weights. Modular invariance implies that $h_i - \bar{h}_i$ is an integer. By

$c_{\mathcal{M}}$ we denote the central charge of the CFT associated to $\mathcal{M}$. The total partition function can then be written as:

$$\begin{aligned}
\mathcal{Z} &= 2(k(k-2))^{1/2} \int_{F_0} \frac{d\tau d\bar{\tau}}{\tau_2} \int_0^1 ds_1 ds_2 \\
& \sum_{w,m=-\infty}^{+\infty} \sum_i q^{h_i} \bar{q}^{\bar{h}_i} e^{4\pi\tau_2(1-\frac{1}{4(k-2)}) - \frac{k\pi}{\tau_2}|(s_1+w)\tau-(s_2+m)|^2 + 2\pi\tau_2 s_1^2} \\
& \frac{1}{|\sin(\pi(s_1\tau - s_2))|^2} \left| \prod_{r=1}^{\infty} \frac{(1-e^{2\pi i r\tau})^2}{(1-e^{2\pi i r\tau - 2\pi i(s_1\tau - s_2)})(1-e^{2\pi i r\tau + 2\pi i(s_1\tau - s_2)})} \right|^2
\end{aligned} \tag{5.2.24}$$

Now we need to disentangle the information hidden in this complicated formula.

5.2.3 Decomposition in characters

We want to connect our partition function computation to expectations from an algebraic analysis for the Hilbert space of the coset theory. To that end we need to manipulate our result further and determine the character contributions of the different affine representations to the partition function. In other words, we have to find the correct Hilbert space to trace over that will reproduce the above partition function. It is appropriate then to first recall some $SL(2,\mathbb{R})$ representation theory. See e.g. [214] for a more complete treatment. The representations of the affine algebras are the modules built on the $SL(2,\mathbb{R})$ representations using the creation modes of the currents. The $SL(2,\mathbb{R})$ representations we will encounter are the (principal) discrete representations with lowest weight $D_j^+ = \{|j,m\rangle : m = j, j+1, j+2, \ldots\}$ where the lowest weight state has J_0^3 eigenvalue $j > 0$ and is annihilated by J_0^-, and similarly for the discrete highest weight representations $D_j^- = \{|j,m\rangle : m = j, j-1, j-2, \ldots\}$. The continuous representations $C_j^\alpha = \{|j,m\rangle : m = \alpha, \alpha \pm 1, \ldots\}$ where $\alpha \in [0,1)$, have an unbounded J_0^3 spectrum and $j = \frac{1}{2} + is$ with s real. The quadratic Casimir of all these representations is $c_2 = -j(j-1)$.

After refreshing our memory on $SL(2,\mathbb{R})$ representations, we return to decompose the partition function into a sum over representations. We will do this in several

steps. We first write the compact boson part in a more recognizable form. Secondly, we expand the partition function into a sum over states. And thirdly we identify contributions from discrete and continuous representations of $SL(2,\mathbb{R})$.

To work towards the spectrum predicted in [170], we first identify the momentum of the compact scalar. The relevant Poisson re-summation is:

$$\sum_{m=-\infty}^{+\infty} e^{-\frac{k\pi}{\tau_2}(m^2-2m((s_1+w)\tau_1-s_2))} = \sqrt{\frac{\tau_2}{k}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\pi\tau_2}{k}(n+\frac{ik}{\tau_2}((s_1+w)\tau_1-s_2))^2} \qquad (5.2.25)$$

where we have re-summed over m and the new integer $n \in \mathbb{Z}$ is the momentum of the scalar. Secondly, after the Poisson re-summation, we expand the infinite products as well as the sin-prefactor in (5.2.24) into an infinite sum of exponential terms. For a state in the $SL(2,\mathbb{R})$ CFT with levels $N, \bar{N}$ and conformal weights $h, \bar{h}$ in the CFT on $\mathcal{M}$ (including reparametrization ghost contributions), the exponent arising from this expansion is:

$$\begin{aligned} \text{exponent}_{expansion} &= 2\pi i\tau_1(N+h-\bar{N}-\bar{h}+(q-\bar{q})s_1) \\ &- 2\pi\tau_2(N+h+\bar{N}+\bar{h}+(q+\bar{q}+1)s_1) - 2\pi i s_2(q-\bar{q}) \end{aligned} \qquad (5.2.26)$$

where q counts the number of $J^+_{n\leq 0}$ minus the number of $J^-_{n<0}$ operators, corresponding to the particular state under examination. A similar definition holds for $\bar{q}$ in terms of the right-moving creation operators. The overall contribution to the exponent is:

$$\begin{aligned} \text{exponent}_{overall} &= 4\pi\tau_2(1-\frac{1}{4(k-2)}) + 2\pi i n s_2 - \frac{\pi\tau_2}{k}n^2 - 2\pi i n\tau_1(w+s_1) \\ &\quad +(2-k)\pi\tau_2 s_1^2 - 2k\pi\tau_2 s_1 w - k\pi\tau_2 w^2. \end{aligned} \qquad (5.2.27)$$

Integrating over s_2 (see (5.2.26) and (5.2.27)) results in the constraint $q-\bar{q}=n$. After substituting $q-\bar{q}=n$, we find the total exponent

$$\begin{aligned} \text{exponent}_{total} &= 2\pi i\tau_1\Big(N+h-\bar{N}-\bar{h}-nw\Big) \\ &\quad -2\pi\tau_2\Big(N+h+\bar{N}+\bar{h}+(q+\bar{q}+1)s_1 \end{aligned}$$

$$-2(1-\frac{1}{4(k-2)})+\frac{n^2}{2k}+\frac{k}{2}w^2+kws_1+\frac{k-2}{2}s_1^2\Big) \quad (5.2.28)$$

The integral over the first holonomy was fairly easy, and gave us one of the expected constraints [170]. It relates the momentum of the compact boson to the $J_0^3 - \bar{J}_0^3$ eigenvalue in the $SL(2,\mathbb{R})$ representation.

The integral over the second holonomy is far less trivial and needs some technical trickery, inspired by the analysis in [199]. It will allow us to separate the contributions from discrete and continuous representations of $SL(2,\mathbb{R})$. We first introduce an auxiliary variable to incorporate a prefactor and the piece of the exponent quadratic in s_1:

$$\sqrt{(k-2)\tau_2}e^{-2\pi\tau_2(\frac{k-2}{2}s_1^2+(kw+1+(q+\bar{q}))s_1)} = \int_{-\infty}^{+\infty} dc\, e^{-\frac{\pi}{(k-2)\tau_2}c^2-2\pi(ic+\tau_2(kw+1+(q+\bar{q})))s_1}.$$

The integration over s_1 is now straightforward:

$$\int_0^1 ds_1 e^{-2\pi s_1(ic+\tau_2(kw+1+(q+\bar{q})))} = \frac{-1}{2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}\left(e^{-2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}-1\right) \quad (5.2.29)$$

Combining it with the quadratic term in c results in the term

$$\frac{-1}{2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}\left(e^{-\frac{\pi}{(k-2)\tau_2}c^2-2\pi(ic+\tau_2(kw+1+(q+\bar{q})))}-e^{-\frac{\pi}{(k-2)\tau_2}c^2}\right) \quad (5.2.30)$$

Discrete representations

Now we observe that the exponent of the first term can be completed to a square if we set $c = 2\tau_2 s - i\tau_2(k-2)$. Shifting the contour of c (for the first term only) from $\mathrm{Im}\, c = 0$ to $\mathrm{Im}\, c = -i\tau_2(k-2)$, picks up residues from the poles of the denominator in the range $-\tau_2(k-2) < \mathrm{Im}\, c < 0$. The poles are located at $c = i\tau_2(kw+1+(q+\bar{q}))$ in the range:

$$-\tau_2(k-2) < \tau_2(kw+1+(q+\bar{q})) < 0. \quad (5.2.31)$$

Now we note that we can interpret the pole contributions to the integral summed over $q, \bar{q}, w, n$, as the trace over a constrained Hilbert space. Consider the product Hilbert space $\hat{\mathcal{D}}_j^+$, the module built on the discrete representation $\mathcal{D}_j^+$ of $SL(2,\mathbb{R})$, and the Hilbert space for the compact boson $\mathcal{H}^{U(1)}$. The first constraint we put on the sum over states is $J_0^3 - \bar{J}_0^3 = n$, namely the constraint we obtained from the s_2-integration. The second constraint determines the quadratic Casimir j of the $SL(2,\mathbb{R})$ representation in terms of the winding number of the compact boson: $J_0^3 + \bar{J}_0^3 = -kw$ or equivalently $kw + 1 + (q + \bar{q}) = 1 - 2j$. One way to see the necessity for this constraint is the fact that the T-duality $(J^3, \bar{J}^3, n, w) \to (\frac{1}{k}J^3, -\frac{1}{k}\bar{J}^3, -w, -n)$ is a symmetry of our partition function (which is reflected in the constraint equations).

The discrete nature of the representations is determined by the fact that the sin-prefactor gives rise to only one kind of operator at level zero, namely J_0^+, and not to J_0^- contributions. Notice moreover that we needn't sum over creation operators for the J^3-current or for the compact boson, since their contributions to the partition function were canceled by the $U(1)$ ghosts. The second constraint $kw + 1 + (q + \bar{q}) = 1 - 2j$, immediately implies (via 5.2.31) the expected bounds on j, the Casimir of the discrete representation:

$$\frac{1}{2} < j < \frac{k-1}{2}. \tag{5.2.32}$$

We emphasize that the upper bound we derived is not the one suggested in [170] but the improved bound[7] derived in [198] for the ungauged $SL(2,\mathbb{R})$ WZW model.[8] Using the constraint we can rewrite the exponent in a familiar form. We obtain a sum over the described Hilbert space $\mathrm{Tr}_{\hat{\mathcal{D}}_j^+ \otimes \hat{\mathcal{D}}_j^+} q^{L_0^{cs}} \bar{q}^{\bar{L}_0^{cs}}$ where the L_0^{cs} operator takes

[7]The improved bound was suggested for the ungauged model on the basis of consistency with the inclusion of spectral flowed representations in [197] and on the basis of fusion rules in [223]. The improved bound was shown to be necessary in the coset model for a tachyon free spectrum in Little String Theory in [183]. Our computation proves this consistency requirement.

[8]As we will see in the following, a continuous spectrum opens up when j reaches either the lower or the upper bound [200, 201].

the standard form:

$$L_0^{cs} = L_0^{SL(2,\mathbb{R})} - L_0^{U(1)}. \tag{5.2.33}$$

The conformal weights of the primary states, which agree with the total exponent after substitution of the values for the poles, are given by:

$$h_{cs} = -\frac{j(j-1)}{k-2} + \frac{(n-kw)^2}{4k} \tag{5.2.34}$$
$$\bar{h}_{cs} = -\frac{j(j-1)}{k-2} + \frac{(n+kw)^2}{4k}. \tag{5.2.35}$$

The summation is over states with the constraints $J_0^3 - \bar{J}_0^3 = n$, $J_0^3 + \bar{J}_0^3 = -kw$ and no contribution from the $J_{n<0}^3$ oscillators. Thus we interpreted the first part of our partition function as a character over a constrained product of an affine discrete $SL(2,\mathbb{R})$ representation times a compact boson. We sum over discrete representations that satisfy the bound (5.2.32). For the parafermion interpretation of this Hilbert space we refer to [226, 197].

We remark that we crucially made use of the periodicity of the Ray-Singer torsion in the u_1-variable in our computation. If we would ignore this periodicity, it is clear from the analysis in [198] that we could identify the winding number w of the compact boson with the parameter w that controls the expansion of the different products in the denominator of the partition function, and therefore with the spectral flow parameter in the $SL(2,\mathbb{R})$-WZW model.[9] This exemplifies in detail the relation uncovered in [197] between spectral flow and the winding of strings, in the coset model.

Continuous representations

We combine now the shifted integral over s of the first term in (5.2.30) with the integral over the second term, in which we re-scale $c = 2\tau_2 s$. Including the summation

[9]Note that this also follows from the fact that the integer holonomies w change the current algebra on the torus according to spectral flow. See e.g. [227].

over winding numbers w, we obtain

$$-\frac{1}{\pi}\sum_{w=-\infty}^{+\infty}\int_{-\infty}^{+\infty}ds\left[\frac{e^{-2\pi\tau_2(N+h+\bar{N}+\bar{h}-2+2\frac{s^2+1/4}{k-2}+\frac{n^2}{2k}+\frac{k}{2}(w+1)^2+(q+\bar{q}))}}{2is+k(w+1)-1+(q+\bar{q})}\right.$$
$$\left.-\frac{e^{-2\pi\tau_2(N+h+\bar{N}+\bar{h}-2+2\frac{s^2+1/4}{k-2}+\frac{n^2}{2k}+\frac{k}{2}w^2)}}{2is+kw+1+(q+\bar{q})}\right] \quad . \qquad (5.2.36)$$

Inspecting the above expression, we observe that the first term of the $w-1$ sector and the second term of the w winding sector share the same exponent after spectral flow of the first by one unit $N,\bar{N}\to N+q,\bar{N}+\bar{q}$. The last operation is based on the isomorphism $\hat{\mathcal{D}}_j^{+,w-1}\cong\hat{\mathcal{D}}_{\frac{k}{2}-j}^{-,w}$ where the second upper index denotes the amount of spectral flow [197], i.e. these are discrete representations defined with respect to the algebra obtained after spectral flow. Combining terms this way we get

$$-\frac{1}{\pi}\sum_{w=-\infty}^{+\infty}\int_{-\infty}^{+\infty}ds\; e^{-2\pi\tau_2(N+h+\bar{N}+\bar{h}-2+2\frac{s^2+1/4}{k-2}+\frac{n^2}{2k}+\frac{k}{2}w^2)}$$
$$\left(\frac{1}{2is+kw-1+(q+\bar{q})}-\frac{1}{2is+kw+1+(q+\bar{q})}\right) \qquad (5.2.37)$$

As in [198], these two terms can be interpreted as representing two halves of a continuous representation with $j=\frac{1}{2}+is$. The first term represents the contribution of a D^- representation (after spectral flow) and the second term still corresponds to a D^+ representation. In particular, note that the second term, when summed over states $(J_0^+\bar{J}_0^+)^r|\psi\rangle$ in $\mathcal{D}^+$, gives rise to a logarithmically divergent sum. We adopt here the regularisation procedure of [198] and introduce a Liouville wall that cuts off the infinite volume otherwise available to the strings in the continuous representation[10]. Thus we obtain a regularised sum over the zero-modes of the following form:

$$-\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{A+r}e^{-r\epsilon}=-\frac{1}{2}\log\epsilon+\frac{1}{2}\frac{d}{dA}\log\Gamma(A),\quad A=is+\frac{1}{2}(kw+1-n). \qquad (5.2.38)$$

[10]A rigorous justification of this procedure would require a precise identification of the coefficient of the exponential suppression after the introduction of a Liouville wall at a finite distance in the target space [198], and a precise treatment of the sum over the J_0^3 charge that is related to the creation and annihilation operators of the J^+ and J^- currents. As in [198], we will find justification for the adopted prescription from an independent scattering amplitude argument.

Similarly, the first term in the integral can be interpreted as an infinite sum over states in a $\mathcal{D}^-$ Hilbert space of the form $(J_0^- \bar{J}_0^-)^r|\psi\rangle$

$$\frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{B-r}e^{-r\epsilon} = -\frac{1}{2}\log\epsilon - \frac{1}{2}\frac{d}{dB}\log\Gamma(-B), \quad B = is + \frac{1}{2}(kw-1+n). \tag{5.2.39}$$

The density of states as a function of s is then found to be

$$\rho(s) = \frac{1}{2\pi}2\log\epsilon + \frac{1}{2\pi i}\frac{d}{2ds}\log\frac{\Gamma(-is+\frac{1}{2}-m)\,\Gamma(-is+\frac{1}{2}+\bar{m})}{\Gamma(+is+\frac{1}{2}+\bar{m})\,\Gamma(+is+\frac{1}{2}-m)}, \tag{5.2.40}$$

where $m = \frac{1}{2}(n-kw)$, $\bar{m} = -\frac{1}{2}(kw+n)$ are the eigenvalues of J_0^3 and $\bar{J}_0^3$. In the above expression we have truncated the range of integration over s to $[0,\infty)$ using the invariance of the exponent under $s \to -s$. Thus, the contribution of the continuous representations of $SL(2,\mathbb{R})$ combined with the momentum and winding modes of the free boson, can be written as

$$\sum_{w,n=-\infty}^{+\infty}\int_0^{+\infty} 2ds\ \rho(s)\mathrm{Tr}_{\hat{\mathcal{C}}_{\frac{1}{2}+is}\otimes\hat{\mathcal{C}}_{\frac{1}{2}+is}} q^{L_0^{cs}}\bar{q}^{\bar{L}_0^{cs}} \tag{5.2.41}$$

where the conformal primaries have weights

$$h_{cs} = \frac{s^2+\frac{1}{4}}{k-2} + \frac{(n-kw)^2}{4k} \tag{5.2.42}$$

$$\bar{h}_{cs} = \frac{s^2+\frac{1}{4}}{k-2} + \frac{(n+kw)^2}{4k} \tag{5.2.43}$$

and the trace over $\hat{\mathcal{C}}_{\frac{1}{2}+is}\otimes\hat{\mathcal{C}}_{\frac{1}{2}+is}$ is subject to the same constraints as before, namely $J_0^3+\bar{J}_0^3 = -kw$, $J_0^3-\bar{J}_0^3 = n$ and the J^3-current and free boson creation operators act trivially.

As in [198], we can perform a consistency check on the density of states by analyzing the phase shift in a scattering experiment. We can introduce a Liouville wall for the continuous representation strings, to cut off the infinite volume available to them[11], and relate the density of states to the phase shift for scattering a string in

[11]In our case the volume divergence is apparent from the pole of the partition function (5.2.24)

the bulk of $SL(2,\mathbb{R})/U(1)$ and then off the Liouville wall [198]. Making use of the fact that the form of the scattering amplitude is the same for the coset theory as for the ungauged $SL(2,\mathbb{R})$ model (see e.g. [183]), we can conclude that the density of states is indeed given by (5.2.40), where we obtain the eigenvalues of the J_0^3 and $\bar{J}_0^3$ operators from the constraint equations on the Hilbert space. This gives an overall consistency check on our regularisation procedure.

at $s_1 = 0 = s_2$. Excising the pole corresponds to introducing the Liouville wall.

