Noncommutative Rational Double Points

by

Daniel Sai-Ping Chan

B.A. Mathematics
University of New South Wales, 1994
B.E. Electrical Engineering
University of New South Wales, 1994

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Signature of Author ....
Department of Mathematics
April 29, 1999

Certified by ....
Michael Artin
Professor of Mathematics
Thesis Supervisor

Accepted by ....
Richard Melrose
Professor of Mathematics
Chairman, Committee for Graduate Students

Science

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Abstract

This thesis has three parts. The first part is a study of dualising complexes for noncommutative complete local rings. We prove a version of local duality for such rings, as well as many other classical results from the commutative theory of local cohomology. The second part is a study of noncommutative analogues of coordinate rings of rational double points. Various properties of these rings are studied including the Gorenstein property, finiteness of representation type and regularity in codimension one. Finally, in the third part, we introduce a multivariable analogue of twisted homogeneous coordinate rings and use it to show that the Rees algebra and tensor product of (amply polarised) twisted homogeneous coordinate rings are noetherian.

Thesis Supervisor: Michael Artin
Title: Professor of Mathematics

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Chapter 0

Introduction

In recent years, noncommutative algebraists have turned their attention to studying noncommutative analogues of specific commutative rings, especially, those arising from algebraic geometry. The goal of this thesis is to study noncommutative local rings which are analogues of the coordinate rings of rational double points.

The class of noncommutative rings in general, is far too large to expect that much of the rich theory of commutative rings extends. For this reason, noncommutative algebraists have long sought conditions one can impose on a noncommutative ring in order that it may enjoy some of the nice properties of commutative rings. Perhaps one of the most novel of these conditions is the existence of a dualising complex. This has been studied in various contexts in [Bj], [L], [Y1] and [VdB1], especially for graded rings. Chapter 1 is a study of dualising complexes for noncommutative local rings. Let $(A, \mathfrak{m})$ be a complete local ring over a field k and $(A^\circ, \mathfrak{m}^\circ)$ be its opposite ring. A dualising complex for A is essentially an element ω of the derived category $D^b(A^\circ \otimes_k A)$ such that the functors $\mathrm{RHom}_A(-,\omega)$ and $\mathrm{RHom}_{A^\circ}(-,\omega)$ induce a duality between $D^b(A)$ and $D^b(A^\circ)$. One reason for studying dualising complexes is that it is a key ingredient in Grothendieck's theory of local duality. The other ingredient is Matlis duality.

Now Matlis duality for noncommutative local rings is more subtle than for commutative rings or noncommutative graded rings. This is because firstly, a priori, there is no reason why the injective hulls of ${}_Ak$ and k_A should agree and secondly, there is no simple analogue of the graded k-linear dual. At the moment, we have established a version of Matlis duality only for complete local rings satisfying certain hypotheses (see (1.1.6)) which we shall assume to hold for the rest of the introduction. Let $E = \varinjlim_p (A/\mathfrak{m}^p)^*$ and define the *Matlis dual* to be the contravariant functor $(-)^\vee : \mathrm{Mod}-A \longrightarrow \mathrm{Mod}-A^\circ$ which maps $M \mapsto \mathrm{Hom}_A(M, E)$. Let $\mathcal{N}$ denote the category of noetherian modules and $\mathcal{A}$ the category of artinian modules. We prove that E is the injective hull of ${}_Ak$ and k_A and that the Matlis dual induces a duality between $\mathcal{N}$ and $\mathcal{A}$.

Let $\Gamma_\mathfrak{m}$ and $\Gamma_{\mathfrak{m}^\circ}$ be the $\mathfrak{m}$-torsion and $\mathfrak{m}^\circ$-torsion functors respectively. A dualising complex is said to be *balanced* if $R\Gamma_\mathfrak{m}(\omega) \simeq A^\vee \simeq R\Gamma_{\mathfrak{m}^\circ}(\omega)$ in $D(A^\circ \otimes_k A)$. Zhang's χ condition, introduced in [AZ], also makes an appearance in the theory of dualising complexes for noncommutative rings. The ring A satisfies the χ condition if $R\Gamma_\mathfrak{m}(M)$ has artinian cohomology for every $M \in \mathcal{N}$. This condition holds automatically for commutative noetherian complete local rings.

