Dynamic Travel Time Models for Pricing and Route Guidance: A Fluid Dynamics Approach

by

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Abstract

This thesis investigates dynamic phenomena that arise in a variety of systems that share similar characteristics. A common characteristic of particular interest in this work is travel time. We wish to address questions of the type: *How long does it take a driver to traverse a route in a transportation network? How long does a unit of product remain in inventory before being sold?* As a result, our goal is not only to develop models for travel times as they arise in a variety of dynamically evolving environments, but also to investigate the application of these models in the contexts of dynamic pricing, inventory management, traffic control and route guidance.

To address these issues, we develop general models for travel times. To make these models more accessible, we describe them as they apply to transportation systems. We propose first-order and second-order fluid models. We enhance these models to account for spillback and bottleneck phenomena. Based on piecewise linear and piecewise quadratic approximations of the departure or exit flows, we propose several classes of travel time functions.

In the area of supply chain, we propose and study a fluid model of pricing and inventory management for make-to-stock manufacturing systems. This model is based on how price and level of inventory affect the time a unit of product remains in inventory. The model applies to *non-perishable* products. Our motivation is based on the observation that in inventory systems, a unit of product incurs a delay before being sold. This delay depends on the level of inventory of this product, its unit price, and prices of competitors. The model includes joint pricing, production and inventory decisions in a competitive capacitated multi-product dynamic environment.

Finally, we consider the anticipatory route guidance problem, an extension of the dynamic user-equilibrium problem. This problem consists of providing messages to drivers, based on forecasts of traffic conditions, to assist them in their path choice decisions. We propose two equivalent formulations that are the first general analytical formulations of this problem. We establish, under weak assumptions, the existence of a solution to this problem.

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To my parents

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation and Contributions

This thesis investigates dynamic phenomena that arise in a variety of systems that share similar characteristics. Such systems include transportation networks as well as supply chain and inventory management systems. Two underlying common characteristics of these systems are that they are dynamic in nature and that there is some form of travel time (delay) incurred in these systems. In particular, due to their service time and the inherent disequilibrium between demand and supply, these systems give rise to dynamic delays. For example, in ground transportation, poor-quality roads and congested traffic conditions cause travelers to experience delays in traversing a network's path. In inventory management systems, a high unit price and a high level of inventory of a product may cause a newly produced unit of that product to incur a delay before it is sold.

As a result, it is important for traffic planners to understand and manage the nature of travelers' delays (costs) in urban and highway transportation systems, and for the supply chain industry to design optimal pricing and inventory management strategies that maximize profits, reduce inventory levels, and effectively manage the delays that products incur before being sold.

Therefore, understanding the nature of the dynamic phenomena arising in these systems, exploring their common characteristics, and designing mechanisms to manage them effectively, have potential for tremendous economic, social and political impacts. This research explores these systems but also exploits the relationship between them.

The travel time models we develop in this research utilize fluid dynamics laws for compressible flow. This allows us to capture a variety of interesting phenomena. For example, in the transportation area, we are able to capture various flow patterns such as the formation and dissipation of queues, drivers' reaction time and response to upstream congestion or decongestion, spillback and bottleneck phenomena.

We investigate the application of these models in two contexts: (i) supply chain management and dynamic pricing, and (ii) advanced traveler information systems (ATIS). Indeed, by interpreting travel times as price/inventory-sojourn-time relationships, we are able to propose a tractable fluid model of pricing and inventory management for make-to-stock manufacturing systems. This model incorporates the delay of price in affecting demand for non-perishable products. Furthermore, in the context of ATIS, we are able to propose the first general analytical formulations of the anticipatory route guidance problem (ARG), an extension of the dynamic traffic user-equilibrium problem. We also establish useful properties of the problem.

Overall, the contributions of this research are the following:

- We develop general analytical models for travel times. Although our results apply to a variety of systems, we focus our exposition on dynamic transportation systems.

- Through these models, we derive closed-form solutions for travel times. These seem to correspond to the ones used in practice.

- We propose enhancements of our models in order to account explicitly for link interactions, spillback, bottleneck phenomena, and capacity constraints.

- We propose and study a fluid model of pricing and inventory management of non-perishable products for make-to-stock manufacturing systems. This model is, to the best of our knowledge, the first model that is based on how price and level of inventory affect the time a unit of product remains in inventory.

- We propose the first general analytical formulations of the Anticipatory Route Guidance problem (ARG). We establish, under weak assumptions, the existence of a solution to the ARG problem.

This research has the potential to significantly impact inventory control and manufacturing as well as transportation planning. In the area of dynamic pricing, we believe that our results will lay the foundations for the use of the delay of price in affecting demand and fluid dynamics models in supply chain and inventory management systems. Furthermore, our results in transportation could play an important role in the development of advanced traveler information systems (ATIS). They could also significantly increase the role of information technology in traffic management.

Our analysis in this research requires an interdisciplinary approach, drawing upon a broad collection of methodologies from areas such as differential equations, functional analysis, and dynamic optimization.

1.2 Thesis Structure

The thesis is organized as follows. In Chapter 2, we propose and study first-order and second-order fluid models for determining travel times. These models capture a variety of flow patterns such as formation and dissipation of queues, drivers' reaction time and response to upstream congestion or decongestion. We consider two simplified models to estimate travel times as functions of the *entrance* flow rates: the Polynomial Travel Time (PTT) Model and Exponential Travel Time (ETT) Model. We propose enhancements of our models in order to account explicitly for spillback and bottleneck phenomena and incorporate inflow, outflow and storage capacity constraints. As a result, we consider two simplified models to estimate travel times as functions of the *exit* flow rates: the Spillback Polynomial Travel Time (SPTT) Model and Spillback Exponential Travel Time (SETT) Model. We propose a general framework for the analysis of the PTT Model and the SPTT Model that reduces the analysis of these models to solving a single ordinary differential equation. Based on piecewise linear and piecewise quadratic approximations of the flow rates, we derive several classes of travel time functions for the separable PTT, SPTT, ETT and SETT models. We further establish a connection between these travel time functions. We extend the analysis of the PTT and SPTT Model to second-order non-separable velocity functions in the case of acyclic networks.

In Chapter 3, we propose and study a fluid model of pricing and inventory management for make-to-stock manufacturing systems. Instead of considering a traditional model that is based on how price affects demand, we consider a new model that relies on how price and level of inventory affect the time a unit of product remains in inventory. The model applies to *non-perishable* products. Our motivation is based on the observation that in inventory systems, a unit of product incurs a delay before being sold. This delay depends on the unit price of the product, prices of competitors, and the level of inventory of this product. Furthermore, we believe that delay data is easy to acquire. It is interesting to notice that this delay is similar to travel times incurred in a transportation network. The model of this paper includes joint pricing, production and inventory decisions in a competitive capacitated multi-product dynamic environment. We apply ideas borrowed from transportation to inventory control and supply chain in order to capture a variety of insightful phenomena that are harder to capture using current models in the literature. In particular, in this chapter, we formulate the Dynamic Pricing Model (DPM) as a continuous-time nonlinear optimization problem. We present a solution algorithm for a discretized version of the model, test it on a small case example, and report on the computational results. Furthermore, we study the analytical properties of the feasible region of the Dynamic Pricing Model in the general case. We also establish, under weak assumptions, the existence of a production/inventory control policy that maximizes the profit of the company under study over the feasible region.

In Chapter 4, we consider the anticipatory route guidance problem (ARG). This problem consists of providing messages, based on forecasts of traffic conditions to drivers, to assist them in their path choice decisions. Guidance becomes inconsistent when the forecasts on which it is based are violated after drivers react to the provided messages. We consider the problem of generating consistent anticipatory guidance that ensures that the messages based on dynamic shortest path criteria do not become self-defeating prophecies. In particular, in this chapter, we start by introducing the notation and the feasibility conditions of the ARG problem. We then provide a variational inequality (VI) formulation of this problem. We also present a fixed-point formulation of the problem and establish the equivalence of the two formulations. We discuss the special case of the Dynamic User-Equilibrium problem. We study the mathematical properties of the problem. Under sufficient conditions on the path flow rate functions and the travel time functions, we establish that the feasible region $F(ARG)$ of the Anticipatory Route Guidance problem is non-empty, and that the FIFO property holds. We show that the conditions we impose are the tightest possible. We establish key properties of the feasible region, as a function of the path flow rate functions, such as boundedness, closedness and convexity. Furthermore, we establish the existence of a solution to the ARG problem.

Finally, in Chapter 5, we provide conclusions and future steps for the study of our models.

Chapter 2

Travel Time Models for Dynamic Transportation Networks

2.1 Introduction and Motivation

2.1.1 Literature Survey

In recent years, traffic congestion has rapidly grown in transportation networks and has become an acute problem. In fact, it is estimated that the presence of congestion costs around $100 billion each year to Americans alone in the form of lost productivity (see Barnhart *et al.* [6]). Therefore, it is critical to investigate and understand its nature and address questions of the type: *how are traffic patterns formed?* and *how can traffic congestion be alleviated?* Answering these questions and designing accurate traffic flow models is important for the development of efficient control strategies.

The way flows circulate in traffic networks, the way queues form and disappear, and the way spillback and shock wave phenomena occur, are striking evidence that traffic flows are similar to gas and water flows. It is therefore natural to use physical laws of fluid dynamics for compressible flow to model traffic flow patterns.

Lighthill and Whitham [49], and Richards [71] introduced the first continuum approximation of traffic flows using kinematic wave theory (see Haberman [34] for a detailed analysis). The dynamic nature of these models gave them instant credibility. Indeed, with the increase of urban and highway congestion, the variations of flow with time are too important to be neglected. Dynamic traffic flow modeling captured the focus of most researchers interested in theoretical or applied research in the transportation area. A variety of dynamic traffic flow models have been proposed in the literature that can be classified in three major categories: microscopic models, macroscopic models and hybrid models.

Microscopic models, or car-following models, have the ability to describe, at a level of detail, the network geometry, the traffic flow and its kinematics and the traffic control logic. Such models enable simulated tests of traffic flow control strategies, and help design safety procedures by better understanding the driver's behavior. In 1950, Reuschell [70] proposed the first car-following model. Pipes [69] and Herman *et al.* [38] extended this model. Gerlough and Huber [32], Bekey *et al.* [7], Papageorgiou ([62], [63]), and Papageorgiou *et al.* ([64], [65]) and references therein provide an extensive analysis of these models.

On the other hand, macroscopic models usually possess mathematical properties that are useful in understanding the properties of a model and in designing solution algorithms to solve instances of this problem. Developing a good understanding of such phenomena is important since they arise not only in transportation systems but also in manufacturing and communication systems. In an attempt to improve modeling accuracy, the model of Lighthill and Whitham [49] was extended by Payne [67] and Whitham [78]. These models are widely applied in practice. However, these models contradict the anisotropic property of traffic flow (see Daganzo [25] for more details) and faced criticism from Daganzo [25], Papageorgiou, Blosseville and Haj-Salem [65], and Heidemann [37].

Finally, hybrid models try to capture, to a certain level, the detail and the realism of microscopic models while allowing for the algorithmic flexibility of analytical models.

Such models include the cell transmission model of Daganzo ([24], [25]) and the model by Smith [75]. However, the discrete nature of these models does not allow for any insightful mathematical analysis.

2.1.2 Objective and Motivation of this Work

The purpose of this work is to address the question of *what is the travel time in a transportation network of a driver in getting from his/her origin to his/her destination.* Practictioners in the transportation area have been using several families of travel time functions. Akcelik ([1], [2]) proposed a polynomial-type travel time function for links at signalized intersections. The BPR function [61], that is used to estimate travel times at priority intersections, is also a polynomial function. Finally, Meneguzzer *et al.* [58] proposed an exponential travel time function for all-way-stop intersections. Our goal is to lay the theoretical foundations for using these polynomial and exponential families of travel time functions in practice. While most analytical models in traffic modeling assume an a priori knowledge of a driver's travel time functions, in this work, travel time is part of the model and comes as an output. To determine the travel time, we examine and further extend the analytical model proposed by Perakis [68]. This model provides a macroscopic fluid dynamics approach to travel times and their connection to the dynamic user-equilibrium problem.

A key advantage in deriving closed-form expressions of travel times is the ability to plug these expressions into optimization problems and variational inequality formulations, in order to solve important research problems such as the dynamic system optimal problem, the dynamic user-equilibrium problem, and, more generally, the anticipatory route guidance problem (the analysis of the latter problem is the focus of Chapter 4). As a result, one can design efficient solution techniques to solve these problems, and gain insight on drivers' behavior and the characteristics of congestion.

The results of our research will enable researchers in the transportation area to develop an alternative theory of equilibrium in transportation problems that is able to make empirically testable predictions of traffic patterns and delays in transportation networks. This research will also allow the transportation community to study both microscopic and macroscopic phenomena of traffic patterns more accurately and give insight into the nature of dynamic and static equilibria. The following list of representative questions could be addressed:
What is the dynamic nature of traffic equilibrium? How are delays dynamically changing? What is the role of information to travelers in the formation of traffic patterns? How can we control what information to release to travelers in order to induce a certain desirable behavior? Furthermore, this research will allow us to address various issues that arise in transportation systems such as *local bottleneck phenomena.* In particular, *if there is a disturbance in the transportation network (like an accident), how is the traffic pattern affected? How should traffic be rerouted?*

A natural motivation for studying these problems arises in transportation planning due to the growing congestion of urban and highway transportation systems worldwide. Time-of-day plays a major role in how these networks are utilized. Studying the traffic flow pattern before, during and after rush hour or when the traffic flow changes in the vicinity of traffic signals and accidents requires the understanding of the dynamic behavior of traffic. Furthermore, the impetus for studying these problems has also been strengthened recently by the fast growing field of Advanced Traveler Information Systems (ATIS).

In this chapter, we state the results and contributions of the first-order polynomial and exponential travel time models of Kachani and Perakis [41], the second-order models of Kachani and Perakis [42], and the models for spillback and bottleneck phenomena of Kachani and Perakis [40].

The main contributions of this chapter are the following:

1. We propose a variety of models for determining travel times in transportation networks.

2. We propose analytical forms (closed-form solutions) of travel times.

3. We capture a variety of flow patterns such as formation and dissipation of queues, drivers' reaction time and response to upstream congestion or decongestion.

4. We account explicitly for splillback, bottleneck phenomena, and link interaction.

5. We incoporate inflow, outflow and storage capacity constraints.

In particular:

- We propose first-order and second-order fluid models for determining travel time functions (Subsection 2.3.1).

- We propose two simplified models to estimate travel times as functions of the *entrance* flow rates: the Polynomial Travel Time (PTT) Model and Exponential Travel Time (ETT) Model (Subsections 2.3.2 and 2.4.1).

- We design enhancements of our models in order to account explicitly for spillback and bottleneck phenomena and to incorporate inflow, outflow and storage capacity constraints (Subsection 2.5.1).

- We propose two simplified models to estimate travel times as functions of the *exit* flow rates: the Spillback Polynomial Travel Time (SPTT) Model and Spillback Exponential Travel Time (SETT) Model (Subsections 2.5.1 and 2.5.2).

- We propose a general framework for the analysis of the PTT Model and the SPTT Model that reduces the analysis of these models to solving a single ordinary differential equation (Subsections 2.4.1 and 2.5.2).

- Based on piecewise linear and piecewise quadratic approximations of the flow rates, we propose several classes of travel time functions for the separable PTT, SPTT, ETT and SETT models (Subsections 2.4.1 and 2.5.2).

- We extend the analysis of the PTT and SPTT Model to second-order non-separable velocity functions in the case of acyclic networks (Subsection 2.4.2 and 2.5.3).

2.2 The Hydrodynamic Theory of Traffic Flow

In Subsection 2.2.1, we summarize the notation that we use throughout the paper. In Subsection 2.2.2, we consider a single link network and introduce the hydrodynamic theory of traffic flow developed by Lighthill and Whitham [49]. In Subsection 2.2.3, we establish a relationship between path and link flows.

2.2.1 Notation

The physical traffic network is represented by a directed network $G = (N, I)$, where N is the set of nodes and I is the set of directed links. Index w denotes an Origin-Destination (O-D) in the set W of origin destination pairs. Index P denotes the set of paths and index P_w denotes the set of paths between O-D w. Moreover, x_p denotes a position on a path p and x_i denotes a position on a link i. Below, we provide the inputs and outputs of the models formulated in the rest of the chapter.

Inputs

Origin-Destination variables:

W	:	number of O-D pairs in the network;
n_w	:	number of paths on O-D pair w;
$d_w(t)$	:	demand rate function on O-D pair w.

Path variables:

$|P|$: number of paths in the network;

L_p : length of path p;

$F_p(0,t)$: flow rate at the entrance of path p at time t;

$F(0,t)$: vector of departure path flow rates.

Link variables:

$|I|$: number of directed links in the network;

L_i : length of link i;

$f_i(0,t)$: departure flow rate on link i at time t;

$f(0,t)$: vector of departure link flow rates;

$u_i(0,t)$: traffic speed at the entrance of link i at time t;

$k_i(0,t)$: traffic density at the entrance of link i at time t;

u_i^{max} : maximum traffic speed on link i;

k_i^{max} : maximum traffic density on link i.

Link-path flow variables:

ip : a link-path pair;

i^-p : predecessor of link i on path p;

δ_{ip} = 1 if link i belongs to path p, and 0 otherwise;

L_{ip} : length from the origin of path p until the beginning of link i.

Time variables:

$[0,T]$: O-D traffic demand period.

Outputs

Path variables:

$F_p(x_p,t)$: flow rate at time t on path p at position x_p;

$T_p(L_p,t)$: traversal time of path p of drivers departing at time t.

Link variables:

$f_i(x_i, t)$	:	flow rate at position x_i on link i at time t;
$T_i(L_i, t)$	:	traversal time of link i of drivers departing at time t;
$u_i(x_i, t)$	:	traffic speed at position x_i on link i at time t;
$k_i(x_i, t)$	:	traffic density at position x_i on link i at time t.

Link-path flow variables:

$T_{ip}(L_{ip}, t)$	:	partial path travel time function from the origin of path p until the entrance of link i of drivers departing at time t.

2.2.2 Hydrodynamic Theory of Traffic Flow on a Single Stretch of Road

In this subsection, we describe the laws of fluid dynamics for compressible flow in a single stretch of road. Lighthill and Whitham [49] introduced these laws. See Haberman [34] for a more detailed analysis.

Let us consider a link of length L. We denote by $\tau = \tau(x, t)$ the travel time to reach position x when departing at time t. The three fundamental traffic variables of fluid dynamics are:

- the flow rate function $f(x, t + \tau)$ that measures, in vehicles per unit of time, that is, the flow rate that crosses point x at time $t + \tau$,

- the density function $k(x, t + \tau)$ that measures, in vehicles per mile, that is, the density rate at point x at time $t + \tau$, and

- the velocity function $u(x, t + \tau)$ that measures, in miles per unit of time, that is, the instantaneous speed at point x at time $t + \tau$.

Two relationships connect these three variables.

$$f(x, t+\tau) = k(x, t+\tau).u(x, t+\tau), \quad \forall x, \tau. \tag{2.1}$$

Assuming that there are no exits in this stretch of road between the entrance position $x = 0$ and the exit position $x = L$, the second relationship expresses a conservation of vehicles in this stretch:

$$\frac{\partial f(x, t+\tau)}{\partial x} + \frac{\partial k(x, t+\tau)}{\partial \tau} = 0. \tag{2.2}$$

If we knew the velocity $u(.)$, then conservation law (2.2) and equation (2.1) would allow us to obtain the flow rate $f(.)$ and as a result the density $k(.)$. Nevertheless the velocity is a consequence of the drivers' behavior. In the mid-1950's Lighthill and Whitham [49] and independently Richards [71], proposed the additional assumption that the velocity at any point depends only on the density. In mathematical terms:

$$u = \widehat{u}(k). \tag{2.3}$$

The function $\widehat{u}$ is empirically measured and is an input to the model.

Several models have been proposed in the literature for the velocity function $\widehat{u}(.)$. Mahmassani and Hernan (1984) proposed a linear model:

$$\widehat{u}(k) = u^{max}(1 - \frac{k}{k^{max}}), \tag{2.4}$$

where they assume that:

- the free flow speed is the maximum speed: $\widehat{u}(0) = u^{max}$,

- at maximum density, the speed is zero: $\widehat{u}(k^{max}) = 0$.

From equations (2.1) and (2.3), we obtain:

$$f(x,t+\tau) = k(x,t+\tau).\hat{u}(k(x,t+\tau)) \tag{2.5}$$

In the case of the linear model of Mahmassani and Hernan, $f(x,t+\tau) = u^{max}.k(x,t+\tau)(1-\frac{k(x,t+\tau)}{k^{max}})$. More generally, there exists a function $g(.)$ such that

$$f(x,t+\tau) = g(k(x,t+\tau)). \tag{2.6}$$

If $g(.)$ is an invertible function, then:

$$k(x,t+\tau) = g^{-1}(f(x,t+\tau)). \tag{2.7}$$

If we further assume that $g(.)$ is differentiable, using the above expression in the conservation law (2.2), we derive:

$$\frac{\partial f(x,t+\tau)}{\partial x} + \frac{\partial g^{-1}(f(x,t+\tau))}{\partial f}.\frac{\partial f(x,t+\tau)}{\partial \tau} = 0. \tag{2.8}$$

Equation (2.8) is a partial differential equation that can be solved using our knowledge of the boundary term $f(0,t)$ corresponding to the entrance flow rate in the stretch of road.

Once we solve this partial differential equation in $f(.)$, we use equation (2.7) to obtain the density function $k(.)$ and subsequently equation (2.3) to obtain the velocity. Using the velocity field equation

$$\frac{dx}{d\tau} = u(x,t+\tau), \tag{2.9}$$

we derive the travel time function $\tau = \tau(x,t)$ using as an initial condition the fact that $\tau(0,t) = 0$.

In the case of a network of multiple links, we will call the velocity function $u_i(.)$ of link i separable if it only depends on the density function k_i.

2.2.3 Relationship between Path and Link Variables

After determining travel time functions on the network's links, we need to determine the travel times to traverse the network's paths. Determining path travel times becomes complicated due to the dynamic nature of traffic. Two approaches have been proposed in the literature to address this problem. The first approach assumes that travelers consider only the current travel time information in the network. That is, travelers compute their path travel time at time t as the sum of all the link travel times along their route, based on the current information available to the travelers at time t. For example, up-to-the-minute radio broadcasts could be a source of such information. This type of travel time function is called instantaneous travel time (see for example Boyce, Ran and Leblanc [17]). The second approach assumes that travelers consider predicted or estimates of travel times. That is, the travel time to traverse a path is the summation of the link travel times that the traveler experiences when he/she reaches each link along the path (see for example Friesz *et al.* [30]). Traveler information systems could provide, for example, such information.

In this paper, we follow the second approach. To illustrate this, let us first consider a network with one path p and two links 1 and 2. We have:

$$T_p(L_p, t) = T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)).$$

Since $L_{2p} = L_1$ and $T_{2p}(L_{2p}, t) = T_1(L_1, t)$, it follows that:

$$T_p(L_p, t) = T_1(L_1, t) + T_2(L_2, t + T_{2p}(L_{2p}, t)).$$

Similarly, if we add a third link 3 to path p, it follows that:

$$T_p(L_p, t) \quad = \quad T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)) +$$

$$\begin{aligned} & T_3(L_3, t + T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)) \\ = \; & T_1(L_1, t) + T_2(L_2, t + T_{2p}(L_{2p}, t)) + T_3(L_3, t + T_{3p}(L_{3p}, t)). \end{aligned}$$

The above formulas easily extend to the general case as follows:

$$T_p(L_p, t) \;=\; \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}.$$

2.3 General Models for Travel Time Functions

The goal of this subsection is to introduce and study second-order travel time models. The term "second-order" reflects the fact that the speed on a link is not only affected by the density, but also by the change in density on this link and its neighbors.

In Subsection 2.3.1, we propose a second-order non-separable model (Model 1) for travel time functions that incorporates the drivers' reactions to upstream congestion or decongestion as well as link interaction. This model generalizes the first-order model proposed by Perakis [68]. In Subsection 2.3.2, we propose two simplified versions of the general model: the Polynomial Travel Time (PTT) Model and the Exponential Travel Time (ETT) Model. The analysis of these two models is the focus of the following sections.

2.3.1 A Second-Order Model

The purpose of this subsection is to model the following two traffic phenomena:

1- Drivers' reaction to upstream congestion or decongestion. In particular, when a driver realizes the formation of a queue upstream, he/she starts slowing down.

Similarly, drivers start accelerating when the queue starts dissipating.

2- Effects on a link of densities as well as variations in densities of neighboring links.

To account for the two phenomena, we replace the speed-density relationship $u_i = \widehat{u}_i(k)$ by $u_i = \overline{u}_i(k, \nabla k)$. The variables k and ∇k contain the term $\frac{\partial k_i}{\partial x_i}$ that allows us to model the reaction of drivers to changes in the link density. They also contain the terms k_j and $\frac{\partial k_j}{\partial x_j}$ for the set of links j in the neighborhood of link i, that allow us to effectively model link interaction. We propose the following general form of the velocity of link i, at position x_i and at time t:

$$\begin{aligned}&\overline{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i, t) - \\ &\frac{\lambda_i(x_i)}{k_i(x_i, t)} \frac{\partial k_i(x_i, t)}{\partial x_i} + \sum_{j \in I} \alpha_{ij}(x_i) k_j(\overline{x}_j, t - \Delta_{ij}) + \\ &\sum_{j \in I} \frac{\beta_{ij}(x_i)}{k_j(\overline{x}_j, t - \Delta_{ij})} \frac{\partial k_j(\overline{x}_j, t - \Delta_{ij})}{\partial x_j},\end{aligned} \tag{2.10}$$

where $\alpha_{ij}(x_i)$ and $\beta_{ij}(x_i)$ are density correlation functions between link i and link j and depend on the position x_i on link i; $\overline{x}_j$ is a fixed position of a detector of density on link j and Δ_{ij} is a propagation time between link i and link j.

The term $-\frac{\lambda_i(x_i)}{k_i(x_i,t)} \frac{\partial k_i}{\partial x_i}$ is borrowed from heat transfer and accounts for the drivers' awareness of upstream and downstream conditions. The heat transfer term $\lambda_i(x_i)$ is a positive term expressed in squared miles per unit of time. The propagation term $\frac{\lambda_i(x_i)}{k_i(x_i,t)} \frac{\partial k_i}{\partial x_i}$ expresses the variation in the speed induced by a variation in the density. For instance, when a queue is expanding on link i, the term $-\frac{\lambda_i(x_i)}{k_i(x_i,t)} \frac{\partial k_i}{\partial x_i}$ is negative and hence the velocity function $u_i(x_i, t)$ decreases.

Model 1 can be formulated as follows:

Model 1

For all $t \in [0, T]$, $p \in P$, and $i \in I$, we have:

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \tag{2.11}$$

$$f_i(x_i, t) = \sum_{p \in P} F_p(x_i, t)\delta_{ip}, \tag{2.12}$$

$$u_i(x_i, t) = \overline{u}_i(k, \nabla k), \tag{2.13}$$

$$f_i(x_i, t) = k_i(x_i, t)u_i(x_i, t), \tag{2.14}$$

$$\frac{\partial f_i(x_i, T_i)}{\partial x_i} + \frac{\partial k_i(x_i, T_i)}{\partial T_i} = 0, \tag{2.15}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \tag{2.16}$$

$$T_i(0, t) = 0. \tag{2.17}$$

When $\lambda_i(.)$, $\alpha_{ij}(.)$ and $\beta_{ij}(.)$ go to 0, the above speed-density relationship becomes $u_i = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i, t)$. The latter corresponds to the first-order model proposed by Perakis [68].

Model 1 is very hard to analyze in its current form. For this reason, in the following subsection, we consider two simplified models of Model 1.

2.3.2 Two Simplified Second-Order Separable Models for Travel Time Functions

Our goal in this subsection is to solve Model 1 and propose specific travel time functions. To achieve this, the first step is to eliminate some of the variables involved in the model. We eliminate the density variables by expressing them as functions of the flow rates. This leads to proposing two simplified versions of Model 1. We impose the following assumptions:

A1 $\overline{u}_i(k, \nabla k)$ is a separable function of the density k_i. Further, $\overline{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i, t) - \frac{\lambda_i(x_i)}{k_i(x_i,t)} \frac{\partial k_i(x_i,t)}{\partial x_i}$, where b_i is a constant.

A2 The term $\frac{1}{(u_i^{max})^2} << 1$.

A3 The term $\lambda_i(x_i)\frac{\partial k_i}{\partial x_i} << 1$.

A4 The link flow rate $f_i(0, t+\tau_i)$ can be approximated through a continuously differentiable function $h_i^t(\tau_i)$ of τ_i.

Lemma 2.1 *Under Assumption (A1), the link density as a function of the link flow rate function and the queue propagation term can be expressed as:*

$$k_i = \frac{1}{2b_i u_i^{max}}(1 - (1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}). \tag{2.18}$$

Proof: Since $\widehat{u}(k_i) = u_i^{max} - b_i(u_i^{max})^2 k_i$, combining the speed-density and the flow-speed-density relationships, we derive $f_i = u_i^{max}k_i - b_i(u_i^{max})^2 k_i^2 - \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}$. By solving in terms of k_i for stable flows, we obtain the result of the lemma.

□

The Polynomial Travel Time (PTT) Model

In this subsection, we consider an approximation of equation (2.18). This approximation enables us to describe the conservation law of cars (2.15) only in terms of the link flow rate functions.

Lemma 2.2 *Under Assumptions (A1)-(A2), the link density as a function of the link flow rate function and the queue propagation term can be expressed as:*

$$k_i = \frac{f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}}{u_i^{max}} + \frac{b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i})^2}{u_i^{max}}. \tag{2.19}$$

Proof: From equation (2.18), $k_i = \frac{1}{2b_i u_i^{max}}(1-(1-4b_i(f_i+\lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}})$. Assumption (A2) and the definition of b_i in Assumption (A1) imply that all the terms of order higher than or equal to 3 in the Taylor expansion of the above equation are negligible. That is, $1 - (1-\epsilon)^{\frac{1}{2}} = \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + O(\epsilon^3)$. The result of the lemma follows.

□

Using the above result, the following theorem provides a partial differential equation that provides a new version of the conservation law (2.15) described only by the link flow rate functions .

Theorem 2.1 *Under Assumptions (A1)-(A3) and equation (2.19), the link flow rate functions f_i are solutions of the second-order partial differential equation:*

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1 + 2b_i f_i} \frac{\partial f_i}{\partial x_i} = \lambda_i(x_i) \frac{\partial^2 f_i}{\partial x_i^2}. \tag{2.20}$$

Assumption (A4) provides a boundary condition and, when $\lambda_i(x_i)$ is non-zero, $f_i(x_i, 0)$, for $x_i \in [0, L_i]$ and $i \in I$, provides an initial condition.

Proof: Replacing the value of k_i from (2.19) in the flow conservation equation gives rise to:

$$\frac{\partial f_i}{\partial x_i} + \frac{1 + 2b_i f_i}{u_i^{max}} \frac{\partial f_i}{\partial t} + \frac{1}{u_i^{max}} \frac{\partial^2 k_i}{\partial t \partial x_i} (\lambda_i(x_i)$$
$$+2b_i \lambda_i(x_i)^2 \frac{\partial k_i}{\partial x_i} + 2b_i \lambda_i(x_i) f_i) + \frac{2b_i \lambda_i(x_i)}{u_i^{max}} \frac{\partial k_i}{\partial x_i} \frac{\partial f_i}{\partial t} = 0.$$

Differentiating the flow conservation equation with respect to x_i leads to $\frac{\partial^2 k_i}{\partial t \partial x_i} = -\frac{\partial^2 f_i}{\partial x_i^2}$. Replacing the above value of $\frac{\partial^2 k_i}{\partial t \partial x_i}$ in the above second-order equation and using Assumption (A3) leads to $\frac{\partial f_i}{\partial x_i} + \frac{1+2b_i f_i}{u_i^{max}} \frac{\partial f_i}{\partial t} - \frac{\lambda_i(x_i)+2b_i \lambda_i(x_i) f_i}{u_i^{max}} \frac{\partial^2 f_i}{\partial x_i^2} = 0$. Dividing each term by $\frac{1+2b_i f_i}{u_i^{max}}$ gives rise to the result of the theorem.

□

Conservation law (2.20) is the basis of our analysis of the PTT Model in the following sections.

The Exponential Travel Time (ETT) Model

In this subsection, we use a different approach. We first eliminate the density variables through equation (2.18), and use this to derive a conservation law. We then approximate this equation to obtain a conservation law in the link flow rate.

Theorem 2.2 *Under Assumption (A1), the link flow rate functions f_i are solutions of the partial differential equation:*

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}.$$

Furthermore, under Assumptions (A2) and (A3), the link flow rate functions f_i are solutions of the second-order partial differential equation:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}. \qquad (2.21)$$

Assumption (A4) provides a boundary condition and, when $\lambda_i(x_i)$ is non-zero, $f_i(x_i, 0)$, $x_i \in [0, L_i]$ and $i \in I$, provides an initial condition.

Proof: Under Assumption (A1), equation (2.18) holds. Differentiating this equation with respect to t gives rise to $\frac{\partial k_i}{\partial t} = \frac{\frac{\partial f_i}{\partial t} + \lambda_i(x_i)\frac{\partial^2 k_i}{\partial x_i \partial t}}{u_i^{max}(1-4b_i(f_i+\lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}}$. Moreover, differentiating the flow conservation equation with respect to x_i leads to $\frac{\partial^2 k_i}{\partial t \partial x_i} = -\frac{\partial^2 f_i}{\partial x_i^2}$. Therefore, it follows that $\frac{\partial k_i}{\partial t} = \frac{\frac{\partial f_i}{\partial t} - \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}}{u_i^{max}(1-4b_i(f_i+\lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}}$. Substituting the above value of $\frac{\partial k_i}{\partial t}$ in the flow conservation equation leads to $\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}$.

Assumption (A2) implies that all the terms of order higher than or equal to 2 in the Taylor expansion of the above equation are negligible. That is, $(1-\epsilon)^{\frac{1}{2}} = 1 - \frac{\epsilon}{2} + O(\epsilon^2)$. Assumption (A3) gives then rise to equation (2.21).

□

Conservation law (2.21) is the basis of our analysis of the ETT Model in the following sections.

Our purpose is to reduce the analysis of the Second-Order ETT Model to the analysis of a known problem in fluid dynamics. This reduction will be achieved in two steps. The first reduction consists of transforming the bottleneck operation of the model to a Burgers equation. In fluid dynamics, Burgers equations are considered to be the simplest form of equations combining both nonlinear propagation effects and diffusive effects. The second reduction consists of a standard reduction of a Burgers' equation to a heat equation.

Equation (2.21) is a second-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. The following result achieves the two-stage reduction outlined above.

Theorem 2.3 *(i) Let $Y_i = u_i^{max}(1 - 2b_i f_i)$. Then, Y_i satisfies $\frac{\partial Y_i}{\partial t} + Y_i \frac{\partial Y_i}{\partial x_i} = \lambda_i \frac{\partial^2 Y_i}{\partial x_i^2}$. (ii) Let Z_i be defined by $Y_i = -2\lambda_i \frac{\frac{\partial Z_i}{\partial x_i}}{Z_i}$. Equation (2.21) reduces to a heat equation of the type*

$$\frac{\partial Z_i}{\partial t} = \lambda_i \frac{\partial^2 Z_i}{\partial x_i^2}, \tag{2.22}$$

Note that $f_i = \frac{1}{2b_i}(1 + 2\frac{\lambda_i}{u_i^{max}} \frac{\frac{\partial Z_i}{\partial x_i}}{Z_i})$. Equation (2.22) is a heat equation. The heat equation has been extensively studied in the literature. The application of these results to our specific problem is the subject of ongoing research.

2.4 Departure-Flow-Based Models

In this subsection, we focus on first-order travel time models. We propose and study two models. Under some approximations on the entrance flow rates, we derive analytical forms of travel time functions. Finally, we compare and discuss these functions.

2.4.1 Analysis of Separable Velocity Functions

In this subsection, we study the PTT and the ETT Models in the case of first-order separable velocity functions. That is, we consider $\lambda_i = 0$. In this case, the velocity functions reduce to $\overline{u}_i(k) = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i, t)$.

In particular, we extensively analyze the PTT Model for piecewise linear and piecewise quadratic functions $h_i^t(T_i)$ (see Assumption (A4)). We show how Model 1 reduces in this case to the analysis of a single ordinary differential equation. We provide families of travel time functions.

We also analyze the ETT Model by approximating the initial flow rate with piecewise linear functions $h_i^t(T_i)$. Moreover, we show why the analysis of the ETT Model is more complex than the one of the PTT Model. Finally, we propose a family of travel time functions.

Furthermore, we summarize our results and show how these families of travel time functions relate. We also provide a numerical analysis of these travel time functions.

Separable PTT Model

In this subsection, we analyze the PTT Model for piecewise linear and piecewise quadratic approximations of departure flow rates. We provide families of travel time functions under a variety of assumptions.

Model Formulation

Theorem 2.1 gives rise to the following formulation:

PTT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1+2b_i f_i} \frac{\partial f_i}{\partial x_i} = 0, \qquad \text{for all } i \in I, \tag{2.23}$$

$$f_i(0, t+T_i) = h_i^t(T_i), \qquad \text{for all } i \in I, \tag{2.24}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \qquad \text{for all } i \in I, \tag{2.25}$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \tag{2.26}$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{2.27}$$

$$T_i(0,t) = 0, \qquad \text{for all } i \in I, \tag{2.28}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{2.29}$$

Equation (2.23) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. Moreover, equation (2.24) provides the boundary condition for this partial differential equation.

If we assume that equations (2.23) and (2.24) possess a continuously differentiable solution f_i, then, equations (2.25) and (2.26) determine the density function k_i and the velocity function u_i. The ordinary differential equation (2.27), under boundary condition (2.28), determines travel times on the network's links. Finally, path travel times follow from equation (2.29). Therefore, if we assume that equations (2.23) and (2.24) possess a continuously differentiable solution f_i, the PTT Model, as formulated by equations (2.23)-(2.29), also possesses a solution.

Remark: Note that equations (2.25) and (2.26) simplify the travel time differential equation (2.27) into

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}. \tag{2.30}$$

Existence of Solution to the PTT Model

The following theorem provides an existence result for a continuously differentiable solution of the PTT Model as formulated above.

Theorem 2.4 *(Perakis [68]) The PTT Model as formulated in equations (25)-(31) possesses a solution if and only if the first derivative of the link flow rate function* $h_i^t(T_i)$ *satisfies the following boundedness condition:*

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_iL_i}. \tag{2.31}$$

A General Framework for the Analysis of the PTT Model

The purpose of this subsection is to provide a general framework for the analysis of the PTT Model that reduces the problem to solving a single ordinary differential equation.

Applying this general framework to piecewise linear departure link flow rate functions will result in an easy derivation of link travel times. Furthermore, applying this framework to piecewise quadratic departure link flow rate functions will provide us with a closed form solution of link travel time functions.

As a first step towards establishing the main result of this subsection, we introduce the classical method of characteristics in fluid dynamics. Haberman [34] provides a detailed analysis of this method. Along the characteristic line that passes through $(x_i, t + T_i)$ with slope $\frac{1+2b_if_i}{u_i^{max}}$, the solution $f_i(x_i, t + T_i)$ of equation (2.23) remains constant. If $(0, t + s_i(x_i, t + T_i))$ denotes the point at which the characteristic line intersects the time axis, we have

$$f_i(x_i, t + T_i) = h_i^t(s_i(x_i, t + T_i)). \tag{2.32}$$

Perakis [68] establishes that

$$s_i(x_i, t + T_i) = \frac{T_i u_i^{max} - x_i - 2b_i x_i h_i^t(s_i(x_i, t + T_i))}{u_i^{max}}. \quad (2.33)$$

We introduce two new variables $m_i(.)$ and $g_i(.)$ defined by $m_i(s_i) = \frac{b_i}{u_i^{max}}(h_i^t(s_i) - A_i)$ and $g_i(x_i, t) = T_i(x_i, t) - \frac{1+b_iA_i}{u_i^{max}} x_i$.

Theorem 2.5 *(General framework) The PTT Model reduces to solving the following ordinary differential equation:*

$$\frac{ds_i}{dx_i} = \frac{-m_i(s_i) - \frac{b_iA_i}{u_i^{max}}}{1 + 2x_i m_i'(s_i)}, \quad (2.34)$$

with $s_i(0) = 0$ as an initial condition. The link flow rate functions and the link travel time functions follow from:

$$f_i(x_i, t) = h_i^t(s_i) \quad (2.35)$$

$$T_i(x_i, t) = s_i + \frac{x_i + 2b_i x_i h_i^t(s_i)}{u_i^{max}}. \quad (2.36)$$

Proof: Introducing g_i, m_i and A_i in equations (2.33) and (2.30), we derive the following two relations:

$$s_i = g_i - 2x_i m_i(s_i) - \frac{b_iA_i}{u_i^{max}} x_i, \quad (2.37)$$

$$\frac{dg_i}{dx_i} = m_i(s_i), \quad (2.38)$$

with $s_i(0) = g_i(0) = m_i(0) = 0$.

From equations (2.37) and (2.38), it follows that

$$\frac{ds_i}{dx_i} = \frac{dg_i}{dx_i} - 2m_i(s_i) - 2x_i \frac{ds_i}{dx_i} m_i'(s_i) - \frac{b_iA_i}{u_i^{max}}$$

$$= \quad -m_i(s_i) - 2x_i \frac{ds_i}{dx_i} m_i'(s_i) - \frac{b_i A_i}{u_i^{max}}.$$

Hence, $\frac{ds_i}{dx_i}(1 + 2x_i m_i'(s_i)) = -m_i(s_i) - \frac{b_i A_i}{u_i^{max}}$. Then the results of the theorem follow.

