

Small unitary representations of the double cover of $SL(m)$

by

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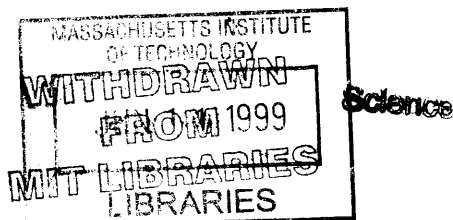
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ABSTRACT

The irreducible representations of the double cover of $\mathrm{SL}(m)$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal are determined. We investigate whether these representations are attached to a nilpotent orbit and use the orbit method to speculate their K types. We conclude with a character formula expressing the representations as a sum of simpler virtual representations.

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To my mother and Mei-Lai

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Chapter 1

Introduction

This thesis investigates the structure of a certain set of representations which are small in the sense that their annihilator in the universal enveloping algebra is maximal. The smallness of these representations make them good candidates for studying via orbit method techniques. As well, their smallness allows them to be written as a sum of virtual representations, leading to a character formula for their K -types.

The representations we consider are the Langlands quotients of the double cover of $\mathrm{SL}(m)$ at infinitesimal character $\frac{1}{2}\rho$. These representations are known to be unitary [18] (Theorem 4.2), and hence are of interest. The infinitesimal character $\frac{1}{2}\rho$ was chosen so that the generalized principal series would be reducible. We expected this would be a good place to look for small representations.

1.1 Outline of the problem

The structure of infinite dimensional representations is often studied by an investigation of their restriction to a maximal compact subgroup. The objective of this thesis is to work towards determining the K type spectrum for those Langlands quotients of the double cover of $\mathrm{SL}(m)$ with infinitesimal character $\frac{1}{2}\rho$ possessing a maximal primitive ideal. Although an explicit description of the K types can only be conjectured at this time, a character formula for the restriction of these Langlands quotients is given. We also wish to understand whether these representations are unipotent and to determine their place in the orbit method.

1.2 Outline of the thesis

The goal in chapter one is to determine which genuine Langlands quotients of the double cover of $\mathrm{SL}(m)$ have a maximal primitive ideal. This takes considerable preparation. We first determine what are the isomorphism classes of genuine discrete series for the subgroups M of cuspidal parabolics $P = MAN$. Next we use the notion of coherent families of virtual (\mathfrak{g}, K) modules and Vogan's theorem on τ invariance to count the number of genuine

Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. We find this number is 4 in the case of the double cover of $SL(2l)$ and is 1 in the case of the double cover of $SL(2l+1)$. In $SL(2l)$ two of the Langlands quotients have minimal parabolics and two have maximal parabolics. For $SL(2l+1)$ the genuine Langlands quotient with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal is the quotient of a principal series representation.

In Chapter 3, we determine the place of those representations of chapter 1, in the orbit method. First we identify the nilpotent orbit the representations are “attached” to. We find there are 2 and 4 genuine admissible orbit data for the nilpotent orbit in the case of the double cover of $SL(2l+1)$ and the double cover of $SL(2l)$ respectively. The orbit method conjectures that under certain conditions, the representations attached to the orbit may be realized as algebraic sections of an algebraic vector bundle. The conditions are satisfied for the orbit in the case of $SL(2l+1)$ and we explicitly determine the K types of the algebraic representation.

In Chapter 4, we prove a character formula for Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. We write the Langlands quotients as a sum of virtual representations. The K types of these virtual representations can be explicitly determined by Blattner’s formula. In deriving the character formula we first determine the lowest K types of the Langlands quotients under consideration. For the Langlands quotients of principal series we use the fact that the lowest K type of these representations are the highest weight of fine representations of the maximal compact subgroup. For the Langlands quotients with maximal parabolic we find the lowest K type directly. We then turn to certain sums of virtual characters which we prove to be irreducible and having maximal primitive ideal. By matching lowest K types we prove the character formula. An explicit multiplicity free formula for the K types of our representations is conjectured.

Chapter 2

Langlands quotients for $\widetilde{\mathrm{SL}}(m)$

2.1 Introduction

In this chapter we explicitly determine the Langlands quotients for $\widetilde{\mathrm{SL}}(m)$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. This result will be crucial for the orbit method results and character formula derivation in chapters 3 and 4. Unless otherwise specified G will be a connected real semisimple Lie group with maximal compact subgroup K and complexified Lie algebra \mathfrak{g} . We write the Cartan decomposition of \mathfrak{g} as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$.

2.2 The group $\widetilde{\mathrm{SL}}(m)$

Let $\mathrm{SL}(m)$ be the group of determinant one real $m \times m$ matrices. We begin with a familiar structure theorem.

Proposition 2.1 (Iwasawa decomposition for $\mathrm{SL}(m)$) *Let $K = \mathrm{SO}(m)$, A the subgroup of $\mathrm{SL}(m)$ of diagonal matrices with positive diagonal entries, and let N be the upper triangular group with 1 in each diagonal entry. Then $\mathrm{SL}(m) = KAN$ in the sense that multiplication $K \times A \times N \rightarrow \mathrm{SL}(m)$ is a diffeomorphism onto.*

The fundamental group of $\mathrm{SL}(m)$, $\pi_1(\mathrm{SL}(m))$, is \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$ depending whether $m = 2$ or $m \geq 3$ respectively. The covers of $\mathrm{SL}(m)$ are the quotients of the universal cover $\widetilde{\mathrm{SL}}(m)$ by subgroups of $\pi_1(\mathrm{SL}(m))$. We note that all such subgroups are central and hence normal in the universal cover so the quotient is a group. Because $\pi_1(\mathrm{SL}(m))$ has a unique subgroup of index 2, $\mathrm{SL}(m)$ has a unique double cover which we denote by $\widetilde{\mathrm{SL}}(m)$.

The subgroup $K = \mathrm{SO}(m)$ in the Iwasawa decomposition contains all of the nontrivial topology of $\mathrm{SL}(m)$. Identifying A and N with one of their two leaves in \widetilde{KAN} allows us to write \widetilde{KAN} as the double cover of $\mathrm{SL}(m)$. The double cover of \widetilde{K} is the group $\mathrm{Spin}(m)$, which we describe next.

2.2.1 The Clifford algebra and $\text{Spin}(m)$

Given a symmetric bilinear form Q on a vector space V , the Clifford algebra, $c(Q)$, is an associative algebra with identity, e_0 , which contains and is generated by V , with the relation $vw + wv = -2Q(v, w)e_0$, for all $v, w \in V$. The Clifford algebra can be constructed quickly by taking the tensor algebra $T^*(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ and setting $c(Q) = T^*(V)/I(Q)$, where $I(Q)$ is the two-sided ideal generated by all elements of the form $v \otimes v + Q(v, v)e_0$.

We will be interested in the case where $V = \mathbb{R}^m$ and Q is a positive definite quadratic form. If e_1, \dots, e_m is the standard basis for $V = \mathbb{R}^m$ then we define $Q(e_i, e_j) = 0$ if $i \neq j$ for $1 \leq i, j \leq m$, and $Q(e_i, e_i) = 1$ for $1 \leq i \leq m$. With this form the Clifford algebra has the following relations: $e_0 e_0 = e_0$, $e_0 e_i = e_i e_0 = e_i$, $e_i e_j + e_j e_i = 0$ ($i \neq j$), and $e_i^2 = -e_0$. The products $e_{i_1} \cdots e_{i_k}$, with $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ together with e_0 form a basis for $c(Q)$ making c a 2^m dimensional vector space.

The Clifford algebra has a natural $\mathbb{Z}/2\mathbb{Z}$ grading given by $c(Q) = c(Q)^{\text{even}} \oplus c(Q)^{\text{odd}}$, where $c(Q)^{\text{even}}$ is spanned by products of an even number of basis elements in V and $c(Q)^{\text{odd}}$ is spanned by products of an odd number. The algebra $c(Q)^{\text{even}}$ is a 2^{m-1} dimension subalgebra of $c(Q)$.

The Clifford algebra has an anti-involution $x \mapsto x^*$, determined by

$$(v_1 \cdots v_r)^* = (-1)^r v_r \cdots v_1$$

for any v_1, \dots, v_r in V . The Clifford algebra also has an involution α which is the identity on $c(Q)^{\text{even}}$ and minus the identity on $c(Q)^{\text{odd}}$; i.e.,

$$\alpha(v_1 \cdots v_r) = (-1)^r v_1 \cdots v_r.$$

$\text{Spin}(Q)$ consists of the following subalgebra of invertible elements in $c(Q)$:

$$\text{Spin}(Q) = \{x \in c(Q)^{\text{even}} : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\}.$$

We will need the following:

Lemma 2.2 *The centralizer of V in $\text{Spin}(Q)$ is $\pm e_0$*

Proof. Let V be an m dimensional vector space over \mathbb{R} with symmetric positive definite bilinear form Q . To prove the claim let $x \in \text{Spin}(Q)$. Suppose that $x \cdot v = v \cdot x$ for all $v \in V$. Letting e_I be a monomial in $\text{Spin}(Q)$, We may write $x = \sum a_I e_I$. We have $e_I e_j = (-1)^{|I|} e_j e_I$ if $j \notin I$ whereas $e_I e_j = (-1)^{|I|-1} e_j e_I$ if $j \in I$. So if $\sum a_I e_I$ centralizes e_j , then necessarily $j \notin I$ whenever $a_I \neq 0$. So the centralizer of V in $c(Q)^{\text{even}}$ is $\mathbb{R} \cdot e_0$. The only elements of $\mathbb{R} \cdot e_0$ in $\text{Spin}(Q)$ are $\pm e_0$. ■

Any x in $\text{Spin}(Q)$ determines an endomorphism $\rho(x)$ of V by

$$\rho(x)(v) = x \cdot v \cdot x^*, v \in V.$$

We have the following:

Proposition 2.3 *Let V be a vector space with dimension at least two having a symmetric bilinear form Q . For $x \in \text{Spin}(Q)$, $\rho(x)$ is in $\text{SO}(Q)$. The mapping*

$$\rho : \text{Spin}(Q) \longrightarrow \text{SO}(Q)$$

is a homomorphism, making $\text{Spin}(Q)$ a connected two-sheeted covering of $\text{SO}(Q)$. The kernel of ρ is $\{\pm e_0\}$.

Proof. It is easy to check that $\rho(x)$ preserves the quadratic form Q (i.e. $Q(\rho(x)(v), \rho(x)(v)) = Q(v, v)$) using the fact that $Q(v, v) = v \cdot v = -v \cdot v^*$, for $v \in V$. This shows that $\rho(\text{Spin}(Q)) \subset \text{SO}(Q)$. From Lemma 2.2, we see that the kernel of ρ is $\pm e_0$. To show that ρ is surjective we use the fact that $\text{SO}(Q)$ is a reflection group, consisting of products of an even number of reflections. If $R \in \text{SO}(Q)$ is written as a product of reflections $R_{v_1} \circ \cdots \circ R_{v_{2r}}$, then the two elements in $\rho^{-1}(R)$ are $\pm v_1 \cdots v_{2r}$. To complete the proof, we must check that $\text{Spin}(Q)$ is connected, or equivalently (using the connectivity of $\text{SO}(Q)$), that the two elements in the kernel of ρ are connected. The elements $\pm e_0$ are connected by the path $e_0 \cos(t) + e_1 e_2 \sin(t)$, $0 \leq t \leq \pi$ in $\text{Spin}(Q)$. ■

2.3 The genuine discrete series of \widetilde{M}

2.3.1 The cuspidal parabolic subgroups of $\widetilde{\text{SL}}(m)$

The Langlands decomposition describes a parabolic subgroup, P , of $\text{SL}(m)$ as a product $P = MAN$. Here $A = \exp \mathfrak{a}$, where \mathfrak{a} is a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{s}$. N is the unipotent radical of P and the Levi subgroup, MA , is the centralizer of A in G . The double cover of the parabolic is written as $\widetilde{P} = \widetilde{MAN}$.

The parabolic subgroups of $\text{SL}(m)$ are in one to one correspondence with the set of subsets of simple roots of $\mathfrak{sl}(m)$. The subgroup M for each parabolic consists of blocks of determinant plus or minus one, $\text{SL}(n_1)^\pm, \text{SL}(n_2)^\pm, \dots, \text{SL}(n_r)^\pm$ along the diagonal of $\text{SL}(m)$, such that $n_1 + \cdots + n_r = m$ and M has determinant one.

A parabolic subgroup P is *cuspidal* if M in the Langlands decomposition of P contains a compact Cartan subgroup. Since M is a product of $\text{SL}(n_i)^\pm$, each n_i must be less than or equal to two in that case. Here we are using the fact that $\text{SL}(n)$ has a compact Cartan subgroup iff n equals 1 or 2.

Next we describe \widetilde{M} for the cuspidal parabolics.

Lemma 2.4 *Let \widetilde{M}_{\min} denote the group \widetilde{M} for the minimal parabolic of $\widetilde{\mathrm{SL}}(m)$. Then \widetilde{M}_{\min} is a finite group of order 2^m equal to all even monomials in $\mathrm{Spin}(m)$.*

Proof. The group M consists of ± 1 along the diagonal with an even number of signs. Hence the order of M is 2^{m-1} . The preimage of a pair of negative signs in the i^{th} and j^{th} diagonal entries are $\pm e_i e_j$ in $\mathrm{Spin}(m)$. The claim follows. ■

Lemma 2.5 *Let ϵ be the nontrivial element in the kernel of the projection homomorphism of $\widetilde{\mathrm{SL}}(2)$ to $\mathrm{SL}(2)$. Let M_o be the identity component of M for a nonminimal parabolic consisting of b $\mathrm{SL}(2)^{\pm}$ diagonal blocks. Then $\widetilde{M}_o = (\widetilde{\mathrm{SL}}(2))^b / A$ where A is the subgroup of $\{1, \epsilon\}^b$ of elements (x_1, \dots, x_b) , with $\prod x_i = 1$. Furthermore, if the $\mathrm{SL}(2)^{\pm}$ diagonal blocks in M are consecutive from the top then e_0 and the set of monomials $e_{i_1}, \dots, e_{i_{2k}}$, having $i_j \in \{2, 4, \dots, 2b, 2b+1, 2b+2, \dots, m\}$ and $i_1 < \dots < i_{2k}$ are a complete set of coset representatives for $\widetilde{M}/\widetilde{M}_o$.*

Proof. In M_o each $\mathrm{SL}(2)^{\pm}$ block has determinant one and all other diagonal entries are one. Thinking of $\mathrm{SL}(2)$ as $\widetilde{\mathrm{SL}}(2)$, we have $(\widetilde{\mathrm{SL}}(2))^b / \{1, \epsilon\}^b$. The description of M_o in the claim follows.

The group M is the product of M_o with the set of diagonal matrices with ± 1 along the diagonal having an even number of signs. Because minus the identity is in $\mathrm{SL}(2)$ we may choose our set to have the first odd diagonal entries equal to 1. The kernel of the projection homomorphism of \widetilde{M} to M , namely ± 1 , is in $\widetilde{\mathrm{SL}}(2)$ so the coefficients of the monomials in $\mathrm{Spin}(m)$ must be positive. The preimage of this set is the finite set of order 2^{m-b-1} in $\mathrm{Spin}(m)$ given in the statement of the lemma. ■

2.3.2 The discrete series of $\widetilde{\mathrm{SL}}(2)$

For a unimodular group G , an irreducible unitary representation π is in the discrete series if every nonzero matrix coefficient $(\pi(g)v_1, v_2)$ is in L^2 . Harish-Chandra showed that a connected semisimple Lie group has a non-empty discrete series exactly when it contains a compact Cartan subgroup. Harish-Chandra parameterized the discrete series by $\lambda \in \mathfrak{t}^*$, dominant with respect to a positive imaginary root system $\Delta^+(\mathfrak{g}, \mathfrak{t})$. We will denote the discrete series with Harish-Chandra parameter λ by δ_λ .

The unimodular group $\widetilde{\mathrm{SL}}(2)$ has a compact Cartan subgroup, $\mathrm{Spin}(2)$, and hence a discrete series δ_λ . Letting $\Delta^+(\mathfrak{g}, \mathfrak{t}) = e_1 - e_2$ and $\lambda = \frac{n}{4}(e_1 - e_2) = (\frac{n}{4}, -\frac{n}{4})$, for a positive integer n , the discrete series $\delta_{(\frac{n}{4}, -\frac{n}{4})}$ has lowest \widetilde{K} type $\frac{n}{2} + 1$. With the opposite choice of positive imaginary roots, the discrete series $\delta_{(-\frac{n}{4}, \frac{n}{4})}$ has lowest \widetilde{K} type $-\frac{n}{2} - 1$.

2.3.3 The genuine discrete series of \widetilde{M}_{min} for minimal parabolics

We will be interested in discrete series of \widetilde{M} , associated with cuspidal parabolics, which do not descend to discrete series of the linear group M . To make this precise we want our discrete series to send the non trivial element of the projection homomorphism $\widetilde{M} \rightarrow M$ to -1 . The kernel of the projection homomorphism is ± 1 , and so we require $\delta(-1) = -1$. Such representations are called *genuine*.

First we will describe the genuine discrete series for \widetilde{M}_{min} , associated with the minimal parabolic. The group is finite and the discrete series are just the finite group representations of \widetilde{M}_{min} . To find these we use the following fact about finite group representations.

Lemma 2.6 *Let G be a finite group of order N , let ρ_1, ρ_2, \dots represent the distinct isomorphism classes of irreducible representations of G .*

(a) *There are finitely many isomorphism classes of irreducible representations, the same as the number of conjugacy classes in the group.*

(b) *Let d_i be the dimension of the irreducible representation ρ_i , and let r be the number of irreducible representations. Then $N = d_1^2 + \dots + d_r^2$.*

Proof. The first part of the claim follows directly from the Peter-Weyl theorem for compact groups which says that the irreducible characters are an orthonormal basis for the space of class functions. The second part of the claim follows from the fact that any irreducible representation ρ_i of G appears in the regular representation $\dim \rho_i$ times. ■

We will use this lemma to prove the following.