Chapter 6

An Off-Shell Digression: Boundary Superstring Field Theory

In the last chapter of the thesis we digress into a topic that is disconnected from the material discussed so far. The reason is that we will be interested in an off-shell formulation of string theory, in sharp contrast to all of our previous investigations which were in the context of the first-quantized on-shell formulation of string theory.

After a general review of string field theory, emphasizing the recent progress in our understanding of open string tachyon condensation, we discuss the bosonic version of boundary string field theory along with a lighting review of the Batalin-Vilkovisky formalism. This will provide the necessary background material for the main subject of this chapter, which is the construction of a similar theory for the superstring, extending Witten's original work. In particular, we will prove a recent conjecture regarding the spacetime action of this theory and we will make a connection with a related conjecture concerning the boundary entropy of supersymmetric quantum field theories in two-dimensional spaces with boundaries.

Our boundary superstring field theory will be shown to be in some sense much simpler that its bosonic counterpart. This is quite different from what happens in the case of the cubic string field theory, where the superstring version is in general more complicated than the bosonic one. The new material presented here is based on [4]. We should mention that an independent work on the same problem appeared

in [228].

6.1 Motivation: string field theory and tachyon condensation

The quest for an off-shell formulation of string theory, also known as string field theory (SFT), is closely related to the problem of tachyons and the fate of the corresponding theories after their condensation. The latter received a great deal of attention from the very early days of string theory (when string theory was known as "dual resonance model"!) [229, 230, 231, 232, 233]; it is one of the great achievements of the recent resurgence of interest in string field theory that some aspects of tachyon condensation in the open string sector have been clarified.

One motivation for constructing a field theory of strings stems from the fact that the first-quantized description of string interactions is clearly perturbative; one has to sum over all Riemann surfaces up to a given genus and the contribution of each surface is weighted by the exponential of the vev of the dilaton, that is the string coupling g_s, to a power that is proportional to the genus (i.e. number of holes of the surface). It is well-known from ordinary quantum field theory however, that the perturbative expansion fails to capture phenomena like quark confinement or instanton corrections to amplitudes, since they are intrinsically non-perturbative. That means simply that even if the perturbation series could somehow be re-summed, these phenomena would still be invisible since their dependence on the coupling constant λ is usually of the form e^{-1/λ^2} which does not admit a series expansion around $\lambda = 0$. In light of the importance of such non-perturbative phenomena in field theory and also the realization that this type of effects are present and quite important in string theory as well, one would welcome any formulation of string theory that goes beyond the perturbative first-quantized approach.

The need for string field theory is also evident from a spacetime point of view; recall that in the first-quantized description of string theory one essentially quan-

tizes a free theory, the 2-dimensional world-sheet CFT, and he assumes that energy eigenstates in this CFT, which are in 1-1 correspondence with local operators on the world-sheet known as vertex operators, are asymptotic states corresponding to on-shell excitations in spacetime. The miracle of string theory is that correlation functions of these vertex operators in the 2-dimensional world-sheet quantum field theory are found to be equivalent to S-matrix elements for the corresponding asymptotic spacetime states, where the relevant interactions include the familiar Yang-Mills and gravitational force and various generalizations.

The correspondence of asymptotic string states with local operators, however, is valid only when the states ($\equiv$ operators) are on-shell, i.e. their left- and right-moving conformal weights are both one for the closed string and similarly for the open. This is necessary for preserving conformal invariance on the world-sheet theory in the presence of the vertex operator. Using conformal invariance we can map the world-sheet with a number of (on-shell) vertex operator insertions, to a world-sheet that describes the propagation of (on-shell) particles from infinity towards a region where they interact. Obviously, without conformal invariance this correspondence is no longer valid and it is not obvious how to define scattering in spacetime in terms of world-sheets.

The requirement of having conformal weights one immediately fixes the momentum of the corresponding state in spacetime. This is due to the fact that for a state with specific quantum numbers, the vertex operator conformal weights $(h, \tilde{h})$ depend on the square of the momentum (i.e. the mass of the particle) and on the form of the vertex operator, which is determined by the quantum number of the state we consider. For example, the graviton with polarization tensor $s_{\mu\nu}$ and momentum k_μ is given by the vertex operator (up to irrelevant numerical factors)

$$\int d^2z \; s_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}. \tag{6.1.1}$$

Conformal invariance of the above expression leads to the condition $k^2 = 0$ since the conformal weights of d^2z are $(-1,-1)$ and have to cancel with those of $\partial X^\mu \bar{\partial} X^\nu$

which are $(1,1)$ and the contribution of $e^{ik\cdot X}$ which is $(k^2\alpha'/4, k^2\alpha'/4)$. In addition, the tensor $s_{\mu\nu}$ is further constrained from conformal invariance to have the gauge invariance properties one expects for the polarization tensor of a graviton. Hence, conformal invariance requires that there are only massless gravitons in string theory and no virtual ones that go off the mass-shell. Generalizing this argument, we conclude that only on-shell particle excitations can be assigned to vertex operators in a way consistent with the symmetries of the first-quantized formulation of string theory.

Since conformal invariance of vertex operators is sine qua non for a spacetime interpretation of the corresponding states, it is obvious that naively making these operators off-shell by extrapolating them to generic momenta is not going to work. Furthermore, the spacetime S-matrix elements that are obtained from on-shell vertex operators obviously do not contain enough information to reconstruct the generic (off-shell) spacetime action. Even though these S-matrices can be used to determine the low-energy effective actions of string theories, extending those actions to include massive modes is bound to be ambiguous up to terms that vanish on-shell, since our starting point contained only on-shell information.

The above comments indicate that one has to extend the first-quantized description of strings and in order to do so a new input - outside the realm of on-shell strings - is required. This input may be simply an ad hoc interaction postulated between strings that is self-consistent and reproduces the S-matrices of first-quantized string theory when particles are on-shell. Or, as we will discuss in more detail when we review boundary string field theory, it may be based on the fact that on-shell string theories correspond to 2-dimensional conformal field theories and accordingly, off-shell strings could be describable by more general 2-dimensional quantum field theories.

The development of string field theory reached its peak with the seminal work of Witten [234] (unfortunately we won't be able to make justice to the huge preceding literature; for more references see for example [235]). In this paper Witten constructed a covariant string field theory of open bosonic strings, known as cubic or Chern-Simons string field theory, which proved to be a powerful tool in addressing the issues related

to the tachyon. The name cubic stands for the cubic interaction that is postulated between string fields; hence, the additional input falls in the first category of the previous paragraph. The mathematical basis of this SFT has also intriguing ties with non-commutative geometry [236, 237], a branch of differential-algebraic geometry that studies the geometry of non-commutative ("quantized") manifolds, and which has also appeared in string theory in the recent non-perturbative developments [79, 80, 81].

The basic object of the cubic SFT is the so-called string field, which can be thought of as a collection of an infinite number of ordinary spacetime fields corresponding to the infinite tower of possible internal string states. We can also think of the string field as a state in the Fock space of the world-sheet Hamiltonian. In the Feynman-Siegel gauge, the leading terms in the expansion of the string field in terms of the basis of states in this Fock space is

$$|\Psi\rangle = (T(x) + A_\mu(x) a^\mu_{-1} + \frac{1}{\sqrt{2}} B_{\mu\nu}(x) a^\mu_{-1} a^\nu_{-1} + \cdots)|0\rangle. \tag{6.1.2}$$

Here, $T, A_\mu, B_{\mu\nu}$ are the spacetime fields corresponding to the tachyon, the gauge field and the antisymmetric tensor, while $|0\rangle$ is related to the standard $SL(2,\mathbb{Z})$ invariant vacuum $\Omega\rangle$ as $|0\rangle = c_1\Omega\rangle$, c being the ghost field. In general, the summation in (6.1.2) is over the infinite spectrum of string states with ghost number one.

The cubic SFT action is given by

$$S = \frac{1}{2\alpha'}\int \Psi * Q\Psi + \frac{g}{3!}\int \Psi * \Psi * \Psi \tag{6.1.3}$$

with Q the open string BRST operator, so that when the interaction term is turned off, i.e. $g = 0$, the equations of motion for Ψ are just the statement of BRST invariance $Q\Psi = 0$. The assumption of [234] is that inclusion of the cubic interaction between the string fields should account for all spacetime interactions between the component fields like $T, A_\mu, B_{\mu\nu}$ etc. In particular, the cubic SFT action (6.1.3) determines an ordinary spacetime action describing the dynamics of the string fields, and S-matrices obtained from it should agree with the ones derived from the first-quantized formulation of string theory.

Quantizing the cubic SFT is straightforward; one has to treat the interaction term as a small perturbation and follow the usual rules of quantum field theory. The requirement however that the corresponding amplitudes agree with the ones obtained from first-quantized strings including quantum corrections, and in particular that the moduli space of higher genus world-sheets is covered fully and only once, is non-trivial. This was shown to be indeed the case in [238]; hence, the equivalence of the quantum cubic SFT on-shell amplitudes with the S-matrices of ordinary string theory is well established.

Notice that this SFT is for open strings only. The other version of open SFT, known as boundary string field theory, will be discussed in more detail in the next section. The construction of closed string field theory has been proved to be substantially more complicated than its open string counterpart (see [239] for the state of the art and [240] for a general introduction to closed SFT). Unlike the open SFT of [234] which was successfully applied to the problem of open string tachyon condensation, the closed SFT [239] has not yet provided any insight on the problem of closed string tachyons and their condensation [1]. Since closed strings contain gravity, which corresponds to fluctuations of the geometry of spacetime, understanding closed string tachyons is certainly bound to give some further insight into the nature of stringy geometry.

Superficially, the main problem of existing versions of closed string field theories is their complexity. However, there are also some arguments, based on general principles such as holography, that seem to suggest that an off-shell formulation of theories that contain quantum gravity may be actually impossible (see for example [243, 244] for some related comments).

The recent interest in open string field theory was fueled by Sen, who made three conjectures regarding the tachyon condensation in open bosonic string theory [245, 246]. Recall that the critical bosonic string theory requires 26 dimensions and it has a tachyon in both the open and closed string sector. The tachyon signals an instability

[1]Very recently, a particular type of closed string tachyons was attacked from other perspectives which follow the philosophy of boundary string field theory; see for example [241, 242, 243].

which may be simply due to a wrong choice of vacuum or, more severely, a sign that the theory is inconsistent and has to be abandoned.

Assuming that the first possibility is correct, there are a number of questions regarding the effects of the condensation and the nature of the theory that describes the stable vacuum. Furthermore, note that the condensation - even though it is triggered by quantum fluctuations - is a classical phenomenon which requires an off-shell formulation of the theory in order to be studied. For example, the form of the tachyon potential cannot be determined just by the on-shell information encoded in first-quantized string theory S-matrices. Since this potential is the foremost quantity of interest regarding tachyon dynamics, one has to resort to string field theory where, at least in principle, this potential is computable.

The importance of Sen's conjectures is that they provide a set of checks for testing the existing versions of open string field theory. The first conjecture is based on the notion of non-BPS D-branes. These are Dp-branes with $p = 0, 2, 4, 6, 8$ in type IIB and $p = 1, 3, 5, 7, 9$ in type IIA superstring theory while they can have $0 \leq p \leq 26$ in bosonic string theory. Their world-volume supports a tachyonic mode arising from the open string sector; hence, the effective field theory has an instability and one expects that the corresponding D-brane will eventually decay. Since these branes are not stable, they cannot be supersymmetric; this is why they are known as non-BPS D-branes.

Sen pointed out that the tachyon arising from quantizing open strings in 26 dimensions is essentially the tachyonic mode on a spacetime filling D25-brane. Condensation of the tachyon to its stable vacuum value corresponds to decay of the D25-brane. It is natural to conjecture that the energy density per unit volume on the D25-brane should equal the energy density stored in the tachyon potential. This automatically provides a way of inter-checking the validity of this conjecture in conjunction with the trustworthiness of the tachyon potential computed with SFT techniques.

The first such computation was performed by Sen and Zwiebach [247], using earlier results on the tachyon potential in the cubic SFT [248, 249]. It was found that indeed there is a remarkable agreement between the D25-brane tension and the height of the

tachyon potential: in the lower-order computation the agreement is 70%, but keeping relevant scalars up to four mass levels above the tachyon it rises to 99%!

Sen's conjecture was further substantiated by impressive numerical computations which showed a 99.91% agreement [250]. Thus, the predictive power of the cubic SFT was established beyond any doubt. In particular, one should be able in principle to construct the string field that corresponds to a D25-brane arising as a fluctuation in the stable vacuum. Since the D-brane is a non-perturbative object, we see that SFT actually contains information about the string spectrum beyond what is found by perturbative means. This is very pleasing as some of the motivation for studying string field theory stems exactly from the fact that it may be the proper framework for understanding non-perturbative effects in string theory.

The second conjecture of Sen relates to the fact that one should be able to construct solitonic type solutions of the tachyon equations of motion which would correspond to D-branes of lower dimensionality. Such configurations would also be unstable since there are no stable D-branes in bosonic string theory. Several solutions of this type were indeed found in [251].

The last conjecture concerns the theory that remains after the tachyon has rolled to its minimum. At this point, the D25-brane has disappeared. Hence, one expects that physical open string degrees of freedom, which were supported on the D25-brane, should be absent. In other words, the open string BRST cohomology should be trivial. Moreover, it is natural to assume that the remnant theory contains closed strings, which were also present in the original one since closed strings can arise in open string loops. However, studying the dynamics of closed strings in the unstable vacuum is quite difficult in practice. Sen's claim is that in the stable vacuum, closed strings and their interactions should be easier to be identified and studied. This is a very strong statement; if true, it implies that open strings are more fundamental that closed ones and, in particular, that open SFT describes all of string theory, open and closed. In other words, open SFT should contain a sub-sector that corresponds to a closed SFT in disguise. It would be very interesting to understand how the construction of [239] connects with these ideas. To summarize, Sen's third conjecture asserts that after

the condensation we are left with the closed string vacuum.

Unfortunately, the last conjecture has been proved to be stubborn to systematic analysis. The main problem is that the string field that minimizes the SFT action (6.1.3) is actually unknown in explicit form. It can be constructed numerically in a finite series form, which is known as level expansion since the string field is truncated at a given mass level, and this is actually the technique used to compute the energy of the tachyon potential. The level expansion however becomes increasingly involved as more massive modes are included and, in any case, it does not seem to be the proper framework to address the possibility of closed strings appearing in the stable vacuum. Numerical evidence for the absence of open string physical states at the stable vacuum was given in [252].

So far, there has not been a definite argument in the literature concerning the existence or not of closed strings when the tachyon condenses. Since the main obstacle is the lack of explicit solutions of the classical cubic SFT, a novel approach was presented in [253] (see also [254] for a review). The idea is to postulate an SFT action describing fluctuations around the stable vacuum, by assuming that indeed the open string BRST cohomology has to be trivial there. The resulting theory, dubbed vacuum SFT, seems to be a promising candidate for understanding the structure of the stable vacuum.

We should mention at this point that extending these results to the superstring, where there can be open string tachyons on non-BPS D-branes or brane-antibrane pairs, and localized closed string tachyons on orbifolds that break all supersymmetry, is not straightforward. In particular, the working version of cubic superstring field theory [255] is substantially more cumbersome that its bosonic cousin. Nevertheless, it provides a framework were non-BPS D-branes or brane-antibrane systems and their tachyons can be analyzed systematically. For example, in [256] it was shown numerically that the tachyonic potential energy agrees to 85% accuracy with the energy stored on the unstable system of D-branes.

In face of the fact that an analytical approach to some of the above issues seems to be very complicated in the framework of cubic-type SFTs, it is natural to seek for

alternatives. As we mentioned earlier, there is another open SFT, known as boundary string field theory (BSFT), constructed by Witten in [257] and further developed in [258, 259, 260], which has certain advantages. For example, in contrast to the cubic string field theory, where an infinite number of fields condense, BSFT allows for the tachyon field to condense alone, while the remaining open string tower continues to have vanishing expectation values [261]. Hence, the analysis of tachyon condensation is simplified considerably [262, 263]. In addition, the exact expressions for the tachyon potential and the descent relations among D-branes, as expected from [245], can be derived in a straightforward manner in this approach (see [264] for a discussion).

The idea behind BSFT is to consider the classical open string field theory as a theory on the space of all boundary interactions on the disk with a fixed conformal world-sheet action in the bulk. This is motivated by the fact that irrelevant operators on the world-sheet, i.e. with conformal weight $h > 1$, correspond to massive spacetime fields, while relevant operators having $h < 1$, correspond to tachyons.

The implementation of this idea relies on the use of the Batalin-Vilkovisky (BV) formalism [265, 266]. This construction was initially presented in [257] for the bosonic string and provides a generic expression for the spacetime action that can be related to the disk partition function in a simple way [2] [258, 259, 260]

$$S_B = Z - \beta^i \frac{\partial}{\partial \lambda^i} Z \tag{6.1.4}$$

with β^i the world-sheet beta function for the boundary coupling λ^i.

In the absence of an analogous construction for the superstring, the authors of [267] conjectured that the corresponding spacetime action S_F in the presence of world-sheet supersymmetry, should be exactly equal to the disk world-sheet partition function Z. This conjecture was motivated by a number of arguments involving the properties of Z at the conformal points, its finiteness, which is guaranteed by the presence of world-sheet supersymmetry, as well as similar proposals [268, 269, 270] in the context of a low-energy effective description. Further evidence for the validity of this choice

[2] To derive this relation one must also assume that ghosts and matter are decoupled.

was provided by the consistency of the results obtained from the analysis of tachyon condensation in superstring theory (see for instance [267, 271, 272]).

The objective of the rest of this chapter is to complete the above picture by repeating the analysis of [257, 258] and applying the BV formalism on the space of supersymmetric world-sheet boundary perturbations on the disk. Our formulation of boundary superstring field theory (SBSFT) will be cast entirely in superspace language. The crucial parts of the construction involve the definition of a fermionic vector field and the definition of the appropriate BV antibracket. The latter presents subtleties associated to the nature of the superconformal ghosts and the existence of different "pictures". Nevertheless, an appropriate antibracket exists and the BV formalism provides a corresponding spacetime action. Furthermore, as was conjectured in [267], this action is exactly equal to the disk partition function.

