1-D Heat Equation and Solutions

3.044 Materials Processing

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The 1-D heat equation for constant k (thermal conductivity) is almost identical to the solute diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p} \tag{1}$$

or in cylindrical coordinates:

$$r \frac{\partial T}{\partial t} = \alpha \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + r \frac{\dot{q}}{\rho c_p} \tag{2}$$

and spherical coordinates:[1]

$$r^2 \frac{\partial T}{\partial t} = \alpha \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + r^2 \frac{\dot{q}}{\rho c_p} \tag{3}$$

The most important difference is that it uses the thermal diffusivity $\alpha = \frac{k}{\rho c_p}$ in the unsteady solutions, but the thermal conductivity k to determine the heat flux using Fourier's first law

$$q_x = -k \frac{\partial T}{\partial x} \tag{4}$$

For this reason, to get solute diffusion solutions from the thermal diffusion solutions below, substitute D for both k and α, effectively setting ρc_p to one.

1-D Heat Conduction Solutions

1. Steady-state
   (a) No generation
      i. Cartesian equation:

$$\frac{d^2 T}{dx^2} = 0$$

Solution:

$$T = Ax + B$$

[1] Most texts simplify the cylindrical and spherical equations, they divide by r and r^2 respectively and product rule the r-derivative apart. This gives

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{\rho c_p}$$

for cylindrical and

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{\rho c_p}$$

for spherical coordinates. I prefer equations 2 and 3 because they are easier to solve.

Flux magnitude for conduction through a plate in series with heat transfer through a fluid boundary layer (analagous to either 1st-order chemical reaction or mass transfer through a fluid boundary layer):

$$|q_x| = \frac{|T_{fl} - T_1|}{\frac{1}{h} + \frac{L}{k}}$$

(T_{fl} is the fluid temperature, analagous to the concentration in equilibrium with the fluid in diffusion; T_1 is the temperature on the side opposite the fluid.)
Dimensionless form:

$$\pi_q = 1 - \frac{1}{1 + \pi_h}$$

where $\pi_q = \frac{q_x L}{k(T_{fl} - T_1)}$ and $\pi_h = \frac{hL}{k}$ (a.k.a. the Biot number).

ii. Cylindrical equation:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$

Solution:

$$T = A \ln r + B$$

Flux magnitude for heat transfer through a fluid boundary layer at R_1 in series with conduction through a cylindrical shell between R_1 and R_2:

$$|r \cdot q_r| = \frac{|T_{fl} - T_2|}{\frac{1}{hR_1} + \frac{1}{k}\ln\frac{R_2}{R_1}}$$

iii. Spherical equation:

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$$

Solution:

$$T = \frac{A}{r} + B$$

(b) Constant generation

i. Cartesian equation:

$$k\frac{d^2T}{dx^2} + \dot{q} = 0$$

Solution:

$$T = -\frac{\dot{q}x^2}{2k} + Ax + B$$

ii. Cylindrical equation:

$$k\frac{d}{dr}\left(r\frac{dT}{dr}\right) + r\dot{q} = 0$$

Solution:

$$T = -\frac{\dot{q}r^2}{4k} + A\ln r + B$$

iii. Spherical equation:

$$k\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + r^2\dot{q} = 0$$

Solution:

$$T = -\frac{\dot{q}r^2}{6k} + \frac{A}{r} + B$$

(c) (Diffusion only, not covered) first-order homogeneous reaction consuming the reactant, so $G = -kC$

i. Cartesian equation:

$$D\frac{d^2C}{dx^2} - kC = 0$$

Solution:

$$C = Ae^{\sqrt{\frac{k}{D}}x} + Be^{-\sqrt{\frac{k}{D}}x}$$

or:

$$C = A\cosh\left(\sqrt{\frac{k}{D}}x\right) + B\sinh\left(\sqrt{\frac{k}{D}}x\right)$$

ii. Cylindrical and spherical solutions involve Bessel functions, but here are the equations:

$$D\frac{d}{dr}\left(r\frac{dC}{dr}\right) - krC = 0$$

$$D\frac{d}{dr}\left(r^2\frac{dC}{dr}\right) - kr^2C = 0$$

2. Unsteady solutions without generation based on the Cartesian equation with constant k and ρc_p:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

where $\alpha = \frac{k}{\rho c_p}$.