□

Piecewise Linear Departure Link Flow Rate Functions

In this subsection, we apply the general framework to simplify the analysis of the piecewise linear approximation of departure flow rates.

We assume that during a time period $[t, t + \Delta]$, travelers make the approximation that the departure link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is,

$$f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \tag{2.39}$$

Over the time period $[0, T]$, this results into a piecewise linear approximation of link departure flow rates as shown in Figure 2-11 of Subsection 2.5.2.

Remark:

Note that equation (2.31) is a necessary and sufficient condition for the existence of solution of the PTT Model. In this case, the condition becomes:

$$B_i(t) > -\frac{u_i^{max}}{2b_i L_i}. \tag{2.40}$$

We call the system of equations (2.23)-(2.29) and (2.39) the Linear PTT Model. Next, we provide a closed form solution for the Linear PTT Model.

Theorem 2.6 *If (2.40) holds, then:*

(i) The Linear PTT Model possesses a solution,

(ii) The link flow rate functions $f_i(x_i, t+T_i)$ are continuously differentiable,

$$f_i(x_i, t+T_i) = \frac{B_i(t)u_i^{max}T_i - B_i(t)x_i + A_i(t)u_i^{max}}{u_i^{max} + 2b_iB_i(t)x_i}, \tag{2.41}$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by:

$$T_i(x_i, t) = \frac{x_i}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_iB_i(t)x_i}{u_i^{max}})^{\frac{1}{2}} - 1). \tag{2.42}$$

Proof: Since $h_i^t(s_i) = A_i + B_is_i$, it follows that $m_i(s_i) = \frac{b_iB_i}{u_i^{max}}s_i$. Replacing in equation (2.34), we obtain

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_iB_i}{u_i^{max}}s_i - \frac{b_iA_i}{u_i^{max}}}{1 + 2x_i\frac{b_iB_i}{u_i^{max}}}.$$

The above equation can be written as the following separable equation:

$$\frac{ds_i}{s_i + \frac{A_i}{B_i}} = -\frac{\frac{b_iB_i}{u_i^{max}}dx}{1 + 2x_i\frac{b_iB_i}{u_i^{max}}}. \tag{2.43}$$

Integrating both parts (see Bender and Orszag [9]) gives rise to $\frac{s_i + \frac{A_i}{B_i}}{\frac{A_i}{B_i}} = \frac{1}{(1+2\frac{b_iB_i}{u_i^{max}}x_i)^{\frac{1}{2}}}$. Therefore it follows that

$$s_i = \frac{A_i}{B_i}(\frac{1}{(1 + 2\frac{b_iB_i}{u_i^{max}}x_i)^{\frac{1}{2}}} - 1). \tag{2.44}$$

Using equations (2.35) and (2.36), the results of the theorem follow.

□

Corollary 2.1 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_i L_i}. \tag{2.45}$$

Then:

(i) The Linear PTT Model possesses a solution,

(ii) The link travel time functions $T_i(x_i, t)$ simplifies as follows:

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2]. \tag{2.46}$$

Proof:

(i) Note that equation (2.45) implies that equation (2.40) holds. From Theorem 2.6, part (i) follows.

(ii) Equation (2.45) justifies why a second order Taylor expansion of equation (2.42) is reasonable. This leads to equation (2.46).

□

Example

To illustrate our results, we consider a network of four links connecting one O/D pair.

The total length of each of the four links is $L_1 = 4$ miles, $L_2 = 5$ miles, $L_3 = 6$ miles and $L_4 = 7.5$ miles, respectively. The speed limit on each link is $u_1^{max} = 40$ miles/hr, $u_2^{max} = 25$ miles/hr, $u_3^{max} = 25$ miles/hr and $u_4^{max} = 30$ miles/hr, respectively. Finally, the maximum density on each link is $k_1^{max} = 200$ cars per mile, $k_2^{max} = 160$ cars per mile, $k_3^{max} = 192$ cars per mile and $k_4^{max} = 250$ cars per mile, respectively.

We illustrate our results using the four-link network example. We consider various choices for $A_i(t)$ and $B_i(t)$.

1) The traveler estimates his/her travel time on link i by assuming that the departure link flow rate $f_i(0, t+T_i) = f_i(0, t)$, that is the flow rate remains constant during the time period $[t, t+\Delta]$. Then, $A_i(t) = f_i(0, t)$ and $B_i(t) = 0$.
2) The traveler assumes that the departure link flow rate is equal to the average of the departure link flow rate over a previous time interval of length h, that is, $f_i(0, t+T_i) = \frac{1}{h}\int_{t-h}^{t} f_i(0, w)dw$. Then, $A_i(t) = \frac{1}{h}\int_{t-h}^{t} f_i(0, w)dw$ and $B_i(t) = 0$.
3) The traveler uses information prior to t as in 2). The traveler considers the departure link flow rate on link i to be

$$f_i(0, t+T_i) = f_i(0, t) + \frac{1}{h}[f_i(0, t) - f_i(0, t-h)]T_i.$$

For this choice, $A_i(t) = f_i(0, t)$ and $B_i(t) = \frac{1}{h}[f_i(0, t) - f_i(0, t-h)]$.
4) The traveler takes into account the first order information of the departure link flow rate function

$$f_i(0, t+T_i) = f_i(0, t) + \frac{df_i(0, t)}{dt}T_i.$$

For this choice, $A_i(t) = f_i(0, t)$ and $B_i(t) = \frac{df_i(0,t)}{dt}$.

Using the first two choices 1) and 2), Corollary 2.1 gives rise to the following travel times

$$\begin{aligned}
T_1(L_1, t) &= \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000],\\
T_2(L_2, t) &= \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000],\\
T_3(L_3, t) &= \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400],\\
T_4(L_4, t) &= \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500],
\end{aligned}$$

Using the latter two choices 3) and 4), the travel times become

$$T_1(L_1, t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000 - \frac{A_1(t)B_1(t)}{1280000}],$$
$$T_2(L_2, t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000 - \frac{A_2(t)B_2(t)}{80000}],$$
$$T_3(L_3, t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400 - \frac{A_3(t)B_3(t)}{80000}],$$
$$T_4(L_4, t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500 - \frac{A_4(t)B_4(t)}{180000}].$$

Piecewise Quadratic Departure Link Flow Rate Functions

In this subsection, we assume that during a time period $[t, t + \Delta]$, travelers make the approximation that the departure link flow rate for subsequent times $t + T_i$ is quadratic in terms of the travel time T_i. That is,

$$f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i + C_i(t)(T_i)^2. \tag{2.47}$$

Over the time period $[0, T]$, this results into a piecewise quadratic approximation of link departure flow rates as shown in Figure 2-12 in Subsection 2.5.2.

Note that equation (2.31), which is a necessary and sufficient condition for existence of a solution, becomes in this case:

$$B_i(t) + 2C_i(t)(t + \Delta) > -\frac{u_i^{max}}{2b_i L_i}. \tag{2.48}$$

We call the system of equations (2.23)-(2.29) and (2.47) the Quadratic PTT Model. Next, we provide a closed form solution to the Quadratic PTT Model. Note that when the quadratic term is neglected (i.e. $C_i = 0$), we capture the previously studied case of piecewise linear departure link flow rate functions.

Let $\alpha_1 = \frac{b_i B_i(t)}{u_i^{max}}$, $\alpha_2 = \frac{b_i A_i(t)}{u_i^{max}}$ and $\alpha_3 = \frac{4b_i C_i(t)}{u_i^{max}}$.

Theorem 2.7 *Assume that*

$$|B_i(t) + 2C_i(t)(t + \Delta)| << \frac{u_i^{max}}{2b_i L_i}. \tag{2.49}$$

Then, the following holds

(i) The Quadratic PTT Model possesses a solution.

(ii) The link characteristic line functions s_i are continuously differentiable and are given by

$$s_i(x_i, t) = \frac{\alpha_2}{\alpha_1} e^{-\alpha_1 x_i + (2\alpha_1^2 + \alpha_2\alpha_3)\frac{x_i^2}{2}} \int_0^{x_i} e^{\alpha_1 t - (2\alpha_1^2 + \alpha_2\alpha_3)\frac{t^2}{2}} (-\alpha_1 + 2\alpha_1^2 t) dt. \tag{2.50}$$

(iii) The third degree Taylor expansion of the link characteristic line functions s_i becomes

$$s_i(x_i, t) = -\frac{\alpha_2}{\alpha_1}(\alpha_1 x_i - \frac{3\alpha_1^2}{2} x_i^2 + (7\alpha_1^3 + 2\alpha_1\alpha_2\alpha_3)\frac{x_i^3}{6}). \tag{2.51}$$

(iv) The third degree Taylor expansion of the link travel time functions $T_i(x_i, t)$ becomes

$$\begin{aligned} T_i(x_i, t) &= \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}} x_i^2 \\ &+ (\frac{11A_i(t)B_i(t)^2(b_i)^3}{6(u_i^{max})^2} - \frac{4A_i(t)^2C_i(t)(b_i)^3}{3(u_i^{max})^2})x_i^3]. \end{aligned} \tag{2.52}$$

Proof: The analysis involved in this proof is very tedious. For the sake of simplicity and brevity, we only include the most important steps of the analysis.

(i) Note that condition (2.49) implies that condition (2.48) holds. Hence, the result of Theorem 2.4 applies. Therefore, the Quadratic PTT Model possesses a solution.

(ii) Equation (2.34) can be rewritten as

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_i}{u_i^{max}}(B_i s_i + C_i s_i^2) - \frac{b_i A_i}{u_i^{max}}}{1 + 2x_i \frac{b_i}{u_i^{max}}(B_i + 2C_i s_i)}.$$

Condition (2.49) allows us to consider a first order Taylor expansion of the denominator in the above equation. Introducing α_i, $i \in \{1,2,3\}$ as defined above, using a Taylor expansion and rearranging terms leads to the following linear ordinary differential equation:

$$\frac{ds_i}{dx_i} - (-\alpha_1 + (2\alpha_1^2 + \alpha_2\alpha_3)x_i)s_i = -\alpha_2(1 - 2\alpha_1 x_i), \tag{2.53}$$

with $s_i(0) = 0$ as an initial condition.

The integrating term $I(x_i)$ of this equation (see Bender and Orszag [9] for more details) can be written as

$$I(x_i) \;\; = \;\; e^{\int_0^{x_i}(\alpha_1 + (2\alpha_1^2 + \alpha_2\alpha_3)t)dt} = e^{\alpha_1 x_i - (2\alpha_1^2 + \alpha_2\alpha_3)\frac{x_i^2}{2}}.$$

Equation (2.50) then follows.

(iii) Let $N(x_i)$ denote the following function:

$$N(x_i) = e^{-\alpha_1 x_i + (2\alpha_1^2 + \alpha_2\alpha_3)\frac{x_i^2}{2}} \int_0^{x_i} e^{\alpha_1 t - (2\alpha_1^2 + \alpha_2\alpha_3)\frac{t^2}{2}}(-\alpha_1 + 2\alpha_1^2 t)dt. \tag{2.54}$$

Tedious analysis leads to $N(0) = 0$, $N^{(1)}(0) = \alpha_1$, $N^{(2)}(0) = -3\alpha_1^2$ and $N^{(3)}(0) = 7\alpha_1^3 + 2\alpha_1\alpha_2\alpha_3$. From

$$N(x_i) = N(0) + N'(0)x_i + N^{(2)}(0)\frac{x_i^2}{2} + N^{(3)}(0)\frac{x_i^3}{6} + o(x_i^3), \tag{2.55}$$

equation (2.51) follows.

(iii) Equations (2.51) and (2.36), with α_i, $i \in \{1,2,3\}$, lead to equation (2.52).

□

Example

We illustrate our results using the four-link network example. We consider two additional choices for $A_i(t)$, $B_i(t)$ and $C_i(t)$.

5) The traveler considers second-order information prior to t. That is, he/she considers the departure link flow rate on link i to be

$$f_i(0,t+T_i) = f_i(0,t) + \frac{df_i(0,t)}{dt}T_i + \frac{1}{2h}[\frac{df_i(0,t)}{dt} - \frac{df_i(0,t-h)}{dt}]T_i^2.$$

For this choice, $A_i(t) = f_i(0,t)$, $B_i(t) = \frac{df_i(0,t)}{dt}$ and $C_i(t) = \frac{1}{2h}[\frac{df_i(0,t)}{dt} - \frac{df_i(0,t-h)}{dt}]$.

6) The traveler considers second-order information of the departure link flow rate function

$$f_i(0,t+T_i) = f_i(0,t) + \frac{df_i(0,t)}{dt}T_i + \frac{1}{2}\frac{d^2 f_i(0,t)}{dt^2}T_i^2.$$

For this choice, $A_i(t) = f_i(0,t)$, $B_i(t) = \frac{df_i(0,t)}{dt}$ and $C_i(t) = \frac{1}{2}\frac{d^2 f_i(0,t)}{dt^2}$.

Using choices 5) and 6), Theorem 2.7 gives rise to

$$T_1(L_1,t) = \frac{1}{10000}[\frac{1}{8}A_1(t) + 1000 - \frac{A_1(t)B_1(t)}{1280000} + \frac{\frac{2A_1^2(t)C_1(t)}{3} - \frac{7A_1(t)B_1^2(t)}{6} + 3B_1^2(t)C_1(t)}{1280000000}],$$

$$T_2(L_2,t) = \frac{1}{10000}[\frac{1}{2}A_2(t) + 2000 - \frac{A_2(t)B_2(t)}{80000} + \frac{\frac{2A_2^2(t)C_2(t)}{3} - \frac{7A_2(t)B_2^2(t)}{6} + 3B_2^2(t)C_2(t)}{32000000}],$$

$$T_3(L_3,t) = \frac{1}{10000}[\frac{1}{2}A_3(t) + 2400 - \frac{A_3(t)B_3(t)}{80000} + \frac{\frac{2A_3^2(t)C_3(t)}{3} - \frac{7A_3(t)B_3^2(t)}{6} + 3B_3^2(t)C_3(t)}{32000000}],$$

$$T_4(L_4,t) = \frac{1}{10000}[\frac{1}{3}A_4(t) + 2500 - \frac{A_4(t)B_4(t)}{180000} + \frac{\frac{2A_4^2(t)C_4(t)}{3} - \frac{7A_4(t)B_4^2(t)}{6} + 3B_4^2(t)C_4(t)}{90000000}].$$

Equation (2.52) provides us with a general family of travel time functions. Below, we will discuss the relationship between this family of travel time functions and the one obtained by the Linear PTT Model.

Separable ETT Model

In this subsection, we study the ETT Model. We show that the analysis of the ETT Model is more complex than the PTT Model, and propose a different class of travel time functions for piecewise linear approximations of departure flow rates.

Model Formulation

Theorem 2.2 gives rise to the following formulation:

ETT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = 0, \quad \text{for all } i \in I, \tag{2.56}$$

$$f_i(0, t + T_i) = h_i^t(T_i), \quad \text{for all } i \in I, \tag{2.57}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \quad \text{for all } i \in I, \tag{2.58}$$

$$u_i = \frac{f_i}{k_i}, \quad \text{for all } i \in I, \tag{2.59}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \quad \text{for all } i \in I, \tag{2.60}$$

$$T_i(0, t) = 0, \quad \text{for all } i \in I, \tag{2.61}$$

$$T_p(L_p, t) = \textstyle\sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \quad \text{for all } p \in P. \tag{2.62}$$

Equation (2.56) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. Moreover, equation (2.57) provides the boundary condition for this equation.

Assuming that equations (2.56) and (2.57) possess a continuously differentiable solution f_i, equations (2.58) and (2.59) determine the density k_i and the velocity u_i. The ordinary differential equation (2.60) under its boundary condition (2.61) determines travel times on the network's links. Finally, path travel times follow from equation (2.62). Therefore, if we assume that equations (2.56) and (2.57) possess a continuously differentiable solution f_i, the ETT Model, as formulated by equations (2.56)-(2.62), also possesses a solution.

Remark: Replacing equations (2.58) and (2.59) in equation (2.60), leads to the same equation as for the PTT Model, that is

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}. \tag{2.63}$$

Existence of Solution to the ETT Model

The following theorem provides an existence result for a continuously differentiable solution of ETT Model as formulated above.

Theorem 2.8 *The ETT Model as formulated in equations (2.56)-(2.57) possesses a solution if and only if the first derivative of the link flow rate function $h_i^t(T_i)$ satisfies the following boundedness condition:*

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_i L_i}(1 - 2b_i h_i^t(T_i))^2. \tag{2.64}$$

Proof:

This result relies on the classical method of characteristics in fluid dynamics. Along the characteristic line that passes through $(x_i, t + T_i)$ with slope $\frac{1}{(1-2b_i f_i)u_i^{max}}$, the solution $f_i(x_i, t + T_i)$ of equation (2.56) remains constant. Let $(0, t + s_i(x_i, t + T_i))$ be the point at which the characteristic line intersects the time axis. We know that:

$$f_i(x_i, t + T_i) = h_i^t(s_i(x_i, t + T_i)). \tag{2.65}$$

Using the slope information, we obtain that

$$\frac{1}{(1 - 2b_i f_i)u_i^{max}} = \frac{t + T_i - (t + s_i(x_i, t + T_i))}{x_i - 0} = \frac{T_i - s_i(x_i, t + T_i))}{x_i}.$$

It follows that

$$s_i(x_i, t + T_i) = T_i - \frac{x_i}{(1 - 2b_i h_i^t(s_i(x_i, t + T_i)))u_i^{max}}. \tag{2.66}$$

Differentiating the above equation with respect to T_i, and rearranging terms, leads to the following expression of $\frac{\partial s_i(x_i,t+T_i)}{\partial T_i}$,

$$\frac{\partial s_i(x_i, t + T_i)}{\partial T_i} = \frac{1}{2b_i x_i \frac{dh_i^t(s_i(x_i,t+T_i))}{ds_i} u_i^{max}(1 - 2b_i h_i^t(T_i))^2 + 1}.$$

Using the method of characteristics, equations (2.56) and (2.57) possess a continuously differentiable solution f_i if and only if $\frac{\partial s_i(x_i,t+T_i)}{\partial T_i} > 0$. This gives rise to equation (2.64).

Therefore, we conclude that the ETT Model possesses a solution if and only if the derivative of $h_i^t(T_i)$ satisfies the boundedness condition

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_i L_i} 2b_i L_i (1 - 2b_i h_i^t(T_i))^2.$$

□

Piecewise Linear Departure Link Flow Rate Functions

In this subsection, we assume that during a time period $[t, t+\Delta]$, travelers make the approximation that the departure link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i (see Figure 2-11 in Subsection 2.5.2). That is,

$$f_i(0, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \tag{2.67}$$

Note that equation (2.64), which is a necessary and sufficient condition for the existence of a solution, becomes in this case:

$$B_i(t) > -\frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t+\Delta))^2. \tag{2.68}$$

We call the system of equations (2.56)-(2.62) and (2.67) the Linear ETT Model. Next, we provide a closed form solution of the Linear ETT Model.

To make our notation more tractable, we introduce variables $\theta_1 = \frac{b_iB_i(t)}{1-2b_iA_i(t)}$, $\theta_2 = \frac{b_iB_i(t)}{u_i^{max}}$ and $\theta_3 = \frac{1+b_iA_i(t)}{u_i^{max}} - \frac{1}{u_i^{max}(1-2b_iA_i(t))}$.

Theorem 2.9 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t+\Delta))^2. \tag{2.69}$$

The following holds,

(i) The Linear ETT Model possesses a solution,

(ii) The link characteristic line functions s_i are continuously differentiable and can be expressed as a function of the link travel travel time functions, that is,

$$s_i(x_i, t) = \frac{T_iu_i^{max}(1 - 2b_iA_i(t)) - x_i}{u_i^{max}(1 - 2b_iA_i(t) + 2b_iB_i(t)T_i}, \tag{2.70}$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by

$$T_i(x_i,t) = \theta_3(\frac{e^{\theta_2 x_i} - 1}{\theta_2}) + \frac{\theta_1 x_i}{\theta_2 (u_i^{max})^2}, \tag{2.71}$$

(iv) If condition (2.45) holds, the link travel time functions $T_i(x_i, t)$ are

$$T_i(x_i,t) \quad = \quad \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}} x_i^2]. \tag{2.72}$$

Proof: The analysis involved in this proof is quite tedious. For the sake of brevity, we only include the key steps of the analysis.

(i) Note that equation (2.69) implies that equation (2.68) holds. Hence, the result of Theorem 2.8 applies. This implies that the Linear ETT Model possesses a solution.

(ii) Using equation (2.67), equation (2.66) can be rewritten as

$$s_i(x_i, t + T_i) = T_i - \frac{x_i}{u_i^{max}(1 - 2b_i A_i(t) - 2b_i B_i(t) s_i(x_i, t + T_i))}. \tag{2.73}$$

Equation (2.73) leads to a second degree polynomial in terms of s_i. Equation (2.69) justifies a first order Taylor expansion of the solution to this polynomial. This Taylor expansion leads to equation (2.70).

(iii) Using equations (2.67) and (2.70), we derive the following expression for the link flow rate functions f_i

$$f_i(x_i, t + T_i) = \frac{B_i(t) T_i u_i^{max} - B_i(t) x_i + A_i(t) u_i^{max}(1 - 2b_i A_i(t))}{u_i^{max}(1 - 2b_i A_i(t) + 2b_i B_i(t) T_i} \tag{2.74}$$

Replacing in equation (2.63) the link flow rate functions we found in equation (2.74) gives rise to the linear ordinary differential equation

$$\frac{dT_i}{dx_i} - (\frac{\theta_1(1-2b_iA_i(t))}{u_i^{max}})T_i = \frac{1 + b_iA_i(t) - \frac{\theta_1 x_i}{u_i^{max}}}{u_i^{max}}, \tag{2.75}$$

with $T_i(0,t) = 0$ as a boundary condition.

The integrating term $I(x_i)$ of this equation (see Bender and Orszag [9] for more details) can be written as

$$I(x_i) = e^{-\frac{\theta_1(1-2b_iA_i(t))}{u_i^{max}}x_i}.$$

Using the boundary condition $T_i(0,t) = 0$, it follows that

$$T_i(x_i,t) = \frac{1}{I(x_i)} \int_0^{x_i} I(w) \frac{1 + b_iA_i(t) - \frac{\theta_1 w}{u_i^{max}}}{u_i^{max}} dw. \tag{2.76}$$

The calculation of the integral in equation (2.76), using an integration by parts, leads to equation (2.71).

(iv) To make the link travel time function T_i more tractable, we assume that condition (2.45) holds. Condition (2.45) allows us to perform a second order Taylor expansion of equation (2.71), which leads to a simpler form,

$$T_i(x_i,t) = (\theta_3 + \frac{\theta_1}{\theta_2(u_i^{max})^2})x_i + \theta_2\theta_3\frac{x_i^2}{2}.$$

Our definition of θ_i, $i \in \{1,2,3\}$ leads to equation (2.72).

□

Example

Let us illustrate our results using the four-link network example. We consider for $A_i(t)$ and $B_i(t)$ the four choices introduced earlier in the subsection.

Using the first two choices 1) and 2), equation (2.72) gives rise to

$$
\begin{aligned}
T_1(L_1,t) &= \frac{1}{10000}[\frac{1}{8}A_1(t)+1000],\\
T_2(L_2,t) &= \frac{1}{10000}[\frac{1}{2}A_2(t)+2000],\\
T_3(L_3,t) &= \frac{1}{10000}[\frac{1}{2}A_3(t)+2400],\\
T_4(L_4,t) &= \frac{1}{10000}[\frac{1}{3}A_4(t)+2500],
\end{aligned}
$$

while for the latter two choices 3) and 4), it follows that

$$
\begin{aligned}
T_1(L_1,t) &= \frac{1}{10000}[\frac{1}{8}A_1(t)+1000-\frac{A_1(t)B_1(t)}{1280000}],\\
T_2(L_2,t) &= \frac{1}{10000}[\frac{1}{2}A_2(t)+2000-\frac{A_2(t)B_2(t)}{80000}],\\
T_3(L_3,t) &= \frac{1}{10000}[\frac{1}{2}A_3(t)+2400-\frac{A_3(t)B_3(t)}{80000}],\\
T_4(L_4,t) &= \frac{1}{10000}[\frac{1}{3}A_4(t)+2500-\frac{A_4(t)B_4(t)}{180000}].
\end{aligned}
$$

It is not a coincidence that the above results exactly match the results we obtained for the PTT model. Below, we will further clarify this similarity.

Equation (2.71) is an exponential family of travel time functions. In the following subsection, we analyze the relationship between the exponential family of travel time functions from this subsection and the one we obtained through the Linear PTT Model and the Quadratic PTT Model.

Examples and Models' Comparison

In summary, we have so far derived two families of travel time functions. The Linear PTT Model which leads to the polynomial family of travel time functions

$$T_i(x_i, t) = \frac{x_i}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_i B_i(t) x_i}{u_i^{max}})^{\frac{1}{2}} - 1), \tag{2.77}$$

and the Linear ETT Model which leads to the exponential family of travel time functions

$$T_i(x_i, t) = \theta_3(\frac{e^{\theta_2 x_i} - 1}{\theta_2}) + \frac{\theta_1 x_i}{\theta_2 (u_i^{max})^2}, \tag{2.78}$$

where, θ_i, $i \in \{1, 2, 3\}$, was defined earlier in the subsection.

It is very important to note that equations (2.77) and (2.78) coincide when $|B_i(t)| << \frac{u_i^{max}}{2b_i L_i}$ holds. That is, they possess the same second order Taylor expansion

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t) b_i) x_i - \frac{A_i(t) B_i(t) (b_i)^2}{2u_i^{max}} x_i^2]. \tag{2.79}$$

We refer to the travel time function above as the limit of the linear PTT and ETT.

This relationship shows that the assumptions made for both the Linear PTT Model and the Linear ETT Model are indeed reasonable.

Furthermore, the Quadratic PTT Model gives rise to a more complicated expression of link travel time functions. The third degree Taylor expansion leads to

$$\begin{aligned} T_i(x_i, t) \quad &= \quad \frac{1}{u_i^{max}}[(1 + A_i(t) b_i) x_i - \frac{A_i(t) B_i(t) (b_i)^2}{2u_i^{max}} x_i^2 \\ &+ \quad (\frac{11 A_i(t) B_i(t)^2 (b_i)^3}{6(u_i^{max})^2} - \frac{4 A_i(t)^2 C_i(t) (b_i)^3}{3(u_i^{max})^2}) x_i^3]. \end{aligned} \tag{2.80}$$

We observe that if the quadratic term is neglected (i.e. $C_i = 0$), then a second order approximation of equation (2.80) leads to equation (2.79) and, as one would expect, we fall in the case of the Linear PTT Model. Hence, it appears that the assumptions made for the Quadratic PTT Model are also reasonable.

Below, we illustrate these families of travel time functions through numerical examples.

Example 1

We first consider a quadratic profile of a link depature flow rate function during a one hour period. This profile is depicted in Figure 2-1. It corresponds in practice to a rush hour period that starts at time zero and ends an hour later. Its peak is attained after 30 minutes. The departure flow rate function is given by

$$f_i(0,t) = 1600 - 6400.(t - 0.5)^2,$$

where t is expressed in hours, and $f_i(0,t)$ is expressed in vehicles per hour.

We derive a piecewise quadratic approximation of the link departure flow rate as follows

$$\begin{aligned} A_i(t) &= 1600 - 6400.(t - 0.5)^2 \\ B_i(t) &= -12800.(t - 0.5) \\ C_i(t) &= -6400 \\ f_i(0, t + T_i) &= A_i(t) + B_i(t).T_i + C_i(t).T_i^2. \end{aligned}$$

We consider two scenarios. The first scenario corresponds to $u_i^{max} = 25$ miles/hr, $k_i^{max} = 175$ vehicles/mile and $L_i = 8$ miles. In this case, $B_i(t)$ is of the same order of

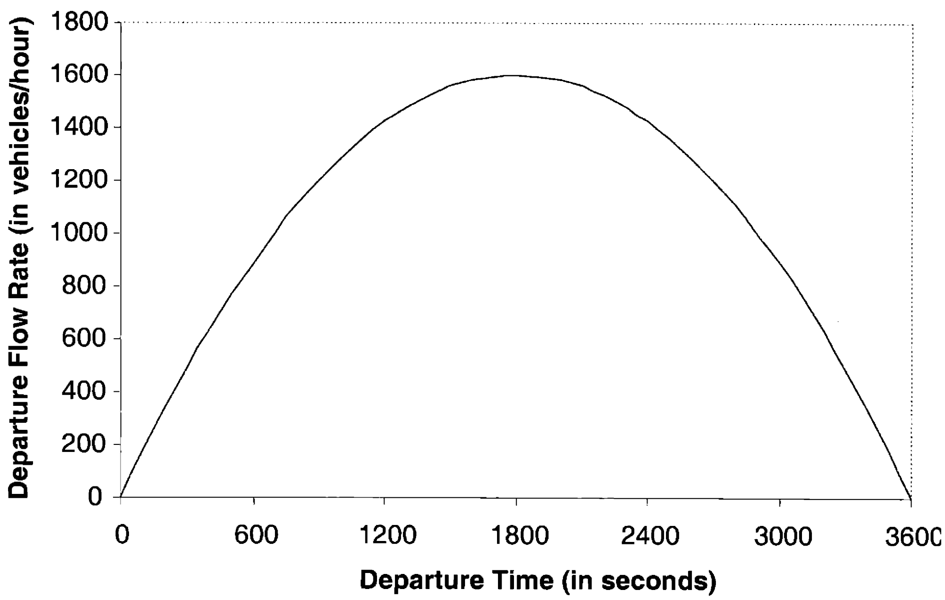

Figure 2-1: Profile of a Link Departure Flow Rate Function During One Hour

magnitude as $\frac{u_i^{max}}{2b_i L_i}$. As a result, we expect the PTT and the ETT models to lead to different travel times.

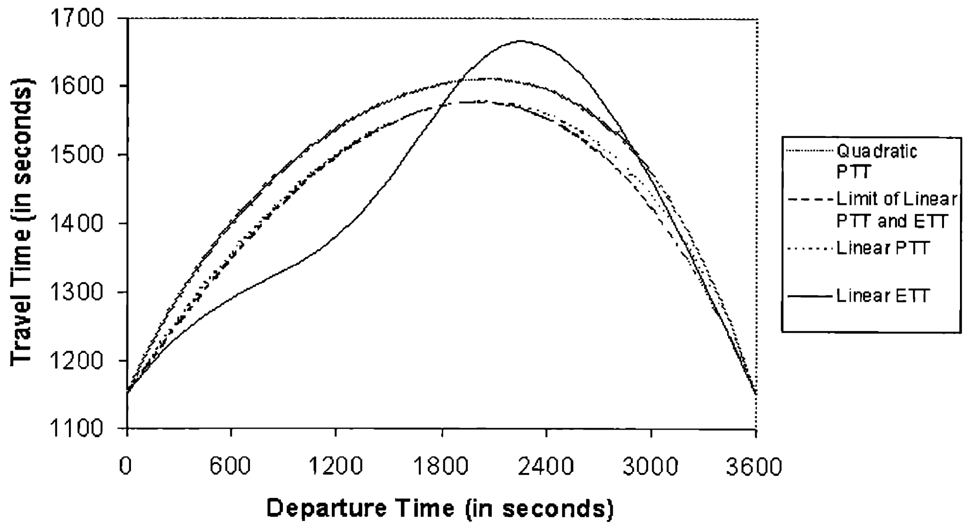

Figure 2-2: Link Traversal Time Functions

We discretize the time interval into intervals of 20 seconds. For each time interval, we compute $A_i(t)$, $B_i(t)$, and $C_i(t)$. We also compute the link traversal times using the expressions of the four travel time functions derived in this section. Figure 2-2 provides a plot of the Linear PTT, the Linear ETT, the limit of the linear PTT and ETT, and the Quadratic PTT travel time functions during the one hour period. Notice that a finer discretization would lead to the same plots. This example leads us to the following observations:

- The travel time functions of the Linear PTT and the limit of the linear PTT and ETT models almost coincide.
- Since the Quadratic PTT Model takes into account the quadratic term $C_i(t)$ in

the departure flow rate function, the travel time of this model is more accurate than the travel time of the Linear PTT Model.

- The Linear PTT, the limit of the linear PTT and ETT, and the Quadratic PTT models lead to a symmetric quadratic shape that is similar to the profile of the link departure flow rate function. However, the Linear ETT model displays an asymmetric behavior.

- For moderate departure flow rates, the Linear ETT Model yields lower travel times than the other three models. However, for high departure flow rates, the Linear ETT Model yileds higher travel times than the other three models. The asymmetric treatment of congestion depicted in Figure 2-2 by the Linear ETT Model seems to correspond to what is experienced in transportation networks. As a result, the Linear ETT Model seems to provide the most realistic travel times.

The second scenario corresponds to $u_i^{max} = 40$ miles/hr, $k_i^{max} = 200$ vehicles/mile and $L_i = 4$ miles. Furthermore, we consider the same discretization intervals of 20 seconds. In this case, $B_i(t)$ is very small compared to $\frac{u_i^{max}}{2b_i L_i}$. As a result, we expect the PTT and the ETT models to yield the same travel time function, that is, the travel time function of the limit of the Linear PTT and ETT. Figure 2-3 illustrates this observation.

Guochun Lin, a Master's student in the Singapore-MIT Alliance Program, analyzed the two scenarios above using two simulation methods. These methods attempt to directly solve Model 1, introduced in Subsection 2.3.1, in the case of linear first-order separable velocity functions. The first simulation method is based on an iterative scheme. The second simulation method utilizes standard techniques used to solve partial differential equations. While he was able to implement the two simulation methods under Scenario 2, only the PDE method was able to provide results under

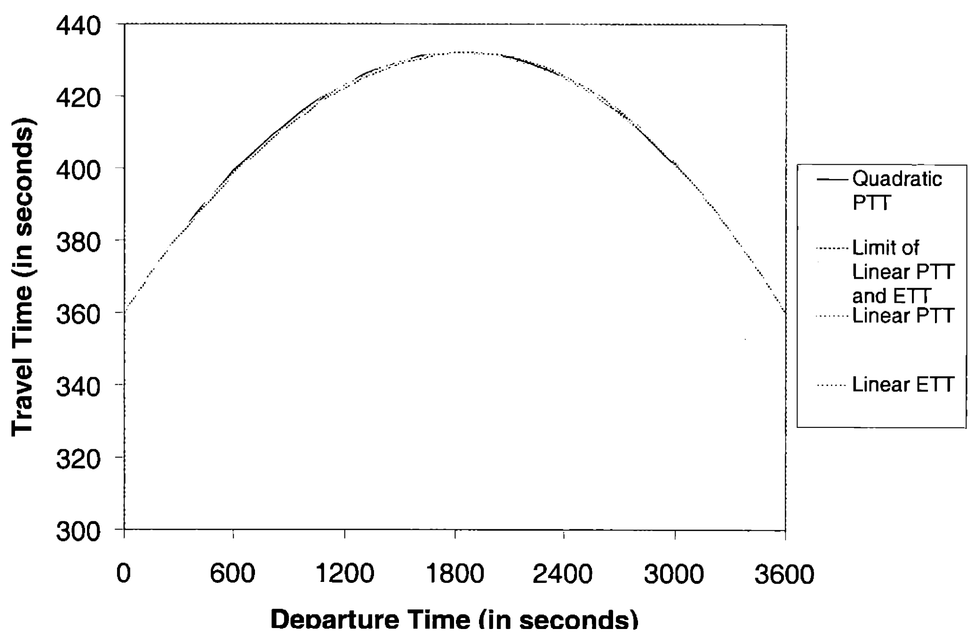

Figure 2-3: Link Traversal Time Functions

Scenario 1. The iterative method was not able to handle the over-saturation appeared in this case.

Figure 2-4 and Figure 2-5 provide a plot of the preliminary results of this work. Note that in both scenarios, the simulation methods provide higher travel times than the PTT and ETT models.

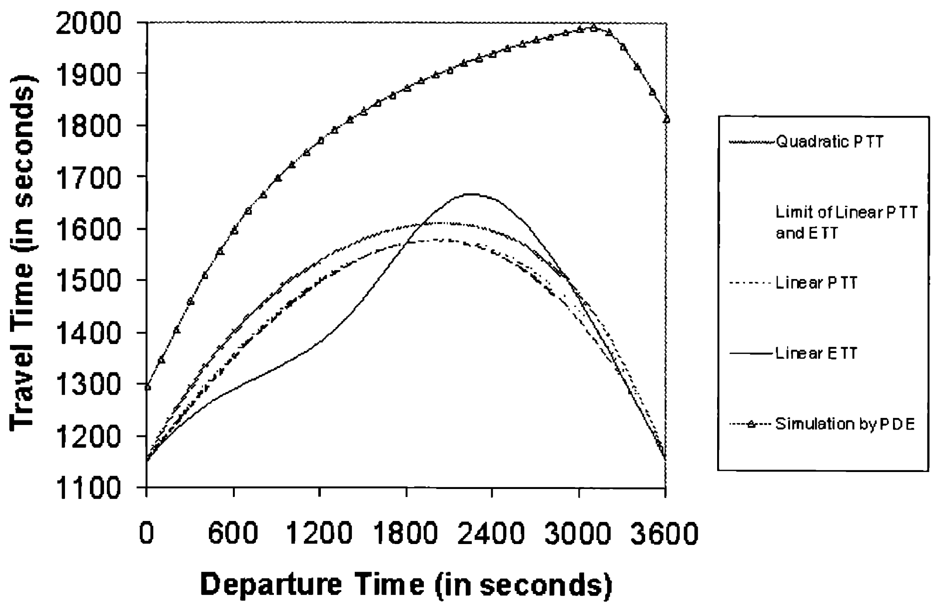

Figure 2-4: Comparison with Simulation Methods under Scenario 1

Example 2

We follow the same approach as in Example 1. We consider a piecewise quadratic profile of a link depature flow rate function during a one hour period. This profile is depicted in Figure 2-6. It corresponds in practice to a rush hour period that starts at time zero and ends an hour later, with two peaks that are attained after 10 minutes

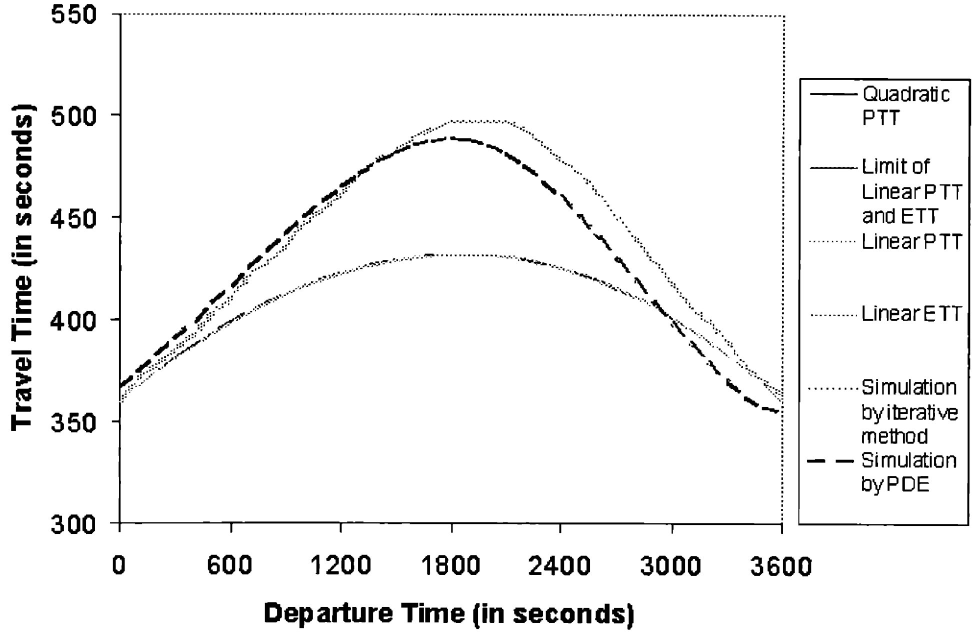

Figure 2-5: Comparison with Simulation Methods under Scenario 2

and 50 minutes respectively. The departure flow rate function is given by

$$\begin{aligned} f_i(0,t) &= 1600 - 57600.(t-\frac{1}{6})^2, \quad \forall t \in [0,\frac{1}{6}) \\ &= 1000 + 5400.(t-0.5)^2, \quad \forall t \in [\frac{1}{6},\frac{5}{6}] \\ &= 1600 - 57600.(t-\frac{5}{6})^2, \quad \forall t \in (\frac{5}{6},1], \end{aligned}$$

where t is expressed in hours, and $f_i(0,t)$ is expressed in vehicles per hour.