Proposition 2.7 *For $\text{SL}(2n+1)$ there are 2^{2n} one-dimensional nongenuine representations and one 2^n -dimensional genuine irreducible representation of \widetilde{M}_{min} .*

Proof. By Lemma 2.4 the order of \widetilde{M}_{min} is 2^{2n+1} . The group \widetilde{M}_{min} consists of all monomials in $\text{Spin}(2n+1)$. The center of this group is $\pm e_0$ and it is not difficult to verify that there are $2^{2n} + 1$ conjugacy classes consisting of each element of the center and \pm each monomial $e_{i_1} \cdots e_{i_{2k}}$ with $i_1 < \dots < i_{2k}$ in $\text{Spin}(2n+1)$. By lemma 2.6 we conclude that there are $2^{2n} + 1$ irreducible representations of \widetilde{M}_{min} . The group M_{min} is an abelian group of order 2^{2n} and so it has 2^{2n} one-dimensional representations. These representations lift to representations of \widetilde{M}_{min} . There is only one way to write 2^{2n+1} as a sum of $2^{2n} + 1$ squares where 2^{2n} of the squares are the square of one, namely $2^{2n+1} = 1^2 + \dots + 1^2 + (2^n)^2$. It follows from lemma 2.6 that \widetilde{M}_{min} has 2^{2n} one-dimensional representations and one 2^n -dimensional irreducible representation. The 2^n -dimensional irreducible representation doesn't descend to a representation of M and hence must be genuine. The one dimensional representations do descend and are not genuine. ■

Proposition 2.8 *For $\widetilde{\mathrm{SL}}(2n)$ there are 2^{2n-1} non genuine one-dimensional representations and two 2^{n-1} -dimensional genuine irreducible representations of \widetilde{M}_{min} .*

Proof. The proof is analogous to the above proposition. The order of \widetilde{M}_{min} is 2^{2n} and it consists of all monomials in $\mathrm{Spin}(2n)$. The center for this group is $\pm e_0$ and $\pm e_1 \cdot e_2 \cdots e_{2n}$ and there are $2^{2n-1} + 2$ conjugacy classes. There are 2^{2n-1} one dimensional representations of the abelian group M_{min} which lift to \widetilde{M}_{min} . There is only one way to write 2^{2n} as a sum of $2^{2n-1} + 2$ squares where 2^{2n-1} of the squares are the square of one, namely $2^{2n} = 1^2 + \dots + 1^2 + (2^{n-1})^2 + (2^{n-1})^2$. The two 2^{n-1} -dimensional irreducible representations don't descend to representations of M and hence must be genuine. The one dimensional representations do descend and are not genuine. ■

We will need to be able to distinguish the two genuine irreducible representations of \widetilde{M}_{min} . This is achieved below.

Proposition 2.9 *The two genuine irreducible representations of \widetilde{M}_{min} in $\widetilde{\mathrm{SL}}(2n)$ are distinguished by their restriction to the center of \widetilde{M}_{min} .*

Proof. The center \widetilde{Z} of \widetilde{M}_{min} consists of the four element abelian group $\{\pm e_0, \pm e_1 \cdots e_{2n}\}$. There are two genuine irreducible representations, ξ_c^+, ξ_c^- of \widetilde{Z} sending $e_N = e_1 \cdots e_{2n}$ to $(\sqrt{-1})^n$ and $-(\sqrt{-1})^n$ respectively. The representations ξ_c^+ and ξ_c^- can be extended to irreducible genuine representations ξ^+ and ξ^- respectively (for example by taking an irreducible component of the induced representation). Because e_N is central in \widetilde{M}_{min} , by Schur's lemma $\xi^+(e_N)$ and $\xi^-(e_N)$ are a scalar times the identity with scalar equal to $\xi_c^+(e_N)$ and $\xi_c^-(e_N)$ respectively. The representations ξ^+ and ξ^- are not isomorphic since their restrictions to \widetilde{Z} are not isomorphic. By lemma 2.8 there are exactly two genuine irreducible representations of \widetilde{M}_{min} , and so these representations must be ξ^+ and ξ^- . ■

In the sequel we will write ξ^\pm for the two genuine irreducible representations of \widetilde{M}_{min} in $\widetilde{\mathrm{SL}}(2n)$. We will also sometimes write ξ^\pm for the single genuine irreducible representations of \widetilde{M}_{min} in $\widetilde{\mathrm{SL}}(2n+1)$ with the understanding that $\xi^+ = \xi^-$.

2.3.4 The genuine discrete series of \widetilde{M} for nonminimal cuspidal parabolics

We will first establish what are the genuine discrete series of $\widetilde{\mathrm{SL}}(2)$. We may write $\mathrm{Spin}(2)$ as $\cos(\frac{t}{2}) + e_1 e_2 \sin(\frac{t}{2})$ for $0 \leq t \leq 4\pi$. The discrete series restricted to $\mathrm{Spin}(2)$, $\delta_{\pm(\frac{n}{4}, -\frac{n}{4})}(\cos(\frac{t}{2}) + e_1 e_2 \sin(\frac{t}{2}))$, have \widetilde{K} types $e^{\pm \frac{(n+2)it}{2}}, e^{\pm \frac{(n+6)it}{2}}, \dots$. Evaluating the discrete series at -1 amounts to setting $t = 2\pi$. Then $\delta_{\pm(\frac{n}{4}, -\frac{n}{4})}(-1)$ equals $e^{\pm(2+n)i\pi}, e^{\pm(n+6)i\pi}, \dots$. It follows that n must be odd for $\delta_{\pm(\frac{n}{4}, -\frac{n}{4})}$ to be genuine.

By Lemma 2.5 $\widetilde{M}_o = \widetilde{\mathrm{SL}(2)}^b / A$ where A is the subgroup of $\{1, \epsilon\}^b$ of elements (x_1, \dots, x_b) , with $\prod x_i = 1$. For a discrete series of \widetilde{M}_o to be genuine it should be genuine for each $\widetilde{\mathrm{SL}(2)}$ in the product, thereby sending the subgroup A to 1.

A discrete series of the product of $\widetilde{\mathrm{SL}(2)}$ is the product of discrete series for each $\widetilde{\mathrm{SL}(2)}$. The genuine discrete series for the identity component of \widetilde{M} has Harish-Chandra parameter $\lambda = (\frac{2l_1+1}{4}, -\frac{2l_1+1}{4}, \frac{2l_2+1}{4}, -\frac{2l_2+1}{4}, \dots)$, for integers l_i .

Let $Z_{\widetilde{M}}(\widetilde{M}_o)$ be the centralizer of \widetilde{M}_o in \widetilde{M} for $\widetilde{\mathrm{SL}(m)}$. We extend $\delta_\lambda|_{Z(\widetilde{M}_o)}$, where $Z(\widetilde{M}_o) = Z_{\widetilde{M}}(\widetilde{M}) \cap \widetilde{M}_o$, to a representation of $Z_{\widetilde{M}}(\widetilde{M}_o)$. In the case where $m = 2l$, by a statement analogous to Proposition 2.9 there are precisely two genuine irreducible representations of $Z_{\widetilde{M}}(\widetilde{M}_o)$ distinguished by their evaluation at the central element $e_N = (e_1 \cdots e_m)$ in $Z_{\widetilde{M}}(\widetilde{M}_o)$. Although these representations are defined in terms of δ_λ we abuse notation and write these two representations of $Z_{\widetilde{M}}(\widetilde{M}_o)$ as ξ^\pm where $\xi^+(e_N) = \sqrt{-1}^l$ and $\xi^-(e_N) = -\sqrt{-1}^l$. In the case where $m = 2l + 1$ by a statement analogous to Proposition 2.7 there is one genuine irreducible representation of $Z_{\widetilde{M}}(\widetilde{M}_o)$ extending $\delta_\lambda|_{Z(\widetilde{M}_o)}$. For convenience, will denote this representation by ξ^\pm with the understanding that $\xi^+ = \xi^-$.

We are finally ready to describe the genuine discrete series for the disconnected group \widetilde{M} . Let δ_λ be a genuine discrete series for \widetilde{M}_o and let ξ^\pm be a genuine representation of $Z_{\widetilde{M}}(\widetilde{M}_o)$. Then $\xi^\pm \otimes \delta_\lambda$ is a genuine discrete series of $Z_{\widetilde{M}}(\widetilde{M}_o) \cdot \widetilde{M}_o$. We denote the genuine discrete series of \widetilde{M} by δ_λ^\pm where $\delta_\lambda^\pm = \mathrm{Ind}_{Z_{\widetilde{M}}(\widetilde{M}_o) \cdot \widetilde{M}_o}^{\widetilde{M}} \xi^\pm \otimes \delta_\lambda$.

2.3.5 Isomorphisms among the genuine discrete series of \widetilde{M} for nonminimal cuspidal parabolics

Due to the disconnectedness of \widetilde{M} for nonminimal cuspidal parabolics there are isomorphisms among the genuine discrete series we need to be aware of. In what follows let b be the number of $\widetilde{\mathrm{SL}(2)}$ blocks in \widetilde{M} . The Harish-Chandra parameter

$$\lambda = \left(\frac{2l_1 + 1}{4}, -\frac{2l_1 + 1}{4}, \frac{2l_2 + 1}{4}, -\frac{2l_2 + 1}{4}, \dots, \frac{2l_b + 1}{4}, -\frac{2l_b + 1}{4} \right),$$

where l_i is an integer, will be abbreviated $(\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_b)$ where λ_i is the positive half integer $|l_i + \frac{1}{2}|$. The genuine discrete series representations of \widetilde{M} in $\widetilde{\mathrm{SL}(m)}$ are $\delta_{(\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_b)}^\pm = \mathrm{Ind}_{Z_{\widetilde{M}}(\widetilde{M}_o) \cdot \widetilde{M}_o}^{\widetilde{M}} \xi^\pm \otimes \delta_{(\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_b)}$.

First a general isomorphism statement about induced spaces.

Lemma 2.10 *Let $H \subset G$ be a subgroup of G and let π be a representation of H . Let $g \in N_G^H$ the normalizer of H in G . Then $\mathrm{Ind}_H^G \pi \cong \mathrm{Ind}_{Hg}^G g \cdot \pi$ where $(g \cdot \pi)(h) = \pi(ghg^{-1})$.*

Proof. By definition $\mathrm{Ind}_H^G \pi$ is the space of functions on G which transform as $f(xh) = \pi(h^{-1})f(x)$ for $x \in G$ and $h \in H$. Our isomorphism will take a function f from $\mathrm{Ind}_H^G \pi$ and send it to a function f' from $\mathrm{Ind}_H^G g \cdot \pi$ where $f'(x) = f(xg^{-1})$. To show that this function is well defined note that $f'(xh) = f(xhg^{-1}) = f(xg^{-1}(ghg^{-1}))$. Since g is in the normalizer of H , $ghg^{-1} \in H$ and $f(xg^{-1}(ghg^{-1})) = \pi(gh^{-1}g^{-1})f(xg^{-1})$ which equals $(g \cdot \pi)(h^{-1})f'(x)$ as required. All steps in the above derivation are reversible so our map is a bijection. Furthermore right translation commutes with left translation by g so our map is an isomorphism. \blacksquare

To apply this proposition below we will let $\pi = \xi^\pm \otimes \delta_{(\lambda_1, \dots, \lambda_b)}$, the subgroup $H = Z_{\widetilde{M}}(\widetilde{M}_o) \cdot \widetilde{M}_0$, and $G = \widetilde{M}$. We will need to find elements $g \in N_G^H$ and understand how they change π under conjugation.

In proving Proposition 2.12 about the isomorphism of genuine discrete series of \widetilde{M} for a non minimal parabolic it is enough to consider the case when the b diagonal $\mathrm{SL}(2)$ blocks in M are consecutive from the top left.

Let

$$\begin{aligned} J_{ev} &= \{j_1, \dots, j_{2r}\} \text{ be a subset of } \{2, 4, \dots, 2b\}, \\ K_{ev} &= \{k_1, \dots, k_{b-2r}\} \text{ be the set } \{2, 4, \dots, 2b\} \setminus \{j_1, \dots, j_{2r}\}, \\ J_{odd} &= \{j_1, \dots, j_{2r-1}\} \text{ be a subset of } \{2, 4, \dots, 2b\}, \\ K_{odd} &= \{k_1, \dots, k_{b-2r+1}\} \text{ be the set } \{2, 4, \dots, 2b\} \setminus \{j_1, \dots, j_{2r}\}. \end{aligned}$$

And let

$$e_{J_{ev}} = e_{j_1} \cdots e_{j_{2r}}, \quad e_{K_{ev}} = e_{k_1} \cdots e_{k_{b-2r}}, \quad e_{J_{odd}} = e_{j_1} \cdots e_{j_{2r-1}}, \quad e_{K_{odd}} = e_{k_1} \cdots e_{k_{b-2r+1}}.$$

Let $t_i = \cos(\frac{\theta_i}{2}) + e_{i-1}e_i \sin(\frac{\theta_i}{2})$ in $\mathrm{Spin}(m)$ for $1 \leq 2i \leq b$ and $0 \leq \theta_i \leq 4\pi$ and write

$$t_{J_{ev}} = t_{j_1} \cdots t_{j_{2r}}, \quad t_{K_{ev}} = t_{k_1} \cdots t_{k_{b-2r}}, \quad t_{J_{odd}} = t_{j_1} \cdots t_{j_{2r-1}}, \quad t_{K_{odd}} = t_{k_1} \cdots t_{k_{b-2r+1}}.$$

To further simplify notation we will write the Harish-Chandra parameter $(\lambda_1, \dots, \lambda_b)$ as $(\lambda_{J_{ev}}, \lambda_{K_{ev}})$ or $(\lambda_{J_{odd}}, \lambda_{K_{odd}})$.

Lemma 2.11 *Let b be the number of copies of $\mathrm{SL}(2)$ in M_o inside $\mathrm{SL}(m)$. The result of conjugating the genuine irreducible representations of $Z_{\widetilde{M}}(\widetilde{M}_o)\widetilde{M}_0$ by $e_{J_{ev}}$ when $(2b = m)$ and by $e_{J_{odd}}e_{2b+1}$, when $2b < m$ is:*

$$e_{J_{ev}} \cdot (\xi^\pm \otimes \delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})}) = \xi^\pm \otimes \delta_{(-\lambda_{J_{ev}}, \lambda_{K_{ev}})} \quad (m = 2b)$$

and

$$e_{J_{odd}}e_{2b+1} \cdot (\xi^\pm \otimes \delta_{(\lambda_{J_{odd}}, \lambda_{K_{odd}})}) = \xi^\pm \otimes \delta_{(-\lambda_{J_{odd}}, \lambda_{K_{odd}})} \quad (m < 2b)$$

In the case where m is odd or $m=2b$, $\xi^+ = \xi^-$.

Proof. We first consider the action of $e_{J_{ev}}$ and $e_{J_{odd}}e_{2b+1}$ on the side of the tensor product involving the discrete series of M_o . We have $(e_{J_{ev}} \cdot \delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})})(t_{J_{ev}} t_{K_{ev}})$ equals $\delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})}(t_{J_{ev}}^{-1} t_{K_{ev}}) = \delta_{(-\lambda_{J_{ev}}, \lambda_{K_{ev}})}(t_{J_{ev}} t_{K_{ev}})$. A similar statement holds for the action of $e_{J_{odd}}e_{2b+1}$.

Note that because ξ^\pm was defined as an extension of $\delta_\lambda|_{Z(M_o)}$ the ξ^\pm in $\xi^\pm \otimes \delta_{(\lambda_{J_{ev}}, \lambda_{K_{ev}})}$ is different from the ξ^\pm in $\xi^\pm \otimes \delta_{(-\lambda_{J_{ev}}, \lambda_{K_{ev}})}$, etc. We apologize for the notation.

The action of $e_{J_{ev}}$ and $e_{J_{odd}}e_{2b+1}$ on ξ^\pm sends ξ^+ to itself and ξ^- to itself since ξ^\pm are distinguished by a central element.

When m is odd then there is just a single extension of $\delta_\lambda|_{Z(M_o)}$ to $Z(M_o) = Z_M(M_o)$. In the case where $m = 2b$, $Z(M_o) = Z_M(M_o)$ so no extension is necessary. ■

From Lemmas 2.10 and 2.11, we immediately have the following isomorphism statement for genuine discrete series of \widetilde{M} for nonminimal cuspidal parabolics.

Proposition 2.12 *Let b be the number of copies of $\widetilde{SL}(2)$ in \widetilde{M} inside $\widetilde{SL}(m)$. We have,*

$$\delta_{(\lambda_1, \dots, \lambda_b)}^{\xi^\pm} \cong \delta_{(\pm\lambda_1, \dots, \pm\lambda_b)}^{\xi^\pm} \text{ if } (\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_b) \text{ has an even number of signs or } 2b < m.$$

In the case where m is odd or $m = 2b$, $\xi^+ = \xi^-$.

2.4 Coherent families and maximal primitive ideals

2.4.1 The infinitesimal character of an irreducible (\mathfrak{g}, K) module

The action of the center of the universal enveloping algebra plays an important role in understanding the structure of irreducible admissible (\mathfrak{g}, K) modules. This action is described by a representation's infinitesimal character, which we make precise below.

Let us fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ (with nilradical \mathfrak{n}). Let \mathfrak{n}^- denote the opposite nilradical, so that $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}$. The Poincare-Birkhoff-Witt theorem gives us a decomposition of the universal enveloping algebra of \mathfrak{g} as $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$.

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra and let $\chi' : Z(\mathfrak{g}) \rightarrow U(\mathfrak{h})$ be the projection on $U(\mathfrak{h})$ in the above decomposition of $U(\mathfrak{g})$. Let ρ be one half the sum of the positive roots given by \mathfrak{b} . We regard $U(\mathfrak{h})$ as the algebra of polynomial functions on \mathfrak{h}^* , and define $T_\rho f(\lambda) = f(\lambda - \rho)$ for $f \in U(\mathfrak{h})$ and $\lambda \in \mathfrak{h}^*$. The Harish-Chandra map from $Z(\mathfrak{g})$ into $U(\mathfrak{h})$ is by definition $\chi = T_\rho \circ \chi'$. Here T_ρ depends on \mathfrak{b} and χ is independent of \mathfrak{b} . For $\lambda \in \mathfrak{h}^*$, let $\chi_\lambda : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ be the composition of χ with evaluation at λ .

