ON STABILITY THEORY*

by

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ABSTRACT

It is found that under mild assumptions, feedback system stability can be concluded if one can 'topologically separate' the infinite-dimensional function space containing the system's dynamical input-output relations into two regions, one region containing the dynamical input-output relation of the 'feedforward' element of the system and the other region containing the dynamical output-input relation of the 'feedback' element. Nonlinear system stability criteria of both the input-output type and the state-space (Lyapunov) type are interpreted in this context. The abstract generality and conceptual simplicity afforded by the topological separation perspective clarifies some of the basic issues underlying stability theory and serves to suggest improvements in existing stability criteria. A generalization of Zames' conic-relation stability criterion is proved, laying the foundation for improved <u>multivariable</u> generalizations of the circle and Popov frequency-domain stability criteria for nonlinear systems.

1. INTRODUCTION

Examining the conditions of Zames' conic relation stability theorem [1] -- a powerful abstract result including among its corollaries the Popov, circle, passivity, and small-gain stability criteria (cf. [2]-[6]) -- we have been struck by the observation that the conditions of the conic relation theorem have an unexpectedly simple interpretation in terms of a topological separation of the space on which the systems input-output relations are defined. Our scrutiny of the classical Lyapunov stability theory (e.g., [7]-[8]) has revealed that a similar interpretation applies to the stability conditions imposed by the Lyapunov theory. Motivated by these discoveries, we have developed a unified theory of stability in which the Lyapunov functions and the contraction mappings of previous theories are replaced by 'separating' functionals. The abstract generality of our approach serves to clarify the roles in stability theory of such basic concepts as extended normed spaces, contraction mappings, and positive-definite, decrescent, radially unbounded Lyapunov functions.

The conceptually simple view of stability theory afforded by topological separations has made clear to us some generalizations Zames' results [1]-[2] -- for example, the 'sector stability criterion' described in this paper. Our sector stability criterion forms the basis for several powerful and useful <u>multivariable</u> generalizations of the circle and Popov stability criteria; these generalizations are described in [9] and are the subject of a forthcoming paper.

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This paper is based primarily on chapters 1-4 of Part II of

M. G. Safonov's Ph.D. dissertation [9].

2. PROBLEM FORMULATION

Our results concern the stability of the following canonical two-subsystem multivariable feedback system (Figure 1a)

$$(\underline{x}, \underline{y}) \in \overline{G}(\underline{u})$$
$$(\underline{y}, \underline{x}) \in \overline{H}(\underline{v})$$

(2.1)

where

$\underline{u} \in U_e$ and $\underline{v} \in V_e$ are disturbance inputs to the system;

$\underline{x} \in X_e$ and $\underline{y} \in Y_e$ are the outputs of the system;

$\overline{G}(\underline{u}) \subset X_e \times Y_e$ and $\overline{H}(\underline{v}) \subset Y_e \times X_e$ are nonlinear relations which are dependent of the disturbance inputs $\underline{u} \in U_e$ and $\underline{v} \in V_e$ respectively;

U_e, V_e, X_e, and Y_e are vector spaces.[1]

It is assumed that $(\underline{0}, \underline{0}) \in \overline{G}(\underline{0})$ and that $(\underline{0}, \underline{0}) \in \overline{H}(\underline{0})$ so that the

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[1] A set $\overline{X}$ is described as a vector space   [10, p. 171] (or, equivalently, as a linear set   [11, pp. 43-44]) if for any two of its elements $\underline{x}_1$ and $\underline{x}_2$, the sum $\underline{x}_1 + \underline{x}_2$ is defined and is an element of $\overline{X}$, and similarly the product $a\underline{x}$ is defined, 'a' being a scalar ; additionally, the following axioms must hold

(1) $(\underline{x}_1 + \underline{x}_2) + \underline{x}_3 = \underline{x}_1 + (\underline{x}_2 + \underline{x}_3)$ (associative addition);

(2) $\underline{x}_1 + \underline{x}_2 = \underline{x}_2 + \underline{x}_1$ (commutative addition);

(3) an element $\underline{0}$ exists in $\overline{X}$ such that $0\underline{x} = \underline{0}$ for all $\underline{x}$ in $\overline{X}$;

(4) $(a_1 + a_2)\underline{x} = a_1\underline{x} + a_2\underline{x}$

(5) $a(\underline{x}_1 + \underline{x}_2) = a\underline{x}_1 + a\underline{x}_2$ (distributive laws)

(6) $(ab)\underline{x} = a(b\underline{x})$ (associative multiplication);

(7) $1\underline{x} = \underline{x}$.

pair $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ is an equilibrium solution of the undisturbed system. The system (2.1) defines a relation between input pairs $(\underline{u}, \underline{v}) \in U_e \times V_e$ and output pairs $(\underline{x}, \underline{y}) \in X_e \times Y_e$; equivalently (2.1) defines a subset of the space $(U_e \times V_e) \times (X_e \times Y_e)$.

The vector spaces U_e, V_e, X_e, and Y_e are assumed to be <u>extended</u> <u>normed spaces</u>, defined in terms of collections of normed spaces U_τ, V_τ, X_τ, and Y_τ and a collection of linear 'projection' operators P_τ as follows.

<u>Definition</u>: Let Z_e be a vector space. If there is a associated with Z_e an interval T and a collection of <u>linear</u> operators P_τ ($\tau \in T$) mapping Z_e into the collection of <u>normed</u> spaces Z_τ ($\tau \in T$) then Z_e is the <u>extended normed space</u> induced by the collection of operators P_τ ($\tau \in T$). If, additionally, each of the spaces Z_τ is an inner-product space then we say that Z_e is the <u>extended inner product space</u> induced by the collection of operators P_τ.[2] A vector space Z_e which is itself a

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[2]One can define a variety of 'extended norm' functionals $||\cdot||_e : Z_e \to R_+ \cup \{\infty\}$ on the space Z_e, e.g., [1]

$$||\underline{z}||_e = \sup_{\tau \in T} ||P_\tau \underline{z}||_{Z_\tau}$$

or [5]

$$||\underline{z}||_e = \lim_{\tau \to (\sup T)} \sup ||P_\tau \underline{z}||_{Z_\tau} \quad . \quad .$$

Since $||\underline{z}||_e$ may in general be infinite, the functional $||\cdot||$ is not necessarily a norm in the usual sense. However, $||\cdot||_e$ does define a norm on the subspace

$$Z \triangleq \{\underline{z} \in Z_e \mid ||\underline{z}||_e < \infty\} \quad .$$

For purposes of stability analysis, we have found that it is not necessary to introduce the extended norm $||\cdot||_e$ or the normed subspace Z since the stability properties of each $\underline{z} \in Z_e$ can be determined from the τ-dependence of $||\underline{z}||_\tau$.

normed space is presumed to be the extended normed space for which $Z_\tau = Z_e$ for all τ, and Z_τ is the identity operator, <u>unless</u> Z_τ and/or P_τ are specifically stated to be otherwise.

Comments:

The foregoing problem formulation is considerably more general than is usual in stability theory. Typically, the interval T represents time and the spaces X_e and Y_e consist of functions mapping T into R^n; the operator P_τ: $(X_e, Y_e, Z_e) \rightarrow (X_\tau, Y_\tau, Z_\tau)$ is typically taken to be the <u>linear truncation operator</u> [1]

$$(P_\tau \cdot \underline{\xi})(t) = \left\{ \begin{array}{ll} \underline{\xi}(t), & \text{if } t \leq \tau \\ \underline{0}, & \text{if } t > \tau \end{array} \right\}.$$ (2.2)

The interval T might typically be taken to be either the non-negative real numbers R_+ (in the case of continuous-time systems) or the set of non-negative integers Z_+ (in the case of discrete-time systems). The disturbance vector spaces U_e and V_e are typically both either R^n (in the case of Lyapunov state-space results [7]) or spaces of functions mapping T into R^n (in the case of 'input-output' stability results [5]).[3]

In the special case in which the relations $\overline{G}(\underline{u})$ and $\overline{H}(\underline{v})$ are induced by disturbance-dependent functional operators $G(\underline{u})$: $Y_e \rightarrow X_e$ and $H(\underline{v})$: $X_e \rightarrow Y_e$, then the system (2.1) may be represented by the equivalent set of feedback equations

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[3]Readers unfamiliar with the notion of a relation, the concept of an extended normed space, the linear truncation operator, or other concepts and definitions associated with 'input-output' stability theory may find it helpful to refer to one of the books [3], [4], [6] or the concise and lucid original exposition of Zames [1].

$$\left. \begin{array}{l} \underline{x} = \mathcal{G}(\underline{u}) \cdot \underline{y} \\[2mm] \underline{y} = \mathcal{H}(\underline{v}) \cdot \underline{x} \end{array} \right\} \tag{2.3}$$

(see Figure 1b).

3. SEPARATION INTERPRETATION OF STABILITY THEORY

To provide motivation and a conceptual framework for the development

that follows, it is instructive to digress at this point by explaining

how to give a simple geometric interpretation to the conic-relation

stability theorem (Zames [1], Theorem 2a) and to one of the principal

theorems of Lyapunov stability theory.[4]

Conic Relation Theorem

The conditions of the conic relation stability theorem involve

'conic' regions of the type (see Figure 2)

$$\text{Cone } (c,r) \triangleq \{(\underline{x},\underline{y}) \in X_e \times Y_e \big| ||(\underline{y}-c\underline{x})||_\tau \leq r||\underline{x}||_\tau$$
$$\text{for all } \tau \in T\} = \{(\underline{x},\underline{y}) \in X_e \times Y_e \big| \langle \underline{y}-(c+r)\underline{x}, \ y-(c-r)\underline{x} \rangle_\tau \leq 0 \text{ for all } \tau \in T\}$$
$$\tag{3.1}$$

where c and r are scalars called the cone <u>center</u> and cone <u>radius</u>,

respectively.[5] It is said that a relation $H \subset X_e \times Y_e$ is <u>inside</u>

Cone (c, r) if

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[4]The circle Popov, passivity, and small-gain stability theorems as well
as the sufficiency part of the Nyquist theorem follow as corollaries
to Zames' conic relation stability theorem; this is demonstrated in [1]-
[2]. These and still other results can be proved via the Lyapunov theory
(cf. [12]).

[5]The notation Cone (c, r) is non-standard; Zames [1] uses the notation
{c-r, c+r}.

$$\overline{H} \subset \text{Cone } (c, r); \tag{3.2}$$

$\underset{\sim}{\overline{H}}$ is <u>strictly inside</u> Cone (c, r) if for some $r' < r$,

$$\underset{\sim}{\overline{H}} \subset \text{Cone } (c, r') \subset \text{Cone } (c, r) \tag{3.3}$$

The notions of <u>outside</u> and <u>strictly outside</u> are defined analogously using in place of Cone (c, r) its complement. A property of conic regions such as (3.1) that is central to the geometric interpretation of Zames' conic-relation stability theorem is that the complement of such a conic region in $X_e \times Y_e$ corresponds to either a conic region (if $c \leq r$) or the complement of a conic region (if $c \geq r$) in $X_e \times Y_e$, as may be seen by comparison of Figure 2a with Figure 2b. For reasons that are not entirely apparent, the conditions of Zames' conic-relation theorem also require that U_e, V_e, X_e, and Y_e be identical extended inner product spaces and that the disturbances $\underline{u} \in X_e$ and $\underline{v} \in Y_e$ enter additively; i.e., if $(\underline{x}_1, \underline{y}_1) \in \overline{H}(\underline{v})$ and $(\underline{y}_2, \underline{x}_2) \in \overline{G}(u)$, then, respectively,

$$(\underline{x}_1 + \underline{v}, \underline{y}_1) \in \underset{\sim}{\overline{H}}(\underline{0}) \tag{3.4}$$

$$(\underline{y}_2 + \underline{u}, x_2) \in \underset{\sim}{\overline{G}}(0) \tag{3.5}$$

(see Figure 3). Subject to these restrictions on the class of systems considered, the conditions of Zames' conic relation theorem state quite simply that a sufficient condition for the feedback system (2.1) to be globally stable is that (for appropriately chosen center and radius parameters c and r), the relation $\overline{G}(0)$ be strictly inside Cone $(c, r) \subset Y_e \times X_e$

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and the relation $\overline{H}(0)$ be inside the region of $X_e \times Y_e$ corresponding to the complement of Cone (c, r) (see Figure 4). The interpretation of the conic relation stability theorem in terms of a <u>topological separation</u> is immediately evident: the interior of Cone (c, r) and the interior of the complement of Cone (c, r) form a <u>topological separation</u> [13] of the space $X_e \times Y_e$ [less the equilibrium point $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ and other points on the boundary of Cone (c, r)] into two disjoint regions, the closure of one region containing all non-zero pairs $(\underline{x}, \underline{y}) \in \overline{H}(0)$ and the interior of the other region containing all non-zero pairs $(\underline{x}, \underline{y})$ such that $(\underline{y}, \underline{x}) \in \overline{G}(0)$.