It is easy to derive a piecewise quadratic approximation of the link departure flow rate as follows

$$\begin{aligned} A_i(t) &= 1600 - 57600.(t-\frac{1}{6})^2, \quad \forall t \in [0,\frac{1}{6}) \\ &= 1000 + 5400.(t-0.5)^2, \quad \forall t \in [\frac{1}{6},\frac{5}{6}] \end{aligned}$$

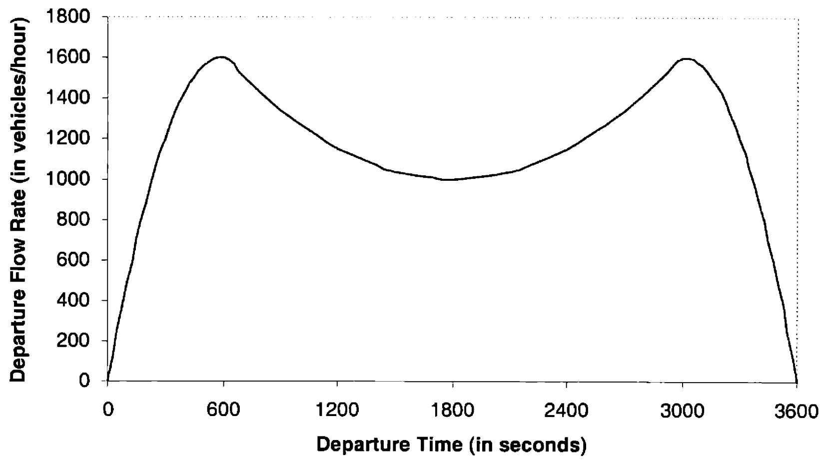

Figure 2-6: Profile of a Link Departure Flow Rate Function During One Hour

$$= \quad 1600 - 57600.(t - \frac{5}{6})^2, \quad \forall t \in (\frac{5}{6}, 1].$$

$$\begin{aligned} B_i(t) &= -115200.(t - \frac{1}{6}), \quad \forall t \in [0, \frac{1}{6}) \\ &= 10800.(t - 0.5), \quad \forall t \in [\frac{1}{6}, \frac{5}{6}] \\ &= -115200.(t - \frac{5}{6}), \quad \forall t \in (\frac{5}{6}, 1]. \end{aligned}$$

$$\begin{aligned} C_i(t) &= -57600, \quad \forall t \in [0, \frac{1}{6}) \\ &= 5400, \quad \forall t \in [\frac{1}{6}, \frac{5}{6}] \\ &= -57600, \quad \forall t \in (\frac{5}{6}, 1]. \end{aligned}$$

It is easy to check that

$$f_i(0, t + T_i) = A_i(t) + B_i(t).T_i + C_i(t).T_i^2.$$

As in Example 1, we consider two scenarios. The first scenario corresponds to $u_i^{max} =$ 30 miles/hr, k_i^{max} = 175 vehicles/mile and L_i = 4 miles. In this case, $B_i(t)$ is of the same order of magnitude as $\frac{u_i^{max}}{2b_i L_i}$. As a result, we expect the PTT and the ETT models to lead to different travel times.

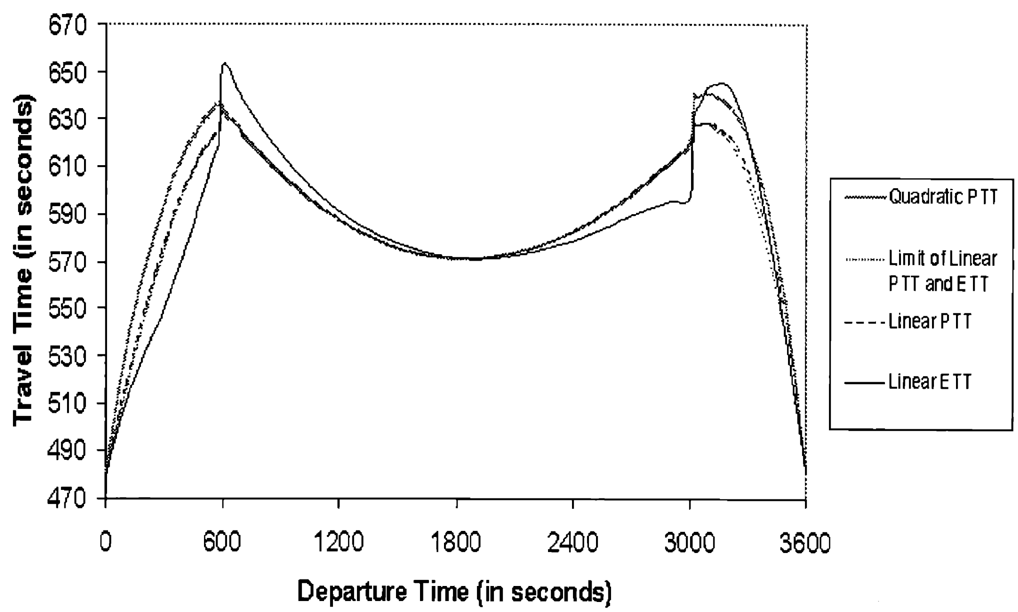

Figure 2-7: Link Traversal Time Functions

We discretize the time interval into intervals of 20 seconds. For each time interval, we compute $A_i(t)$, $B_i(t)$, and $C_i(t)$. We also compute the link traversal times using the expressions of the four travel time functions derived in this section. Figure 2-7 provides a plot of the Linear PTT, the Linear ETT, the limit of the linear PTT and ETT, and the Quadratic PTT travel time functions during the one hour period.

Notice once again that a finer discretization would lead to the same plots.

The observations we made in the first scenario of Example 1 apply here as well. In particular, the travel times of the Quadratic PTT Model are more accurate than the travel times of the Linear PTT Model. Furthermore, the Linear ETT Model seems to provide the most realistic travel times.

The second scenario corresponds to $u_i^{max} = 40$ miles/hr, $k_i^{max} = 200$ vehicles/mile and $L_i = 4$ miles. Furthermore, we consider the same discretization intervals of 20 seconds. In this case, $B_i(t)$ is very small compared to $\frac{u_i^{max}}{2b_i L_i}$. As a result, as in Example 1, we expect the PTT and the ETT models to yield the same travel time function, that is, the travel time function of the limit of the Linear PTT and ETT. Figure 2-8 illustrates this observation.

This concludes our analysis of the separable case. In the following section, we study the non-separable case of this problem.

2.4.2 A Non-Separable Model

In this section, we generalize the Polynomial Travel Time Model (PTT Model) to the case of non-separable velocity functions. We show how the results obtained for the separable case extend to the non-separable case. The proofs are similar to the ones of Subsection 2.4.1.

In order to ease the transition to the non-separable PTT Model, we first consider the case of a two-link network. After that, we extend our results to the more general case of acyclic networks.

Two Links Interaction

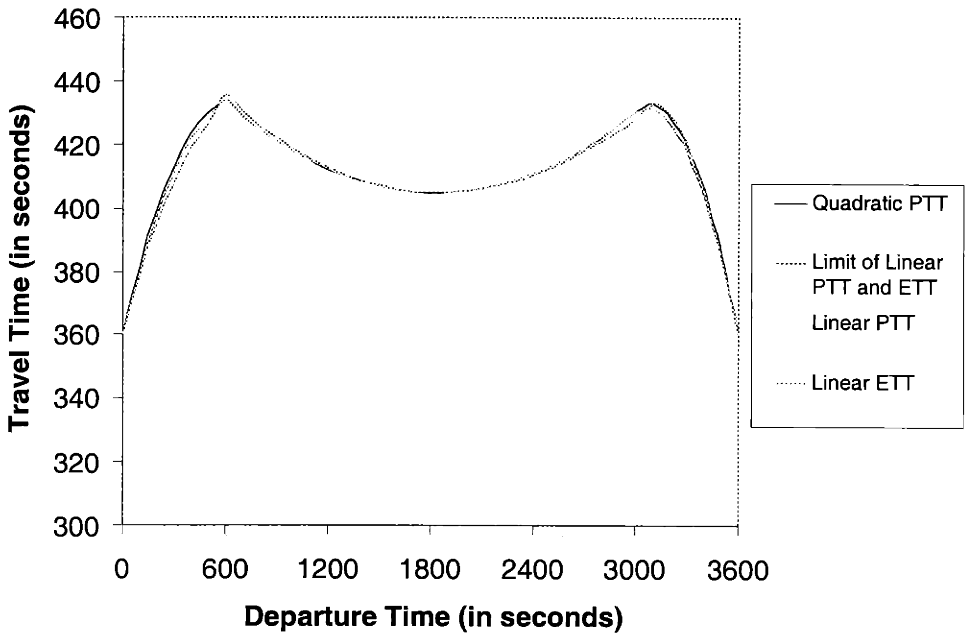

Figure 2-8: Link Traversal Time Functions

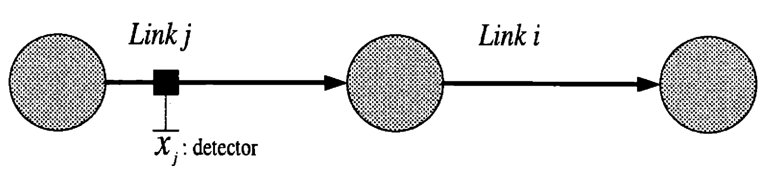

Figure 2-9: A network with two links

In this subsection, we consider the case of two links: link 1 and link 2, as shown in Figure 2-9. We consider the case of non-separable velocity functions. We model the two-link network interaction by considering that the velocity of link 2, at position x_2 and at time t, can be expressed as in Subsection 2.3.1 by

$$u_2 = \widehat{u}(k_2(x_2,t)) = u_2^{max} - b_2(u_2^{max})^2 k_2(x_2,t) + \alpha_{21}(x_2)R_{21}(\overline{x}_1, t-\Delta_{21}), \quad (2.81)$$

where $\alpha_{21}(x_2)$ is the density correlation function between link 2 and link 1 and depends on the position x_2 on link 2; R_{21} is a function of k_1 and ∇k_1; $\overline{x}_1$ is a fixed position of a detector of density on link 1; and Δ_{21} is a propagation time between link 1 and link 2.

For the sake of simplicity, let us consider $R_{21}(.) = k_1(.)$. Moreover, for the sake of simplifying notation, we introduce the term $J_{21} = \frac{\alpha_{21}(x_2)}{u_2^{max}} k_1(\overline{x}_1, t-\Delta_{21})$.

Lemma 2.2 from Subsection 2.3.2, that relates the density on a link to the link flow rate, extends to this case as well. Using a similar proof, we derive

$$k_2 = \frac{f_2}{u_2^{max}(1+J_{21})} + \frac{b_2(f_2 u_2^{max})^2}{(u_2^{max}(1+J_{21}))^3}. \quad (2.82)$$

Furthermore, through similar arguments as in Subsection 2.4.1, we show that the general framework of Theorem 2.5 (see equation (2.34)) leads to the conservation law $\frac{\partial f_2}{\partial x_2} + \frac{1}{u_2^{max}}(\frac{1}{1+J_{21}} + \frac{2b_2 f_2}{(1+J_{21})^3})\frac{\partial f_2}{\partial T_2} = 0$.

Therefore, the Non-Separable PTT Model for link 2 becomes:

Non-Separable Polynomial Travel Time Model

For all $t \in [0,T]$,

$$\frac{\partial f_2}{\partial x_2} + \frac{1}{u_2^{max}}(\frac{1}{1+J_{21}} + \frac{2b_2 f_2}{(1+J_{21})^3})\frac{\partial f_2}{\partial T_2} = 0,$$
$$f_2(0, t+T_2) = h_2^t(T_2),$$

$$k_2 = \frac{f_2}{u_2^{max}(1+J_{21})} + \frac{b_2(f_2 u_2^{max})^2}{(u_2^{max}(1+J_{21}))^3},$$
$$u_2 = \frac{f_2}{k_2},$$
$$\frac{dT_2(x_2,t)}{dx_2} = \frac{1}{u_2},$$
$$T_2(0,t) = 0.$$

Similarly to the separable case, we can express the link flow rate function $f_2(x_2, t+T_2)$ as

$$f_2(x_2, t+T_2) = \frac{B_2(t)u_2^{max}T_2 - \frac{B_2(t)x_2}{1+J_{21}} + A_2(t)u_2^{max}}{u_2^{max} + \frac{2b_2B_2(t)x_2}{(1+J_{21})^3}}, \tag{2.83}$$

and consider a linear ordinary differential equation for determining link travel times

$$\begin{aligned}\frac{dT_2}{dx_2} &- \frac{b_2B_2(t)}{(1+J_{21})^3u_2^{max} + 2b_2x_2B_2(t)}T_2 \\ &- \frac{\frac{b_2x_2B_2(t)}{1+J_{21}} + b_2A_2(t)u_2^{max} + (1+J_{21})^2u_2^{max}}{u_2^{max}((1+J_{21})^3u_2^{max} + 2b_2x_2B_2(t))} = 0.\end{aligned} \tag{2.84}$$

The complexity of equation (2.84) depends on the complexity of the density correlation function $\alpha_{21}(x_2)$ expressed through the term J_{21}. Notice that we can establish similar results as in Subsection 2.4.1 if $\alpha_{21}(x_2)$ is a constant. However, deriving analytical closed form solutions is more complex if $\alpha_{21}(x_2)$ is linear in x_2 and too difficult in other cases. If $\alpha_{21}(x_2)$ is neither constant nor linear, numerical methods seem to be the only approach to solve differential equation (2.84) and determine travel times.

We are now ready to extend our results to acyclic networks.

Acyclic Networks

In this subsection, we consider an acyclic network. The acyclicity assumption will enable us to extend the results of Subsection 2.4.1 to the case of non-separable velocity functions. We model link interactions by assuming that the velocity of link i, at position x_i and at time t, can be expressed as in Subsection 2.3.1 by

$$\widehat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i + \sum_{j \in B(i)} \alpha_{ij}(x_i) R_{ij}(\overline{x}_j, t - \Delta_{ij}), \tag{2.85}$$

where $B(i)$ is the set of predecessors of link i.

A predecessor of a link i is any link that comes before i on a path. It does not restrict to only the immediate parent of a link. Note that since we consider the case of acyclic networks, we can talk of predecessors of a link as shown in Figure 5. Note as well that the results we will establish for the case where we consider the set of predecessors, also apply to the case where we consider the set of successors instead.

For the sake of simplicity, let us consider $R_{ij}(.) = k_j(.)$. Moreover, for the sake of simplifying notation, we introduce $J_i = 1 + \sum_{j \in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x}_j, t - \Delta_{ij})$.

Therefore, the Non-Separable PTT Model becomes:

Non-Separable Polynomial Travel Time Model

For all $t \in [0, T]$,

$$\frac{\partial f_i}{\partial x_i} + \frac{1}{u_i^{max}}\left(\frac{1}{J_i} + \frac{2b_i f_i}{J_i^3}\right)\frac{\partial f_i}{\partial T_i} = 0, \qquad \text{for all } i \in I, \tag{2.86}$$

$$f_i(0, t + T_i) = h_i^t(T_i), \qquad \text{for all } i \in I, \tag{2.87}$$

$$k_i = \frac{f_i}{u_i^{max} J_i} + \frac{b_i f_i^2}{u_i^{max} J_i^3}, \qquad \text{for all } i \in I, \tag{2.88}$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \tag{2.89}$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{2.90}$$

$$T_i(0, t) = 0, \qquad \text{for all } i \in I, \tag{2.91}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \quad \text{for all } p \in P. \tag{2.92}$$

Next, we show how the general framework (see Subsection 2.4.1) for analyzing the Separable PTT Model, extends to the non-separable model.

General Framework for Constant Density Correlation Functions

In this subsection, we assume that the density correlation function $\alpha_{ij}(x_i)$ between link i and link j is a constant function of x_i. In this case, J_i is also a constant function of x_i.

As in Subsection 2.4.1, we introduce two new variables $m_i(.)$ and $g_i(.)$ defined by $m_i(s_i) = \frac{b_i}{u_i^{max}}(h_i^t(s_i) - A_i)$ and $g_i(x_i, t) = T_i(x_i, t) - \frac{1+b_iA_i}{u_i^{max}}x_i$.

Theorem 2.10 *(A General Framework for Constant Density Correlation Functions) If the density correlation functions are constant, the Non-Separable PTT Model reduces to solving the ordinary differential equation:*

$$\frac{ds_i}{dx_i} = \frac{-m_i(s_i) - \frac{b_iA_i}{u_i^{max}}}{J_i^3 + 2x_i m_i'(s_i)}, \tag{2.93}$$

with $s_i(0) = 0$ as an initial condition. The link flow rate functions and the link travel time functions follow from:

$$f_i(x_i, t) = h_i^t(s_i) \tag{2.94}$$

$$T_i(x_i, t) = s_i + x_i \frac{\frac{1}{J_i} + \frac{2b_i h_i^t(s_i)}{J_i^3}}{u_i^{max}}. \tag{2.95}$$

Proof:

In the non-separable case, equations (2.33) and (2.30) become

$$s_i(x_i, t + T_i) = \frac{T_i u_i^{max} - \frac{x_i}{J_i} - \frac{2b_i x_i h_i^t(s_i(x_i, t+T_i))}{J_i^3}}{u_i^{max}},$$

$$\text{and,}\quad \frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i} = \frac{\frac{1}{J_i} + \frac{b_i f_i}{J_i^3}}{u_i^{max}}.$$

After introducing g_i, m_i and A_i, we derive the following two relations:

$$s_i = g_i - \frac{2x_i m_i(s_i)}{J_i^3} - \frac{b_i A_i}{u_i^{max} J_i^3} x_i, \tag{2.96}$$

$$\frac{dg_i}{dx_i} = \frac{m_i(s_i)}{J_i^3}, \tag{2.97}$$

with $s_i(0) = g_i(0) = m_i(0) = 0$.

From equations (2.96) and (2.97), it follows that

$$\begin{aligned} \frac{ds_i}{dx_i} &= \frac{dg_i}{dx_i} - \frac{2m_i(s_i)}{J_i^3} - \frac{2x_i \frac{ds_i}{dx_i} m_i'(s_i)}{J_i^3} - \frac{b_i A_i}{u_i^{max} J_i^3} \\ &= -\frac{m_i(s_i)}{J_i^3} - \frac{2x_i}{J_i^3}\frac{ds_i}{dx_i} m_i'(s_i) - \frac{b_i A_i}{u_i^{max} J_i^3}. \end{aligned}$$

Hence, $\frac{ds_i}{dx_i}(1 + \frac{2x_i m_i'(s_i)}{J_i^3}) = -\frac{m_i(s_i)}{J_i^3} - \frac{b_i A_i}{u_i^{max} J_i^3}$. The results of the theorem follow.

□

We now assume that during a time period $[t, t+\Delta]$, travelers make the approximation that the link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is, $f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i$.

The following theorem is an extension of Theorem 2.6 to the non-separable case with constant density correlation functions.

Theorem 2.11 *If* $B_i > -\frac{u_i^{max} J_i^3}{2b_i L_i}$ *holds, then:*

(i) The Non-Separable Linearized PTT Model possesses a solution,

(ii) The link flow rate functions $f_i(x_i, t + T_i)$ *are continuously differentiable, and we have:*

$$f_i(x_i, t + T_i) = \frac{B_i(t) u_i^{max} T_i - \frac{B_i(t) x_i}{J_i} + A_i(t) u_i^{max}}{u_i^{max} + \frac{2b_i B_i(t) x_i}{J_i^3}},$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by:

$$T_i(x_i,t) = \frac{x_i}{u_i^{max} J_i} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_i B_i(t) x_i}{u_i^{max} J_i^3})^{\frac{1}{2}} - 1).$$

Proof: The proof is fairly similar to the one of Theorem 2.6.

Since $h_i^t(s_i) = A_i + B_i s_i$, it follows that $m_i(s_i) = \frac{b_i B_i}{u_i^{max}} s_i$. Replacing in equation (2.93), we obtain

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_i B_i}{u_i^{max}} s_i - \frac{b_i A_i}{u_i^{max}}}{J_i^3 + 2x_i \frac{b_i B_i}{u_i^{max}}}.$$

The above equation can be written as the following separable equation:

$$\frac{ds_i}{s_i + \frac{A_i}{B_i}} = -\frac{\frac{b_i B_i}{u_i^{max}} dx}{J_i^3 + 2x_i \frac{b_i B_i}{u_i^{max}}}. \tag{2.98}$$

Integrating both parts gives rise to $\frac{s_i + \frac{A_i}{B_i}}{\frac{A_i}{B_i}} = \frac{1}{(1+2\frac{b_i B_i}{u_i^{max} J_i^3} x_i)^{\frac{1}{2}}}$. Therefore it follows that

$$s_i = \frac{A_i}{B_i}(\frac{1}{(1 + 2\frac{b_i B_i}{u_i^{max} J_i^3} x_i)^{\frac{1}{2}}} - 1). \tag{2.99}$$

Using equations (2.94) and (2.95), we easily derive the results of the theorem.

□

Note that when the density correlation functions are set to zero, we have $J_i = 1$. The results of Theorem 2.11 then reduce to the results of Theorem 2.6 introduced in Subsection 2.4.1.

Linear Density Correlation Functions

In this subsection, we consider the more complex case of linear density correlation functions. That is, for every link $j \in B(i)$, $\alpha_{ij}(x_i) = a_{ij} + b_{ij} x_i$. In addition to the acyclicity assumption we imposed on the network, we further assume that the influence of neighboring links only has a first order effect. This translates into $\sum_{j\in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x_j}, t - \Delta_{ij}) << 1$. Therefore, we can make the following first-order approximation

$$J_i^3 \approx 1 + 3 \sum_{j\in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x_j}, t - \Delta_{ij}). \tag{2.100}$$

For every integer n, let $\theta_{in} = n \sum_{j\in B(i)} b_{ij} k_j$ and $\gamma_{in} = u_i^{max} + n \sum_{j\in B(i)} a_{ij} k_j$. The following result provides a linear ordinary differential equation satisfied by link travel time functions T_i for the case of linear density correlation functions.

Theorem 2.12 *If* $B_i > -\frac{u_i^{max} J_i^3}{2b_i L_i}$ *holds, then:*

(i) The Linear PTT Model possesses a solution,

(ii) The link travel time functions $T_i(x_i, t)$ *satisfy*

$$\frac{dT_i}{dx_i} - \frac{b_i B_i(t)}{\gamma_{i3} + (\theta_{i3} + 2b_i B_i(t))x_i} T_i - \frac{\frac{b_i x_i B_i(t)}{J_i} + b_i A_i(t) u_i^{max} + u_i^{max} J_i^2}{u_i^{max}(u_i^{max} J_i^3 + 2b_i x_i B_i(t))} = 0, \tag{2.101}$$

Proof: Equation (2.101) is an easy extension of equation (2.84). We can obtain a complicated closed form solution of equation (2.101). Its derivation is too complicated and involves several integration by parts that we do not include for the sake of simplicity.

□

2.5 Spillback Models

2.5.1 A General Travel Time Model and Two Approximation Models

A General Travel Time Model

In this subsection, we propose a general model for travel time functions that directly accounts for spillback and bottleneck phenomena.

Assumption (A4), introduced in Subsection 2.3.2, provides an approximation of link entrance flow rates by continuously differentiable functions. In this subsection, we consider instead an approximation of the flow rates at the exit of a link. That is, we replace Assumption (A4) by the following:

A5 For all t, the exit link flow rate $f_i(L_i, t + \tau_i)$ can be approximated by $h_i^t(\tau_i)$, a continuously differentiable function of τ_i.

Our analysis is based on a family of first-order velocity functions. We consider the general case of non-separable velocity functions and model link interactions as in Subsection 2.4.2 by considering that the velocity of link i, at position x_i and at time t, can be expressed as a function $\widehat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i + \sum_{j \in B(i)} \alpha_{ij}(x_i) R_{ij}(\overline{x}_j, t - \Delta_{ij})$, where b_i is a constant; $\alpha_{ij}(x_i)$ is the density correlation function between link i and link j and depends on the position x_i on link i; R_{ij} is a function of k_j and ∇k_j; $\overline{x}_j$ is a fixed position of a detector of density on link j; Δ_{ij} is a propagation time between link j and link i; and $B(i)$ denotes a set of links neighboring link i. In Subsection 2.5.2, we consider separable velocity functions (e.g. $\alpha_{ij}(.) = 0$). In Subsection 2.5.3, we revisit the more general case of non-separable velocity functions for acyclic networks.

Below, we provide a general model:

Model 2

For all $t \in [0, T]$, we have:

$$\frac{\partial f_i(x_i,T_i)}{\partial x_i} + \frac{\partial k_i(x_i,T_i)}{\partial T_i} = 0, \qquad \text{for all } i \in I, \tag{2.102}$$

$$f_i(L_i, t+T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \tag{2.103}$$

$$f_i(0,t) \leq C_i^{in}(t), \qquad \text{for all } i \in I, \tag{2.104}$$

$$f_i(x_i,t) = k_i(x_i,t)u_i(x_i,t), \qquad \text{for all } i \in I, \tag{2.105}$$

$$u_i = \widehat{u}_i(k, \nabla k), \qquad \text{for all } i \in I, \tag{2.106}$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{2.107}$$

$$T_i(0,t) = 0, \qquad \text{for all } i \in I, \tag{2.108}$$

$$T_p(L_p,t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip},t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{2.109}$$

Notice in Model 2 that given exit link flow rate functions $f_i(L_i,t)$, $i \in I$, inflow and outflow link capacity functions $C_i^{in}(t)$ and $C_i^{out}(t)$, $i \in I$, and a density velocity relationship $u_i = \widehat{u}_i(k, \nabla k)$, $i \in I$, the Dynamic Travel Time Problem is the problem of determining $f_i(x_i,t)$ $i \in I$, $u_i(x_i,t)$ $i \in I$, $k_i(x_i,t)$ $i \in I$, $T_i(x_i,t)$ $i \in I$, and $T_p(x_p,t)$ $p \in P$ as functions of x_i $i \in I$, x_p $p \in P$ and $t \in [0,T]$.

Modeling Flexibility

As in the case of Model 2 and as we will illustrate next for the SPTT and the SETT models, our models rely on Assumption (A5), that is, that at departure time t, the exit flow rates $f_i(L_i, t+\tau_i)$ on link i of length L_i can be approximated by piecewise continuously differentiable functions $h_i^t(\tau_i)$. This analysis differs from the one in Kachani and Perakis [41] where the same assumption was made on the entrance (i.e. position 0 on link i) link flow rates $f_i(0, t+\tau_i)$. While the two assumptions seem to be similar, the assumption in this subsection enables us to:

- Account directly for spillback and bottleneck phenomena. Spillback of queues is a common phenomenon in transportation networks. Such phenomena might occur in the case of congested traffic (e.g. rush hour) or accidents, where queues form and may backward propagate from a link to its upstream link. This happens whenever the head of a queue reaches the head of a link and the inflow capacity of that link is lower than the flow coming from upstream. The assumption we make on the exit link flow rates $f_i(L_i, t)$ at the tail of link i will enable us to take into account link inflow rate and outflow rate capacities more easily. Our previous model accounts for spillback and bottleneck phenomena only implicitly via approximations.

- Determine travel times not only for drivers who are entering a link but also for drivers who are already in the link. In the latter case, we may need to update our estimates of travel times due to a significant change in traffic conditions. Indeed, once we determine the travel time for a driver entering a link, if spillback occurs while this driver is still in the link, then the travel time might change dramatically. Hence there is a need to re-compute it based on the new traffic conditions. To the best of our knowledge, the current literature of macroscopic analytical dynamic travel time models does not address these issues, and as a result does not allow the computation of travel times for drivers who are already in a link.

Moreover, capacity constraints (2.104) and (2.103) enable us to realistically model the inflow and outflow link capacity rates $C_i^{in}(t)$ and $C_i^{out}(t)$. These capacities are functions of time as they may depend on traffic conditions. Furthermore, the hydrodynamic theory of traffic flow implicitly accounts for the link storage capacity rate f_i^{max}. This storage capacity, also called road capacity (see Haberman [34]), is positive and always bounded from above by $u_i^{max} k_i^{max}$.

To illustrate our modeling flexibility, let us consider the case of a three-link network (as shown in Figure 1) with two paths: $p_1 = (i_1, i_2)$ and $p_2 = (i_1, i_3)$. We assume that during a time period $[t, t + \Delta]$, travelers on link i_1 make the approximation that the exit link flow rate for subsequent times $t + T_{i_1}$ is linear in terms of the travel time T_{i_1}.

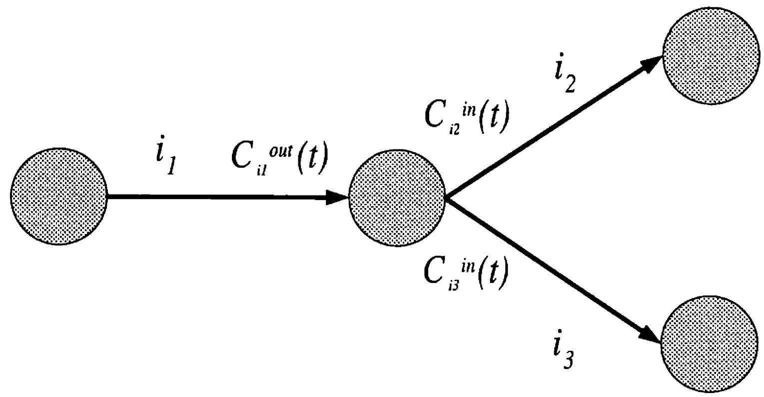

Figure 2-10: A three-link network example

That is,

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = h^t_{i_1}(T_{i_1}) = A_{i_1}(t) + B_{i_1}(t)T_{i_1}.$$

We also assume that during the time period $[t, t + \Delta]$, the inflow and outflow link capacity rates $C_i^{in}(t)$ and $C_i^{out}(t)$ are constant.

The simplest case would be to assume that the exit link flow rate is constant (e.g. $B_{i_1}(t) = 0$). For instance we can have:

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = A_{i_1}(t) = min(C_{i_1}^{out}(t), min(f_{i_2}(0, t), C_{i_2}^{in}(t)) + min(f_{i_3}(0, t), C_{i_3}^{in}(t))).$$

Another situation could correspond to the case where the exit flow rate on link i_1 is below the outflow capacity rate $C_{i_1}^{out}(t)$, the departure flow rate on link i_2 is below the inflow capacity rate $C_{i_2}^{in}(t)$, while the departure flow rate on link i_3 is below the inflow capacity rate $C_{i_3}^{in}(t)$. In this case, drivers in link i_1 might consider that the exit flow rate is linear in the following form:

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = C_{i_3}^{in}(t) + f_{i_2}(0, t) + \frac{df_{i_2}(0, t)}{dt} T_{i_1}.$$

A third situation could correspond to the case where the exit flow rate on link i_1 is below the outflow capacity rate $C_{i_1}^{out}(t)$, and the departure flow rates on links i_2 and i_3 are respectively below the inflow capacity rates $C_{i_2}^{in}(t)$ and $C_{i_3}^{in}(t)$. In this case, drivers in link i_1 might consider that the exit flow rate is linear in the following form:

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = f_{i_1}(L_{i_1}, t) + \frac{df_{i_1}(L_{i_1}, t)}{dt} T_{i_1}.$$

Model 2 is hard to analyze in its current form. For this reason, as in Section 2.4, we consider two simplified models in the case of separable velocity functions (where $\alpha_{ij}(.) = 0$). This will give rise to the Separable Spillback Polynomial Travel Time Model (Separable SPTT Model) and the Separable Spillback Exponential Travel Time Model (Separable SETT Model). In Subsection 2.5.2, we provide a detailed analysis of the two models.

Separable Spillback Polynomial Travel Time Model

Below, we introduce the SPPT Model. To explicitly account for spillback and bottleneck phenomena in this model, we introduce a boundary condition at the end of the link (see Equation (2.111)). This boundary condition will allow us to solve partial differential equation (2.110). Theorem 2.1 gives rise to the following formulation:

SPTT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1+2b_i f_i} \frac{\partial f_i}{\partial x_i} = 0, \qquad \text{for all } i \in I, \qquad (2.110)$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \qquad (2.111)$$

$$f_i(0,t) \leq C_i^{in}(t), \quad \text{for all } i \in I, \tag{2.112}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \quad \text{for all } i \in I, \tag{2.113}$$

$$u_i = \frac{f_i}{k_i}, \quad \text{for all } i \in I, \tag{2.114}$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \quad \text{for all } i \in I, \tag{2.115}$$

$$T_i(0,t) = 0, \quad \text{for all } i \in I, \tag{2.116}$$

$$T_p(L_p,t) = \textstyle\sum_{i\in I} T_i(L_i, t + T_{ip}(L_{ip},t))\delta_{ip}, \quad \text{for all } p \in P. \tag{2.117}$$

As we discussed above, equation (2.111) allows us to explicitly account for spillback and bottleneck phenomena. This analysis differs from the one in Section 2.4 where a similar assumption, $f_i(0, t + T_i) = h_i^t(T_i)$, was made on the entrance link flow rates $f_i(0,t)$. Equation (2.110) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model.

If we assume that equations (2.110) and (2.111) possess a continuously differentiable solution f_i, then, equations (2.113) and (2.114) determine the density function k_i and the velocity function u_i. The ordinary differential equation (2.115), under boundary condition (2.116), determines travel times on the network's links. Finally, path travel times follow from equation (2.117). Therefore, if we assume that equations (2.110) and (2.111) possess a continuously differentiable solution f_i, the SPTT Model, as formulated by equations (2.110)-(2.117), also possesses a solution.

Remark: Note that equations (2.113) and (2.114) simplify the travel time differential equation (2.115) to

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}. \tag{2.118}$$

Separable Spillback Exponential Travel Time Model

Theorem 2.2 gives rise to the following formulation:

SETT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = 0, \qquad \text{for all } i \in I, \tag{2.119}$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \tag{2.120}$$

$$f_i(0, t) \leq C_i^{in}(t), \qquad \text{for all } i \in I, \tag{2.121}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \qquad \text{for all } i \in I, \tag{2.122}$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \tag{2.123}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{2.124}$$

$$T_i(0, t) = 0, \qquad \text{for all } i \in I, \tag{2.125}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{2.126}$$

Equation (2.119) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. Moreover, equation (2.120) provides the boundary condition for this equation. It explicitly accounts for spillback and bottleneck phenomena by assuming that the exit link flow rates $f_i(L_i, t)$ can be approximated by continuously differentiable functions $h_i(t)$.

Assuming that equations (2.119) and (2.120) possess a continuously differentiable solution f_i, equations (2.121) and (2.123) determine the density k_i and the velocity u_i. The ordinary differential equation (2.124) under its boundary condition (2.125) determines travel times on the network's links. Finally, path travel times follow from equation (2.126). Therefore, if we assume that equations (2.119) and (2.120) possess a continuously differentiable solution f_i, the SETT Model, as formulated by equations

(2.119)-(2.126), also possesses a solution.

Remark: Replacing equations (2.121) and (2.123) in equation (2.124), leads to the same equation as for the SPTT Model, that is

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1+b_i f_i}{u_i^{max}}. \tag{2.127}$$

2.5.2 Analysis of Separable Velocity Functions

In this subsection, we study the SPTT and the SETT Models in further details.

In particular, we extensively analyze the SPTT Model for piecewise linear and piecewise quadratic functions $h_i^t(T_i)$. We show how Model 2 reduces in this case to the analysis of a single ordinary differential equation. We provide families of travel time functions.

Furthermore, we analyze the SETT Model by approximating the initial flow rate with piecewise linear functions $h_i^t(T_i)$. Moreover, we show why the analysis of the SETT Model is more complex than the one of the SPTT Model. In addition, we propose a family of travel time functions. Finally, we summarize our results and show how the families of travel time functions we propose relate.

Separable SPTT Model

Below, we analyze the SPTT Model for piecewise linear and piecewise quadratic approximations of exit flow rates. We provide families of travel time functions under a variety of assumptions.

A General Framework for the Analysis of the SPTT Model

The purpose of this subsection is to provide a general framework for the analysis of the SPTT Model that reduces the problem to solving a single ordinary differential equation.

As a first step towards establishing the main result of this subsection, we introduce the classical method of characteristics in fluid dynamics. Haberman [34] provides a detailed analysis of this method. Along the characteristic line that passes through $(x_i, t+T_i)$ with slope $\frac{1+2b_i f_i}{u_i^{max}}$, the solution $f_i(x_i, t+T_i)$ of equation (2.110) remains constant. If $(L_i, t + s_i(x_i, t+T_i))$ denotes a point on this characteristic line, we have

$$f_i(x_i, t+T_i) = h_i^t(s_i(x_i, t+T_i)). \tag{2.128}$$

Note that this definition of the characteristic function $s_i(.)$ differs from the one of Haberman [34] and Perakis [68]. Indeed, in this subsection, to explicitly account for spillback and bottleneck phenomena, the boundary condition is on the exit link flow rates $f_i(L_i, t)$. Therefore, we consider that $(L_i, t + s_i(x_i, t+T_i))$ is a point of the characteristic line. In Perakis [68], the boundary condition was on the departure link flow rates $f_i(0, t)$ instead and as a result, $(0, t + s_i(x_i, t+T_i))$ was considered as a point of the characteristic line. By using similar arguments as in [68], one can easily establish that

$$s_i(x_i, t+T_i) = \frac{T_i u_i^{max} - (x_i - L_i) - 2b_i(x_i - L_i)h_i^t(s_i(x_i, t+T_i))}{u_i^{max}}. \tag{2.129}$$

We introduce two new variables $m_i(.)$ and $g_i(.)$ defined by $m_i(s_i) = \frac{b_i}{u_i^{max}}(h_i^t(s_i) - A_i)$ and $g_i(x_i, t) = T_i(x_i, t) - \frac{1+b_i A_i}{u_i^{max}}(x_i - L_i)$.

Theorem 2.13 *(General framework) The SPTT Model reduces to solving the following ordinary differential equation:*

$$\frac{ds_i}{dx_i} = \frac{-m_i(s_i) - \frac{b_i A_i}{u_i^{max}}}{1 + 2(x_i - L_i)m_i'(s_i)}, \tag{2.130}$$

with $s_i(x_0,t) = \frac{(L_i-x_0)(1+2b_iA_i)}{u_i^{max}-2b_iB_i(L_i-x_0)}$ *as an initial condition, and where* x_0 *denotes the position of a driver on link* i*. The link flow rate functions and the link travel time functions follow from:*

$$f_i(x_i,t) = h_i^t(s_i) \tag{2.131}$$

$$T_i(x_i,t) = s_i + \frac{(x_i-L_i)+2b_i(x_i-L_i)h_i^t(s_i)}{u_i^{max}}. \tag{2.132}$$

Proof: Introducing g_i, m_i and A_i in equations (2.129) and (2.118), we derive the following two relations:

$$s_i = g_i - 2x_im_i(s_i) - \frac{b_iA_i}{u_i^{max}}x_i, \tag{2.133}$$

$$\frac{dg_i}{dx_i} = m_i(s_i). \tag{2.134}$$

From equations (2.133) and (2.134), it follows that

$$\begin{aligned}\frac{ds_i}{dx_i} &= \frac{dg_i}{dx_i} - 2m_i(s_i) - 2x_i\frac{ds_i}{dx_i}m_i'(s_i) - \frac{b_iA_i}{u_i^{max}} \\ &= -m_i(s_i) - 2x_i\frac{ds_i}{dx_i}m_i'(s_i) - \frac{b_iA_i}{u_i^{max}}.\end{aligned}$$

Hence, $\frac{ds_i}{dx_i}(1+2x_im_i'(s_i)) = -m_i(s_i) - \frac{b_iA_i}{u_i^{max}}$. Then the results of the theorem follow.

□

Piecewise Linear Exit Link Flow Rate Functions

In this subsection, we apply the general framework to simplify the analysis of the piecewise linear approximation of exit flow rates.

We assume that during a time period $[t, t+\Delta]$, drivers make the approximation that the exit link flow rate for subsequent times $t+T_i$ is linear in terms of the travel time

T_i. That is,

$$f_i(L_i, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \qquad (2.135)$$

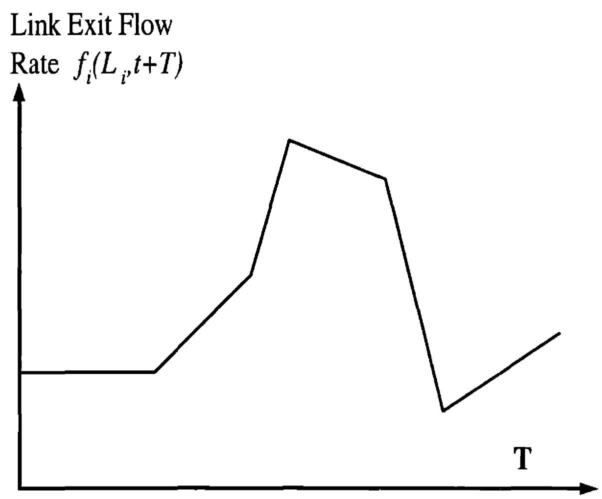

Figure 2-11: A possible profile of approximated exit flow rates

Over the time period $[0, T]$, this results into a piecewise linear approximation of link exit flow rates as shown in Figure 2-11.

Remark:

Note that using the method of characteristics, a necessary and sufficient condition for the existence of solution of the SPTT Model in this case is:

$$B_i(t) < \frac{u_i^{max}}{2b_iL_i}. \qquad (2.136)$$

We call the system of equations (2.110)-(2.117) and (2.135) the Linear SPTT Model. Next, we provide a closed form solution for the Linear SPTT Model.