We have the following result of Harish-Chandra [2],

Theorem 2.13 (*Harish-Chandra*) *Every homomorphism from $Z_M(M_o)(\mathfrak{g})$ to \mathbb{C} is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$. Furthermore, $\chi_\lambda = \chi_{\lambda'}$ iff there is a $w \in W(\mathfrak{g}, \mathfrak{h})$ such that $\lambda' = w\lambda$.*

The center $Z(\mathfrak{g})$ is known to act by scalars on any irreducible (\mathfrak{g}, K) module X . The corresponding homomorphism $\chi_\lambda : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ defined by $\chi_\lambda(z)x = z \cdot x$, ($x \in X, z \in Z(\mathfrak{g})$) is called the *infinitesimal character* of X . We often abuse notation and say X has infinitesimal character λ . By Theorem 2.13 we see that elements in the Weyl group orbit of λ all define the same infinitesimal character.

2.4.2 The Langlands quotient of a generalized principal series representation

Let $P = MAN$ be a cuspidal parabolic of G , δ_λ a discrete series for M , and ν an irreducible character on A with $\mathrm{Re} \nu$ weakly dominant with respect to P .

The *Langlands quotient*, $J_P(\delta_\lambda \otimes \nu)$, is the largest completely reducible quotient of $\mathrm{Ind}_P(\delta_\lambda \otimes \nu)$. The infinitesimal character of $J_P(\delta_\lambda \otimes \nu)$ is $\lambda + \nu$. If ν is weakly antidominant with respect to \widetilde{P} , $J_P(\delta \otimes \nu)$ is the maximal submodule of $\mathrm{Ind}_{\widetilde{P}}(\delta_\lambda \otimes \nu)$ and is called a *Langland subrepresentation*.

The parameter $\lambda \in (\mathfrak{m} \cap \mathfrak{k})^*$ and ν are called the *Harish-Chandra parameter*, and the *continuous parameter* respectively. These two pieces of data together, $\gamma = (\lambda, \nu)$, called the *Langlands parameter*, specify the Langlands quotient $J_P(\delta_\lambda \otimes \nu)$. If δ_λ is the genuine discrete series δ_λ^\pm we will often write the Langlands quotient as $J_P^\pm(\gamma)$. When $\xi^+ = \xi^-$ we write $J_P(\gamma)$. Note that λ doesn't distinguish between δ_λ^\pm so the Langlands parameter doesn't uniquely specify the Langlands quotient here.

We will be concerned with the case where $G = \widetilde{\mathrm{SL}(m)}$, and δ_λ is a genuine discrete series representation of \widetilde{M} , and $\lambda + \nu = \frac{1}{2}\rho$.

The following theorem places the Langlands quotient at the heart of representation theory of real groups [7].

Theorem 2.14 (*Langlands Classification theorem*) *Let G be a connected semisimple Lie group and let X be an irreducible (\mathfrak{g}, K) module. Then there is a cuspidal parabolic subgroup $P = MAN$ of G , a discrete series representation δ_λ of M , and an irreducible character ν of A weakly positive with respect to P , such that X is equivalent to a summand of $J_P(\delta \otimes \nu)$. Furthermore, the pair $(P, \delta \otimes \nu)$ is unique up to conjugation in G when ν is strictly positive with respect to P .*

In the case where P is a minimal parabolic, the Langlands quotient, $J_P(\delta \otimes \nu)$, is called a *principal series* representation. For nonminimal parabolics it is called a *principal, gener-*

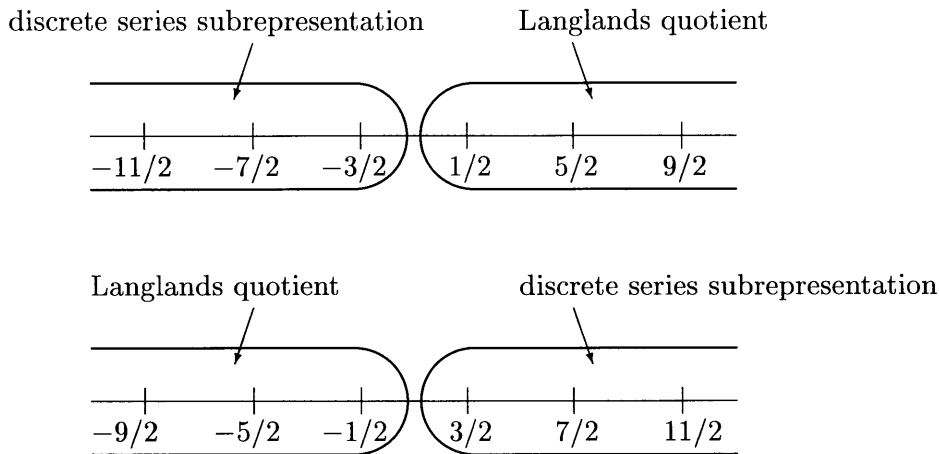


Figure 2.1: The four genuine irreducible representations of $\widetilde{\mathrm{SL}(2)}$ are shown inside of the principal series $Ind_{P_{min}}(\xi^+ \otimes \frac{1}{2}\rho)$ and $Ind_{P_{min}}(\xi^- \otimes \frac{1}{2}\rho)$.

alized series.

2.4.3 An example: The Langlands quotients for $\widetilde{\mathrm{SL}(2)}$ with infinitesimal character $\frac{1}{2}\rho$

The group $\widetilde{\mathrm{SL}(2)}$ has two cuspidal parabolics, namely the minimal parabolic, \widetilde{P}_{min} , and the whole group, \widetilde{P}_{max} . Let ξ^\pm be the two genuine irreducible representations of \widetilde{M}_{min} and $\delta_{(\frac{1}{4}, -\frac{1}{4})}$, $\delta_{(-\frac{1}{4}, \frac{1}{4})}$ the genuine irreducible representations of \widetilde{M}_{max} with Harish-Chandra parameter $\pm\frac{1}{2}\rho$. There are four Langlands quotients of $\widetilde{\mathrm{SL}(2)}$ with infinitesimal character $\frac{1}{2}\rho$: the two quotients of principal series $J_{P_{min}}(\xi^+ \otimes \frac{1}{2}\rho)$, and $J_{P_{min}}(\xi^- \otimes \frac{1}{2}\rho)$, and two discrete series with Harish-Chandra parameter $\pm\frac{1}{2}\rho$. The two discrete series can be realized as subquotients in $J_{P_{min}}(\xi^\pm \otimes -\frac{1}{2}\rho)$. Figure 2.1 shows how the \widetilde{K} types for the two principal series contain the \widetilde{K} types of their Langlands quotient and submodule.

2.4.4 Coherent families of virtual (\mathfrak{g}, K) modules

For every irreducible (\mathfrak{g}, K) module of G there is the notion of a coherent family of virtual (\mathfrak{g}, K) modules based at that representation. The properties of coherent families will be important for all that follows. We begin with some definitions.

Let $V(\mathfrak{g}, K)$ be the Grothendieck group of virtual (\mathfrak{g}, K) modules. A *virtual (\mathfrak{g}, K) module* is a linear combination of irreducible (\mathfrak{g}, K) modules with integer coefficients. We will also need a lattice of weights. For this let $H \subseteq G$ be a Cartan subgroup of G and let \hat{H} be the irreducible characters of H . The weight lattice in \hat{H} is the subgroup Λ of \hat{H} consisting of weights of finite dimensional representations of G . By identifying Λ with a subgroup of \mathfrak{h}^* we write $\xi + \lambda \in \mathfrak{h}^*$.

Fix a weight $\xi \in \mathfrak{h}^*$ and write $\xi + \Lambda = \{\xi + \lambda \mid \lambda \in \Lambda\}$. A *coherent family of virtual (\mathfrak{g}, K) modules* (based on H and $\xi + \Lambda$) is a map $\Phi : \xi + \Lambda \rightarrow V(\mathfrak{g}, K)$ satisfying

- a) $\Phi(\xi + \lambda)$ has infinitesimal character $\xi + \lambda \in \mathfrak{h}^*$, and
- b) If F is any finite dimensional representation of G with weights $\Delta(F)$ then

$$\phi(\xi + \lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Phi(\xi + (\lambda + \mu)).$$

From this point we will abuse notation and say that a coherent family Φ is based at an irreducible (\mathfrak{g}, K) module. Furthermore we will embed \hat{H} in \mathfrak{h}^* and write $\xi + \lambda \in \mathfrak{h}^*$.

We have the following result stating the existence of a coherent family based at a representation [7].

Proposition 2.15 *Given an irreducible (\mathfrak{g}, K) module X of regular infinitesimal character ξ , there exists a unique coherent family Φ on $\xi + \Lambda$ such that $\Phi(\xi) = X$*

The members of the coherent family we will be most interested in are parameterized by a cone in the weight lattice. Let $R(\xi) = \{\alpha \in \Delta(\xi) : 2\frac{\langle \alpha, \xi \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}\}$ be the *integral roots with respect to ξ* and let $R^+(\xi) = \{\alpha \in R(\xi) : \langle \xi, \alpha \rangle > 0\}$ be the *positive integral roots with respect to ξ* . The integral roots are a root system in \mathfrak{h} . The *integral Weyl group*, denoted $W(R(\xi))$, is a subgroup of the Weyl group $W(\Delta(\xi))$, generated by reflections s_α about the positive integral roots. We say that a weight is *dominant regular* if it lies properly inside the dominant Weyl chamber (i.e. the weight doesn't lie on a wall). The relationship between the dominant $\Delta^+(\frac{1}{2}\rho)$ Weyl chamber and the dominant $R^+(\frac{1}{2}\rho)$ Weyl chamber in \mathfrak{h}^* for $\mathfrak{sl}(3)$ are illustrated in figure 2.2.

We will be primarily interested only in those members of a coherent family parameterized by a cone in a lattice. The following result can be found in [7].

Proposition 2.16 *A coherent family based at an irreducible (\mathfrak{g}, K) module with regular infinitesimal character ξ has the property that if $\gamma \in \xi + \Lambda$ is dominant regular with respect to $R^+(\xi)$ then $\Phi(\gamma)$ is an irreducible (\mathfrak{g}, K) module.*

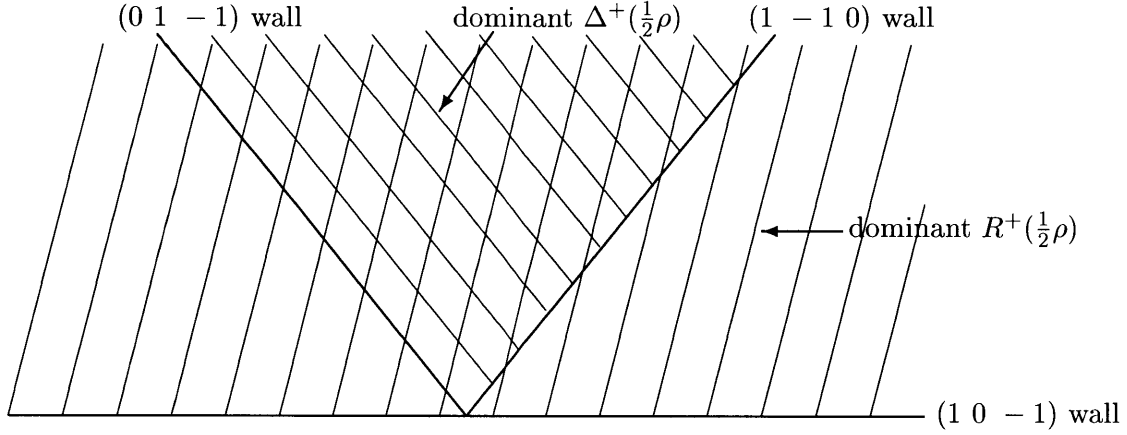


Figure 2.2: Dominant $\Delta^+(\frac{1}{2}\rho)$ and $R^+(\frac{1}{2}\rho)$ chambers in \mathfrak{h}^* for $\mathfrak{sl}(3)$.

To describe the behavior of the coherent family outside the $R^+(\xi)$ dominant Weyl chamber we need to introduce the idea of a (\mathfrak{g}, K) module's τ invariant.

2.4.5 Translation functors and τ invariance

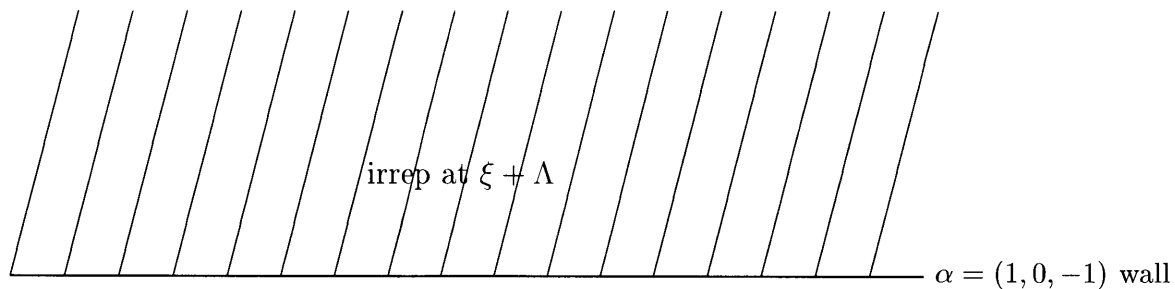
We now introduce Zuckerman's exact translation functor, $\Psi_{\xi+\lambda}^{\xi}$ from the category of Harish-Chandra modules with regular infinitesimal character ξ to the category of Harish-Chandra modules with infinitesimal character $\xi + \lambda \in \xi + \Lambda$.

We define a projection functor, P_{λ} , on the category of Harish-Chandra modules. Let X be a Harish-Chandra module, then $P_{\lambda}X$ is the maximal subspace of X on which $z - \chi_{\lambda}(z)$ acts nilpotently for every $z \in Z(\mathfrak{g})$. Given any Harish-Chandra module we have $X = \bigoplus_{\gamma \in \mathfrak{h}^*} P_{\gamma}X$, where the sum is finite and each $P_{\gamma}X$ is a Harish-Chandra submodule of X . Let F^{λ} be a finite dimensional representation with extremal weight λ . We define the *Zuckerman translation functor* to be $\Psi_{\xi+\lambda}^{\xi}(X) = P_{\xi+\lambda}(F^{\lambda} \otimes P_{\xi}(X))$.

We use Zuckerman functors to define the τ invariant of a Harish-Chandra module, X , with regular infinitesimal character ξ . Let $R^+\xi$ be a positive set of integral roots with respect to ξ and let α be a simple root in $R^+\xi$. Let Φ be a coherent family based at X . The *Borho-Jantzen-Duflo τ invariant* of X consists of the simple integral roots α such that $\Psi_{\xi+\lambda}^{\xi}(X) = 0$ whenever $\xi + \lambda$ lies on an α wall. Another way to say this is that for every $\lambda \in \Lambda$ such that $\langle \alpha, \xi + \lambda \rangle = 0$ and $\langle \beta, \xi + \lambda \rangle > 0$, for $\beta \in R^+(\xi) - \alpha$, $\Phi(\xi + \lambda)$ is zero.

From the definition of the τ invariant of X it is clear that any irreducible (\mathfrak{g}, K) module in the coherent family based at X has the same τ invariant.

The following alternative definition of τ invariant will be useful to us [2].



$-\text{irrep}$ if $\alpha \in \tau(\Phi(\xi))$
 $\text{irrep} + \dots$ if $\alpha \notin \tau(\Phi(\xi))$.

Figure 2.3: Picture for $\mathfrak{sl}(3)$: $R^+(\xi)$ dominant regular lattice points in $\xi + \Lambda$ parameterize irreducible representations in the coherent family. There is a dichotomy in the case of non $R^+(\xi)$ dominant regular lattice points.

Proposition 2.17 *Let X be an irreducible (\mathfrak{g}, K) module with regular infinitesimal character ξ . Let α be a simple root of $R^+(\xi)$. Then $\alpha \in \tau(X)$ iff $\Phi(s_\alpha\gamma) = -\Phi(\gamma)$, for all $\gamma \in \xi + \Lambda$.*

If X is a virtual (\mathfrak{g}, K) module with regular infinitesimal character γ we say that a simple root $\alpha \in R^+(\gamma)$ is in the τ invariant of X if it is the τ invariant of each irreducible term of X . Next we have a proposition describing the coherent family based at ξ outside of the $R^+(\xi)$ dominant integral chamber [7].

Proposition 2.18 *Fix a simple root $\alpha \in R^+(\xi)$. Let $\gamma \in \xi + \Lambda$ be a member of the weight lattice based at ξ with $\langle \gamma, \beta \rangle > 0$ for all $\beta \in R^+(\xi)$. We have the following dichotomy.*

- (a) $\Phi(s_\alpha\gamma) = -\Phi(\gamma)$, if $\alpha \in \tau(X)$
- (b) $\Phi(s_\alpha\gamma) = \Phi(\gamma) + \text{character of a representation}$, if $\alpha \notin \tau(X)$

Figure 2.3 illustrates a coherent family based at an irreducible (\mathfrak{g}, K) module for $\mathfrak{g} = \mathfrak{sl}(3)$.

In computing the τ invariants of a Langlands quotient we will make use of a theorem by Vogan [7]. We will need the following definition. Suppose α is a real root for a Θ -stable Cartan H . Let $\varphi_\alpha : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{g}$ be an injection so that $H_\alpha = \varphi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$, $\varphi_\alpha(-{}^t X) = \theta\varphi_\alpha(X)$. Set $m_\alpha = \exp(\varphi_\alpha \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}) \in T$.