If $\mathcal{C}$ is a subcategory of $\mathrm{Mod}-A$ which is closed under extensions, we let $D^b_\mathcal{C}(A)$ be the subcategory of $D^b(A)$ consisting of complexes with cohomology in Obj $\mathcal{C}$. The main result of chapter 1 is an existence theorem for balanced dualising complexes and a version of local duality:

Theorem. *A complete local ring A has a balanced dualising complex ω if and only if A and A° satisfy χ and $\Gamma_\mathfrak{m}$ and $\Gamma_{\mathfrak{m}^\circ}$ have finite cohomological dimension. If these conditions hold then $\omega \simeq R\Gamma_\mathfrak{m}(A)^\vee$ and local duality holds i.e. $R\Gamma_\mathfrak{m}(M)^\vee \simeq RHom_A(M,\omega)$.*

The proof of this theorem is based on Van den Bergh's proof of the analogous theorem for graded rings in [VdB1]. A similar result has been obtained independently by Wu and Zhang.

We next investigate examples of complete local rings with balanced dualising complexes. These include the complete regular local rings of dimension two which were studied by Artin and Stafford in [AS]. These are shown to be precisely the complete local domains with balanced dualising complex which have global dimension two. To generate more examples, we use the existence criteria above to deduce the existence of balanced dualising complexes from the existence of balanced dualising complexes of related rings. More specifically, let $A \longrightarrow B$ be a local homomorphism of rings such that B is finite as a right A-module and as a left A-module. Then,

- If A has a balanced dualising complex then so does B.
- If B has a balanced dualising complex and $A \longrightarrow B$ is split as as a morphism of right A-modules and as left A-modules, then B has a balanced dualising complex.

Using local cohomology, many concepts in commutative algebra can be described purely homologically. For example, the depth and dimension of a finitely generated module M are respectively the minimum and maximum of the set $\{i|R^i\Gamma_{\mathfrak{m}}(M) \neq 0\}$. Maximal Cohen-Macaulay modules can then be defined as those for which the depth equals the cohomological dimension of $\Gamma_{\mathfrak{m}}$. These definitions can be carried over for arbitrary noncommutative local rings. The rest of chapter 1 is concerned with showing how some of the commutative theory extends to complete local rings with balanced dualising complexes. In particular, we prove the Auslander-Buchsbaum formula. The key to these results hinges on the fact that the commutative results can be proved by purely homological means.

Consider the commutative power series ring in two variables $k[[u, v]]$ and let G be a finite subgroup of $SL(V)$ where $V = ku + kv$. Then G induces an action on $k[[u, v]]$ and the resulting invariant ring $k[[u, v]]^G$ is the coordinate ring of a rational double point. One naive way to generalise rational double points to the noncommutative setting is to replace $k[[u, v]]$ by its noncommutative analogues which are the complete regular local rings of dimension two. The resulting objects are called *special quotient surface singularities* and are the subject of study in chapter 2. We classify these singularities and verify the following properties for a special quotient surface singularity A,

- A is Auslander-Gorenstein of injective dimension two and has a balanced dualising complex $\omega \simeq L[-2]$ where L is an invertible A-bimodule.
- The number of isomorphism classes of indecomposable maximal Cohen-Macaulay A-modules is finite.

Hence, the results of chapter 1 concerning complete local rings with balanced dualising complexes apply to special quotient surface singularites. Finally, let $\mathcal{T}$ denote the category of $\mathfrak{m}$-torsion modules. For most special quotient surface singularities A, we show that the quotient category $(\text{Mod}-A)/\mathcal{T}$ has finite injective dimension. In the commutative case, this last property corresponds to the fact that rational double points are regular in codimension one.