Theorem 2.14 *If (2.136) holds, then:*

(i) The Linear SPTT Model possesses a solution,

(ii) The link flow rate functions $f_i(x_i, t + T_i)$ are continuously differentiable,

$$f_i(x_i, t+T_i) = \frac{B_i(t)u_i^{max}T_i - B_i(t)(x_i - L_i) + A_i(t)u_i^{max}}{u_i^{max} + 2b_iB_i(t)(x_i - L_i)}, \tag{2.137}$$

(iii) The link travel time $T_i(x_0, x_i, t)$ to reach position x_i of a driver initially at position x_0 at time t is given by:

$$T_i(x_0, x_i, t) = \frac{(x_i - x_0)}{u_i^{max}} + (\frac{L_i - x_0}{u_i^{max}} + \frac{A_i(t)}{B_i(t)})((1 + \frac{x_i - x_0}{\frac{u_i^{max}}{2b_iB_i} - (L_i - x_0)})^{\frac{1}{2}} - 1). \tag{2.138}$$

Proof: Since $h_i^t(s_i) = A_i + B_is_i$, it follows that $m_i(s_i) = \frac{b_iB_i}{u_i^{max}}s_i$. Replacing in equation (2.130), we obtain

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_iB_i}{u_i^{max}}s_i - \frac{b_iA_i}{u_i^{max}}}{1 + 2(x_i - L_i)\frac{b_iB_i}{u_i^{max}}}.$$

The above equation can be written as the following separable equation:

$$\frac{ds_i}{s_i + \frac{A_i}{B_i}} = -\frac{\frac{b_iB_i}{u_i^{max}}dx_i}{1 + 2(x_i - L_i)\frac{b_iB_i}{u_i^{max}}}. \tag{2.139}$$

Integrating both parts using $s_i(x_0, t) = \frac{(L_i - x_0)(1 + 2b_iA_i)}{u_i^{max} - 2b_iB_i(L_i - x_0)}$ (see Bender and Orszag [9]) gives rise to

$$\frac{s_i + \frac{A_i}{B_i}}{\frac{(L_i - x_0) + \frac{A_iu_i^{max}}{B_i}}{u_i^{max} - 2b_iB_i(L_i - x_0)}} = \frac{1 - 2(L_i - x_i)\frac{b_iB_i}{u_i^{max}}}{(1 - 2(L_i - x_i)\frac{b_iB_i}{u_i^{max}})^{\frac{1}{2}}}.$$

Therefore it follows that

$$s_i(x_i,t) = \frac{(L_i - x_0) + \frac{A_i u_i^{max}}{B_i}}{u_i^{max} - 2b_i B_i (L_i - x_0)} \frac{1}{(1 + \frac{x_i - x_0}{\frac{u_i^{max}}{2b_i B_i} - (L_i - x_0)})^{\frac{1}{2}}} - \frac{A_i}{B_i}. \tag{2.140}$$

Using equations (2.131) and (2.132), the results of the theorem follow.

□

Corollary 2.2 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_i L_i}. \tag{2.141}$$

Then:

(i) The Linear SPTT Model possesses a solution,

(ii) The link travel time functions $T_i(x_0, x_i, t)$ simplify as follows:

$$\begin{aligned} T_i(x_0,x_i,t) = & \quad \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) \\ & - \quad \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2. \end{aligned} \tag{2.142}$$

Proof:

(i) Note that equation (2.141) implies that equation (2.136) holds. Hence, part (i) follows.

(ii) Relationship (2.141) allows us to use a second order Taylor expansion of equation (2.138). This leads to equation (2.142).

□

Piecewise Quadratic Exit Link Flow Rate Functions

In this subsection, we assume that during a time period $[t, t+\Delta]$, drivers make the approximation that the exit link flow rate for subsequent times $t + T_i$ is quadratic in terms of the travel time T_i. That is,

$$f_i(L_i, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i + C_i(t)(T_i)^2. \tag{2.143}$$

Over the time period $[0, T]$, this results into a piecewise quadratic approximation of link exit flow rates as shown in Figure 2-12.

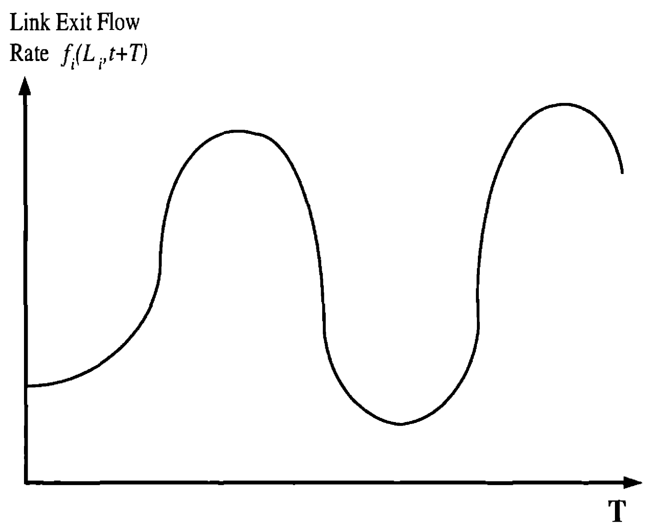

Figure 2-12: A possible profile of approximated exit flow rates

Note that a necessary and sufficient condition for existence of a solution, is in this case:

$$B_i(t) + 2C_i(t)(t+\Delta) < \frac{u_i^{max}}{2b_iL_i}. \tag{2.144}$$

We call the system of equations (2.110)-(2.117) and (2.143) the Quadratic SPTT Model. Next, we provide a closed form solution to the Quadratic SPTT Model. Note that when the quadratic term is neglected (i.e. $C_i = 0$), we capture the previously studied case of piecewise linear exit link flow rate functions.

Theorem 2.15 *Assume that*

$$|B_i(t) + 2C_i(t)(t+\Delta)| << \frac{u_i^{max}}{2b_i L_i}. \tag{2.145}$$

Then, the following holds

(i) The Quadratic SPTT Model possesses a solution.

(ii) The third degree Taylor expansion of the link travel time functions $T_i(x_i, t)$ becomes

$$\begin{aligned} T_i(x_0, x_i, t) = \quad & \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) \\ - \quad & \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2 \\ + \quad & (\frac{11A_i(t)(B_i(t))^2(b_i)^3}{6(u_i^{max})^3} + \frac{11(B_i(t))^3(b_i)^3(L_i - x_0)}{6(u_i^{max})^4} \\ - \quad & \frac{4(A_i(t))^2 C_i(t)(b_i)^3}{3(u_i^{max})^3} - \frac{2A_i(t)B_i(t)C_i(t)(b_i)^3(L_i - x_0)}{(u_i^{max})^4})(x_i - x_0)^3. \end{aligned} \tag{2.146}$$

Proof: The analysis involved in this proof is very tedious and very similar to the one in Subsection 2.4.1. We do not include it for the sake of brevity.

Separable SETT Model

In this subsection, we study the SETT Model. We show that the analysis of the SETT Model is more complex than the SPTT Model, and propose a different class of travel time functions for piecewise linear approximations of exit flow rates.

Piecewise Linear Exit Link Flow Rate Functions

We assume that during a time period $[t, t+\Delta]$, drivers make the approximation that the exit link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time

T_i (see Figure 2-11). That is,

$$f_i(L_i, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \qquad (2.147)$$

A necessary and sufficient condition for the existence of a solution, is in this case:

$$B_i(t) < \frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t+\Delta))^2. \qquad (2.148)$$

We call the system of equations (2.119)-(2.126) and (2.147) the Linear SETT Model. Next, we provide a closed form solution of the Linear SETT Model.

To make our notation more tractable, we introduce variables $\theta_1 = \frac{b_iB_i(t)}{1-2b_iA_i(t)}$, $\theta_2 = \frac{b_iB_i(t)}{u_i^{max}}$ and $\theta_3 = \frac{1+b_iA_i(t)+\frac{\theta_1(L_i-x_0)}{u_i^{max}}}{u_i^{max}} - \frac{1}{u_i^{max}(1-2b_iA_i(t))}$.

Theorem 2.16 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t+\Delta))^2. \qquad (2.149)$$

The following holds,

(i) The Linear SETT Model possesses a solution,

(ii) The link characteristic line functions $s_i(.)$ are continuously differentiable and can be expressed as a function of the link travel time functions, that is,

$$s_i(x_i, t) = \frac{T_iu_i^{max}(1 - 2b_iA_i(t)) - (x_i - L_i)}{u_i^{max}(1 - 2b_iA_i(t) + 2b_iB_i(t)T_i)}, \qquad (2.150)$$

(iii) The link travel time $T_i(x_0, x_i, t)$ to reach position x_i of a driver initially at position x_0 at time t is given by:

$$T_i(x_i, t) = \theta_3(\frac{e^{\theta_2(x_i-x_0)} - 1}{\theta_2}) + \frac{\theta_1(x_i - x_0)}{\theta_2(u_i^{max})^2}, \qquad (2.151)$$

(iv) If condition (2.141) holds, the link travel time function $T_i(x_0, x_i, t)$ is

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) - \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2. \quad (2.152)$$

Proof: The analysis involved in this proof is quite tedious and similar to [41]. For the sake of brevity, we do not include it.

Summary and Models Comparison

In summary, we have so far derived two families of travel time functions. The Linear SPTT Model which leads to the polynomial family of travel time functions

$$T_i(x_0, x_i, t) = \frac{(x_i - x_0)}{u_i^{max}} + (\frac{L_i - x_0}{u_i^{max}} + \frac{A_i(t)}{B_i(t)})((1 + \frac{x_i - x_0}{\frac{u_i^{max}}{2b_i B_i} - (L_i - x_0)})^{\frac{1}{2}} - 1), \quad (2.153)$$

and the Linear SETT Model which leads to the exponential family of travel time functions

$$T_i(x_0, x_i, t) = \theta_3(\frac{e^{\theta_2(x_i - x_0)} - 1}{\theta_2}) + \frac{\theta_1(x_i - x_0)}{\theta_2(u_i^{max})^2}, \quad (2.154)$$

where, θ_i, $i \in \{1, 2, 3\}$ defined above.

It is very important to note that equations (2.153) and (2.154) coincide when $|B_i(t)| << \frac{u_i^{max}}{2b_i L_i}$ holds. That is, they possess the same second order Taylor expansion

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) - \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2. \quad (2.155)$$

This relationship seems to indicate that the assumptions made for both the Linear SPTT Model and the Linear SETT Model are indeed reasonable.

Furthermore, the Quadratic SPTT Model gives rise to a more complicated expression of link travel time functions. The third degree Taylor expansion leads to

$$
\begin{aligned}
T_i(x_0, x_i, t) = & \quad \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) \qquad (2.156) \\
& - \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2 \\
& + (\frac{11A_i(t)(B_i(t))^2(b_i)^3}{6(u_i^{max})^3} + \frac{11(B_i(t))^3(b_i)^3(L_i - x_0)}{6(u_i^{max})^4} \\
& - \frac{4(A_i(t))^2C_i(t)(b_i)^3}{3(u_i^{max})^3} - \frac{2A_i(t)B_i(t)C_i(t)(b_i)^3(L_i - x_0)}{(u_i^{max})^4})(x_i - x_0)^3.
\end{aligned}
$$

We observe that

- If the quadratic term is neglected (i.e. $C_i = 0$), then a second order approximation of equation (2.156) leads to equation (2.155) and, as one would expect, we fall in the case of the Linear SPTT Model. Hence, it appears that the assumptions made for the Quadratic SPTT Model are also reasonable.

- If the constant term is neglected (i.e. $A_i = 0$), equation (2.156) provides us with a non-zero third order degree term.

This concludes our analysis of the separable case. In the following subsection, we study the non-separable case of this problem.

2.5.3 A Non-Separable Model

In this subsection, we generalize the Spillback Polynomial Travel Time Model (SPTT Model) we presented in Subsection 2.5.1 to the case of non-separable velocity functions. We show how the results obtained for the separable case extend to the non-separable case.

In this subsection, we consider an acyclic network. The acyclicity assumption will enable us to extend the results of subsubsection 2.5.2 to the case of non-separable velocity functions. We model link interactions by considering that the velocity of link i, at position x_i and at time t, can be expressed as in Subsection 2.5.1 by

$$\widehat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i + \sum_{j \in B(i)} \alpha_{ij}(x_i) R_{ij}(\overline{x}_j, t - \Delta_{ij}), \qquad (2.157)$$

where $\alpha_{ij}(x_j)$ is the density correlation function between link j and link i and depends on the position x_j on link j; R_{ij} is a function of k_j and ∇k_j; $\overline{x}_j$ is a fixed position of a detector of density on link j; Δ_{ij} is a propagation time between link j and link i; and $B(i)$ is the set of predecessors of link i (See Figure 2-9 in Subsection 2.4.2 for an illustration of the notation in the case of a two-link-network).

For the sake of simplicity, let us consider, as in Subsection 2.4.2, that $R_{ij}(.) = k_j(.)$. Moreover, for the sake of simplifying notation, we similarly introduce $J_i = 1 + \sum_{j \in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x}_j, t - \Delta_{ij})$.

The Non-Separable SPTT Model becomes:

Non-Separable Spillback Polynomial Travel Time Model

For all $t \in [0, T]$,

$$\frac{\partial f_i}{\partial x_i} + \frac{1}{u_i^{max}}\left(\frac{1}{J_i} + \frac{2b_i f_i}{J_i^3}\right)\frac{\partial f_i}{\partial T_i} = 0, \qquad \text{for all } i \in I, \qquad (2.158)$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \qquad (2.159)$$

$$f_i(0, t) \leq C_i^{in}(t), \qquad \text{for all } i \in I, \qquad (2.160)$$

$$k_i = \frac{f_i}{u_i^{max} J_i} + \frac{b_i f_i^2}{u_i^{max} J_i^3}, \qquad \text{for all } i \in I, \qquad (2.161)$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \qquad (2.162)$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{2.163}$$

$$T_i(0,t) = 0, \qquad \text{for all } i \in I, \tag{2.164}$$

$$T_p(L_p,t) = \sum_{i\in I} T_i(L_i, t + T_{ip}(L_{ip},t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{2.165}$$

Note that boundary condition (2.159) allows us to explicitly account for spillback and bottleneck phenomena.

We assume that the density correlation function $\alpha_{ij}(x_i)$ between link i and link j is a constant function of x_i. In this case, J_i is also a constant function of x_i.

We also assume that during a time period $[t, t+\Delta]$, drivers make the approximation that the link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is, $f_i(0, t+T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i$.

The following theorem is an extension of Theorem 4 to the non-separable case with constant density correlation functions.

Theorem 2.17 *If $B_i < \frac{u_i^{max} J_i^3}{2b_i L_i}$ holds, then:*
(i) The Non-Separable Linearized SPTT Model possesses a solution,
(ii) The link flow rate functions $f_i(x_i, t+T_i)$ are continuously differentiable, and we have:

$$f_i(x_i, t+T_i) = \frac{B_i(t)u_i^{max}T_i - \frac{B_i(t)(x_i - L_i)}{J_i} + A_i(t)u_i^{max}}{u_i^{max} + \frac{2b_i B_i(t)(x_i-L_i)}{J_i^3}},$$

(iii) The link travel time $T_i(x_0, x_i, t)$ to reach position x_i of a driver initially at position x_0 at time t is given by:

$$T_i(x_0,x_i,t) = \frac{(x_i - x_0)}{u_i^{max} J_i} + \left(\frac{L_i - x_0}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}\right)\left(\left(1 + \frac{x_i - x_0}{\left(\frac{u_i^{max}}{2b_i B_i} - (L_i - x_0)\right)J_i^3}\right)^{\frac{1}{2}} - 1\right). \tag{2.166}$$

Proof: The proof is similar to the one of Theorem 2.14.

Note that when the density correlation functions are set to zero, it follows that $J_i = 1$. The results of Theorem 2.17 then reduce to the results of Theorem 2.14 from subsubsection 2.5.2.

2.5.4 Connection with the Dynamic Network Equilibrium Problem

As in Perakis [68], we assume in this subsection that the drivers in the network operate under Wardrop's user-optimizing principle. The user-equilibrium property holds if no driver can decrease his/her travel time by making a unilateral decision to change his/her travel path. We further assume that every user has full information over the departure time period $[0, T]$.

The dynamic user-equilibrium problem is then the problem of determining a distribution of the demand rate functions $d_w(t)$ of each O/D pair w in the set of paths P_w, so that for each time $t \in [0, T]$, the user-equilibrium property holds (see Bernstein *et al.* [11] and [10]).

The following three equations express the equilibrium conditions in a dynamic setting

$$T_{1_w}(L_{1_w}, t) = \ldots = T_{m_w}(L_{m_w}, t) \leq T_{m_w+1}(L_{m_w+1}, t) \leq \ldots \leq T_{n_w}(L_{n_w}, t) \quad ,$$

for all $w \in W$ and $t \in [0, T]$, satisfying:

$$F_{1_w}(0,t), \ldots, F_{m_w}(0,t) > 0, F_{m_w+1}(0,t) = \ldots = F_{n_w}(0,t) = 0, \text{ for all } w \in W,$$
$$\textstyle\sum_{p \in P_w} F_p(0,t) = d_w(t), \text{ for all } w \in W,$$

and subject to any model from Sections 2.3, 2.4 and 2.5.

In the new model we introduce, we will not assume that we know the travel time functions in advance. Rather, we will include travel times by incorporating a dynamic travel time model.

For the sake of generality, we present the dynamic user-equilibrium model that corresponds to Model 1 introduced in Subsection 2.3.1.

A User-Equilibrium Model

$$T_{1_w}(L_{1_w}, t) = ... = T_{m_w}(L_{m_w}, t) \leq T_{m_w+1}(L_{m_w+1}, t) \leq ... \leq T_{n_w}(L_{n_w}, t) \quad (2.167)$$

for all $w \in W$ and $t \in [0, T]$, satisfying for all $t \in [0, T]$:

$$F_{1_w}(0, t), ..., F_{m_w}(0, t) > 0, F_{m_w+1}(0, t) = ... = F_{n_w}(0, t) = 0, \text{ for all } w \in W \quad (2.168)$$

$$\textstyle\sum_{p \in P_w} F_p(0, t) = d_w(t), \text{ for all } w \in W, \quad (2.169)$$

$$T_p(L_p, t) = \textstyle\sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \text{ for all } p \in P, \quad (2.170)$$

$$f_i(x_i, t) = \textstyle\sum_{p \in P} F_p(x_i, t)\delta_{ip}, \text{ for all } i \in I, \quad (2.171)$$

$$u_i(x_i, t) = \overline{u}_i(k, \nabla k), \quad (2.172)$$

$$f_i(x_i, t) = k_i(x_i, t)u_i(x_i, t), \text{ for all } i \in I, \quad (2.173)$$

$$\tfrac{\partial f_i(x_i, T_i)}{\partial x_i} + \tfrac{\partial k_i(x_i, T_i)}{\partial T_i} = 0, \text{ for all } i \in I, \quad (2.174)$$

$$\tfrac{dT_i(x_i, t)}{dx_i} = \tfrac{1}{u_i}, \text{ for all } i \in I, \quad (2.175)$$

$$T_i(0, t) = 0, \text{ for all } i \in I. \quad (2.176)$$

The user-equilibrium problem seeks to determine the path flow rates $F_p(0, t)$, $p \in P$, the link flow rates $f_i(0, t)$, $i \in I$, and the densities $k_i(0, t)$, $i \in I$, that satisfy equation (2.167) and the feasibility conditions (2.168) to (2.176).

Therefore, the analysis of the dynamic travel time problems proposed in this subsection is relevant to the analysis of the dynamic user-equilibrium (DUE) problem. In particular, deriving closed-form solutions for travel time functions allows us to solve the DUE problem directly. In Chapter 4, we examine and study the anticipatory route guidance problem, which is an extension of the dynamic user-equilibrium problem.

Chapter 3

A Fluid Model of Dynamic Pricing and Inventory Management for Make-to-Stock Manufacturing Systems

3.1 Introduction and Motivation

3.1.1 Introduction

In this chapter, we propose and study a fluid model of pricing and inventory management for make-to-stock manufacturing systems. Instead of considering a traditional model that relies on how price affects demand, we consider a new model that is based on how price and level of inventory affect the time a unit of product remains in inventory. The model applies to *non-perishable* products.

Our motivation is based on the observation that in inventory systems, a unit of product incurs a delay before being sold. This delay depends on the unit price of the product, prices of competitors, and the level of inventory of this product. Furthermore, we believe that delay data is easy to acquire. This delay is similar to travel times incurred in a transportation network.

The model of this chapter includes joint pricing, production and inventory decisions in a competitive multi-product environment. We apply ideas borrowed from transportation to inventory control and supply chain in order to capture a variety of insightful phenomena that are harder to capture using current models in the literature.

3.1.2 Motivation

In recent years, pricing has become very important in a variety of areas including airline revenue management, inventory control and supply chain management. For instance, in the airline industry, revenue management has demonstrated its potential to dramatically improve revenue. Smith *et al.* discuss in [74] how revenue management enabled American Airlines to increase its yearly revenue by nearly 5%, which led to a $1.4 billion profit improvement over a period of three years. Moreover, the rapid development of information technology, the Internet and E-commerce has had a very strong influence on the development of pricing.

As a result, pricing theory has been extensively studied by researchers from a variety of fields. These fields include, among others, economics (see for example, [79]), marketing (see for example, [50]), telecommunications (see for example, [44], [45], [66]), and revenue management and supply chain management (see for example, [5], [13], [20], [28], [31], [47], [83]). The paper by McGill and Van Ryzin [57], and the references therein, provide a thorough review of revenue management and pricing models.

As the nature of pricing is becoming more dynamic and tactical, companies are faced with the challenge of reacting to and taking advantage of these changes. A study by McKinsey and Company on the cost structure of Fortune 1000 companies in the year 2000 shows that pricing is a more powerful lever than variable cost, fixed cost or sales volume improvements. An improvement of 1% in pricing yields an average of 8.6%

in operating margin improvement (see Figure 3-1). Therefore, companies' ability to survive in this very competitive environment depends on the development of efficient pricing models.

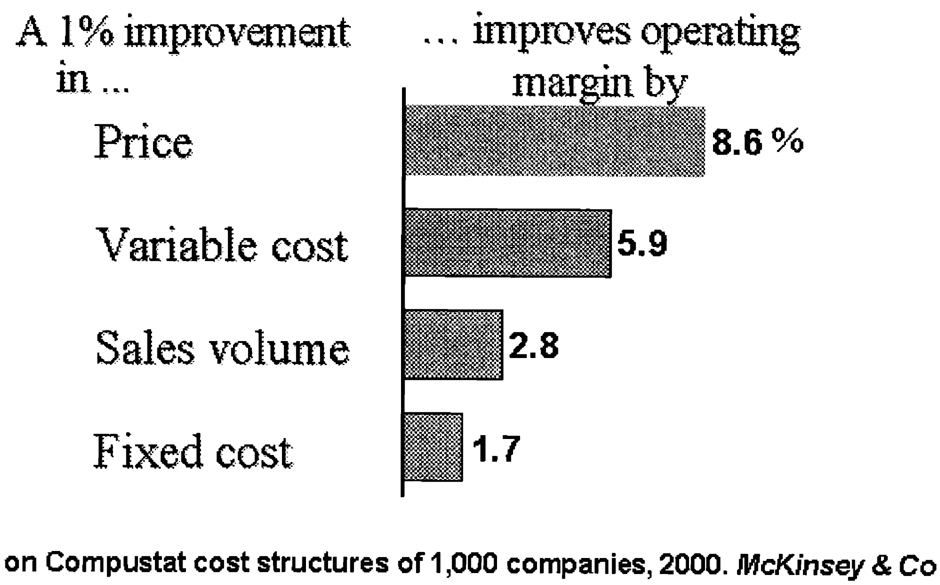

Figure 3-1: Price as a powerful lever to improve profitability

Make-to-stock manufacturing is the standard for a very large number of industries such as retail (see Ha [33] and Wein [77] for more details on make-to-stock models). Furthermore, a motivation for the use of fluid models is that these models have shown to provide good production and inventory policies in a variety of settings, as illustrated in Avram, Bertsimas and Ricard [4], Bertsimas and Paschalidis [12], Harrison [36], and Meyn [59]. Nevertheless, these models do not address the pricing aspect of the problem.

In this chapter, we consider (i) a multi-product and dynamic environment, (ii) a dynamic production capacity shared amongst all products, (iii) the presence of competition, and (iv) non-perishable products. We address the joint pricing, production and inventory problem, without assuming any fixed relationship between price and inventory. Subsequently, for better numerical tractability, we study the model assuming a specific price-inventory relationship.

Instead of considering a traditional model that assumes an a priori relationship between price and demand with fixed parameters, we consider a model that relies on how price and level of inventory affect the time a unit of product remains in inventory. We refer to this time spent in inventory as delay or sojourn time.

The impetus of considering delay data is motivated from: (1) The widespread recording, by barcode readers, of entrance times and exit times of products in inventory systems, which makes this delay data available. (2)The delay data being internal and easily extractable from data warehouses, as opposed to demand data, which is external, and therefore not controlled by the manufacturer. As a result, there are issues of missing data when dealing with demand estimation, which is not as much present when dealing with delay and inventory data. (3) In an environment where price does not vary a lot with time, the estimation of the relationship between price and demand, which is used as an input to the pricing models in the literature, can be quite inaccurate. However, because of the moderate to high variability of inventories with time, the estimation of the relationship between inventory level and sojourn time can be more accurate.

Furthermore, unlike the pricing models described above, we consider a continuous-time formulation of the problem. We provide insightful analytical properties of this model. In addition, in an effort to provide a numerical example and to establish a link between tradititional demands models and the delay model of this paper, we consider a discretized version of the model.

The structure of the chapter is as follows. In Section 3.2, we provide the notation and some definitions. In Section 3.3, we formulate the Dynamic Pricing Model as a continuous-time nonlinear optimization problem. In Section 3.4, we present a solution algorithm for a discretized version of the model, test it on a small case example, and report on the computational results. In Section 3.5, we consider the general Dynamic Pricing Model. In particular, we study the analytical properties of its feasible region, and establish, under weak assumptions, the existence of a pricing/production/inventory control policy that maximizes the profit of the company under study over the feasible region. Finally, in Section 3.6, we provide conclusions and discuss future steps.

3.2 Notation and Definitions

In this section, we present the notation and some definitions that we use throughout the chapter.

3.2.1 Notation

In this chapter, we study a multi-product inventory system that we represent conceptually by a directed network with two nodes O and D, and n links joining these two nodes. Node O represents the arrival of a product to the warehouse and node D represents the delivery of this product to the customer. Each link joining O and D corresponds to a distinct product that the company is selling and is indexed by i, $i \in \{1, ..., n\}$. We assume that the company under study is a Stackelberg leader, and as a result is a price setter. Therefore, competitors' prices are functions of the price of the company under study. These functions can be estimated in practice using regression on the competitors' prices and the prices of the company under study, as illustrated in Subsection 3.3.2. Below, we describe the inputs and the outputs of the Dynamic Pricing Model. Figure 3-2 provides a network illustration of the notation introduced below.

Inputs of the Dynamic Pricing Model

Product variables:

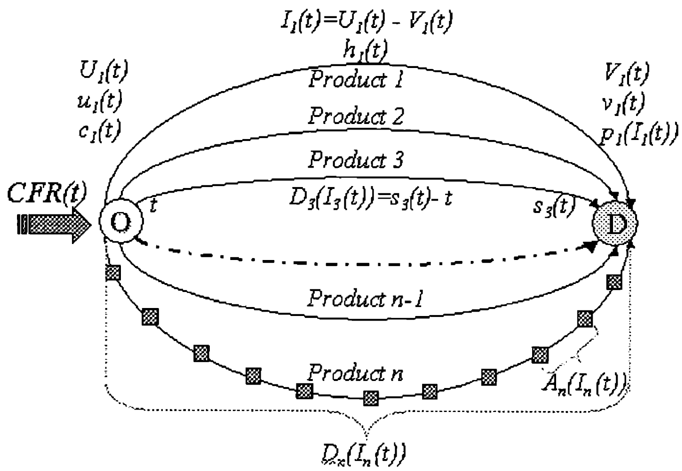

Figure 3-2: Network representation of the multi-product inventory system

$CFR(t)$	=	Shared production capacity rate at time t;
$p_i^c(p_i(.))$	=	$(p_{i,j}^c(p_i(.))_{j\in\{1,\ldots,J(i)\}})$, vector of price functions of companies competing on product i;
$D_i(I_i) = T_i(I_i, p_i, p_i^c)$	:	product sojourn time function, that is the total time a newly produced unit of product i, spends in the inventory system, given an inventory I_i, a unit price $p_i(I_i)$, and a set of competitors' price functions $p_i^c(.)$;
$A_i(I_i)$	:	average product delay function, that is the average time needed to sell a newly produced unit of product i (i.e. $A_i(I_i) = \frac{D_i(I_i)}{I_i}$);

B_{1i} : a lower bound on the derivative $D_i'(.)$ of the product sojourn time function $D_i(.)$;

B_{2i} : an upper bound on the derivative $D_i'(.)$ of the product sojourn time function $D_i(.)$;

$c_i(t)$: production cost of product i at time t;

$h_i(t)$: inventory holding cost of product i at time t.

Time variables:

t : index for continuous time;

$[0, T]$: production period. After time T, the company under study ceases producing.

Outputs of the Dynamic Pricing Model

Product variables:

$U_i(t)$: cumulative production flow of product i during interval $[0, t]$;

$u_i(t)$: production flow rate of product i at time t;

$V_i(t)$: cumulative sales flow of product i during interval $[0, t]$;

$v_i(t)$: sales flow rate of product i at time t;

$I_i(t)$: inventory (number of units of product) i at time t;

$p_i(I_i(t))$: sales price of one unit of product i given an inventory $I_i(t)$;

$s_i(t)$: exit time of a production flow of product type i entering at time t $(s_i(t) = t + D_i(I_i(t)))$.

Time variables:

$[0, T_\infty]$: analysis period. It is the interval of time from when the first unit of product is produced to the first instant all products have been sold.

Notice that the control variables are the production flow rates $u_i(.)$ and the unit price functions $p_i(I_i)$.

3.2.2 Definitions

The following definitions express different types of First In First Out (FIFO) properties. The FIFO property will play a key role in the analysis of our model in Section 3.5.

Definition 1 *A product verifies the FIFO property if and only if:*

$$\forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 \leq t_2, \text{ then: } s_i(t_1) \leq s_i(t_2). \tag{3.1}$$

The above property expresses that a newly produced unit of product cannot be sold before its predecessors. Similarly, a product verifies the Strict FIFO property if and only if the product exit time function is striclty increasing.

Definition 2 *A product verifies the strong FIFO property if and only if:*

$$\exists\, \theta > 0 \text{ such that } \forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 < t_2, \text{ then: } s_i(t_2) - s_i(t_1) \geq \theta (t_2 - t_1). \tag{3.2}$$

3.3 Formulation of the Dynamic Pricing Model

3.3.1 Modeling Assumptions

Before formulating the model, we describe the setting and the assumptions.

We consider a competitive setting where:

A1) The company under study is a Stackelberg leader (a monopoly is a special case of a Stackelberg leader).

A2) There are multiple products. These products are non-perishable.

A3) The total production capacity rate is bounded by a non-negative capacity flow rate function $CFR(.)$.

A4) There is no substitution between products.

A5) The company under study faces holding costs but no setup costs.

A6) The demand is deterministic.

A7) The unit price $p_i(.)$ is a function of the inventory I_i.

Assumption A7 allows us to consider a variety of models for the unit price $p_i(.)$ as a function of the inventory. Examples of such models include linear functions of the type $p_i(I_i) = p_i^{max} - \frac{p_i^{max} - p_i^{min}}{C_i} I_i$ as well as nonlinear functions of the type $p_i(I_i) = \frac{p_i^{max}}{(\frac{p_i^{max}}{p_i^{min}} - 1)\frac{I_i}{C_i} + 1}$, where C_i denotes the inventory capacity, p_i^{max} the maximum allowable price, and p_i^{min} the minimum allowable price. Notice that the unit price function $p_i(I_i(t))$ depends on time only through the time-dependence of the inventory $I_i(t)$. Furthermore, the examples we state consider the case where the unit price decreases as inventory increases.

We consider a sojourn time function $D_i(I_i(t)) = T_i(I_i(t), p_i(I_i(t)), p_i^c(p_i(I_i(t))))$ that represents the total time it takes to sell, at time t, a newly produced unit of product i, given a level of inventory $I_i(t)$, a unit price $p_i(I_i(t))$ and a set of competitors' prices $p_i^c(p_i(I_i(t)))$. Notice that the product sojourn time function $D_i(I_i(t))$ resembles the time to traverse a link in a transportation network.

3.3.2 Estimation of Sojourn Times in Practice

A few companies such as *Amazon*.com utilize the sojourn time information to control their inventory levels and adjust their pricing policies. A key motivation for introducing sojourn times in the model of this chapter is the availability of sojourn time information in almost every company's data warehouse. Indeed, as a unit of product enters the inventory system, a barcode reader records its entrance time. When this unit is sold, a barcode reader records its exit time. The lag between the entrance time and the exit time is the sojourn time.

Below, we describe how to estimate the sojourn times $D_i(I_i)$ in practice:

- Extract entrance times t_i and exit times $s_i(t_i)$ of units of product i from the data warehouse and record sojourn times $\widehat{D}_i(t_i) = s_i(t_i) - t_i$.

- Record the inventory levels $I_i(t_i)$ and the unit prices $p_i(t_i)$ at entrance times t_i.

- Fit the triplets $(I_i(t_i)\ ,\ p_i(t_i)\ ,\ \widehat{D}_i(t_i)\ )$ into a parametric function $\overline{D}_i(I_i(t_i), p_i(t_i))$.

- Assume a parametric shape for the unit price function $p_i(I_i)$ and plug it in $\overline{D}_i(I_i(t_i), p_i(t_i))$ to derive the sojourn time function $D_i(I_i)$.

Notice that since the vector of competitors' price functions $p_i^c(p_i(.))$ is assumed to be a function of the unit price function $p_i(.)$ of the company under study, it follows that the function $\overline{D}_i(I_i(t_i), p_i(t_i))$ takes into account the effect of competition.

Finally, notice that the estimation procedure outlined above is easy to implement. The parameters of the sojourn time functions $D_i(I_i)$ can be recalibrated regularly to account for changes in customer behavior and in competitors' pricing policies.

3.3.3 Model Formulation

We are now ready to propose a continuous-time analytical model for the dynamic pricing problem. We take a fluid dynamics approach by expressing link dynamics, flow conservation, flow propagation and boundary constraints. This formulation resembles the formulation of the Dynamic Network Loading (DNL) model used in the context of the dynamic traffic equilibrium problem (see Friesz *et al.* [30], Wu *et al.* [81], Xu *et al.* [82], and Kachani [39] for more details).

Link dynamics equations

The link dynamics equations express the relationship between the flow variables of a link. That is, the change in inventory at time t is the difference between the production and the sales flow rates:

$$\frac{dI_i(t)}{dt} = u_i(t) - v_i(t), \qquad \forall i \in \{1, ..., n\}. \tag{3.3}$$

Flow propagation equations

The flow propagation equations below describe the flow progression over time. Note that a production flow entering link i at time t will be sold at time $s_i(t) = t + D_i(I_i(t))$. Therefore, by time t, the cumulative sales flow of link i should be equal to the integral of all production inflow rates which would have entered link i at some earlier time ω and would have been sold by time t. This relationship is expressed through the following equation:

$$V_i(t) = \int_{\omega \in W} u_i(\omega) d\omega, \quad \forall i \in \{1, ..., n\}, \quad \text{where } W = \{\omega : s_i(\omega) \leq t\}. \tag{3.4}$$

If the product exit time functions $s_i(.)$ are continuous and satisfy the strict First-In-First-Out (FIFO) property, then the flow propagation equations (3.4) can be equivalently rewritten as

$$V_i(t) = \int_0^{s_i^{-1}(t)} u_i(\omega) d\omega, \qquad \forall i \in \{1, ..., n\}. \tag{3.5}$$

Notice that $s_i^{-1}(t)$ is the time at which a unit of product i needs to be produced so that it is sold at time t. Furthermore, under the strict FIFO condition, a unit of product i, entering the queue at time t, will be sold only after the units of product i, that entered the queue before it, are all sold. In mathematical terms, this is equivalent to the product exit time functions $s_i(.)$ being strictly increasing. As a result, defining the production time $s_i^{-1}(t)$ makes sense.

Boundary equations

Since we assume that the network is empty at $t = 0$, we impose the following boundary conditions

$$U_i(0) = 0, \quad V_i(0) = 0, \quad I_i(0) = 0, \qquad \forall i \in \{1, ..., n\}. \tag{3.6}$$

Note that it is not necessary to assume that $I_i(0) = 0$. Instead, we could assume that $I_i(0) = I_{i0} > 0$. However, we consider zero-level inventories at $t = 0$ for simplicity of notation.

Capacity constraint

We assume that at each time t, the total production flow rate is no more than the total capacity flow rate $CFR(t)$. This can be expressed as:

$$\sum_{i=1}^{n} u_i(t) \leq CFR(t). \tag{3.7}$$

Non-negativity conditions

We further assume that the production flow rate functions $u_i(.)$ are non-negative:

$$u_i(.) \geq 0, \quad \forall i \in \{1, ..., n\}. \tag{3.8}$$

Objective function

The objective of the company is to maximize its profits. Profits are obtained by substracting production costs and inventory costs from sales. As a result, the objective function can be expressed as the sum over all products of the difference between the revenue of sales and the cost of both production and inventory:

$$\sum_{i=1}^{n}\int_{0}^{T_\infty}[p_i(I_i(t))v_i(t) - c_i(t)u_i(t) - h_i(t)I_i(t)]dt. \tag{3.9}$$

In summary, the continuous-time Dynamic Pricing Model (DPM) is formulated as maximizing objective function (3.9) subject to constraints (3.3)-(3.8). In general, the DPM Model is a continuous-time non-linear optimization problem. The non-linearity of the model comes from the unit price as a function of the inventory, as well as the integral equation (3.4). In this formulation, the known variables are the product sojourn time functions $D_i(.)$, the production and inventory costs $c_i(.)$ and $h_i(.)$, and the total capacity flow rate function $CFR(.)$. The unknown variables we wish to determine are $u_i(t)$, $v_i(t)$, $U_i(t)$, $V_i(t)$, $I_i(t)$ and $p_i(I_i)$. Notice that integral equation (3.4), which connects the production to the sales schedules through the delays incurred in the system due to price and inventory, makes this problem hard to solve.

Dynamic Pricing Model:

$$\begin{aligned}
\text{Maximize } \sum_{i=1}^{n}\int_{0}^{T_\infty}[p_i(I_i(t))v_i(t) \quad &- \quad c_i(t)u_i(t) - h_i(t)I_i(t)]dt \\
s.t. \quad \frac{dI_i(t)}{dt} = u_i(t) \quad &- \quad v_i(t), \quad \forall i \in \{1, ..., n\} \\
V_i(t) = \int_{\omega\in W} u_i(\omega)d\omega, \quad \forall i \in \{1, ..., n\} \quad &, \quad \text{where} \quad W = \{\omega : s_i(\omega) \le t\} \\
U_i(0) = 0, \quad V_i(0) = 0 \quad &, \quad I_i(0) = 0, \quad \forall i \in \{1, ..., n\} \\
\sum_{i=1}^{n} u_i(t) \quad &\le \quad CFR(t), \\
u_i(.) \ge 0, \quad \forall i \in \{1, ..., n\} \quad &, \quad CFR(.) \ge 0.
\end{aligned}$$

Notice that the model is general enough to account for the case where the FIFO property, defined above, is not necessarily verified (notice that Equation (3.4) does not assume that the FIFO property holds). Below, we investigate when the FIFO property holds. We examine conditions on the product sojourn time functions $D_i(.)$ and on the production flow rates $u_i(.)$. When the FIFO property holds, the model becomes more tractable.

In the remainder of the chapter, we will denote by $F(DPM)$ the feasible region of the DPM Model. In Section 3.5, we study the analytical properties of this region.

In Section 3.4, we examine the solution of discretized version of the Dynamic Pricing Model. In Section 3.5, we illustrate how our results extend to the general case.

3.4 Solution Algorithm and Computational Results

In this section, we consider a discretized version of the Dynamic Pricing Model. This discretization allows us to study its solution. In particular, we propose and analyze a relaxation algorithm, illustrate this algorithm on an example, and report some preliminary computational results.

In the following two subsections, we do not make a direct assumption on the shape of the sojourn time function. Instead, we provide a connection with traditional demand models.

3.4.1 A Pricing Model

We consider a special case of unit price and demand arrival rate functions. We first need to define the following primitive quantities that are the essential data for our model. Let $\overline{p}_i^{min}$ denote a minimum allowable reference price. Let $\overline{\lambda}_i^{max}$ denote its corresponding demand arrival rate, and $\overline{p}_i^{max}$ denote its corresponding reservation price, that is, the minimum price for which there is no demand for product i. These three quantities are input data in the model.

Moreover, in addition to Assumptions A1-A7 that we considered in Section 3.3, we assume the following, for every product i:

A8) The unit price function $p_i(I_i)$ is linear in terms of the inventory level I_i (see Figure 3-3). As we discussed in Section 3.3, we assume that

$$p_i(I_i) = p_i^{max} - \frac{p_i^{max} - p_i^{min}}{C_i} I_i, \tag{3.10}$$

where C_i denotes the storage capacity, p_i^{min} denotes the minimum allowable price, and p_i^{max} denotes the reservation price. Notice that this function is decreasing in terms of inventory.