Theorem 2.19 (Vogan) *Let G be a semisimple Lie group, with a Θ -invariant Cartan subgroup $H = TA$, and let $P = MAN$ be a cuspidal parabolic subgroup. Let $J_P(\delta_\lambda \otimes \nu)$ be a Langlands quotient having nonsingular Langlands parameter $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$ (i.e. $\langle \gamma, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\gamma)$). Suppose that $\alpha \in R^+(\gamma)$ is simple in $\Delta^+(\gamma)$ and $2\langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$.*

a) *If α is a real root (i.e. $\Theta(\alpha) = -\alpha$), then a necessary and sufficient condition for $\alpha \in \tau(J_P(\delta_\lambda \otimes \nu))$ is that the eigenvalues of $\delta_\lambda(m_\alpha)$ be of the form $\epsilon_\alpha \exp(\pm 2\pi i \langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle)$ where ϵ_α is ± 1 depending on a condition described in [2].*

b) *If α is a complex root (i.e. $\Theta(\alpha) \neq \pm \alpha$) then a necessary and sufficient condition for $\alpha \in \tau(J_P(\delta_\lambda \otimes \nu))$ is that $\Theta\alpha \notin R^+(\gamma)$.*

c) *If α is compact imaginary then $\alpha \in \tau(J_P(\delta_\lambda \otimes \nu))$. If α is noncompact imaginary then $\alpha \notin \tau(J_P(\delta_\lambda \otimes \nu))$.*

Corollary 2.20 *Let $G = \widetilde{\text{SL}}(m)$, $H = TA$ be a Θ -invariant Cartan subgroup, and $P = MAN$ be a cuspidal parabolic subgroup. Let $J_P(\delta_\lambda \otimes \nu)$ be a genuine Langlands quotient having nonsingular Langlands parameter $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$. Then no real root can be in the τ invariant of $J_P(\delta_\lambda \otimes \nu)$.*

Proof. Suppose that $\alpha \in R^+(\gamma)$ is simple in $\Delta^+(\gamma)$ and $2\langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$. Then $\epsilon_\alpha \exp(\pm 2\pi i \langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle)$ is ± 1 . However, m_α is a monomial in $\text{Spin}(m)$ and hence $m_\alpha^2 = -1$. We have $\delta_\lambda(-1) = -1$ since δ_λ is genuine and it follows that $\delta_\lambda(m_\alpha)$ has eigenvalues $\pm \sqrt{-1}$. Therefore the eigenvalues of $\delta_\lambda(m_\alpha)$ are not of the form $\pm \exp(\pm 2\pi i \langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle)$. ■

2.4.6 Maximal primitive ideals

The *primitive ideal* of an irreducible (\mathfrak{g}, K) module, X , is the annihilator of X in the universal enveloping algebra $U(\mathfrak{g})$. We say that a primitive ideal, $\text{ann}(X)$, has infinitesimal character λ if X does. A theorem of Duflo [3] says that every primitive ideal of infinitesimal character λ is the annihilator of an irreducible highest weight module $L(\lambda')$ where $\lambda' \in \mathfrak{h}^*$ is in the Weyl group orbit of λ . Here $L(\lambda')$ is the unique irreducible quotient of the Verma module $M(\lambda') = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda' - \rho}$ with highest weight $\lambda' - \rho$ for a fixed Cartan subalgebra contained in a Borel subalgebra \mathfrak{b} . There are exactly $|W/W^\lambda|$ non-isomorphic simple highest weight modules with infinitesimal character λ , namely $L(w\lambda)$ for $w \in W$. It follows that the number of primitive ideals with a given infinitesimal character is finite. We denote this set by $\text{Prim}_\lambda U(\mathfrak{g})$. If λ' is dominant and regular integral than $L(\lambda')$ is a finite dimensional representation and its annihilator is the unique *maximal primitive ideal* having infinitesimal character λ . The annihilator of the irreducible Verma module, $L(w_0\lambda') = M(w_0\lambda')$, where w_0 is the longest element in $W(R^+(\lambda))$ is the unique *minimal primitive ideal* with infinitesimal character λ . We shall denote the maximal, and minimal primitive ideals with

infinitesimal character λ by $I^{\max}(\lambda)$ and $I^{\min}(\lambda)$.

Duflo, Borho-Jantzen showed that the “almost” minimal primitive ideals with a given infinitesimal character λ (i.e. primitive ideals properly containing only the minimal primitive ideal), denoted by $I^{\min}(s_\alpha, \lambda)$ are parameterized by the simple integral roots $\alpha \in R^+\lambda$. Similarly, the “almost” maximal primitive ideals (i.e. primitive ideals properly contained in only the maximal primitive ideal), denoted by $I^{\max}(s_\alpha, \lambda)$ are parameterized by the simple integral roots $\alpha \in R^+\lambda$.

Proposition 2.21 (*B-J-Duflo*) *Let X be an irreducible (\mathfrak{g}, K) module with regular infinitesimal character λ . The following conditions on a simple integral root $\alpha \in R^+(\lambda)$ are equivalent.*

- a) $\Psi_\lambda^{\lambda+\alpha} X = 0$ (definition for $\alpha \in \tau(X)$)
- b) $\mathrm{ann} X \supseteq I^{\min}(s_\alpha, \lambda)$
- c) $\mathrm{ann} X \not\subseteq I^{\max}(s_\alpha, \lambda)$

Corollary 2.22 *The τ invariant is an order preserving map from $\mathrm{Prim}_\lambda U(\mathfrak{g})$ to subsets of simple roots in $R^+\lambda$. A primitive ideal is maximal iff every simple root of $R^+\lambda$ is in τ .*

Proof. Suppose I_1 and I_2 are primitive ideals with $I_1 \subseteq I_2$. By Proposition 2.21, $\tau(I_1) \subseteq \tau(I_2)$. Furthermore, since the maximal primitive ideal contains the “almost” minimal primitive ideals, its τ invariant contains all of the simple roots of $R^+(\lambda)$. To prove the converse, suppose that I is not a maximal primitive ideal, then $I \neq \subset I^{\max}(\lambda)$. It follows that $I \subseteq I^{\max}(s_\alpha, \lambda)$ for some α , and hence $\alpha \notin \tau(I)$. ■

2.5 Langlands quotients with a maximal primitive ideal

We wish to determine the Langlands quotients for $\widetilde{\mathrm{SL}(m)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal (i.e. where all simple integral roots with respect to $\frac{1}{2}\rho$ are τ invariants for our representation). Theorem 2.19 provides a way to check that an integral root α is a τ invariant. However, one cannot apply Theorem 2.19 directly unless the integral root α is a simple root for $\Delta^+(\frac{1}{2}\rho)$. Unfortunately for $\frac{1}{2}\rho = (\frac{m-1}{4}, \frac{m-1}{4} + \frac{1}{2}, \dots, -\frac{m-1}{4})$ none of the integral roots of $R^+(\frac{1}{2}\rho)$ are simple roots in $\Delta^+(\frac{1}{2}\rho)$.

We propose to find the Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal, by first determining how many of them there are. It suffices to determine the number of Langlands quotients with maximal primitive ideal at an infinitesimal character in the dominant regular $R^+(\frac{1}{2})$ chamber. This follows from the fact that

irreducible members of a coherent family based at infinitesimal character $\frac{1}{2}\rho$ all have the same τ invariants.

We need to find an infinitesimal character $\frac{1}{2}\rho + \lambda$ in the dominant regular integral Weyl chamber based at $\frac{1}{2}\rho$ with respect to which all simple integral roots are simple in $\Delta(\frac{1}{2}\rho + \lambda)$.

Proposition 2.23 *Let $G = \widetilde{\mathrm{SL}}(2l)$, $l \geq 1$ and let γ be a Weyl group representative of the infinitesimal character $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$, where*

$$\begin{aligned} p_0 &= \frac{6l+3}{4} \text{ and } p_i = p_0 + i \\ n_0 &= -\frac{6l+3}{4} \text{ and } n_i = n_0 - i, \text{ for } i \in \{0, 1, \dots, l-1\} \end{aligned}$$

The weight γ lies in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber based at $\frac{1}{2}\rho$ and the simple integral roots in $R^+(\gamma)$ are simple in $\Delta^+\gamma$.

Proof. γ is a representative of the infinitesimal character $\frac{1}{2}\rho + (2l, -2l, 2l, -2l, \dots, 2l, -2l)$. It is easy to check that p_i and n_i have the values given in the claim. The p_i are all positive and differ from one another by an integer. Similarly, the n_i are all negative and differ from one another by an integer. Because of the descending ordering $p_{l-1} > p_{l-2} > \dots > p_0 > n_0 > \dots > n_{l-1}$ of its entries, γ lies in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber based at $\frac{1}{2}\rho$ and $\{e_1 - e_2, \dots, e_{l-1} - e_l, e_{l+1} - e_{l+2}, \dots, e_{2l-1} - e_{2l}\}$ are simple integral roots for both $R^+(\gamma)$ and $\Delta^+\gamma$. Taking another Weyl group representative of γ , say $w \cdot \gamma$ makes $w \cdot \{e_1 - e_2, \dots, e_{l-1} - e_l, e_{l+1} - e_{l+2}, \dots, e_{2l-1} - e_{2l}\}$ simple integral roots for $R^+(w \cdot \gamma)$ and $\Delta^+(w \cdot \gamma)$. ■

Proposition 2.24 *Let $G = \widetilde{\mathrm{SL}}(2l+1)$, $l \geq 1$, and let γ' be a Weyl group representative of the infinitesimal character $(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{3l-1})$, where*

$$\begin{aligned} p'_0 &= \frac{3l}{2} \text{ and } p'_j = p'_0 + j \text{ for } j \in \{0, 1, \dots, l\} \\ n'_0 &= -\frac{3l+1}{2} \text{ and } n'_i = n'_0 - i, \text{ for } i \in \{0, 1, \dots, l-2, 3l-1\} \end{aligned}$$

The weight γ' lies in the dominant regular $R^+(\frac{1}{2}\rho)$ chamber based at $\frac{1}{2}\rho$ and the simple integral roots in $R^+(\gamma')$ are simple in $\Delta^+\gamma'$.

Proof. The statement is less symmetric looking than Proposition 2.23 because of the odd number of entries in γ' . The proof however is no different than Proposition 2.23 except that γ' is a representative of the infinitesimal character $\frac{1}{2}\rho + (2l, -2l, 2l, -2l, \dots, 2l, -4l, 2l)$. Professor Vogan has informed me that translating by the $(l, -(l+1), \dots, -(l+1), l)$ would have made the formulas prettier. Due to time constraints we have chosen however to stick with $(2l, -2l, 2l, -2l, \dots, 2l, -4l, 2l)$. ■

For the purpose of counting the number of maximal primitive ideals we will study the Langlands quotients having infinitesimal character $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$ in the even case and $(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{3l-1})$ in the odd case. The following will help us determine which Weyl group representatives of an infinitesimal character can be Langlands parameters.

Lemma 2.25 *With n_i, p_i as in Proposition 2.23 the inequalities, $n_i \geq \frac{p_j+n_k}{2}$ and $\frac{p_j+n_k}{2} \geq p_i$ are impossible for any $i, j, k \in \{0, 1, \dots, l-1\}$.*

Proof. To prove the first inequality it suffices to show that $\frac{p_0+n_{l-1}}{2} > n_0$ since $p_0+n_{l-1} = \min\{p_j + n_k : j, k \in \{0, \dots, l-1\}\}$ and $n_0 = \max\{n_i : i \in \{0, \dots, l-1\}\}$. Indeed, $\frac{p_0+n_{l-1}}{2} = \frac{p_0+n_0-l+1}{2} = \frac{-l+1}{2} > \frac{-6l-3}{4} = n_0$

To prove the second inequality it suffices to show that $\frac{p_{l-1}+n_0}{2} < p_0$ since $p_{l-1} + n_0 = \max\{p_j + n_k : j, k \in \{0, \dots, l-1\}\}$ and $p_0 = \min\{p_i : i \in \{0, \dots, l-1\}\}$. Indeed, $\frac{p_{l-1}+n_0}{2} = \frac{p_0+l-1+n_0}{2} = \frac{l-1}{2} < \frac{6l+3}{4} = p_0$. ■

Lemma 2.26 *With n'_i, p'_i as in Proposition 2.24 the inequalities, $n'_i \geq \frac{p'_j+n'_k}{2}$ and $\frac{p'_j+n'_k}{2} \geq p'_i$ are impossible for all $p_i, p_j \in \{p_0, \dots, p_l\}$ and all $n_i, n_k \in \{n_0, \dots, n_{l-2}, n_{3l-1}\}$.*

Proof. To prove the first inequality it suffices to show that $\frac{p'_0+n'_{3l-1}}{2} > n'_0$. Indeed, $\frac{p'_0+n'_{3l-1}}{2} = \frac{-3l+1-\frac{1}{2}}{2} = -\frac{3l}{2} + \frac{1}{4} > -\frac{3l}{2} - \frac{1}{2} = n'_0$

To prove the second inequality it suffices to show that $\frac{p'_l+n'_0}{2} < p'_0$. Indeed, $\frac{p'_l+n'_0}{2} = \frac{l-\frac{1}{2}}{2} < \frac{3l}{2} = p_0$. ■

We are now ready to count the number of Langlands quotients having maximal primitive ideal. We start by considering Langlands quotients with minimal parabolics. This is the easiest case.

Proposition 2.27 *Langlands quotients of principal series representations with infinitesimal character $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$ or $(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{3l-1})$, as defined in Proposition 2.23 and Proposition 2.24, do not have a maximal primitive ideal.*

Proof. The real form of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}(m)$ associated with a minimal parabolic is a diagonal matrix with real entries. It follows that all integral roots are real. For the infinitesimal characters given in the claim, all integral roots are simple in $\Delta^+\gamma$ (Proposition 2.23 and Proposition 2.24), and by Corollary 2.20 cannot be a τ invariant. ■

Next we consider Langlands quotients with nonminimal parabolics. We begin with a lemma about Langlands parameters for Langlands quotients with a nonminimal parabolic.

Lemma 2.28 *Let (a_1, \dots, a_m) be a Langlands parameter of a genuine Langlands quotient with cuspidal parabolic $\widetilde{P} = \widetilde{MAN}$. If the linear group M has an $\mathrm{SL}(2)$ block along the $\{j, j+1\}$ diagonal entries then $a_j - a_{j+1} \notin \mathbb{N}$. When immediately followed by another $\mathrm{SL}(2)$ block the Langlands parameter must satisfy $\frac{a_j+a_{j+1}}{2} \geq \frac{a_{j+2}+a_{j+3}}{2}$, otherwise $\frac{a_j+a_{j+1}}{2} \geq a_{j+2}$.*

If the $\mathrm{SL}(2)$ block along the $\{j, j+1\}$ diagonal entries is immediately preceded by an $\mathrm{SL}(2)$ block the Langlands parameter must satisfy $\frac{a_{j-2}+a_{j-1}}{2} \geq \frac{a_j+a_{j+1}}{2}$, otherwise $a_{j-1} \geq \frac{a_j+a_{j+1}}{2}$. If no $\mathrm{SL}(2)$ block begins or ends at the $\{j, j+1\}$ diagonal entries then $a_j \geq a_{j+1}$.

Proof. Suppose that M has an $\mathrm{SL}(2)$ block along the $\{j, j+1\}$ diagonal entries. Then entries j and $j+1$ of the Harish-Chandra parameter are $(\dots, \frac{a_j-a_{j+1}}{2}, -\frac{(a_j-a_{j+1})}{2}, \dots)$ and entries j and $j+1$ of the continuous parameter, ν are $(\dots, \frac{a_j+a_{j+1}}{2}, \frac{a_j+a_{j+1}}{2}, \dots)$. We must have $a_j - a_{j+1} \notin \mathbb{N}$ for our Harish-Chandra parameter to belong to a genuine discrete series. For ν to be the continuous parameter of a Langlands quotient it must be weakly dominant with respect to \tilde{P} . As a result we must have the $(j+2)$ entry of the continuous parameter be less than or equal to the $j+1$ entry. When our $\mathrm{SL}(2)$ block is immediately followed by another $\mathrm{SL}(2)$ block this translates to the condition that $\frac{a_j+a_{j+1}}{2} \geq \frac{a_{j+2}+a_{j+3}}{2}$. When our $\mathrm{SL}(2)$ block is not immediately followed by another $\mathrm{SL}(2)$ block we have $\frac{a_j+a_{j+1}}{2} \geq a_{j+2}$. The other parts of the lemma follow in the same way. ■

Proposition 2.29 *Let $G = \widetilde{\mathrm{SL}(2l)}$ and let $\tilde{P} = \tilde{M}AN$ be a non minimal cuspidal parabolic subgroup of G . With notation as in Proposition 2.23 let $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$ be the infinitesimal character of a Langlands quotient with parabolic \tilde{P} . The diagonal $\mathrm{SL}(2)$ blocks in the linear group M must be consecutive and centered in the matrix. For the Langlands quotient to have a maximal primitive ideal, the number of blocks, b , must equal $l-1$ or l .*

Proof. Our Langlands parameter is in the Weyl group orbit of the infinitesimal character $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$. By Lemma 2.28 for there to be a $\mathrm{SL}(2)$ block in the $\{j, j+1\}$ diagonal entry of M the difference between the j and $j+1$ entry of the Langlands parameter must be a half integer. It follows that one of the two entries must be positive and the other one negative. By Lemma 2.25, if the $\mathrm{SL}(2)$ block in M is not immediately followed by another $\mathrm{SL}(2)$ block then the $j+2$ entry of the Langlands parameter must be negative. If there is another $\mathrm{SL}(2)$ block further down the diagonal of M then we would have a situation where $n_i \geq \frac{p_j+n_k}{2}$ or $\frac{p_j+n_k}{2} \geq p_i$ for some $i, j, k \in \{0, 1, \dots, l-1\}$. This is impossible by Proposition 2.27. If the $\mathrm{SL}(2)$ block in the $\{j, j+1\}$ diagonal entry of M is not immediately preceded by another $\mathrm{SL}(2)$ block then the $j-1$ entry of the Langlands parameter must be positive. Were there to be another $\mathrm{SL}(2)$ block further up the diagonal of M then we again would have a situation where $n_i \geq \frac{p_j+n_k}{2}$ or $\frac{p_j+n_k}{2} \geq p_i$ for some $i, j, k \in \{0, 1, \dots, l-1\}$. It follows that the $\mathrm{SL}(2)$ blocks must be consecutive in M .

Suppose the $\mathrm{SL}(2)$ blocks lie along the j through $j+2b-1$ diagonal entries in M . Each pair of entries between j and $j+2b+1$ in the Langlands parameter contains a positive and a negative entry. Furthermore, entries 1 through $j-1$ of the Langlands parameter are descending positive numbers while entries $j+2b+2$ through $2l$ are descending negative numbers. Since there are an equal number of positive and negative entries in the Langlands parameters, the $\mathrm{SL}(2)$ blocks must be centered in M (i.e. $b = l-j$).