Lyapunov Stability Theorem

In the continuous-time state-space stability problems typically attacked by Lyapunov methods, the system under consideration is often given in the form (see Figure 5)

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}(t), t); \qquad \underline{x}(0) = \underline{x}_0 \tag{3.6}$$

where

$$t \in R_+ \overset{\Delta}{=} [0, \infty]$$

$$\underline{x}(t) \in R^n \text{ for all } t \in R_+$$

This system can be interpreted in terms of the system (2.3) as

$$
\left.
\begin{array}{l}
\underline{x} \equiv [\underline{x}(t)]_{t \in R_+} = \left(\underline{x}_0 + \int_0^t \underline{y}(\tau) d\tau \right)_{t \in R_+} = G(\underline{u})\underline{y} \\[2ex]
\underline{y} \equiv [\underline{y}(t)]_{t \in R_+} = \left(\underline{f}[\underline{x}(t), t] \right)_{t \in R_+} = H(\underline{v})\underline{x} \\[2ex]
\underline{v} = \underline{0} \in V = \{\underline{0}\} \\[2ex]
\underline{u} = \underline{x}_0 \in U = R^n
\end{array}
\right\} \tag{3.7}
$$

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where $T = R_+$, $\underline{x} \in X_e$ and $\underline{y} \in Y_e$ and where X_e and Y_e are the extended normed spaces

$$X_e = \{\underline{x}: \ R_+ \rightarrow R^n \mid \underline{x} \text{ is once differentiable}\} \tag{3.8}$$

$$Y_e = \{\underline{y}: \ R_+ \rightarrow R^n \ \} \tag{3.9}$$

induced by the identity operator

$$P_\tau \ \underline{x} \ = \ \underline{x} \text{ for all } \underline{x} \tag{3.10}$$

mapping X_e and Y_e into the normed spaces

$$X_\tau = X_e \tag{3.11}$$

with norm

$$||\underline{x}||_\tau = \sup_{t \in [0, \ \tau]} ||\underline{x}(t)||_{R^n} \tag{3.12}$$

and

$$Y_\tau = Y_e \tag{3.13}$$

with (degenerate) norm

$$||\underline{y}||_\tau \equiv 0. \tag{3.14}$$

One of the main theorems of Lyapunov stability ([8], Theorem 4.1) states that if in some neighborhood of the origin of the state space R^n there exists a positive-definite decrescent Lyapunov function $V(\underline{x}, \ t)$ such that its derivative $\nabla V(\underline{x}, \ t)\underline{f}(\underline{x}, \ t) + \frac{\partial V}{\partial t} (\underline{x}, \ t)$ is negative semi-definite in this neighborhood, then the solution $\underline{x}(t) \equiv \underline{0}$ is stable in the sense of Lyapunov; i.e., for every constant $\epsilon > 0$, there exists a constant $\delta(\epsilon) > 0$ such that $||\underline{x}_0||_{R^n} \le \delta(\epsilon)$ implies $||\underline{x}||_\tau \le \epsilon$ for all τ.

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The stipulation that $V(\underline{x}, t)$ be positive definite ensures that every pair $(\underline{x}, \underline{y})$ satisfying $\underline{x} = \mathcal{G}(\underline{0}) \cdot \underline{y}$ lies <u>inside</u> the subset of $X_e \times Y_e$,

$$\{(\underline{x}, \underline{y}) \mid \int_0^\tau [\nabla V(\underline{x}, t)]^T \underline{y}(t) + \frac{\partial V}{\partial t}(\underline{x}, t) dt$$

$$= V(x(\tau)) - V(x_0) = V(x(\tau)) \geq 0 \text{ for all } \tau \in T\}. \tag{3.15}$$

The condition (3.15) ensures that, in some neighborhood of the origin of $X_e \times Y_e$, every pair $(\underline{x}, \underline{y})$ satisfying $\underline{y} = \underset{\sim}{H}(\underline{0}) \cdot \underline{x}$ lies 'outside' the set (3.15), i.e., is contained in the set

$$\{(\underline{x}, \underline{y}) \mid \int_0^\tau \nabla V(\underline{x}(t)) \underline{y}(t) + \frac{\partial V}{\partial t}(\underline{x}(t), t) dt \leq 0 \text{ for all } \tau \in T\}. \tag{3.16}$$

The results of this paper show that it is more than just a coincidence that the conditions of such powerful stability results as the conic-relation stability theorem and the foregoing Lyapunov stability theorem correspond to the existence of a topological separation. The results show in essence that one can use any such partitioning of $X_e \times Y_e$ into two disjoint regions, provided that the 'distance' between the two regions is positive and increases as the 'distance' from the equilibrium increases. What constitutes suitable measure of 'distance' is the subject of our main results in §5.

4. <u>NOTATION AND TERMINOLOGY</u>

In this section some of the standard terminology from the stability literature (e.g., [1], [7]) is reviewed and, where necessary, <u>generalized</u> so as to be applicable to the broad class of stability problems admitted by our abstract problem formulation.

Relations (cf. [1])

A relation $\overline{R}$ is any set of the form $\overline{R} \subset X \times Y$; i.e., a relation is any subset of the Cartesian product of any two sets. A relation $\overline{R} \subset X \times Y$ can be represented equivalently as a mapping of subsets of X into subsets of Y and in this regard is merely a generalization of the notion of a function mapping X into Y. Some operations involving relations are defined below.

Image: The image $\overline{R}[A]$ of a set $A \subset X$ under a relation $\overline{R} \subset X \times Y$ is the subset of Y

$$\overline{R}[A] \triangleq \{\underline{y} \mid (\underline{x}, \underline{y}) \in \overline{R} \text{ for some } \underline{x} \in A\} . \tag{4.1}$$

For $\underline{x}_0 \in X$, we may denote $\overline{R}[\{\underline{x}_0\}]$ using the abbreviated notation $\overline{R} \cdot \underline{x}_0$ or $\overline{R}\underline{x}_0$.

Inverse: The inverse of a relation $\overline{R} \subset X \times Y$ is the relation $\overline{R}^{\,I} \subset Y \times X$

$$\overline{R}^{\,I} \triangleq \{(\underline{y}, \underline{x}) \in Y \times X \mid (\underline{x}, \underline{y}) \in \overline{R}\} ; \tag{4.2}$$

clearly, the inverse always exists.

Composition Product: The composition product of the relations $\overline{R}_1 \subset X \times Y$ followed by $\overline{R}_2 \subset Y \times Z$ is the relation $\overline{R}_2 \circ \overline{R}_1 \subset X \times Z$

$$\overline{R}_2 \circ \overline{R}_1 \triangleq \{(\underline{x}, \underline{z}) \in X \times Z \mid \text{ there exists } \underline{y} \in Y \text{ such that}$$

$$(\underline{x}, \underline{y}) \in \overline{R}_1 \text{ and } (y, z) \in \overline{R}_2 . \tag{4.3}$$

Sum: If $\overline{R}_1$, $\overline{R}_2 \subset X \times Y$ and if addition is defined on Y, then the sum of the relations $\overline{R}_1$ and $\overline{R}_2$ is the relation

$$\overline{R}_1 + \overline{R}_2 \triangleq \{(\underline{x}, \underline{y}) \in X \times Y \mid \underline{x} \in X \text{ and } \underline{y} = \underline{y}_1 + \underline{y}_2$$

for some $\underline{y}_1 \in \overline{R}_1 \underline{x}$ and $\underline{y}_2 \in \overline{R}_2 \underline{x} \}$. (4.4)

Graph: If G is a function mapping of points $\underline{x} \in X$ into points $G\underline{x} \in Y$, then the graph of G is the relation

$$\overline{G} \equiv \text{Graph } (G) \triangleq \{(\underline{x}, \underline{y}) \in X \times Y \mid$$

$$\underline{x} \in X \text{ and } \underline{y} = G \underline{x} \}$$

Stability Terminology

Class K (cf. [7, p. 7]):

A function ϕ mapping the non-negative real numbers R_+ into non-negative real numbers R_+ is defined to be in class K, denoted $\phi \in K$, if ϕ is continuous, strictly increasing and $\phi(0) = 0$.

Positive Definite; Decrescent; Radially Unbounded: Let X_e be an extended normed space and T be the associated interval; let S be a subset of X_e containing the point $\underline{x} = \underline{0}$; a functional $\eta: S \times T \to R$ is said to be positive definite on S if both

    i)   for some ϕ in class K, all $\underline{x} \in S$ and all $\tau \in T$

$$\eta(\underline{x}, \tau) \geq \phi \left(\|\underline{x}\|_\tau \right) \tag{4.6}$$

        and,

    ii)   for all $\tau \in T$

$$\eta(\underline{0}, \tau) = 0 \tag{4.7}$$

a functional $\eta: X_e \times T \to R$ is said to be decrescent on S if for some ϕ in class K and all $\tau \in T$

$$0 \leq \eta(\underline{x}, \tau) \leq \phi \left(\|\underline{x}\|_\tau \right). \tag{4.8}$$

A functional $\eta: X_e \times T \to R$ is said to be <u>radially unbounded on</u> S if

there exists a continuous non-decreasing function $\phi: R_+ \to R_+$ with

$\lim\limits_{\alpha \to \infty} \phi(\alpha) = \infty$ such that for all $\tau \in T$ and all $\underline{x} \in S$

$$\eta(\underline{x}, \tau) \geq \phi(||\underline{x}||_\tau). \tag{4.9}$$

<u>Neighborhood</u>: For any extended normed space X_e, any set $A \subset X_e$, and

any non-negative number α, the neighborhood $N(A; \alpha)$ is the set

$$N(A; \alpha) \triangleq \{x \in X_e | \text{ for some } \underline{x}_0 \in A,$$

$$||(\underline{x} - \underline{x}_0)|| \leq \alpha \text{ for all } \tau \in T\} \tag{4.10}$$

If A consists of a single point, say x_0, we may use the abbreviated

notation $N(\underline{x}_0; \alpha) \triangleq N(\{\underline{x}_0\}; \alpha)$. A set S is said to be a <u>neighborhood</u>

of a set A if for some $\varepsilon > 0$

$$N(A; \varepsilon) \subset S; \tag{4.11}$$

if $\underline{x}_0$ is a point, the set S is said to be a <u>neighborhood</u> of $\underline{x}_0$ if for

some $\varepsilon > 0$ $N(\underline{x}_0; \varepsilon) \subseteq S$.

$||(\underline{x}, \underline{y})||$: for any normed spaces X, Y and any $(\underline{x}, \underline{y}) \in X \times Y$, the notation

$||(\underline{x}, \underline{y})||$ is in this paper defined to mean:

$$||(\underline{x}, \underline{y})|| \triangleq (||\underline{x}||^2 + ||\underline{y}||^2)^{1/2}; \tag{4.12}$$

clearly, (2.10) defines a norm on $X \times Y$. So, for example, $X_e \times Y_e$ is

an extended normed space with associated norm $||(\underline{x}, \underline{y})||_\tau = (||\underline{x}||_\tau^2 + ||\underline{y}||_\tau^2)^{1/2}$.

<u>Gain</u>; <u>Incremental Gain</u>: Suppose X_e and Y_e are extended normed spaces;

let $\tilde{F} \subset X_e \times Y_e$. If for some scalar $k < \infty$ and for all $\underline{x} \in X_e$ and all $\tau \in T$,

$$\tilde{F}\,\underline{x} \subset N(\{\underline{0}\};\ k||\underline{x}||_\tau).\tag{4.13}$$

then $\tilde{F}$ has <u>finite gain</u>; the smallest k for which (2.11) is satisfied

is called the <u>gain</u> of $\tilde{F}$. If for some $k < \infty$, all $\underline{x}_1$, $\underline{x}_2 \in X_e$, and all $\tau \in T$,

$$\tilde{F}\,\underline{x}_2 \subset N(\tilde{F}\,\underline{x}_1;\ k||\underline{x}_1 - \underline{x}_2||_\tau),\tag{4.14}$$

then $\tilde{F}$ has <u>finite incremental gain</u>; the smallest such k is called the

<u>incremental gain</u> of $\tilde{F}$. A function $\tilde{F}: X_e \to Y_e$ is said to have finite

(incremental) gain if the relation Graph $(\tilde{F})$ has finite (incremental) gain.

<u>Bounded</u>; <u>Stable</u>; <u>Finite-Gain Stable</u>: Let X_e and Y_e be extended normed

spaces; let $\tilde{F} \subset X_e \times Y_e$; let $A \subset Y_e$. If there exist neighborhood continuous

of A, say S, and a non-decreasing function $\phi: R_+ \to R_+$ such that for all $\underline{x} \in S$

$$\tilde{F}\,\underline{x} \subset N[A;\ \phi(||\underline{x}||_\tau)]\tag{4.15}$$

then $\tilde{F}$ is <u>bounded in S about the set A</u>; if $\phi \in K$ then we say $\tilde{F}$ is <u>stable</u>

<u>about the set A</u>; if ϕ is linear (i.e., ϕ of the form $\phi(||\underline{x}||_\tau) = k||\underline{x}||_\tau$),

then we say that $\tilde{F}$ is finite-gain stable about the set A. If, in the

foregoing, the neighborhood S can be taken to be the entire space X_e,

then 'bounded in S' becomes simply <u>bounded</u>, 'stable' becomes <u>globally</u>

<u>stable</u> and finite-gain stable becomes <u>globally finite-gain stable.</u> When

$A = \{\underline{0}\}$, then we say simply that $\tilde{F}$ is <u>bounded in S</u>, or <u>globally stable</u>,

or <u>globally finite-gain stable</u>, respectively (i.e., we omit the phrase

'about the set A').

Comment:

It is <u>necessary</u> that stability be defined here, because there is no standard definition in the literature. The definition varies in subtle ways from author to author (cf. [1], [4], [14]), and even among papers with a common author (e.g., [1] and [14]). The motivation for the present choice of definition is two-fold. First, the definition is more flexible than previous definitions of stability in that

i) inputs need not enter additively, and

ii) by allowing discussion of stability about an arbitrary set, the definition permits one, in principle, to address certain special issues in stability theory, e.g., the stability of time-varying functions or sets of time-varying functions such as the limit cycles of autonomous systems. Second, the definition meshes well with the classical notion of stability in the sense of Lyapunov (cf. [8]), coinciding when the magnitude of the system input is taken to be the Euclidean norm of its initial state and the state trajectory is presumed to lie in the extended normed space X_e defined in (3.8) - (3.11).

Comment:

It is noteworthy that in the case of linear systems, the definitions of bounded, stable, and finite-gain stable coincide: for such systems ϕ can always be taken to be linear: e.g., in (2.13), pick any $\underline{x}_0$ with $||\underline{x}_0|| \neq 0$ and replace $\phi(||\underline{x}||_\tau)$ by $\phi'(||\underline{x}||_\tau) = [||\underline{x}||_\tau \div \phi(||\underline{x}_0||_\tau)]$ (cf. Theorem 5.4 in [15]). Consequently, when speaking of <u>linear</u> systems, the terms <u>bounded</u>, <u>stable</u>, and <u>finite-gain stable</u> may be used interchangeably.