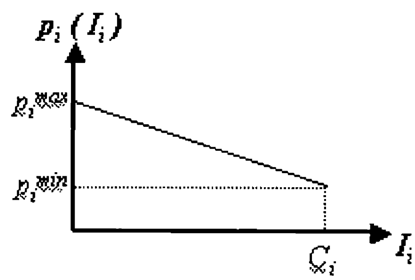

Figure 3-3: Linear unit price function $p_i(I_i)$

A9) The reservation price p_i^{max} is a function of the minimum allowable price p_i^{min}. To illustrate this assumption, we consider the example of two retail stores competing on a product i. If the minimum price of Store 1 is lower than that of Store 2 (that is, $p_{i,1}^{min} < p_{i,2}^{min}$), then Store 1 will be perceived by customers as cheaper. As a result, Store 1 can from time to time take advantage of this perception by having slightly higher prices than Store 2. This observation illustrates that the reservation price of Store 1 is higher than that of Store 2 (that is $p_{i,1}^{max} > p_{i,2}^{max}$). Therefore, p_i^{max} is a decreasing function of p_i^{min}. However, due to customers' sensitivity to high prices, the difference (in absolute value) between the reservation prices of the two stores needs to be smaller than the difference (in absolute values) between their minimum allowable prices. This can be achieved by assuming that the difference between reservation prices is a concave function of the difference between minimum allowable prices. We consider the following reservation price function that verifies the condition above.

$$p_i^{max}(p_i^{min}) = \overline{p}_i^{max} + sign(\overline{p}_i^{min} - p_i^{min}).|\overline{p}_i^{min} - p_i^{min}|^{\frac{1}{4}}, \tag{3.11}$$

where $sign(x) = 1$ when $x \geq 0$, and $sign(x) = -1$ when $x < 0$. Note that the exponent term $\frac{1}{4}$ can be replaced by any real $r \in (0,1)$ (since $|\phi| \to |\phi|^r$ is concave for $r \in (0,1)$).

A10) We assume that the storage capacity for each of the n products of the firm under study is allocated so that the firm is able to satisfy the maximum demand rate λ_i^{max} within a fixed period of time δ that is the same for all products. In mathematical terms $C_i = \lambda_i^{max}.\delta$, $\forall i \in \{1, ..., n\}$. Quantity δ represents the minimum reserve time.

So far, we imposed assumptions on how pricing relates to inventory and the functional form we consider for the pricing in the model. Our goal in this section is to model the delay of a product waiting in inventory. To achieve this, we consider average delay functions $A_i(I_i(t))$ of the hyperbolic form $\frac{\delta}{I_i(t)}$, where δ is the minimum reserve time, and $A_i(I_i(t))$ and $A_i(I_i(t))$ is defined in Subsection 3.2.1. This seems to imply that as inventory increases, we price so that the average delay of a product decreases. In what follows, we impose assumptions on the demand arrival rate (as done traditionally in the literature) and demonstrate how these assumptions give rise to these hyperbolic average product delay functions.

3.4.2 Delay and Demand Models

We assume the following, for every product i:

A11) To every p_i^{min} corresponds an arrival rate λ_i^{max}. This maximum arrival rate is a hyperbolic function of the form

$$\lambda_i^{max}(p_i^{min}) = \overline{\lambda}_i^{max}.\frac{\overline{p}_i^{min}}{p_i^{min}}. \tag{3.12}$$

See Allen [3] and Tirole [76] for more details.

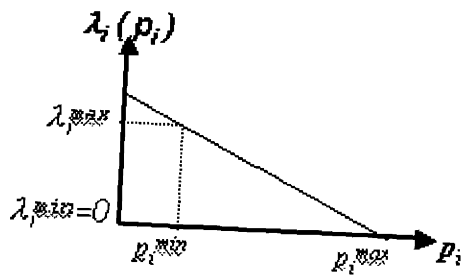

Figure 3-4: Linear arrival rates $\lambda_i(p_i)$

A12) A non-homogenous renewal demand arrival process, with rate $\lambda_i(p_i, p_i^{min})$ that is linear as a function of the price p_i (see Figure 3-4). Similarly to [3], [26], [76], and [80], we assume that

$$\lambda_i(p_i, p_i^{min}) = \lambda_i^{max}(p_i^{min}).\frac{p_i^{max}(p_i^{min}) - p_i}{p_i^{max}(p_i^{min}) - p_i^{min}}. \tag{3.13}$$

Notice that when the inventory level hits its capacity level (i.e. $I_i = C_i$), then we charge the minimum allowable price (i.e. $p_i(I_i) = p_i^{min}$), and we target the maximum arrival rate (i.e. $\lambda_i(p_i) = \lambda_i^{max}$). On the other hand, when the inventory level is zero (i.e. $I_i = 0$), then we charge the reservation price (i.e. $p_i(I_i) = p_i^{max}$), and we target a zero arrival rate (i.e. $\lambda_i(p_i) = 0$). Figure 3-5 illustrates Assumptions A9 and A11-A12.

In practice, given a minimum allowable reference price $\overline{p}_i^{min}$, its correponding demand arrival $\overline{\lambda}_i^{max}$ is readily available in a datawarehouse. Furthermore, its corresponding reservation price $\overline{p}_i^{min}$ can be estimated through customers' surveys. Therefore, the parametric functions in Assumptions A8-A12 can be estimated in practice.

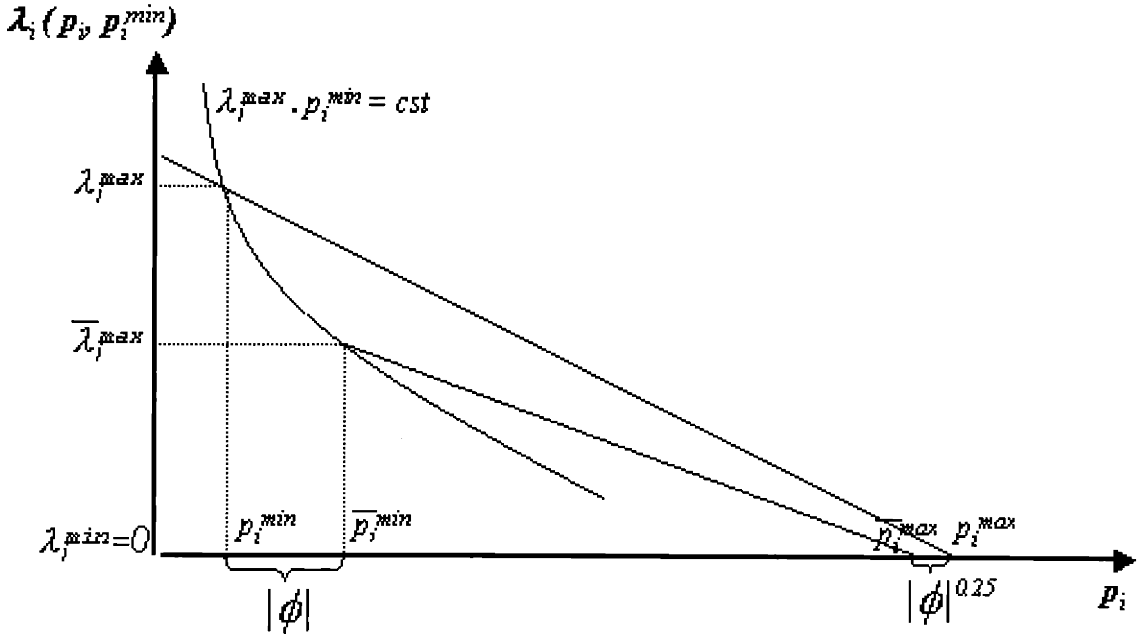

Figure 3-5: Model for arrival rates, maximum arrival rates, and reservation prices

In order to provide a connection between demand and delay models, we consider the approximation that Little's law holds for every time t. That is, the inventory level $I_i(t) = \lambda_i(p_i(I_i(t))).D_i(I_i(t))$. We view $I_i(.)$ as the average length of the queue, $\lambda_i(.)$ as the arrival demand rate, and $D_i(.)$ as the average waiting time in the queue. As a result, this approximation views the sojourn time $D_i(I_i(t))$ as the expected value of $I_i(t)$ interarrivals of the renewal process, that is $\frac{I_i(t)}{\lambda_i(p_i(I_i(t)))}$.

Notice that this approximation looks at the average state of the system. Indeed, the expression $\lambda_i(p_i(I_i(t))) = \frac{I_i(t)}{D_i(I_i(t))}$ describes the average arrival rate for product i. As a result, it is indifferent about how far or close some of the $I_i(t)$ units of product i in the inventory system are from being sold at time t. However, by looking at the system from the perspective of the sojourn time, that is the waiting in the system, the fluid model formulated in Section 3.3 captures the dynamics of the $I_i(t)$ units of product i in detail. Furthermore, quantity $v_i(t)$ describes the selling rate of product i exactly and not on average. As a result, the two approaches are different and in general, $\lambda_i(p_i(I_i(t))) \neq v_i(t)$. Nevertheless, in what follows, we will attempt to gain some insight on the relationship between the two approaches. Figure 3-7 illustrates this relationship in the case of the test example of Subsection 3.4.6.

In fact, the next lemma shows that the total amount sold in the analysis period $[0, T_\infty]$ is the same as the cumulative demand.

Lemma 3.1 *For constant product sojourn time functions $D_i(I_i(t)) = \theta_i$, the total cumulative demand is equal to the total cumulative sales:*

$$\int_0^{T_\infty} \lambda_i(p_i(I_i(t)))dt = V_i(T_\infty). \tag{3.14}$$

Proof:

Since $\lambda_i(p_i(I_i(t))) = \frac{I_i(t)}{D_i(I_i(t))}$, it follows that

$$\begin{aligned}
\int_0^{T_\infty} \lambda_i(p_i(I_i(t)))dt &= \frac{1}{\theta_i}.\int_0^{T_\infty} I_i(t)dt \\
&= \frac{1}{\theta_i}.[\int_0^{T_\infty} U_i(t)dt - \int_{\theta_i}^{T_\infty} V_i(t)dt] \\
&= \frac{1}{\theta_i}.[\int_0^{T_\infty} U_i(t)dt - \int_{\theta_i}^{T_\infty} U_i(t-\theta_i)dt] \text{ (from equation (3.5))} \\
&= \frac{1}{\theta_i}.[\int_0^{T+\theta_i} U_i(t)dt - \int_0^{T} U_i(t)dt] \text{ (since } T_\infty = s_i(T) = T+\theta_i) \\
&= \frac{1}{\theta_i}.\int_T^{T+\theta_i} U_i(t)dt \\
&= \frac{1}{\theta_i}.\theta_i.U_i(T) \text{ (since production ends at time } T) \\
&= V_i(T_\infty) \text{ (from equation (3.5)).}
\end{aligned}$$

□

3.4.3 Discretized DPM Model

Below, we discretize the time space by introducing $N = \lfloor \frac{T}{\delta} \rfloor$. We consider $N+1$ intervals of time of length δ and assume that for every discretization interval index

$j \in \{0, 1, ..., N\}$ and for every time $t \in [j\delta, (j+1)\delta)$, the following variables in each interval are constant: $CFR(t) = CFR_j$, $u_i(t) = u_{ij}$, $c_i(t) = c_{ij}$, and $h_i(t) = h_{ij}$. The decision variables are the production levels u_{ij} for every product i and for every discretization interval index j, as well as the unit price function parameter p_i^{min}.

Remarks:

- Notice that we can have a finer discretization by choosing intervals of length $\frac{\delta}{M}$, where M is a positive integer that represents the discretization accuracy. This does not add any complexity in the formulation of the discretized model and in the solution algorithm. Furthermore, the computational burden increases linearly with M. However, for the sake of clarity and brevity, in what follows we choose $M = 1$.

- As discussed in Subsection 3.2.1, in addition to prices, the control variables are the production levels.

Let $u = ((u_{ij})_{(i \in \{1,\dots,n\}, j \in \{0,\dots,N\})})$ and $p^{min} = ((p_i^{min})_{(i \in \{1,\dots,n\})})$.

Proposition 3.1 *Under Assumptions A8-A12, the solution of the Dynamic Pricing Model is equivalent to solving the following quadratic optimization problem:*

Discretized Quadratic Pricing Model (DQPM):

$$Min_{u,p^{min}} \sum_{i=1}^{n} (k_i [\sum_{j=0}^{N-1} u_{ij} u_{ij+1} + \sum_{j=0}^{N} u_{ij}^2] + \sum_{j=0}^{N} g_{ij} u_{ij})$$

$$\sum_{i=1}^{n} u_{ij} \leq CFR_j, \quad \forall j \in \{0, 1, ..., N\}$$

$$u_{ij} \geq 0, \quad \forall i \in \{1, 2, ..., n\}, \quad \forall j \in \{0, 1, ..., N\}$$

$$\text{where} \quad g_{ij} = -\delta(p_i^{max} - c_{ij} - \tfrac{h_{ij} + h_{ij+1}}{2}\delta), \quad k_i = \tfrac{\epsilon_i \delta^2}{2},$$

$$\text{and } \epsilon_i = \frac{p_i^{max} - p_i^{min}}{C_i}.$$

Proof:

The capacity constraint above follows directly from its continuous analogue (3.7) in

Section 3.3. Moreover, the non-negativity constraint also follows from its continuous analogue (3.8). Next, we establish that the objective function in the DPM Model (see equation (3.9)) simplifies to the quadratic objective of the DQPM Model formulated above. Notice that we converted the problem to a minimization problem by changing the signs. As a result, in what follows, we will illustrate that the optimal objective value of the original DPM Model is the opposite of the optimal objective value of the DQPM Model.

For $j \in \{0, 1, ..., N\}$, and $t \in [j\delta, (j+1)\delta)$, the previous assumptions together with relations (3.3)-(3.8) imply that $U_i(t) = u_{ij}.(t - j\delta) + \delta. \sum_{l=0}^{j-1} u_{il}$, $v_i(t) = u_{ij-1}$, and $I_i(t) = u_{ij}.(t - j\delta) + u_{ij-1}.((j+1)\delta - t)$ (3.14b). Furthermore, replacing the unit price as a function of the inventory in equation (3.9) yields the following objective function:

$$
\begin{aligned}
Obj &= -Min \sum_{i=1}^{n} \int_0^{T_\infty} -p_i(I_i(t))v_i(t) + c_i(t)u_i(t) + h_i(t)I_i(t)dt \\
&= -Min \sum_{i=1}^{n} \int_0^{T_\infty} \epsilon_i I_i(t)v_i(t) - (p_i^{max} v_i(t) - c_i(t)u_i(t) - h_i(t)I_i(t))dt.
\end{aligned}
\tag{3.15}
$$

Moreover, replacing $v_i(t)$ and $I_i(t)$ from (3.14b) gives rise to:

$$
\begin{aligned}
\epsilon_i \int_0^{T_\infty} I_i(t)v_i(t)dt &= \epsilon_i \sum_{j=1}^{N+1} \int_{j\delta}^{(j+1)\delta} (j+1)\delta u_{ij-1}^2 - j\delta u_{ij-1}u_{ij} + u_{ij-1}(u_{ij} - u_{ij-1})tdt \\
&= \epsilon_i \delta^2 [\sum_{j=0}^{N-1} (j+2)u_{ij}^2 - \sum_{j=0}^{N-1} (j+1)u_{ij}u_{ij+1} \\
&\quad +0.5 \sum_{j=0}^{N-1} (2j+3)u_{ij}(u_{ij+1} - u_{ij}) + 0.5u_{iN}^2] \\
&= k_i[\sum_{j=0}^{N-1} u_{ij}u_{ij+1} + \sum_{j=0}^{N} u_{ij}^2],
\end{aligned}
\tag{3.16}
$$

where $k_i = \frac{\epsilon_i \delta^2}{2}$.

Furthermore, notice that $\int_0^{T_\infty} p_i^{max} v_i(t)dt = \delta.p_i^{max}.\sum_{j=0}^{N} u_{ij}$ (3.16b). In addition, it is easy to see that $\int_0^{T_\infty} c_i(t)u_i(t)dt = \delta.\sum_{j=0}^{N} c_{ij}.u_{ij}$ (3.16c). Replacing $I_i(t)$ from (3.14b) gives rise to:

$$\begin{aligned}\int_0^{T_\infty} h_i(t)I_i(t)dt &= \sum_{j=0}^{N+1}\int_{j\delta}^{(j+1)\delta} h_{ij}[(j+1)\delta u_{ij-1} - j\delta u_{ij} + (u_{ij} - u_{ij-1})t]dt \\ &= \delta^2[0.5h_{i0}u_{i0} + \sum_{j=0}^{N-1}(j+2)h_{ij+1}u_{ij} - \sum_{j=0}^{N-1}(j+1)h_{ij+1}u_{ij+1} \\ &\quad +0.5\sum_{j=0}^{N-1}(2j+3)h_{ij+1}(u_{ij+1} - u_{ij}) + 0.5h_{iN+1}u_{iN}] \\ &= \frac{\delta^2}{2}\sum_{j=0}^{N}(h_{ij} + h_{ij+1})u_{ij}. \end{aligned} \tag{3.17}$$

Replacing equations (3.16), (3.16b), (3.16c), and (3.17) in (3.15) gives rise to the result of the proposition.

□

3.4.4 An Iterative Relaxation Algorithm

In this subsection, we focus on the solution of the DQPM Model introduced in Proposition 3.1. In particular, we propose a solution algorithm that determines optimal production levels for a fixed unit price function (that is, when p_i^{min}, and as a result p_i^{max}, are fixed). In the next subsection, we also will illustrate how to incorporate pricing decisions in the solution algorithm.

Below, we propose a solution algorithm that applies the iterative relaxation scheme of Dafermos [21] and Nagurney [60], and the equilibration scheme of Dafermos and Sparrow ([22] and [23]), to the pricing problem. This solution method is an equilibration approach that is extensively used in static traffic equilibrium problems (see Florian and Hearn [29], and Sheffi [73] for more details).

To illustrate this equilibration approach, we need to introduce some additional notation. We define C_{ij} as the opposite of the marginal profit of product i for discretization interval index j. In mathematical terms:

$$C_{ij} = \frac{-\partial Obj}{\partial u_{ij}} = 2k_i u_{ij} + k_i(u_{ij+1} + u_{ij-1}) + g_{ij}, \tag{3.18}$$

where Obj is defined in equation (3.15).

We define m_{ij} as the opposite of the marginal profit of product i for discretization interval index j at a zero production level. That is, $m_{ij} = k_i(u_{ij+1}+u_{ij-1})+g_{ij}$. Therefore, $C_{ij} = m_{ij} + 2k_{ij}$. We further introduce the upperscript index k to denote the number of iterations of the algorithm. Hence, at iteration k, our goal is to determine, for every product i, and for every discretization interval index j, the production levels u_{ij}^k. Moreover, we introduce $C_{ij}^k = 2k_i u_{ij}^k + m_{ij}^k$, where $m_{ij}^k = k_i(u_{ij+1}^{k-1} + u_{ij-1}^k) + g_{ij}$. Note that the production levels u_{ij}^k will be computed in increasing order of the discretization interval index j in the algorithm. Hence, in the expression of m_{ij}^k, u_{ij-1} is evaluated at iteration k while u_{ij+1} is evaluated at iteration $k-1$.

We introduce an $n \times (N+1)$ matrix with elements $order(i,j)$ that sorts the opposite of the zero-production marginal profits m_{ij}^k, $i \in \{1, 2, ..., n\}$ in a non-decreasing order. That is, for j and k fixed, $m_{order(1,j)j}^k \leq m_{order(2,j)j}^k \leq \cdots \leq m_{order(n,j)j}^k$.

At every iteration k, the equilibration algorithm computes for every discretization interval index j, an index l_j and a vector α_{ij}^k by equilibrating the corresponding opposite of the marginal profits, that is,

$$\begin{aligned} C_{order(1,j)j}^k = ... = C_{order(l_j,j)j}^k = \alpha_{order(l_j,j)j}^k &\leq C_{order(l_j+1,j)j}^k, ..., \leq C_{order(n,j)j}^k \\ u_{order(1,j)j}^k > 0, ..., u_{order(l_j,j)j}^k &> 0, \quad \sum_{i=1}^{l_j} u_{order(i,j)j}^k = CFR_j \\ u_{order(l_j+1,j)j}^k = u_{order(l_j+2,j)j}^k &= ... = u_{order(n,j)j}^k = 0. \end{aligned}$$

Figure 3-6 provides a network representation of the equilibration algorithm described above. There is an interesting analogy between this equilibration algorithm and static traffic equilibrium (see [29], [73]). Indeed, for a fixed discretization interval index j, we select (i) which products we should produce, and (ii) how much of each of the selected products we should be producing, so that all selected products have equal and minimum opposite marginal profit. In static traffic equilibrium, we select (i) which paths should be used, and (ii) how much traffic should flow on each of the selected paths, so that all selected paths have equal and minimum travel times.

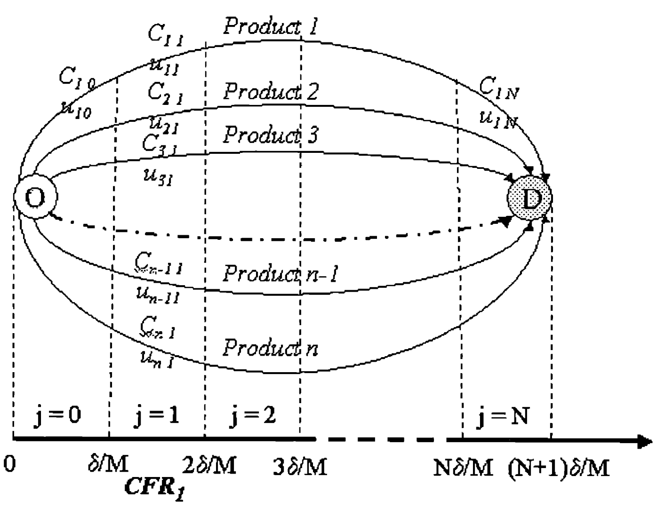

Figure 3-6: Network representation of the discretized Dynamic Pricing Model

Let ϵ be our tolerance level:

Iterative Relaxation Algorithm:

Step 0: for every time index $j \in \{0, 1, ..., N\}$ and product $i \in \{1, 2, ..., n\}$

$$u^0_{ij} = \frac{CFR_j}{n}.$$

Step k: for every time index $j \in \{0, 1, ..., N\}$,

$$\text{Let } \alpha^k_{order(i,j)j} = \frac{CFR_j + \sum_{p=1}^{i} \frac{m^k_{order(p,j)j}}{2k_{order(p,j)}}}{\sum_{p=1}^{i} \frac{1}{2k_{order(p,j)}}}.$$

If $eq_j = argmin\{i \text{ such that } \alpha^k_{order(i,j)j} \leq m^k_{order(i+1,j)j}\}$ exists, then

set $l_j = eq_j$.

Otherwise, set $l_j = n$.

If $i > l_j$, then $u^k_{order(i,j)j} = 0$.

Otherwise, $u^k_{order(i,j)j} = \frac{\alpha^k_{order(l_j,j)j} - m^k_{order(i,j)j}}{2k_{order(i,j)}}$.

Convergence criterion:

If for all $j \in \{0, 1, ..., N\}$ and $i \in \{1, 2, ..., n\}$,

all $u^k_{ij} = 0$ satisfy $C^k_{ij} \geq \alpha^k_{order(l_j,j)j} - \epsilon$, then stop.

Otherwise, set $k = k + 1$ and go to Step k.

Below, we establish that this algorithm converges to the optimal solution of the DQPM Model.

Theorem 3.2 *The Iterative Relaxation Algorithm converges to the unique optimal solution of the DQPM Model.*

Proof:

The Iterative Relaxation Algorithm is based on considering a separable approximation of C_{ij} (equation (3.18)) in terms of production levels u_{ij}. For j fixed, let vector $W_j(u) = (2.u_{1j}, 2.u_{2j}, ..., 2.u_{nj})$ and let Z_j be the $n \times n$ matrix defined by

$$Z_j = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & . & 0 \\ 1 & 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 1 & 0 & 1 & . & . & . & . & . \\ . & 0 & 1 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & 0 & . \\ . & . & . & . & . & 1 & 0 & 1 & 0 \\ . & . & . & . & . & 0 & 1 & 0 & 1 \\ 0 & 0 & . & . & . & . & 0 & 1 & 0 \end{bmatrix}$$

Notice that matrix $Z_j = \nabla_u[(\frac{m_{ij}}{k_i})_{i\in\{1,\dots,n\}}]$ is the Jacobian matrix of the opposite of the zero-production marginal profits at time j and $W_j(u) = (\frac{C_{ij}-m_{ij}}{k_i})_{i\in\{1,\dots,n\}}$. We will use the following result from Nagurney [60] to prove that the Iterative Relaxation Algorithm converges.

Lemma 3.2 *([21], [60]) Assume that there exist a scalar $\gamma > 0$ and a scalar $\lambda \in (0,1)$ such that:*
(F1) For every $i \in \{1, ..., n\}$, $\frac{\partial W_j(u)}{\partial u_{ij}} \geq \gamma$.
(F2) $|||Z_j||| \leq \lambda.\gamma$, (where $|||Z_j|||$ denotes the maximum eigenvalue of matrix Z_j).
Then, the Iterative Relaxation Algorithm converges to an optimal solution.

We will show that vector $W_j(u)$ and matrix Z_j verify conditions (F1) and (F2) of Lemma 3.2 with $\gamma = 2$.

For every $i \in \{1, ..., n\}$, notice that $\frac{\partial W_j(u)}{\partial u_{ij}} = 2$. Hence, condition (F1) is verified with $\gamma = 2$. Since Z_j is a symmetric matrix, it has at most n distinct eigenvalues. To show that condition (F2) is verified, it suffices to show that every eigenvalue α of matrix Z_j is strictly less than 2.

If α is an eigenvalue of matrix Z_j, there exists a vector $x^\alpha \neq 0$ such that

$$Z_j x^\alpha = \alpha x^\alpha. \tag{3.19}$$

We will assume that $\alpha \geq 2$ and try to reach a contradiction. If x^α verifies equation (3.19), then $-x^\alpha$ also verifies it. Hence, without loss of generality, we can assume that $x_1^\alpha \geq 0$. Using an induction over the rows of equation (3.19), it follows that $0 \leq x_1^\alpha \leq x_2^\alpha \leq ... \leq x_n^\alpha$. Therefore, x^α is a non-negative vector.

Summing up the rows of the vectors in both sides of equation (3.19) gives rise to

$$x_1^\alpha + x_n^\alpha + 2.\sum_{i=2}^{n-1} x_i^\alpha = \alpha \sum_{i=1}^{n} x_i^\alpha.$$

Therefore, $(\alpha - 1)(x_1^\alpha + x_n^\alpha) + (\alpha - 2).\sum_{i=2}^{n-1} x_i^\alpha = 0$. Since $\alpha \geq 2$ and x^α is non-negative, it follows that $x_1^\alpha = x_n^\alpha = 0$. Through an induction argument over the rows of equation (3.19), it follows that $x_2^\alpha = x_3^\alpha = ... = x_{n-1}^\alpha = 0$. Hence, $x^\alpha = 0$, which contradicts our earlier assumption that $x^\alpha \neq 0$.

Therefore, $\alpha < 2$, which in turn implies that $|||Z_j||| < 2$. Therefore, there exists a scalar $\lambda \in (0,1)$ such that condition (F2) is verified. Lemma 3.2 implies that the Iterative Relaxation Algorithm converges to an optimal solution.

Furthermore, the quadratic terms of the objective function can be rewritten as

$$\begin{aligned} Q(u) &= \sum_{i=1}^{n} \frac{k_i}{2}[\sum_{j=0}^{N-1}(u_{ij}^2 + 2u_{ij}u_{ij+1} + u_{ij+1}^2) + u_{i0}^2 + u_{iN}^2] \\ &= \sum_{i=1}^{n} \frac{k_i}{2}[\sum_{j=0}^{N-1}(u_{ij} + u_{ij+1})^2 + u_{i0}^2 + u_{iN}^2]. \end{aligned}$$

Notice that $Q(u)$ is a strictly convex quadratic function in terms of the production levels u_{ij}. Hence, the DQPM Model has a unique optimal solution. Therefore, the Iterative Relaxation Algorithm converges to the unique optimal solution.

□

Notice that the iterative relaxation algorithm belongs to a family of linearly converging algorithms (see [22], [60] for more details). It easy to see that by enhancing the algorithm with a binary search, each iteration of the algorithm requires computations of the order of $N.n.log(n)$.

3.4.5 Determining Optimal Production/Pricing Policies

In this subsection, we show how we can extend the Iterative Relaxation Algorithm to determine both the optimal production levels and the optimal pricing policies.

We introduce the minimum allowable price parameter ϕ defined as $\phi = p_i^{min} - \overline{p}_i^{min}$. Assumptions A7-A11 give rise to the following relations:

$$\begin{aligned}
p_i^{min} &= \overline{p}_i^{min} + \phi, \\
p_i^{max} = \overline{p}_i^{max} &- sign(\phi)|\phi|^{\frac{1}{4}}, \\
\lambda_i^{max} = \overline{\lambda}_i^{max} &. \frac{\overline{p}_i^{min}}{\overline{p}_i^{min} + \phi}, \\
p_i(I_i) = \overline{p}_i^{max} - sign(\phi)|\phi|^{\frac{1}{4}} &- \frac{\overline{p}_i^{max} - \overline{p}_i^{min} - sign(\phi)|\phi|^{\frac{1}{4}} - \phi}{\delta.\overline{\lambda}_i^{max}.\frac{\overline{p}_i^{min}}{\overline{p}_i^{min}+\phi}}.I_i.
\end{aligned}$$

As a result, in the formulation of the DQPM Model in Proposition 3.1, the objective function depends on the minimum allowable price parameter ϕ through the parameters k_i and g_{ij} that can be rewritten as:

$$\begin{aligned}
k_i = \frac{\delta}{2} &. \frac{\overline{p}_i^{max} - \overline{p}_i^{min} - sign(\phi)|\phi|^{\frac{1}{4}} - \phi}{\overline{\lambda}_i^{max}.\frac{\overline{p}_i^{min}}{\overline{p}_i^{min}+\phi}}, \quad \text{and} \\
g_{ij} = -\delta(\overline{p}_i^{max} - sign(\phi)|\phi|^{\frac{1}{4}} &- c_{ij} - \frac{h_{ij} + h_{ij+1}}{2}\delta).
\end{aligned}$$

Hence, for every value of the minimum allowable price parameter ϕ, we can perform the Iterative Relaxation Algorithm (IRA) and obtain an optimal production policy

$u(\phi)$ that yields a profit $IRA_{opt}(\phi)$. Therefore, solving the overall DQPM Model is equivalent to maximizing $IRA_{opt}(\phi)$ for $\phi \in (-\overline{p}_i^{min}, \overline{p}_i^{max} - \overline{p}_i^{min})$ such that $p_i^{min}(\phi) \leq p_i^{max}(\phi)$. Notice that this problem is a one-dimensional maximization problem.

As a result, by embedding the Iterative Relaxation Algorithm in a line search procedure for the one-dimensional objective function $IRA_{opt}(\phi)$, we are able to solve the Discretized Dynamic Pricing Model and determine optimal production levels and pricing policies.

3.4.6 Test Example

In this subsection, we apply the Iterative Relaxation Algorithm in a small test example. We consider 5 products and 10 discretization intervals (i.e. $n = 5$ and $N = 9$). We use as inputs the minimum allowable reference prices $\overline{p}_i^{min}$ and their corresponding reservation prices $\overline{p}_i^{max}$ outlined in Table 3.1, the shared capacity flow rate vector CFR outlined in Table 3.2, the production costs c_{ij} and the holding costs h_{ij} provided in Tables 3.3 and 3.4 respectively.

$\overline{p}_i^{max}$	$\overline{p}_i^{min}$	
16.25	12.56	*Product 1*
13.25	9.56	*Product 2*
13.25	9.56	*Product 3*
13.25	9.56	*Product 4*
14.85	11.16	*Product 5*

Table 3.1: Minimum allowable unit reference prices and their corresponding reservation prices

We first assume that the unit price function is fixed (that is, $p_i^{min} = \overline{p}_i^{min}$ and $p_i^{max} = \overline{p}_i^{max}$). We apply the steps of the Iterative Relaxation Algorithm outlined

Discretization Interval Index	0	1	2	3	4	5	6	7	8	9
CFR_j	19	21	23	25	27	29	31	33	35	37

Table 3.2: Shared capacity flow rate per discretization interval

	0	1	2	3	4	5	6	7	8	9
c_{1j}	8.5938	8.7500	8.8698	8.9709	9.0599	9.1405	9.2144	9.2833	9.3481	9.4093
c_{2j}	5.4000	5.6209	5.7905	5.9333	6.0593	6.1731	6.2778	6.3751	6.4667	6.5533
c_{3j}	4.1698	4.4405	4.6481	4.8231	4.9772	5.1167	5.2449	5.3642	5.4762	5.5823
c_{4j}	4.9209	5.2333	5.4731	5.6751	5.8533	6.0142	6.1622	6.3000	6.4293	6.5517
c_{5j}	7.6599	8.0093	8.2772	8.5033	8.7022	8.8823	9.0478	9.2017	9.3465	9.4833

Table 3.3: Production costs c_{ij}

	0	1	2	3	4	5	6	7	8	9
h_{1j}	1.6037	1.5135	1.4865	1.4236	1.4107	1.3568	1.3504	1.3013	1.2987	1.2528
h_{2j}	1.4138	1.2862	1.2482	1.1591	1.1409	1.0647	1.0556	0.9861	0.9825	0.9176
h_{3j}	1.2331	1.0770	1.0302	0.9213	0.8989	0.8057	0.7944	0.7095	0.7049	0.6254
h_{4j}	1.0574	0.8770	0.8230	0.6972	0.6714	0.5637	0.5507	0.4526	0.4474	0.3556
h_{5j}	0.8846	0.6830	0.6226	0.4820	0.4532	0.3326	0.3182	0.2085	0.2027	0.1000

Table 3.4: Holding costs h_{ij}

in the previous subsection. For our computations, we used a PC with a Pentium III, 366 MHz, 128 MB RAM, and implemented the algorithm in MATLAB. We chose a tolerance level of $\epsilon = 10^{-9}$ in the convergence criterion. In this example, the algorithm converged in 102 iterations. The running time was 4.2 seconds. Table 3.5 provides the optimal production levels. The optimal profit associated with these production levels is \$1,539.

	0	1	2	3	4	5	6	7	8	9
u_{1j}	0	0	0	0	1.1412	0.0627	3.7224	0.0544	6.2184	0
u_{2j}	0	0.1635	2.7467	0.8509	4.8769	1.4403	6.3578	2.0120	7.7943	2.5721
u_{3j}	13.6804	11.6337	13.5364	14.4020	12.9029	17.1538	11.6776	19.9317	10.4424	22.7265
u_{4j}	5.3196	9.2028	6.7169	9.7471	7.7320	10.3432	8.2041	11.0020	8.6957	11.7015
u_{5j}	0	0	0	0	0.3469	0	1.0381	0	1.8492	0

Table 3.5: Optimal production levels

Furthermore, Figure 3-7 illustrates, in this example, the corresponding demand rate $\lambda_3(t)$ and the corresponding sales flow rate $v_3(t)$ for product 3. Note that, as established in Lemma 3.1, while the profiles of the demand rate and the sales flow rate are different, the two areas under the curves of $\lambda_3(t)$ and $v_3(t)$, depicted in Figure 3-7, are equal.

Next, we incorporate the pricing aspect in the algorithm as we described in Subsection 3.4.4. We perform a line search procedure by varying the minimum allowable price parameter ϕ. For every value of ϕ, we run the Iterative Relaxation Algorithm to obtain an optimal objective value $Obj(\phi)$.

Figures 3-8 and 3-9 show that the profit of the company under study, for this instance of the Discretized Dynamic Pricing Model, is a quasi-concave function of the slope of the unit price function, and a quasi-concave function of the minimum allowable price

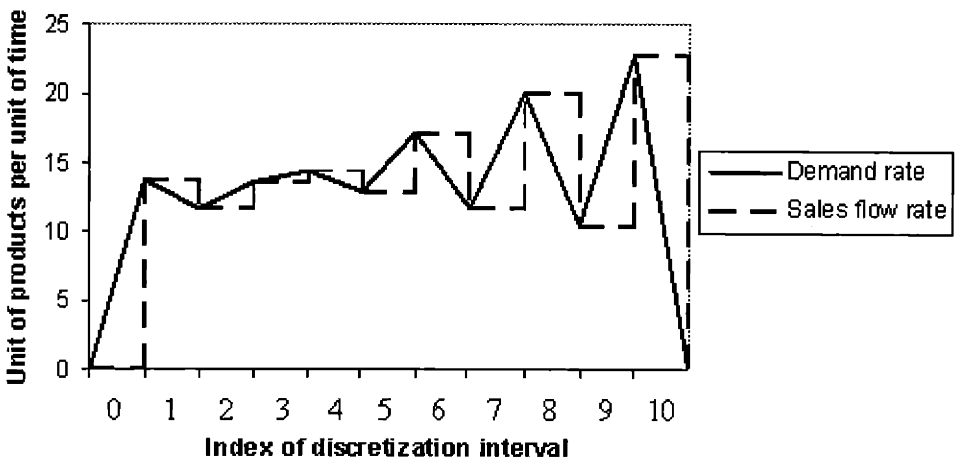

Figure 3-7: Profile of Demand Rate and Sales Flow Rate of Product 3

of product 1. Notice that the optimal profit is attained for a slope of 0.025 at a value of \$1,549.5.

3.5 The General Dynamic Pricing Fluid Model

In this section, we consider the General Dynamic Pricing Model without imposing any of the assumptions of Section 3.4. In particular, in Subsection 3.5.1 we examine the analytical properties of the feasible region of the model. Furthermore, in Subsection 3.5.2, we establish, under weak assumptions, the existence of a pricing/production/inventory control policy that maximizes the profit of the company under study over the feasible region.

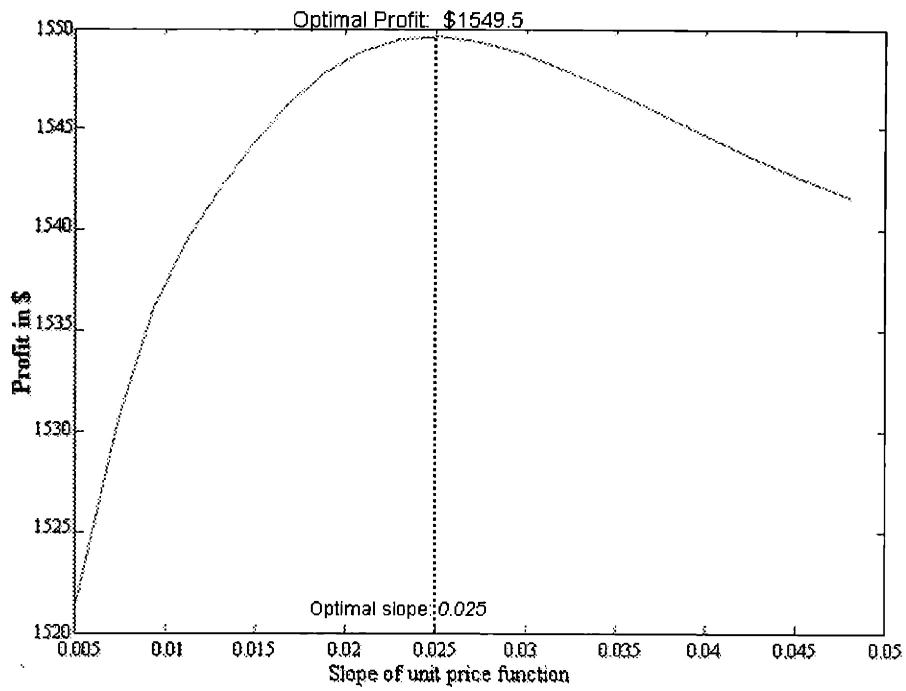

Figure 3-8: Optimal profit as a function of the slope of the unit price function

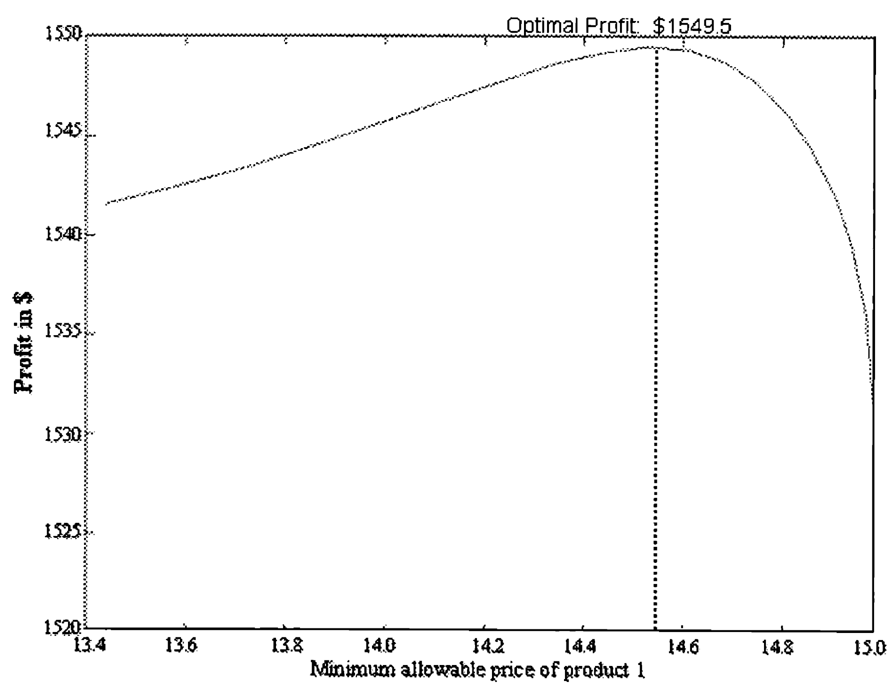

Figure 3-9: Optimal profit as a function of the minimum allowable price of product 1

3.5.1 Properties of the Feasible Region of the Dynamic Pricing Model

Preliminary Results

In the model presented in Section 3.3, the production flow rate functions $u_i(.)$ are control variables. In an effort to establish very general results, we assume that these functions are Lebesgue integrable. A function is said to be Lebesgue integrable if the set of points where this function is discontinuous is Lebesgue negligible. A set is Lebesgue negligible if its Lebesgue measure is 0.