Before delving into the details of our construction, we review first the bosonic BSFT.

6.2 Review of bosonic boundary string field theory (BSFT)

The world-sheet σ-model approach to string theory [269] suggests that spacetime fields should be viewed as generalized coupling "constants" of two-dimensional world-sheet interactions. If we think of the string field as a collection of spacetime fields, this picture implies that the string field simply encodes the data of a two-dimensional field theory. It is thus natural to expect that string field theory can be formulated on the "space of all two-dimensional field theories".

Boundary string field theory (BSFT) [3] is an open string field theory based on this idea. At the classical level its precise formulation relies on the application of the BV formalism [4] on the configuration space of all two-dimensional field theories on

[3] BSFT was originally known as Background Independent Open String Field Theory since its formulation is independent of the choice of a (on-shell) closed string background.

[4] The application of the BV formalism has also been very useful in the formulation of the open

the disk with arbitrary boundary interaction terms and a fixed conformal world-sheet action in the bulk.

6.2.1 A synopsis of the Batalin-Vilkovisky formalism

In the following we review some basic features of the BV formalism [265, 266] emphasizing those that play a prominent role in the construction of BSFT. For a more detailed exposition on the BV formalism see [274, 275]

One starts with a field configuration space $\mathcal{M}$ equipped with a supermanifold structure, a non-degenerate closed odd two-form ω and a $U(1)$ symmetry under which ω has charge -1. We call this charge (BV) ghost number. The two-form ω gives rise to an antibracket, in the same way that a symplectic form gives rise to a Poisson bracket. In local coordinates u^I, the antibracket of two functions F and G is given by

$$\{F, G\} = \frac{\partial_r F}{\partial u^I} \omega^{IJ} \frac{\partial_l G}{\partial u^J}. \tag{6.2.5}$$

The focal point of the BV formalism is the master equation

$$\{S, S\} = 0 \tag{6.2.6}$$

for the master action S. This equation guarantees the gauge invariance of the system. Notice that because of the fermionic nature of ω the master equation is not trivially zero.

One way to satisfy the master equation identically is by choosing an appropriate fermionic vector field V on $\mathcal{M}$ that has ghost number 1 and satisfies the equation

$$i_V \omega = dS. \tag{6.2.7}$$

Borrowing the language of the Hamiltonian formalism, we would say that V is the

cubic string field theory [273, 235] and closed string field theory [239].

"Hamiltonian vector field" of the action S. As usual, $i_V\omega$ denotes the inner product of the vector V with the form ω and d is simply the exterior derivation on $\mathcal{M}$. The master equation will be satisfied identically if and only if V is nilpotent, i.e. $V^2 = 0$ and there exists at least one point where $V = 0$ [257]. The latter is a very natural requirement because the points of vanishing V are precisely the extrema of the spacetime action where the classical equations of motion are satisfied.

Moreover, the exterior derivative of equation (6.2.7) gives

$$d(i_V\omega) = 0. \tag{6.2.8}$$

Given the closedness of the antibracket, it is not hard to show that this is precisely the statement that ω is invariant under infinitesimal transformations generated by V, i.e. that

$$\mathcal{L}_V\omega = (di_V + i_V d)\omega = 0. \tag{6.2.9}$$

Hence, by the Poincare lemma, the property that the antibracket is V-invariant guarantees the existence of an action S given by (6.2.7). This is true, of course, only locally on the space of theories.

To summarize, the strategy for constructing a gauge invariant open string spacetime action involves two steps. First we define a nilpotent fermionic vector field V of ghost number 1 that plays the role of the "Hamiltonian" vector field of the action S and second we appropriately define a V-invariant odd symplectic two-form ω with ghost number -1. The master action S is then determined by (6.2.7).

We move on now to a brief discussion of the bosonic boundary string field theory.

6.2.2 Summary of BSFT

In the bosonic BSFT [257] the above construction works as follows. The configuration space $\mathcal{M}_B$ consists, roughly speaking, of all possible two-dimensional world-sheet theories with a fixed conformal action $\mathcal{S}_{\text{bulk}}$ in the bulk of the disk (at the classical

level) and a boundary action $\mathcal{S}_{\text{bdy}}$ with arbitrary interactions on the boundary. After choosing, for instance, the standard flat action for the bulk, the dynamics of the two-dimensional field theories we want to consider are given by

$$
\begin{aligned}
&\mathcal{S} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdy}} = \\
&\frac{1}{4\pi\alpha'}\int_D d\sigma^1 d\sigma^2 \sqrt{h}(h^{ab}\partial_a X^\mu \partial_b X_\mu) + \\
&\frac{1}{2\pi}\int_D d\sigma^1 d\sigma^2 \sqrt{h}\, b^{ab}\partial_a c_b + \frac{1}{2\pi}\oint_{\partial D} d\tau \mathcal{V}
\end{aligned}
\tag{6.2.10}
$$

where D is the disk world-sheet, h^{ab} a rotationally invariant metric and τ a periodic coordinate on the boundary $\partial D = S^1$. Different points on the configuration space correspond to different choices of the boundary operator $\mathcal{V}$.

The vector field V is associated to the flow generated on $\mathcal{M}_B$ by the bulk BRST charge Q. Under this definition it is straightforward to check that V has the required properties. It is nilpotent and has ghost number 1. Furthermore, when the classical equations of motion are satisfied, BRST invariance is restored and V is vanishing as expected.

The construction of the odd symplectic two-form ω is more involved. [257] first defines ω on-shell and later extends the definition off-shell. This extension, however, involves a subtle redefinition of the actual degrees of freedom of the theory and in fact it leads to an enlargement of the actual space of theories. This subtle point comes about in the following way. Tangent vectors to an on-shell submanifold of $\mathcal{M}_B$ are spin one primary fields $\delta\mathcal{V}$. Since the BRST transformation of such fields should leave the world-sheet action invariant we deduce that there must be some operator O of ghost number 1, such that

$$
\{Q, \delta\mathcal{V}\} = \partial_\tau \mathcal{O}. \tag{6.2.11}
$$

On-shell, O can be uniquely determined [5] from $\delta\mathcal{V}$. It satisfies the following two

[5] Up to a total derivative that has no effect on the world-sheet theory.

equations

$$\delta\mathcal{V} = b_{-1}O \tag{6.2.12}$$

and

$$\{Q, O\} = 0. \tag{6.2.13}$$

Off-shell, however, the second equation is no longer valid and the first one, which still makes sense, determines O only up to terms of the form $b_{-1}(...)$. Since O seems to be more fundamental than $\delta\mathcal{V}$ in some respects, it was proposed in [257] to consider an enlarged space of theories determined not only by the world-sheet action (6.2.10) but also by a ghost number 1 local operator O satisfying equation (6.2.12). Thus, it seems more appropriate to view $\mathcal{M}_B$ as the space of the operators O and not as the space of the boundary perturbation operators $\mathcal{V}$.

Hence, given two vectors $\delta_1 O$ and $\delta_2 O$ at a point O of the enlarged configuration space, we define the odd symplectic form ω by

$$\omega(\delta_1 O, \delta_2 O) = (-)^{\epsilon(\delta_1 O)} \oint_{\partial D} d\tau_1 d\tau_2 \langle \delta_1 O(\tau_1) \delta_2 O(\tau_2) \rangle \tag{6.2.14}$$

with the correlation function being computed in the world-sheet theory with boundary interaction $\mathcal{V} = b_{-1}O$. This definition is slightly different from that of [257] by a sign factor. This factor is introduced in order to get the correct exchange property

$$\omega(\delta_1 O, \delta_2 O) = (-)^{(\epsilon_1+1)(\epsilon_2+1)+1} \omega(\delta_2 O, \delta_1 O) \tag{6.2.15}$$

where $\epsilon_i = \epsilon(\delta_i O)$. For a similar definition in the context of closed string field theory see for example [240]. Notice also that in the above expressions we still define the statistics of the arguments of ω as the natural statistics of the corresponding $\delta\mathcal{V}$ fields. For the precise off-shell definition of b_{-1} and a proof that (6.2.14) has the required properties, we refer the reader to [257].

Now that we have established the needed BV structure we can easily write down an expression for the master action S_B (up to an irrelevant sign) by using equation (6.2.7) and the explicit form of the fermionic vector field $V(O) = \{Q, O\}$

$$dS_B = \oint_{\partial D} d\tau_1 d\tau_2 \langle dO(\tau_1)\{Q, O\}(\tau_2)\rangle. \tag{6.2.16}$$

Under the simplifying assumption that ghosts and matter are decoupled one can set $O = c\mathcal{V}$. In that case, using different approaches, it was proved in [258, 259, 260] that

$$S_B = Z - W^i \frac{\partial}{\partial \lambda^i} Z \tag{6.2.17}$$

where a generic expansion of the boundary operator $\mathcal{V} = \sum_i \lambda^i \mathcal{V}_i$ has been implied. W^i is a vector field on $\mathcal{M}_B$. More precisely, it was identified up to second order in conformal perturbation theory [259, 260] with the beta function β^i, which corresponds to the world-sheet RG flow of the coupling λ^i.

6.3 Boundary superstring field theory (SBSFT)

6.3.1 The BV formalism of SBSFT

Boundary superstring field theory is formulated on the space $\mathcal{M}_F$ of all world-sheet supersymmetric two-dimensional field theories on the superdisk with the usual NSR action in the bulk.

In order to have a manifestly supersymmetric formalism we use a superspace notation. In particular, the world-sheet action takes the following form

$$\begin{aligned}
&\mathcal{S}_{NSR} = \mathcal{S}_{\text{bulk}} + \mathcal{S}_{\text{bdy}} = \\
&\frac{1}{4\pi\alpha'} \int d^2z\, d^2\theta D_{\bar\theta} \mathbf{X}^\mu D_\theta \mathbf{X}_\mu + \\
&\frac{1}{2\pi} \int d^2z\, d^2\theta B D_{\bar\theta} C + \int d\tau d\theta \mathcal{V}.
\end{aligned} \tag{6.3.18}$$

Our conventions follow those of [7, 276] and for the ghost and antighost superfields respectively we set

$$C(z,\theta) = c(z) + \theta\gamma(z) \tag{6.3.19}$$

$$B(z,\theta) = \beta(z) + \theta b(z). \tag{6.3.20}$$

The bottom and upper components of the superfield boundary perturbation will be generically given in the form

$$\mathcal{V}(\tau,\theta) = D(\tau) + \theta U(\tau). \tag{6.3.21}$$

For example, in the case of a single unstable non-BPS D9 brane in type IIA superstring theory, the tachyon perturbation is given by the world-sheet action

$$\mathcal{S}_{\text{bdy}} = \int_{\partial D} d\tau d\theta (\Gamma D\Gamma + \Gamma T(\mathbf{X})) \tag{6.3.22}$$

with $\Gamma = \eta + \theta F$ an auxiliary boundary fermion [267, 261, 277]. In that case

$$\mathcal{V} = \Gamma D\Gamma + \Gamma T(\mathbf{X}). \tag{6.3.23}$$

Furthermore, we can express the superconformal ghosts in a bosonized form [276]

$$\beta(z) = e^{-\phi(z)}\partial\xi(z), \qquad \gamma(z) = e^{\phi(z)}\eta(z) \tag{6.3.24}$$

where ξ is a fermion of dimension 0 and η a fermion of dimension 1. Since this language will be very useful for the subsequent analysis, let us briefly recall a few relevant facts.

The zero-mode of ξ does not enter the above bosonized expressions and this results in a multiplicity of physically equivalent vacuum states that lead to different irreducible representations of the superconformal algebra, known as "pictures". A

vertex operator with a factor $e^{q\phi}$ is by definition in the q picture.

When calculating on-shell amplitudes, pictures can be used in a more or less arbitrary manner as long as the total superghost number of the insertions is -2. In terms of the bosonized form of the ghosts this condition implies a total ϕ-charge -2. For example, one may include two vertex operators in the -1 picture and the rest in the 0 picture [276]. Switching between different pictures can be achieved by the use of the picture changing operator

$$X = Q \cdot \xi = -\partial\xi c + e^{\phi} T_F^m - \partial\eta b e^{2\phi} - \partial(\eta b e^{2\phi}) \tag{6.3.25}$$

which increases the ϕ-charge by 1 or the use of the inverse picture changing operator [278, 279]

$$Y = -\partial\xi c e^{-2\phi} \tag{6.3.26}$$

which decreases the ϕ-charge by 1.

The freedom of moving a picture changing operator inside an amplitude is not, however, a valid off-shell operation. This is an important subtlety of the superstring case and must be taken into account in the following manipulations.

The definition of V

In complete analogy to the bosonic case, it is again natural to associate the fermionic vector field V to the flow generated on $\mathcal{M}_F$ by the bulk BRST charge Q. The only extra subtlety in the superstring case is that we choose $\mathcal{M}_F$ as a space of superfields. Hence, a sensible definition of a vector field should generate flows that respect this property. More precisely, this property is satisfied if and only if the generator of the corresponding flow anticommutes with the generator of world-sheet SUSY. In our case, this is true since

$$\{Q, G_{-1/2}\} = 0. \tag{6.3.27}$$

Moreover, one can check that V also inherits the rest of the required properties. It is nilpotent, because Q is nilpotent, and has ghost number 1. We should emphasize that by ghost number we mean here the BV ghost number that coincides with the total ghost and superghost number of the bc and $\beta\gamma$ systems respectively. The details of this definition of V are the same as those of the bosonic case and we refer the reader to [257].

The definition of ω

We are looking for an appropriate two-form on the configuration space $\mathcal{M}_F$ of supersymmetric boundary perturbations on the superdisk. In order to grow the right intuition about this form we first consider what happens on-shell. Let us also simplify the situation further by assuming that ghosts and matter are decoupled, so that the boundary interaction $\mathcal{V} = D + \theta U$ has no ghost dependence. In the NS sector the -1 picture vertex operator corresponding to $\mathcal{V}$ will be of the form [6]

$$\Lambda = -ce^{-\phi}D. \tag{6.3.28}$$

We obtain a 0 picture representation of this operator by acting with the picture changing operator X

$$X \cdot \Lambda = \gamma D - cU. \tag{6.3.29}$$

This expression is precisely the upper component of the superfield

$$G = C\mathcal{V} \tag{6.3.30}$$

which is the natural supersymmetric generalization of the corresponding bosonic expression $O = c\mathcal{V}$. We propose that G should be considered as the fundamental object of boundary superstring field theory. This is actually analogous to the classical string

[6] The vertex operators given by Λ correspond to the so-called strongly physical states. For a related discussion see [279] .

field of the modified cubic superstring field theory [280, 281, 282] where the basic object is a 0 picture, ghost number 1 operator.

Given two superfield tangent vectors $\delta_1 \mathcal{V}$ and $\delta_2 \mathcal{V}$ we define the odd symplectic two-form ω in the superstring case as a two-point function at the perturbed point $\mathcal{V}$ in the following way

$$\omega(\delta_1 G, \delta_2 G) = (-)^{\epsilon(\delta_1 G)} \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \langle Y(\tau_1)\delta_1 G(\tau_1, \theta_1) Y(\tau_2)\delta_2 G(\tau_2, \theta_2)\rangle \tag{6.3.31}$$

with $\delta_1 G$ and $\delta_2 G$ given by (6.3.30). The main difference with respect to the bosonic definition are the two insertions of the inverse picture changing operator Y in front of the 0 picture superfields. Their presence is required because a non-vanishing expectation value should have a total superghost number -2.

Expression (6.3.30), however, is only valid on-shell and only under the assumption of ghost-matter decoupling. Nevertheless, we can rewrite it as

$$\mathcal{V} = b_{-1} G. \tag{6.3.32}$$

As was explained in [257], this form also makes sense off-shell. Hence, for the off-shell definition of the antibracket, we propose to consider the defining relation (6.3.31) but with tangent vectors $\delta_1 G$ and $\delta_2 G$ given now (implicitly) by (6.3.32) and not by (6.3.30).

Equation (6.3.32) does not define the superfield G uniquely. As in the bosonic case, we circumvent this problem by considering an enlarged space of theories $\mathcal{M}_F$ determined not only by the world-sheet action (6.3.18), but also by a 0 picture, ghost number 1 local superfield G satisfying (6.3.32). In some sense, we consider the superfields G as the fundamental degrees of freedom of the theory. Nevertheless, we still define the statistics of the arguments of ω as the natural statistics of the corresponding $\delta\mathcal{V}$ fields.

We also want to emphasize that contrary to our on-shell experience from the first quantized description of string theory, the position of the picture changing operators in the definition of ω is important. We cannot move them freely inside the correlator

because this is not a valid off-shell operation.

The next step is to verify that ω given by (6.3.31) has all the required properties, i.e. that it is a V-invariant non-degenerate odd symplectic two-form with BV ghost number -1.

Since the total ghost number of the vacuum is -1 (-3 from the bc system and +2 from the $\beta\gamma$ system) and the ghost number of both insertions $Y \cdot \delta_i G$ is 0, we deduce that ω has BV ghost number -1, as expected.

The statistics $\epsilon(\omega)$, on the other hand, is given by the sum

$$\epsilon(\delta_1 G) + \epsilon(\delta_2 G) + 2\epsilon(Y) + 2 = \epsilon(\delta_1 G) + \epsilon(\delta_2 G) \pmod 2 \tag{6.3.33}$$

with the extra +2 in the left-hand side coming from the two θ integrations. Since a non-vanishing correlator requires $\epsilon(\delta_1 G) + \epsilon(\delta_2 G) = 1 \pmod 2$ we get

$$\epsilon(\omega) = 1 \pmod 2 \tag{6.3.34}$$

and therefore ω is odd.

In order to show the non-degeneracy of the antibracket let us go on-shell. In that case ω vanishes for BRST exact insertions and therefore we may regard it as a two-form on the space of classical solutions. It is non-degenerate because it is related to the Zamolodchikov metric g on the space of conformal field theories. We can prove this by setting $\delta_1 G = C\mathcal{V}$ and $\delta_2 G = C\partial C\mathcal{W}$, with V and W two spin 1/2 primary matter superfields. It follows that $\omega(\delta_1 G, \delta_2 G) \propto g(D_V, D_W)$, where D_V and D_W are the bottom components of the matter superfields. Thus, the non-degeneracy of ω follows from the non-degeneracy of the Zamolodchikov metric in complete analogy to the bosonic situation [257].