5. FUNDAMENTAL STABILITY THEOREM

An abstract result which provides an aggregate characterization of the set feedback system outputs achievable with a specific system input is now stated. We refer to this result as our <u>fundamental stability theorem</u> because stability tests of both the input-output type and the Lyapunov state-space type can be derived from this result. The stability implications the result are found to have a simple interpretation in terms of 'topological separation' of the product space $X_e \times Y_e$ on which the systems dynamical relations are defined.

<u>Theorem 5.1 (Fundamental Stability Theorem)</u>: Let S be a subset of $X_e \times Y_e$. Suppose that real-valued functionals $d(\underline{x}, \underline{y}, \tau)$, $\eta_1(\underline{x}, \underline{y}, \tau)$, $\eta_2(u, \tau)$, $\eta_3(\underline{v}, \tau)$ and $\eta_4(\underline{x}, \underline{y}, \tau)$ can be found such that for each $\tau \in T$

$$\overline{G}(\underline{u}) \cap S \subset \{(\underline{x}, \underline{y}) \mid d(\underline{x}, \underline{y}, \tau)$$
$$\geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau)\} \tag{5.1}$$

$$\overline{H}(\underline{v}) \cap S \subset \{(\underline{y}, \underline{x}) \mid d(\underline{x}, \underline{y}, \tau)$$
$$\leq \eta_3(\underline{v}, \tau) - \eta_4(\underline{x}, \underline{y}, \tau)\}. \tag{5.2}$$

Then every solution $(\underline{x}, \underline{y})$ of (2.1) in S satisfies the inequality

$$\eta_{out}(\underline{x}, \underline{y}, \tau) \leq \eta_{in}(\underline{u}, \underline{v}, \tau) \tag{5.3}$$

for all $\tau \in T$, where

$$\eta_{out} \triangleq \eta_1 + \eta_4 \tag{5.4}$$

$$\eta_{in} \triangleq \eta_2 + \eta_3 \tag{5.5}$$

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<u>Proof</u>: Suppose $(\underline{x}, \underline{y})$ is a solution of (2.1) and that $(\underline{x}, \underline{y}) \in S$. Then,

$$(\underline{x}, \underline{y}) \in \overline{G}(\underline{u}) \cap \overline{H}^I(\underline{v}) \cap S. \tag{5.6}$$

By (5.1) - (5.2) if follows that for all $\tau \in T$

$$d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau) \tag{5.7}$$

$$d(\underline{x}, \underline{y}, \tau) \leq \eta_3(\underline{v}, \tau) - \eta_4(\underline{x}, \underline{y}, \tau) \tag{5.8}$$

Substracting (5.7) from (5.6) and adding η_{in} to both sides yields

$$\begin{aligned}
\eta_{in}(\underline{u}, \underline{v}, \tau) \geq\ & (\eta_1(\underline{x}, \underline{y}, \tau) + \eta_4(\underline{x}, \underline{y}, \tau) \\
& - \eta_2(\underline{u}, \tau) - \eta_3(\underline{v}, \tau) + \eta_{in}(\underline{u}, \underline{v}, \tau)) \\
=\ & \eta_{out}(\underline{x}, \underline{y}, \tau)
\end{aligned} \tag{5.9}$$

which proves Theorem 5.1 ☐

The importance of Theorem 5.1 is that it provides an aggregate characterization of the set of output pairs $(\underline{x}, \underline{y})$ in S that are achievable by input pairs $(\underline{u}, \underline{v})$ in sets of the form $\{(\underline{u}, \underline{v}) \mid \eta_{in}(\underline{u}, \underline{v}, \tau) \leq$ constant$\}$. By imposing the additional restriction that the functional η_{out} is positive definite on S, this inequality may be used to establish the stability properties of the system (2.1) by establishing a $(\underline{u}, \underline{v})$-dependent bound on $||(\underline{x}, \underline{y})||_\tau$. When, additionally, $S = X_e \times Y_e$ then the stability properties thus determined are <u>global</u>. The following corollary to Theorem 5.1 illustrates this.

<u>Corollary 5.1 (Boundedness & Global Stability)</u>:

a) (Boundedness) If in Theorem 5.1

    i) $S = X_e \times Y_e$,

    ii) η_{out} is positive definite and radially unbounded, and

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iii) η_{in} is bounded,

then the system (2.1) is bounded.

    b) (Global Stability)   If in Theorem 5.1

        i) $S = X_e \times Y_e$,

        ii) η_{out} is positive definite and radially unbounded, and

        iii) η_{in} is decrescent,

then the system (2.1) is globally stable.

    c) (Global Finite-Gain Stability) If in Theorem 5.1

        i) $S = X_e \times Y_e$

        ii) for some strictly positive constants ε_1, ε_2, and α and

            for all $(\underline{x}, \underline{y})$ $X_e \times Y_e$

$$\left.\begin{array}{l} \eta_{out}(\underline{x}, \underline{y}, \tau) \geq \varepsilon_1 (||(\underline{x}, \underline{y})||_\tau)^\alpha \\[2mm] \eta_{in}(\underline{u}, \underline{v}, \tau) \leq \varepsilon_2 (||(\underline{u}, \underline{v})||_\tau)^\alpha , \end{array}\right\} \tag{5.10}$$

then the null solution of (2.1) is globally finite-gain stable.

<u>Proof</u>: We prove each of the results (a) - (c) in sequence.

    a) Since η_{out} is positive-definite radially-unbounded and since η_{in} is bounded, there exist continuous non-decreasing functions $\phi_{out} : R_+ \to R_+$ and $\phi_{in} : R_+ \to R_+$ with $\phi_{out}(0) = 0$ and ϕ_{out} strictly increasing and $\lim_{\alpha \to \infty} \phi_{out}(\alpha) = \infty$ such that

$$\phi_{out}(||(\underline{x}, \underline{y})||_\tau) \leq \eta_{out}(\underline{x}, \underline{y}, \tau) \tag{5.11}$$

and

$$\phi_{in}(\underline{u}, \underline{v}, \tau) \leq \eta_{in}(||(\underline{u}, \underline{v},)||_\tau) . \tag{5.12}$$

Evidently ϕ_{out}^{-1} exists and is decrescent, so

$$||(\underline{x}, \underline{y})||_\tau \leq \phi(||(\underline{u}, \underline{v})||_\tau) \tag{5.13}$$

where $\phi = \phi_{out}^{-1} \circ \phi_{in}$. Clearly ϕ is continuous and non-decreasing, so the system (2.1) is bounded.

b) Taking ϕ_{out}, ϕ_{in} and ϕ as in (a), it follows that

$$||(x, y)||_\tau \leq \phi(||(\underline{u}, \underline{v})||_\tau) \tag{5.14}$$

Since η_{out} is positive definite and radially unbounded and since η_{in} is decrescent, it follows that ϕ_{out}^{-1} and ϕ_{in} are decrescent and hence $\phi = \phi_{out}^{-1} \circ \phi_{in}$ is likewise decrescent. It follows from (5.14) that the null solution of (2.1) is globally stable.

c) From (5.10) and the inequality (5.3), it follows that

$$\varepsilon_1(||(\underline{x}, \underline{y})||_\tau)^\alpha \leq \eta_{out}(\underline{x}, \underline{y}, \tau) \leq \eta_{in}(\underline{u}, \underline{v}, \tau)$$
$$\leq \varepsilon_2(||(\underline{u}, \underline{v})||_\tau)^\alpha \tag{5.15}$$

and hence

$$||(\underline{x}, \underline{y})||_\tau \leq \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1/\alpha} ||(\underline{u}, \underline{v})||_\tau . \tag{5.16}$$

It follows that the null solution of (2.1) is globally finite-gain stable. □

Comments:

The stability conditions of Corollary 5.1 may be interpreted and conceptually motivated in terms of a 'topological separation'.

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For simplicity we consider only the case of global stability (part (b) of Corollary 5.1) -- a similar interpretation is possible for the other parts of Corollary 5.1. We further assume for simplicity that $\eta_4 \equiv 0$ so that $\eta_{out} = \eta_1$ -- this entails no loss of generality since every case may be reduced to this by substituting $d + \eta_4$ for d. For each $\tau \in T$, the functional $d(\underline{x}, \underline{y}, \tau)$ serves to 'topologically separate' the space $X_e \times Y_e$ into two disjoint regions, viz. the region where $d(\underline{x}, \underline{y}, \tau) > 0$ and the region where $d(\underline{x}, \underline{y}, \tau) \leq 0$, the set $\{(\underline{x}, \underline{y}) \mid d(\underline{x}, \underline{y}, \tau) = 0\}$ forming the boundary. The positive-definiteness of η_{out} ensures the undisturbed relation $\overline{H}(\underline{0})$ lies entirely in the region where $d(\underline{x}, \underline{y}, \tau) \leq 0$ and that every non-zero point of the undisturbed relation $\overline{G}^I(\underline{0})$ lies in the complementary region where $d(\underline{x}, \underline{y}, \tau) > 0$. Consequently, the null solution $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ is the unique solution of the undisturbed system (2.1) -- this is prerequisite to global stability. We can vizualize $|d(\underline{x}, \underline{y}, \tau)|$ as defining the τ-dependent distance of each point $(\underline{x}, \underline{y}) \in X_e \times Y_e$ from the boundary set $\{(\underline{x}, \underline{y}) \mid d(\underline{x}, \underline{y}, \tau)\} = 0$, the sign of $d(\underline{x}, \underline{y}, \tau)$ determining on which side of the boundary the point lies. The positive definiteness and radial unboundedness of η_{out} ensures for every $(\underline{x}, \underline{y})$ $(\underline{x}, \underline{y}) \in \overline{G}^I(\underline{0})$ that this distance is positive and grows unboundedly as $||(\underline{x}, \underline{y})||_\tau$ increases. In this conceptual framework, the quantity $\eta_2(\underline{u}, \tau)$ is simply an upper bound on the "distance" that $\overline{G}^I(\underline{u})$ shifts toward the boundary as a consequence of the disturbance, $\underline{u}$. Similarly, the 'distance' of $\overline{H}(\underline{0})$ from the boundary is non-positive and $\eta_3(\underline{v}, \tau)$ is an upper bound on the distance that $\overline{H}(\underline{v})$ shifts toward the boundary as

a consequence of $\underline{v}$. Because solutions of (1.1) must lie in the set $\overline{G}^I(\underline{u}) \cap \overline{H}(\underline{v})$, we see that the "distance" of $\eta(\underline{x}, \underline{y}, \tau)$ must be less than the sum $\eta_2(\underline{u}, \tau) + \eta_3(\underline{v}, \tau)$, i.e.,

$$\eta_{out}(\underline{x}, \underline{y}, \tau) \leq \eta_2(\underline{u}, \tau) + \eta_3(\underline{v}, \tau) \triangleq \eta_{in}(\underline{u}, \underline{v}, \tau) \qquad (5.17)$$

Because the "distance" $\eta_{out}(\underline{x}, \underline{y}, \tau)$ increases unboundedly as $||(\underline{x}, \underline{y})||_\tau$ increases, it is evident from (3.4) that $||(\underline{x}, \underline{y})||_\tau$ is bounded; this is illustrated in Figure 6. The fact that η_{in} and, hence, η_2 and η_3 are decrescent ensures that vanishly small $\underline{u}$ and $\underline{v}$ will produce vanishly small shifts $\phi_1(\underline{u}, \tau)$ and $\phi_2(\underline{v}, \tau)$ in the respective relations $\overline{G}^I(\underline{u})$ and $\overline{H}(\underline{v})$. This establishes the global stability of the system (5.1).

In view of the foregoing, one may loosely interpret Corollary 5.1 as saying that stability can be assured if one can find some real-valued functional (viz. $d(\underline{x}, \underline{y}, \tau)$) which separates the set $\overline{G}^I(\underline{0}) \cup \overline{H}(\underline{0})$ less the pair $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ into the component parts $\overline{G}^I(\underline{0}) - \{(\underline{0}, \underline{0})\}$ and $\overline{H}(\underline{0})$. The conditions that η_{in} be decrescent and that η_{out} be positive definite may be viewed as technical conditions that are imposed to rule-out 'peculiar' situations in which either the amount of separation fails to grow with distance from the origin in the $X_e \times Y_e$ 'plane' or in which the system is ill-posed in the sense that small disturbances $(\underline{u}, \underline{v})$ produce disproportiately large changes in the input-output relations $G(\underline{u})$ and $\overline{H}(\underline{v})$.

It is noteworthy that Theorem 5.1 and Corollary 5.1 make no reference to loop transformations, multipliers, contraction mappings, or any of the

other mathematical paraphernalia usually associated with input-output

stability results (cf. [3] - [6]). This is a consequence of the fact

that, in contrast to most previous input-output stability criteria, no

fixed-point theorems (e.g., the contraction mapping theorem) are used in

the proof of Corollary 5.1 and Theorem 5.1. This underscores the fact

that <u>existence</u> of solutions -- and existence is always assured when fixed-

point theorems are employed -- is not central to the issue of stability.

Rather, in stability analysis, we are concerned primarily with ascertaining

that all existing solutions are stable. Existence of solutions, which

relates to the 'well-posedness' of the system equations, can be deduced

from entirely separate considerations [4, pp. 93-101].[5]

The Lyapunov stability result discussed in §3 is a special case

of Corollary 5.1 in which the separating functional $d(\underline{x}, \underline{y}, \tau)$ is taken

to be

$$d(\underline{x}, \underline{y}, \tau) = \int_0^\tau \nabla V(\underline{x}(t), t)y(t) + \frac{\partial V}{\partial t}(x(t), t)dt$$

where $V(\underline{x}, \tau)$ is the 'Lyapunov' function. With this choice of $d(\underline{x}, \underline{y}, \tau)$

and with $\underset{\sim}{G}(\underline{u})$ and $\underset{\sim}{H}(\underline{v})$ as in (3.7), it follows

i) $d(\underline{x}, \underline{y}, \tau) = \int_0^\tau \nabla V(x(t)\dot{x}(t) + \frac{\partial V}{\partial t}(x(t), t)dt$

$= V(\underline{x}(\tau), \tau) - V(\underline{x}(0), 0)$

for all $(\underline{x}, \underline{y})$ satisfying

$\underline{x} = \underset{\sim}{G}(\underline{x}_0) \cdot \underline{y} = \int_0^\tau y(t)dt + \underline{x}_0$

---

[5]Well-posedness tests, based on considerations other than stability, are
provided in [4]. However, it should be noted that (in contrast to the view
taken here and elsewhere in the literature, cf. [1], [6], [14]), reference
[4] <u>defines</u> well-posedness to be prerequisite to any discussion of
stability or instability.

ii) $d(\underline{x}, \underline{y}, \tau) = \int_0^\tau \nabla V(\underline{x}(t)) f(\underline{x}(t), t) + \frac{\partial V}{\partial t}(\underline{x}(t), t) dt$

for all $(\underline{x}, \underline{y})$ satisfying

$\underline{y}(t) = f(\underline{x}(t), t)$.