Finally we show that to have a maximal primitive ideal, then b must equal $l-1$ or l . Indeed if this were not the case then either a simple integral root would be a real root or the

positive entry of the first $\mathrm{SL}(2)$ block would be greater than one of the preceding positive entries (for example $(p_2, p_0, p_1, n_1, n_0, n_2)$). By Corollary 2.20, a simple integral root cannot be real if the Langlands quotient has a maximal primitive idea. As well, the second scenario is ruled out by an easy tau invariant computation. ■

Proposition 2.30 *Let $G = \mathrm{SL}(\widetilde{2l+1})$ and let $\widetilde{P} = \widetilde{M}AN$ be a non minimal cuspidal parabolic subgroup of G . Let $(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{l-1})$ be the infinitesimal character of a Langlands quotient with parabolic \widetilde{P} , with notation as in Proposition 2.24. The diagonal $\mathrm{SL}(2)$ blocks in the linear group M must be consecutive and centered along the lower $2l$ diagonal entries of M . For the Langlands quotient to have a maximal primitive ideal, the number of blocks, b , must equal l .*

Proof. The proof is almost identical to the proof of Proposition 2.29. Because the Langlands parameter contains $l+1$ positive entries, the $\mathrm{SL}(2)$ blocks in the linear group M must be one off the center of the diagonal. ■

Proposition 2.31 *In the notation of Proposition 2.23, there are four Langlands quotients of $\widetilde{\mathrm{SL}}(2l)$ with infinitesimal character $(p_{l-1}, p_{l-2}, \dots, p_0, n_0, n_1, \dots, n_{l-1})$ having a maximal primitive ideal.*

Proof. By Proposition 2.29 there are only two subgroups \widetilde{M} we need to consider, namely when the linear group M contains $b = l-1$ and $b = l$ diagonal $\mathrm{SL}(2)$ blocks centered in M . In the case where $b = l-1$ by Proposition 2.12 the genuine discrete series of \widetilde{M} are, up to isomorphism, $\delta_{(\lambda_1, \dots, \lambda_b)}^{\xi^\pm}$, where λ_i are positive half integers. In the case where $b = l$ by Proposition 2.12 the genuine discrete series of \widetilde{M} are, up to isomorphism, $\delta_{(\lambda_1, \dots, \lambda_b)}^{\xi^\pm}$ and $\delta_{(\lambda_1, \dots, -\lambda_b)}^{\xi^\pm}$, where λ_i are positive half integers and $\xi^+ = \xi^-$.

We first prove the proposition for $\widetilde{\mathrm{SL}}(4)$. Let M contain one $\mathrm{SL}(2)$ block along the second and third diagonal entries. The Langlands parameters we are considering are permutations of (p_1, p_0, n_0, n_1) . Because the Harish-Chandra parameter is a positive half integer, we must have a positive and a negative entry in the second and third entries respectively of the Langlands parameter. Furthermore, by Lemma 2.25 the first entry of the Langlands parameter must be positive and the fourth entry negative. Any permutation with these restrictions is a Langlands parameter for our parabolic. However requiring the simple integral roots to be τ invariant pins things down. The root, $e_1 - e_2$ is simple and integral for the Langlands parameter (p_1, p_0, n_0, n_1) and $\Theta(e_1 - e_2) = -e_1 + e_3 \in \Delta^-(\gamma)$. Also for the other simple integral root, $\Theta(e_3 - e_4) = -e_2 + e_4 \in \Delta^-(\gamma)$. Hence $e_1 - e_2$ and $e_3 - e_4$ are τ invariants and so the Langlands quotient with Langlands parameter (p_1, p_0, n_0, n_1) has a maximal primitive ideal. It is easy to check that the three other possible Langlands parameters don't have both simple roots as τ invariants. We have two genuine Langlands quotients with this Langlands parameter, $J_{P_0=1}^\pm(p_1, p_0, n_0, n_1)$.

Next we consider the case for $\widetilde{\mathrm{SL}}(4)$ where M contain two $\mathrm{SL}(2)$ blocks. By Proposition 2.12, up to isomorphism of the discrete series of \widetilde{M} , the first and second entries of the Langlands parameter must be positive and negative respectively, but the third and fourth entries can be positive or negative in any order. There are four permutations of (p_1, n_0, p_0, n_1) where positive entries can be interchanged and negative entries can be interchanged. The simple integral roots are $\pm(e_1 - e_3)$ and $\pm(e_2 - e_4)$. Suppose that $e_1 - e_3$ is simple. Then $\Theta(e_1 - e_3) = -e_2 + e_4$. Hence for $e_1 - e_3$ to be a τ invariant, $e_2 + e_4$ must be a simple root. It follows that the Langlands quotient with Langlands parameter (p_1, n_0, p_0, n_1) has τ invariants $e_1 - e_3$ and $e_2 - e_4$. If on the otherhand $e_3 - e_1$ is simple, then to be a τ invariant, $e_4 - e_2$ would have to be simple. However by Lemma 2.28, (p_0, n_1, p_1, n_0) isn't a Langlands parameter because $\frac{p_0+n_1}{2} < \frac{p_1+n_0}{2}$. Switching the third and fourth entries in the above argument shows (p_1, n_0, n_1, p_0) is also a Langlands parameter with simple integral roots which are τ invariant. The two resulting Langlands quotients with maximal primitive ideal are $J_{\widetilde{P}_{b=2}}(p_1, n_0, p_0, n_1)$ and $J_{P_{b=2}}(p_1, n_0, n_1, p_0)$.

For $\mathrm{SL}(2l)$, investigation of Langlands parameters whose simple integral roots are τ invariants does not require us to look at more than two integral roots at a time, and hence the above arguments directly generalize. The four Langlands quotients with a maximal primitive ideal are $J_{\widetilde{P}_{b=l-1}}^{\pm}(p_{l-1}, p_{l-2}, n_0, \dots, p_0, n_{l-2}, n_{l-1})$ and $J_{\widetilde{P}_{b=l}}(p_{l-1}, n_0, p_{l-2}, n_1, \dots, p_0, n_{l-1})$, and $J_{\widetilde{P}_{b=l}}(p_{l-1}, n_0, p_{l-2}, n_1, \dots, p_1, n_{l-2}, n_{l-1}, p_0)$. ■

Proposition 2.32 *In the notation of Proposition 2.24, there is one Langlands quotient of $\mathrm{SL}(2l+1)$ with infinitesimal character $(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{3l-1})$ having a maximal primitive ideal.*

Proof. The proof is almost identical to Proposition 2.31. The Langlands quotient with maximal primitive ideal is $J_{\widetilde{P}_{b=l}}(p'_l, p'_{l-1}, n'_0, \dots, n'_{l-2}, p'_0, n'_{3l-1})$. ■

Corollary 2.33 $\widetilde{\mathrm{SL}}(2l)$ has four representations with maximal primitive ideals with infinitesimal character $\frac{1}{2}\rho$, and $\mathrm{SL}(2l+1)$ has one representation with maximal primitive ideals with infinitesimal character $\frac{1}{2}\rho$.

Proof. This follows from what was said just prior to Proposition 2.23. ■

We now identify the Langlands quotients at infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. To do this we will need to translate our Langlands quotient in the coherent family. We begin with a statement that tells us how to do this.

Lemma 2.34 *Let G be a semisimple group, with Cartan subgroup H and parabolic subgroup MAN . Let $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$ be a Langlands parameter. Let $\mathrm{Ind}_{MAN}(V_\gamma)$ be a generalized principal series representation. Let $\gamma' = (\lambda', \nu') \in \mathfrak{h}^*$ be an extremal weight of a finite dimensional representation $F^{\gamma'}$ of G . Then $\Psi_{\gamma+\gamma'}^\gamma \mathrm{Ind}_{MAN}^G(\gamma) = \mathrm{Ind}_{MAN}^G(\gamma + \gamma')$.*

Proof. One has $\Psi_{\gamma+\gamma'}^\gamma \mathrm{Ind}_{MAN}^G(\gamma) = \mathrm{Ind}_{MAN}^G(\Psi_{\gamma+\gamma'}^\gamma(\gamma))$ following an argument of Zuckerman [4]. Then $\Psi_{\gamma+\gamma'}^\gamma(V_\gamma) = p_{\gamma'}(V_\gamma \otimes F^{\gamma'}|_{MAN}) = V_{\gamma+\gamma'}$. ■

Next we need to introduce the notion of induction-in-stages [5].

Lemma 2.35 (*Induction-in-stages*) *Let G be a semisimple Lie group with parabolic subgroup MAN . Let $\mathrm{Ind}_{MAN}^G(\xi, \lambda)$ be a generalized principal series representation with Langlands parameter (ξ, λ) . Given a parabolic subgroup $M'A'N' \supseteq MAN$ we have the following induction-in-stages formula,*

$$\mathrm{Ind}_{MAN}^G(\xi, \nu) = \mathrm{Ind}_{M'A'N'}^G([\mathrm{Ind}_{M'A''N''}^{M'}(\xi, \nu|_{\mathfrak{a}''})], \nu|_{\mathfrak{a}'}),$$

where $M'' = M \cap M'$, $A'' = A \cap M'$, $N'' = N \cap M'$ and $\mathfrak{a}'' =$ orthogonal complement of \mathfrak{a}' in \mathfrak{a} , so that $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$.

Theorem 2.36 *The Langlands quotients of $\widetilde{\mathrm{SL}}(2l)$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal are $J_{P_{min}}^\pm(\frac{1}{2}\rho)$, $J_{P_{b=l}}(\frac{1}{2}\rho)$, and $J_{P_{b=l}}(s_{(e_{2l-1}-e_{2l})}\frac{1}{2}\rho)$.*

Proof. To simplify the notation we will write P_l instead of $P_{b=l}$. For each of the representations in the claim there exists a coherent family based at that representation. We will translate each representation to the member of the coherent family with infinitesimal character $\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)$. The translated representation will be a Langlands quotient from Proposition 2.31 having a maximal primitive ideal.

We start with the representations $J_{P_{min}}^\pm(\frac{1}{2}\rho)$. By Corollary 2.34,

$$\Psi_{\frac{1}{2}\rho+(2l,-2l,\dots,2l,-2l)}^{\frac{1}{2}\rho} \mathrm{Ind}_{P_{min}}^\sim(\xi^\pm, \frac{1}{2}\rho) = \mathrm{Ind}_{P_{min}}^\sim(\xi^\pm, \frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)).$$

Because Ind is an exact functor $\Psi_{\frac{1}{2}\rho+(2l,-2l,\dots,2l,-2l)}^{\frac{1}{2}\rho} (J_{P_{min}}^\pm(\frac{1}{2}\rho))$ is an irreducible quotient in $\mathrm{Ind}_{P_{min}}^\sim(\xi^\pm, \frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l))$. However, because the continuous parameter $\nu = \frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)$ is not weakly dominant with respect to P_{min} , it is not a Langlands quotient.

To find the irreducible quotient in $\mathrm{Ind}_{P_{min}}^\sim(\xi^\pm, \frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l))$ it is helpful to write this representation in another way using the induction-in-stages formula of Lemma 2.35. Let $P_{min} = M_{min}AN$ be the Iwasawa decomposition of the minimal parabolic, and let $P_{l-1} = M'_{l-1}A'N'$ be the Iwasawa decomposition of the parabolic where M'_{l-1} has $l-1$ $\mathrm{SL}(2)$ blocks centered along the diagonal. Letting $A'' = A \cap M'_{l-1}$ and $N'' = N \cap M'_{l-1}$, the subgroup $MA''N''$ consists of $l-1$ upper triangular 2×2 blocks centered along the diagonal. In the notation of Proposition 2.23 we write $\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l) = (p_{l-1}, n_0, p_{l-2}, \dots, n_{l-2}, p_0, n_{l-1})$. We have the identity

$$\mathrm{Ind}_{P_{min}}^\sim(\xi^\pm, \nu) = \mathrm{Ind}_{P_{l-1}}^\sim(\mathrm{Ind}_{MA''N''}^\sim(\xi^\pm, \nu|_{\mathfrak{a}''}), \nu|_{\mathfrak{a}'})$$

where

$$\begin{aligned}\nu &= (p_{l-1}, n_0, p_{l-2}, n_1, \dots, p_0, n_{l-1}), \\ \nu|_{\mathfrak{a}''} &= (0, \frac{n_0-p_{l-2}}{2}, -\frac{(n_0-p_{l-2})}{2}, \dots, \frac{n_{l-2}-p_0}{2}, -\frac{(n_{l-2}-p_0)}{2}, 0), \\ \nu|_{\mathfrak{a}'} &= (p_{l-1}, \frac{n_0+p_{l-2}}{2}, \frac{n_0+p_{l-2}}{2}, \dots, \frac{n_{l-2}+p_0}{2}, \frac{n_{l-2}+p_0}{2}, n_{l-1}).\end{aligned}$$

The irreducible quotient of $\text{Ind}_{\widetilde{P_{min}}}(\xi^\pm, \nu)$ is the irreducible quotient of $\text{Ind}_{\widetilde{P}_{l-1}}$ (irreducible quotient of $\text{Ind}_{\widetilde{M}A''N''}(\xi^\pm, \nu|_{\mathfrak{a}''})$). This follows from the fact that Ind is exact, but Ind of an irreducible quotient isn't necessarily irreducible. The irreducible quotient of $\text{Ind}_{\widetilde{M}A''N''}(\xi^\pm, \nu|_{\mathfrak{a}''})$ is the genuine discrete series $\delta_{(p_{l-2}-n_0, \dots, p_0-n_{l-1})}^\pm$. It follows that the Langlands quotient $J_{P_{l-1}}(p_{l-1}, p_{l-2}, n_0, \dots, p_0, n_{l-2}, n_{l-1})$ is the irreducible quotient of $\text{Ind}_{\widetilde{P_{min}}}(\xi^\pm, \nu)$. We know from Proposition 2.31 that this representation has a maximal primitive ideal.

Proving that the Langlands quotients having nonminimal parabolic have a maximal primitive ideal is much easier since their translation to an irreducible with infinitesimal character $\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)$ is a Langlands quotient. We have

$$\Psi_{\frac{1}{2}\rho+(2l, -2l, \dots, 2l, -2l)}^{\frac{1}{2}\rho} \text{Ind}_{\widetilde{P}_l}(\frac{1}{2}\rho) = \text{Ind}_{\widetilde{P}_l}(\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)).$$

As $\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)$ is dominant with respect to \widetilde{P}_l , $\Psi_{\frac{1}{2}\rho+(2l, -2l, \dots, 2l, -2l)}^{\frac{1}{2}\rho} J_{\widetilde{P}_l}(\frac{1}{2}\rho)$ is a Langlands quotient in $\text{Ind}_{\widetilde{P}_l}(\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l))$. From Proposition 2.31 we know that this Langlands quotient has a maximal primitive ideal. Similarly we find that $\Psi_{\frac{1}{2}\rho+(2l, -2l, \dots, 2l, -2l)}^{\frac{1}{2}\rho} J_{\widetilde{P}_l}(s_{e_{2l-1}-e_{2l}}\frac{1}{2}\rho) = J_{\widetilde{P}_l}(s_{e_{2l-1}-e_{2l}}(\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -2l)))$ which we showed in Proposition 2.31 to have a maximal primitive ideal. ■

Theorem 2.37 *The Langlands quotient of $\widetilde{\text{SL}(2l+1)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal is $J_{P_{min}}(\frac{1}{2}\rho)$.*

Proof. The proof is analogous to the proof that $J_{\widetilde{P}_{min}}^\pm(\frac{1}{2}\rho)$ has a maximal primitive ideal. We choose a coherent family based at this representation and translate to the irreducible representation having infinitesimal character $\frac{1}{2}\rho + (2l, -2l, \dots, 2l, -4l, 2l)$. Using an induction-in-stages argument we find that

$$\Psi_{\frac{1}{2}\rho+(2l, -2l, \dots, -4l, 2l)}^{\frac{1}{2}\rho} J_{P_{min}}(\frac{1}{2}\rho) = J_{P_l}(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{3l-1})$$

where $J_{P_l}(p'_l, p'_{l-1}, \dots, p'_0, n'_0, n'_1, \dots, n'_{l-2}, n'_{3l-1})$ is the Langlands quotient shown to have a maximal primitive ideal in Proposition 2.32. ■

Chapter 3

The orbit method picture

3.1 Introduction

Because the Langlands quotients considered in chapter 1 have a maximal primitive ideal, they provide a good paradigm for how unitary representations can be “attached” to nilpotent coadjoint orbits. By saying that a representation is attached to a nilpotent orbit, we roughly mean that the associated variety of the annihilator of the representation is the closure of the nilpotent orbit. There may be several representations attached in this sense. The orbit method conjectures that when the orbit satisfies a certain condition on the codimension of its boundary then the set of representations attached to the orbit is parameterized by the set of admissible orbit data. With each admissible orbit datum, the orbit method gives a realisation of the locally finite K types of the attached representation.

To establish notation, we will let G denote a real semisimple Lie group with Lie algebra \mathfrak{g}_0 . $G_{\mathbb{C}}$ will be the complexification of G with Lie algebra \mathfrak{g} . K will be a maximal compact subgroup of G with complexification $K_{\mathbb{C}}$.

Let $\Theta_{\mathbb{C}}$ be a Cartan involution on \mathfrak{g} fixing \mathfrak{k} , the Lie algebra of $K_{\mathbb{C}}$. If $Z = X + \sqrt{-1}Y$ in \mathfrak{g} , then $\Theta_{\mathbb{C}}(Z) = -(\overline{\sigma Z})^t = -\overline{X}^t - \sqrt{-1}\overline{Y}^t$, where σ is complex conjugation with respect to the real form (i.e. $\sigma(X + \sqrt{-1}Y) = X - \sqrt{-1}Y$). Let \mathfrak{k} and \mathfrak{s} be the 1 and -1 eigenspace of $\Theta_{\mathbb{C}}$.

In the case where $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C})$ and the real form is $\mathfrak{sl}(m)$, then $\Theta_{\mathbb{C}}Z = -Z^t$. We have \mathfrak{k} and \mathfrak{s} equal to the skew symmetric and symmetric matrices in $\mathfrak{sl}(m, \mathbb{C})$ respectively.

3.2 Attaching a nilpotent orbit

We outline below how one finds the nilpotent orbit “attached” to our Langlands quotients, having a maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$.