To prove that $d\omega = 0$, let us introduce local coordinates λ^i on $\mathcal{M}_F$. We can then write a generic superfield tangent vector δG in terms of the expansion $\delta G = \sum_i \lambda^i \delta_i G$. By definition, we have

$$d\omega(\delta_i G, \delta_j G, \delta_k G) = \frac{\partial}{\partial \lambda^i}\omega(\delta_j G, \delta_k G) \pm (\text{cyclic permutations}). \tag{6.3.35}$$

The derivative with respect to λ^i gives

$$\frac{\partial}{\partial\lambda^i}\omega(\delta_j G, \delta_k G) = \tag{6.3.36}$$
$$\int \prod_{\beta=1}^{3} d\tau_\beta d\theta_\beta (-)^{\epsilon_j} \Big\langle \Big(b_{-1}\delta_i G(\tau_1, \theta_1)\Big) Y(\tau_2)\delta_j G(\tau_2, \theta_2) Y(\tau_3)\delta_k G(\tau_3, \theta_3)\Big\rangle$$

with $\epsilon_i = \epsilon(\delta_i G)$. Hence, written explicitly, equation (6.3.35) takes the form

$$d\omega(\delta_i G, \delta_j G, \delta_k G) = \tag{6.3.37}$$
$$(-)^{\epsilon_j} \int \prod_{\beta=1}^{3} d\tau_\beta d\theta_\beta \Big(\Big\langle (b_{-1}\delta_i G)(Y\delta_j G)(Y\delta_k G)\Big\rangle +$$
$$(-1)^{\epsilon_i}\Big\langle (Y\delta_i G)(b_{-1}\delta_j G)(Y\delta_k G)\Big\rangle + (-1)^{\epsilon_i+\epsilon_j}\Big\langle (Y\delta_i G)(Y\delta_j G)(b_{-1}\delta_k G)\Big\rangle\Big).$$

The fact that the above expression vanishes follows from the Q invariance of the unperturbed correlators and the invariance under b_{-1}. The details of the relevant calculation can be found in appendix D.

The final property we have to check is V-invariance, i.e. $d(i_V\omega) = 0$. More explicitly, one must show that

$$\frac{\partial}{\partial\lambda^i} \int \prod_{\beta=1}^{3} d\tau_\beta d\theta_\beta (-)^{\epsilon_j} \Big\langle Y(\tau_1)\delta_j G(\tau_1, \theta_1) Y(\tau_2)[Q, G](\tau_2, \theta_2)\Big\rangle \pm (i \leftrightarrow j) = 0. \tag{6.3.38}$$

For comments on this proof we refer the reader again to appendix D.

This concludes our discussion of the BV structure of the boundary superstring field theory. We now employ the above formalism to investigate the master action.

6.3.2 The relation between S_F and Z

The spacetime action of SBSFT (again up to an irrelevant sign factor) follows directly from equation (6.2.7), the definition of the vector field V and the definition of the

odd symplectic form ω

$$dS_F = \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \langle Y(\tau_1) dG(\tau_1, \theta_1) Y(\tau_2)[Q, G](\tau_2, \theta_2)\rangle. \qquad (6.3.39)$$

Furthermore, since we make the assumption that ghosts and matter are decoupled, we may set $G = C\mathcal{V}$, in which case the action takes the form

$$dS_F = \int d\tau_1 d\tau_2 d\theta_1 d\theta_2 \langle Y(\tau_1) C(\tau_1, \theta_1) d\mathcal{V}(\tau_1, \theta_1) Y(\tau_2)[Q, C\mathcal{V}](\tau_2, \theta_2)\rangle. \quad (6.3.40)$$

In the following discussion we perform an explicit calculation of this expression and demonstrate how it relates to the superdisk partition function Z.

Let us start by making the simplifying assumption that $\mathcal{V}$ has a definite scaling dimension h. The commutator $[Q, C\mathcal{V}]$ is given more explicitly by

$$[Q, C\mathcal{V}] = \{Q, C\}\mathcal{V} - C\{Q, \mathcal{V}\}. \qquad (6.3.41)$$

Moreover, the following two equations hold [276]

$$\{Q, C\} = C\partial_\tau C - \frac{1}{4}(D_\theta C)(D_\theta C) \qquad (6.3.42)$$

$$\{Q, \mathcal{V}\} = C\partial_\tau \mathcal{V} - \frac{1}{2}(D_\theta C)(D_\theta \mathcal{V}) + h(\partial_\tau C)\mathcal{V}. \qquad (6.3.43)$$

Substituting them back into equation (6.3.41) gives

$$[Q, C\mathcal{V}] = [(1-h)C\partial_\tau C - \frac{1}{4}(D_\theta C)(D_\theta C)]\mathcal{V} + \frac{1}{2}C(D_\theta C)(D_\theta \mathcal{V}). \qquad (6.3.44)$$

An explicit calculation of the normal ordered expression $Y[Q, C\mathcal{V}]$ in components (see appendix E for details) gives

$$Y[Q, C\mathcal{V}] = \qquad (6.3.45)$$
$$\Big((1-h)Yc\partial_\tau c - \frac{1}{4}Y\gamma^2\Big)\mathcal{V} + \frac{1}{2}\theta(Y\gamma^2 - Yc\partial_\tau c)D_\theta\mathcal{V} + \Big(h - \frac{1}{2}\Big)\theta c\partial_\tau c e^{-\phi} D.$$

Non-vanishing amplitudes require three insertions of the c ghost and therefore only the last term in the above expression contributes to the action. Hence

$$dS_F = \tag{6.3.46}$$
$$\left(\frac{1}{2} - h\right) \oint_{\partial D} d\tau_1 d\tau_2 \langle c(\tau_1)c(\tau_2)\partial_\tau c(\tau_2)\rangle_{bc} \langle e^{-\phi(\tau_1)} e^{-\phi(\tau_2)}\rangle_{\beta\gamma} \langle dD(\tau_1)D(\tau_2)\rangle_m .$$

Since

$$\langle c(\tau_1)c(\tau_2)\partial_{\tau_2} c(\tau_2)\rangle = 2(\cos(\tau_1 - \tau_2) - 1) = -4\sin^2\left(\frac{\tau_1 - \tau_2}{2}\right) \tag{6.3.47}$$

and

$$\langle e^{-\phi(\tau_1)} e^{-\phi(\tau_2)}\rangle = \frac{1}{2} \frac{1}{\sin(\frac{\tau_1 - \tau_2}{2})} \tag{6.3.48}$$

we take

$$dS_F = (2h - 1) \oint_{\partial D} d\tau_1 d\tau_2 \sin\left(\frac{\tau_1 - \tau_2}{2}\right) \langle dD(\tau_1)D(\tau_2)\rangle . \tag{6.3.49}$$

For a generic perturbation $\mathcal{V}$ parametrized by couplings λ^i and operators $\mathcal{V}_i$ of conformal weight h_i

$$\mathcal{V} = \sum_i \lambda^i \mathcal{V}_i \tag{6.3.50}$$

and the above equation becomes

$$\frac{\partial S_F}{\partial \lambda^i} = \left(h_j - \frac{1}{2}\right) \lambda^j G_{ij}(\lambda) \tag{6.3.51}$$

where

$$G_{ij} = 2 \oint_{\partial D} d\tau_1 d\tau_2 \sin\left(\frac{\tau_1 - \tau_2}{2}\right) \langle D_i(\tau_1) D_j(\tau_2)\rangle . \tag{6.3.52}$$

These equations should be compared to equations (2.9) and (2.10) of reference [263]

for the bosonic case.

For a unitary theory, the integral appearing in equation (6.3.52) is expected to give a positive definite metric G_{ij}. Indeed, the Hilbert space of a unitary theory should have, by definition, a positive definite norm given by the time ordered expectation value. In particular, for a Hilbert space of odd excitations, this statement implies that we should be able to write an expectation value of the form $\langle D(x_1)D(x_2)\rangle$ as $\text{sign}(x_1 - x_2)f_D(x_1 - x_2)$, with f_D being a positive function. Combining this extra sign factor with the sine in the integral of (6.3.52) gives a manifestly positive form to the metric G_{ij}.

Also, equation (6.3.51) cannot be correct in general because $(\frac{1}{2} - h_j)\lambda^j$ is not a covariant expression on the space of theories. The relevant argument goes exactly the same way as in the bosonic case [263]. The correct covariant generalization of (6.3.51) is given by

$$\frac{\partial S_F}{\partial \lambda^i} = \beta^j G_{ij}. \tag{6.3.53}$$

Thus, along the RG flow

$$\frac{\partial S_F}{\partial \log t} = -\beta^i \frac{\partial S_F}{\partial \lambda_i} = -\beta^i \beta^j G_{ij} \tag{6.3.54}$$

where t is an RG (length) scale parameter. Since G_{ij} is positive definite, S_F is a monotonically decreasing function and also stationary at the conformal points. Furthermore, as we show in the following discussion, S_F equals to the disk partition function Z. Therefore, S_F can be identified with the boundary entropy and the above analysis agrees very nicely with the conjecture of [283, 284] in the context of boundary CFT.

In order to prove the conjectured relation $S_F = Z$, we make use of the "two-systems" approach introduced in [258]. According to this, we assume that the matter system consists of two decoupled subsystems with partition functions Z_1 and Z_2. Thus, the combined matter partition function Z equals the product $Z_1 Z_2$ and the

expansion (6.3.50) takes the form

$$\mathcal{V} = \sum_i \lambda^i V_i + \sum_k \rho^k \tilde{V}_k \tag{6.3.55}$$

with λ^i couplings for the first system and ρ^k couplings for the second system. Accordingly, (6.3.49) becomes

$$dS_F = \tag{6.3.56}$$
$$\oint_{\partial D} d\tau_1 d\tau_2 \sin\left(\frac{\tau_1 - \tau_2}{2}\right)\Big\langle dD(\tau_1)\Big(\sum_i (2h_i - 1)\lambda^i D_i(\tau_2) + \sum_k (2h_k - 1)\rho^k \tilde{D}_k(\tau_2)\Big)\Big\rangle.$$

After substituting into this equation the full expansion of dD we get two kinds of terms. One involves two point functions of the form $\langle D_i D_j\rangle$ and $\langle \tilde{D}_k \tilde{D}_l\rangle$. The other involves mixed terms of the form $\langle D_i \tilde{D}_k\rangle$. Since the two systems are decoupled these mixed correlators factorize into a product of two one-point functions, which are identically zero because the bottom components D are fermionic. Moreover, even if these terms were not zero, the resulting expression would involve the integral of $\sin(\frac{\tau_1-\tau_2}{2})$ over the circle and hence it would still be vanishing. The bosonic case, on the other hand, involved several non-vanishing mixed terms and these were responsible for the extra term with the β function on the right side of (6.2.17) . Thus, we find that

$$dS_F = \oint_{\partial D} d\tau_1 d\tau_2 \sin\left(\frac{\tau_1 - \tau_2}{2}\right)\Big(\sum_{i,j}(2h_i - 1)\lambda^i d\lambda^j \langle D_j(\tau_1) D_i(\tau_2)\rangle +$$
$$\sum_{k,l}(2h_l - 1)\rho^l d\rho^k \langle\ \tilde{D}_k(\tau_1)\tilde{D}_l(\tau_2)\rangle\Big). \tag{6.3.57}$$

After setting

$$a = \sum_j a_j(\lambda) d\lambda^j = \tag{6.3.58}$$
$$\sum_j d\lambda^j \Big[\oint_{\partial D} d\tau_1 d\tau_2 \sin\left(\frac{\tau_1 - \tau_2}{2}\right)\sum_i (2h_i - 1)\lambda^i \langle D_j(\tau_1) D_i(\tau_2)\rangle_1\Big]$$

$$\tilde{a} = \sum_k \tilde{a}_k(\rho) d\rho^k = \tag{6.3.59}$$
$$\sum_k d\rho^k \Big[\oint_{\partial D} d\tau_1 d\tau_2 \sin\left(\frac{\tau_1 - \tau_2}{2}\right) \sum_l (2h_l - 1)\rho^l \langle \tilde{D}_k(\tau_1)\tilde{D}_l(\tau_2)\rangle_2 \Big]$$

we can write equation (6.3.57) in a compact form as

$$dS_F = aZ_2 + Z_1\tilde{a}. \tag{6.3.60}$$

The one-forms a and $\tilde{a}$ can be related to the partition functions Z_1 and Z_2 by using the fact that $d^2S_F = 0$. Setting to zero the coefficient of $d\lambda^i \wedge d\rho^k$ in the expression of d^2S_F gives

$$dZ_1\tilde{a} - adZ_2 = 0. \tag{6.3.61}$$

Since the two systems are decoupled, this implies the existence of a non-zero constant g such that

$$a = gdZ_1, \quad \tilde{a} = gdZ_2. \tag{6.3.62}$$

g cannot be zero because ω is non-degenerate. In fact, as we show in the next subsection, conformal perturbation theory fixes the value of g to 1. Thus, putting everything together gives

$$dS_F = d(Z_1 Z_2) \tag{6.3.63}$$

or

$$S_F = Z. \tag{6.3.64}$$

The above derivation of the equality between the master action and the superdisk partition function was carried under the assumption that the matter system consists of two decoupled subsystems. However, as pointed out in [258] , this restriction is

not necessary. One can consider any matter system and carry out the above analysis by adding an auxiliary decoupled system. At the end of the calculation the auxiliary system can be suppressed by setting its couplings to a fixed value.

6.3.3 Some calculations in conformal perturbation theory

In the previous subsection we used a general argument to show the relation

$$\partial_i S_F = g \partial_i Z \tag{6.3.65}$$

where g is a constant factor. We examine now this relation from the point of view of conformal perturbation theory up to 3rd order on the bare couplings λ^i of the boundary perturbations. In particular, we consider the following expansion of equations (6.3.51) and (6.3.52) around the conformal point

$$\begin{aligned}\partial_i S_F = 2(h_j - \frac{1}{2})\Big(\lambda^j \oint d\tau_1 d\tau_2 \sin\frac{\tau_1-\tau_2}{2}\langle D_i(\tau_1)D_j(\tau_2)\rangle_0 + \\ \lambda^j\lambda^k \oint d\tau_1 d\tau_2 d\tau_3 \sin\frac{\tau_1-\tau_2}{2}\langle D_i(\tau_1)D_j(\tau_2)U_k(\tau_3)\rangle_0\Big)\end{aligned} \tag{6.3.66}$$

and compare it to the corresponding expansion of the disk partition function

$$\begin{aligned}&\partial_i Z = \\ &\lambda^j \oint d\tau_1 d\tau_2 \langle U_i(\tau_1)U_j(\tau_2)\rangle_0 + \frac{1}{2}\lambda^j\lambda^k \oint d\tau_1 d\tau_2 d\tau_3 \langle U_i(\tau_1)U_j(\tau_2)U_k(\tau_3)\rangle_0.\end{aligned} \tag{6.3.67}$$

According to (6.3.65), we should be able to verify the following two equation

$$2(h_j - \frac{1}{2})\oint d\tau_1 d\tau_2 \sin\frac{\tau_1-\tau_2}{2}\langle D_i(\tau_1)D_j(\tau_2)\rangle_0 = g\oint d\tau_1 d\tau_2 \langle U_i(\tau_1)U_j(\tau_2)\rangle_0 \tag{6.3.68}$$

in second order, and

$$\begin{aligned}2(h_j - \frac{1}{2})\oint d\tau_1 d\tau_2 d\tau_3 \sin\frac{\tau_1-\tau_2}{2}\langle D_i(\tau_1)D_j(\tau_2)U_k(\tau_3)\rangle_0 + (j \leftrightarrow k) = \\ g\oint d\tau_1 d\tau_2 d\tau_3 \langle U_i(\tau_1)U_j(\tau_2)U_k(\tau_3)\rangle_0.\end{aligned} \tag{6.3.69}$$

in third order.

For the second order computation, the explicit form of the correlators is as follows

$$\langle D_i(\tau_1)D_j(\tau_2)\rangle_0 = \frac{c\delta_{h_i,h_j}}{\left|\sin\frac{\tau_1-\tau_2}{2}\right|^{2h}}\mathrm{sign}(\tau_1-\tau_2), \tag{6.3.70}$$

$$\langle U_i(\tau_1)U_j(\tau_2)\rangle_0 = \frac{b\delta_{h_i,h_j}}{\left|\sin\frac{\tau_1-\tau_2}{2}\right|^{2h+1}}, \tag{6.3.71}$$

with $h = h_i = h_j$. After evaluating the relevant integrals by using the general expression

$$\int_0^{2\pi}\frac{d\tau_1}{2\pi}\frac{d\tau_2}{2\pi}\left|\sin\frac{\tau_1-\tau_2}{2}\right|^z = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\frac{z}{2})}{\Gamma(1+\frac{z}{2})}, \tag{6.3.72}$$

equation (6.3.68) becomes

$$2(h-\frac{1}{2})c\frac{\Gamma(1-h)}{\Gamma(\frac{3}{2}-h)} = gb\frac{\Gamma(-h)}{\Gamma(\frac{1}{2}-h)} \Rightarrow 2ch = gb. \tag{6.3.73}$$

This equation, with $g = 1$, is a consequence of the SUSY Ward identities on the world-sheet. For a short derivation see appendix F.