Evidently when V is positive definite and decrescent and when $V(\underline{x}, t) f(x, t) + \frac{\partial V}{\partial t}(\underline{x}, t) \leq 0$ for all $\underline{x}$ and t, then the conditions of Corollary 5.1 are satisfied with

$$\eta_1(\underline{x}, \tau) = V(\underline{x}(\tau), \tau)$$

and with

$$\eta_2(\underline{x}_0) = V(\underline{x}_0, 0)$$

$$\eta_3 = \eta_4 \equiv 0.$$

This establishes that Lyapunov stability results can be treated as corollaries to Theorem 5.1.

Not surprisingly, Zames' powerful conic relation stability theorem can also be shown to be a corollary to Theorem 5.1. To prove this and to demonstrate the power of our results, a generalization of Zames' conic relation theorem is now developed.

6. THE SECTOR STABILITY CRITERION

The conic relation stability theorem of Zames is generalized in this section to permit the utilization of the more flexible definition of sector which follows.

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<u>Definition (Sector)</u>: Let X_e and Y_e be extended normed spaces and let Z_e be an extended inner product space. For each $\tau \in T$ let

$$\underset{\sim}{F}(\underline{x}, \underline{y}, \tau) \triangleq \langle \underset{\sim 11}{F}\underline{y} + \underset{\sim 12}{F}\underline{x},\ \underset{\sim 21}{F}\underline{y} + \underset{\sim 22}{F}\underline{x}\rangle_\tau \tag{6.1}$$

where $\underset{\sim ij}{F}\underline{0} = \underline{0}$ (i, j = 1, 2) and $\underset{\sim 11}{F}$, $\underset{\sim 21}{F}$: $Y_e \to Z_e$ and $\underset{\sim 12}{F}$, $\underset{\sim 22}{F}$: $X_e \to Y_e$. Then the <u>sector</u> of $\underset{\sim}{F}$ is defined to be

$$\text{Sector } (\underset{\sim}{F}) \triangleq \{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid \text{for all } \tau \in T$$

$$\underset{\sim}{F}(\underline{x}, \underline{y}, \tau) \leq 0\}. \tag{6.2}$$

For notational convenience, the functional $\underset{\sim}{F}$ will be denoted equivalently by the 2×2 array

$$\underset{\sim}{F} \triangleq \begin{bmatrix} \underset{\sim 11}{F} & \underset{\sim 12}{F} \\ \underset{\sim 21}{F} & \underset{\sim 22}{F} \end{bmatrix} \tag{6.3}$$

<u>Definition (inside, outside, strictly inside, strictly outside)</u>: A subset $\underset{\sim}{\overline{A}}$ of $X_e \times Y_e$ is said to be <u>inside</u> Sector $(\underset{\sim}{F})$ if $\underset{\sim}{\overline{A}} \subset$ Sector $(\underset{\sim}{F}) \triangleq$ $\{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid \text{for all } \tau \in T\ \ \underset{\sim}{F}(\underline{x}, \underline{y}, \tau) \leq 0\}$; $\overline{A}$ is said to be <u>strictly inside</u> Sector $(\underset{\sim}{F})$ if for some $\varepsilon > 0$, $\underset{\sim}{\overline{A}} \subset \{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid \text{for all } \tau \in T$ $\underset{\sim}{F}(\underline{x}, \underline{y}, \tau) \leq -\varepsilon ||(\underline{x}, \underline{y})||_\tau^2\}$; $\overline{A}$ is said to be <u>outside</u> Sector $(\underset{\sim}{F})$ if $\underset{\sim}{\overline{A}} \subset \{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid \text{for all } \tau \in T\ \ \underset{\sim}{F}(\underline{x}, \underline{y}, \tau) \geq 0\}$; and, $\overline{A}$ is said to be <u>strictly outside</u> Sector $(\underset{\sim}{F})$ if for some $\varepsilon > 0$, $\underset{\sim}{\overline{A}} \subset \{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid$ for all $\tau \in T$, $\underset{\sim}{F}(\underline{x}, \underline{y}, \tau) \geq \varepsilon ||(\underline{x}, \underline{y})||_\tau^2\}$.

We now state a stability result based on Corollary 5.1 that

employs sectors to accomplish the requisite 'topological separation'

of the space $X_e \times Y_e$. The proof, which involves a straightforward --

but tedious -- verification of the conditions of Corollary 5.1, is

in Appendix A.

<u>Theorem 6.1 (Sector Stability Criterion)</u>: Let $\underset{\sim}{F}$ be a 2×2 array as

in (6.3); let the $\underset{\sim ij}{F}$ (i, j = 1, 2) have finite incremental gain; let

the mappings of $\underline{u} \in U$ into $\overline{\underset{\sim}{G}}(\underline{u})$ and $\underline{v} \in V$ into $\overline{\underset{\sim}{H}}(\underline{v})$ be bounded (respectively,

globally stable; respectively globally finite-gain stable) about the

respective sets $\overline{\underset{\sim}{G}}(\underline{0})$ and $\overline{\underset{\sim}{H}}(\underline{0})$. If $\overline{\underset{\sim}{G}}^I(\underline{0})$ is <u>strictly inside</u> Sector $(\underset{\sim}{F})$

and if $\overline{\underset{\sim}{H}}(\underline{0})$ is <u>outside</u> Sector $(\underset{\sim}{F})$, then system (1.1) is bounded

(respectively, globally stable; respectively, globally finite-gain stable).

<u>Proof</u>: See Appendix A.

<u>Comments</u>:

The requirement in Theorem 6.1 that the functions mapping of $\underline{u} \in U$

into $\overline{\underset{\sim}{G}}(\underline{u})$ and $\underline{v} \in V$ into $\overline{\underset{\sim}{H}}(\underline{v})$ be bounded about the sets $\overline{\underset{\sim}{G}}(\underline{0})$ and $\overline{\underset{\sim}{H}}(\underline{0})$

should not be confused with the more restrictive requirement that the

subsystems $\overline{\underset{\sim}{G}}$ and $\overline{\underset{\sim}{H}}$ be 'open-loop bounded', i.e., that the mappings of $(\underline{u}, \underline{y})$

into $\overline{\underset{\sim}{G}}(\underline{u})\underline{y}$ and $(\underline{v}, \underline{x})$ into $\overline{\underset{\sim}{H}}(\underline{v})x$ be bounded about $\{(0, 0)\}$. For example,

if the disturbances $\underline{u}$ and $\underline{v}$ enter additively as in Figure 3 -- and this

is the only case considered in the majority of the input-output stability

literature -- then the boundedness requirement placed on the mappings

$\overline{G}(\cdot)$ and $\overline{H}(\cdot)$ in Theorem 6.1 is automatically satisfied (with finite gain!).

Thus, <u>the boundedness restriction on the mappings</u> $\overline{G}(\cdot)$ <u>and</u> $\overline{H}(\cdot)$ <u>is actually</u> <u>very mild</u>; it can be viewed as a sort of well-posedness condition on the feedback equations, ensuring that small disturbances do not produce unboundedly large dislocations of the dynamical relations in the $X_e \times Y_e$-'plane' -- cf. [4, p. 90] condition WP.4.

Comparison of the definition of Sector $(\overline{F})$ with Zames' conic sector (3.1) shows that

$$\text{Cone } (c, r) \quad = \quad \text{Sector } \left(\begin{bmatrix} 1 & -(c + r) \\ 1 & -(c - r) \end{bmatrix} \right) \quad .$$

Zames' conic relation stability criterion is a special case of Theorem 6.1 that results when sectors of this form are employed and the class of systems considered is restricted to the additive-input type depicted in Figure 3.

Some Properties of Sectors:

Zames' [1, App. A] demonstrates that his conic sectors have several properties which make them especially well-suited to feedback system stability analysis. Our more general sectors have similar properties, some of which are enumerated in the following lemma.

Lemma 6.2 (Sector Properties):

Let $F_{\sim ij}$ and $F_{\sim ij}^{(k)}$ be operators mapping into extended inner product spaces Z_e and $Z_e^{(k)}$ respectively; let $F_{\sim ij}\underline{0} = \underline{0}$ and $F_{\sim ij}^{(k)}\underline{0} = \underline{0}$; let $\overline{A}_\sim$, $\overline{B}_\sim$ and $\overline{A}_\sim^{(k)}$ be relations on extended normed spaces; let $(\underline{0}, \underline{0}) \in \overline{A}_\sim$,

$\overline{B}$, $\underset{\sim}{A}^{(k)}$; let a and b be scalers with ab > 0; let $\underset{\sim}{M}$ and $\underset{\sim}{M}^*$ be operators with the property that $<\underset{\sim}{M}\underline{z}_1,\ \underline{z}_2>_\tau\ =\ <\underline{z}_1,\ \underset{\sim}{M}^*\underline{z}_2>_\tau$ for all $\underline{z}_1,\ \underline{z}_2 \in Z_e$ and all $\tau \in T$. Then the following properties hold:

i) (Complimentary Sector)

$\overline{\underset{\sim}{A}}$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right)$

$\Longleftrightarrow$

$\overline{\underset{\sim}{A}}$ <u>outside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ -\underset{\sim}{F}_{21} & -\underset{\sim}{F}_{22} \end{bmatrix}\right)$; (6.4)

furthermore, (6.4) holds with <u>inside</u> and <u>outside</u> replaced respectively by <u>strictly inside</u> and <u>strictly outside</u>.

ii) (Multiplier)

$$\text{Sector}\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{M}\cdot\underset{\sim}{F}_{21} & \underset{\sim}{M}\cdot\underset{\sim}{F}_{22} \end{bmatrix}\right) = \text{Sector}\left(\begin{bmatrix} a\ \underset{\sim}{M}^*\cdot\underset{\sim}{F}_{11} & a\ \underset{\sim}{M}^*\cdot\underset{\sim}{F}_{12} \\ b\ \underset{\sim}{F}_{21} & b\ \underset{\sim}{F}_{22} \end{bmatrix}\right)$$ (6.5)

iii) (inverse Relation)

$\overline{\underset{\sim}{A}}$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right)$

$\Longleftrightarrow$

$\overline{\underset{\sim}{A}}^I$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{12} & \underset{\sim}{F}_{11} \\ \underset{\sim}{F}_{22} & \underset{\sim}{F}_{21} \end{bmatrix}\right)$. (6.6)

Furthermore, (6.6) holds with <u>inside</u> replaced by <u>strictly inside</u> throughout.

iv) (Sums of Relations) If $\overline{\underset{\sim}{B}}$ = Graph $(\underset{\sim}{B})$ and if $\underset{\sim}{F}_{11}$ and $\underset{\sim}{F}_{21}$ are linear, then

$\overline{A}$ inside Sector
$$\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right)$$

$\Longrightarrow$

$(\overline{A} - \overline{B})$ inside Sector
$$\left(\begin{bmatrix} F_{11} & (F_{11} \cdot B + F_{12}) \\ F_{21} & (F_{21} \cdot B + F_{22}) \end{bmatrix} \right) \qquad (6.7)$$

If $(\overline{A} - \overline{B})$ has finite gain, then (6.7) holds with <u>inside</u> replaced by <u>strictly inside</u> throughout.

v) (Composition Products of Relations)

    a) If $\overline{B}$ = Graph (B), then

      $\overline{B} \circ \overline{A}$ inside Sector
$$\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right)$$

      $\Longrightarrow$

      $\overline{A}$ inside Sector
$$\left(\begin{bmatrix} F_{21} \cdot B & F_{12} \\ F_{21} \cdot B & F_{22} \end{bmatrix} \right) \qquad (6.8)$$

    b) If $\overline{A}$ = Graph (A) and if A^{-1} exists, then

      $\overline{B} \circ \overline{A}$ inside Sector
$$\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right)$$

      $\Longrightarrow$

      $\overline{B}$ inside Sector
$$\left(\begin{bmatrix} F_{11} & F_{12} \cdot A^{-1} \\ F_{21} & F_{22} \cdot A^{-1} \end{bmatrix} \right) \qquad (6.9)$$

    c) If $\overline{A}$ = Graph (A), then

      $\overline{B}$ inside Sector
$$\left(\begin{bmatrix} F_{11} \cdot A & F_{12} \\ F_{21} \cdot A & F_{22} \end{bmatrix} \right)$$

      $\Longrightarrow$

      $\overline{A} \circ \overline{B}$ inside Sector
$$\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right) \qquad (6.10)$$

Furthermore, if $\underset{\sim}{\bar{A}}$ has finite gain, then (6.8) - (6.10) hold with <u>inside</u>

replaced by <u>strictly inside</u> throughout.

     vi)   (Composites of Relations)

Suppose $\underset{\sim}{\bar{A}} = \{((\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}), (\underline{y}^{(1)}, \ldots, \underline{y}^{(n)}) \mid (\underline{x}^{(k)}, \underline{y}^{(k)}) \in \underset{\sim}{\bar{A}}^{(k)}$

for all $k = 1, \ldots, n\}$; suppose $\underset{\sim}{F}_{ij}(\underline{\xi}^{(1)}, \ldots, \underline{\xi}^n) = (\underset{\sim}{F}_{ij}^{(1)}\underline{\xi}^{(1)}, \ldots, \underset{\sim}{F}_{ij}\underline{\xi}^{(n)})$

for $(i, j = 1, 2)$; and suppose that $Z_e = Z_e^{(1)} \times \ldots \times Z_e^{(n)}$ and that

$$<(\underline{z}_1^{(1)}, \ldots, \underline{z}_1^{(n)})>_\tau = \sum_{k=1}^{n} \underline{z}_1^{(k)}, \underline{z}_2^{(k)} >_\tau$$

Then,

    $\underset{\sim}{\bar{A}}$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix} \right)$

$\Longleftrightarrow$

    $\underset{\sim}{\bar{A}}^{(k)}$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11}^{(k)} & \underset{\sim}{F}_{12}^{(k)} \\ \underset{\sim}{F}_{21}^{(k)} & \underset{\sim}{F}_{22}^{(k)} \end{bmatrix} \right)$ for all $k = 1, \ldots, n$.