Definition 3 *A solution to a problem is unique (or respectively differentiable) almost everywhere if and only if the set of points where this solution is not unique (or respectively not differentiable) is Lebesgue negligible.*

Later in this chapter, we show that the cumulative flow variables are differentiable and the solution to the problem is unique "almost everywhere". We refer to "almost everywhere" by "a.e.".

A *diffeomorphism* is a continuously differentiable function that has a continuously differentiable inverse. The following lemma gives sufficient conditions for a function to be a diffeomorphism. This result will be used to establish, under certain assumptions, that the Dynamic Pricing Model leads to product exit time functions that are diffeormorphisms.

Lemma 3.3 *Let $g(.)$ be a continuously differentiable function on $[0,T]$. If for every scalar $x \in [0,T]$ $g'(x) \neq 0$, then $g(.)$ is invertible on $[0,T]$, its inverse function $g^{-1}(.)$ is continuously differentiable on $[min(g(0),g(T)), max(g(0),g(T))]$ and, $g^{-1\prime}(x) = \frac{1}{g'(g^{-1}(x))}$.*

Proof: Since $g(.)$ is a continuously differentiable function, then $g'(.)$ is continuous. Since for every $x \in [0,T]$, $g'(x) \neq 0$, then $g'(.)$ has a constant sign. Hence, $g(.)$ is either strictly increasing or strictly decreasing. Since every strictly monotone function is invertible, it follows that $g(.)$ is invertible. Let $g^{-1}(.)$ denote the inverse function of $g(.)$. Then, $g(g^{-1}(x)) = x$. If we differentiate both sides of the above equality, we obtain: $g^{-1\prime}(x)g'(g^{-1}(x)) = 1$. Since $g'(x) \neq 0$ on [0,T], $g'(g^{-1}(x)) \neq 0$. It follows that $g^{-1\prime}(x) = \frac{1}{g'(g^{-1}(x))}$.

□

Remark: In the proof of Theorems 3.3 and 3.4, we use Lemma 3.3 where $g(.)$ is replaced with the product exit time function $s_i(.)$.

Lemma 3.4 *Let $f(.)$ be a continuous and strictly increasing function on interval $[a, b]$. For $x \in [f(a), f(b)]$, the set $W_x = \{w \in [a, b] | f(w) \leq x\}$ is the interval $[a, f^{-1}(x)]$.*

Proof: The proof follows easily.

Lemma 3.5 *If $g_i(.)$ is a continuously differentiable function over a compact set $[0, Y]$, then, there exists a scalar $\overline{B}_{2i}$ such that: $\overline{B}_{2i} = Max\{g_i'(x), x \in [0, Y]\}$.*

Proof:

Since $g_i(.)$ is continuously differentiable, $g_i'(.)$ is continuous over the compact set [0,Y]. Therefore, $g_i'(.)$ attains its maximum.

□

Remark: In the proof of Theorems 3.3 and 3.4, we use Lemma 3.5 where $g_i(.)$ is replaced with the product sojourn time function $D_i(.)$ and x is replaced with the inventory level I_i.

Unifying Analysis for Non-linear and Linear Sojourn Time Functions

Under sufficient conditions on the production flow rate functions and the sojourn time functions, we prove in this subsection that the feasible region of the Dynamic Pricing Model ($F(DPM)$) is not empty. Furthermore, we provide a unifying analysis of the $F(DPM)$ region for both non-linear and linear sojourn time functions. We also show that if the conditions of Theorem 3.4 are violated, then the FIFO property is also violated. In this sense, the conditions of Theorem 3.4 are tight.

Theorem 3.3 *If the pair $(D_i(.), u_i(.))$ satisfies the following conditions:*
(A1) The product sojourn time function $D_i(.)$ is continuously differentiable, and there exist two non-negative scalars B_{1i} and B_{2i} such that for every inventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.
(A2) The production flow rate function $u_i(.)$ is Lebesgue integrable, non-negative and bounded from above by $\overline{M}_i = \frac{M_i}{1+B_{1i}M_i}$ on $[0,T]$, where M_i is a positive scalar.
(A3) $M_i \leq \frac{1}{B_{2i}-B_{1i}}$.
Then, the feasible region $F(DPM)$ has the following properties:
(1) $F(DPM)$ is well defined (that is, the product inventory $I_i(.)$, the sales flow rate $v_i(.)$, and the cumulative variables can be uniquely determined by the product sojourn time function $D_i(.)$ and the production rate $u_i(.)$ on the analysis period $[0, T_\infty]$).
(2) The Strong FIFO property holds.

Proof:

Before providing the proof of Theorem 3.3, we establish additional preliminary results. Condition (A3) of Theorem 3.3 can either hold as an equality or as a strict inequality. The following lemma shows that the proof of Theorem 3.3 can be reduced to an easier proof where condition (iii) can be replaced by $B_{2i} - B_{1i} = \frac{1}{M_i}$.

Lemma 3.6 *In Theorem 3.3, one can assume that $B_{2i} - B_{1i} = \frac{1}{M_i}$.*

Proof: If $B_{2i} - B_{1i} < \frac{1}{M_i}$, let $B_{3i} = B_{1i} + \frac{1}{M_i}$. For every scalar I_i, it follows that $0 \leq B_{1i} \leq D_i'(I_i) < B_{3a}$, since $B_{2i} < B_{3i}$.

□

Consider the following sequence of time instants defined by: $t_0 = 0$, $t_1 = s_i(t_0)$ and $t_{j+1} = s_i(t_j)$. We prove the results of Theorem 3.3 by induction over the index j of interval $[t_j, t_{j+1})$. We first establish that the induction proof is valid.

Lemma 3.7 *For every non-negative integer j, $t_{j+1} - t_j \geq D_i(0) > 0$. Furthermore, there exists an integer n, such that $T \in [t_n, t_{n+1})$.*

Proof: For a given non-negative integer j, $t_{j+1} = s_i(t_j) = t_j + D_i(I_i(t_j))$. Therefore, $t_{j+1} - t_j = D_i(I_i(t_j))$. Since $D_i(.)$ is a nondecreasing function and since for every t, $I_i(t) \geq 0$, it follows that $D_i(I_i(t_j)) \geq D_i(0)$. Since by assumption $D_i(0) > 0$, it follows that $t_{j+1} - t_j \geq D_i(0) > 0$.

If $n_0 = \lceil \frac{T}{D_i(0)} \rceil$, it follows that $t_{n_0} \geq T$. Hence, $\text{Max}\{j | t_j \leq T\}$ exists. Let $n = \text{Max}\{j | t_j \leq T\}$. It follows that $n \leq n_0$ and $T \in [t_n, t_{n+1})$.

□

Let Y be defined by $Y = \int_0^T u_i(w)dw$. Y represents the total number of units of product i that are produced. Since the product delay function $D_i(.)$ is continuously differentiable and bounded, using Lemma 3.5, there exists $\overline{B}_{2i}$ such that $\overline{B}_{2i} = Max\{g_i'(x), x \in R\}$. Below is a series of three lemmas that we need in the induction proof of Theorem 3.3. The following lemma shows that there exists a constant θ that will serve to construct a lower bound on the product exit time function $s_i(.)$.

Lemma 3.8 *For every $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$ and for every $t \in [0, T]$, it follows that $\theta + \widehat{B}_{2i} u_i(t) \in (0, 1]$, where $\theta = \frac{1 + (B_{1i} - \widehat{B}_{2i})M_i}{1 + B_{1i} M_i}$.*

Proof: Since for every $I_i \in [0, Y]$, $D_i'(I_i) < B_{2i}$, and since $\overline{B}_{2i} = Max\{D_i'(I_i), I_i \in [0, Y]\}$, it follows that $\overline{B}_{2i} < B_{2i}$. Let $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$. From condition (A2) of Theorem 3.3, $0 \leq u_i(t) \leq \frac{M_i}{1+B_{1i}M_i}$. Therefore,

$$1 \leq 1 + \widehat{B}_{2i}u_i(t) \leq 1 + \frac{\widehat{B}_{2i}M_i}{1 + B_{1i}M_i}.$$

By subtracting $\frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i}$ from each side of these inequalities, it follows that:

$$1 - \frac{\widehat{B}_{2i}M_i}{1 + B_{1i}M_i} \leq 1 - \frac{\widehat{B}_{2i}M_i}{1 + B_{1i}M_i} + \widehat{B}_{2i}u_i(t) \leq 1.$$

Using Lemma 3.6, we can assume that $B_{2i} = B_{1i} + \frac{1}{M_i}$. Since $\widehat{B}_{2i} < B_{2i}$, it follows that $\widehat{B}_{2i} < B_{1i} + \frac{1}{M_i}$. Hence, $\frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} < 1$. Thus, $1 - \frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} > 0$. Therefore,

$$0 < 1 - \frac{\widehat{B}_{2i}M_i}{1 + B_{1i}M_i} \leq \frac{1 + (B_{1i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i} + \widehat{B}_{2i}u_i(t) \leq 1.$$

It follows that $0 < \theta + \widehat{B}_{2i}u_i(t) \leq 1$.

□

Lemma 3.9 *For every $t \in [t_0, t_1)$, the set $W = \{w | s_i(w) \leq t\}$ is empty and hence $V_i(t) = 0$.*

Proof: For every $t \in [t_0, t_1)$ and for every $w \in [0, t]$,

$$s_i(w) = w + D_i(I_i(w)) \geq \ 0 + D_i(0) \geq t_1 > t.$$

Hence, the set W defined by $W = \{\omega : s_i(\omega) \leq t\}$ is empty for $t \in [t_0, t_1)$. From equation (3.4), it follows that $V_i(t) = 0$, for $t \in [t_0, t_1)$.

□

The following lemma is needed in the induction step of the proof of Theorem 3.3.

Lemma 3.10 *For $t \in [t_{j+1}, t_{j+2})$,*

$$\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta} \leq \frac{M_i}{1 + B_{1i}M_i}, \quad \textit{(where } \theta = \tfrac{1+(B_{1i}-\widehat{B}_{2i})M_i}{1+B_{1i}M_i}\textit{).} \tag{3.20}$$

Proof: Let $t \in [t_{j+1}, t_{j+2})$. From condition (A2) of Theorem 3.3, $u_i(s_i^{-1}(t)) \leq \overline{M}_i = \frac{M_i}{1+B_{1i}M_i}$. By multiplying each side of the inequality by $B_{2i} - \widehat{B}_{2i}$, we obtain that

$$\begin{aligned}(B_{2i} - \widehat{B}_{2i})u_i(s_i^{-1}(t)) &\leq \frac{(B_{2i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i} \\ &\leq \frac{1 + (B_{1i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i}, \quad \text{(from Lemma 3.6)} \\ &\leq \theta, \quad \text{(by definition of } \theta).\end{aligned}$$

Since $\theta > 0$ and $\widehat{B}_{2i}u_i(s_i^{-1}(t)) \geq 0$, it follows that: $\frac{B_{2i}u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta} \geq 1$. Hence, from Lemma 3.6, we obtain that $\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta} \geq \frac{1}{B_{2i}} \geq \frac{M_i}{1+B_{1i}M_i}$.

□

We are now ready to provide an induction proof that establishes the results of Theorem 3.3.

Proof of Theorem 3.3:

The induction proof is over the index j of interval $[t_j, t_{j+1})$. The induction hypothesis for interval $[t_j, t_{j+1})$ is that the following properties hold:

(i) $s_i(.)$ is differentiable (a.e.) and continuous over $[t_j, t_{j+1})$, and $s_i'(t) \geq \widehat{B}_{2i}u_i(t) + \theta > 0$;

(ii) $v_i(.)$ is differentiable (a.e.) over $[t_j, ti + 1)$;

(iii) For every $t \in [t_j, t_{j+1})$, $v_i(t) \leq M_i'$; and

(iv) the $F(DPM)$ has a solution on $[0, t_{j+1})$ and this solution is unique (a.e.).

We first examine the base case on interval $[t_0, t_1)$. We then assume that the induction hypothesis holds for $[t_j, t_{j+1})$ and prove that it holds for $[t_{j+1}, t_{j+2})$. The proof of the base case is an easy application of Lemma 3.9.

We assume that the production flow rate functions $u_i(.)$ are given. Hence, they are unique (a.e.). Since the integral operator is unique, the integral $U_i(.)$ is also unique (a.e.). In order to prove the uniqueness (a.e.) of a solution to the $F(DPM)$ on each interval of the induction, it remains to prove that $V_i(.)$ and $s_i(.)$ are unique (a.e.) on these intervals. Then by uniqueness of the differentiation operator, $v_i(.)$ is unique (a.e.). Furthermore, using equations (3.3) and (3.6), it follows that $I_i(.) = U_i(.) - V_i(.)$. Hence, $I_i(.)$ is unique (a.e.).

Base Case: Time interval $[t_0, t_1)$.
For $t \in [t_0, t_1)$,

$$\begin{aligned} I_i(t) &= U_i(t) - V_i(t) \\ &= U_i(t) - 0 \qquad\qquad (\ V_i(t) = 0 \text{ , from Lemma 3.9 }) \\ &= \int_0^t u_i(w)dw. \end{aligned}$$

Hence, $I_i(.)$ is differentiable (a.e.) and continuous on $[t_0, t_1)$. Moreover, since the integral operator is unique and $U_i(.)$ is unique (a.e.), it follows that $I_i(.)$ is unique (a.e.). Furthermore, the function $s_i(t) = t + D_i(I_i(t))$ is continuous, since $I_i(.)$ and $D_i(.)$ are continuous. The exit time function $s_i(.)$ is differentiable and unique (a.e.) on $[t_0, t_1)$, since $I_i(.)$ is differentiable and unique (a.e.) and $D_i(.)$ is differentiable and unique. By differentiating each term in the expression of $s_i(t)$, we obtain, for $t \in [t_0, t_1)$:

$$\begin{aligned} s_i'(t) &= 1 + D_i'(I_i(t))\frac{dI_i(t)}{dt} \\ &= 1 + u_i(t)D_i'(I_i(t)). \end{aligned}$$

Since $u_i(t) \geq 0$ and $D_i'(I_i(t)) \geq 0$, it follows that $s_i'(t) \geq 1$. Using Lemma 3.8, we obtain: $s_i'(t) \geq \theta + \widehat{B}_{2i}u_i(t) > 0$. Therefore, the product exit time function $s_i(.)$ is differentiable (a.e.) and strictly increasing on $[t_0, t_1)$. Furthermore, from Lemma 3.9, $V_i(.) = 0$. Thus, $V_i(.)$ is both differentiable and unique (a.e.) over $[t_0, t_1)$ and for $t \in [t_0, t_1)$, $v_i(.)$ is unique (a.e.) and $v_i(t) = 0 \leq M_i'$. Hence, the $F(DPM)$ has a solution on interval $[0, t_1)$ and this solution is unique (a.e.).

Induction Step: Time interval $[t_{j+1}, t_{j+2})$.

From the induction hypothesis, we know that the product exit time function $s_i(.)$ is differentiable and unique (a.e.), continuous and strictly increasing on $[t_j, t_{j+1})$. From Lemma 3.3, it follows that $s_i^{-1}(.)$ is differentiable and unique (a.e.), and continuous over $[s_i(t_j), s_i(t_{j+1})) = [t_{j+1}, t_{j+2})$. Using Equation (3.5) in the model formulation, it follows that:

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad Y_i(t) = \int_0^{s_i^{-1}(t)} u_i(w)dw.$$

Since $s_i^{-1}(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, $V_i(.)$ is differentiable and unique (a.e.). By differentiating $V_i(.)$, we obtain: $v_i(t) = (s_i^{-1})'(t)u_i(s_i^{-1}(t))$. Using Lemma 3.3, $s_i^{-1\prime}(t) = \frac{1}{s_i'(s_i^{-1}(t))}$. Therefore,

$$v_i(t) = \frac{u_i(s_i^{-1}(t))}{s_i'(s_i^{-1}(t))}. \tag{3.21}$$

Furthermore, from the induction hypothesis, $s_i'(s_i^{-1}(t)) \geq \theta + \widehat{B}_{2i} > 0$. Hence, $0 < \frac{1}{s_i'(s_i^{-1}(t))} \leq \frac{1}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta}$. Since for every $t \in [t_{j+1}, t_{j+2})$, $0 \leq u_i(s_i^{-1}(t)) \leq M_i'$, it follows that:

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad v_i(t) \leq \frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta}. \tag{3.22}$$

Using Lemma 3.10, it follows that $v_i(.)$ is unique (a.e.) and $v_i(t) \leq M_i'$. Therefore, both the production and sales flow rate functions $u_i(.)$ and $v_i(.)$ are bounded from above by M_i'. This shows that if an upper bound is verified at the entrance of a link, it is also maintained at its exit.

If $t_{j+1} \geq T$, the induction ends and the proof is complete. Otherwise, $t_{j+1} < T$, and

$$I_i(t) = \int_0^t u_i(w)dw - \int_0^t v_i(w)dw.$$

Hence, $I_i(.)$ is differentiable (a.e.) and continuous on $[t_{j+1}, t_{j+2})$. Moreover, since the integral operator is unique and both $u_i(.)$ and $v_i(.)$ are unique (a.e.), it follows that $I_i(.)$ is unique (a.e.). Furthermore, the function $s_i(t) = t + D_i(I_i(t))$ is continuous, since both $I_i(.)$ and $D_i(.)$ are continuous. The exit time function $s_i(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, since $I_i(.)$ is differentiable and unique (a.e.) and $D_i(.)$ is differentiable and unique. By differentiating each term in $s_i(t)$, we obtain:

$$s_i'(t) = 1 + D_i'(I_i(t))\frac{dI_i(t)}{dt} = 1 + (u_i(t) - v_i(t))D_i'(I_i(t)), \quad \text{(from equation (3.3)).}$$

We discuss two cases: $u_i(t) - v_i(t) \geq 0$ and $u_i(t) - v_i(t) < 0$. First, we consider the case $u_i(t) - v_i(t) \geq 0$. Since $D_i'(I_i(t)) \geq B_{1i} \geq 0$, from Lemma 3.8, it follows that:

$$s_i'(t) \geq 1 \geq \widehat{B}_{2i}u_i(t) + \theta > 0.$$

Now, consider the case $u_i(t) - v_i(t) < 0$. Since $D_i'(I_i(t)) \leq \overline{B}_{2i} \leq \widehat{B}_{2i}$, it follows that:

$$s_i'(t) \geq 1 + \widehat{B}_{2i}(u_i(t) - v_i(t)).$$

Since $v_i(t) \leq M' = \frac{M_i}{1+B_{1i}M_i}$, we obtain:

$$s_i'(t) \ \geq \ 1 + \widehat{B}_{2i}u_i(t) - \frac{\widehat{B}_{2i}}{1 + B_{1i}M_i}M$$

$$
\begin{aligned}
s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \frac{1 + (B_{1i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i} \\
s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \theta > 0.
\end{aligned}
$$

Hence, we have showed that properties (i)-(iii) of the induction hypothesis hold on interval $[t_{j+1}, t_{j+2})$. Furthermore, we have proved that the $F(DPM)$ has a solution on interval $[0, t_{j+2})$ and that this solution is unique (a.e.).

□

Next, we show that the induction terminates after a finite number steps. This means that a construction algorithm, based on the induction proof of Theorem 3.3, will determine a feasible point of the $F(DPM)$ region in a finite number of steps.

Lemma 3.11 *The induction terminates after a finite number of steps, i.e. T_∞ is finite.*

Proof: Let $n_0 = Max\{n \in N, t_n \leq T\}$. From Lemma 3.7, n_0 exists and $T \in [t_{n_0}, t_{n_0+1})$. The induction proof, at all steps $i \leq n_0$, ensures that $s_i(.)$ is continuous and strictly increasing over $[0, t_{n_0+1})$. Hence, $Max\{s_i(t), t \in [0,T]\} = s_i(T)$ exists and is finite. Since $T_\infty = Max\{s_i(t), t \in [0,T]\}$, it follows that T_∞ is finite. Hence the induction terminates after a finite number of steps.

□

Remark: Below, we provide an intuitive interpretation of conditions similar to Conditions (A1)-(A3). In what follows, we provide a unifying analysis for both linear and nonlinear sojourn time functions. Moreover, Corollary 3.1 shows how linear sojourn time functions can be interpreted as a limit case of nonlinear sojourn time functions and why linear sojourn time functions lead to stronger results.

Theorem 3.4 *If the pair $(D_i(.), u_i(.))$ satisfies the following conditions:*

(B1) The product sojourn time function $D_i(.)$ is continuously differentiable, and there exist two non-negative constants B_{1i} and B_{2i} such that for every iventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.

(B2) The production flow rate function $u_i(.)$ is Lebesgue integrable, non-negative and bounded from above by a positive real number M_i on $[0, T]$.

(B3) $M_i \leq \frac{1}{B_{2i}-B_{1i}}$.

Then, the feasible region $F(DPM)$ has the following properties:

(1) $F(DPM)$ is well-defined, (that is, the product inventory $I_i(.)$, the sales flow rate $v_i(.)$, and the cumulative variables can be uniquely (a.e.) determined by the product sojourn time function $D_i(.)$ and the production rate $u_i(.)$ on the analysis period $[0, T_\infty]$).

(2) The Strong FIFO property holds.

Proof:

If $B_{1i} = 0$, then $M_i' = M_i$. In this case, both Theorem 3.3 and Theorem 3.4 have the same conditions and provide the same result of existence and uniqueness (a.e.) of a solution to the $F(DPM)$. Next, we only consider the case where $B_{1i} > 0$.

Since Theorem 3.3 and 3.4 have in common the first and the third conditions, using Lemma 3.6, one can assume $B_{2i} = B_{1i} + \frac{1}{M_i}$ in the proof of Theorem 3.4. In the proof to follow, we will assume that: $B_{1i} > 0$ and $B_{2i} = B_{1i} + \frac{1}{M_i}$.

Consider now the following sequence of time instants defined by: $t_0 = 0$, $t_1 = s_i(t_0)$ and $t_{j+1} = s_i(t_j)$. We prove the results of Theorem 3.4 by induction over the index j of interval $[t_j, t_{j+1})$. Let Y be the defined by $Y = \int_0^T u_i(w)dw$. Below, we provide two preliminary results that we use in the proof of Theorem 3.4.

Lemma 3.12 *There exists $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$ such that $\theta_j = \frac{1+(B_{1i}-\widehat{B}_{2i})M}{1+B_{1i}M+\sum_{k=2}^{i}(\widehat{B}_{2i}M)^k} \in (0,1)$.*

Proof: From condition (B1) of Theorem 3.4, $\forall I_i \in [0, Y]$, $D_i'(I_i) < B_{2i}$. Using Lemma 3.5, $\overline{B}_{2i} < B_{2i}$.

Let $\widehat{B}_{2i} = Max(\frac{\overline{B}_{2i}+B_{2i}}{2}, \frac{\frac{1}{M_i}+B_{2i}}{2})$. From Lemma 3.6, we can assume that $B_{2i} = B_{1i} + \frac{1}{M_i}$. Since $B_{1i} > 0$, it follows that $B_{2i} > \frac{1}{M_i}$. Hence, $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$.

Since $\widehat{B}_{2i} < B_{2i}$, it follows that $\widehat{B}_{2i} - B_{1i} < B_{2i} - B_{1i}$. Using Condition (B3) of Theorem 3.4, we obtain that $\widehat{B}_{2i} - B_{1i} < \frac{1}{M_i}$. Thus, $1 + (B_{1i} - \widehat{B}_{2i})M > 0$. Since B_{1i}, $\widehat{B}_{2i}$ and M are positive, it follows that the denominator of θ_j is greater than 1, and hence $\theta_j > 0$. Furthermore,

$$1 + (B_{1i} - \widehat{B}_{2i})M < 1 + B_{1i}M < 1 + B_{1i}M + \sum_{k=2}^{i} (\widehat{B}_{2i}M)^k.$$

Since $\theta_j = \frac{1+(B_{1i}-\widehat{B}_{2i})M}{1+B_{1i}M+\sum_{k=2}^{i}(\widehat{B}_{2i}M)^k}$, it follows that $\theta_j < 1$.

□

Lemma 3.13 is essential for the induction step in the proof of Theorem 3.4.

Lemma 3.13 *For any interval index j, and for every $t \in [t_{j+1}, t_{j+2})$,*

$$\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta_j} \leq \alpha_{i+1}, \tag{3.23}$$

where $\alpha_j = \frac{M\sum_{k=0}^{i-1}(\widehat{B}_{2i}M)^k}{1+B_{1i}M+\sum_{k=2}^{i}(\widehat{B}_{2i}M)^k}$.

Proof: By replacing α_{i+1} with its value given above, inequality (3.23) is equivalent to:

$$\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta_j} \leq \frac{M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^i}{1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^i}.$$

Therefore, $u_i(s_i^{-1}(t))(1+B_{1i}M+\sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k) \leq (u_i(s_i^{-1}(t))\widehat{B}_{2i}+\theta_j)M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k$. Through algebraic manipulations of the above expression, inequality (3.23) can be equivalently rewritten as

$$u_i(s_i^{-1}(t))(1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k) \leq u_i(s_i^{-1}(t))\sum_{k=1}^{i+1}(\widehat{B}_{2i}M)^k + \theta_j M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k.$$

Hence, $u_i(s_i^{-1}(t))(1+B_{1i}M) \leq u_i(s_i^{-1}(t))\widehat{B}_{2i}M + \theta_j M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k$. Thus, it follows that:

$$u_i(s_i^{-1}(t))(1 + (B_{1i} - \widehat{B}_{2i})M) - \theta_j M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k \leq 0.$$

Using Lemma 3.6, $1 + (B_{1i} - \widehat{B}_{2i})M = (B_{2i} - \widehat{B}_{2i})M$. Since $\widehat{B}_{2i} \leq B_{2i} = B_{1i} + \frac{1}{M_i}$, it follows that: $\frac{\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k}{1+B_{1i}M+\sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k} \leq 1$. Hence, we obtain:

$$u_i(s_i^{-1}(t))(B_{2i} - \widehat{B}_{2i})M - (1 + (B_{1i} - \widehat{B}_{2i})M)M \leq 0.$$

Thus, $u_i(s_i^{-1}(t))(B_{2i} - \widehat{B}_{2i}) - (B_{2i} - \widehat{B}_{2i})M \leq 0$. By dividing each term of the inequality by the positive scalar $B_{2i} - \widehat{B}_{2i}$, it follows that $u_i(s_i^{-1}(t)) \leq M$. Using Condition (B3) of Theorem 3.4, we verify the inequality.

□

We are now ready to provide an induction proof that establishes Theorem 3.4.

Proof of Theorem 3.4:

Recall that the induction proof is over the index j of interval $[t_j, t_{j+1})$. The induction hypothesis for interval $[t_j, t_{j+1})$ is that the following properties hold:

(i) $s_i(.)$ *is differentiable (a.e.) and continuous over* $[t_j, t_{j+1})$, *and* $s_i'(t) \geq \widehat{B}_{2i}u_i(t)+\theta_j$;

(ii) $V_i(.)$ *is differentiable (a.e.) over* $[t_j, t_{j+1})$;

(iii) $\forall t \in [t_j, t_{j+1})$, $v_i(t) \leq \alpha_j$; *and*

(iv) the $F(DPM)$ *has a solution on* $[0, t_{j+1})$ *and this solution is unique (a.e.).*

Base Case: Time interval $[t_0, t_1)$.

From Lemma 3.9, for every $t \in [t_0, t_1)$, $V_i(t) = 0$. The proof of this Base Case is similar to the proof of the first Base Case of Theorem 3.3. As a result, we do not provide it here.

Induction Step: Time interval $[t_{j+1}, t_{j+2})$.

From the induction hypothesis, we know that the link exit time function $s_i(.)$ is differentiable and unique (a.e.), continuous, and strictly increasing on $[t_j, t_{j+1})$. From Lemma 3.3, it follows that $s_i^{-1}(.)$ is differentiable and unique (a.e.), and continuous over $[s_i(t_j), s_i(t_{j+1})) = [t_{j+1}, t_{j+2})$. Using equation 3.5 in the model formulation, it follows that

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad V_i(t) = \int_0^{s_i^{-1}(t)} u_i(w)dw.$$

Since $s_i^{-1}(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, $V_i(.)$ is differentiable and unique (a.e.). By differentiating $V_i(.)$, we obtain: $v_i(t) = (s_i^{-1})'(t)u_i(s_i^{-1}(t))$. Using Lemma 3.3, it follows that:

$$v_i(t) = \frac{u_i(s_i^{-1}(t))}{s_i'(s_i^{-1}(t))}.$$

Furthermore, from the induction hypothesis, $s_i'(s_i^{-1}(t)) \geq \theta_j + \widehat{B}_{2i}u_i(s_i^{-1}(t)) > 0$. Hence, $0 < \frac{1}{s_i'(s_i^{-1}(t))} \leq \frac{1}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta_j}$. Note that for every $t \in [t_{j+1}, t_{j+2})$, $0 \leq u_i(s_i^{-1}(t)) \leq M$. Thus,

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad v_i(t) \leq \frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta_j}. \tag{3.24}$$

Using Lemma 3.13, it follows that $v_i(.)$ is unique (a.e.) and $v_i(t) \leq \alpha_{i+1}$.

If $t_{j+1} \geq T$, the induction ends and the proof is complete. Otherwise, $t_{j+1} < T$, and

$$I_i(t) = \int_0^t u_i(w)dw - \int_0^t v_i(w)dw.$$

Hence, $I_i(.)$ is differentiable (a.e.) and continuous on $[t_{j+1}, t_{j+2})$. Moreover, since the integral operator is unique and both $u_i(.)$ and $v_i(.)$ are unique (a.e.), it follows that $I_i(.)$ is unique (a.e.). Furthermore, the function $s_i(t) = t + D_i(I_i(t))$ is continuous, since both $I_i(.)$ and $D_i(.)$ are continuous. The exit time function $s_i(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, since $I_i(.)$ is differentiable and unique (a.e.), and $D_i(.)$ is differentiable and unique. By differentiating each term in $s_i(t)$, we obtain:

$$\begin{aligned} s_i'(t) &= 1 + D_i'(I_i(t))\frac{dI_i(t)}{dt} \\ &= 1 + (u_i(t) - v_i(t))D_i'(I_i(t)). \end{aligned}$$

We discuss two cases: $u_i(t) - v_i(t) \geq 0$ and $u_i(t) - v_i(t) < 0$. First, consider the case $u_i(t) - v_i(t) \geq 0$. Since $D_i'(I_i(t)) \geq B_{1i} \geq 0$, it follows that $s_i'(t) \geq 1 > \theta_j$. Now, consider the case $u_i(t) - v_i(t) < 0$. Since $D_i'(I_i(t)) \leq \overline{B}_{2i} \leq \widehat{B}_{2i}$, it follows that: $s_i'(t) \geq 1 + \widehat{B}_{2i}(u_i(t) - v_i(t))$. Since $v_i(t) \leq \alpha_{i+1}$, we obtain:

$$\begin{aligned} s_i'(t) &\geq 1 + \widehat{B}_{2i}u_i(t) - \widehat{B}_{2i}\frac{M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k}{1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k}, \\ s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \frac{1 + (B_{1i} - \widehat{B}_{2i})M}{1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k}, \\ s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \theta_{i+1} > 0. \end{aligned}$$

Hence, we have showed that properties (i)-(iii) of the induction hypothesis hold on interval $[t_{j+1}, t_{j+2})$. Furthermore, we have proved that the $F(DPM)$ has a solution on interval $[0, t_{j+2})$ and that this solution is unique (a.e.). The proof of Theorem 3.4 is now complete.

□

From Lemma 3.11, T_∞ is finite and the induction terminates after a finite number of steps. This means that a construction algorithm, based on the induction proof of Theorem 3.4, will determine a feasible point of the $F(DPM)$ region in a finite number of steps.

Remarks:

- Intuitively, Conditions (B1)-(B3) are the minimal conditions to ensure that the FIFO property is verified. Indeed, $B_{2i} - B_{1i}$ represents the maximum variation of the sojourn time in terms of inventory. During a time interval Δt of inventory decrease, the variation of inventory $I_i(t) - I_i(t + \Delta t)$ is bounded by the quantity $M_i.\Delta t$. Therefore, the variation of sojourn time $D_i(I_i(t)) - D_i(I_i(t + \Delta t))$ is bounded by $(B_{2i} - B_{1i}).M_i.\Delta t$. Using Condition (B3), this variation is also bounded by Δt. Hence, $s_i(t) = t+D_i(I_i(t)) \leq t+\Delta t+D_i(I_i(t+\Delta t)) = s_i(t+\Delta t)$, which is the FIFO property. Moreover, during a time interval Δt of inventory increase, since the sojourn time functions are non-decreasing, $D_i(I_i(t)) \leq D_i(I_i(t + \Delta t))$. Thereofre, $s_i(t) \leq s_i(t + \Delta t)$, which is the FIFO property.

- Notice that if the product sojourn time function $D_i(.)$ is linear, then conditions (B1)-(B3) of Theorem 3.4 simplify significantly. Indeed, in this case, $D_i'(.) = cst = B_{1i}$. Moreover, for any arbitrarily small positive scalar ϵ, by introducing $B_{2i} = B_{1i} + \epsilon$, Condition (B1) of Theorem 3.4 is verified. Furthermore, since Condition (B3) can be rewritten as $M_i \leq \frac{1}{\epsilon}$, M_i can be arbitrarily large. Therefore, the following corollary follows.

Corollary 3.1 *If the pair $(D_i(.), u_i(.))$ satisfies the following conditions:*
(C1) The product sojourn time function $D_i(.)$ is linear and non-negative.

(C2) The production flow rate function $u_i(.)$ is Lebesgue integrable and non-negative. Then, conditions (B1)-(B3) of Theorem 3.4 also hold.

In summary, the results of this subsection establish that by constraining the production capacity with the maximum variation of sojourn time with inventory:

- The effect of the variation of inventory with time can be limited, so that the FIFO property holds.

- When the FIFO property holds, the feasible region $F(DPM)$ is non-empty, and we can uniquely determine the sales flow rate and inventory in terms of the production flow rate.

Tightness of the Conditions of Theorem 3.4

In this subsection, we illustrate using a counter-example that conditions (B1)-(B3) in Theorem 3.4 are tight.

Theorem 3.5 *For any arbitrarily small positive scalar δ, there exist a product sojourn time function $D_i(.)$ and a production flow rate function $u_i(.)$ that verify the following conditions*
(D1) $D_i(.)$ is continuously differentiable and nondecreasing;
(D2) $u_i(.)$ is non-negative, Lebesgue integrable and bounded from above by M_i;
(D3) $\frac{1}{M_i} < Max\{D_i'(I_i), I_i \in R\}$ - $Min\{D_i'(I_i), I_i \in R\} \leq \frac{1}{M_i} + \delta$,
violating the FIFO property.

Proof: To show this, we will construct a production flow rate function $u_i(.)$ and a product sojourn time function $D_i(.)$ such that $(u_i(.), D_i(.))$ verify conditions (D1)-(D3) of Theorem 3.5, violating the FIFO property.

Let δ and M_i be any positive scalars, B_{1i} and β be any non-negative scalars, and ϵ and α be any two positive scalars such that: $\epsilon < \alpha$. Let ω be a positive scalar such that $\omega \in (\alpha, 2\alpha - \epsilon)$.

We first construct the product sojourn time function $D_i(.)$. We define $D_i(.)$ on three contiguous intervals: $[0, (\alpha-\epsilon)M_i]$, $((\alpha-\epsilon)M_i, \alpha M_i)$ and $[\alpha M_i, +\infty)$. On the first and third intervals, $D_i(.)$ is affine with a slope on its first affine piece less than the slope on its second affine piece. On the second interval, $D_i(.)$ is an exponential, nondecreasing and continuously differentiable function. Let I_{i1}, y_{i1}, I_{i2}, y_{i2}, γ_{i1} and γ_{i2} be given by:

$$\begin{aligned}
I_{i1} &= (\alpha - \epsilon)M_i & \text{and,} \quad & y_{i1} = D_i(I_{i1}) = \alpha + B_{1i}(\alpha - \epsilon)M_i,\\
I_{i2} &= \alpha M_i & \text{and,} \quad & y_{i2} = D_i(I_{i2}) = \beta + (B_{1i} + \tfrac{1}{M_i} + \delta)\alpha M_i,\\
\gamma_{i1} &= \frac{B_{1i}+\frac{1}{M_i}+\delta}{y_{i2}} - \frac{2}{I_{i2}-I_{i1}} & \text{and,} \quad & \gamma_{i2} = \frac{B_{1i}}{y_{i1}} - \frac{2}{I_{i1}-I_{i2}}.
\end{aligned}$$

Consider the following product sojourn time function $D_i(.)$:

$$D_i(I_i) = \begin{cases} \alpha + B_{1i}I_i & \text{, on } [0, I_{i1}] \\ y_{i2}(\frac{I_i-I_{i1}}{I_{i2}-I_{i1}})^2 e^{\gamma_{i1}(I_i-I_{i2})} + y_{i1}(\frac{I_i-I_{i2}}{I_{i1}-I_{i2}})^2 e^{\gamma_{i2}(I_i-I_{i1})} & \text{, on } (I_{i1}, I_{i2}) \\ \beta + (B_{1i} + \frac{1}{M_i} + \delta)I_i & \text{, on } [I_{i2}, +\infty). \end{cases}$$

Notice that $D_i(.)$ is continuously differentiable and nondecreasing on $[0, +\infty)$.

Consider the production flow rate function $u_i(.)$ given by:

$$u_i(t) = \begin{cases} M_i & \text{, if} \quad t \in [0, \omega), \\ 0 & \text{, if} \quad t \in [\omega, +\infty). \end{cases}$$

Notice that functions $u_i(.)$ and $D_i(.)$, as defined above, verify conditions (D1)-(D3) of Theorem 3.5.

In what follows, we solve constraints (3.3)-(3.8) of the $F(DPM)$ on intervals $[0, \alpha - \epsilon)$ and $[\alpha, \omega]$. That is, we express the variables $U_i(.)$, $v_i(.)$, $V_i(.)$, $I_i(.)$, and $s_i(.)$ in terms of the data above. Then, we show that the FIFO property is violated at $t = \omega$.

Notice that for $t \in [0, \alpha - \epsilon)$, $u_i(t) = M_i$. Hence, $U_i(t) = M_i t$. Furthermore, since $\alpha - \epsilon \le \alpha = D_i(0) = t_1$, it follows that $V_i(t) = 0$. Thus, $I_i(t) = U_i(t) - V_i(t) = M_i t$.

For $t \in [\alpha, \omega]$, there exists $z \in [0, \alpha - \epsilon)$ such that $s_i(z) = t$. Hence $z + D_i(I_i(z)) = t$. Thus, $z + \alpha + B_{1i} M_i z = t$. It follows that $z = s_i^{-1}(t) = \frac{t-\alpha}{1+B_{1i}M_i}$. Using equation (3.5) of the DPM formulation, that describes the relationship between the cumulative sales and the production flow rate, we obtain

$$V_i(t) \quad = \quad \int_0^{s_i^{-1}(t)} u_i(w)dw = \int_0^{\frac{t-\alpha}{1+B_{1i}M_i}} M_i dw = \frac{(t-\alpha)}{1+B_{1i}M_i} M_i.$$

Hence, $v_i(t) = \frac{M_i}{1+B_{1i}M_i}$. Therefore, we obtain $I_i(t) = M_i t - \frac{t-\alpha}{1+B_{1i}M_i} M_i = \frac{\alpha M_i + B_{1i} M_i^2 t}{1+B_{1i}M_i}$. Since $t \ge \alpha$, it follows that $I_i(t) \ge \alpha M_i$. Hence, by definition of $D_i(.)$, it follows that $D_i'(I_i(t)) = (B_{1i} + \frac{1}{M_i} + \delta)$.

Next, we show that $s_i'(\omega) < 0$. Indeed, since $s_i'(t) = 1 + D_i'(I_i(t))(u_i(t) - v_i(t))$, it follows that

$$\begin{aligned} s_i'(\omega) &= 1 + D_i'(I_i(\omega))(u_i(\omega) - v_i(\omega)) \\ &= 1 + (B_{1i} + \frac{1}{M_i} + \delta)(0 - \frac{M_i}{1+B_{1i}M_i}) = -\delta \frac{M_i}{1+B_{1i}M_i} < 0. \end{aligned}$$

This implies that the exit time function $s_i(.)$ is strictly decreasing at $t = \omega$. Hence, the FIFO property is violated for $t = \omega$.

□

Properties of the Feasible Region

In this subsection, we present some properties of the $F(DPM)$. These properties are not only useful in understanding the structure of the model, but will also be useful in Subsection 3.5.2 in order to prove the existence of a solution to the Dynamic Pricing Model.

Let $D = (D_1, ..., D_n)$, $p = (p_1, ..., p_n)$, and $u = (u_1, ..., u_n)$ denote respectively a vector of product sojourn time functions, a vector of unit price functions, and a vector of production flow rate functions. $(D(.), p(.), u(.))$ is feasible if each component $(D_i, p_i, u_i(.))$ verifies conditions (B1)-(B3) of Theorem 3.4 as well as capacity equation (3.8). In this case, using Theorem 3.4, the product inventory functions $I_i(.)$, the sales flow rates $v_i(.)$, and the cumulative variables can be uniquely determined through the product sojourn time functions $D_i(.)$, the unit price functions $p_i(.)$, and the production rates $u_i(.)$ on the analysis period $[0, T_\infty]$.

Proposition 3.6 *Assume that for every product i, the unit price function p_i is bounded by a scalar p_i^{max}. Then, the feasible region $F(DPM)$ is non-empty and bounded.*

Proof:

First, notice that $(D(.), p(.), 0)$ lies in the feasible region $F(DPM)$.