Similarly, in third order we have the correlators

$$\langle D_i(\tau_1)D_j(\tau_2)U_k(\tau_3)\rangle_0 = \tag{6.3.74}$$
$$\frac{C_{ijk}\ \mathrm{sign}(\tau_1-\tau_2)}{\left|\sin\frac{\tau_1-\tau_2}{2}\right|^{h_i+h_j-h_k-\frac{1}{2}}\left|\sin\frac{\tau_2-\tau_3}{2}\right|^{h_j+h_k-h_i+\frac{1}{2}}\left|\sin\frac{\tau_1-\tau_3}{2}\right|^{h_i+h_k-h_j+\frac{1}{2}}},$$

$$\langle U_i(\tau_1)U_j(\tau_2)U_k(\tau_3)\rangle_0 = \tag{6.3.75}$$
$$\frac{B_{ijk}}{\left|\sin\frac{\tau_1-\tau_2}{2}\right|^{h_i+h_j-h_k+\frac{1}{2}}\left|\sin\frac{\tau_2-\tau_3}{2}\right|^{h_k+h_j-h_i+\frac{1}{2}}\left|\sin\frac{\tau_1-\tau_3}{2}\right|^{h_i+h_k-h_j+\frac{1}{2}}}.$$

By using the general expression

$$\int_0^{2\pi} \frac{d\tau_1}{2\pi}\frac{d\tau_2}{2\pi}\frac{d\tau_3}{2\pi}\left|\sin\frac{\tau_1-\tau_2}{2}\right|^a\left|\sin\frac{\tau_2-\tau_3}{2}\right|^b\left|\sin\frac{\tau_1-\tau_3}{2}\right|^c = \tag{6.3.76}$$
$$\frac{1}{\pi^{3/2}}\frac{\Gamma(\frac{1}{2}(1+a))\Gamma(\frac{1}{2}(1+b))\Gamma(\frac{1}{2}(1+c))\Gamma(1+\frac{1}{2}(a+b+c))}{\Gamma(1+\frac{1}{2}(a+b))\Gamma(1+\frac{1}{2}(b+c))\Gamma(1+\frac{1}{2}(a+c))}.$$

we can evaluate all terms in (6.3.69) . The result of this calculation for $g = 1$ is

$$-2B_{ijk}\Big(\frac{1}{2}-h_i\Big) = \tag{6.3.77}$$
$$\Big(\frac{1}{2}-h_i-h_j-h_k\Big)\Big(C_{ijk}\Big(\frac{1}{2}+h_k-h_i-h_j\Big)+C_{ikj}\Big(\frac{1}{2}+h_j-h_i-h_k\Big)\Big).$$

Since $C_{ijk} = C_{ikj}$, the above equation becomes

$$B_{ijk} = -\Big(\frac{1}{2}-h_i-h_j-h_k\Big)C_{ijk}. \tag{6.3.78}$$

Again, we can verify this relation, as well as the symmetry properties of the constants C_{ijk}, by using the SUSY Ward identities. The relevant details can be found in appendix F.

Another important statement of the previous subsection was the identification of the world-sheet partition function Z with the boundary entropy of [283, 284]. We can check this identification by an explicit conformal perturbation theory calculation, showing the decrease of the partition function under the renormalization group flow. We perturb the world-sheet action by some relevant operator very close to marginality and as a result the world-sheet theory flows towards a nearby conformal fixed point, where we can still use perturbation theory to calculate the new value of the partition function. A similar calculation can be found in appendix E of reference [284].

Thus, consider the boundary perturbation $\mathcal{V}$ with conformal weight $h = 1/2 - y$ and $0 < y \ll 1/2$. For simplicity, we assume that the RG flow is closed under this perturbation and that there is no mixing with other fields. The corresponding beta

function up to second order equals

$$\beta(\lambda) = -y\lambda - \frac{B}{\pi}\lambda^2, \tag{6.3.79}$$

with $B = B_{\mathcal{VVV}}$ the 3-point function constant appearing in (6.3.75). This β function implies the existence of a nearby fixed point given by

$$\beta(\lambda^*) = 0 \Rightarrow \lambda^* = -\frac{y\pi}{B} \ll 1. \tag{6.3.80}$$

In order to verify (6.3.79), consider an ultraviolet (length scale) cut-off l, an RG length scale parameter t and the boundary perturbation for bare coupling λ written in the form

$$S_{\text{bdy}} = \left(\frac{t}{l}\right)^y \lambda \oint \frac{d\tau}{2\pi} U(\tau). \tag{6.3.81}$$

In this relation, λ, τ and U are dimensionless.

Expanding the partition function to quadratic order in the coupling and using the OPE

$$U(\tau_1)U(\tau_2) \sim \frac{B}{|\sin\frac{\tau_1-\tau_2}{2}|^{h+1/2}} U(\tau_2) \tag{6.3.82}$$

gives

$$\begin{aligned}
e^{S_{\text{bdy}}} &= 1 + \left(\frac{t}{l}\right)^y \lambda \oint \frac{d\tau}{2\pi} U(\tau) + \frac{1}{2}\left(\frac{t}{l}\right)^{2y} \lambda^2 \oint \frac{d\tau_1}{2\pi}\frac{d\tau_2}{2\pi} U(\tau_1)U(\tau_2) = \\
&1 + \left(\frac{t}{l}\right)^y \lambda \oint \frac{d\tau}{2\pi} U(\tau) + \frac{1}{2}\left(\frac{t}{l}\right)^{2y} \lambda^2 \oint \frac{d\tau_1}{2\pi} U(\tau_1) \oint \frac{d\tau_2}{2\pi} \frac{B}{|\sin\frac{\tau_1-\tau_2}{2}|^{h+1/2}} = \\
&1 + \left(\frac{t}{l}\right)^y \lambda \oint \frac{d\tau}{2\pi} U(\tau) + \frac{1}{2}\left(\frac{t}{l}\right)^{2y} \lambda^2 \frac{B}{\sqrt{\pi}} \frac{\Gamma(\frac{y}{2})}{\Gamma(\frac{1+y}{2})} \oint \frac{d\tau}{2\pi} U(\tau) \sim \\
&1 + \left(\frac{t}{l}\right)^y \lambda \oint \frac{d\tau}{2\pi} U(\tau) + \left(\frac{t}{l}\right)^{2y} \lambda^2 \frac{B}{y\pi} \oint \frac{d\tau}{2\pi} U(\tau).
\end{aligned} \tag{6.3.83}$$

In the last step we have set $y \sim 0$. Considering now these terms as a correction to

the initial perturbation, we find

$$\frac{\delta\lambda(t)}{\delta \ln t} = y\lambda(t) + \frac{B}{\pi}\lambda(t)^2, \tag{6.3.84}$$

which gives precisely the beta function (6.3.79).

In terms of the bare coupling $\lambda = \lambda(l)$, the partition function up to 3rd order is given by

$$\begin{aligned} Z = 1 + \frac{1}{2}\Big(\frac{t}{l}\Big)^{2y}\lambda^2 \oint \frac{d\tau_1}{2\pi}\frac{d\tau_2}{2\pi}\langle U(\tau_1)U(\tau_2)\rangle_0 + \\ \frac{1}{6}\Big(\frac{t}{l}\Big)^{3y}\lambda^3 \oint \frac{d\tau_1}{2\pi}\frac{d\tau_2}{2\pi}\frac{d\tau_3}{2\pi}\langle U(\tau_1)U(\tau_2)U(\tau_3)\rangle_0. \end{aligned} \tag{6.3.85}$$

After substituting the CFT expressions (6.3.71) (with normalization $b = 1$) and (6.3.75) we obtain

$$Z = 1 + \frac{1}{2}\Big(\frac{t}{l}\Big)^{2y}\lambda^2 \frac{1}{\sqrt{\pi}}\frac{\Gamma(y-\frac{1}{2})}{\Gamma(y)} + \frac{1}{6}\Big(\frac{t}{l}\Big)^{3y}\lambda^3 B\frac{1}{\pi\sqrt{\pi}}\frac{(\Gamma(\frac{1}{2}y))^3\Gamma(-\frac{1}{2}+\frac{3}{2}y)}{(\Gamma(y))^3}, \tag{6.3.86}$$

which in the $y \sim 0$ limit simplifies to

$$Z = 1 - y\lambda^2\Big(\frac{t}{l}\Big)^{2y} - \frac{8}{3\pi}B\lambda^3\Big(\frac{t}{l}\Big)^{3y}. \tag{6.3.87}$$

This equation can be re-expressed in terms of the renormalized coupling $\lambda(t)$ by solving the β function equation (6.3.84) . The solution gives

$$\lambda = \Big(\frac{t}{l}\Big)^{-y}\frac{\lambda(t)}{1 - \frac{\lambda(t)}{\lambda^*}(1-(\frac{t}{l})^{-y})}. \tag{6.3.88}$$

Expanding this expression up to second order in $\lambda(t)$ and substituting the result into (6.3.87) gives

$$Z = 1 - y\lambda(t)^2 - \frac{2}{3\pi}\lambda(t)^3 B. \tag{6.3.89}$$

Thus, in the IR limit where $\lambda(t) \to \lambda^*$, the total change of the disk partition function

between the UV and IR fixed points becomes

$$\delta Z = -\frac{\pi^2 y^3}{3B^2}. \tag{6.3.90}$$

This result was also obtained in [284]. As expected, we find that Z decreases under the renormalization group flow.

6.4 Discussion

We have found that on a formal level the construction of the (classical) boundary string field theory for the NS sector of the superstring is more or less parallel to the analogous construction of the bosonic case. However, in the superstring case there are certain extra subtle points. The first is the requirement to preserve world-sheet supersymmetry. We satisfy it by using a superspace formalism that is manifestly supersymmetric. We should stress here that in our case (NS boundary interaction terms on the disk) the global world-sheet supersymmetry is "softly" broken as a consequence of the antiperiodic boundary conditions on the disk [268]. A second subtlety in the superstring case arises from the involved nature of the superconformal ghosts. Vertex operators can be chosen in different pictures and this choice involves the insertion of appropriate picture changing operators inside the correlation functions.

After defining the appropriate BV structure the spacetime action is determined by (6.3.39). Under the additional assumption of ghost and matter decoupling, this action takes a simpler and more appropriate for calculations form. Using the two-systems analysis of [258] and conformal perturbation theory up to third order we verified that the master action is non other than the disk partition function, exactly as it was conjectured in [267]. This identification also suggests that the spacetime action can also be thought of as the boundary entropy of [283, 284]; it takes the right value at the conformal points and decreases along the RG flow.

Notice that the spacetime action of boundary superstring field theory is simpler than its bosonic counterpart. This is in contrast to what happens in cubic string field

theory, whose sypersymmetric generalization is substantially more involved. From this point of view, SBSFT seems to provide a more tractable framework to address questions regarding tachyon dynamics in open superstring theories.

This construction of boundary superstring field theory is not at all complete however. First of all, the above analysis has to be extended appropriately to include the R sector. Secondly, the presented formalism is plagued by the same problems that characterize the analogous construction of the bosonic case. Very simply put, the construction is too formal. The space of all theories (with varying local boundary interaction terms) gives rise to serious ultraviolet divergences, especially when one tries to add non-renormalizable boundary interaction terms. These terms correspond to higher massive excitations of the open string and they certainly have to be included in any acceptable formulation of string field theory.

In order to tackle these divergences an appropriate cut-off has to be introduced. The cut-off should respect the rotational invariance, the b_{-1} Ward identities and V-invariance (i.e. the invariance under the closed BRST charge Q) [7]. In the superstring case it should also respect world-sheet supersymmetry. A cut-off can be chosen to respect all of the above symmetries except for V-invariance. Because of this, at the end of the calculation one would like to remove the cut-off in such a way that the V-invariance of the antibracket is restored. The relevant discussion of [285] for certain integrable boundary interactions in the bosonic case revealed that the removal of the cut-off presented difficulties. It is not clear however whether this poses an insurmountable obstacle in making sense of the notion of a space of open string theories with local boundary interactions. One expects similar difficulties in the superstring case as well.

Because of such problems the question of whether this formalism can provide a rigorous and full formulation of open string field theory is still open. It might be possible, however, that a more careful application of the BV formalism could provide the needed resolution of the above subtleties.

[7]Given the previous invariances, V-invariance is equivalent to the statement that the boundary interaction does not modify the BRST charge. See [260, 285] for further comments on this point.

Appendix A

Some explicit computations for $M(G)$

A.1 Preliminary definitions

We begin with a few rudimentary definitions [102]. Let H be a subgroup of G and let $g \in G$. For any cocycle $\alpha \in Z^2(G, \mathbb{C}^*)$ we define an induced action $g \cdot \alpha \in Z^2(gHg^{-1}, \mathbb{C}^*)$ thereon as $g \cdot \alpha(x, y) = \alpha(g^{-1}xg, g^{-1}yg), \ \forall \ x, y \in gHg^{-1}$. Now, it can be proved that the mapping

$$c_g : M(H) \to M(gHg^{-1}), \quad c_g(\alpha) := g \cdot \alpha$$

is a homomorphism, which we call cocycle conjugation by g.

On the other hand we have an obvious concept of restriction: for $S \subseteq L$ subgroups of G, we denote by $\mathrm{Res}_{L,S}$ the restriction map $M(L) \to M(S)$. Thereafter we define stability as:

Definition A.1.9 *Let H and K be arbitrary subgroups of G. An element $\alpha \in M(H)$ is said to be K-stable if*

$$\mathrm{Res}_{H, gHg^{-1} \cap H}(\alpha) = \mathrm{Res}_{gHg^{-1}, gHg^{-1} \cap H}(c_g(\alpha)) \ \ \forall \ \ g \in K.$$

The set of all K-stable elements of $M(H)$ will be denoted by $M(H)^K$ and it forms a subgroup of $M(H)$ known as the K-stable subgroup of $M(H)$.

When $K \subseteq N_G(H)$ all the above concepts[1] coalesce and we have the following important lemma:

Lemma A.1.2 ([102] p299) *If H and K are subgroups of G such that $K \subseteq N_G(H)$, then $M(H)^K$ is the K-stable subgroup of $M(H)$ with respect to the action of K on $M(H)$ induced by the action of K on H by conjugation. In other words,*

$$M(H)^K = \{\alpha \in M(H),\ \ \alpha(x,y) = c_g(\alpha)(x,y) \ \ \forall \ \ g \in K, \ \ \forall \ \ x,y \in H\}.$$

Finally let us present a useful class of subgroups:

Definition A.1.10 *A subgroup H of a group G is called a Hall subgroup of G if the order of H is coprime with its index in G, i.e.* $\gcd(|H|, |G/H|) = 1$.

For these subgroups we have:

Theorem A.1.11 ([102] p334) *If N is a normal Hall subgroup of G. Then*

$$M(G) \cong M(N)^{G/N} \times M(G/N).$$

The above theorem is really a corollary of a more general case of semi-direct products:

Theorem A.1.12 ([103] p33) *Let $G = N \rtimes T$ with $N \triangleleft G$, then*
(i) $M(G) \cong M(T) \times \tilde{M}(G)$;
(ii) The sequence $1 \to H^1(T, N^*) \to \tilde{M}(G) \overset{\text{Res}}{\to} M(N)^T \to H^2(T, N^*)$ *is exact, where* $\tilde{M}(G) := \ker \text{Res}_{G,N}$, $N^* := \text{Hom}(N, \mathbb{C}^*)$ *and* $H^{i=1,2}(T, N^*)$ *is the cohomology defined with respect to the conjugation action by T on N^*.*

Part (ii) of this theorem actually follows from the Lyndon-Hochschild-Serre spectral sequence.

[1] $N_G(H)$ is the normalizer of H in G, i.e., the set of all elements $g \in G$ such that $gHg^{-1} = H$. When H is a normal subgroup of G we obviously have $N_G(H) = G$.

One clarification is needed at hand. Let us define the first A-valued cohomology group for G, which we shall utilise later in our calculations. Here the 1-cocycles are the set of functions $Z^1(G,A) := \{f : G \to A | f(xy) = (x \cdot f(y))f(x) \quad \forall x, y \in G\}$, where A is being acted upon ($x \cdot A \to A$ for $x \in G$) by G as a $\mathbb{Z}G$-module. These are known as crossed homomorphisms. On the other hand, the 1-coboundaries are what is known as the principal crossed homomorphisms, $B^1(G,A) := \{f_{a \in A}(x) = (x \cdot a)a^{-1}\}$ from which we define $H^1(G,A) := Z^1(G,A)/B^1(G,A)$.

Note that in our definition of $H^2(G,A)$ in the main text, the action of G on A (as in the case of the Schur multiplier) is taken to be trivial. We must be careful in the ensuing however, to compute with respect to non-trivial actions such as conjugation. In our case the conjugation action of $t \in T$ on $\chi \in \text{Hom}(N, \mathbb{C}^*)$ is given by $\chi(tnt^{-1})$ for $n \in N$.

A.2 The Schur multiplier of Δ_{3n^2}

A.2.1 Case I: $\gcd(n,3) = 1$

Thus equipped, we can now use theorem A.1.11 at our ease to compute the Schur multipliers the first case of the finite groups Δ_{3n^2}. Recall that $\mathbb{Z}_n \times \mathbb{Z}_n \triangleleft \Delta(3n^2)$ or explicitly

$$\Delta_{3n^2} \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3.$$

Our crucial observation is that when $\gcd(n,3) = 1$, $\mathbb{Z}_n \times \mathbb{Z}_n$ is in fact a normal Hall subgroup of Δ_{3n^2} with quotient group $\mathbb{Z}_3$. Whence theorem A.1.11 can be immediately applied to this case when n is coprime to 3:

$$M(\Delta_{3n^2}) = (M(\mathbb{Z}_n \times \mathbb{Z}_n))^{\mathbb{Z}_3} \times M(\mathbb{Z}_3) = (M(\mathbb{Z}_n \times \mathbb{Z}_n))^{\mathbb{Z}_3},$$

by recalling that the Schur multiplier of all cyclic groups is trivial and that of $\mathbb{Z}_n \times \mathbb{Z}_n$ is $\mathbb{Z}_n$ [102]. But, $\mathbb{Z}_3 \subseteq N_{\Delta_{3n^2}}(\mathbb{Z}_n \times \mathbb{Z}_n) = \Delta_{3n^2}$, and hence by lemma A.1.2 it suffices to compute the $\mathbb{Z}_3$-stable subgroup of $\mathbb{Z}_n$ by cocycle conjugation.

Let the quotient group $\mathbb{Z}_3$ be $\langle z|z^3 = \mathbb{I}\rangle$ and similarly, if $x, y, x^n = y^n = \mathbb{I}$ are the generators of $\mathbb{Z}_n \times \mathbb{Z}_n$, then a generic element thereof becomes $x^a y^b$, $a, b = 0, \ldots, n-1$. The group conjugation by z on such an element gives

$$z^{-1}x^a y^b z = x^b y^{-a-b} \qquad z x^a y^b z^{-1} = x^{-a-b} y^a. \tag{A.2.1}$$

It is easy now to check that if α is a generator of the Schur multiplier $\mathbb{Z}_n$, we have an induced action

$$c_z(\alpha)(x^a y^b, x^{a'} y^{b'}) := \alpha(z^{-1}x^a y^b z, z^{-1}x^{a'} y^{b'} z) = \alpha(x^b y^{-(a+b)}, x^{b'} y^{-(a'+b')})$$

by lemma A.1.2.