(6.11)

Furthermore, (6.11) holds if <u>inside</u> is replaced by <u>strictly inside</u>

throughout.

    vii) Properties (i) - (iv) hold if throughout <u>inside</u> is replaced by

<u>outside</u> and <u>strictly inside</u> is replaced by <u>strictly outside</u>.

<u>Proof</u>: See Appendix B.

<u>Comment</u>:

    Properties (i) and (ii) of Lemma 6.2 provide a parameterization

of the various representations of a sector. Property (iii) establishes

the relation between sectors containing the inverse relation $\underset{\sim}{\bar{A}}^I$. Properties

(iv) - (vi) permit the characterization of a sector containing the

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relation of a complicated multivariable system using knowledge of
sectors associated with subsystems comprising the system and knowledge
of the interconnection structure of the system -- these properties
have application to the analysis of the stability of multivarible systems.

7. DISCUSSION

Our <u>fundamental stability theorem</u> (Theorem 5.1) is of significance
largely because it permits a clear understanding of the basic issues
involved in stability analysis; this is enabled by the relatively high
level of abstraction in our formulation of the stability problem. In
particular, Corollary 5.1 makes it clear that if one can find a 'topological
separation' of the product space $X_e \times Y_e$ such that the undisturbed
relation $\overline{G}^I(\underline{0})$ lies in one component of the separated space and the
undisturbed relation $\overline{H}(\underline{0})$ lies 'strictly' in the other component, then
under mild conditions closed-loop stability can be concluded. The
conceptual simplicity of this abstract view of stability theory may
prove to be of considerable pedagogical value, since it is possible
to relate the conditions of the various input-output and Lyapunov
stability criteria to the simple geometric notion of topological separation.
Also, the conceptual simplicity of the topological separation viewpoint
serves to elucidate the general approach that must be taken to generate
new results such as the sector stability criterion (Theorem 6.1). Our
<u>sector stability criterion</u> leads fairly directly to powerful new
<u>multivariable</u> generalizations of the circle stability criterion
for nonlinear systems, as is demonstrated in [9].

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One of the implications of using sets (such as the components
of a topological separation) to bound the dynamical relations of a
system is the possibility of determining the qualitative behavior of
a system -- and even bounding its quantitative behavior -- using only
a coarse knowledge of the system. The stability margins (e.g., gain
margin and phase margin) of a simplified model of a feedback system can
be thusly characterized, providing a measure of robustness against the
destabilizing effects of modeling errors -- this is demonstrated in
[9]. Also, bounds on a poorly defined or imprecisely modelled system's
transient response can be found using the inequality (5.3), allowing
one to deduce the relative degree of exponential stability or instability.

Appendix A: Proof of 6.1

We apply Corollary 5.1, taking $d(\underline{x}, \underline{y}, \tau)$ to be the map

$$d(\underline{x}, \underline{y}, \tau) = \begin{cases} 0 & , \text{ if } ||(\underline{x},\underline{y})||_\tau = 0 \\ \dfrac{-1}{||(\underline{x}, \underline{y})||_\tau} \underset{\sim}{F}(\underline{x}, \underline{y}, \tau) & , \text{ if } ||(\underline{x},\underline{y})||_\tau \neq 0 \end{cases} \qquad (A1)$$

We begin by establishing (5.1). Let $\underline{u} \in \mathcal{U}$ be fixed and take $(\underline{x}, \underline{y}) \in \overline{G}^I(\underline{u})$. Applying the Schwartz inequality, we have that for some $\varepsilon > 0$, for every $(\underline{x}_0, \underline{y}_0) \in \overline{G}^I(\underline{0})$, and every $\tau \in T$

$$||(\underline{x}, \underline{y})||_\tau \cdot d(\underline{x}, \underline{y}, \tau) = -\langle (\underset{\sim}{F}_{11}\underline{y} + \underset{\sim}{F}_{12}\underline{x}), (\underset{\sim}{F}_{21}\underline{y} + \underset{\sim}{F}_{22}\underline{x}) \rangle_\tau$$

$$= ||(\underline{x}_0, \underline{y}_0)||_\tau \cdot d(\underline{x}_0, \underline{y}_0, \tau)$$

$$- \langle (\underset{\sim}{F}_{11}\underline{y}_0 + \underset{\sim}{F}_{12}\underline{x}_0), [(\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{21}\underline{y}_0) + (\underset{\sim}{F}_{22}\underline{x} - \underset{\sim}{F}_{22}\underline{x}_0)] \rangle_\tau$$

$$- \langle [(\underset{\sim}{F}_{11}\underline{y} - \underset{\sim}{F}_{11}\underline{y}_0) + (\underset{\sim}{F}_{12}\underline{x} - \underset{\sim}{F}_{12}\underline{x}_0)], (\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{22}\underline{x}) \rangle_\tau$$

$$\geq \varepsilon ||(\underline{x}_0, \underline{y}_0)||_\tau^2$$

$$- ||(\underset{\sim}{F}_{11}\underline{y}_0 + \underset{\sim}{F}_{12}\underline{x}_0)||_\tau \cdot || [\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{21}\underline{y}_0)$$
$$+ (\underset{\sim}{F}_{22}\underline{x} - \underset{\sim}{F}_{22}\underline{x}_0)] ||_\tau$$

$$- || [(\underset{\sim}{F}_{11}\underline{y} - \underset{\sim}{F}_{11}\underline{y}_0) + (\underset{\sim}{F}_{12}\underline{x} - \underset{\sim}{F}_{12}\underline{x}_0)] ||_\tau$$
$$\cdot ||(\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{22}\underline{x})||_\tau$$

$$\geq \varepsilon ||(\underline{x}_0, \underline{y}_0)||_\tau^2$$

$$- (k||(\underline{x}_0, \underline{y}_0)||_\tau) \cdot (k||[(\underline{x}, \underline{y}) - (\underline{x}_0, \underline{y}_0)]||_\tau)$$

$$- (k||[(\underline{x}, \underline{y}) - (\underline{x}_0, \underline{y}_0)||_\tau) \cdot (k||(\underline{x}, \underline{y})||_\tau) \qquad (A2)$$

where the latter inequality follows with $k<\infty$ an upper bound on the gain and incremental gain of $\underset{\sim}{F}_{ij}$ (i, j = 1, 2). Since by hypothesis the map of $\underline{u} \in \mathcal{U}$ into $\overline{G}(\underline{u})$ is bounded, there exist a continuous increasing function $\rho_1 \colon R_+ \to R_+$ and a point $(\underline{x}_0^{(1)}, \underline{y}_0^{(1)}) \in \overline{\underset{\sim}{G}}^I(\underline{0})$ such that for every $\tau \in T$

$$||[(\underline{x}, \underline{y}) - (\underline{x}_0^{(1)}, \underline{y}_0^{(1)})]||_\tau \le \rho_1(||\underline{u}||_\tau). \qquad (A3a)$$

Also, for all $\tau \in T$, there exists a point $(\underline{x}_0^{(2)}, \underline{y}_0^{(2)}) \in \overline{\underset{\sim}{G}}^I(\underline{0})$ such that

$$||[(\underline{x}, \underline{y}) - (\underline{x}_0^{(2)}, \underline{y}_0^{(2)})]||_\tau \le ||(\underline{x}, \underline{y})||_\tau , \qquad (A3b)$$

namely the point $(\underline{x}_0^{(2)}, \underline{y}_0^{(2)}) = (\underline{0}, \underline{0})$. From (A3a, b) it follows that there exists an $(\underline{x}_0, \underline{y}_0) \in \overline{\underset{\sim}{G}}^I(\underline{0})$ such that for all $\tau \in T$

$$||[(\underline{x}, \underline{y}) - (\underline{x}_0, \underline{y}_0)]||_\tau \le \min \{||(\underline{x}, \underline{y})||_\tau, \rho_1(||\underline{u}||_\tau)\}, \qquad (A4)$$

$$||(\underline{x}, \underline{y})||_\tau - \rho_1(||\underline{u}||_\tau) \le ||(\underline{x}_0, \underline{y}_0)||_\tau \le 2 \cdot ||(\underline{x}, \underline{y})||_\tau \qquad (A5)$$

and, from the former inequality in (A5), it follows that for all $\tau \in T$

$$||(\underline{x}_0, \underline{y}_0)||_\tau^2 \ge ||(\underline{x}, \underline{y})||_\tau(-2\rho_1(||\underline{u}||_\tau) + ||(\underline{x}, \underline{y})||_\tau). \qquad (A6)$$

Substituting (A4), (A5), and (A6) into (A2) and dividing by $||(\underline{x}, \underline{y})||_\tau$, it follows that for all $(\underline{x}, \underline{y}) \in \overline{\underset{\sim}{G}}^I(\underline{u})$ and all $\tau \in T$,

$$d(\underline{x}, \underline{y}, \tau) \geq \varepsilon(||(\underline{x}, \underline{y})||_\tau - 2\rho_1(||\underline{u}||_\tau))$$
$$- 2k^2\rho_1(||\underline{u}||_\tau)$$
$$- k^2\rho_1(||\underline{u}||_\tau). \tag{A7}$$

Taking $\phi_1, \phi_2 : R_+ \to R_+$, $\eta_1(\underline{x}, \underline{y}, \tau)$ and $\eta_2(\underline{u}, \tau)$ to be

$$\phi_1(\alpha) \triangleq \varepsilon \cdot \alpha \tag{A8}$$

$$\phi_2(\alpha) \triangleq (2\varepsilon + 3k^2)\rho_1(\alpha) \tag{A9}$$

$$\eta_1(\underline{x}, \underline{y}, \tau) = \phi_1(||(\underline{x}, \underline{y})||_\tau) \tag{A10}$$

$$\eta_2(\underline{u}, \tau) = \phi_2(||\underline{u}||_\tau) \tag{A11}$$

we see that $\phi_1 \in K$ is linear and radially unbounded and that for all $(\underline{x}, \underline{y}) \in \overline{\underset{\sim}{G}}^I(\underline{u})$ and all $\tau \in T$

$$d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau) \tag{A12}$$

which establishes (5.1).

Proceeding in an analogous fashion to establish (5.2), we have that for some $\rho_2 : R_+ \to R_+$, every $(\underline{x}, \underline{y}) \in \overline{H}(\underline{v})$, and every $\tau \in T$

$$d(\underline{x}, \underline{y}, \tau) \leq \eta_3(\underline{v}, \tau) - \eta_4(\underline{x}, \underline{y}, \tau) \tag{A13}$$

where $\eta_3(\underline{v}, \tau) = \phi_3(||\underline{v}||_\tau) = 3k^2\rho_2(||v||_\tau)$, $\quad \eta_4(\underline{x}, \underline{y}, \tau) \equiv 0$;

as before $k < \infty$ is an upper bound on the gain and incremental gain of $\underset{\sim}{F}_{ij}$ $(i, j = 1, 2)$ and ρ_2, like ρ_1, is continuous and increasing.

Thus,

$$\eta_{out}(\underline{x}, \underline{y}, \tau) = \varepsilon ||\underline{x}, \underline{y}||_\tau \tag{A14}$$

and

$$\eta_{in}(\underline{u}, \underline{v}, \tau) = (2\varepsilon + 3k^2)\rho_1(||\underline{u}||_\tau) + 3k^2\rho_2(||\underline{v}||_\tau). \tag{A15}$$

Clearly η_{out} is positive definite, radially unbounded and satisfies the

constraint imposed by (5.10) with $\alpha = 1$. Since η_{in} is clearly bounded,

it follows from Corollary 5.1-a that (2.1) is bounded. If additionally

the maps taking $\underline{u}$ into $\overline{G}(\underline{u})$ and $\underline{v}$ into $\overline{H}(\underline{v})$ are globally stable (globally

finite-gain stable) about the respective sets $\overline{G}(\underline{0})$ and $\overline{H}(\underline{0})$, then ρ_1

and ρ_2 may be taken to be in class K (may be taken to linear) from which

it follows that η_{in} is decrescent (satisfied (5.10) with $\alpha = 1$) and

global stability (global finite-gain stability) of (2.1) follows from

Corollary 5.1-b (Corollary 5.1-c).

Appendix B: Proof of Lemma 6.2

We prove properties (i) - (vii) in sequence.

Proof of Property (i):

$\underset{\sim}{\overline{A}}$ inside (strictly inside) Sector $\left(\begin{bmatrix} F_{\sim 11} & F_{\sim 12} \\ F_{\sim 21} & F_{\sim 22} \end{bmatrix} \right)$ (B1)

$\Longleftrightarrow$

$\{ <F_{\sim 11}\underline{y} + F_{\sim 12}\underline{x}, \; F_{\sim 21}\underline{y} + F_{\sim 22}\underline{x}>_\tau \leq -\varepsilon || (\underline{x}, \; \underline{y}) ||_\tau^2$

for some $\varepsilon \geq 0$ $(\varepsilon > 0)$, all $\tau \in T$, and all $(\underline{x}, \; \underline{y}) \in \underset{\sim}{\overline{A}}$ (B2)

$\Longleftrightarrow$

$\{ <F_{\sim 11}\underline{y} + F_{\sim 12}\underline{x}, \; -F_{\sim 21}\underline{y} - F_{\sim 22}\underline{x}>_\tau \geq \varepsilon || (\underline{x}, \; \underline{y}) ||_\tau^2$

for some $\varepsilon \geq 0$ $(\varepsilon > 0)$, all $\tau \in T$ and all $(\underline{x}, \; \underline{y}) \in \underset{\sim}{\overline{A}}$ (B3)

$\Longleftrightarrow$

$\underset{\sim}{\overline{A}}$ outside (strictly outside) Sector $\left(\begin{bmatrix} F_{\sim 11} & F_{\sim 12} \\ F_{\sim 21} & -F_{\sim 22} \end{bmatrix} \right)$ (B4)

This proves property (i).