Further, let CFR denote the minimum total capacity, i.e. $CFR = min_{t \in [0,T]}(CFR(t))$. We assume, without loss of generality, that $CFR > 0$, and show that we can construct a feasible solution $(D(.), p(.), u(.))$ with $u(.) \neq 0$. Given a vector of product sojourn time functions $D(.)$, and a vector of unit price functions $p(.)$, let M denote the scalar $M = min(CFR, \frac{1}{max(\frac{1}{CFR}, (B_{2i} - B_{1i})_{\{i: B_{2i} - B_{1i} > 0\}})})$. Let $(\alpha_1, ..., \alpha_n)$ denote a finite sequence of non-negative scalars such that $\sum_{i=1}^{n} \alpha_i = 1$. For every $i \in \{1, ..., n\}$, and for every $t \in [0, T]$, let $u_i(t) = \alpha_i M$. It follows that vector $(D(.), p(.), u(.))$ as well as every vector $(D(.), p(.), \overline{u}(.))$, with $0 \leq \overline{u}(.) \leq u(.)$, are feasible.

Moreover, from the proof of Theorem 3.4, it follows that the flow rate functions $u_i(.)$ and v_i are bounded by CFR, the cumulative flow rate functions $U_i(.)$, $V_i(.)$ and $I_i(.)$ are bounded by $CFR.T_\infty$, and the sojourn time functions $D_i(.)$ and the exit time functions $s_i(.)$ are bounded by T_∞. Furthermore, by assumption, the unit price functions $p_i(.)$ are bounded. Therefore, the $F(DPM)$ region is bounded.

□

Proposition 3.7 *If vectors $(D(.), p(.), u(.))$ and $(D(.), q(.), w(.))$ are feasible, then, for every $\lambda \in [0, 1]$, vector $(D(.), \lambda p(.) + (1-\lambda)q(.), \lambda u(.) + (1-\lambda)w(.))$ is also feasible. In this sense, the feasible region $F(DPM)$ is convex.*

Proof:

We assume that $(D(.), p(.), u(.))$ and $(D(.), q(.), w(.))$ are feasible. For any $\lambda \in [0, 1]$, it is easy to see that $(D(.), \lambda p(.) + (1-\lambda)q(.), \lambda u(.) + (1-\lambda)w(.))$ verifies conditions (B1)-(B3) of Theorem 3.4 as well as capacity equation (3.7).

□

Proposition 3.8 *If a sequence $(p^j(.))_{j \in \aleph}$ of vectors of unit price functions converges to $(p(.))$, and a sequence $(u^j(.))_{j \in \aleph}$ of vectors of production flow rates converges to $(u(.))$, and, if for every j, vector $(D(.), p^j(.), u^j(.))$ is feasible, then, the limit $(D(.), p(.), u(.))$ is also feasible. In this sense, the $F(DPM)$ region is closed.*

Proof:

Let us assume that for all $j \in \aleph$, vectors $(D(.), p^j(.), u^j(.))$ are feasible. Then, it is easy to see that the limit $(D(.), p(.), u(.))$ verifies conditions (B1)-(B3) of Theorem 3.4 as well as capacity equation (3.7).

□

3.5.2 Existence of an Optimal Production/Inventory Control Policy

A Variational Inequality Formulation for the Dynamic Pricing Model

In this subsection, we formulate the DPM Model as a variational inequality problem. Using this variational inequality formulation, in the next subsection, we will establish the existence of an optimal production/inventory control policy under weak assumptions.

The DPM Model introduced in Section 3.3 can be summarized as the problem of finding a vector $e^*(.) = (u_i^*(.), v_i^*(.), I_i^*(.), p_i^*(.))_{i \in \{1,\dots,n\}} \in F(DPM)$ that maximizes the objective function:

$$
\begin{aligned}
F(e(.)) &= \sum_{i=1}^{n} \int_0^{T_\infty} f_i(e_i(t))dt \\
&= \sum_{i=1}^{n} \int_0^{T_\infty} p_i(I_i(t))v_i(t) - c_i(t)u_i(t) - h_i(t)I_i(t)dt.
\end{aligned}
$$

If $G_i(t, e_i(t))$ denotes the gradient $\nabla f_i(e_i(t))$ of $f_i(e_i(t))$, then notice that

$$
G_i(t, e_i(t)) = (-c_i(t), p_i(I_i(t)), p_i'(I_i(t)v_i(t) - h_i(t), v_i(t)).
$$

Therefore, the DPM Model is equivalent to solving the following variational inequality problem: Find a vector $e^* \in F(DPM)$ satisfying

$$
\begin{aligned}
&\textstyle\sum_{i=1}^{n} \int_0^{T_\infty} (-c_i(t), p_i(I_i^*(t)), p_i'(I_i^*(t)v_i^*(t) - h_i(t), v_i(t)). \\
&(u_i(t) - u_i^*(t), v_i(t) - v_i^*(t), I_i(t) - I_i^*(t), p_i(I_i(t)) - p_i^*(I_i^*(t)))^T dt \leq 0, \quad (3.25)
\end{aligned}
$$

for all vectors $e(.) \in F(DPM)$.

Variational inequality (3.25) can be written in compact form as: Find a vector $e^* \in F(DPM)$, such that for all vectors $e(.) \in F(DPM)$,

$$
< G(e^*), e^* - e > \leq 0, \qquad (3.26)
$$

where $< x, y >$ denotes the scalar product $\sum_{i=1}^{n} \int_0^{T_\infty} x_i(t).y_i(t)dt$ of two vectors x and y.

In what follows, we will refer to $G(.)$ as the Dynamic Pricing Map.

Properties of the Dynamic Pricing Map

In this subsection, we establish some properties of the Dynamic Pricing Map $G(.)$. These properties will be useful in establishing that the general Dynamic Pricing Model has a solution. We first introduce a definition from functional analysis (for more details, see Kirillov [46], Kolmogorov and Fomin [48], and Rudin [72]).

Definition 4 *(Weak Continuity):*

(i) A sequence $(u_n)_{n\in\aleph}$ in a normed space is said to converge weakly to u, if, for every bounded linear map $LM(.)$, $(LM(u_n))_{n\in\aleph}$ converges to $LM(u)$.

(ii) A map MP from a normed space to another is said to be weakly continuous if, for every sequence of functions $(u_n)_{n\in\aleph}$ weakly converging to u, the sequence $(||MP(u_n) - MP(u)||)_{n\in\aleph}$ converges to 0.

We establish the weak continuity of the Dynamic Pricing Map.

Theorem 3.9 *If the price inventory functions $p_i(I_i)$ are continuously differentiable and bounded from above by scalars p_i^{max}, then conditions (B1)-(B3) of Theorem 3.4 imply that the Dynamic Pricing Map $G(.)$ is weakly continuous.*

Proof:

The proposition below summarizes some results from functional analysis that are useful to prove Theorem 3.9 (for more details, see Kirillov [46], and Kolmogorov and Fomin [48]).

Proposition 3.10 *[46], [48]*

(i) If f and g are two weakly continuous maps, then the maps $f + g$, $f.g$ and $f(g)$ are weakly continuous.

(ii) If f is a weakly continuous map on the set of real numbers and has a constant sign, then the map $\frac{1}{f}$ is weakly continuous.

(iii) The integral operator from the space of bounded functions on $L^1([0,T_\infty])$ to $L^2([0,T_\infty])$ defined as $u(.) \mapsto \int_0^t u(w)dw$ is weakly continuous.

Proof of Theorem 3.9: Property (iii) of Proposition 3.10 implies that $u_i(.) \mapsto U_i(.)$ is weakly continuous. We will prove by induction over the time intervals $[t_j, t_{j+1})$ (defined in the proof of Theorem 3.4), that the maps $u_i(.) \mapsto V_i(.)$, $u_i(.) \mapsto I_i(.)$, $u_i(.) \mapsto s_i(.)$, $u_i(.) \mapsto v_i(.)$, $u_i(.) \mapsto s_i^{-1}(.)$ and $u_i(.) \mapsto (s_i^{-1})'(.)$ are weakly continuous.

We first need to establish a preliminary result.

Lemma 3.14 *Under conditions (B1)-(B3) of Theorem 3.4, if the product exit time operator $u_i \mapsto s_i(.)$ is weakly continuous on the interval $[t_j, t_{j+1})$, then its inverse operator $u_i \mapsto s_i^{-1}(.)$ is weakly continuous on the interval $[t_{j+1}, t_{j+2})$.*

Proof: We assume that the product exit time operator $u_i \mapsto s_i(.)$ is weakly continuous on the interval $[t_j, t_{j+1})$.

From the proof of Theorem 3.4 in Subsection 3.5.1, we know that for every $t \in [t_j, t_{j+1})$, $s_i'(t) \leq \theta_j$, where $\theta_j \in (0, 1)$ as defined in Lemma 3.12. Hence, $s_i^{-1}(.)$ is Lipschitz continuous on $[t_{j+1}, t_{j+2})$ with parameter $\frac{1}{\theta_j}$.

Furthermore, for every $t \in [t_j, t_{j+1})$,

$$\begin{aligned} s_i'(t) = 1 + D_i'(I_i(t))(u_i(t) - v_i(t)) &\leq 1 + D_i'(I_i(t))u_i(t), \\ &\leq 1 + B_{2i}M_i. \end{aligned}$$

Hence, $s_i(.)$ is Lipschitz continuous on $[t_j, t_{j+1})$ with parameter $1 + B_{2i}M_i$.

Let $(u_i^k(.))_{k\in\mathbb{N}}$ denote a weakly converging sequence of product flow rate functions to $u_i(.)$. Let $s_i^k(.)$ denote the product exit time function corresponding to $u_i^k(.)$.

Furthermore,

$$\begin{aligned} \int_{t_{i+1}}^{t_{i+2}} |(s_i^k)^{-1}(w) - s_i^{-1}(w)|^2 dw &= \int_{s_i(t_i)}^{s_i(t_{i+1})} |(s_i^k)^{-1}(w) - s_i^{-1}(w)|^2 dw \\ &= \int_{t_i}^{t_{i+1}} |(s_i^k)^{-1}(s_i(w)) - s_i^{-1}(s_i(w))|^2 s_i'(w) dw \end{aligned}$$

$$
\begin{aligned}
&= \int_{t_i}^{t_{i+1}} |(s_i^k)^{-1}(s_i(w)) - w|^2 s_i'(w)dw \\
&= \int_{t_i}^{t_{i+1}} |(s_i^k)^{-1}(s_i(w)) - (s_i^k)^{-1}(s_i^k(w))|^2 s_i'(w)dw \\
&\leq \frac{1 + B_{2i}M_i}{\theta_j^2} \int_{t_i}^{t_{i+1}} |s_i^k(w) - s_i(w)|^2 dw.
\end{aligned}
$$

Since $u_i \mapsto s_i(.)$ is weakly continuous on the interval $[t_j, t_{j+1})$, it follows that $u_i \mapsto s_i^{-1}(.)$ is weakly continuous on the interval $[t_{j+1}, t_{j+2})$.

□

Induction Proof:

Base Case: Time interval $[t_0, t_1)$.

On $[t_0, t_1)$, $v_i(t) = V_i(t) = 0$ and $I_i(t) = U_i(t)$. Hence, the maps $u_i(.) \mapsto V_i(.)$, $u_i(.) \mapsto I_i(.)$, and $u_i(.) \mapsto v_i(.)$ are weakly continuous. Furthermore, since $s_i(t) = t + D_i(I_i(t))$, and $D_i(.)$ are continuous functions (and therefore weakly continuous), using property (i) of Proposition 3.10, it follows that the map $u_i(.) \mapsto s_i(.)$ is weakly continuous. Using Lemma 3.14 and property (i) of Proposition 3.10, it follows that the map $u_i(.) \mapsto s_i^{-1}(.)$ is also weakly continuous on $[t_1, t_2)$.

Moreover $(s_i^{-1})'(t) = \frac{1}{s_i'(s_i^{-1}(t))} = \frac{1}{1+D_i'(I_i(s_i^{-1}(t)))(u_i(s_i^{-1}(t))-v_i(s_i^{-1}(t))}$. Using properties (i) and (ii) of Proposition 3.10, we obtain that $u_i(.) \mapsto (s_i^{-1})'(.)$ is also weakly continuous on $[t_1, t_2)$.

Induction Step: Time interval $[t_{j+1}, t_{j+2})$. From the induction hypothesis, we know that the maps $u_i(.) \mapsto s_i^{-1}(.)$ and $u_i(.) \mapsto (s_i^{-1})'(.)$ are weakly continuous on $[t_{j+1}, t_{j+2})$. Since $v_i(w) = u_i(s_i^{-1}(w)).(s_i^{-1})'(w)$, using property (i) of Proposition 3.10, it follows that the map $u_i(.) \mapsto v_i(.)$ is weakly continuous. Property (iii) of Proposition 3.10 implies that $u_i(.) \mapsto V_i(.)$ is weakly continuous. Since $I_i(.) = U_i(.) - V_i(.)$ and $s_i(t) = t + D_i(I_i(t))$, property (i) of Proposition 3.10 implies that $u_i(.) \mapsto I_i(.)$ and $u_i(.) \mapsto s_i(.)$ are weakly continuous.

Using Lemma 3.14, it follows that $u_i(.) \mapsto s_i^{-1}(.)$ is also weakly continuous on interval $[t_{j+2}, t_{j+3})$. Moreover, since $(s_i^{-1})'(t) = \frac{1}{1+D_i'(I_i(s_i^{-1}(t)))(u_i(s_i^{-1}(t))-v_i(s_i^{-1}(t))}$, using properties (i) and (ii) of Proposition 3.10, we obtain that $u_i(.) \mapsto (s_i^{-1})'(.)$ is also weakly continuous on $[t_{j+2}, t_{j+3})$. The induction proof is now complete. Since the price inventory function $p_i(I_i)$ is continuously differentiable, both $p_i(I_i)$ and $p_i'(I_i)$ are continuous (and hence weakly continuous). Property (i) of Proposition 3.10 implies that the Dynamic Pricing Map $G(.)$ is weakly continuous.

□

We now define the notion of pseudo-monotonicity introduced by Brezis [18] and show that the Dynamic Pricing Map $G(.)$ is pseudo-monotone.

Definition 5 *(Pseudo-monotonicity) A bounded map MP is pseudo-monotone over X if, whenever a sequence $(u^k)_{k\in\aleph} \in X^{\aleph}$ weakly converging to u satisfies limsup $< MP(u^k), u^k - x > \leq 0$, $\forall x \in X$, it also satisfies liminf $< MP(u^k), u^k - x > \geq < MP(u), u - x >$, $\forall x \in X$.*

Lemma 3.15 *The Dynamic Pricing Map $G(.)$ is pseudo-monotone over the $F(DPM)$ region.*

Proof: Notice that $G(.)$ is weakly continuous on the $F(DPM)$ region, and from Proposition 1, the $F(DPM)$ region is bounded. Therefore, $G(.)$ is a bounded map. Let $diam(F(DPM))$ denote the diameter of the $F(DPM)$ region and let $(e_k)_{k\in\aleph}$ denote a sequence of elements of the $F(DPM)$ region weakly converging to e. Then, for $y \in F(DPM)$,

$$\begin{aligned} < G(e_k) - G(e), e_k - y > &\leq ||G(e_k) - G(e)||.||e_k - y|| \\ &\leq diam(F(DPM)).||G(e_k) - G(e)||. \end{aligned}$$

Since $G(.)$ is weakly continuous on the $F(DPM)$, it follows that the sequence $(||G(e_k) - G(e)||)_{k\in\aleph}$ converges to 0. Hence $lim_{k\to\infty} < G(e_k) - G(e), e_k - y >= 0$. It follows that:

$$\begin{aligned} lim_{k\to\infty} < G(e_k), e_k - y > &= lim_{k\to\infty} < G(e), e_k - y > \\ &= < G(e), e - y > . \end{aligned}$$

Hence, the Dynamic Pricing Map $G(.)$ is pseudo-monotone over the $F(DPM)$ region.

□

Existence of An Optimal Solution for the Dynamic Pricing Model

In this subsection, we establish one of the fundamental results of this chapter. That is, we illustrate that under weak assumptions, the DPM Model possesses an optimal solution.

Theorem 3.11 *Assume that the following conditions hold:*
(E1) The price inventory functions $p_i(I_i)$ are continuously differentiable and bounded from above by scalars p_i^{max}.
(E2) The product sojourn time functions $D_i(.)$ are continuously differentiable, and there exist two non-negative constants B_{1i} and B_{2i} such that for every inventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.
(E3) The shared capacity flow rate function $CFR(.)$ is Lebesgue integrable, non-negative and bounded from above by a positive real number CFR on $[0,T]$.
(E4) $CFR \leq \sum_1^n \frac{1}{B_{2i}-B_{1i}}$.
Then, the Dynamic Pricing Model has an optimal solution.

Proof: Under conditions (E1)-(E4), Theorem 3.9 holds, that is, the Dynamic Pricing Map $G(.)$ is weakly continuous. Using Lemma 3.15, it follows that the Dynamic Pricing Map $G(.)$ is pseudo-monotone over the $F(DPM)$ region. From Propositions 3.6-3.8, the $F(DPM)$ region is non-empty, bounded, closed and convex. Using Lemma

3.16 with $K = F(DPM)$, $A(.) = G(.)$ and $z = 0$, and the variational inequality formulation (3.26), it follows that the Dynamic Pricing Model has an optimal solution.

Lemma 3.16 *(Brezis [18], [19])*

Let K be an non-empty, bounded, convex and closed set. Let $A(.)$ denote a map from K to L that is pseudo-monotone map over K. Then, for every vector $z \in L$, there exists a vector $e^* \in K$ such that $< A(e^*), e - e^* > \geq < z, e - e^* >$ is verified for every vector $e \in K$.

For more details on the above lemma, see [18], [19], [52], [53], and [54].

□

Chapter 4

The Anticipatory Route Guidance Problem: Formulations and Analysis

4.1 Introduction and motivation

4.1.1 Introduction

The anticipatory route guidance problem (ARG), an extension of the dynamic traffic user-equilibrium problem, consists of providing messages to drivers, based on forecasts of traffic conditions, to assist them in their path choice decisions. Guidance becomes inconsistent when the forecasts on which it is based are violated after drivers react to the provided messages.

In this chapter, we consider the problem of generating consistent anticipatory guidance that ensures that the messages based on dynamic shortest path criteria do not become self-defeating prophecies. We design a framework for the analysis of the ARG problem based on a fixed-point formulation of the problem.

We also provide an infinite-dimensional variational inequality (VI) formulation. These equivalent formulations are to the best of our knowledge the first general analytical formulations of this problem. Furthermore, we establish, under weak assumptions, the existence of a solution to the ARG problem.

4.1.2 Motivation

An important characteristic of road traffic congestion is its randomness. Data suggest that roughly 60% of congestion-related delays on urban freeways in the U.S. are due to specific random incidents such as accidents, vehicle breakdowns and the like (see Lindley [51] for more details). Even without such incidents, congestion has a random component that comes from the variability in demand patterns and in network performance. Because of this randomness, a driver's past experience can be an unreliable basis for predicting the conditions associated with various travel options, and as a result, for making good travel choices.

Advanced traveler information systems (ATIS) attempt to provide tripmakers with data intended to help them make better travel decisions. In this chapter, such data will be referred to as *messages.* Messages may have an arbitrary content. They may be available to all tripmakers (for example by radio or television broadcasts) or only to some: for example, those who pass near a particular infrastructure (such as variable message signs) or who have special receivers. Tripmakers, of course, may react to the messages in any way they choose.

Traveler information systems may be distinguished on the basis of the type of information they provide in messages. Static systems furnish information that changes only infrequently, such as locations of and directions to trip attractions such as cultural centers or restaurants. *Reactive* systems estimate prevailing travel conditions from real-time measurements and provide messages directly based on these estimates: for example, information about current travel times. *Predictive* or *anticipatory* systems use real-time measurements to forecast travel conditions in the near-term future (up to a few hours), and present messages based on these predictions. As a result, a trip-maker can make a decision based on what conditions would be at network locations at the time he/she would actually be there, rather than on (possibly very different) currently prevailing conditions.

This chapter focuses on predictive traveler information systems that provide messages intended to facilitate drivers' path choice decisions before and during a trip. The messages may inform drivers about anticipated traffic conditions on different available paths, or (based on these conditions) recommend a specific path to follow, or both. Such systems are sometimes called route guidance systems.

If only a few drivers receive route guidance messages, they may benefit from it by making better path choice decisions. Nevertheless, the choices they make will not impact overall network traffic conditions. On the other hand, when more drivers receive guidance, their reactions to the guidance may have a significant effect on traffic conditions. The key issue in generating guidance messages based on traffic condition forecasts is to ensure that drivers' reactions to the guidance do not invalidate the forecasts and render the guidance irrelevant or worse. Messages predicting impending congestion on one road, for example, may cause drivers to switch *en masse* to a parallel road less able to accommodate them, leaving the original road free flowing and overall producing worse traffic conditions. Guidance is *consistent* when the forecasts on which it is based are indeed experienced by drivers after they react to it.

Generation of anticipatory guidance clearly requires the application of some kind of a traffic prediction model. Deterministic traffic assignment models assume that drivers departing from their origin have full information about network conditions on the paths available to them, choose one accordingly, and follow it unswervingly to their destination. Stochastic assignment models account for driver perception errors by assuming that drivers at the origin choose a path based on a randomly perturbed version of actual network conditions. A full information assumption underlies even these models in the sense that, as the magnitude of the perception error decreases, the stochastic equilibrium path choices increasingly resemble those that would be made in a full information deterministic setting.

However, the applicability of the full information assumption to general route guidance modeling is questionable. Consider a guidance system consisting of a variable message sign (VMS) located somewhere on a network. Drivers leaving their origins make a path choice based on assumptions about traffic conditions that may be more or less accurate. Drivers whose path choice happens to take them by the VMS receive guidance messages and may decide to switch to another path for the remainder of their trip; those who do not presumably pursue the path they chose earlier to their destination. The information available to a driver, and the resulting en route path switches, depend on the path taken through the network and the guidance information available at locations along that path. This path dependency of information availability and driver behavior has no counterpart in full information models.

Some network-level analyses of route guidance systems have assumed that the effect of guidance will be to establish conventional equilibrium conditions (see Kaufman *et al.* [43] and Engelson [27] for more details). Others have modeled guidance effects via a reduction in the perception error of guided drivers using a stochastic assignment model (see Lotan and Koutsopoulos [55] and Hamerslag and van Berkum [35] for more details). These approaches may be appropriate if guidance (in some cases, perfect guidance) is available to drivers at all decision points. More generally, however, where a guidance system provides limited information at certain network locations, there is no reason to expect that the resulting flow patterns and traffic conditions would correspond to those of a conventional network equilibrium solution.

A number of traffic simulation models, such as DYNASMART by Mahmassani and co-workers (Mahmassani *et al.* [56]), MITSIM and DynaMIT by Ben-Akiva and co-workers (Ben-Akiva *et al.* [8] and Bottom [14]), can represent a variety of guidance technologies, including limited-range systems such as VMS. These models attempt to achieve consistency, often using methods that explicitly or implicitly determine a fixed point. However, to the best of our knowledge, no analytical results are available regarding their solutions.

Bovy and van der Zijpp [16], and Bottom [14] proposed a time-dependent framework for the Anticipatory Route Guidance (ARG) Problem. The variables are the network conditions, the path splitting rates at control nodes (i.e. the rates of flow splits between paths at control nodes) and the guidance messages. The relationships are the dynamic network loading map, which transforms the path splits into network conditions (see Kachani [39] for a detailed analysis); the guidance map, which transforms the network conditions into guidance messages; and the routing map, which transforms guidance messages into path splits. These three relationships can then be combined into three alternative composite maps that model the ARG Problem and that lead to three equivalent fixed-point formulations. Finding consistent guidance is equivalent to finding a fixed point of a composite map. However, since these composite maps are discontinuous, the standard existence results of fixed-point theory do not apply in this case. As a result, it is still an open question whether the ARG Problem even possesses a solution.

One of our main goals in this chapter is to formulate and establish the existence of a solution to the ARG problem under weak assumptions. To achieve this, we propose the first analytical formulation of the ARG Problem. We hope these results will lay the foundations for the use of numerical methods for solving fixed-point formulations and variational inequality formulations of this problem in practice. Such numerical results can be found in Bottom, Kachani and Perakis [15].

The chapter is organized as follows. In Section 4.2, we start by introducing the notation and the feasibility conditions of the ARG problem. We then provide a variational inequality (VI) formulation of this problem. We also present a fixed-point formulation of the problem and establish equivalence of the two formulations. We discuss two special cases: the static ARG problem and the Dynamic User-Equilibrium problem. In Section 4.3, we study the mathematical properties of the problem. Under sufficient conditions on the path flow rate functions and the travel time functions, we establish that the feasible region $F(ARG)$ of the Anticipatory Route Guidance problem is non-empty, and that the FIFO property holds. We provide a generic counterexample illustrating that the assumptions we imposed, to ensure that FIFO holds, are the tightest possible. We establish key properties of the feasible region, as a function of the path flow rate functions, such as boundedness, closedness and convexity. Finally, we establish the existence of a solution to the ARG problem.

4.2 Problem Formulations

In Subsection 4.2.1, we introduce the notation we use throughout the chapter. In Subsection 4.2.2, we state the feasibility conditions of the ARG problem. In Subsection 4.2.3, we then provide a variational inequality (VI) formulation of the problem. Furthermore, in Subsection 4.2.4, we connect this formulation with a fixed-point formulation. In Subsection 4.2.5, we establish equivalence of the two formulations. Finally, in Subsection 4.2.6, we discuss how the model simplifies in the case of the Dynamic User-Equilibrium problem.

4.2.1 Notation

In this section, we introduce the necessary notation to formulate the ARG problem. The notation we introduce is quite tedious. However, much of the contribution of this work is in identifying the minimum set of notation that enable us to formulate the ARG Problem analytically.

The physical traffic network is represented conceptually by a directed network $G = (N, A)$, where N is the set of nodes and A is the set of directed links. N_1 denotes the set of origin nodes, N_2 the set of control nodes (i.e. nodes at which vehicle messaging display systems are placed), and P the set of paths.

In practice, the number of control nodes in a transportation network is small relative to the total number of nodes. As a result of their path choice decision, drivers might not go through control nodels. Therefore, we distinguish between two categories of drivers. Drivers in the first category do not receive any information. These drivers have an estimate of flows in the network that they utilize to select their paths. On the other hand, drivers in the second category go through control nodes. As a result, these drivers receive full information about the network traffic conditions.

In the following, the index r denotes an origin node, the index s denotes a destination node and the index p denotes a path between Origin-Destination (O-D) pair (r, s). The subset of paths between O-D pair (r, s) is denoted by K_{rs}. Below, we provide the inputs and outputs of the ARG problem.

Inputs of the ARG problem

Path variables:

$\|P\|$	:	number of paths in the network;
$RS(p)$	:	(O-D) pair associated with path p;
p^1	:	first link of path p;
p^l	:	last link of path p;
$f_p(t)$	:	departure flow rate on path p at time t;
$f^{RS(p)}(t)$	:	departure flow rate on O-D pair $RS(p)$ at time t;
f	:	vector of path departure flow rate functions $(f_p(.))_{p\in P}$ for all times t;
M_p	:	upper bound on the departure path flow rate function $f_p(.)$.

Link variables:

$head(a)$: head node of link a;

$tail(a)$: tail node of link a;

$D_a(y)$: travel time function of link a, where y is the number of vehicles on link a;

B_{1a} : lower bound on the derivative $D'_a(.)$ of the travel time function $D_a(.)$;

B_{2a} : upper bound on the derivative $D'_a(.)$ of the travel time function $D_a(.)$.

Link-path flow variables:

(a, p) : a link-path pair;

δ_{ap} = 1 if link a belongs to path p, and 0 otherwise.

Time variables:

t : index for continuous time;

$[0, T]$: O-D traffic demand period. After time T, the flow rate functions are zero.

Outputs of the ARG problem

Path variables:

$f^{n,RS(p)}(t)$: flow rate on O-D pair $RS(p)$ traversing node n at time t;

$\beta_{np}(t)$: path splitting rate of path p at node n at time t;

β : vector of path splitting rate functions $\beta_{np}(.)$;

$S_p(t, f)$: travel time on path p of a vector of flows $f(.)$ departing at time t.

Link variables:

M_a : upper bound on the entrance link flow rate function $u_a(.)$;

$u_a(t)$: entrance flow rate on link a at time t;

$v_a(t)$: exit flow rate on link a at time t;

$U_a(t)$: cumulative entrance flow on link a during time interval $[0, t]$;

$V_a(t)$: cumulative exit flow on link a during time interval $[0, t]$;

$X_a(t)$: load (number of vehicles) of link a at time t;

$s_a(t)$: exit time of a flow entering link a at time t

$= t + D_a(X_a)(t))$.

Link-path flow variables:

$u_{ap}(t)$: entrance flow rate on link a traveling on path p at time t;

$v_{ap}(t)$: exit flow rate on link a traveling on path p at time t;

$U_{ap}(t)$: cumulative entrance flow on link a traveling on path p during time interval $[0, t]$;

$V_{ap}(t)$: cumulative exit flow on link a traveling on path p during time interval $[0, t]$;

$X_{ap}(t)$: partial link load on link a induced by flow on path p at time t.

Time variables:

$[0, T_\infty]$: analysis period. It is the interval of time from the instant when flows enter the network to the first instant when all flows exit the network.

Below, we perform a network transformation that will enable us to define decision variables that are equivalent to the path-node splitting rates $\beta_{np}(t)$. This network transformation is the basis of all our results in this chapter.

We divide every path $p \in (r, s)$ that goes through a control node into subpaths, in the following manner. Subpaths can either (i) originate at r and end at the first control node on path p, or (ii) originate at a control node and end at the following control node on path p, or (iii) originate at the last control node on path p and end at s. Let P_1 denote the set of subpaths we have just created. Note that we allow P_1 to contain several copies of the same subpath. However, these copies come from different paths. Let P_2 denote the set of paths that do not go through control nodes (but might originate at control nodes).

Let $\overline{P} = P_1 \cup P_2$. Also, let $\overline{P}_1$ be the set of subpaths that originate at control nodes and hence receive information about the network conditions. It follows that $\overline{P}_1 = \{p \in \overline{P} | head(p^1) \in N_2\}$, where $head(p^1)$ denotes the head of the first link of path p, that is the origin node of path p. Let $\overline{P}_2 = \overline{P} \setminus \overline{P}_1$.

Let $\widehat{p}(p)$ denote the path containing subpath p (for example, this could be path p itself if $p \in P_2$). $s(p)$ will denote the first subpath on path p (this could be path p itself if $p \in P_2$). Finally, $f_{p\widehat{p}(p)}(t)$ denotes the subpath flow rate at time t on subpath p of path $\widehat{p}(p)$.

In the example of Figure 4-1, we consider a network of 9 nodes and one O-D pair (O, D). Applying the network transformation defined above on this network, we obtain:

$$\overline{P}_1 = \{(1,3,6,D),(1,4),(1,4),(4,6,D),(4,6,D),(4,7,D),(4,7,D)\},$$
$$\overline{P}_2 = \{(O,1),(O,1),(O,1),(O,2,4),(O,2,4),(O,2,5,7,D)\}.$$

Notice for instance that there are two copies of subpath $(4,6,D)$ in $\overline{P}_1$ since both path $(O,1,4,6,D)$ and path $(O,2,4,6,D)$ contain this subpath.

4.2.2 Feasibility Conditions of the ARG Problem

The objective of this subsection is to present an analytical formulation of the feasibility conditions of the ARG problem. Each point in $F(ARG)$, the feasible region of the ARG problem, is obtained as a solution of a dynamic network loading (DNL) problem.

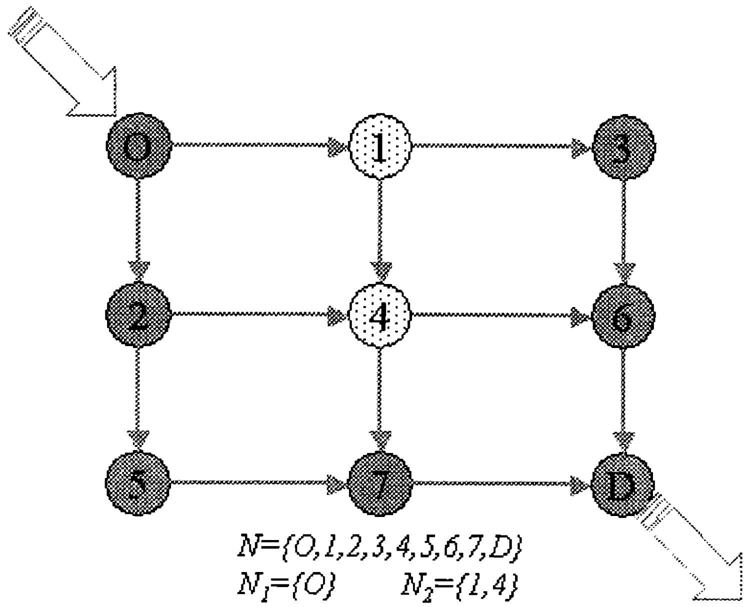

Figure 4-1: Network Example

In the context of the ARG problem, the DNL problem consists of determining the time-varying network flows and travel times that result from the movement of given origin-destination flows over the network in accordance with particular splitting rates at the origins and at intermediate control nodes. For a fixed set of origin-destination flows, the ARG DNL problem can thus be viewed as a map from the domain of path splitting rates to the range of network flows and travel times.

Similar to the DNL maps by Friesz *et al.* [30], by Wu *et al.* [81] and by Kachani [39] in the context of the Dynamic User-Equilibrium problem (DUE), the DNL map of this chapter is formulated as a system of equations expressing link dynamics, flow conservation, flow propagation, non-negativity and boundary constraints. Unlike the DUE DNL problem, in which flows departing the origin on a particular path always remain on that path, the ARG DNL problem allows flow to change from one path to another at intermediate locations. Furthermore, the ARG DNL map has two added features: (i) the model is formulated in terms of subpath flow rates instead of path flow rates, and (ii) the path-node splitting rates are explicitly used.

Note that, for every node and every time instant, the set of feasible splitting rates form a simplex. Therefore, the set of all feasible splitting rates is a product of simplices. The ARG feasible region, $F(ARG)$, is then the set of flows and travel times that result as the splitting rates vary across their feasible region.

Link dynamics equations

The link dynamics equations express the relationship between the flow variables of a link. They are given by:

$$\frac{dX_{ap}(t)}{dt} = u_{ap}(t) - v_{ap}(t), \qquad \forall (r,s), \forall p \in K_{rs}, \forall a \in p. \tag{4.1}$$

Flow conservation equations

For every link a that has a head node which is neither an origin node nor a control node (i.e. $head(a) \in N \setminus (N_1 \cup N_2)$), the flow conservation equations can be expressed as

$$u_{ap}(t) = v_{a'p}(t), \tag{4.2}$$

where a' is the link preceeding link a on path p.

For every link a that has a head node n which is an origin node (i.e. $n \in N_1$) and for all paths p originating at n, the flow conservation equations can be expressed as

$$\begin{aligned} u_{ap}(t) &= f_{s(p)p}(t) \\ &= \beta_{np}(t) f^{RS(p)}(t), \end{aligned} \tag{4.3}$$

where the (O-D) pair departure flow rates $f^{RS(p)}(t)$ are given.

For every link a that has a head node n which is a control node (i.e. $n \in N_2$) and for all paths p that do not originate at n, the flow conservation equations can be expressed as

$$\begin{aligned} u_{ap}(t) &= f_{p\widehat{p}(p)}(t) \\ &= \beta_{np}(t) f^{n,RS(p)}(t), \end{aligned} \tag{4.4}$$

where $f^{n,RS(p)}(t) = \sum_{\overline{p} \in RS(p), \overline{a} \in \overline{p} | tail(\overline{a})=n} v_{\overline{ap}}(t)$.

The path-node splitting rates $\beta_{np}(t)$ (or equivalently the subpath flow rates $f_{p\widehat{p}(p)}(t)$) are the unknown variables in the ARG Problem. In fact, notice that there is a one-to-one mapping between the path-node splitting rates $\beta_{np}(t)$ (exogenous variables) and the subpath flow rates $f_{p\widehat{p}(p)}(t)$ (endogenous variables). Furthermore, one can directly compute one quantity from the other, provided that the O-D pair flow rates $f^{RS(p)}(t)$ are given and the node/O-D pair flow rates $f^{n,RS(p)}(t)$ are computed.

Link-path flow relationships

The following relationships express that the link flow variables are the sum of their corresponding link-path variables:

$$u_a(.) = \textstyle\sum_{p|a \in p} u_{ap}(.), \qquad v_a(.) = \sum_{p|a \in p} v_{ap}(.),$$

$$U_a(.) = \textstyle\sum_{p|a \in p} U_{ap}(.), \qquad V_a(.) = \sum_{p|a \in p} V_{ap}(.), \tag{4.5}$$

$$X_a(.) = \textstyle\sum_{p|a \in p} X_{ap}(.), \qquad \forall (r,s), \forall p \in K_{rs}, \forall a \in p.$$

Flow propagation equations

Flow propagation equations are used to describe the flow progression over time. Note that a flow entering link a at time t will exit the link at time $s_a(t)$. Therefore, by time t, the cumulative exit flow of link a should be equal to the integral of all inflow rates which would have entered link a at some earlier time ω and exited link a by time t. This relationship is expressed by the following equation:

$$V_{ap}(t) = \int_{\omega \in W} u_{ap}(\omega) d\omega, \qquad \forall (r,s), \forall p \in K_{rs}, \forall a \in p, \tag{4.6}$$

where $W = \{\omega : s_a(\omega) \leq t\}$.

If the link exit time functions $s_a(.)$ are continuous and satisfy the strict FIFO property, then the flow propagation equations (4.6) can be equivalently rewritten as

$$V_{ap}(t) = \int_0^{s_a^{-1}(t)} u_{ap}(\omega) d\omega, \qquad \forall (r,s), \forall p \in K_{rs}, \forall a \in p. \tag{4.7}$$

The strict FIFO condition implies that a car entering link a at time t, will exit only after the cars that entered link a before it, have all exited. In mathematical terms, this is equivalent to the link exit time functions $s_a(.)$ being strictly increasing.

Link exit time functions $s_a(t)$ are obtained from link travel time functions using the following definitional constraint:

$$s_a(t) = t + D_a(X_a(t)).$$

Valid expressions of $D_a(X_a(t))$ can be found in Kachani and Perakis [40], [41] and [42].

Non-negativity constraints

We further assume that the departure path flow rates are non-negative:

$$f_p(.) \geq 0 \qquad \forall (r,s), \forall p \in K_{rs}. \tag{4.8}$$

Boundary equations

Since we assume that the network is empty at $t = 0$, the following boundary conditions are required:

$$U_{ap}(0) = 0, \quad V_{ap}(0) = 0, \quad X_{ap}(0) = 0, \quad \forall (r,s), \forall p \in K_{rs}, \forall a \in p. \tag{4.9}$$

Notice that the above formulation of the DNL map is general enough to account for the case where the FIFO property, defined above, is not necessarily verified (notice that Equation (4.6) does not assume that the FIFO property holds). In Section 4.3, we investigate when the FIFO property holds. We examine conditions on the link travel time functions $D_a(.)$ and on the departure path flow rates $f_p(.)$. When the FIFO property holds, the model becomes more tractable.

In the remainder of the chapter, we will denote by $F(ARG)$ the feasible region of the ARG problem. In the next subsection, we provide a variational inequality formulation of the ARG problem.

4.2.3 A Variational Inequality Formulation

Similarly to the work of Friesz *et al.* [30] on variational inequality formulations for the Dynamic User-Equilibrium Problem in terms of the *path flow rates*, we can formulate the ARG Problem in terms of the *subpath flow rates.*

As we discussed in Subsection 4.2.1, we have two types of drivers in the network:

- Drivers who travel on subpaths $p \in \overline{P}_2$ do not receive any information. These drivers have a certain estimate $\widehat{f}$ of the vector of flows in the network. We assume that these drivers utilize these flow estimates to estimate the vector of path travel times $S(t, \widehat{f})$. Such estimate could be the travel times experienced by these users in the past (e.g. the day before or the week before) or a weighted average of past experiences. We further assume that these drivers use this estimate to select paths that minimize their travel times. The simplest case corresponds to the vector of free-flow path travel times $S(t, 0)$. In this case, drivers follow the static shortest path from their origin to their destination.

- Drivers who travel on subpaths $p \in \overline{P}_1$ (that originate at control nodes) receive full information about the network traffic conditions. We assume that these drivers behave "rationally" in the sense that they select paths that minimize their flow-dependent dynamic travel times.

In what follows, we consider these two categories of drivers traveling in the same network. As a result of the above two categories of drivers, solving the ARG problem is equivalent to solving the following variational inequality problem: Find a vector of time-dependent flows $f^\star \in F(ARG)$ satisfying

$$\begin{aligned}
&\sum_{r,s} \sum_{p \in \overline{P}_2} \int_0^{T_\infty} S_p(w, \widehat{f})(f_{p\widehat{p}(p)}(w) - f^\star_{p\widehat{p}(p)}(w))dw + \\
&\sum_{r,s} \sum_{p \in \overline{P}_1} \int_0^{T_\infty} S_p(w, f^\star)(f_{p\widehat{p}(p)}(w) - f^\star_{p\widehat{p}(p)}(w))dw \quad \geq 0, \qquad \forall f \in F(ARG).
\end{aligned} \tag{4.10}$$

Notice that the first part of this variational inequality formulation describes the first type of drivers while the second part describes the second type of drivers. For the sake of simplicity of notation, in the remainder of the chapter, we will denote by $< h, g >= \sum_{r,s} \sum_{p \in \overline{P}} \int_0^{T_\infty} h_p(w) g_p(w) dw$.