However, we have a well-known result:

Proposition A.2.3 *For the group $\mathbb{Z}_n \times \mathbb{Z}_n$, the explicit generator of the Schur multiplier is given by*

$$\alpha(x^a y^b, x^{a'} y^{b'}) = \omega_n^{ab'-a'b}.$$

Consequently, $\alpha(x^b y^{-(a+b)}, x^{b'} y^{-(a'+b')}) = \alpha(x^a y^b, x^{a'} y^{b'})$ whereby making the c_z-action trivial and causing $(M(\mathbb{Z}_n \times \mathbb{Z}_n)^{\mathbb{Z}_3} \cong M(\mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_n$. From this we conclude part I of our result: $M(\Delta_{3n^2}) = \mathbb{Z}_n$ for n coprime to 3.

A.2.2 Case II: $\gcd(n, 3) \neq 1$

Here the situation is much more involved. Let us appeal to part (ii) of theorem A.1.12. We let $N = \mathbb{Z}_n \times \mathbb{Z}_n$ and $T = \mathbb{Z}_3$ as above and define $U := \mathrm{Hom}(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C}^*))$; the exact sequence then takes the form

$$1 \to H^1(\mathbb{Z}_3, U) \to \tilde{M}(\Delta_{3n^2}) \to \mathbb{Z}_n \to H^2(\mathbb{Z}_3, U) \tag{A.2.2}$$

using the fact that the stable subgroup $M(\mathbb{Z}_n \times \mathbb{Z}_n)^{\mathbb{Z}_3} \cong \mathbb{Z}_n$ as shown above. Some explicit calculations are now called for.

As for U, it is of course isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$ since for an Abelian group A,

$\mathrm{Hom}(A, \mathbb{C}^*) \cong A$ ([103] p17). We label the elements thereof as $(p,q)(x^a y^b) := \omega_n^{ap+bq}$, taking $x^a y^b \in \mathbb{Z}_n \times \mathbb{Z}_n$ to $\mathbb{C}^*$.

We recall that the conjugation by $z \in \mathbb{Z}_3$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ is (A.2.1). Therefore, by the remark at the end of the previous subsection, z acts on U as: $(z \cdot (p,q))(x^a y^b) := (p,q)(z(x^a y^b)z^{-1}) = \omega_n^{a'p+b'q}$ with $a' = -a-b$ and $b' = a$ due[2] to (A.2.1), whence

$$z \cdot (p,q) = (q-p, -p), \qquad \text{for } (p,q) \in U. \tag{A.2.3}$$

Some explicit calculations are called for. First we compute $H^1(\mathbb{Z}_3, U)$. Z^1 is generically composed of functions such that $f(z) = (p,q)$ (and also $f(\mathbb{I}) = \mathbb{I}$ and $f(z^2) = (z \cdot f(z))f(z)$ by the crossed homomorphism condition, and is subsequently equal to $(q, p+q)$ by (A.2.3). Since no further conditions can be imposed, $Z^1 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Now B^1 consists of all functions of the form $(z \cdot (p,q))(p,q)^{-1} = (q-2p, -p-q)$, these are to be identified with the trivial map in Z^1. We can re-write these elements as $(p' := q-2p, -p'-3p) = (\omega_n^a \omega_n^{-b})^{p'} (\omega_n^b)^{-3p}$, and those in Z^1 we re-write as $(\omega_n^a \omega_n^{-b})^{p'} (\omega_n^b)^{q'}$ as we are free to do. Therefore if $\gcd(3,n) = 1$, then $H^1 := Z^1/B^1$ is actually trivial because in mod n, $3p$ also ranges the full $0, \cdots, n-1$, whereas if $\gcd(3,n) \neq 1$ then $H^1 := Z^1/B^1 \cong \mathbb{Z}_3$.

The computation for $H^2(\mathbb{Z}_3, U)$ is a little more involved, but the idea is the same. First we determine Z^2 as composed of $\alpha(z_1, z_2)$ constrained by the cocycle condition (with respect to conjugation which differs from (3.3.18) where the trivial action was taken)

$$\alpha(z_1, z_2)\alpha(z_1 z_2, z_3) = (z_1 \cdot \alpha(z_2, z_3))\alpha(z_1, z_2 z_3) \qquad z_1, z_2, z_3 \in \mathbb{Z}_3.$$

Again we only need to determine the following cases: $\alpha(z,z) := (p_1, q_1); \alpha(z^2, z^2) := (p_2, q_2); \alpha(z^2, z) := (p_3, q_3); \alpha(z, z^2) := (p_4, q_4)$. The cocycle constraint gives $(p_1, q_1) = (q_4, -q_3)$; $(p_2, q_2) = (-q_3 - q_4, -q_4); (p_3, q_3) = (-q_4, q_3); (p_4, q_4) = (p_4, q_4)$, giving $Z^2 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. The coboundaries are given by $(\delta t)(z_1, z_2) = (z_1 \cdot t(z_2))t(z_1)t(z_1 z_2)^{-1}$

[2]Note that we must be careful to let the order of conjugation be the opposite of that in the cocycle conjugation.

(for any mapping $t : \mathbb{Z}_3 \to \mathbb{Z}_n \times \mathbb{Z}_n$ which we define to take values $t(z) = (r_1, s_1)$ and $t(z^2) = (r_2, s_2))$), making $(\delta t)(z, z) = (s_1 - r_2, -r_1 + s_1 - s_2)$; $(\delta t)(z^2, z^2)(-s_2 + r_2 - r_1, r_2 - s_1)$; $(\delta t)(z^2, z) = (-s_1 + r_2, r_1 - s_1 + s_2)$; $(\delta t)(z, z^2) = (s_2 - r_2 + r_1, s_1 - r_2)$. Now, the transformation $r_2 = s_1 + q_4; r_1 = s_1 - s_2 - p_4 + q_4$ makes this set of values for B^2 completely identical to those in Z^2, whence we conclude that $B^2 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. In conclusion then $H^2 := Z^2/B^2 \cong \mathbb{I}$.

The exact sequence (A.2.2) then assumes the simple form of

$$1 \to \left\{ \begin{array}{ll} \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{I}, & \gcd(n,3) = 1 \end{array} \right\} \to \tilde{M}(G) \to \mathbb{Z}_n \to 1,$$

which means that if n does not divide 3, $\tilde{M}(G) \cong \mathbb{Z}_n$, and otherwise $\tilde{M}(G)/\mathbb{Z}_3 \cong \mathbb{Z}_n$. Of course, in conjunction with part (i) of theorem A.1.12, we immediately see that the first case makes part I of our discussion (when $\gcd(n,3) = 1$) a special case of our present situation.

On the other hand, for the remaining case of $\gcd(n,3) \neq 1$, we have $M(\Delta_{3n^2})/\mathbb{Z}_3 \cong \mathbb{Z}_n$, which means that $M(\Delta_{3n^2})$, being an Abelian group, can only be $\mathbb{Z}_{3n}$ or $\mathbb{Z}_n \times \mathbb{Z}_3$. The exponent of the former is $3n$, while the later (since 3 divides n), is n, but by theorem 3.3.4, the exponent squared must divide the order, which is $3n^2$, whereby forcing the second choice.

Therefore in conclusion we have:

$$M(\Delta_{3n^2}) = \left\{ \begin{array}{ll} \mathbb{Z}_n \times \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{Z}_n, & \gcd(n,3) = 1 \end{array} \right.$$

as reported in table (3.3.22).

A.3 The Schur multiplier of Δ_{6n^2}

Recalling that n is even, we have $\Delta_{6n^2} \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$ with $\mathbb{Z}_n \times \mathbb{Z}_n$ normal and thus we are once more aided by theorem A.1.12.

We let $N := \mathbb{Z}_n \times \mathbb{Z}_n$ and $T := S_3$ and the exact sequence assumes the form

$$1 \to H^1(S_3, U) \to \tilde{M}(\Delta_{6n^2}) \to (\mathbb{Z}_n)^{S_3} \to H^2(S_3, U)$$

where $U := \mathrm{Hom}(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C}^*)$ as defined in the previous subsection.

By calculations entirely analogous to the case for Δ_{3n^2}, we have $(\mathbb{Z}_n)^{S_3} \cong \mathbb{Z}_2$. This is straightforward to show. Let $S_3 := \langle z, w | z^3 = w^2 = \mathbb{I}, zw = wz^2 \rangle$. We see that it contains $\mathbb{Z}^3 = \langle z | z^3 = \mathbb{I} \rangle$ as a subgroup, which we have treated in the previous section. In addition to (A.2.1), we have

$$w^{-1} x^a y^b w = x^{-1-b} y^b = w x^a y^b w^{-1}.$$

Using the form of the cocycle in proposition A.2.3, we see that $c_w(\alpha) = \alpha^{-1}$. Remembering that $c_z(\alpha) = \alpha$ from before, we see that the S_3-stable part of consists of α^m with $m = 0$ and $n/2$ (recall that in our case of $\Delta(6n^2)$, n is even), giving us a $\mathbb{Z}_2$.

Moreover we have $H^1(S_3, U) \cong \mathbb{I}$. This is again easy to show. In analogy to (A.2.3), we have

$$w \cdot (p, q) = (-q, q - p), \qquad \text{for } (p, q) \in U,$$

using which we find that Z^1 consists of $f : S_3 \to U$ given by $f(z) = (l_1, 3k_2 - l_1)$ and $f(w) = (2k_2, k_2)$. In addition B^1 consists of $f(z) = (k - 2l, -l - k)$ and $f(w) = (-2l, -l)$. Whence we see instantly that H^1 is trivial.

Now in fact $H^2(S_3, U) \cong \mathbb{I}$ as well. The exact sequence then forces immediately that $\tilde{M}(\Delta_{6n^2}) \cong \mathbb{Z}_2$. Moreover, since $M(S_3) \cong \mathbb{I}$ (q.v. e.g. [102]), by part (i) of theorem A.1.12, we conclude that

$$M(\Delta_{6n^2}) \cong \mathbb{Z}_2$$

as reported in table (3.3.22).

Appendix B

Intransitive subgroups of $SU(3)$

The computation of the Schur multipliers for the non-abelian intransitive subgroups of $SU(3)$ involves some subtleties related to the precise definition and construction of the groups.

Let us consider the case of combining the generators of $\mathbb{Z}_n$ with these of $\hat{\mathcal{D}}_m$ to construct the intransitive subgroup $< \mathbb{Z}_n, \hat{\mathcal{D}}_m >$. We can take the generators of $\hat{\mathcal{D}}_m$ to be

$$\alpha = \begin{pmatrix} \omega_{2m} & 0 & 0 \\ 0 & \omega_{2m}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that of $\mathbb{Z}_n$ to be

$$\gamma = \begin{pmatrix} \omega_n & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-2} \end{pmatrix}.$$

The group $< \mathbb{Z}_n, \hat{\mathcal{D}}_m >$ is not in general the direct product of $\mathbb{Z}_n$ and $\hat{\mathcal{D}}_m$. More specifically, when n is odd $< \mathbb{Z}_n, \hat{\mathcal{D}}_m > = \mathbb{Z}_n \times \hat{\mathcal{D}}_m$. For n even however, we notice that $\alpha^m = \beta^2 = \gamma^{n/2}$. Accordingly, we conclude that $< \mathbb{Z}_n, \hat{\mathcal{D}}_m > = (\mathbb{Z}_n \times \hat{\mathcal{D}}_m)/\mathbb{Z}_2$ for n even where the central $\mathbb{Z}_2$ is generated by $\gamma^{n/2}$. Actually the conditions are more refined: when $n = 2(2k+1)$ we have $\mathbb{Z}_n = \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}$ and so $(\mathbb{Z}_2 \times \hat{\mathcal{D}}_m)/\mathbb{Z}_2 = \mathbb{Z}_{2k+1} \times \hat{\mathcal{D}}_m$. Thus the only non-trivial case is when $n = 4k$.

This subtlety in the group structure holds for all the cases where $\mathbb{Z}_n$ is combined

with binary groups $\widehat{G}$. When $n \bmod 4 \neq 0$, $< \mathbb{Z}_n, \widehat{G} >$ is the direct product of $\widehat{G}$ with either $\mathbb{Z}_n$ or $\mathbb{Z}_{n/2}$. For $n \bmod 4 = 0$ it is the quotient group $(\mathbb{Z}_n \times \widehat{G})/\mathbb{Z}_2$. In summary

$$< \mathbb{Z}_n, \widehat{G} >= \begin{cases} \mathbb{Z}_n \times \widehat{G} & n \bmod 2 = 1 \\ \mathbb{Z}_{n/2} \times \widehat{G} & n \bmod 4 = 2 \\ (\mathbb{Z}_n \times \widehat{G})/\mathbb{Z}_2 & n \bmod 4 = 0 \end{cases} .$$

The case of $\mathbb{Z}_n$ combined with the ordinary dihedral group D_{2m} is a bit different however. The matrix forms of the generators are

$$\alpha = \begin{pmatrix} \omega_m & 0 & 0 \\ 0 & \omega_m^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \omega_n & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-2} \end{pmatrix}$$

where α and β generate D_{2m} and γ generates $\mathbb{Z}_n$.

From these we notice that when both n and m are even, $\alpha^{m/2} = \gamma^{n/2}$ and $< \mathbb{Z}_n, D_{2m} >$ is not a direct product. After inspection, we find that

$$< \mathbb{Z}_n, D_{2m} >= \begin{cases} \mathbb{Z}_n \times D_{2m} & m \bmod 2 = 1 \\ \mathbb{Z}_n \times D_{2m} & m \bmod 2 = 0, n \bmod 2 = 1 \\ \mathbb{Z}_{n/2} \times D_{2m} & m \bmod 2 = 0, n \bmod 4 = 2 \\ (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 & m \bmod 2 = 0, n \bmod 4 = 0 \end{cases} .$$

The Schur multipliers of the direct product cases are immediately computable by consulting theorem 3.3.5. For example, $M(\mathbb{Z}_n \times \hat{\mathcal{D}}_m) \cong M(\mathbb{Z}_n) \times M(\hat{\mathcal{D}}_m) \times (\mathbb{Z}_n \otimes \hat{\mathcal{D}}_m)$ by theorem 3.3.5, the last term of which in turn equates to $\mathrm{Hom}(\mathbb{Z}_n, \hat{\mathcal{D}}_m/\hat{\mathcal{D}}'_m)$. This is $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_{\gcd(n,2)} \times \mathbb{Z}_{\gcd(n,2)}$ for m even and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_4) \cong \mathbb{Z}_{\gcd(n,4)}$ for m odd. By similar token, we have that $M(\mathbb{Z}_n \times D_{2m})$ for even m is $\mathbb{Z}_2 \times \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_{\gcd(n,2)} \times \mathbb{Z}_{\gcd(n,2)}$ and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2) \cong \mathbb{Z}_{\gcd(n,2)}$ for odd m. Likewise $M(\mathbb{Z}_n \times E_{6,7,8}) = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_{3,2,1})$.

Appendix C

Character tables

We here present for reference, the ordinary character tables of the groups as well as the covering groups thereof, of the examples which we studied in chapter 3.

Σ_{60}

1	12	12	15	20
1	1	1	1	1
3	$-\omega_5^2 - \omega_5^{-2}$	$-\omega_5 - \omega_5^{-1}$	-1	0
3	$-\omega_5 - \omega_5^{-1}$	$-\omega_5^2 - \omega_5^{-2}$	-1	0
4	-1	-1	0	1
5	0	0	1	-1

Σ_{60}^*

1	1	12	12	12	12	30	20	20
1	1	1	1	1	1	1	1	1
3	3	$-\omega_5^2 - \omega_5^{-2}$	$-\omega_5^2 - \omega_5^{-2}$	$-\omega_5 - \omega_5^{-1}$	$-\omega_5 - \omega_5^{-1}$	-1	0	0
3	3	$-\omega_5 - \omega_5^{-1}$	$-\omega_5 - \omega_5^{-1}$	$-\omega_5^2 - \omega_5^{-2}$	$-\omega_5^2 - \omega_5^{-2}$	-1	0	0
4	4	-1	-1	-1	-1	0	1	1
5	5	0	0	0	0	1	-1	-1
2	-2	$-\omega_5^2 - \omega_5^{-2}$	$\omega_5^2 + \omega_5^{-2}$	$-\omega_5 - \omega_5^{-1}$	$\omega_5 + \omega_5^{-1}$	0	-1	1
2	-2	$-\omega_5 - \omega_5^{-1}$	$\omega_5 + \omega_5^{-1}$	$-\omega_5^2 - \omega_5^{-2}$	$\omega_5^2 + \omega_5^{-2}$	0	-1	1
4	-4	1	-1	1	-1	0	1	-1
6	-6	-1	1	-1	1	0	0	0

Σ_{168}

1	21	42	56	24	24
1	1	1	1	1	1
3	-1	1	0	a	$\bar{a}$
3	-1	1	0	$\bar{a}$	a
6	2	0	0	-1	-1
7	-1	-1	1	0	0
8	0	0	-1	1	1