Proof of Property (ii):

$(\underline{x}, \; \underline{y}) \in$ Sector $\left(\begin{bmatrix} F_{\sim 11} & F_{\sim 12} \\ \underset{\sim}{M} \cdot F_{\sim 21} & \underset{\sim}{M} \cdot F_{\sim 22} \end{bmatrix} \right)$ (B5)

$\Longleftrightarrow$

$\{ <F_{\sim 11}\underline{y} + F_{\sim 12}\underline{x}, \; \underset{\sim}{M} \cdot F_{\sim 21}\underline{y} + \underset{\sim}{M} \cdot F_{\sim 22}\underline{x}>_\tau \leq 0$

for all $\tau \in T$ (B6)

$\Longleftrightarrow$

$$\langle F_{11}\underline{y},\ M \cdot F_{21}\underline{y}\rangle_\tau + \langle F_{11}\underline{y},\ M \cdot F_{22}\underline{x}\rangle_\tau + \langle F_{12}\underline{x},\ M \cdot F_{21}\underline{y}\rangle_\tau + \langle F_{12}\underline{x},\ M \cdot F_{22}\underline{x}\rangle_\tau$$

$$\leq\ 0 \text{ for all } \tau \in T \tag{B7}$$

$\Longleftrightarrow$

$$\{\frac{1}{ab} \langle a M^* \cdot F_{11}\underline{y},\ b F_{21}\underline{y}\rangle_\tau + \frac{1}{ab} \langle a M^* \cdot F_{11}\underline{y},\ b F_{22}\underline{x}\rangle_\tau$$

$$+ \frac{1}{ab} \langle a M^* \cdot F_{12}\underline{x},\ b F_{21}\underline{y}\rangle_\tau + \frac{1}{ab} \langle a M^* \cdot F_{12}\underline{x},\ b F_{22}\underline{x}\rangle_\tau\}$$

$$\leq\ 0 \text{ for all } \tau \in T \tag{B8}$$

$\Longleftrightarrow$

$$\langle a M^* \cdot F_{11}\underline{y} + a M^* \cdot F_{12}\underline{x},\ b F_{21}\underline{y} + b F_{22}\underline{x}\rangle_\tau$$

$$\leq\ 0 \text{ for all } \tau \in T \tag{B9}$$

$\Longleftrightarrow$

$$(\underline{x},\ \underline{y}) \in \text{Sector}\left(\begin{bmatrix} a\ M^* \cdot F_{11} & +\ a\ M^* \cdot F_{12} \\ b F_{21} & b F_{22} \end{bmatrix}\right) \tag{B10}$$

From (B5) - (B10), property (ii) follows.

Proof of Property (iii):

$$\overline{A} \underline{\text{ inside (strictly inside) Sector}}\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right) \tag{B11}$$

$\Longleftrightarrow$

$$\langle F_{11}\underline{y} + F_{12}\underline{x},\ F_{21}\underline{y} + F_{22}\underline{x}\rangle_\tau \leq\ -\varepsilon ||(\underline{x},\ \underline{y})||_\tau^2$$

$$\text{for some } \varepsilon \geq 0\ (\varepsilon > 0), \quad \text{all } \tau \in T, \text{ and all } (\underline{x},\underline{y}) \in \overline{A} \tag{B12}$$

$$\Longleftrightarrow$$

$$\langle \underset{\sim}{F}_{12}\underline{x} + \underset{\sim}{F}_{11}\underline{y}, \; \underset{\sim}{F}_{22}\underline{x} + \underset{\sim}{F}_{21}\underline{y}\rangle_\tau \leq -\varepsilon ||(\underline{y}, \; \underline{x})||_\tau^2$$

for some $\varepsilon \geq 0$ $(\varepsilon > 0)$, all $\tau \in T$, and all $(\underline{y},\underline{x}) \in \underset{\sim}{\overline{A}}^I$　　　　(B13)

$$\Longleftrightarrow$$

$\underset{\sim}{\overline{A}}^I$ inside (strictly inside) Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{12} & \underset{\sim}{F}_{11} \\ \underset{\sim}{F}_{22} & \underset{\sim}{F}_{21} \end{bmatrix} \right)$　　　　(B14)

This proves Property (iii).

Proof of Property (iv):

Let $k = $ Gain $(\underset{\sim}{\overline{A}} - \underset{\sim}{\overline{B}})$. Then,

$\underset{\sim}{\overline{A}}$ inside (strictly inside) Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix} \right)$　　　　(B15)

$$\Longleftrightarrow$$

$$\langle \underset{\sim}{F}_{11}\underline{y} + \underset{\sim}{F}_{12}\underline{x}, \; \underset{\sim}{F}_{21}\underline{y} + \underset{\sim}{F}_{22}\underline{x}\rangle_\tau \leq -\varepsilon ||(\underline{x}, \; \underline{y})||_\tau^2$$

for all $(\underline{x}, \; \underline{y}) \in \underset{\sim}{\overline{A}}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$ and all $\tau \in T$　　　　(B16)

$$\Longleftrightarrow$$

$$\langle \underset{\sim}{F}_{11}\underline{y} - \underset{\sim}{F}_{11} \cdot \underset{\sim}{B} \; \underline{x} + \underset{\sim}{F}_{11} \cdot \underset{\sim}{B} \; \underline{x} + \underset{\sim}{F}_{12}\underline{x},$$
$$\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{21} \cdot \underset{\sim}{B} \; \underline{x} + \underset{\sim}{F}_{21} \cdot \underset{\sim}{B} \; \underline{x} + \underset{\sim}{F}_{22}\underline{x}\rangle_\tau \leq -\varepsilon ||(\underline{x}, \; \underline{y})||_\tau^2$$

for all $(\underline{x}, \; \underline{y}) \in \underset{\sim}{\overline{A}}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$ and all $\tau \in T$　　　　(B17)

$$\Longleftrightarrow$$

$$\langle \underset{\sim}{F}_{11}(\underline{y} - \underset{\sim}{B} \; \underline{x}) + (\underset{\sim}{F}_{11} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{11})\underline{x},$$
$$\underset{\sim}{F}_{21}(\underline{y} - \underset{\sim}{B} \; \underline{x}) + (\underset{\sim}{F}_{21} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{22})\underline{x}\rangle_\tau \leq -\varepsilon ||(\underline{x}, \; \underline{y})||_\tau^2$$

for all $(\underline{x}, \; \underline{y}) \in \underset{\sim}{\overline{A}}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$　　　　(B18)

$$\Longleftrightarrow \langle \underset{\sim}{F}_{11}\underline{\tilde{y}} + (\underset{\sim}{F}_{11} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{12})\underline{x}, \; \underset{\sim}{F}_{21}\underline{\tilde{y}} + (\underset{\sim}{F}_{21} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{22})\underline{x}\rangle_\tau$$
$$\leq -\varepsilon ||(\underline{x},\underline{\tilde{y}} + \underset{\sim}{B}\underline{x})||_\tau^2 \leq -\varepsilon ||\underline{x}||_\tau^2 \leq \frac{-\varepsilon}{1+k^2} ||(\underline{x},\underline{\tilde{y}})||_\tau^2$$

for all $(\underline{x},\underline{\tilde{y}}) \in (\underset{\sim}{\overline{A}} - \underset{\sim}{\overline{B}})$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$　　　　(B19)

$\Rightarrow$

$(\bar{\underset{\sim}{A}} - \bar{\underset{\sim}{B}})$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & (\underset{\sim}{F}_{11} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{12}) \\ \underset{\sim}{F}_{21} & (\underset{\sim}{F}_{21} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{22}) \end{bmatrix} \right)$

and, provided $k < \infty$ and (B15) holds with the parenthetical <u>strictly inside</u>,

$(\bar{\underset{\sim}{A}} - \bar{\underset{\sim}{B}})$ <u>strictly inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & (\underset{\sim}{F}_{11} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{12}) \\ \underset{\sim}{F}_{21} & (\underset{\sim}{F}_{21} \cdot \underset{\sim}{B} + \underset{\sim}{F}_{22}) \end{bmatrix} \right)$ (B20)

This proves property (iv).

<u>Proof of Property (v)</u>:

Let $k = \text{Gain } (\bar{\underset{\sim}{A}})$.

(a) $\bar{\underset{\sim}{B}} \circ \bar{\underset{\sim}{A}}$ <u>inside</u> (strictly inside) Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix} \right)$ (B21)

$\Longleftrightarrow$

$\langle \underset{\sim}{F}_{11}\underline{y} + \underset{\sim}{F}_{12}\underline{x}, \ \underset{\sim}{F}_{21}\underline{y} + \underset{\sim}{F}_{22}\underline{x} \rangle_\tau \leq -\varepsilon \| (\underline{x}, \ \underline{y}) \|_\tau^2$

for all $(\underline{x}, \ \underline{y}) \in \bar{\underset{\sim}{B}} \circ \bar{\underset{\sim}{A}}$, some $\varepsilon \geq 0$ ($\varepsilon > 0$), and all $\tau \in T$ (B22)

$\Longleftrightarrow$

$\langle \underset{\sim}{F}_{11}\underset{\sim}{B}\tilde{\underline{y}} + \underset{\sim}{F}_{12}\underline{x}, \ \underset{\sim}{F}_{21}\underset{\sim}{B}\tilde{\underline{y}} + \underset{\sim}{F}_{22}\underline{x} \rangle_\tau$

$\leq -\varepsilon \| (\underline{x}, \ \underset{\sim}{B}\tilde{\underline{y}}) \|_\tau^2 \leq -\varepsilon \| \underline{x} \|_\tau^2$

$\leq \frac{-\varepsilon}{1 + k^2} \| (\underline{x}, \ \tilde{\underline{y}}) \|_\tau^2$

for all $(\underline{x}, \ \tilde{\underline{y}}) \in \bar{\underset{\sim}{A}}$, some $\varepsilon \geq 0$ ($\varepsilon > 0$) and all $\tau \in T$ (B23)

$\Longleftrightarrow$

$\bar{\underset{\sim}{A}}$ <u>inside</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} \cdot \underset{\sim}{B} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} \cdot \underset{\sim}{B} & \underset{\sim}{F}_{22} \end{bmatrix} \right)$

and, provided k<∞ and (B21) holds with the parenthetical <u>strictly inside</u>,

$$\underset{\sim}{\overline{A}} \ \underline{\text{strictly inside}} \ \text{Sector} \quad \left(\begin{bmatrix} \underset{\sim11}{F} \cdot \underset{\sim}{B} & \underset{\sim12}{F} \\ \underset{\sim21}{F} \cdot \underset{\sim}{B} & \underset{\sim22}{F} \end{bmatrix} \right) \tag{B24}$$

(b) $\underset{\sim}{\overline{B}} \circ \underset{\sim}{\overline{A}} \ \underline{\text{inside (strictly inside)}} \ \text{Sector}$ $\quad \left(\begin{bmatrix} \underset{\sim11}{F} & \underset{\sim12}{F} \\ \underset{\sim21}{F} & \underset{\sim22}{F} \end{bmatrix} \right)$ (B25)

$\Longleftrightarrow$

$$\langle \underset{\sim11}{F}\underline{y} + \underset{\sim12}{F} \cdot \underset{\sim}{A}^{-1} \cdot \underset{\sim}{A}\underline{x}, \ \underset{\sim21}{F}\underline{y} + \underset{\sim22}{F} \cdot \underset{\sim}{A}^{-1} \cdot \underset{\sim}{A}\underline{x} \rangle_\tau \leq -\epsilon || (\underline{x}, \ \underline{y}) ||_\tau^2$$

for all $(\underline{x}, \ \underline{y}) \in \underset{\sim}{\overline{B}} \circ \underset{\sim}{\overline{A}}$, some $\epsilon \geq 0$ ($\epsilon > 0$), and all $\tau \in T$. (B26)

$\Longleftrightarrow$

$$\langle \underset{\sim11}{F}\underline{y} + \underset{\sim12}{F}\underset{\sim}{A}^{-1}\underset{\sim}{\tilde{x}}, \ \underset{\sim21}{F}\underline{y} + \underset{\sim22}{F}\underset{\sim}{A}^{-1}\underset{\sim}{\tilde{x}} \rangle_\tau \leq -\epsilon || (\underset{\sim}{A}^{-1}\underset{\sim}{\tilde{x}}, \ y) ||_\tau^2$$

$$\leq -\epsilon(||\underline{y}||_\tau^2 + ||\underset{\sim}{A}^{-1}\underset{\sim}{\tilde{x}}||_\tau^2)$$

$$\leq -\epsilon(||y||_\tau^2 + \frac{1}{k^2}||\underset{\sim}{\tilde{x}}||_\tau^2)$$

$$\leq -\epsilon \cdot \min\{1, \ 1/k^2\} \cdot ||(\underset{\sim}{\tilde{x}}, \ y)||_\tau^2$$

for all $(\underset{\sim}{\tilde{x}}, \ \underline{y}) \in \underset{\sim}{\overline{B}}$, some $\epsilon \geq 0$ ($\epsilon > 0$) and all $\tau \in T$ (B27)

$\Longrightarrow$

$\underset{\sim}{\overline{B}} \ \underline{\text{inside}} \ \text{Sector}$ $\quad \left(\begin{bmatrix} \underset{\sim11}{F} & \underset{\sim12}{F} \cdot \underset{\sim}{A}^{-1} \\ \underset{\sim21}{F} & \underset{\sim22}{F} \cdot \underset{\sim}{A}^{-1} \end{bmatrix} \right)$ (B28)

and, provided k<∞ and (B25) holds with the parenthetical <u>strictly inside</u>,

$\underset{\sim}{\overline{B}} \ \underline{\text{strictly inside}} \ \text{Sector}$ $\quad \left(\begin{bmatrix} \underset{\sim11}{F} & \underset{\sim12}{F} \cdot \underset{\sim}{A}^{-1} \\ \underset{\sim21}{F} & \underset{\sim22}{F} \cdot \underset{\sim}{A}^{-1} \end{bmatrix} \right)$ (B29)