In chapter 3, we examine another approach for generalising rational double points to the noncommutative setting. The idea here is to control the associated graded ring. The starting point is the associated graded ring of a complete regular local ring of dimension two which we recall is isomorphic to either $\overline{B}_{\overline{q}} = k\langle\overline{u}, \overline{v}\rangle/(\overline{v}\,\overline{u} - \overline{q}\,\overline{u}\,\overline{v})$ or $\overline{B}_J = k\langle\overline{u}, \overline{v}\rangle/(\overline{v}\,\overline{u} - \overline{u}\,\overline{v} - \overline{v}^2)$. Another natural candidate for a noncommmutative rational double point is a complete local ring A equipped with a filtration F such that $\text{gr}_F A \simeq \overline{B}^G$ where G is a finite group acting on $\overline{B} = \overline{B}_q$ or $\overline{B}_J$ via a subgroup of SL_2. All special quotient surface singularities are of this kind. In chapter 3 we study the special case where $\text{gr}_F A \simeq \overline{B}_{\overline{q}}^G$ and G is a cyclic group acting diagonally on V. We call these rings *q-singularities of type* A_{d-1} where d is the order of the group G and $q = \overline{q}^d$. The main result of chapter 3 is

Theorem. *Suppose the only power of q which is a d-th root of unity is 1. Then any q-singularity A, of type A_{d-1} is a special quotient surface singularity. Moreover, A is isomorphic to the quotient of $k\langle\langle x, y, z\rangle\rangle$ with defining relations:*

$$yx = qxy + xg \quad , \quad zx = (qy + g)^d \quad , \quad zy = qyz + gz \quad , \quad y^d = xz$$

for some $g \in k\langle\langle x, y, z\rangle\rangle$.

Chapter 4 is completely independent from the rest of the thesis. We study a multivariable analogue of Van den Bergh's notion of a twisted homogeneous coordinate ring which is in turn, a noncommutative analogue of the homogeneous coordinate ring one finds in algebraic geometry. These twisted multihomogeneous coordinate rings are used to yield information about twisted homogeneous coordinate rings. Perhaps the most interesting result of this kind is,

Theorem. *Let* X *and* Y *be projective schemes over an algebraically closed field* k *and* L, M *ample invertible bimodules on* X, Y *respectively. Let* $R = B(X; L)$ *and* $S = B(Y; M)$ *be the corresponding twisted homogeneous coordinate rings and* $\mathfrak{m}$ *be the augmentation ideal of* R*. Then the Rees algebra* $R[\mathfrak{m}t]$ *and tensor product* $R \otimes_k S$ *are noetherian.*

We refer the reader to the introduction of chapter 4 for more information.

Chapter 1

Noncommutative Complete Local Rings

In this chapter, we study a class of noncommutative complete local rings to which Grothendieck's theory of local cohomology generalises. The two main concepts are the dualising complex and local duality. In the noncommutative graded case, these have been studied in the work of Yekutieli [Y1] and Van den Bergh [VdB1]. The well-known graded-local analogy of commutative algebra applies here too and permits a simple translation of their results to noncommutative complete local rings.

Throughout this chapter, let k be a (commutative) field which we consider as a base field.

1.1 Matlis Duality

Let A be a k-algebra. The functor $* = \mathrm{Hom}_k(-,k)$ defines a duality between the category of finite dimensional right A-modules and finite dimensional left A-modules. Matlis duality extends this to a duality between noetherian right A-modules and artinian left A-modules. Our approach to Matlis duality will go via the filtered and continuous dual.

A *filtration* F on a vector space (over k) M, is a chain $\ldots F^2M \subseteq F^1M \subseteq F^0M \subseteq \ldots$ of subspaces of M. Following convention, we will often use the same symbol, usually F, to denote filtrations on different vector spaces. Let M be a vector space with a filtration F. We say that M is *complete* if the natural map $M \longrightarrow \varprojlim_p M/F^pM$ is an isomorphism and *cocomplete* if the natural map $\varinjlim_p F^pM \longrightarrow M$ is an isomorphism. We say that M is *locally finite* if the successive quotients $F^pM/F^{p+1}M$ are finite dimensional over k. Recall that two filtrations F, G on a vector space M are said to be *equivalent* if there exists an integer r such that $F^{p+r}M \subseteq G^pM \subseteq F^{p-r}M$ for all integers p. If F and G are equivalent filtrations on M then (M,F) is complete, cocomplete or locally finite if and only if (M,G) is.