Σ_{168}^*

1	1	42	42	42	56	56	24	24	24	24
1	1	1	1	1	1	1	1	1	1	1
3	3	-1	1	1	0	0	a	a	$\bar{a}$	$\bar{a}$
3	3	-1	1	1	0	0	$\bar{a}$	$\bar{a}$	a	a
6	6	2	0	0	0	0	-1	-1	-1	-1
7	7	-1	-1	-1	1	1	0	0	0	0
8	8	0	0	0	-1	-1	1	1	1	1
4	-4	0	0	0	1	-1	$-a$	a	$-\bar{a}$	$\bar{a}$
4	-4	0	0	0	1	-1	$-\bar{a}$	$\bar{a}$	$-a$	a
6	-6	0	$-\sqrt{2}$	$\sqrt{2}$	0	0	-1	1	-1	1
6	-6	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	-1	1	-1	1
8	-8	0	0	0	-1	1	1	-1	1	-1

$$a := \frac{-1 + \sqrt{7}i}{2}$$

Σ_{1080}

1	1	1	45	45	45	72	72	72	72	72	72	90	90	90	120	120
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	$3\bar{A}$	$3A$	$-A$	$-\bar{A}$	-1	X	Y	Z	W	$\bar{Z}$	$\bar{W}$	$\bar{A}$	A	1	0	0
3	$3\bar{A}$	$3A$	$-A$	$-\bar{A}$	-1	Y	X	W	Z	$\bar{W}$	$\bar{Z}$	$\bar{A}$	A	1	0	0
3	$3A$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	X	Y	$\bar{Z}$	$\bar{W}$	Z	W	A	$\bar{A}$	1	0	0
3	$3A$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	Y	X	$\bar{W}$	$\bar{Z}$	W	Z	A	$\bar{A}$	1	0	0
5	5	5	1	1	1	0	0	0	0	0	0	-1	-1	-1	2	-1
5	5	5	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	2
6	$6\bar{A}$	$6A$	$2A$	$2\bar{A}$	2	1	1	$\bar{A}$	$\bar{A}$	A	A	0	0	0	0	0
6	$6A$	$6\bar{A}$	$2\bar{A}$	$2A$	2	1	1	A	A	$\bar{A}$	$\bar{A}$	0	0	0	0	0
8	8	8	0	0	0	X	Y	Y	X	Y	X	0	0	0	-1	-1
8	8	8	0	0	0	Y	X	X	Y	X	Y	0	0	0	-1	-1
9	9	9	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	0	0
9	$9\bar{A}$	$9A$	A	$\bar{A}$	1	-1	-1	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	$\bar{A}$	A	1	0	0
9	$9A$	$9\bar{A}$	$\bar{A}$	A	1	-1	-1	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	A	$\bar{A}$	1	0	0
10	10	10	-2	-2	-2	0	0	0	0	0	0	0	0	0	1	1
15	$15\bar{A}$	$15A$	$-A$	$-\bar{A}$	-1	0	0	0	0	0	0	$-\bar{A}$	$-A$	-1	0	0
15	$15A$	$15\bar{A}$	$-\bar{A}$	$-A$	-1	0	0	0	0	0	0	$-A$	$-\bar{A}$	-1	0	0

$A := \omega_3$;
$B := \omega_5$;
$C := \omega_{15}$;
$X := -B - \bar{B}$;
$Y := -B^2 - \bar{B}^2$;
$Z := -C - C^4$;
$W := -\bar{C}^2 - C^7$;

Σ^*_{1080}

$D := B + \bar{B}, E := B^2 + \bar{B}^2, F := \bar{C} + \bar{C}^4, G := C^2 + \bar{C}^7, H := \omega_{24}, J := \bar{H}^7 - H^{11}, K := \bar{H}^5 - H$

1	1	1	1	1	1	90	90	90	72	72	72	72	72	72	72	72	72	72	72	72	90	90	90	90	90	90	120	120	120	120
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	$3\bar{A}$	$3\bar{A}$	$3A$	$3A$	$-A$	$-\bar{A}$	-1	X	X	Y	Y	Z	Z	W	W	$\bar{Z}$	$\bar{Z}$	$\bar{W}$	$\bar{W}$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
3	3	$3\bar{A}$	$3\bar{A}$	$3A$	$3A$	$-A$	$-\bar{A}$	-1	Y	Y	X	X	W	W	Z	Z	$\bar{W}$	$\bar{W}$	$\bar{Z}$	$\bar{Z}$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
3	3	$3A$	$3A$	$3\bar{A}$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	X	X	Y	Y	$\bar{Z}$	$\bar{Z}$	$\bar{W}$	$\bar{W}$	Z	Z	W	W	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
3	3	$3A$	$3A$	$3\bar{A}$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	Y	Y	X	X	$\bar{W}$	$\bar{W}$	$\bar{Z}$	$\bar{Z}$	W	W	Z	Z	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
5	5	5	5	5	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	2	2	-1	-1
5	5	5	5	5	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	2	2
6	6	$6\bar{A}$	$6\bar{A}$	$6A$	$6A$	$2A$	$2\bar{A}$	2	1	1	1	1	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	A	A	A	A	0	0	0	0	0	0	0	0	0	0
6	6	$6A$	$6A$	$6\bar{A}$	$6\bar{A}$	$2\bar{A}$	$2A$	2	1	1	1	1	A	A	A	A	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	0	0	0	0	0	0	0	0	0	0
8	8	8	8	8	8	0	0	0	X	X	Y	Y	Y	Y	X	X	Y	Y	X	X	0	0	0	0	0	0	-1	-1	-1	-1
8	8	8	8	8	8	0	0	0	Y	Y	X	X	X	X	Y	Y	X	X	Y	Y	0	0	0	0	0	0	-1	-1	-1	-1
9	9	9	9	9	9	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	0	0	0	0
9	9	$9\bar{A}$	$9\bar{A}$	$9A$	$9A$	A	$\bar{A}$	1	-1	-1	-1	-1	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	$-A$	$-A$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
9	9	$9A$	$9A$	$9\bar{A}$	$9\bar{A}$	$\bar{A}$	A	1	-1	-1	-1	-1	$-A$	$-A$	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
10	10	10	10	10	10	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
15	15	$15\bar{A}$	$15\bar{A}$	$15A$	$15A$	$-A$	$-\bar{A}$	-1	0	0	0	0	0	0	0	0	0	0	0	0	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	-1	-1	0	0	0	0
15	15	$15A$	$15A$	$15\bar{A}$	$15\bar{A}$	$-\bar{A}$	$-A$	-1	0	0	0	0	0	0	0	0	0	0	0	0	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	-1	-1	0	0	0	0
4	-4	-4	4	-4	4	0	0	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	1	-1	-2	2
4	-4	-4	4	-4	4	0	0	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	-2	2	1	-1
6	-6	$-6A$	$6A$	$-6\bar{A}$	$6\bar{A}$	0	0	0	-1	1	-1	1	$-A$	A	$-A$	A	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	J	$-J$	K	$-K$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
6	-6	$-6A$	$6A$	$-6\bar{A}$	$6\bar{A}$	0	0	0	-1	1	-1	1	$-A$	A	$-A$	A	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-J$	J	$-K$	K	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
6	-6	$-6\bar{A}$	$6\bar{A}$	$-6A$	$6A$	0	0	0	-1	1	-1	1	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-A$	A	$-A$	A	$-K$	K	$-J$	J	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
6	-6	$-6\bar{A}$	$6\bar{A}$	$-6A$	$6A$	0	0	0	-1	1	-1	1	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-A$	A	$-A$	A	K	$-K$	J	$-J$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
8	-8	-8	8	-8	8	0	0	0	D	X	E	Y	E	Y	D	X	E	Y	D	X	0	0	0	0	0	0	-1	1	-1	1
8	-8	-8	8	-8	8	0	0	0	E	Y	D	X	D	X	E	Y	D	X	E	Y	0	0	0	0	0	0	-1	1	-1	1
10	-10	-10	10	-10	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	1	-1	1	-1
10	-10	-10	10	-10	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	1	-1	1	-1
12	-12	$-12A$	$12A$	$-12\bar{A}$	$12\bar{A}$	0	0	0	X	D	Y	E	$\bar{Z}$	$\bar{F}$	$\bar{W}$	G	Z	F	W	$\bar{G}$	0	0	0	0	0	0	0	0	0	0
12	-12	$-12A$	$12A$	$-12\bar{A}$	$12\bar{A}$	0	0	0	Y	E	X	D	$\bar{W}$	G	$\bar{Z}$	$\bar{F}$	W	$\bar{G}$	Z	F	0	0	0	0	0	0	0	0	0	0
12	-12	$-12\bar{A}$	$12\bar{A}$	$-12A$	$12A$	0	0	0	X	D	Y	E	Z	F	W	$\bar{G}$	$\bar{Z}$	$\bar{F}$	$\bar{W}$	G	0	0	0	0	0	0	0	0	0	0
12	-12	$-12\bar{A}$	$12\bar{A}$	$-12A$	$12A$	0	0	0	Y	E	X	D	W	$\bar{G}$	Z	F	$\bar{W}$	G	$\bar{Z}$	$\bar{F}$	0	0	0	0	0	0	0	0	0	0

$\Delta_{6\cdot 2^2} =$

1	3	6	6	8
1	1	1	1	1
1	1	-1	-1	1
2	2	0	0	-1
3	-1	-1	1	0
3	-1	1	-1	0

$\Delta^*_{6\cdot 2^2} =$

1	1	6	6	6	12	8	8
1	1	1	1	1	1	1	1
1	1	1	-1	-1	-1	1	1
2	2	2	0	0	0	-1	-1
3	3	-1	-1	-1	1	0	0
3	3	-1	1	1	-1	0	0
2	-2	0	$-e^{\frac{i}{4}\pi} - e^{\frac{3i}{4}\pi}$	$e^{\frac{i}{4}\pi} + e^{\frac{3i}{4}\pi}$	0	-1	1
2	-2	0	$e^{\frac{i}{4}\pi} + e^{\frac{3i}{4}\pi}$	$-e^{\frac{i}{4}\pi} - e^{\frac{3i}{4}\pi}$	0	-1	1
4	-4	0	0	0	0	1	-1

$\Delta_{6\cdot 4^2} =$

1	3	3	3	6	12	12	12	12	32
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	-1	-1	-1	-1	1
2	2	2	2	2	0	0	0	0	-1
3	3	-1	-1	-1	-1	1	1	-1	0
3	3	-1	-1	-1	1	-1	-1	1	0
3	-1	$-1-2i$	$-1+2i$	1	-1	i	$-i$	1	0
3	-1	$-1+2i$	$-1-2i$	1	-1	$-i$	i	1	0
3	-1	$-1-2i$	$-1+2i$	1	1	$-i$	i	-1	0
3	-1	$-1+2i$	$-1-2i$	1	1	i	$-i$	-1	0
6	-2	2	2	-2	0	0	0	0	0

$\Delta^*_{6\cdot 4^2} =$

1	1	3	3	6	6	12	24	12	12	12	12	24	32	32
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
2	2	2	2	2	2	2	0	0	0	0	0	0	-1	-1
3	3	3	3	-1	-1	-1	-1	1	1	1	1	-1	0	0
3	3	3	3	-1	-1	-1	1	-1	-1	-1	-1	1	0	0
3	3	-1	-1	$-1-2i$	$-1+2i$	1	-1	i	i	$-i$	$-i$	1	0	0
3	3	-1	-1	$-1+2i$	$-1-2i$	1	-1	$-i$	$-i$	i	i	1	0	0
3	3	-1	-1	$-1-2i$	$-1+2i$	1	1	$-i$	$-i$	i	i	-1	0	0
3	3	-1	-1	$-1+2i$	$-1-2i$	1	1	i	i	$-i$	$-i$	-1	0	0
6	6	-2	-2	2	2	-2	0	0	0	0	0	0	0	0
2	-2	-2	2	0	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$	$-i\sqrt{2}$	$i\sqrt{2}$	0	-1	1
2	-2	-2	2	0	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$	$i\sqrt{2}$	$-i\sqrt{2}$	0	-1	1
4	-4	-4	4	0	0	0	0	0	0	0	0	0	1	-1
6	-6	2	-2	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0
6	-6	2	-2	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0

$\Delta_{3\cdot 4^2} =$

1	3	3	3	3	3	16	16
1	1	1	1	1	1	1	1
1	1	1	1	1	1	ω_3	$\bar{\omega_3}$
1	1	1	1	1	1	$\bar{\omega_3}$	ω_3
3	-1	-1	3	-1	-1	0	0
3	1	1	-1	$-1-2i$	$-1+2i$	0	0
3	1	1	-1	$-1+2i$	$-1-2i$	0	0
3	$-1-2i$	$-1+2i$	-1	1	1	0	0
3	$-1+2i$	$-1-2i$	-1	1	1	0	0

$\Delta^*_{3\cdot 4^2} =$

1	1	1	1	12	12	6	6	12	12	16	16	16	16	16	16	16	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	ω_3
3	3	3	3	-1	-1	3	3	-1	-1	0	0	0	0	0	0	0	0
3	3	3	3	1	1	-1	-1	$-1-2i$	$-1+2i$	0	0	0	0	0	0	0	0
3	3	3	3	1	1	-1	-1	$-1+2i$	$-1-2i$	0	0	0	0	0	0	0	0
3	3	3	3	$-1-2i$	$-1+2i$	-1	-1	1	1	0	0	0	0	0	0	0	0
3	3	3	3	$-1+2i$	$-1-2i$	-1	-1	1	1	0	0	0	0	0	0	0	0
2	-2	2	-2	0	0	2	-2	0	0	$-\omega_3$	ω_3	ω_3	$-\omega_3$	$-\bar{\omega}_3$	$-\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
2	-2	2	-2	0	0	2	-2	0	0	$-\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$-\omega_3$	$-\omega_3$	ω_3	ω_3
2	-2	2	-2	0	0	2	-2	0	0	-1	1	1	-1	-1	-1	1	1
6	-6	6	-6	0	0	-2	2	0	0	0	0	0	0	0	0	0	0
4	$4i$	-4	$-4i$	0	0	0	0	0	0	$\bar{\omega}_3$	$-\bar{\omega}_{12}$	$\bar{\omega}_{12}$	$-\bar{\omega}_3$	ω_3	$-\omega_3$	$\bar{\omega}_{12}^5$	$-\bar{\omega}_{12}^5$
4	$4i$	-4	$-4i$	0	0	0	0	0	0	ω_3	$-\bar{\omega}_{12}^5$	$\bar{\omega}_{12}^5$	$-\omega_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$\bar{\omega}_{12}$	$-\bar{\omega}_{12}$
4	$4i$	-4	$-4i$	0	0	0	0	0	0	1	$-i$	i	-1	1	-1	i	$-i$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	$\bar{\omega}_3$	$\bar{\omega}_{12}$	$-\bar{\omega}_{12}$	$-\bar{\omega}_3$	ω_3	$-\omega_3$	$-\bar{\omega}_{12}^5$	$\bar{\omega}_{12}^5$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	ω_3	$\bar{\omega}_{12}^5$	$-\bar{\omega}_{12}^5$	$-\omega_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$-\bar{\omega}_{12}$	$\bar{\omega}_{12}$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	1	i	$-i$	-1	1	-1	$-i$	i

$\Delta_{3\cdot 5^2} =$

1	3	3	3	3	3	3	3	3	25	25
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	E	$\bar{E}$
1	1	1	1	1	1	1	1	1	$\bar{E}$	E
3	A	A	B	B	C	D	$\bar{C}$	$\bar{D}$	0	0
3	A	A	B	B	$\bar{C}$	$\bar{D}$	C	D	0	0
3	B	B	A	A	$\bar{D}$	C	D	$\bar{C}$	0	0
3	B	B	A	A	D	$\bar{C}$	$\bar{D}$	C	0	0
3	C	$\bar{C}$	D	$\bar{D}$	B	A	B	A	0	0
3	$\bar{D}$	D	C	$\bar{C}$	A	B	A	B	0	0
3	$\bar{C}$	C	$\bar{D}$	D	B	A	B	A	0	0
3	D	$\bar{D}$	$\bar{C}$	C	A	B	A	B	0	0

$\Delta^*_{3\cdot 5^2} =$

1	1	1	1	1	15	15	15	15	15	15	15	15	25	25	25	25	25	25	25	25	25	25
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	E	E	E	E	E	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$
1	1	1	1	1	1	1	1	1	1	1	1	1	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	E	E	E	E	E
3	3	3	3	3	A	A	B	B	C	D	$\bar{C}$	$\bar{D}$	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	A	A	B	B	$\bar{C}$	$\bar{D}$	C	D	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	B	B	A	A	$\bar{D}$	C	D	$\bar{C}$	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	B	B	A	A	D	$\bar{C}$	$\bar{D}$	C	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	C	$\bar{C}$	D	$\bar{D}$	B	A	B	A	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	$\bar{D}$	D	C	$\bar{C}$	A	B	A	B	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	$\bar{C}$	C	$\bar{D}$	D	B	A	B	A	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	D	$\bar{D}$	$\bar{C}$	C	A	B	A	B	0	0	0	0	0	0	0	0	0	0
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	-1	$-\bar{F}$	$-\bar{F}^2$	$-\bar{F}^2$	$-F$	-1	$-F$	$-\bar{F}^2$	$-\bar{F}^2$	$-\bar{F}$
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}^7$	$-\bar{G}$	$-\bar{G}^4$	$-G^2$	$-\bar{E}$	$-\bar{G}^7$	$-G$	$-\bar{G}^4$	$-\bar{G}^2$
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^2$	$-\bar{G}^4$	$-G$	$-\bar{G}^7$	$-E$	$-G^2$	$-\bar{G}^4$	$-\bar{G}$	$-\bar{G}^7$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	-1	$-F$	$-\bar{F}^2$	$-\bar{F}^2$	$-\bar{F}$	-1	$-\bar{F}$	$-\bar{F}^2$	$-\bar{F}^2$	$-F$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-E$	$-G^2$	$-\bar{G}^4$	$-\bar{G}$	$-\bar{G}^7$	$-\bar{E}$	$-\bar{G}^2$	$-\bar{G}^4$	$-G$	$-\bar{G}^7$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^7$	$-G$	$-\bar{G}^4$	$-\bar{G}^2$	$-E$	$-\bar{G}^7$	$-\bar{G}$	$-\bar{G}^4$	$-G^2$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	-1	$-\bar{F}^2$	$-F$	$-\bar{F}$	$-\bar{F}^2$	-1	$-\bar{F}^2$	$-\bar{F}$	$-F$	$-\bar{F}^2$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}$	$-G^2$	$-\bar{G}^7$	$-\bar{G}^4$	$-\bar{E}$	$-G$	$-\bar{G}^2$	$-\bar{G}^7$	$-\bar{G}^4$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^4$	$-\bar{G}^7$	$-\bar{G}^2$	$-G$	$-E$	$-\bar{G}^4$	$-\bar{G}^7$	$-G^2$	$-\bar{G}$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	-1	$-\bar{F}^2$	$-\bar{F}$	$-F$	$-\bar{F}^2$	-1	$-\bar{F}^2$	$-F$	$-\bar{F}$	$-\bar{F}^2$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}^4$	$-\bar{G}^7$	$-G^2$	$-\bar{G}$	$-\bar{E}$	$-\bar{G}^4$	$-\bar{G}^7$	$-\bar{G}^2$	$-G$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-G$	$-\bar{G}^2$	$-\bar{G}^7$	$-\bar{G}^4$	$-E$	$-\bar{G}$	$-G^2$	$-\bar{G}^7$	$-\bar{G}^4$

$A := -\omega_5 - \bar{\omega}_5, B := -\omega_5^2 - \bar{\omega}_5^2, C := \bar{\omega}_5 - 2\bar{\omega}_5^2, D := 2\omega_5 + \bar{\omega}_5^2; E := \omega_3, F := \bar{\omega}_5, G := \omega_{15}.$

Appendix D

Proofs of the properties of the SBFST antibracket

By definition we have

$$
\begin{aligned}
&d\omega(\delta_i G, \delta_j G, \delta_k G) = \\
&(-)^{\epsilon_j} \int \prod_{\beta=1}^{3} d\tau_\beta d\theta_\beta \Big(\Big\langle (b_{-1}\delta_i G)(Y\delta_j G)(Y\delta_k G) \Big\rangle + \\
&(-1)^{\epsilon_i} \Big\langle (Y\delta_i G)(b_{-1}\delta_j G)(Y\delta_k G) \Big\rangle + (-1)^{\epsilon_i+\epsilon_j} \Big\langle (Y\delta_i G)(Y\delta_j G)(b_{-1}\delta_k G) \Big\rangle \Big),
\end{aligned}
\tag{D.0.1}
$$

with $\epsilon_i = \epsilon(\delta_i G)$.