The above infinite-dimensional variational inequality formulation can be rewritten in a more compact form as: Find a vector of flows $f^\star = (f^\star_{|\overline{P}_1}, f^\star_{|\overline{P}_2}) \in F(ARG)$ satisfying

$$< S(\widehat{f}), f_{|\overline{P}_2} - f^\star_{|\overline{P}_2} > + < S(f^\star_{|\overline{P}_1}, f^\star_{|\overline{P}_2}), f_{|\overline{P}_1} - f^\star_{|\overline{P}_1} > \quad \geq 0,$$

$$\forall f = (f_{|\overline{P}_1}, f_{|\overline{P}_2}) \in F(ARG). \tag{4.11}$$

In summary, the continuous-time Anticipatory Route Guidance problem (ARG) is equivalent to the variational inequality formulation (4.11) subject to the system of equations (4.1)-(4.9). In general, the ARG problem is a continuous-time non-linear optimization problem. The non-linearity of the model comes from the path flow dependence of the path travel times in the variational inequality, as well as the integral equation (4.6). In this formulation, the input parameters consist of the link travel times $D_a(X_a)$ as functions of the number of vehicles in the link, and the (O-D) pair departure flow rates $f^{rs}(t)$. The unknown variables we wish to determine are the link and path entrance flow rates $u_a(.)$ and $f_p(.)$, the link exit flow rates $v_a(.)$, the link cumulative entrance and exit flows $U_a(.)$ and $V_a(.)$, the link loads $X_a(.)$, and the link and path exit time functions $s_a(.)$ and $S_p(.)$. Notice that due to the integral equation (4.6), this problem is hard to solve.

Furthermore, notice that solving Variational Inequality (4.11) is equivalent to solving the following two variational inequalities in sequence: Find a vector of flows $f^{\star}_{|\overline{P}_2} \in F(ARG)_{|\overline{P}_2}$ satisfying

$$< S(\hat{f}), f_{|\overline{P}_2} - f^{\star}_{|\overline{P}_2} > \quad \geq 0 \quad \forall f_{|\overline{P}_2} \in F(ARG)_{|\overline{P}_2}. \tag{4.12}$$

Then, find a vector of flows $f^{\star}_{|\overline{P}_1} \in F(ARG)_{|\overline{P}_1}$ satisfying

$$< S(f^{\star}_{|\overline{P}_1}, f^{\star}_{|\overline{P}_2}), f_{|\overline{P}_1} - f^{\star}_{|\overline{P}_1} > \quad \geq 0, \quad \forall f_{|\overline{P}_1} \in F(ARG)_{|\overline{P}_1}. \tag{4.13}$$

In the next subsection, we provide an alternative formulation of the ARG problem based on a fixed-point approach to the problem.

4.2.4 A Fixed-Point Formulation

In this subsection we argue that the ARG problem can also be formulated as a fixed point problem in the subpath flow rates. The following subsection will show that the fixed point and variational inequality formulations are equivalent.

The fixed-point approach solves the following two sub-problems in sequence:

Sub-problem 1: Drivers who do not receive any information, follow shortest time paths based on travel times determined from some default (and fixed) estimate of the vector $\widehat{f}$ of network flows. For these drivers, the fixed-point approach attempts to find flows that are a fixed-point solution of the travel time minimization problem with $S(t, \widehat{f})$ as the vector of path travel times. In mathematical terms, this sub-problem is equivalent to: Find a vector of flows $f^{\star}_{|\overline{P}_2} \in F(ARG)_{|\overline{P}_2}$ satisfying

$$f^{\star}_{|\overline{P}_2} \in ArgMin_{\ f_{|\overline{P}_2} \ \in \ F(ARG)_{|\overline{P}_2}} < S(\widehat{f}), f_{|\overline{P}_2} > . \tag{4.14}$$

Sub-problem 2: Drivers who receive full information about the network traffic conditions, follow shortest time paths based on travel times determined from the actual flows of both uninformed and other informed drivers. For these drivers, the fixed-point approach attempts to find flows that are a fixed-point solution of the flow-dependent travel time minimization problem. In mathematical terms, this sub-problem is equivalent to: Find a vector of flows $f^{\star}_{|\overline{P}_1} \in F(ARG)_{|\overline{P}_1}$ satisfying

$$f^{\star}_{|\overline{P}_1} \in ArgMin_{\ f_{|\overline{P}_1} \ \in \ F(ARG)_{|\overline{P}_1}} < S(f^{\star}_{|\overline{P}_1}, f^{\star}_{|\overline{P}_2}), f_{|\overline{P}_1} > . \tag{4.15}$$

4.2.5 Relationship between the Variational Inequality and the Fixed-Point Formulations

In this subsection, we establish equivalence of the variational inequality formulation in Subsection 4.2.3 and the fixed-point formulation in Subsection 4.2.4.

Proposition 4.1 *The variational inequality formulation in Subsection 4.2.3 and the fixed-point formulation in Subsection 4.2.4 are equivalent.*

Proof:

It is easy to see that Equation (4.14) in Subsection 4.2.4 is equivalent to Equation (4.12) defined in Subsection 4.2.3. Furthermore, Equation (4.15) in Subsection 4.2.4 is also equivalent to Equation (4.13) defined in Subsection 4.2.3. Since solving Equation (4.12) and Equation (4.13) in sequence is equivalent to solving the variational inequality formulation, the result of the proposition follows.

□

4.2.6 A Special Case: The Dynamic User-Equilibrium Problem

The ARG Problem is a generalization of the Dynamic User-Equilibrium (DUE) Problem. As we discussed in Section 4.1, the DUE problem assumes that flows departing the origin on a particular path always remain on that path. However, in the ARG problem, we allow flow to change from one path to another at intermediate locations. Furthermore, the DUE Problem assumes that all drivers while departing from their origin have full information about the network conditions on the paths available to them (i.e the set of control nodes is exactly is the set of origin nodes). As a result, these drivers choose a path that minimizes their flow-dependent travel time.

In this special case, the ARG problem simplifies significantly. Instead of two types of drivers as discussed in Subsection 4.2.3, the only type of drivers we have in this case are drivers who receive full information. As a result, all paths belong to $\overline{P}_1$, the set $\overline{P}_2$ is empty, and the network transformation introduced in Subsection 4.2.1 is no more needed. Furthermore, out of the three types of conservation equations (4.2), (4.4) and (4.5), only equations (4.2) and (4.4) apply in this case. Finally, the variational inequality formulation (4.11) reduces in this case to: Find a vector of flows $f^\star \in F(DUE)$ satisfying

$$< S(f^\star), f - f^\star > \quad \geq 0, \quad \forall f \in F(DUE). \tag{4.16}$$

Variational inequality formulation (4.16) can be interpreted as the problem of determining feasible path flows $f^\star$ so that, at each time t, Wardrop's first principle is verified. In other terms, it consists of determining feasible path flows $f^\star$ so that, at every time t, and for every origin-destination pair (r, s), all used paths belonging to (r, s) have equal and minimum travel times. Over the past decade, the DUE Problem has attracted the attention of many researchers in the transportation area interested in both theoretical and applied research. We hope that our work in this chapter on the ARG problem will contribute to this literature.

4.3 Mathematical Properties of the ARG Problem

In this section, we examine the mathematical properties of the ARG problem. In Subsection 4.3.1, we introduce some definitions and preliminary results. In Subsection 4.3.2, we establish key properties of the feasible region $F(ARG)$. In Subsection 4.3.3, we establish the existence of a solution to the ARG problem.

4.3.1 Definitions

In this subsection, we present some important definitions. In particular, the following three definitions express three different types of First In First Out (FIFO) properties. These definitions are similar to the ones introduced for product sales functions in Subsection 3.2.2 of Chapter 3. The FIFO property will play a key role in the analysis of our model in Subsection 4.3.2.

A link verifies the FIFO property if and only if the link exit time function is non-decreasing. This means that a car that enters link a at time t cannot exit before the cars that entered earlier. In particular,

Definition 6 *(FIFO 1): A link verifies the FIFO property if and only if:*

$$\forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 \leq t_2, \text{ then: } s_a(t_1) \leq s_a(t_2). \tag{4.17}$$

A link verifies the strict FIFO property if and only if the link exit time function is strictly increasing. This means that, a car on a link a cannot exit before or at the same time as other cars that entered the same link earlier. In particular,

Definition 7 *(FIFO 2): A link verifies the strict FIFO property if and only if:*

$$\forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 < t_2, \text{ then: } s_a(t_1) < s_a(t_2). \tag{4.18}$$

Definition 8 *(FIFO 3): A link verifies the strong FIFO property if and only if:*

$$\exists\, \theta > 0 \text{ such that } \forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 < t_2, \text{ then: } s_a(t_2) - s_a(t_1) \geq \theta(t_2 - t_1). \tag{4.19}$$

In the model presented in Section 4.2, the path departure flow rate functions $f_p(.)$ are control variables. In an effort to establish general results, we assume that these functions are Lebesgue integrable.

4.3.2 Properties of the Feasible Region of the ARG Problem

In this subsection, we establish key properties of the feasible region $F(ARG)$ of the ARG problem. In particular, we summarize results due to Kachani [39] that determine the tighest assumptions for the $F(ARG)$ region to be non-empty and for the FIFO property to hold. We start with a network of one link. Then, we show how the results extend to a general network. We also establish the boundedness, closedness and convexity of the $F(ARG)$ region in terms of the path flow rates $f_p(.)$.

Network of One Link

Unifying Analysis of Non-Linear and Linear travel time Functions

In this subsection, we consider a network of one link a. Below, we report on a result that establishes that the feasible region of the ARG problem is not empty. This result provides a unifying analysis for both linear and non-linear travel time functions. Corollary 4.1 shows how linear travel time functions can be interpreted as a limit case of non-linear travel time functions and why linear travel time functions lead to stronger results.

Theorem 4.2 *[39] If the pair $(D_a(.), u_a(.))$ satisfies the following conditions:*
(A1) The link travel time function $D_a(.)$ is continuously differentiable, and there exist two non-negative constants B_{1a} and B_{2a} such that for every link load X_a, $0 \leq B_{1a} \leq D'_a(X_a) < B_{2a}$.
(A2) The link entrance flow rate function $u_a(.)$ is Lebesgue integrable, non negative and bounded from above by a positive real number M_a on $[0, T]$.
(A3) $M_a \leq \frac{1}{B_{2a}-B_{1a}}$.
Then, the feasible region $F(ARG)$ has the following properties:
(1) $F(ARG)$ is well defined, that is, the link load $X_a(.)$, the exit flow rate $v_a(.)$, and the cumulative variables can be uniquely determined by the link travel time function $D_a(.)$ and the link entrance flow rate $u_a(.)$ on the analysis period $[0, T_\infty]$.

(2) The Strong FIFO property holds.

Remarks:

- Intuitively, Conditions (A1)-(A3) are the minimal conditions to ensure that the FIFO property is verified. Indeed, $B_{2a} - B_{1a}$ represents the maximum variation of travel time in terms of X_a (that is, the total number of vehicles in the link). During a time interval Δt of decrease in the number of vehicles in the link, the variation in the number of vehicles $X_a(t) - X_a(t+\Delta t)$ is bounded by the quantity $M_a.\Delta t$. Therefore, the variation of travel time $D_a(X_a(t)) - D_a(X_a(t+\Delta t))$ is bounded by $(B_{2a} - B_{1a}).M_a.\Delta t$. Using Condition (A3), it is also bounded by Δt. Hence, $s_a(t) = t + D_a(X_a(t)) \leq t + \Delta t + D_a(X_a(t + \Delta t)) = s_a(t + \Delta t)$. Therefore, the FIFO property is verified in this case. On the other hand, during a time interval Δt of increase in the number of vehicles, since the travel time functions are non-decreasing, $D_a(X_a(t)) \leq D_a(X_a(t + \Delta t))$. Therefore, $s_a(t) \leq s_a(t + \Delta t)$, and the FIFO property is also verified in this case.

- Notice that if the link travel time function $D_a(.)$ is linear, then conditions (A1)-(A3) of Theorem 4.2 simplify significantly. Indeed, in this case, $D_a'(.) = cst = B_{1a}$. Moreover, for any arbitrarily small positive scalar ϵ, by introducing $B_{2a} = B_{1a} + \epsilon$, Condition (A1) of Theorem 4.2 is verified. Furthermore, since Condition (A3) can be rewritten as $M_a \leq \frac{1}{\epsilon}$, M_a can be arbitrarily large. Therefore, the following corollary follows.

Corollary 4.1 *[39] If the pair $(D_a(.), u_a(.))$ satisfies the following conditions:*
(B1) The link travel time function $D_a(.)$ is linear and non-negative.
(B2) The link entrance flow rate function $u_a(.)$ is Lebesgue integrable and non-negative.
Then, conditions (A1)-(A3) of Theorem 4.2 also hold.

In summary, Theorem 4.2 establishes that by constraining the link entrance capacity to the variation of travel time with respect to the number of vehicles in the link, we limit the effect of the variation of the number of vehicles with respect to time, and ensure that the FIFO property holds. Furthermore, when the FIFO property holds, the feasible region $F(ARG)$ is non-empty and we can uniquely determine the exit flow rate in terms of the link entrance flow rate.

Tightness of the Conditions of Theorem 4.2

Below, we illustrate using a counter-example that conditions (A1)-(A3) in Theorem 4.2 are tight.

Theorem 4.3 *For any arbitrarily small positive scalar δ, there exist a link travel time function $D_a(.)$ and a production flow rate function $u_a(.)$ that verify the following conditions*
(C1) $D_a(.)$ is continuously differentiable and nondecreasing;
(C2) $u_a(.)$ is non-negative, Lebesgue integrable and bounded from above by M_a;
(C3) $\frac{1}{M_a} < Max\{D'_a(X_a), X_a \in R\}$ - $Min\{D'_a(X_a), X_a \in R\} \leq \frac{1}{M_a} + \delta$,
and that violate the FIFO property.

Proof: The proof of this result is the same as the proof of Theorem 3.5 in Subsection 3.5.1.

Extension to a General Network

In this subsection, we generalize the results we obtained for a single link network to the case of a general network. The following theorem illustrates this generalization.

Theorem 4.4 *[39]*
Assume that for every (O-D) pair (r, s), the (O-D) pair departure flow rate function $f^{rs}(.)$ is Lebesgue integrable, non-negative, and bounded from above by M_{rs}. Then, there exists a vector $(\widetilde{M}_a)_{a \in A} > 0$ such that:

If the link travel time functions $(D_a(.))_{a\in A}$ *satisfy the following conditions:*

(D1) $D_a(.)$ *is continuously differentiable,*

(D2) $\forall x \in R$, $0 \leq B_{1a} \leq D_a^{'}(x) < B_{2a}$, *and,*

(D3) $B_{2a} - B_{1a} \leq \frac{1}{\tilde{M}_a}$,

then, the feasible region $F(ARG)$ *has the following properties:*

(1) $F(ARG)$ *is well defined (that is, the link entrance flow rates* $u_a(.)$*, the link exit flow rates* $v_a(.)$*, the link cumulative entrance and exit flows* $U_a(.)$ *and* $V_a(.)$*, the link loads* $X_a(.)$*, and the link and path exit time functions* $s_a(.)$ *and* $S_p(.)$*, can be uniquely determined by the link travel time functions* $D_a(.)$ *and the path departure flow rates* $f_p(.)$ *on the analysis period* $[0, T_\infty]$*),*

(2) The Strong FIFO property holds.

Non-Emptiness, Boundedness, Convexity and Closedness of the Feasible Region

In this subsection, we present some properties of the $F(ARG)$ region. These properties are not only useful in understanding the structure of the model, but will also be useful in Subsection 4.3.3 for proving the existence of a solution to the ARG problem.

Let $D = (D_a)_{a\in A}$ and $f = (f_p)_{p\in P}$ denote a vector of link travel time functions and a vector of path departure flow rate functions respectively. Vector $(D(.), f(.))$ is feasible if it verifies conditions (D1)-(D3) of Theorem 4.4. In this case, using Theorem 4.4, the link entrance flow rates $u_a(.)$, the link exit flow rates $v_a(.)$, the link cumulative entrance and exit flows $U_a(.)$ and $V_a(.)$, the link loads $X_a(.)$, and the link and path exit time functions $s_a(.)$ and $S_p(.)$, can be uniquely determined by the link travel time functions $D_a(.)$ and the path departure flow rates $f_p(.)$ on the analysis period $[0, T_\infty]$.

Proposition 4.5 *The feasible region $F(ARG)$ is non-empty and bounded.*

Proof: This result follows directly from Theorem 4.4.

Proposition 4.6 *If vectors $(D(.), \beta_1(.).f(.))$ and $(D(.), \beta_2(.).f(.))$ are feasible, then, for every $\lambda \in [0, 1]$, vector $(D(.), (\lambda\beta_1(.) + (1 - \lambda)\beta_2(.)).f(.))$ is also feasible. In this sense, the feasible region $F(ARG)$ is convex.*

Since the two vectors β and f have the same dimension, note that $\beta(.).f(.)$ denotes the vector whose elements are obtained from multiplying each component $\beta_{np}(.)$ of the splitting rate $\beta(.)$ with its correponding node/O-D pair flow rates $f^{n,RS(p)}(.)$.

Proof: The result of this proposition follows from the observation that, for every node and every time instant, the set of feasible splitting rates form a simplex. Therefore, the set of all feasible splitting rates is a product of simplices, which is convex.

Proposition 4.7 *If a sequence $(f^j(.))_{j \in \aleph}$ of vectors of path departure flow rates converges to $(f(.))$, and, if for every j, vector $(D(.), f^j(.))$ is feasible, then, the limit $(D(.), f(.))$ is also feasible. In this sense, the $F(ARG)$ region is closed.*

Proof: The proof is the same as the proof of Proposition 3.8 in Subsection 3.5.1.

4.3.3 Existence of a Solution to the Anticipatory Route Guidance Problem

In this subsection, we study key properties of the ARG problem that enable us to establish the existence of a solution to the ARG problem. To this end, we follow the same approach as in Subsection 3.5.2 of Chapter 3.

As established in Subsection 4.2.3, the ARG problem is equivalent to solving the following two variational inequalities in sequence: Find a vector of flows $f^*_{|\overline{P}_2} \in$

$F(ARG)_{|\overline{P}_2}$ satisfying

$$< S(\widehat{f}), f_{|\overline{P}_2} - f^*_{|\overline{P}_2} > \quad \geq 0 \quad \forall f_{|\overline{P}_2} \in F(ARG)_{|\overline{P}_2}. \tag{4.20}$$

Then, find a vector of flows $f^*_{|\overline{P}_1} \in F(ARG)_{|\overline{P}_1}$ satisfying

$$< S(f^*_{|\overline{P}_1}, f^*_{|\overline{P}_2}), f_{|\overline{P}_1} - f^*_{|\overline{P}_1} > \quad \geq 0, \quad \forall f_{|\overline{P}_1} \in F(ARG)_{|\overline{P}_1}. \tag{4.21}$$

The existence of a vector of flows $f^*_{|\overline{P}_2} \in F(ARG)_{|\overline{P}_2}$ satisfying variational inequality formulation (4.20) follows immediately from the continuity of the scalar product $< S(\widehat{f}), . >$ in (4.20), and the non-emptiness, boundedness, convexity and closedness results of the $F(ARG)$ region established in the previous subsection.

However, the complexity of establishing the existence of a solution to the ARG problem lies in proving the existence of a vector of flows $f^*_{|\overline{P}_1} \in F(ARG)_{|\overline{P}_1}$ satisfying variational inequality formulation (4.21). We will refer to the functional operator $S(.)_{|\overline{P}_1} = S(., f^*_{|\overline{P}_2})$ as the ARG operator.

Properties of the ARG Operator

In this subsection, we establish some properties of the ARG operator $S(.)_{|\overline{P}_1}$. These properties will be useful in establishing that the ARG problem has a solution.

We establish the weak continuity of the ARG operator. Weak continuity is defined in Subsection 3.5.2.

Theorem 4.8 *Conditions (D1)-(D3) of Theorem 4.4 imply that the ARG operator* $S(.)_{|\overline{P}_1}$ *is weakly continuous.*

Proof: The proof of this result is the same as the proof of Theorem 3.9 in Subsection 3.5.2.

Lemma 4.1 *The ARG operator $S(.)_{|\overline{P}_1}$ is pseudo-monotone over the $F(ARG)$ region.*

Proof: The proof of this result is the same as the proof of Theorem 3.15 in Subsection 3.5.2.

Existence of a Solution to the ARG problem

In this subsection, we establish one of the fundamental results of this chapter. That is, we illustrate that under weak assumptions, the ARG problem has a solution.

Theorem 4.9 *Under conditions (D1)-(D3) of Theorem 4.4, the ARG problem has a solution.*

Proof: The proof of this result is the same as the proof of Theorem 3.11 in Subsection 3.5.2.

Chapter 5

Conclusions and Future Steps

5.1 Summary of Contributions and Future Steps

5.1.1 Travel Time Models for Dynamic Transportation Networks

In Chapter 2, we took a fluid dynamics approach to determine the delay (travel time) of a traveler in traversing a network's link. The main contributions were the following:

1. We proposed a variety of models for determining travel times in transportation networks.

2. We proposed analytical forms (closed-form solutions) of travel times.

3. We captured a variety of flow patterns such as formation and dissipation of queues, drivers' reaction time and response to upstream congestion or decongestion.

4. We accounted explicitly for splillback, bottleneck phenomena, and link interaction.

5. We incoporated inflow, outflow and storage capacity constraints.

In particular:

- We proposed first-order and second-order fluid models for determining travel time functions.
- We proposed two simplified models to estimate travel times as functions of the *entrance* flow rates: the Polynomial Travel Time (PTT) Model and Exponential Travel Time (ETT) Model.
- We designed enhancements of our models in order to account explicitly for spillback and bottleneck phenomena and to incorporate inflow, outflow and storage capacity constraints.
- We proposed two simplified models to estimate travel times as functions of the *exit* flow rates: the Spillback Polynomial Travel Time (SPTT) Model and Spillback Exponential Travel Time (SETT) Model.
- We proposed a general framework for the analysis of the PTT Model and the SPTT Model that reduces the analysis of these models to solving a single ordinary differential equation.
- Based on piecewise linear and piecewise quadratic approximations of the flow rates, we proposed several classes of travel time functions for the separable PTT, SPTT, ETT and SETT models.
- We extended the analysis of the PTT and SPTT Model to second-order non-separable velocity functions in the case of acyclic networks.

Continuing this work, we intend to examine other fluid dynamics models. For example, consider a different model for relating speed and density. Moreover, we will investigate alternate approaches including queuing models. We wish to connect these models with the dynamic user-equilibrium problem. We plan to investigate the solution to this problem and propose algorithms for computing the solution to our models.

We also intend to perform a numerical study for realistic networks using the models and the analysis that we already performed in order to show how a numerical solution approach compares to an analytical one.

5.1.2 A Fluid Model of Pricing and Inventory Management for Make-to-Stock Manufacturing Systems

In Chapter 3, we took a fluid dynamics approach to determine optimal pricing and inventory policies for make-to-stock manufacturing systems. The main contributions were the following:

- We proposed a fluid model of pricing and inventory management for make-to-stock manufacturing systems. A key novel characteristic of this model is that it incorporates the delay of price and level of inventory in affecting demand for *non-perishable* products. We formulated the model as a continuous-time non-linear optimization problem.

- We analyzed the feasible region ($F(DPM)$) of the general DPM Model. In particular, we provided a unifying analysis for both linear and non-linear product delay functions. Under sufficient conditions on the production flow rate functions and the product sojourn time functions, we established that the ($F(DPM)$) region is non-empty, and that the FIFO property holds. We showed that in the case of linear product sojourn time functions, the assumptions we imposed for non-linear product sojourn time functions simplify significantly. We showed that the conditions we imposed are the tightest possible. We established key properties of the feasible region, as a function of the production flow rate functions $u_i(.)$, such as boundedness, closedness and convexity.

- We established under weak assumptions the existence of an optimal production/inventory control policy that maximizes the profit of the firm over the feasible region.

- We proposed a solution algorithm that solves the DPM Model in special cases of demand arrival rates λ_i, production cost functions $c_i(t)$, inventory cost functions $h_i(t)$, unit price functions $p_i(I_i(t))$, and shared production capacity $CFR(.)$.

In summary, some of the insights obtained from the analysis of Chapter 3 are the following:

- We considered a fluid model that describes the selling rate of a unit of product through its sojourn time in the system. Our motivation is based on the belief that delay (sojourn time) data is easier to acquire than demand data. This approach allowed us to describe the system in greater detail by accounting explicitly how each unit of product waits in inventory before being sold.

- We derived an optimal pricing/production/inventory control policy in a capacitated environment for a discretized version of the model. This policy is dynamic and is based on an equilibration of the marginal profit of products.

- Our model connects and is consistent with traditional demand models.

- Furthermore, we generalized our approach without considering a time discretization. We established key properties of the general Dynamic Pricing Model that allowed us to establish that the general model also has a solution.

Continuing this work, we intend to devise solution algorithms to solve more general instances of the discretized version of the DPM model. We also intend to investigate the extension of the model to incorporate product substitution and bundled offerings.

We aim to consider stochastic delay functions. Finally, we intend to benchmark our approach against industry practices and current literature in application areas such as retail.

We hope that the results of this research will lay the foundations for the use of the delay of price and level of inventory in affecting demand in supply chain and inventory management systems.

5.1.3 The Anticipatory Route Guidance Problem: Formulations and Analysis

In Chapter 4, we also took a fluid dynamics approach to study the anticipatory route guidance problem. The main contributions were the following:

- We proposed a variational inequality (VI) formulation, the first general analytical formulation of this problem.

- We presented a fixed-point formulation of the problem and established equivalence of the two formulations.

- We studied the feasible region $F(ARG)$ of the ARG problem. We provided a unifying analysis for both linear and non-linear travel time functions. Under sufficient conditions on the path flow rate functions and the travel time functions, we established that the $F(ARG)$ region is non-empty, and that the FIFO property holds. We showed that the conditions we imposed are the tightest possible. We established key properties of the feasible region, as a function of the path flow rate functions, such as boundedness, closedness and convexity.

- We established under weak assumptions the existence of a solution to the ARG problem.

Continuing this work, we intend to devise solution algorithms to solve general instances of the discretized version of the ARG problem. We also intend to investigate the extension of the model to incorporate the effects of inaccurate guidance messages. Furthermore, we aim to consider and study stochastic travel time functions. Finally, we intend to apply the results of our work in practice.

5.2 Conclusions

This research has the potential to significantly impact inventory control and manufacturing as well as transportation planning. In the area of dynamic pricing, we believe that our results will lay the foundations for the use of the delay of price in affecting demand and fluid dynamics models in supply chain and inventory management systems. Furthermore, our results in transportation could play an important role in the development of ATIS.

Our analysis in this research required an interdisciplinary approach, drawing upon a broad collection of methodologies from areas such as differential equations, functional analysis and dynamic optimization. Finally, this research addresses problems arising in many diverse application areas including dynamic pricing, revenue management, inventory control, transportation planning and management, air traffic flow management, routing messages in communication networks, mechanical systems and electrical power systems.

Bibliography

[1] R. Akcelik. Capacity of a Shared Lane. *Australian Road Research Board Proceedings*, 14(2):228–241, 1988.

[2] R. Akcelik. Travel Time Functions for Transport Planning Purposes: Davidson's Function, its Time-Dependent Form and an Alternative Travel Time Function. *Australian Road Research Board Proceedings*, 21(3):49–59, 1991.

[3] C. L. Allen. The Framework of Price Theory. *Wadsworth Publishing, California*, 1967.

[4] F. Avram, D. Bertsimas, and M. Ricard. Optimization of Multiclass Fluid Queueing Networks: a Linear Control Approach. *Proceedings of the IMA, in F.P. Kelly and R. Williams (eds.), Springer-Verlag*, pages 199–234, 1995.

[5] R. C. Baker and T. L. Urban. A Deterministic Inventory System with an Inventory-Level-Dependent Demand Rate. *Journal of the Operational Research Society*, 39(9):823–831, 1988.

[6] C. Barnhart, P. Ioannou, G. Nemhauser, and H. Richardson. A Report of the National Science Foundation on the Workshop on Planning, Design, Management and Control of Transportation Systems. 1998.

[7] A.G. Bekey, G.O. Burnham, and J. Seo. Control Theoretic Models of Human Drivers in Car Following. *Hum. Factors*, 19:399–413, 1977.

[8] M. Ben-Akiva, M. Bierlaire, J. A. Bottom, H. Koutsopoulos, and R. Mishalani. Development of a Route Guidance Generation System for Real-Time Application. *In Papageorgiou and Pouliezos*, pages 433–438, 1997.

[9] C.M. Bender and S.A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill Book, Inc., 1978.

[10] D. Bernstein, T.L. Friesz, R.L. Tobin, and B-W. Wie. A variational control formulation of the simultaneous route and departure-time choice equilibrium problem. *Transportation and traffic theory, ed. C.F. Daganzo, Elsevier Science Publishers B.V.*, 1993.

[11] D. Bernstein and T.L. Smith. Equilibria for networks with lower semicontinuous costs: with an application to congestion pricing. *Transportation Science*, 28:221–235, 1994.

[12] D. Bertsimas and I. Paschalidis. Probabilistic Service level Guarantees in Make-to-Stock Manufacturing Systems. *Operations Research*, 49(1):119–133, 2001.

[13] G. Bitran and S. Mondschein. Periodic Pricing of Seasonal Products in Retailing. *Management Science*, 43(1):64–79, 1997.

[14] J. A. Bottom. Consistent Anticipatory Route Guidance. *Ph.D. Thesis, Center for Transportation Studies, Massachusetts Institute of Technology*, 2000.

[15] J. A. Bottom, S. Kachani, and G. Perakis. The Anticipatory Route Guidance Problem: Formulations, Analysis and Computational Results. *Submitted for Publication*, 2002.

[16] P. H. Bovy and N. J. Van Der Zijpp. The Close Connection between Dynamic Traffic Management and Driver Information Systems. In Richard Emmerink and Peter Nijkamp, editors,. *Behavioural and Network Impacts of Driver Information Systems. Ashgate Publishing Company,*, pages 355–370, 1999.

[17] D. Boyce, B. Ran, and L. Leblanc. Solving an Instantaneous Dynamic User-Optimal Route Choice Model. *Transportation Science*, 29(2):128–142, 1995.

[18] H. Brezis. Equations and Inequations Non Lineaires dans les Espaces Vectoriels en Dualite. *Annales Institut Fourier*, 18:115–176, 1968.

[19] H. Brezis. Inequations Variationnelles Asssociees a des Operateurs d'Evolution. *NATO Summer School, Venise*, 1968.

[20] LMA. Chan, D. Simchi-Levi, and J. Swann. Flexible Pricing Strategies to Improve Supply Chain Performance. *Working Paper*, 2000.

[21] S. C. Dafermos. Traffic Equilibrium and Variational Inequalities. *Transportation Science*, 14:42–54, 1980.

[22] S. C. Dafermos and S. C. Sparrow. The traffic assignment problem for a general network. *Journal of Research, National Bureau of Standards*, 73B:91–118, 1969.

[23] S. C. Dafermos and S. C. Sparrow. Traffic equilibrium and variational inequalities. *Transportation Science*, 14:42–54, 1980.

[24] C. Daganzo. The Cell Transmission Model: A Dynamic Representation of Highway Traffic Consistent with the Hydrodynamic Theory. *Transportation Research B*, 28B:269–287, 1994.

[25] C. Daganzo. The Cell Transmission Model, Part ii: Network Traffic. *Transportation Research B*, 29B:79–93, 1995.

[26] R. Dorfman. Prices and Markets. *Third Edition, Foundations of Modern Economic Series. Prentice-Hall, New Jersey*, 1978.

[27] L. Engelson. Self-Fulfilling and Recursive Forecasts - An Analytical Perspective for Driver Information Systems. *In Proceedings of the 8th IATBR Meeting, Texas*, 1997.

[28] A. Federgruen and A. Heching. Combined Pricing and Inventory Control Under Uncertainty. *Operations Research*, 47(3):454–475, 1999.

[29] M. Florian and D. Hearn. Network Equilibria. *Handbook of Operations Research and Management Science*, 8, 1995.

[30] T. L. Friesz, D. Bernstein, T. E. Smith, R. L. Tobin, and B. W. Wie. A Variational Inequality Formulation of the Dynamic Network User Equilibrium Problem. *Operations Research*, 41(1):179–191, 1993.

[31] G. Gallego and G. J. Van Ryzin. Optimal Dynamic Pricing of Inventories with Stochastic Demand Over Finite Horizons. *Management Science*, 40:990–1020, 1993.

[32] D. L. Gerlough and M. J. Huber. Traffic Flow Theory: A Special Report. Special Report no. 165. *Transportation Research Board, Washington, DC*, 1975.

[33] A. Y. Ha. Optimal Dynamic Scheduling Policy for a Make-to-Stock Production System. *Operations Research*, 45(1):42–53, 1997.

[34] R. Haberman. Mathematical Models, Mechanical Vibrations, Population Dynamics and Traffic Flow. *Prentice-Hall, Engelwood Cliffs, New Jersey*, 1977.

[35] R. Hamerslag and E. C. van Berkum. Effectiveness of Information Systems in Networks with and without Congestion. *Transportation Research Record 1306, TRB, National Research Council, Washington, D. C.*, pages 14–21, 1991.

[36] M. Harrison. Balanced Fluid Models of Multiclass Queueing Networks: a Heavy Traffic Conjecture. *Proceedings of the IMA, in F.P. Kelly and R. Williams (eds.), Springer-Verlag*, 71:1–20, 1995.

[37] D. Heidemann. Some Critical Remarks on a Class of Traffic Flow Models. *Transporation Research B, 33*, 2:153–155, 1999.

[38] R. Herman, E. Montroll, R. Potts, and R. Rothery. Traffic Dynamics: Analysis of Stability in Car-Following. *Operations Research*, 7:86–106, 1959.

[39] S. Kachani. Analytical Dynamic Traffic Flow Models: Formulation, Analysis and Solution Algorithms. *Master's Thesis, Operations Research Center, Massachusetts Institute of Technology*, 2000.

[40] S. Kachani and G. Perakis. A Fluid Model of Spillback and Bottleneck Phenomena for Determining Dynamic Travel Times. *Submitted for Publication*, 2001.

[41] S. Kachani and G. Perakis. Modeling Travel Times in Dynamic Transportation Networks; A Fluid Dynamics Approach. *Submitted for Publication*, 2001.

[42] S. Kachani and G. Perakis. Second-Order Fluid Dynamics Models for Travel Times in Dynamic Transportation Networks. *Proceedings of the 4th International IEEE Conference on Intelligent Transportation Systems*, 2001.

[43] D. E. Kaufman, R. L. Smith, and K. E. Wunderlich. An Itererative Routing/Assignment Method for Anticipatory Real-Time Route Guidance. *IN VNIS91*, pages 693–700, 1991.

[44] F.P. Kelly. On Tariffs, Policing and Admission Control for Multiservice Networks. *Operations Research Letters*, 15:1–9, 1994.

[45] F.P. Kelly, A.K. Maulloo, and D.K.H. Tan. Rate Control for Communication Networks: Shadow Prices, Proportional Fairness and Stability. *Journal of the Operational Research Society*, 49:237–252, 1998.

[46] A. A. Kirillov. Theorems and Problems in Functional Analysis. *Springer-Verlag, New-York*, 1982.

[47] A. J. Kleywegt. An Optimal Control Problem of Dynamic Pricing. *Working Paper, Georgia Institute of Technology*, 2001.

[48] A. N. Kolmogorov and S. V. Fomin. Elements of the Theory of Functions and Functional Analysis. *Rochester, N.Y., Graylock Press*, 1957.

[49] M.J. Lighthill and G.B. Whitham. On Kinematic Waves ii. A Theory of Traffic Flow on Long Crowded Roads. *Proceedings of the royal society of London, Series A*, 229:317–345, 1955.

[50] G. Lilien, P. Kotler, and K. Moorthy. Marketing Models. *Prentice Hall, NJ*, 1992.

[51] J. Lindley. Urban Freeway Congestion: Quantification of the Problem and Effectiveness of Potential Solutions. *ITE Journal*, 1987.

[52] J. L. Lions. Sur Quelques Proprietes des Solutions d'Inequations Variationnelles. *C. R. Acad. Sc. Paris, 267*, pages 631–633, 1968.

[53] J. L. Lions. Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires. *Dunod, Gauthiers-Villars, Paris*, 1969.

[54] J. L. Lions. Some Remarks on Variational Inequalities. *Int. Conference on Functional Analysis, Tokyo*, 1969.

[55] T. Lotan and H. Koutsopoulos. Approximate Reasoning Models for Route Choice Behavior in the Presence of Information. *In C. Daganzo, editor, Proceedings of the 12th international symposium on transporation and traffic theory (ISTS), Berkeley, CA*, pages 71–88, 1993.

[56] H. Mahmassani and R. Herman. Dynamic User Equilibrium Departure Time and Route Choice on Idealized Traffic Arterials. *Transportation Science, 18, 4, 362-384*, 1984.

[57] J. McGill and G. Van Ryzin. Focused Issue on Yield Management in Transportation. *Transportation Science*, 33(2), 1999.

[58] C. Meneguzzer, D.E. Boyce, N. Rouphail, and A. Sen. Implementation and Evaluation of an Asymmetric Equilibrium Route Choice Model Incorporating Intersection-Related Travel Times. *Report to Illinois Department of Transportation, Urban Transportation Center, University of Illinois, Chicago*, 1990.

[59] S. P. Meyn. Stability and Optimization of Queueing Networks and their Fluid Models. *Proceedings of the Summer Seminar on the Mathematics of Stochastic Manufacturing Systems, Virginia*, pages 17–21, 1996.

[60] A. Nagurney. *Network Economics: A Variational Inequality Approach.* Kluwer Academic Publisher, Norwell, MA, 1993.

[61] "Bureau of Public Roads". Traffic Assignment Manual. *US Dep. of Commerce, Washington, DC*, 1964.

[62] M. Papageorgiou. Applications of Automatic Control Concepts to Traffic Flow Modeling and Control. *Springer Verlag, Berlin*, 1983.

[63] M. Papageorgiou. Some Remarks on Macroscopic Flow Modeling. *Transportation Research A, 32*, 5:323–329, 1998.

[64] M. Papageorgiou, J.M. Blosseville, and H. Haj-Salem. Macroscopic Modeling of Traffic Flow on the Boulevard Peripherique in Paris. *Transportation Research B, 23*, pages 29–47, 1989.

[65] M. Papageorgiou, J.M. Blosseville, and H. Haj-Salem. Modeling and Real-Time Control of Traffic Flow on the Southern Part of Boulevard Peripherique in Paris-Part i: Modeling. *Transportation Research A, 24*, pages 345–359, 1990.

[66] I. Paschalidis and J. Tsitsiklis. Congestion-Dependent Pricing of Network Services. *Technical Report*, 1998.

[67] H. J. Payne. Models of Freeway Traffic and Control. in G.A. Bekey, editor. *Mathematical Models of Public Systems, volume 1 of Simulation Councils Proc. Ser.*, pages 51–60, 1971.

[68] G. Perakis. The Dynamic User-Equilibrium Problem through Hydrodynamic Theory. *Working paper, Operations Research Center, MIT*, 1997.

[69] L. A. Pipes. An Operational Analysis of Traffic Dynamics. *J. Appl. Phys.*, 24:274–281, 1953.

[70] R. Reuschel. Fahrzeugbewegungen in der Kolonne bei Gleichformig Beschleunigtem Verzogertem Leitfahrzeug. *Z. Osterr. Ing. Arch. Ver.*, 95:52–62, 1950.

[71] P.I. Richards. Shock Waves on the highway. *Operations Research*, 4:42–51, 1956.

[72] W. Rudin. Principles of Mathematical Analysis. *McGraw-Hill, New-York*, 1976.

[73] Y. Sheffi. *Urban Transportation Networks.* Prentice-Hall, Englewood, NJ, 1985.

[74] B. Smith, J. Leimkuhler, R. Darrow, and J. Samuels. Yield Management at American Airlines. *Interfaces*, 1:8–31, 1992.

[75] M. J. Smith. A New Dynamic Traffic Model and the Existence and Calculation of Dynamic User Equilibria on Congested Capacity-Constraint Road Networks. *Transportation Research*, 27B(1):49–63, 1993.

[76] J. Tirole. The Theory of Industrial Organization. *The MIT Press, Massachusetts*, 1988.

[77] L. Wein. Dynamic Scheduling of a Multiclass Make-to-Stock Queue. *Operations Research*, 40(4):724–735, 1992.

[78] G. B. Whitham. Linear and Non-Linear waves. *John Wiley and Sons, New York*, 1974.

[79] R. Wilson. Nonlinear Pricing. *Oxford University Press*, 1993.

[80] H. Wold and L. Jureen. Demand Analysis. *A study in econometrics, Wiley Publications in Statistics*, 1953.

[81] J. H. Wu, Y. Chen, and M. Florian. The Continuous Dynamic Network Loading Problem: A Mathematical Formulation and Solution Method. *Transportation Research B*, 32(3):173–187, 1995.

[82] Y. W. Xu, J. H. Wu, M. Florian, P. Marcotte, and D. L. Zhu. Advances in the Continuous Dynamic Network Loading Problem. *Transportation Science*, 33:341–353, 1999.

[83] P. H. Zipkin. Foundations of Inventory Management. *Irwin McGraw-Hill*, 1999.