(a) Uniform initial condition $T = T_i$ (or $T = T_\infty$), constant boundary condition $T = T_s$ at $x = 0$, semi-infinite body; or step function initial condition in an infinite body.

Solution is the error function or its complement:

$$\frac{T - T_s}{T_i - T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\frac{T - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Semi-infinite criterion:

$$\frac{L}{2\sqrt{\alpha t}} \geq 2$$

Note: this also applies to a "diffusion couple", where two bodies of different temperatures (or concentrations) are joined at $x = 0$ and diffuse into each other; the boundary condition there is halfway between the two initial conditions. This works only if the (thermal) diffusivities are the same.

(b) Fixed quantity of heat/solute diffusing into a (semi-)infinite body (same semi-infinite criterion as 2a), no flux through $x = 0$, initial condition $T = T_i$ everywhere except a layer of thickness δ if semi-infinite or 2δ if fully infinite where $T = T_0$.

Short-time solution consists of erfs at the interfaces, like a diffusion couple.

Long-time solution is the shrinking Gaussian:

$$T = T_i + \frac{(T_0 - T_i)\delta}{\sqrt{\pi\alpha t}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$

(c) (Neither covered nor required) Uniform initial condition $T = T_i$, constant boundary condition $T = T_s$ at $x = 0$ and $x = L$ (or zero-flux boundary condition $q_x = -k\partial T/\partial x = 0$ at $x = L/2$), finite body; *or* periodic initial condition (we've covered sine and square waves) in an infinite body.

Solution is the Fourier series:

$$T = T_s + (T_i - T_s)\sum_{n=0}^{\infty} a_n \exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)$$

For a square wave or uniform IC in a finite body, $a_n = \frac{4}{n\pi}$ for odd n, zero for even n, T_s is the average temperature for a periodic situation or the boundary condition for a finite layer, L is the half period of the wave or the thickness of the finite layer.

The $n = 1$ term dominates when $\frac{\pi^2\alpha}{L^2}t \geq 1$.

(d) (Neither covered nor required) Uniform initial condition $T = T_\infty$, constant flux boundary condition at $x = 0$ $q_x = -k\frac{dT}{dx} = q_0$, semi-infinite body (same semi-infinite criterion as 2a).

Solution:

$$T = T_\infty + \frac{q_0}{k}\left[2\sqrt{\frac{\alpha t}{\pi}}\exp\left(-\frac{z^2}{4\alpha t}\right) - z\left(1 - \text{erf}\frac{z}{2\sqrt{\alpha t}}\right)\right]$$

(e) (Neither covered nor required) Uniform initial condition $T = T_\infty$, heat transfer coefficient boundary condition at $x = 0$ $q_x = -k\frac{dT}{dx} = h(T_{fl} - T)$, semi-infinite body (same semi-infinite criterion as 2a).

Solution:

$$\frac{T - T_{fl}}{T_\infty - T_{fl}} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\cdot\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

3. Moving body

If a body is moving relative to a frame of reference at speed u_x and conducting heat only in the direction of motion, then the equation in that reference frame (for constant properties) is:

$$\frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

Note that this is the diffusion equation with the substantial derivative instead of the partial derivative, and nonzero velocity only in the x-direction. Recall the definition of the substantial derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$$

Applied to temperature with $u_y = u_z = 0$:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x}$$

Therefore:

$$\frac{DT}{Dt} = \alpha\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

When this reaches steady-state, so $\frac{\partial T}{\partial t} = 0$, then the solution in the absence of generation is

$$T = A + Be^{u_x x/\alpha}$$