Let us consider the following Ward identities for b_{-1}

$$
\begin{gathered}
\langle b_{-1}(\delta_i G(Y\delta_j G)(Y\delta_k G))\rangle = 0 \Rightarrow \\
\langle (b_{-1}\delta_i G)(Y\delta_j G)(Y\delta_k G)\rangle + (-)^{\epsilon_i}\langle \delta_i G(b_{-1}Y\delta_j G)(Y\delta_k G)\rangle + \\
(-)^{\epsilon_i+\epsilon_j}\langle \delta_i G(Y\delta_j)G(b_{-1}Y\delta_k G)\rangle = 0,
\end{gathered}
$$

$$
\begin{gathered}
\langle b_{-1}((Y\delta_i G)\delta_j G(Y\delta_k G))\rangle = 0 \Rightarrow \\
\langle (b_{-1}Y\delta_i G)\delta_j G(Y\delta_k G)\rangle + (-)^{\epsilon_i}\langle (Y\delta_i G)(b_{-1}\delta_j G)(Y\delta_k G)\rangle + \\
(-)^{\epsilon_i+\epsilon_j}\langle (Y\delta_i G)\delta_j G(b_{-1}Y\delta_k G)\rangle = 0,
\end{gathered}
$$

$$\langle b_{-1}((Y\delta_i G)(Y\delta_j G)\delta_k G)\rangle = 0 \Rightarrow$$
$$\langle (b_{-1}Y\delta_i G)(Y\delta_j G)\delta_k G\rangle + (-)^{\epsilon_i}\langle (Y\delta_i G)(b_{-1}Y\delta_j G)\delta_k G\rangle +$$
$$(-)^{\epsilon_i+\epsilon_j}\langle (Y\delta_i G)(Y\delta_j G)(b_{-1}\delta_k G)\rangle = 0.$$

Adding the last three identities, separating the appropriate terms and integrating gives

$$d\omega(\delta_i G, \delta_j G, \delta_k G) =$$
$$(-)^{\epsilon_j+1}\int \prod_{\beta=1}^{3} d\tau_\beta d\theta_\beta \Big((-)^{\epsilon_i}\langle \delta_i G(b_{-1}Y\delta_j G)(Y\delta_k G)\rangle + (-)^{\epsilon_i+\epsilon_j}\langle \delta_i G(Y\delta_j G)(b_{-1}Y\delta_k G)\rangle +$$
$$\langle (b_{-1}Y\delta_i G)\delta_j G(Y\delta_k G)\rangle + (-)^{\epsilon_i+\epsilon_j}\langle (Y\delta_i G)\delta_j G(b_{-1}Y\delta_k G)\rangle +$$
$$\langle (b_{-1}Y\delta_i G)(Y\delta_j G)\delta_k G\rangle + (-)^{\epsilon_i}\langle (Y\delta_i G)(b_{-1}Y\delta_j G)\delta_k G\rangle \Big).$$

We have three pairs of terms, each of them labelled by the same statistics factor in front. These pairs are actually vanishing. To see that, let us consider for example the pair

$$\langle (b_{-1}Y\delta_i G)\delta_j G(Y\delta_k G)\rangle + \langle (b_{-1}Y\delta_i G)(Y\delta_j G)\delta_k G\rangle. \tag{D.0.2}$$

For the unperturbed correlator $\langle ...\rangle_0$ we can write the following two Ward identities for the BRST charge Q:

$$\Big\langle \xi(0)Q\Big((b_{-1}Y\delta_i G)(\tau_1,\theta_1)(\xi Y\delta_j G)(\tau_2,\theta_2)(Y\delta_k G)(\tau_3,\theta_3)e^{\int d\tau d\theta \mathcal{V}}\Big)\Big\rangle_0 =$$
$$\Big\langle X(0)(b_{-1}Y\delta_i G)(\tau_1,\theta_1)(\xi Y\delta_j G)(\tau_2,\theta_2)(Y\delta_k G)(\tau_3,\theta_3)\Big\rangle \Rightarrow$$
$$\langle \xi(0)(Qb_{-1}Y\delta_i G)(\xi Y\delta_j G)(Y\delta_k G)\rangle +$$
$$(-)^{1+\epsilon_i}\langle \xi(0)(b_{-1}Y\delta_i G)\delta_j G(Y\delta_k G)\rangle + (-)^{\epsilon_i}\langle \xi(0)(b_{-1}Y\delta_i G)(\xi(QY\delta_j G))(Y\delta_k G)\rangle +$$
$$(-)^{\epsilon_i+\epsilon_j}\langle \xi(0)(b_{-1}Y\delta_i G)(\xi Y\delta_j G)(QY\delta_k G)\rangle +$$
$$(-)^{\epsilon_i+\epsilon_j+\epsilon_k}\langle \xi(0)(b_{-1}Y\delta_i G)(\xi Y\delta_j G)(Y\delta_k G)[Q, e^{\int d\tau d\theta \mathcal{V}}]\rangle_0 =$$
$$\langle X(0)(b_{-1}Y\delta_i G)(\xi Y\delta_j G)(Y\delta_k G)\rangle$$

and

$$\Big\langle \xi(0)Q\Big((b_{-1}Y\delta_i G)(\tau_1,\theta_1)(Y\delta_j G)(\tau_2,\theta_2)(\xi Y\delta_k G)(\tau_3,\theta_3)e^{\int d\tau d\theta \mathcal{V}}\Big)\Big\rangle_0 =$$

$$
\begin{gathered}
\Big\langle X(0)(b_{-1}Y\delta_i G)(\tau_1,\theta_1)(Y\delta_j G)(\tau_2,\theta_2)(\xi Y\delta_k G)(\tau_3,\theta_3)\Big\rangle \Rightarrow \\
\langle \xi(0)(Qb_{-1}Y\delta_i G)(Y\delta_j G)(\xi Y\delta_k G)\rangle + \\
(-)^{1+\epsilon_i}\langle \xi(0)(b_{-1}Y\delta_i G)(QY\delta_j G)(\xi Y\delta_k G)\rangle + (-)^{1+\epsilon_i+\epsilon_j}\langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)\delta_k G\rangle + \\
(-)^{\epsilon_i+\epsilon_j}\langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)(\xi(QY\delta_k G))\rangle + \\
(-)^{\epsilon_i+\epsilon_j+\epsilon_k}\langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)(\xi Y\delta_k G)[Q, e^{\int d\tau d\theta \mathcal{V}}]\rangle_0 = \\
\langle X(0)(b_{-1}Y\delta_i G)(Y\delta_j G)(\xi Y\delta_k G)\rangle .
\end{gathered}
$$

We have explicitly inserted a ξ insertion in the center of the disk to saturate the zero mode of the $\xi\eta$ system [7]. This insertion is absent from similar expressions in the main text, but its presence is always implied. The left hand side of these Ward identities is obtained by pushing the BRST current contour onto the boundary and the right hand side by shrinking it to zero radius around the center of the disk.

Solving in the above identities for $\langle \xi(0)(b_{-1}Y\delta_i G)\delta_j G(Y\delta_k G)\rangle$ and $\langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)\delta_k G\rangle$ and adding the resulting expressions gives

$$
\begin{gathered}
\langle \xi(0)(b_{-1}Y\delta_i G)\delta_j G(Y\delta_k G)\rangle + \langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)\delta_k G\rangle = \\
(-)^{\epsilon_i}\langle \xi(0)(Qb_{-1}Y\delta_i G)(\xi Y\delta_j G)(Y\delta_k G)\rangle + (-)^{\epsilon_i+\epsilon_j}\langle \xi(0)(Qb_{-1}Y\delta_i G)(Y\delta_j G)(\xi Y\delta_k G)\rangle + \\
\langle \xi(0)(b_{-1}Y\delta_i G)(\xi(QY\delta_j G))(Y\delta_k G)\rangle + (-)^{\epsilon_j+1}\langle \xi(0)(b_{-1}Y\delta_i G)(QY\delta_j G)(\xi Y\delta_k G)\rangle + \\
(-)^{\epsilon_i+1}\langle X(0)(b_{-1}Y\delta_i G)(\xi Y\delta_j G)(Y\delta_k G)\rangle + \\
(-)^{\epsilon_j}\langle \xi(0)(b_{-1}Y\delta_i G)(\xi Y\delta_j G)(QY\delta_k G)\rangle + \langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)(\xi(QY\delta_k G))\rangle + \\
(-)^{\epsilon_j+\epsilon_k}\langle \xi(0)(b_{-1}Y\delta_i G)(\xi Y\delta_j G)(Y\delta_k G)[Q, e^{\int d\tau d\theta \mathcal{V}}]\rangle_0 + \\
(-)^{\epsilon_k}\langle \xi(0)(b_{-1}Y\delta_i G)(Y\delta_j G)(\xi Y\delta_k G)[Q, e^{\int d\tau d\theta \mathcal{V}}]\rangle_0 + \\
(-)^{\epsilon_j+\epsilon_i+1}\langle X(0)(b_{-1}Y\delta_i G)(Y\delta_j G)(\xi Y\delta_k G)\rangle .
\end{gathered}
$$

Since, the position of the ξ insertions is irrelevant, we can move appropriately the ξ insertion (on the boundary) in the above expressions and after integrating over τ and θ we take

$$
\int \prod_{\beta=1}^{3} d\tau_\beta d\theta_\beta \Big(\langle (b_{-1}Y\delta_i G)\delta_j G Y\delta_k G\rangle + \langle (b_{-1}Y\delta_i G)Y\delta_j G\delta_k G\rangle \Big) = 0. \quad \text{(D.0.3)}
$$

Continuing in the same fashion for the other two pairs we conclude that $d\omega = 0$.

The proof of V-invariance of ω goes in a similar way. The only addition is the use of the identity $\{Q, b_{-1}\} = v^a\partial_a$, where $v^a\partial_a$ is the generator of rotations on the disk, as well as the use of the identity

$$\int d\tau d\theta v^a \partial_a (Y\delta G) = 0. \tag{D.0.4}$$

Appendix E

Calculation of formula (6.3.45)

In this appendix we would like to show equation (6.3.45). From (6.3.44) we have

$$[Q, C\mathcal{V}] = [(1-h)C\partial_\tau C - \frac{1}{4}(D_\theta C)(D_\theta C)]\mathcal{V} + \frac{1}{2}C(D_\theta C)(D_\theta \mathcal{V}). \tag{E.0.1}$$

In components, the above equation involves the expressions [1]

$$C\partial_\tau C = c\partial_\tau c + \theta(\gamma\partial_\tau c - c\partial_\tau \gamma), \tag{E.0.2}$$

$$(D_\theta C)(D_\theta C) = \gamma^2 + 2\theta\gamma\partial_\tau c, \tag{E.0.3}$$

$$CD_\theta C = c\gamma + \theta(\gamma^2 - c\partial_\tau c). \tag{E.0.4}$$

Since $Y = -\partial\xi c e^{-2\phi}$ we easily deduce by using the relevant OPEs that

$$Y\gamma = -ce^{-\phi}. \tag{E.0.5}$$

[1] All products of fields appearing here are normal ordered in the usual CFT fashion, i.e. $AB(w) = \oint \frac{dz}{2\pi i}\frac{1}{z-w}A(z)B(w)$.

Hence, acting with Y on the above equations gives the following expressions

$$YC\partial_\tau C = Yc\partial_\tau c + \theta Y\gamma\partial_\tau c = Yc\partial_\tau c - \theta c\partial_\tau c e^{-\phi}, \tag{E.0.6}$$

$$Y(D_\theta C)(D_\theta C) = Y\gamma^2 + 2\theta Y\gamma\partial_\tau c = Y\gamma^2 - 2\theta c\partial_\tau c e^{-\phi}, \tag{E.0.7}$$

$$YCD_\theta C = \theta(Y\gamma^2 - Yc\partial_\tau c). \tag{E.0.8}$$

Combining these equations with (E.0.1) gives

$$\begin{aligned}
&Y[Q, C\mathcal{V}] = \qquad\qquad \text{(E.0.9)}\\
&\Big((1-h)(Yc\partial_\tau c - \theta c\partial_\tau c e^{-\phi} - \frac{1}{4}Y\gamma^2 + 2\frac{1}{4}\theta c\partial_\tau c e^{-\phi}\Big)\mathcal{V} + \frac{1}{2}\theta(Y\gamma^2 - Yc\partial_\tau c)D_\theta\mathcal{V} =\\
&\Big((1-h)Yc\partial_\tau c - \frac{1}{4}Y\gamma^2\Big)\mathcal{V} + \frac{1}{2}\theta(Y\gamma^2 - Yc\partial_\tau c)D_\theta\mathcal{V} + \Big(h - \frac{1}{2}\Big)\theta c\partial_\tau c e^{-\phi}\mathcal{V} =\\
&\Big((1-h)Yc\partial_\tau c - \frac{1}{4}Y\gamma^2\Big)\mathcal{V} + \frac{1}{2}\theta(Y\gamma^2 - Yc\partial_\tau c)D_\theta\mathcal{V} + \Big(h - \frac{1}{2}\Big)\theta c\partial_\tau c e^{-\phi}D,
\end{aligned}$$

i.e. equation (6.3.45) .

Appendix F

Relating the coefficients of 2- and 3-point functions from SUSY Ward identities

In this appendix we use the SUSY Ward identities on the real line in order to demonstrate relations between the coefficients of certain 2-point and 3-point functions at the conformal point.

F.1 2-point functions

The SUSY Ward identity for the 2-point functions on the real line reads

$$\begin{aligned}&\delta\langle U_i(x_1)D_j(x_2)\rangle_0 = 0 \Rightarrow \\ &\langle \partial_{x_1} D_i(x_1)D_j(x_2)\rangle_0 + \langle U_i(x_1)U_j(x_2)\rangle_0 = 0,\end{aligned} \tag{F.1.1}$$

where δ denotes an infinitesimal SUSY transformation.

Plugging in (F.1.1) the CFT expressions

$$\langle D_i(x_1)D_j(x_2)\rangle_0 = \frac{c\delta_{h_i,h_j}}{|x_1 - x_2|^{2h}}\mathrm{sign}(x_1 - x_2) \tag{F.1.2}$$

and

$$\langle U_i(x_1)U_j(x_2)\rangle_0 = \frac{c\delta_{h_i,h_j}}{|x_1-x_2|^{2h+1}} \tag{F.1.3}$$

with $h = h_i = h_j$, we conclude that

$$2ch = b. \tag{F.1.4}$$

F.2 3-point functions

The relevant Ward identity is

$$\delta\langle U_i(x_1)D_j(x_2)U_k(x_3)\rangle_0 \Rightarrow \tag{F.2.5}$$
$$\langle \partial_{x_1}D_i(x_1)D_j(x_2)U_k(x_3)\rangle_0 + \langle U_i(x_1)U_j(x_2)U_k(x_3)\rangle_0 - \langle U_i(x_1)D_j(x_2)\partial_{x_3}D_k(x_3)\rangle_0 = 0.$$

Substituting for the CFT expressions

$$\langle U_i(x_1)U_j(x_2)U_k(x_3)\rangle_0 = \tag{F.2.6}$$
$$\frac{B_{ijk}}{|x_1-x_2|^{h_i+h_j-h_k+\frac{1}{2}}|x_2-x_3|^{h_k+h_j-h_i+\frac{1}{2}}|x_3-x_1|^{h_i+h_k-h_j+\frac{1}{2}}},$$

$$\langle D_i(x_1)D_j(x_2)U_k(x_3)\rangle_0 = \tag{F.2.7}$$
$$\frac{C_{ijk}\ \ \mathrm{sign}(x_1-x_2)}{|x_1-x_2|^{h_i+h_j-h_k-\frac{1}{2}}|x_2-x_3|^{h_j+h_k-h_i+\frac{1}{2}}|x_3-x_1|^{h_i+h_k-h_j+\frac{1}{2}}}$$

gives

$$B_{ijk}(x_1-x_3) = \tag{F.2.8}$$
$$C_{ijk}\Big(h_i+h_j-h_k-\frac{1}{2}\Big)(x_1-x_3) + C_{ijk}\Big(h_k+h_i-h_j+\frac{1}{2}\Big)(x_1-x_2) +$$
$$+C_{jki}\Big(h_j+h_k-h_i-\frac{1}{2}\Big)(x_1-x_3) - C_{jki}\Big(h_i+h_k-h_j+\frac{1}{2}\Big)(x_3-x_2).$$

This equation must be valid for any value of the worldsheet variables x_1, x_2, x_3. Hence,

$$B_{ijk} = 2h_i C_{ijk} + C_{jki}\left(h_k + h_j - h_i - \frac{1}{2}\right), \tag{F.2.9}$$

$$B_{ijk} = C_{ijk}(h_i + h_j - h_k) + 2h_k C_{jki}, \tag{F.2.10}$$

$$C_{ijk} = C_{jki}. \tag{F.2.11}$$

Equivalently, the first two equations give

$$B_{ijk} = -C_{ijk}\left(\frac{1}{2} - h_i - h_j - h_k\right). \tag{F.2.12}$$

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