Let $I \subset U(\mathfrak{g})$ be a primitive ideal. Then the quotient ring $U(\mathfrak{g})/I$ is a finitely generated $U(\mathfrak{g})$ -module. The natural grading on $U(\mathfrak{g})$ defines a filtration on $U(\mathfrak{g})/I$ and makes the associated graded algebra $\text{gr}(U(\mathfrak{g})/I)$. Regarding $S(\mathfrak{g})$ as an algebra of polynomial functions on \mathfrak{g} we define the *associated variety* of I to be $\mathcal{V}(I) = \{\lambda \in \mathfrak{g}^* | p(\lambda) = 0 \text{ whenever } p \in \text{gr } I\}$.

The orbit of an element of \mathfrak{g}^* under the action of $\text{Ad}^*(G_{\mathbb{C}})$ is called a *coadjoint nilpotent orbit* or *nilpotent orbit* if its closure is a cone. Formally,

$$\mathcal{N}^* = \{\lambda \in \mathfrak{g}^* | \text{for all } t \in \mathbb{C}^\times, t\lambda \in G_{\mathbb{C}} \cdot \lambda\}.$$

We have the following result (Corollary 4.7 [12]).

Proposition 3.1 *Let $I \subset U(\mathfrak{g})$ be a primitive ideal. The associated variety $\mathcal{V}(I)$ is the closure of a single coadjoint nilpotent orbit in \mathfrak{g}^* .*

Because of the identification of \mathfrak{g} with \mathfrak{g}^* , we will not distinguish between nilpotent coadjoint orbits and nilpotent adjoint orbits. If the primitive ideal I is maximal and has infinitesimal character λ then we write $\mathcal{O}_{\mathbb{C}}(\lambda)$ for the nilpotent orbit whose closure is $\mathcal{V}(I)$. We wish to identify $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$ for $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C})$ in the case where I is a maximal primitive ideal.

One ingredient for identifying the nilpotent orbit is the Springer correspondence between nilpotent orbits and Weyl group representations. The nilpotent orbits for $\mathfrak{sl}(m, \mathbb{C})$ are in one to one correspondence with the partitions of m . To a partition of $m = [m_1, \dots, m_d]$ we associate the nilpotent orbit

$$\mathcal{O}_{[m_1, \dots, m_d]} = \text{SL}(m, \mathbb{C}) \cdot \begin{bmatrix} J_{m_1} & & & \\ & \ddots & & \\ & & J_{m_d} & \\ & & & \ddots \end{bmatrix}, \text{ where } J_{m_i} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \\ & & & 1 & 0 \end{bmatrix}_{m_i \times m_i}.$$

The irreducible representations of the symmetric group on m letters, S_m , are in one to one correspondence with the partitions of m (partitions of m correspond to the conjugacy classes of S_m which in turn correspond to the irreducible representations of S_m).

We will need the following characterization of the representations of the symmetric group due to Young [13].

Proposition 3.2 *Let $[m_1, \dots, m_d]$ be a partition of m and $[m_1, \dots, m_d]^t = [f_1, \dots, f_d]$ the conjugate partition. A representation of S_m is characterized by two properties: its restriction to the subgroup $\prod_{i=1}^d S_{m_i}$ contains the trivial representation, while its restriction to $\prod_{i=1}^d S_{f_i}$ contains a copy of the sign representation.*

The Weyl group for $\mathfrak{sl}(m, \mathbb{C})$ is S_m . The integral Weyl group with respect to $\frac{1}{2}\rho$ is $S_l \times S_l$ and $S_{l+1} \times S_l$ for $m = 2l$ and $m = 2l + 1$ respectively. This group acts on a coherent family of virtual (\mathfrak{g}, K) modules by the *coherent continuation representation*, defined as follows. Given a coherent family Φ based at an irreducible (\mathfrak{g}, K) module, X , with regular infinitesimal character γ , then $s_\alpha \Phi(\lambda) = \Phi(s_\alpha \lambda)$ for $\alpha \in R^+(\gamma)$ a simple integral root and $\lambda \in \gamma + \Lambda$. If X has a maximal primitive ideal then by Proposition 2.17, $s_\alpha \Phi(\gamma) = -\Phi(\gamma)$.

We wish to extend this representation of the integral Weyl group to the entire Weyl group. There is no well defined representation of the Weyl group on the coherent family of (\mathfrak{g}, K) modules for $\widetilde{\mathrm{SL}}(m)$. There is however an integral Weyl group equivariant map between the virtual (\mathfrak{g}, K) modules having a maximal primitive ideal and the symmetric algebra, $S^d(\mathfrak{h}^*)$, where d is the number of positive integral roots [15](Theorem 4.2.2). The integral Weyl group acts by the sign representation on $\prod_{\alpha_i \in R^+} \alpha_i \in S^d(\mathfrak{h}^*)$. This characterizes a Weyl group representation on $S^d(\mathfrak{h}^*)$ with conjugacy classes $[n, n]^t$ or $[n + 1, n]^t$ by Proposition 3.2 (verification that S_2^l contains the trivial representation is left to the reader). By the Springer Correspondence this Weyl group representation corresponds to the nilpotent orbit $\mathcal{O}_{[n, n]^t}$ or $\mathcal{O}_{[n+1, n]^t}$. Rossman shows that this is the nilpotent orbit $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$ [14].

3.3 $K_{\mathbb{C}}$ orbits

In connection with determining admissible orbit data we will be concerned with action of $K_{\mathbb{C}}$ on $(\mathfrak{g}/\mathfrak{k})^*$ (i.e $K_{\mathbb{C}}$ orbits). The corresponding nilpotent cone is $\mathcal{N}_{\mathfrak{k}}^* = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$.

We have the following result [12] (Corollary 5.20).

Proposition 3.3 *Let $\lambda \in \mathfrak{g}^*$. The intersection $\mathcal{O}_{\mathbb{C}}(\lambda) \cap (\mathfrak{g}/\mathfrak{k})^*$ is a finite union of $K_{\mathbb{C}}$ orbits.*

We will need to explicitly determine the $K_{\mathbb{C}}$ orbits associated with $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$. The Kostant-Sekiguchi correspondence relates $K_{\mathbb{C}}$ orbits on nilpotent elements in \mathfrak{s} to G orbits on nilpotent elements in \mathfrak{g}_o . By finding the decomposition of $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho) \cap \mathfrak{g}_o$ into G orbits, we may use the Kostant-Sekiguchi correspondence to determine the $K_{\mathbb{C}}$ orbits.

To find the $\mathrm{SL}(m)$ nilpotent orbits we begin by first finding the $\mathrm{GL}(m)$ nilpotent orbits. We note that adjoint orbits of a linear group and its double cover coincide so we will not write $\widetilde{\mathrm{SL}}(m)$ orbits.

Proposition 3.4 *The $\mathrm{GL}(m)$ nilpotent orbits correspond to the partitions of m . To a partition of $m = [m_1, \dots, m_d]$ we associate the nilpotent orbit*

$$\mathcal{O}_{[m_1, \dots, m_d]} = \text{GL}(m) \cdot \begin{bmatrix} J_{m_1} & & \\ & \ddots & \\ & & J_{m_d} \end{bmatrix}, \text{ where } J_{m_i} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \\ & & & 1 & 0 \end{bmatrix}_{m_i \times m_i}.$$

Proof. This follows directly from the theorem on rational canonical form. \blacksquare

Each $\text{GL}(m)$ nilpotent orbit as given in Proposition 3.4 may decompose as a sum of $\text{SL}(m)$ nilpotent orbits. For example we have

$$\text{GL}(2) \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{SL}(2) \cdot \begin{bmatrix} 1 \\ \end{bmatrix} \cup \text{SL}(2) \cdot \begin{bmatrix} -1 \\ \end{bmatrix}$$

The next statement indicates that example generalizes to $\text{SL}(2n)$.

Proposition 3.5 *The $\text{SL}(2n+1)$ nilpotent orbits are the same as the $\text{GL}(2n+1)$ nilpotent orbits. The $\text{GL}(2n)$ nilpotent orbits splits as a union of at most two $\text{SL}(2n)$ nilpotent orbits. If the rows of the partition are all even, then there are two $\text{SL}(m)$ orbits, otherwise there is just one.*

Proof. Let $X \in \text{GL}(2n)$ and J be a nilpotent matrix in Jordan canonical form. Conjugating J by X and by $\frac{X}{\sqrt[2n]{\det X}}$ are equivalent and $\frac{X}{\sqrt[2n]{\det X}} \in \text{SL}(2n+1)$. For the second part of the claim define $\text{GL}(2n)^+ = \{x \in \text{GL}(2n) : \det X > 0\}$ and $\text{GL}(2n)^- = \{x \in \text{GL}(2n) : \det X < 0\}$. Indeed, the action of $\text{GL}(2n)^+$ and $\text{SL}(2n)$ coincide by the above

argument. Further we have $\text{GL}(2n)^- = \text{GL}(2n)^+ \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$. It follows that the

action of $\text{GL}(2n)^-$ and

$\text{SL}(2n) \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$ coincide. For the last part of the claim one must determine

whether $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \cdot J \in \text{SL}(2n) \cdot J$. This we leave to the reader. \blacksquare

Let $J_{[m_1, \dots, m_d]}$ be a nilpotent element of $\text{SL}(m)$ in canonical Jordan form.

Corollary 3.6 *The $\mathrm{SL}(2n)$ nilpotent orbits are $\mathrm{SL}(2n) \cdot J_{[m_1, \dots, m_d]}$ and*

$$\mathrm{SL}(2n) \cdot \left(\left[\begin{array}{cccc} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array} \right] J_{[m_1, \dots, m_d]} \left[\begin{array}{cccc} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{array} \right] \right)$$

for all partitions $[m_1, \dots, m_d]$ of $2n$. The $\mathrm{SL}(2n+1)$ nilpotent orbits are $\mathrm{SL}(2n+1) \cdot J_{[m_1, \dots, m_d]}$ for all partitions $[m_1, \dots, m_d]$ of $2n+1$.

Proof. This follows immediately from the proof of Proposition 3.5. \blacksquare

Next we use the Kostant-Sekiguchi correspondence [12] to find the corresponding $K_{\mathbb{C}}$ orbits.

Theorem 3.7 (*Kostant-Sekiguchi Correspondence*) *Let $\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}$ be a homomorphism respecting notions of complex conjugation in $\mathfrak{sl}(2, \mathbb{C})$ and \mathfrak{g} and respecting Cartan involution. Then there is a one to one correspondence between G nilpotent orbits and $K_{\mathbb{C}}$ nilpotent orbits given by,*

$$G \cdot \phi \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \longleftrightarrow K_{\mathbb{C}} \cdot \phi \left[\begin{array}{cc} \frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{array} \right]$$

An important example is for $\mathfrak{g} = \mathfrak{sl}(2)$. Here ϕ is the identity map and the $\mathrm{SL}(2)$ orbits $\mathrm{SL}(2) \cdot \left[\begin{array}{cc} & 1 \\ 1 & \end{array} \right] \cup \mathrm{SL}(2) \cdot \left[\begin{array}{cc} & -1 \\ -1 & \end{array} \right]$ corresponds to the $\mathrm{SO}(2, \mathbb{C})$ orbits $\mathrm{SO}(2, \mathbb{C}) \cdot \left[\begin{array}{cc} \frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{array} \right] \cup \mathrm{SO}(2, \mathbb{C}) \cdot \left[\begin{array}{cc} \frac{1}{2} & \frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{array} \right]$.

Corollary 3.8 *For $\widetilde{\mathrm{SL}(2l)}$,*

$$\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho) \cap (\mathfrak{g}/\mathfrak{k})^* = K_{\mathbb{C}} \cdot \frac{1}{2} \left[\begin{array}{cccc} 1 & -i & & \\ -i & -1 & & \\ & & 1 & -i \\ & & -i & -1 \\ & & & \ddots \\ & & & & 1 & -i \\ & & & & -i & -1 \end{array} \right] \cup K_{\mathbb{C}} \cdot \frac{1}{2} \left[\begin{array}{cccc} 1 & i & & \\ i & -1 & & \\ & & 1 & -i \\ & & -i & -1 \\ & & & \ddots \\ & & & & 1 & -i \\ & & & & -i & -1 \end{array} \right]$$

For $\widetilde{\mathrm{SL}(2l+1)}$,

A representation ξ of $K_{\mathbb{C}}^x$ is called *admissible* if the differential of ξ is $\frac{1}{2}d\gamma(x)$. We have $\det \text{Ad}^*(k)|_{(\mathfrak{k}/\mathfrak{k}^x)^*} = (\det \text{Ad}(k)|_{\mathfrak{k}/\mathfrak{k}^x})^{-1}$ and $\text{tr ad}^*(k)|_{(\mathfrak{k}/\mathfrak{k}^x)^*} = -\text{tr ad}(k)|_{\mathfrak{k}/\mathfrak{k}^x}$. Hence the test for admissibility is

$$d\xi(k) = -\frac{1}{2}\text{ad}(k)|_{\mathfrak{k}/\mathfrak{k}^x} \text{ for all } k \in \mathfrak{k}^x$$

Lemma 3.10 *For $\mathfrak{k} = \mathfrak{so}(m, \mathbb{C})$, $\text{tr ad}k|_{\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^x} = 0$ for all $k \in \mathfrak{k}_{\mathbb{C}}^x$.*

Proof. We have the Levi decomposition $\mathfrak{so}(2l + \epsilon, \mathbb{C})^x = \mathfrak{so}(l, \mathbb{C}) \oplus u^x$ for $\epsilon \in \{0, 1\}$. For $l > 2$, $\mathfrak{so}(l, \mathbb{C})$ is semisimple so its one dimensional representation $d\gamma$ must be zero. It follows that $\text{ad}(k)$ is traceless since ad -nilpotent elements are. One can check directly that the claim holds true for the case $l = 1$ and $l = 2$. ■

It follows from Lemma 3.10 that an irreducible representation of $\widetilde{\text{SO}}(m, \mathbb{C})^x$ is admissible if its differential is zero.

Lemma 3.11 *Let π be an irreducible algebraic representation of $K_{\mathbb{C}}^x = \text{O}(l, \mathbb{C}) \cdot U^x$. Then U^x acts by the trivial representation.*

Proof. Let $\pi : K_{\mathbb{C}}^x \rightarrow GL(V)$. Then $\pi(U^x)$ is a unipotent algebraic group in $GL(V)$. By Engel's theorem $\exists v \in V$ fixed by U^x . Because U^x is a normal subgroup, V^{U^x} is a nonzero $K_{\mathbb{C}}^x$ invariant subspace of V . By the irreducibility of π , $V^{U^x} = V$, so U^x acts trivially. ■

Theorem 3.12 *There are 4 genuine admissible $\text{Spin}(2l, \mathbb{C})$ orbit data, and 2 genuine admissible $\text{Spin}(2l + 1, \mathbb{C})$ orbit data for $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$.*

Proof. Let π be an irreducible representation of $K_{\mathbb{C}}^x$. By Lemma 3.11, π acts by the trivial representation on U^x . For π to be admissible, by Lemma 3.10, its differential must be zero. This forces π to be the trivial representation on $\text{SO}(l, \mathbb{C})$. Hence we may think of π as an irreducible representation of $\{\pm 1, \pm e_1 e_2\}$. There are two such genuine representations, π^+ and π^- defined by $\pi^+(e_1 e_2) = \sqrt{-1}$ and $\pi^-(e_1 e_2) = -\sqrt{-1}$. Because there are two $\text{Spin}(2l, \mathbb{C})$ orbits, $\mathcal{O}_{[l, l]^t}$ has 4 admissible orbit data (π^{\pm}, x) and (π^{\pm}, x') . Because there is one $\text{Spin}(2l + 1, \mathbb{C})$ orbits $\mathcal{O}_{[l, l]^t}$ has 2 admissible orbit data (π^{\pm}, x) . ■

3.5 Orbit method prediction of K types

The following statement lies at the heart of attaching representations to nilpotent orbits [12] (Conjecture 12.1).

Conjecture 3.13 *Suppose X is an irreducible unipotent Harish-Chandra module and \mathcal{O} a nilpotent coadjoint orbit with $\mathcal{V}(\text{gr Ann}X) = \bar{\mathcal{O}}$ and assume the codimension of $\partial\bar{\mathcal{O}}$ in $\bar{\mathcal{O}}$ is*

at least 4. Then there is an element $x \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{k})^*$ and an admissible representation π of the stabilizer $K_{\mathbb{C}}^x$ such that, as a representation of $K_{\mathbb{C}}$,

$$X \cong \text{Ind}_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}}(\pi)$$

We wish to use Conjecture 3.13 to determine the K types of our Langlands quotient with maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$ attached to $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$. We first check whether $\mathcal{O}_{\mathbb{C}}(\frac{1}{2}\rho)$ satisfies the codimension hypothesis of the conjecture.

Lemma 3.14 *The codimension of the boundary of the complex nilpotent orbit $\mathcal{O}_{[l,l]^t}$ of $\mathfrak{sl}(2l, \mathbb{C})$ is 2 and the codimension of the boundary of the complex nilpotent orbit $\mathcal{O}_{[l+1,l]^t}$ of $\mathfrak{sl}(2l+1, \mathbb{C})$ is 4.*

Proof. We use the formula of Corollary 6.1.4 in [13] to compute the dimension of a complex nilpotent orbit of $\mathfrak{sl}(m)$. The boundary of $\mathcal{O}_{[l,l]^t}$ is $\mathcal{O}_{[l+1,l-1]^t}$. It has codimension 2. The boundary of $\mathcal{O}_{[l+1,l]^t}$ is $\mathcal{O}_{[l+2,l-1]^t}$. It has codimension 4. \blacksquare

The codimension condition of Conjecture 3.13, is satisfied for the nilpotent orbit $\mathcal{O}_{[l+1,l]^t}$ of $\mathfrak{sl}(2l+1, \mathbb{C})$. Then the algebraic representation in Conjecture 3.13 is our Langlands quotient $J_{P_{\min}}(\frac{1}{2}\rho)|_K$ from chapter one thought of as a representation of $K_{\mathbb{C}}$. We note that $\mathcal{O}_{[l+1,l]^t}$ has two admissible orbits but $\widetilde{\text{SL}}(m)$ has only one Langlands quotient with maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$. We thus have an example where the orbit method overestimates the number of irreducible representations attached to an orbit. Indeed Torasso observed this for the case of $\widetilde{\text{SL}}(3)$ [17].