(c) $\overline{\underset{\sim}{B}}$ <u>inside</u> (<u>strictly inside</u>) Sector $\left(\begin{bmatrix} F_{\sim 11} \cdot \underset{\sim}{A} & F_{\sim 12} \\ F_{\sim 21} \cdot \underset{\sim}{A} & F_{\sim 22} \end{bmatrix} \right)$ (B30)

$\Longleftrightarrow$

$$\langle F_{\sim 11} \cdot \underset{\sim}{A}\underline{y} + F_{\sim 12}\underline{x},\ F_{\sim 21} \cdot \underset{\sim}{A}\underline{y} + F_{\sim 22}\underline{x} \rangle_\tau \leq -\varepsilon \, ||(\underline{x},\ \underline{y})||^2_\tau$$

for all $(\underline{x},\ \underline{y}) \in \overline{\underset{\sim}{B}}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$.   (B31)

$\Longleftrightarrow$

$$\langle F_{\sim 11}\underline{\tilde{y}} + F_{\sim 12}\underline{x},\ F_{\sim 21}\underline{\tilde{y}} + F_{\sim 22}\underline{x} \rangle_\tau \leq -\varepsilon \, ||\underline{x},\ \underline{y}||^2_\tau$$

$$= -\varepsilon(||\underline{x}||^2_\tau + ||\underline{y}||^2_\tau) \leq -\varepsilon\left(||\underline{x}||^2_\tau + \frac{1}{k^2} \cdot ||\underline{\tilde{y}}||^2_\tau\right)$$

$$\leq -\varepsilon \cdot \min\{1,\ 1/k^2\} \cdot ||(\underline{x},\ \underline{\tilde{y}})||^2_\tau$$

for all $(\underline{x},\ \underline{\tilde{y}}) \in \overline{\underset{\sim}{A}} \circ \overline{\underset{\sim}{B}}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$ (B32)

$\Longrightarrow$

$\overline{\underset{\sim}{A}} \circ \overline{\underset{\sim}{B}}$ <u>inside</u> Sector $\left(\begin{bmatrix} F_{\sim 11} & F_{\sim 12} \\ F_{\sim 21} & F_{\sim 22} \end{bmatrix} \right)$

and, provided $k < \infty$ and (B30) holds with the parenthetical <u>strictly inside</u>,

$\overline{\underset{\sim}{A}} \circ \overline{\underset{\sim}{B}}$ <u>strictly inside</u> Sector $\left(\begin{bmatrix} F_{\sim 11} & F_{\sim 12} \\ F_{\sim 21} & F_{\sim 22} \end{bmatrix} \right)$ (B33)

This proves Property (v).

<u>Proof of Property (vi)</u>:

$\overline{\underset{\sim}{A}}^{(k)}$ <u>inside</u> (<u>strictly inside</u>) Sector $\left(\begin{bmatrix} F_{\sim 11}^{(k)} & F_{\sim 12}^{(k)} \\ F_{\sim 21}^{(k)} & F_{\sim 22}^{(k)} \end{bmatrix} \right)$

for all $k = 1, \ldots, n$ (B34)

$\Longleftrightarrow$

$$\langle F_{\sim 11}^{(k)} \underline{y}^{(k)} + F_{\sim 12}^{(k)} \underline{x}^{(k)},\ F_{\sim 21}^{(k)} \underline{y}^{(k)} + F_{\sim 22}^{(k)} \underline{x} \rangle_\tau \leq -\varepsilon^{(k)} ||(\underline{x}^{(k)}, \underline{y}^{(k)})||^2_\tau$$

for all $(\underline{x}^{(k)},\ \underline{y}^{(k)}) \in \overline{\underset{\sim}{A}}^{(k)}$, some $\varepsilon^{(k)} \geq 0$ $(\varepsilon^{(k)} > 0)$, all $\tau \in T$,

and all $k = 1, \ldots, n$ (B35)

$$\sum_{k=1}^{n} <\underset{\sim}{F}_{11}^{(k)} \underline{y}^{(k)} + \underset{\sim}{F}_{12}^{(k)} \underline{x}^{(k)}, \underset{\sim}{F}_{21}^{(k)} \underline{y}^{(k)} + \underset{\sim}{F}_{22}^{(k)} \underline{x}^{(k)}$$

$$\leq -\min\{\epsilon^{(k)} \mid k=1, \ldots, n\} \cdot \sum_{k=1}^{n} ||(\underline{x}^{(k)}, \underline{y}^{(k)})||_{\tau}^{2}$$

$$= -\min\{\epsilon^{(k)} \mid k=1, \ldots, n\} \cdot ||((x^{(1)}, \ldots, \underline{x}^{(n)}), (\underline{y}^{(1)}, \ldots, \underline{y}^{(n)}))||_{\tau}^{2}$$

for all $((x^{(1)}, \ldots, x^{(n)}), (\underline{y}^{(1)}, \ldots, \underline{y}^{(n)})) \in \overline{A}$, for some $\underset{\sim}{\epsilon}^{(k)} \geq 0$

$(\epsilon^{(k)} > 0)$ $k = 1, \ldots, n$, and for all $\tau \in T$ \hfill (B36)

$$\Longleftrightarrow$$

$\underset{\sim}{\overline{A}}$ <u>inside</u> <u>(strictly inside)</u> Sector $\left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix} \right)$ \hfill (B37)

This proves Property (vi).

<u>Proof of Property (vii)</u>:

This follows directly from property (i).

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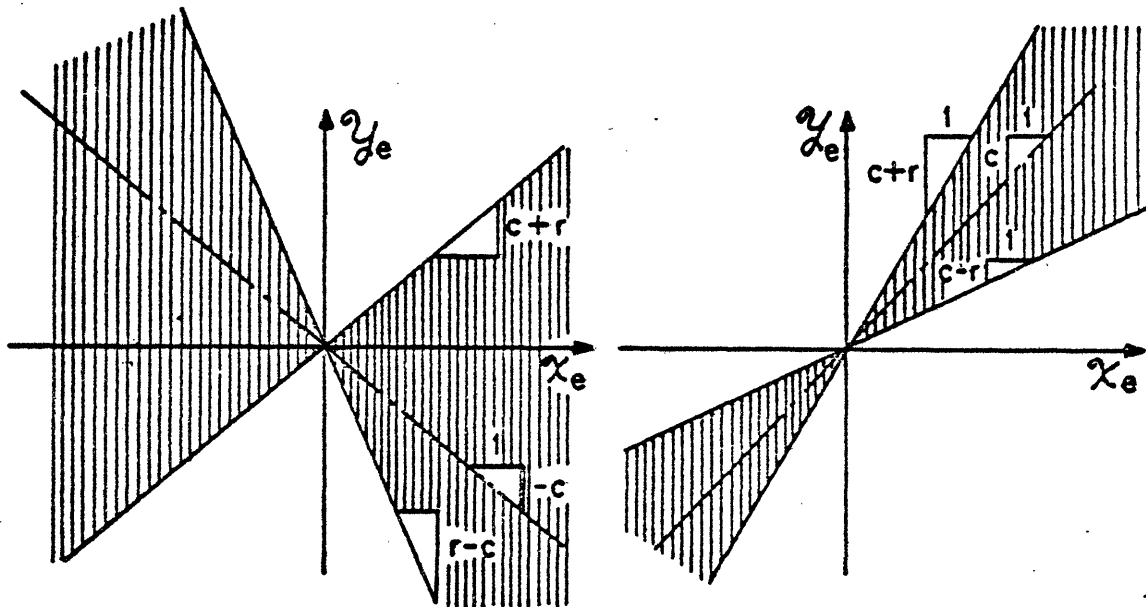[15] W. Rudin, Real and Complex Analysis. New York: McGraw-Hill, 1974.
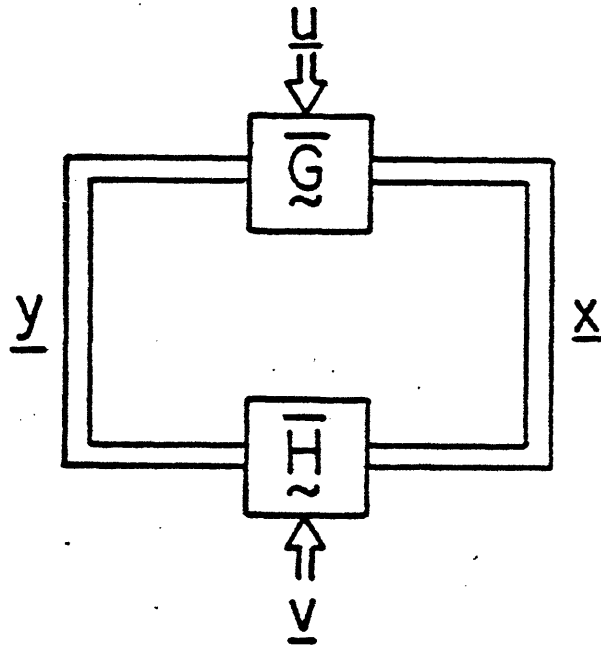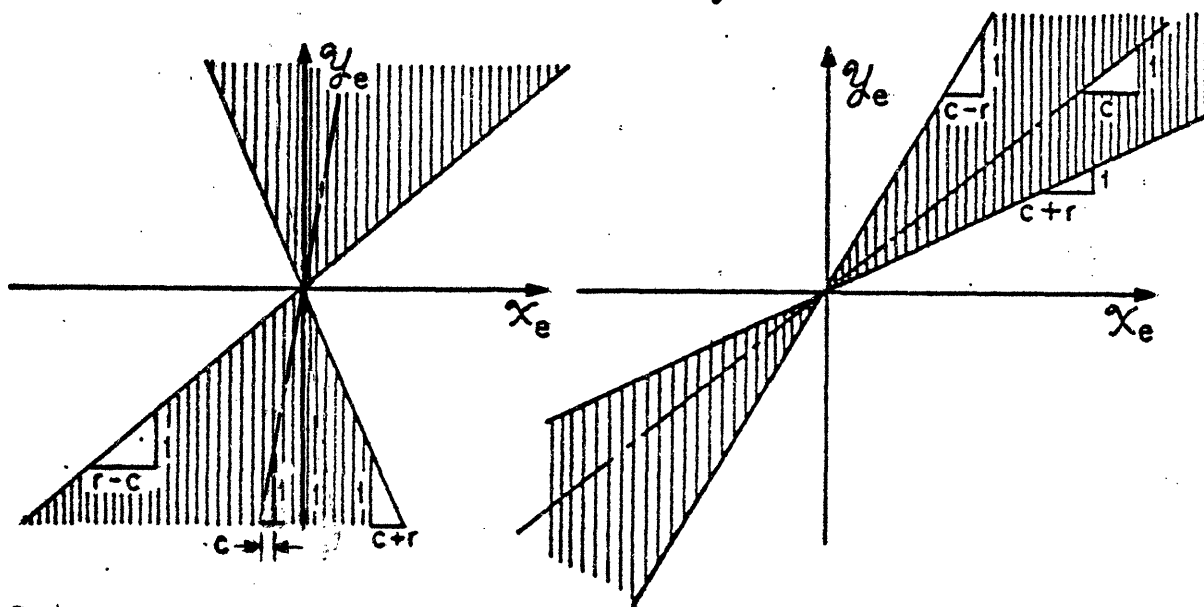
(a) Canonical two-subsystem multivariable feedback system
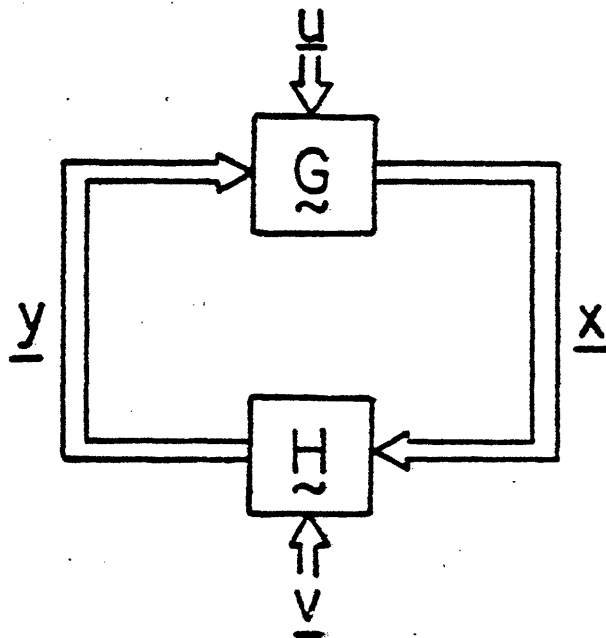
(b) Special case: $\underset{\sim}{G}(\underline{u})$ and $\underset{\sim}{H}(\underline{v})$ operators