Given a subspace N of M, F induces a filtration on N by $F^pN = F^pM \cap N$. Similarly, F induces a filtration on the quotient M/N by $F^p(M/N) = (F^pM + N)/N$. Note that equivalent filtrations induce equivalent filtrations on subspaces and quotients.

Let M^F be the vector space $\bigcup_p (M/F^pM)^*$ where the union takes place in M^*. The *filtered dual* of M is defined to be the vector space M^F equipped with the filtration $F^pM^F = (M/F^{-p}M)^*$. It will also be denoted by M^F.

Let A be a *filtered ring*, by which we mean a ring with a filtration F such that $F^0A = A$ and $(F^pA)(F^qA) \subseteq F^{p+q}A$ for all $p, q \in \mathbb{Z}$. Note that the F^pA are ideals. A right A-module M is said to be a *filtered A-module* or to have an *A-filtration* if it has a filtration F such that $(F^pM)(F^qA) \subseteq F^{p+q}M$ for all $p, q \in \mathbb{Z}$. Note that in this case, the F^pM are submodules. Furthermore, the filtrations induced on submodules or quotients of a filtered A-module are A-filtrations.

Proposition 1.1.1 *Let M be a vector space over k with a filtration F.*

i. *If M is a filtered right A-module then M^F is a filtered left A-module.*

ii. *If M is locally finite then so is M^F. If M is complete and cocomplete then so is M^F.*

iii. *If G is an equivalent filtration on M then $M^F = M^G$ (as subspaces of M^*) and F and G induce equivalent filtrations on this vector space.*

iv. *(exactness) Consider an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ of vector spaces. Let F induce filtrations on L and N. Then $0 \longrightarrow N^F \longrightarrow M^F \longrightarrow L^F \longrightarrow 0$ is exact. Furthermore, the filtrations on L^F, N^F are those induced from M^F.*

Proof. To prove (i), first note that the $(M/F^pM)^*$ (for $p \in \mathbb{Z}$) are left A-modules, so their union M^F also is. Let $f \in F^qM^F = (M/F^{-q}M)^*$ and $a \in F^pA$. It will be convenient to view f as an element of M^* which annihilates $F^{-q}M$. The inclusion $(F^{-p-q}M)(F^pA) \subseteq F^{-q}M$ ensures that $af \in M^*$ annihilates $F^{-p-q}M$ and so is an element of $(M/F^{-p-q}M)^*$. Hence $(F^pA)(F^qM^F) \subseteq F^{p+q}M^F$ and (i) holds.

For (ii), observe first that M^F is cocomplete by construction. We next prove local finiteness. Consider the exact sequence

$$0 \longrightarrow F^pM/F^qM \longrightarrow M/F^qM \longrightarrow M/F^pM \longrightarrow 0$$

Applying the exact functor * shows that

$$(F^pM/F^qM)^* = (M/F^qM)^*/(M/F^pM)^*$$

so M^F is locally finite if M is. It remains to show that M^F is complete.

$$M^F = \varinjlim_q(\varinjlim_p F^pM/F^qM)^* = \varinjlim_q \varprojlim_p (F^pM/F^qM)^* = \varinjlim_q \varprojlim_p (M/F^qM)^*/(M/F^pM)^*$$

On the other hand,

$$\varprojlim_p M^F/F^{-p}M^F = \varprojlim_p \varinjlim_q (M/F^qM)^*/(M/F^pM)^*$$

Hence we have to show that the limits commute in this case. This follows from

Lemma 1.1.2 *Given a sequence $\ldots \subset V^1 \subset V^0 \subset V^{-1} \subset \ldots$ of vector spaces over k, the natural map,*

$$\varinjlim_q \varprojlim_p V^q/V^p \longrightarrow \varprojlim_p \varinjlim_q V^q/V^p$$

is an isomorphism.