In hope that one of the algebraic representations given in the conjecture is our Langlands quotient, we will proceed to determine the $K_{\mathbb{C}}$ types of the algebraic representation. To investigate the K types of $\text{Ind}_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}}(\pi)$ it will be convenient to use the transitivity property of induction. Let $\{x, y, h\}$, be a $\mathfrak{sl}(2, \mathbb{C})$ triple and \mathfrak{k}^x as above Lemma 3.9. By Lemma 3.8.4 in [13], the sum of the non-negative $\text{ad}(h)$ weight spaces is a parabolic subalgebra of \mathfrak{k} containing \mathfrak{k}^x . This parabolic subalgebra is $\mathfrak{gl}(l, \mathbb{C}) \oplus \mathfrak{u}$. Here $\mathfrak{gl}(l, \mathbb{C})$ is a diagonal subalgebra of \mathfrak{k} containing $\mathfrak{o}(l, \mathbb{C})$. At the group level we write the parabolic subgroup as $\widetilde{\text{GL}}(l, \mathbb{C}) \cdot U$. Because the unipotent groups U and U^x play no role in the representation, we will omit them from the notation. By the transitivity property of induction we have,

$$\text{Ind}_{K_{\mathbb{C}}^x}^{\widetilde{K}_{\mathbb{C}}}(\pi) = \text{Ind}_{\widetilde{\text{GL}}(l, \mathbb{C})}^{\widetilde{K}_{\mathbb{C}}} \text{Ind}_{\widetilde{\text{O}}(l, \mathbb{C})}^{\widetilde{\text{GL}}(l, \mathbb{C})} \pi$$

To analyse the K types we will use the Borel-Weil Theorem [5]

Theorem 3.15 *(Borel-Weil Theorem) For a compact connected Lie group K , if $\lambda \in \mathfrak{k}^*$ is dominant and analytically integral and ξ_{λ} , denotes the corresponding homomorphic one-dimensional representation of B , then a realization of an irreducible representation of K with highest weight λ is the space $\text{Ind}_B^{K_{\mathbb{C}}} \xi_{\lambda}$.*

We see that to determine the K types of $\text{Ind}_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}^{\mathbb{C}}}(\pi^{\pm})$ it suffices to determine the irreducible representations ξ_{λ} of $\widetilde{\text{GL}}(l, \mathbb{C})$ in $\text{Ind}_{\widetilde{O}(l, \mathbb{C})}^{\widetilde{\text{GL}}(l, \mathbb{C})} \pi^{\pm}$

Proposition 3.16 *For admissible representations π^{\pm} of $K_{\mathbb{C}}^x$ the K types of $\text{Ind}_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}^{\mathbb{C}}}(\pi^+)$ are $(2k_1 + \frac{1}{2}, \dots, 2k_l + \frac{1}{2})$ and the K types of $\text{Ind}_{K_{\mathbb{C}}^x}^{K_{\mathbb{C}}^{\mathbb{C}}}(\pi^-)$ are $(2k_1 + \frac{3}{2}, \dots, 2k_l + \frac{3}{2})$ for $k_1 \geq \dots \geq k_l \geq 0$.*

Proof. The $\det^{\frac{1}{2}}$ cover of $\text{GL}(l, \mathbb{C})$, is $\widetilde{\text{GL}}(l, \mathbb{C}) = \{(g, z) \in \text{GL}(l, \mathbb{C}) \times \mathbb{C}^{\times} : \det g = z^2\}$. The subgroup $\widetilde{O}(l, \mathbb{C}) = \{(x, y) : \det x = z^2\}$ has 2 genuine characters: $\det^{\frac{1}{2}}, \det^{\frac{3}{2}}$. We note that $\det^{\frac{1}{2}} = \pi^+$ and $\det^{\frac{3}{2}} = \pi^-$ in the statement of the proposition.

We have the identity $\text{Ind}_{\widetilde{O}(l, \mathbb{C})}^{\widetilde{\text{GL}}(l, \mathbb{C})} (\det^{\frac{1}{2}})^a \cong (\det^{\frac{1}{2}})^a \otimes \text{Ind}_{\widetilde{O}(l, \mathbb{C})}^{\widetilde{\text{GL}}(l, \mathbb{C})} 1$ for $a \in \{1, 3\}$. $\text{Ind}_{\widetilde{O}(l, \mathbb{C})}^{\widetilde{\text{GL}}(l, \mathbb{C})} 1$ consists of algebraic functions on $\widetilde{\text{GL}}(l, \mathbb{C})$ containing an $\widetilde{O}(l, \mathbb{C})$ fixed vector. By Helgason's theorem on spherical representations [19] (theorem 4.12) the K types of $\text{Ind}_{\widetilde{O}(l, \mathbb{C})}^{\widetilde{\text{GL}}(l, \mathbb{C})} 1$ are $(2k_1, \dots, 2k_n)$. Twisting by $\det^{\frac{1}{2}}$ adds $(\frac{1}{2}, \dots, \frac{1}{2})$ and twisting by $\det^{\frac{3}{2}}$ adds $(\frac{3}{2}, \dots, \frac{3}{2})$ proving the claim. \blacksquare

Chapter 4

A character formula

4.1 Lowest K types

In chapter one we showed that there are four Langlands quotients of $\widetilde{\mathrm{SL}(2l)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. For $\widetilde{\mathrm{SL}(2l+1)}$ we showed that there is a single Langlands quotient with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. We maintain the same notation as chapter 1 except that we suppose that G is split (the group G being split allows us to consider a real root space decomposition of \mathfrak{g}). Our aim in this chapter is to give a character formula for the K types of these Langlands quotients. We turn now to the determination of their lowest K types.

We can assign a non-negative real number to each $\pi \in \hat{K}$, roughly the length of the highest weight of π . The *lowest K types* of a non-zero (\mathfrak{g}, K) module, X , are the irreducible representations of K , with non zero multiplicity in X , minimal with respect to this “norm” on \hat{K} .

We begin by determining the lowest K types for those Langlands quotients induced from a minimal parabolic subgroup. To do this will involve a discussion of fine representations of K as presented in [9].

For each positive root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ let $\varphi_\alpha : \mathfrak{sl}(2) \rightarrow \mathfrak{g}$ be an injection so that $H_\alpha = \varphi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}$, $\varphi_\alpha(-{}^tX) = \theta\varphi_\alpha(X)$, and $X_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ lies in the α root space of \mathfrak{a} in \mathfrak{g} . Let $Z_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k}$. Note that the bracket relations on H_α , $X_{\pm\alpha}$ and the relation $\theta X_\alpha = X_{-\alpha}$ only determines $X_{\pm\alpha}$, and hence Z_α , up to a sign. We say that an irreducible representation μ of K is *fine* if $\mu(iZ_\alpha)$ has eigenvalues between -1 and 1 for each positive root α . (We abuse notation and denote the differential of μ as μ).

Proposition 4.1 *Let $G = \widetilde{\mathrm{SL}(m)}$ for $m = 2n$ or $m = 2n + 1$. The fine representations of*

$K = \text{Spin}(m)$ have highest weight (m_1, m_2, \dots, m_n) with $|m_1| \leq 1$.

Proof. For each root α of $\mathfrak{sl}(m)$ we determine which representations μ of $\mathfrak{so}(m)$ evaluated at iZ_α take eigenvalues between -1 and 1 . We will see below that we only need to consider the root $\alpha = e_1 - e_2$. The roots $e_1 - e_2$ and $e_2 - e_1$ of $\mathfrak{sl}(m)$ have the unit matrices E_{12} , E_{21} respectively as their root vectors. Together with their bracket, $[E_{12}, E_{21}]$, these root vectors span a copy of $\mathfrak{sl}(2)$ sitting inside of $\mathfrak{sl}(m)$. $Z_{e_1 - e_2}$ equals $\pm(E_{12} - E_{21})$ and spans a copy of $\mathfrak{so}(2)$ sitting inside of \mathfrak{k} .

We would like to extend the action of the Weyl group of $\mathfrak{sl}(m)$ to Z_α , and establish their conjugacy under the Weyl group. We identify the analytic Weyl group of $\widetilde{\text{SL}(m)}$ with the analytic Weyl group of the linear group. The analytic Weyl group of $\text{SL}(m)$ is the normalizer of \mathfrak{h} in $\text{SO}(m)$ modulo the centralizer of \mathfrak{h} in $\text{SO}(m)$. This consists of all monomial matrices (i.e. matrices with one nonzero entry in every row and column) in $\text{SO}(m)$, modulo all diagonal matrices with entries ± 1 having an even number of negative signs. Coset representatives are $n \times n$ permutation matrices with the nonzero entry in the first column negated if the permutation matrix has determinant minus one. The Weyl group acts on \mathfrak{h} by conjugation. Because $E_{12} - E_{21}$ is not in the Cartan subalgebra of $\mathfrak{sl}(2)$ the Weyl group action on $E_{12} - E_{21}$ is only well defined if we identify $E_{12} - E_{21}$ with $E_{21} - E_{12}$. This sign problem occurs because different coset representatives of the identity element of the Weyl group map $E_{12} - E_{21}$ to different signs of $E_{12} - E_{21}$. For example in $\mathfrak{sl}(3, \mathbb{R})$ the identity matrix and the diagonal matrix with 1 in the first diagonal entry and -1 in the last two diagonal entries map $E_{12} - E_{21}$ to $E_{12} - E_{21}$ and $E_{21} - E_{12}$ respectively. We can extend the action to $Z_{e_1 - e_2}$ however since $Z_{e_1 - e_2}$ is defined only up to a sign. In this way we get a well defined action of the Weyl group on Z_α for every positive root α . The Weyl group permutes Z_α according to the formula $wZ_\alpha w^{-1} = Z_{w\alpha}$, where w is an element of the Weyl group. To see this one can first use the bracket relations on $H_\alpha, X_{\pm\alpha}$, together with the formula $wH_\alpha w^{-1} = H_{w\alpha}$ to show that $wX_\alpha w^{-1}$ is a root vector of $w\alpha$ ($wX_\alpha w^{-1}$ makes sense since X_α is only defined up to a sign). Each root space being one dimensional implies that the Weyl group maps Z_α into a multiple of $Z_{w\alpha}$. Then $wZ_\alpha w^{-1} = Z_{w\alpha}$ follows from the Weyl group consisting of monomial matrices with entries ± 1 . Because the Weyl group acts transitively on the roots of $\mathfrak{sl}(m)$ the Z_α are all conjugate to one another by an element of the Weyl group. This indicates that to check the fineness of a representation μ of $\mathfrak{so}(m)$ it suffices to check that the eigenvalues of $\mu(iZ_{e_1 - e_2})$ are between -1 and 1 .

If $\lambda = (m_1, m_2, \dots, m_n)$ is a highest weight of a representation μ of $\mathfrak{so}(m)$, then for $m = 2n$, the lowest weight is $(-m_1, -m_2, \dots, -m_n)$ and for $m = 2n + 1$ the lowest weight is $(-m_1, -m_2, \dots, -m_{n-1}, m_n)$. These weights are in the $\mathfrak{so}(m)$ Weyl group orbit of the highest weight of μ . By the theorem of highest weight, the weights of μ are $\lambda - \sum \alpha$ for positive roots α . It is important to note that the weights of μ have a smaller e_1 coefficient than that of the highest weight. Since $iZ_{e_1 - e_2}$ is i times an element of the Cartan subalgebra of $\mathfrak{so}(m)$, the weights of μ applied to $(iZ_{e_1 - e_2})$ are the eigenvalues of $\mu(iZ_{e_1 - e_2})$. Applying a weight $\beta = (l_1, \dots, l_n)$ to $iZ_{e_1 - e_2}$ picks off the first coefficient up to a sign. In other words, $\beta(iZ_{e_1 - e_2}) = l_1 e_1(iZ_{e_1 - e_2}) = \pm l_1$ is an eigenvalue of $\mu(iZ_{e_1 - e_2})$. This implies that the e_1 coefficient of the highest and lowest weights of μ give a bound on the size of the eigenvalues

of $\mu(iZ_{e_1-e_2})$ namely the eigenvalues necessarily have absolute value less than m_1 . ■

Taking into account what the dominant analytically integral forms are for $\text{Spin}(m)$ we immediately get the following result.

Corollary 4.2 *The fine representations of $\text{Spin}(2n)$ have highest weights: $(1, 0, \dots, 0)$, $(1, 1, 0, \dots, 0)$, \dots , $(1, \dots, 1, 0, 0)$, $(1, \dots, \pm 1)$, $(\frac{1}{2}, \dots, \frac{1}{2})$ and $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$. The fine representations of $\text{Spin}(2n+1)$, have highest weights $(1, 0, \dots, 0)$, $(1, 1, 0, \dots, 0)$, \dots , $(1, \dots, 1)$ and $(\frac{1}{2}, \dots, \frac{1}{2})$.*

Let $\text{Ind}_{MAN}(\delta \otimes \nu)$ be a generalized principal series of a semisimple Lie group G . Its K types are independent of ν since we have the equality $\text{Ind}_{MAN}^{KAN}(\delta \otimes \nu)|_K = \text{Ind}_{M \cap K}^K(\delta)$. Let $A(\delta)$ be the set of lowest K types of $\text{Ind}_{MAN}(\delta \otimes \nu)$.

We have the following theorems of Vogan [9].

Theorem 4.3 *Suppose G is split and P is minimal and $\delta \in \hat{M}$, then $A(\delta)$ consists of fine representations of K .*

It follows from Theorem 4.3 that the lowest K type of the principal series of $\widetilde{\text{SL}(m)}$ are fine representations. A generalized principal series can have many irreducible constituents, known as *Langlands subquotients*. The next result states that the lowest K types of these quotients taken together is precisely the set of lowest K types of the generalized principal series.

Theorem 4.4 *(Corollary 4.6 in [10]) Let $\text{Ind}_P(\delta, \nu)$ be a generalized principal series of a connected split semisimple Lie group. Each subquotient of $\text{Ind}_P(\delta, \nu)$ has lowest K type in $A(\delta)$ and every $\mu \in A(\delta)$ is the lowest K type for a Langlands subquotient of $\text{Ind}_P(\delta, \nu)$. The number of Langlands subquotients is one unless ν annihilates some real root.*

We now determine the lowest K types of the Langlands quotients having a minimal parabolic.

Lemma 4.5 *With notation as below Proposition 2.9 let μ^+ and μ^- be fine representations of $\widetilde{K} = \text{Spin}(2l)$ whose restriction to \widetilde{M} contains ξ^+ and ξ^- respectively. Then μ^+ and μ^- have highest weights $(\frac{1}{2}, \dots, \frac{1}{2})$ and $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ respectively.*

Proof. Because μ^+ and μ^- are fine representations of \widetilde{K} their highest weight must be among the set of l highest weights given in Corollary 4.2. The restriction of μ^+ and μ^- to \widetilde{M} contains a genuine irreducible representations so μ^+ and μ^- must send -1 to the scalar matrix -1 . This rules out the possibility that they have highest weight with integer coefficients. It remains to determine which of the half integer highest weight representations contains ξ^+ and which contains ξ^- . Recall that ξ^+ and ξ^- are differentiated by their evaluation at the central element e_I . We have $\xi^+(e_I) = \sqrt{-1}^l$ and $\xi^-(e_I) = -\sqrt{-1}^l$. Indeed,

$\mu_{(\frac{1}{2}, \dots, \frac{1}{2})}(e_I) = (\exp \sqrt{-1} \frac{\pi}{2})^l = \sqrt{-1}^l = \xi^+(e_I)$ and $\mu_{(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})}(e_I) = (\exp \sqrt{-1} \frac{\pi}{2})^{l-2} = -\sqrt{-1}^l = \xi^-(e_I)$ proving the claim. \blacksquare

From Lemma 4.5 we immediately have the following statements.

Corollary 4.6 *The Langlands quotients $J_{P_{\min}}^+(\frac{1}{2}\rho)$ and $J_{P_{\min}}^-(\frac{1}{2}\rho)$ for $\widetilde{\mathrm{SL}(2l)}$ have lowest K types $(\frac{1}{2}, \dots, \frac{1}{2})$ and $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$, respectively.*

Corollary 4.7 *The lowest K type for the Langlands quotient $J_{P_{\min}}(\frac{1}{2}\rho)$ of $\mathrm{SL}(2l+1)$ is $(\frac{1}{2}, \dots, \frac{1}{2})$.*

Next we determine the lowest K types for the Langlands quotients of $\widetilde{\mathrm{SL}(2l)}$ having nonminimal parabolic. Letting α_0 be the simple root $\alpha_0 = e_{2l-1} - e_{2l}$ and s_{α_0} the simple reflection with respect to α_0 . We showed in Chapter 1 that $J_{P_l}(\frac{1}{2}\rho)$, and $J_{P_l}(s_{\alpha_0}\frac{1}{2}\rho)$ have a maximal cuspidal parabolic subgroup in $\widetilde{\mathrm{SL}(2l)}$. Let $\Delta_M^+(\frac{1}{2}\rho)$ be the positive roots in $\Delta^+(\frac{1}{2}\rho)$ which lie in M (i.e. the imaginary roots) and ρ_M half the sum of the roots in $\Delta_M^+(\frac{1}{2}\rho)$. We have the following result.