Figure 1

(a) Conic regions in $\mathcal{X}_e \times \mathcal{Y}_e$

(b) Conic regions in $\mathcal{Y}_e \times \mathcal{X}_e$
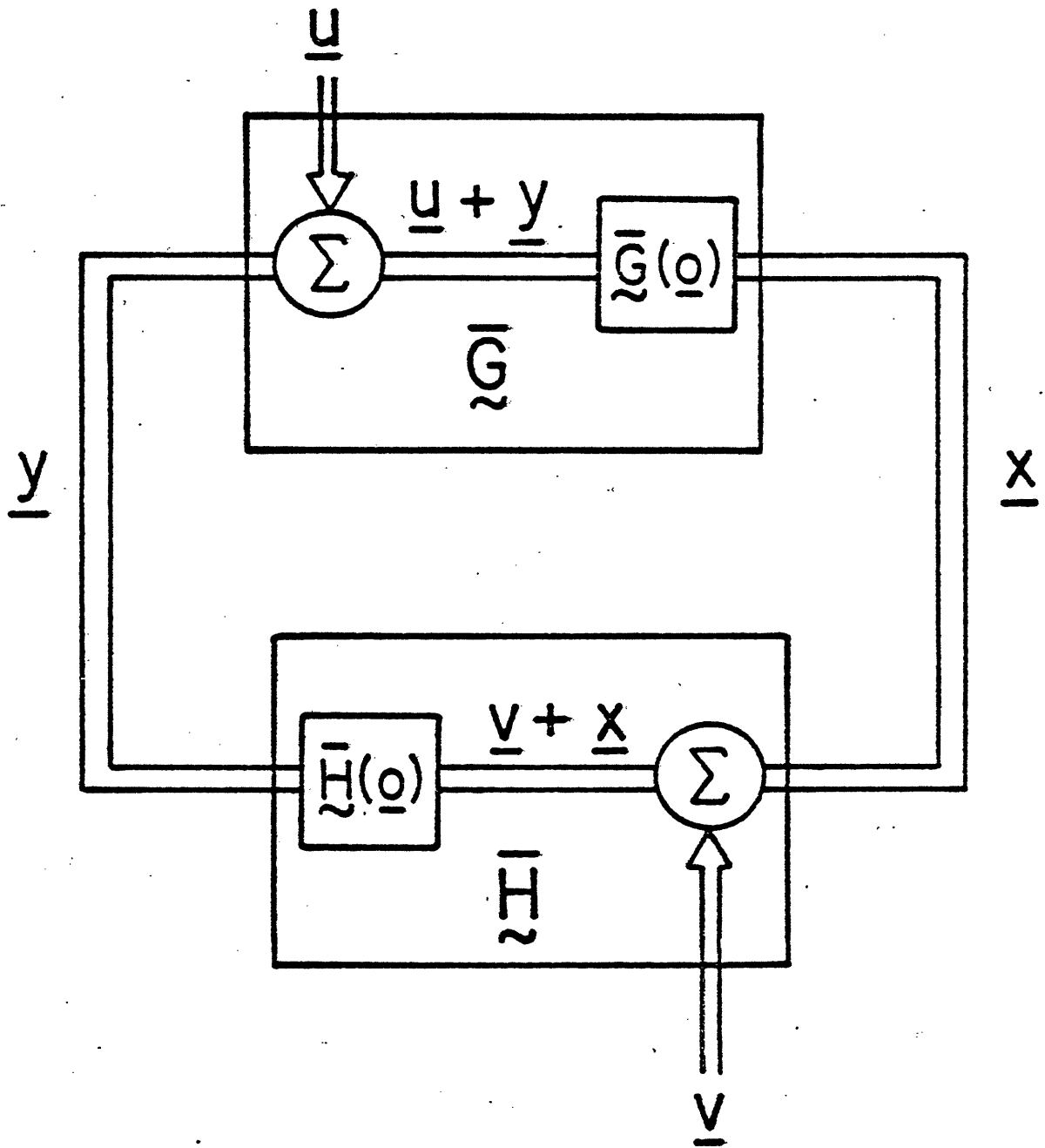
Fig. 2.   Conic regions with center c and radius r.

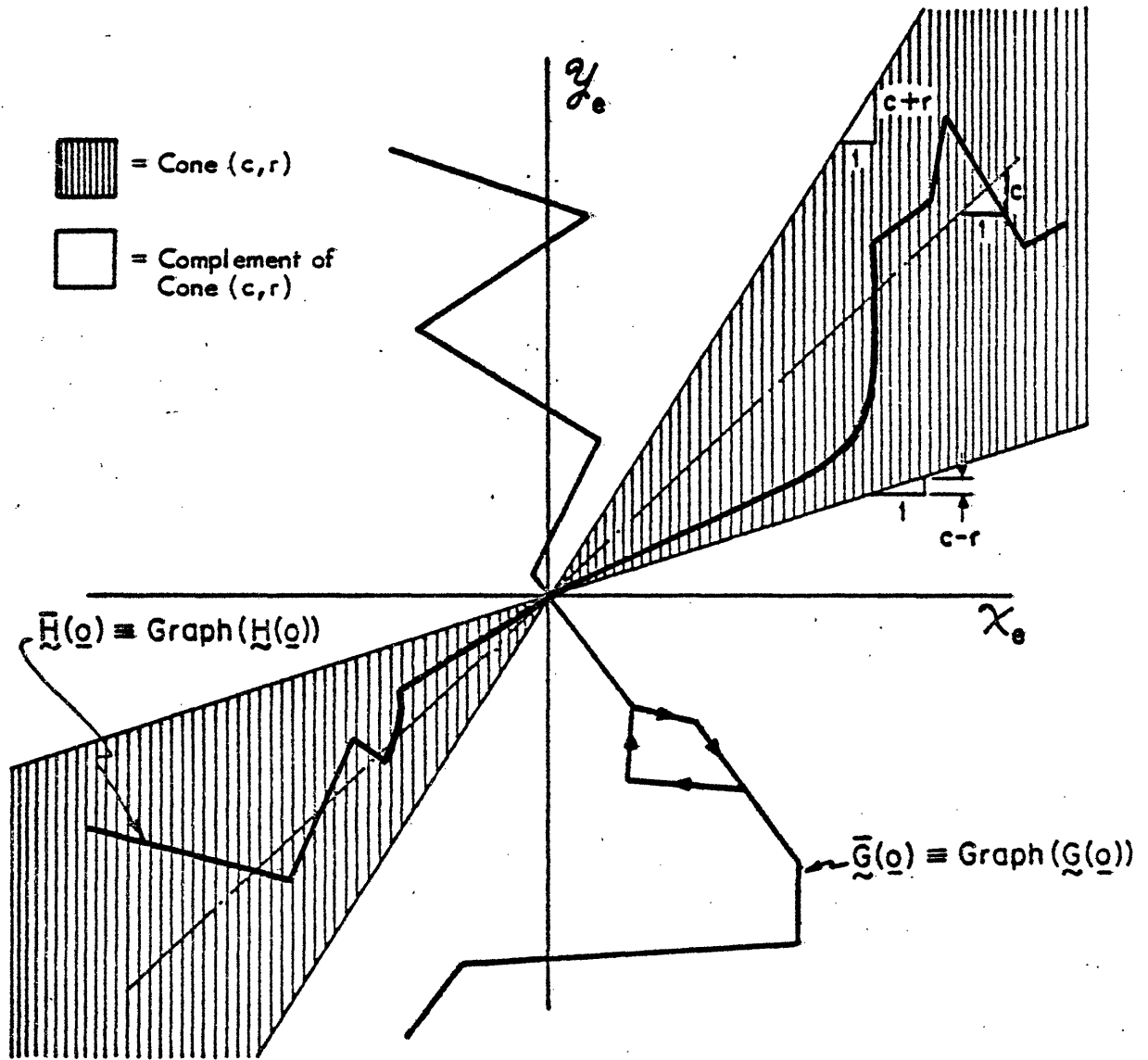**Fig. 3.** Feedback system with disturbances entering additively.

Fig. 4. Two-dimensional geometric interpretation of Zames' conic relation theorem.
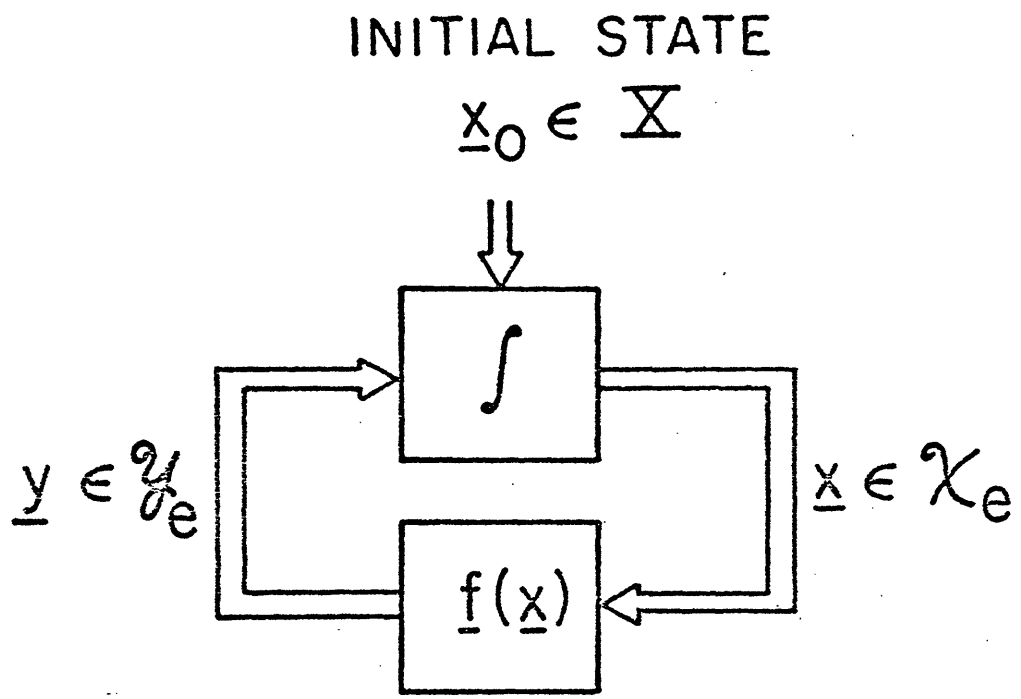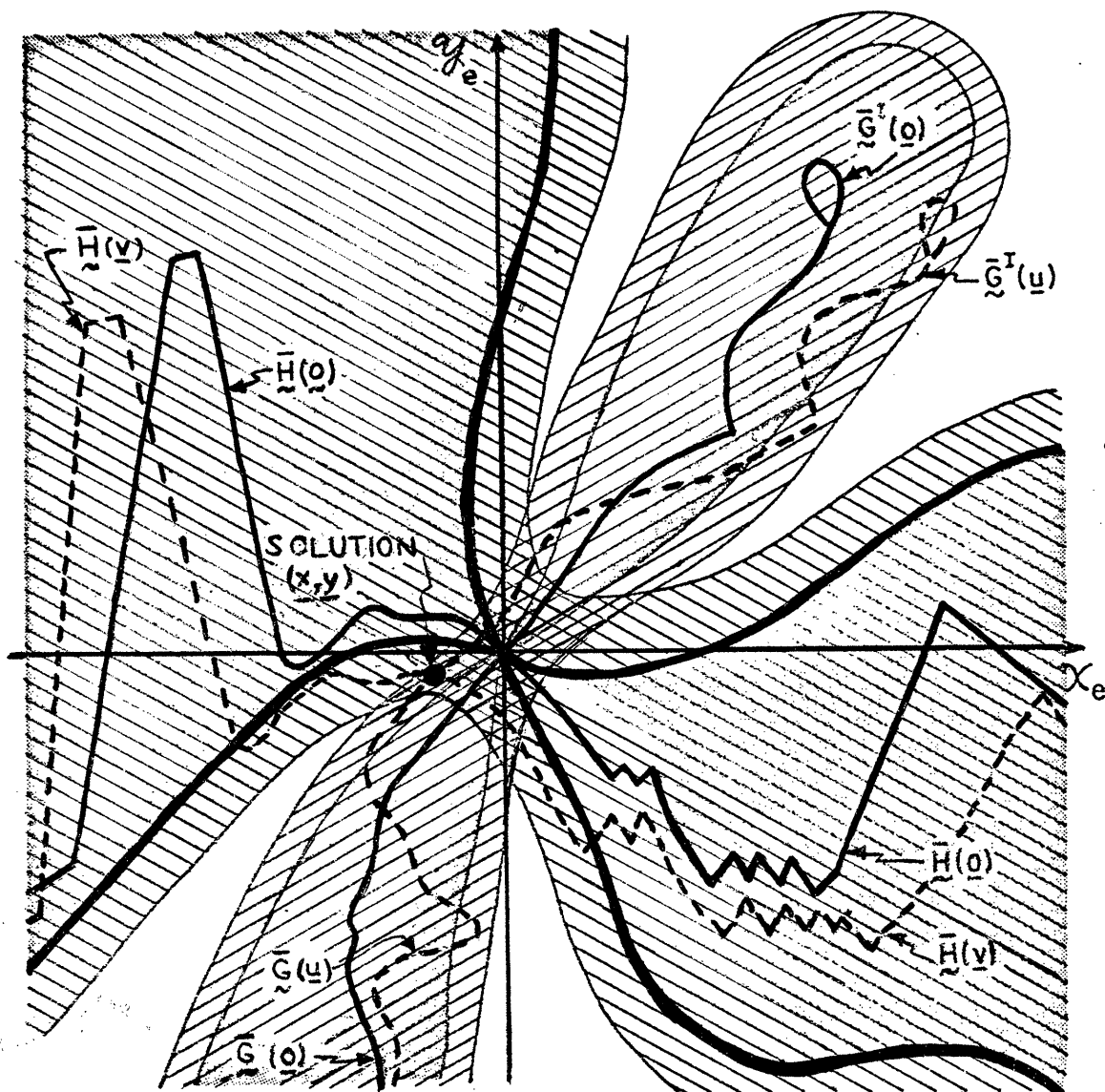
INITIAL STATE

$$\underline{x}_0 \in \underline{X}$$

$$\underline{y} \in \mathcal{Y}_e \qquad\qquad \underline{x} \in \mathcal{X}_e$$

Fig. 5.   Feedback representation of systems considered by Lyapunov methods.

▬▬▬	$d(\underline{x}, \underline{y}, \tau) = 0$	(the 'boundary')
▨	$d(\underline{x}, \underline{y}, \tau) \leq 0$	(region containing $\overline{H}(\underline{0})$)
▨	$d(\underline{x}, \underline{y}, \tau) \leq \eta_e(\underline{v}, \tau)$	(region containing $\overline{H}(\underline{v})$)
▨	$d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau)$	(region containing $\overline{G}^I(\underline{0})$)
▨	$d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau)$	(region containing $\overline{G}^I(\underline{v})$)
▨	$\eta_3(\underline{v}, \tau) \geq d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau)$	

(region containing $\overline{H}(\underline{v}) \cap \overline{G}^I(\underline{u})$)

Fig. 6   Visualization of the conditions of Theorem 5.1 in the "$X_e \times Y_e$-plane" -- the solution $(\underline{x}, \underline{y})$ must lie in the region containing both $\overline{H}(\underline{v})$ and $\overline{G}^I(\underline{u})$.