Proposition 4.8 *Let $\tilde{P} = \tilde{M}AN$ be a maximal cuspidal parabolic subgroup of $\widetilde{\mathrm{SL}(2l)}$. Let $\gamma = (\lambda, \nu)$ be a Langlands parameter. For $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\rho_M = (\rho_1, \dots, \rho_l)$, the lowest K type for the generalized principal series $\mathrm{Ind}_{\tilde{P}}(\lambda, \nu)$ has extremal weight $(\lambda_1 + \rho_1, \dots, \lambda_l + \rho_l)$.*

Proof. For determining K types it suffices to restrict the generalized principal series to \tilde{K} . We have $\mathrm{Ind}_{\tilde{P}}(\lambda, \nu)|_{\tilde{K}} = \mathrm{Ind}_{\tilde{M} \cap \tilde{K}}^{\tilde{K}}(\delta_\lambda|_{\tilde{M} \cap \tilde{K}})$. The discrete series $\delta_{(\lambda_1, \dots, \lambda_l)}$ of \tilde{M} has $M \cup K$ types with highest weight $(\lambda_{k_1}, \dots, \lambda_{k_l})$ where $\lambda_{k_i} = \lambda_i + \rho_i + \mathrm{sign}(\rho_i)2k_i$,

$1 \leq i \leq l$, for integers $k_i \geq 0$. We have $\delta_\lambda|_{\tilde{M} \cap \tilde{K}} = \sum_{k_1, \dots, k_l \geq 0} (\lambda_{k_1}, \dots, \lambda_{k_l})$. Then,

$$\mathrm{Ind}_{\tilde{M} \cap \tilde{K}}(\sum_{k_1, \dots, k_l \geq 0} (\lambda_{k_1}, \dots, \lambda_{k_l})) = \sum_{k_1, \dots, k_l \geq 0} \mathrm{Ind}_{\tilde{M} \cap \tilde{K}}(\lambda_{k_1}, \dots, \lambda_{k_l}).$$

To prove the proposition it then suffices to show that $\mathrm{Ind}_{\tilde{M} \cap \tilde{K}}(\lambda_{k_1}, \dots, \lambda_{k_l})$ has lowest K type with extremal weight $(\lambda_{k_1}, \dots, \lambda_{k_l})$. By Frobenius reciprocity, the irreducible representation of \tilde{K} with extremal weight (a_1, \dots, a_l) is not in $\mathrm{Ind}_{\tilde{M} \cap \tilde{K}}(\lambda_{k_1}, \dots, \lambda_{k_l})$ if $a_i < \lambda_i$ for $1 \leq i \leq l$.

It remains to show that $\mathrm{Ind}_{\tilde{M} \cap \tilde{K}}(\lambda_{k_1}, \dots, \lambda_{k_l})$ has K type $(\lambda_{k_1}, \dots, \lambda_{k_l})$. This follows easily from Frobenius reciprocity. \blacksquare

Corollary 4.9 *With notation as in Proposition 4.8 the lowest K types of $J_{P_l}(\frac{1}{2}\rho)$, and $J_{P_l}(s_{\alpha_0}\frac{1}{2}\rho)$ for $\widetilde{\mathrm{SL}(2l)}$ have highest weight $(\frac{3}{2}, \dots, \frac{3}{2})$ and $(\frac{3}{2}, \dots, -\frac{3}{2})$ respectively.*

Proof. The generalized principal series $\text{Ind}_{P_l}(\frac{1}{2}\rho)$, and $\text{Ind}_{P_l}(s_{\alpha_0}\frac{1}{2}\rho)$ have a unique lowest K type since P_l doesn't contain any real roots (see Theorem 4.4). Hence, the claim will follow by showing that they have lowest K types $(\frac{3}{2}, \dots, \frac{3}{2})$ and $(\frac{3}{2}, \dots, -\frac{3}{2})$ respectively. For Langlands parameter $(\lambda, \nu) = \frac{1}{2}\rho$, we have $\lambda = (\frac{1}{2}, \dots, \frac{1}{2})$ and $\rho_M = (1, \dots, 1)$. Similarly for Langlands parameter $(\lambda, \nu) = s_{\alpha_0}\frac{1}{2}\rho$, we have $\lambda = (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ and $\rho_M = (1, \dots, 1, -1)$. The claim then follows immediately from Proposition 4.8. \blacksquare

4.2 Characters of virtual representations

By a *virtual representation* we will mean a formal finite combination of irreducible representations with integer coefficients. In this section we will express the Langlands quotients having maximal primitive ideal and infinitesimal character $\frac{1}{2}\rho$ as a sum of virtual representations. Blattner's formula can be applied directly to compute the K types of the virtual representations [11]. This provides a tractable way for determining the K types for our Langlands quotients.

We begin by introducing the notion of the continued fundamental series for a semisimple connected Lie group G [11]. Let $T \subset K$ be a maximal torus, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be Cartan decomposition of \mathfrak{g} , and $A = \text{Cent}_{\mathfrak{p}}(T)$. Let $H = \text{Cent}_G(T) = TA$ be a maximally compact Cartan. Let $MA = \text{Cent}_G(A)$ be a Levi subgroup. The subgroup M given by $MA = \text{Cent}_G(A)$ has a compact Cartan subgroup, T , and hence discrete series representations. Let Ψ be a Θ stable positive root system for \mathfrak{h} in \mathfrak{g} . Define Ψ_M to be the roots of M in Ψ (i.e. the positive imaginary roots). For $\lambda \in \mathfrak{t}^*$, dominant with respect to Ψ_M , we write $\Theta_M(\Psi_M, \lambda)$ for the character of the discrete series δ_λ of M with Harish Chandra parameter λ . Let μ be an extremal weight of a finite dimensional representation of M . Thinking of δ_λ as a virtual $(\mathfrak{m}, M \cap K)$ module we can form the coherent family Φ_M based at δ_λ . We define $\Theta_M(\Psi_M, \lambda + \mu)$ to be the character of the virtual representation $\Phi_M(\lambda + \mu)$. For $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$, we define the character of a *continued fundamental series* for G to be $\Theta(\Psi, \gamma) = \text{Ind}_{MAN}^G(\Theta_M(\Psi_M, \lambda) \otimes \nu \otimes 1)$. Here MAN is a maximal cuspidal parabolic subgroup of G and $\Theta_M(\Psi_M, \lambda)$ is notation for the corresponding virtual $(\mathfrak{m}, K \cap M)$ module. We have the following result [10]

Theorem 4.10 *If $\gamma = (\lambda, \nu) \in \mathfrak{h}^*$ is strictly dominant for Ψ then $\Theta(\Psi, \gamma)$ is the character of the Langlands quotient with Langlands parameter γ . Letting Φ be a coherent family based at this irreducible representation, and μ be the extremal weight of a finite dimensional representation of G , then $\Theta(\Psi, \gamma + \mu)$ is the character of the virtual (\mathfrak{g}, K) module $\Phi(\gamma + \mu)$.*

There is a theory of coherent families of characters of virtual (\mathfrak{g}, K) modules paralleling the theory of coherent families of virtual (\mathfrak{g}, K) modules described in chapter 1. We will often blur the distinction between characters of virtual representations, $\Theta(\Psi, \gamma)$, and the virtual representations themselves.

We have the following result describing the K types of the virtual characters for $\widetilde{\text{SL}}(2)$.

Lemma 4.11 For $\widetilde{\mathrm{SL}(2)}$ the K types of $\Theta(\Psi, \lambda)$ are $\lambda + \rho + \mathrm{sign}(\rho)(2k)$ for nonnegative integers k .

Let Ψ be the standard positive roots for $\mathfrak{sl}(2l)$ or $\mathfrak{sl}(2l+1)$ and define $w_0 = s_{(e_1-e_2)}s_{(e_3-e_4)} \cdots s_{(e_{2l-1}-e_{2l})}$ and $s_{\alpha_0} = s_{(e_{2l-1}-e_{2l})}$. We will abbreviate various sums of virtual characters as follows.

$$\begin{aligned} \Theta_{(\frac{3}{2}, \dots, \frac{3}{2})} &= \frac{1}{l!} \sum_{w \in W(R^+(\frac{1}{2}\rho))} (-1)^w \Theta(\Psi, w \cdot \frac{1}{2}\rho); \\ \Theta_{(\frac{1}{2}, \dots, \frac{1}{2})} &= \frac{1}{l!} \sum_{w \in W(R^+(w_0 \frac{1}{2}\rho))} (-1)^w \Theta(\Psi, w \cdot w_0 \frac{1}{2}\rho); \\ \Theta_{(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})} &= \frac{1}{l!} \sum_{w \in W(R^+(s_{\alpha_0} w_0 \frac{1}{2}\rho))} (-1)^w \Theta(s_{\alpha_0} \Psi, w \cdot s_{\alpha_0} \frac{1}{2}\rho); \\ \Theta_{(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})} &= \frac{1}{l!} \sum_{w \in W(R^+(s_{\alpha_0} \frac{1}{2}\rho))} (-1)^w \Theta(s_{\alpha_0} \Psi, w \cdot s_{\alpha_0} \frac{1}{2}\rho). \end{aligned}$$

We have the following result about lowest K types of a sum of virtual representations. Note that the lowest K type of a virtual representation may have negative multiplicity.

Lemma 4.12 *With notation as above,*

1. $\Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}$ has lowest K type $(\frac{3}{2}, \dots, \frac{3}{2})$ with multiplicity 1.
2. $\Theta_{(\frac{1}{2}, \dots, \frac{1}{2})}$ has lowest K type $(\frac{1}{2}, \dots, \frac{1}{2})$ with multiplicity ± 1 .
3. $\Theta_{(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})}$ has lowest K type $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ with multiplicity ± 1 .
4. $\Theta_{(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})}$ has lowest K type $(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})$ with multiplicity 1.

Proof. Let λ be an element in the integral Weyl group orbit of $\frac{1}{2}\rho$. We have $\Theta(\Psi, \lambda)|_{\widetilde{K}} = \mathrm{Ind}_{\widetilde{M} \cap \widetilde{K}}^{\widetilde{K}} \Theta_M(\Psi_M, \lambda|_{\mathfrak{t}^*})|_{\widetilde{M} \cap \widetilde{K}}$. Let Ψ_M^j be the root $\pm(e_{2j-1} - e_{2j})$ in Ψ_M so $\Psi_M = \{\Psi_M^1, \dots, \Psi_M^l\}$. Also let $\lambda|_{\mathfrak{t}^*} = (\lambda_1, \dots, \lambda_l)$. Restricting $\Theta_M(\Psi_M, \lambda|_{\mathfrak{t}^*})$ to the maximal torus \widetilde{T} gives us $\Theta_M(\Phi_M, \lambda|_{\mathfrak{t}^*})|_{\widetilde{T}} = (\Theta(\Psi_M^1, \lambda_1), \dots, \Theta(\Psi_M^l, \lambda_l))$. From Lemma 4.11 we know that the K types of $\Theta(\Psi_M^j, \lambda_j)$ are $(\lambda_j + \rho_M^j + \mathrm{sign}(\rho_M^j)2k_j)$ for nonnegative integers k_j . Using Frobenius reciprocity we find that $\Theta(\Psi, \lambda)$ has lowest K type with extremal weight $\lambda + \rho_M$. Furthermore, among the sum of virtual representations $\sum_{w \in W(R^+(\lambda))} -1^w \Theta(*, w\lambda)$, the virtual representation $\Theta(*, \lambda)$ corresponding to $w = 1$ has the lowest K type. It follows that $\sum_{w \in W(R^+(\lambda))} -1^w \Theta(*, w\lambda)$ has lowest K type $\lambda + \rho_M$. Applying this to the sums of virtual representations in the claim gives the desired lowest K type.

Note that there are $l!$ elements in the integral Weyl group orbit, $W(R^+(\frac{1}{2}\rho)) \cdot (\frac{1}{2}\rho)$, all with the same sign, whose restriction to \mathfrak{t}^* equals $\frac{1}{2}\rho|_{\mathfrak{t}^*}$. Hence, the lowest K type of $\sum_{w \in W(R^+(\frac{1}{2}\rho))} (-1)^w \Theta(\Psi, w \cdot \frac{1}{2}\rho)$ has multiplicity $\pm l!$. The multiplicity of the lowest K type of the other sums of virtual representations is found analogously. Since the lowest K type of $\sum_{w \in W(R^+(\frac{1}{2}\rho))} (-1)^w \Theta(\Psi, w \cdot \frac{1}{2}\rho)$ lies in the discrete series character $\Theta(\Psi, \frac{1}{2}\rho)$ it must have positive multiplicity $l!$. Similarly the lowest K type for $\sum_{w \in W(R^+(s_{\alpha_0} \frac{1}{2}\rho))} (-1)^w \Theta(s_{\alpha_0} \Psi, w \cdot$

$s_{\alpha_0} \frac{1}{2}\rho$) must have positive multiplicity $l!$. It turns out that the lowest K types of the other sums of virtual representations in the claim have positive multiplicity $l!$ but we will have to wait till Theorem 4.14 to see this. ■

A character version of Proposition 2.17 says that if $\Theta(\gamma)$ is an irreducible character, then a simple root $\alpha \in R^+(\gamma)$ is a τ invariant iff $\Theta(s_\alpha\gamma) = -\Theta(\gamma)$. If $\Theta(\Psi, \gamma)$ is a virtual character, a simple root $\alpha \in R^+(\gamma)$ is a τ invariant iff $\Theta(\Psi, s_\alpha\gamma) = -\Theta(\Psi, \gamma)$. Indeed it is clear that if each genuine representation in $\Theta(\Psi, \gamma)$ has α in its τ invariant then $\Theta(\Psi, s_\alpha\gamma) = -\Theta(\Psi, \gamma)$. To show the converse we note that we can't have a situation whereby the virtual representation equals $X - Y$, and $s_\alpha X = Y$ since if $\alpha \notin \tau(X)$ then by Proposition 2.18 $s_\alpha X = X +$ (nonzero representation).

We observe the following.

Lemma 4.13 *The sums of virtual representations of Lemma 4.12 have a maximal primitive ideal.*

Proof. Indeed, the sums of virtual representations were constructed to have the property that s_α acts by the sign representation, for every simple integral root. ■

Theorem 4.14 *With the notation introduced above Lemma 4.12 we have the following character formulas for Langlands quotients of $\widetilde{\text{SL}(2l)}$ with infinitesimal character $\frac{1}{2}\rho$, having a maximal primitive ideal:*

1. $J_{P_1}(\frac{1}{2}\rho) = \Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}$
2. $J_{P_{min}^+}(\frac{1}{2}\rho) = \Theta_{(\frac{1}{2}, \dots, \frac{1}{2})}$
3. $J_{P_{min}^-}(\frac{1}{2}\rho) = \Theta_{(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})}$
4. $J_{P_1}(s_{\alpha_0} \frac{1}{2}\rho) = \Theta_{(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})}$.

Proof. First we show that $\Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}$ is an irreducible representation. This sum of virtual representations has infinitesimal character $\frac{1}{2}\rho$ and by Lemma 4.13 has a maximal primitive ideal. It must therefore be an integer linear combination of the four Langlands quotients with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal. In Lemma 4.12 we showed that $\Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}$ has lowest K type $(\frac{3}{2}, \dots, \frac{3}{2})$ with multiplicity 1. Because the four Langlands quotients under consideration have lowest K types $(\frac{3}{2}, \dots, \frac{3}{2})$, $(\frac{1}{2}, \dots, \frac{1}{2})$, $(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})$, and $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$, our sum of virtual representations must be irreducible and equal to the Langlands quotient with lowest K type $(\frac{3}{2}, \dots, \frac{3}{2})$, namely $J_{P_1}(\frac{1}{2}\rho)$. The same argument shows that $J_{P_1}(s_{\alpha_0} \frac{1}{2}\rho) = \Theta_{(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})}$.

We use a coherent continuation argument to show that the other two sums of virtual representations are in fact irreducible representations. Consider a coherent family based at the irreducible character $\Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}$. If $\mu = (-\frac{1}{2}, \frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2})$ is the extremal weight of a finite dimensional representation of $\widetilde{\mathrm{SL}(2l)}$. Then $\frac{1}{2}\rho + \mu = w_0 \frac{1}{2}\rho$ is dominant with respect to the integral roots with respect to $\frac{1}{2}\rho$. Using properties of translation functors on virtual characters [7] (Theorem 4.6) and [10] (Lemmas 5.3 – 5.5) we have $\Psi_{\frac{1}{2}\rho + \mu}^{\frac{1}{2}\rho}(\Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}) = \Theta_{(\frac{1}{2}, \dots, \frac{1}{2})}$. The translated irreducible representation is necessarily irreducible by Proposition 2.16. Similarly we have $\Psi_{s_{\alpha_0} \frac{1}{2}\rho + \mu}^{s_{\alpha_0} \frac{1}{2}\rho}(\Theta_{(\frac{3}{2}, \dots, \frac{3}{2}, -\frac{3}{2})}) = \Theta_{(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})}$. The dominance of $s_{\alpha_0} \frac{1}{2}\rho + \mu$ also proves the irreducibility of $\Theta_{(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})}$. Finally, matching lowest K types proves the claim. ■

Theorem 4.15 *With the notation introduced above Lemma 4.12 we have the following character formula for the Langlands quotient of $\widetilde{\mathrm{SL}(2l+1)}$ with infinitesimal character $\frac{1}{2}\rho$ having a maximal primitive ideal:*

$$J_{P_{\min}}(\frac{1}{2}\rho) = \Theta_{(\frac{1}{2}, \dots, \frac{1}{2})}$$

Proof: The proof is identical to the proof that $J_{P_l}(\frac{1}{2}\rho) = \Theta_{(\frac{3}{2}, \dots, \frac{3}{2})}$ in Theorem 4.14. ■

The character formula of this chapter we believe comes close to giving away the K types of the Langlands quotients under investigation. Based on our findings for $\widetilde{\mathrm{SL}(m)}$ with $2 \leq m \leq 6$ we conclude this thesis with a conjecture saying that the weights of our small unitary representations are multiplicity free.

Conjecture 4.16 *The Langlands quotients with minimal parabolic, $J_{P_{\min}}^{\pm}(\frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2l)}$ have K types $(\frac{1}{2} + 2k_1, \dots, \frac{1}{2} + 2k_{l-1}, \pm(\frac{1}{2} + 2k_l))$ where $k_1 \geq k_2 \geq \dots \geq k_l \geq 0$*

The Langlands quotients with maximal cuspidal parabolic, $J_{P_l}(\frac{1}{2}\rho)$ and $J_{P_l}(s_{\alpha_0} \frac{1}{2}\rho)$, of $\widetilde{\mathrm{SL}(2l)}$, have K types $(\frac{3}{2} + 2k_1, \dots, \frac{3}{2} + 2k_{l-1}, \pm(\frac{3}{2} + 2k_l))$ where $k_1 \geq k_2 \geq \dots \geq k_l \geq 0$

The Langlands quotient with minimal parabolic, $J_{P_{\min}}(\frac{1}{2}\rho)$ of $\widetilde{\mathrm{SL}(2l+1)}$ has K types $(\frac{1}{2} + 2k_1, \dots, \frac{1}{2} + 2k_l)$ where $k_1 \geq k_2 \geq \dots \geq k_l \geq 0$.

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