To see (iii), note first that $M^F = M^G$ because $\{F^pM\}$ and $\{G^pM\}$ are cofinal. Now choose r so that $F^{p+r}M \subseteq G^pM \subseteq F^{p-r}M$ for every $p \in \mathbb{Z}$. This induces a sequence of surjections

$$M/F^{p+r}M \longrightarrow M/G^pM \longrightarrow M/F^{p-r}M$$

which in turn gives

$$(M/F^{p+r}M)^* \supseteq (M/G^pM)^* \supseteq (M/F^{p-r}M)^*$$

Hence,

$$F^{-p-r}M^F \supseteq G^{-p}M^G \supseteq F^{-p+r}M^F$$

and we see that F and G do indeed induce equivalent filtrations on $M^F = M^G$.

To prove (iv), note that since filtrations are induced, we have an exact sequence,

$$0 \longrightarrow F^pL \longrightarrow F^pM \longrightarrow F^pN \longrightarrow 0$$

which in turn yields the exact sequence,

$$0 \longrightarrow L/F^pL \longrightarrow M/F^pM \longrightarrow N/F^pN \longrightarrow 0$$

Applying the exact functors $(-)^*$ and $\varinjlim$ gives exactness of the dualised sequence. We omit the proof of the last assertion.

We define a category fil $-$ A by setting the objects to be right A-modules M, equipped with an equivalence class of A-filtrations such that with respect to any, and hence all filtrations in the class, M is complete, cocomplete and locally finite. The filtrations in the equivalence class are said to be *associated* to M. A morphism between two objects M and N in fil $-$ A is a module homomorphism $f : M \longrightarrow N$ such that there exists a filtration F associated to M and a filtration F associated to N with $f(F^pM) \subset F^pN$. We will call such a morphism *filtered.* We omit the proof of the next result which gives an alternative characterisation of filtered morphisms.

Proposition 1.1.3 *Let F be a filtration associated to $M \in$ fil $-$ A and F be a filtration associated to $N \in$ fil $-$ A. A module homomorphism $f : M \longrightarrow N$ is filtered if and only if there exists $r \in \mathbb{Z}$ such that $f(F^{p+r}M) \subseteq F^pN$ for all $p \in \mathbb{Z}$.*

The composition of filtered morphisms is again filtered so fil-A is indeed a category. Proposition 1.1.1(ii) allows us to define unambiguously

Definition 1.1.4 *Let (M, F) be an object in fil-A. We define the* continuous dual $M^\vee$ *of M to be the module M^F equipped with the equivalence class of filtrations containing F.*

Proposition 1.1.5 *(duality) The continuous dual $(-)^\vee$: fil $-$ $A \longrightarrow$ fil $-$ A° is a contravariant functor such that the natural morphism $M \longrightarrow M^{\vee\vee}$ is an isomorphism.*

Proof. The previous proposition shows that the continuous dual of an object in fil-A is an object in fil-A°. Given a morphism $f : M \longrightarrow N$ in fil-A, pick associated filtrations F on M and N so that $f(F^pM) \subseteq F^pN$ for all $p \in \mathbb{Z}$. Then there is a natural map

$$F^pN^\vee = (N/F^{-p}N)^* \longrightarrow (M/F^{-p}M)^* = F^pM^\vee$$

This defines a morphism $f^\vee : N^\vee \longrightarrow M^\vee$ in fil-A° so $(-)^\vee$ is indeed a contravariant functor. To see that the bidual is isomorphic to the identity note that,

$$F^qM^{\vee\vee}/F^pM^{\vee\vee} = (M^\vee/F^{-q}M^\vee)^*/(M^\vee/F^{-p}M^\vee)^* \cong (F^{-p}M^\vee/F^{-q}M^\vee)^*$$

$$\cong ((M/F^pM)^*/(M/F^qM)^*)^* \cong (F^qM/F^pM)^{**} \cong F^qM/F^pM$$

Taking $\varprojlim \varinjlim$ of both sides yields the desired isomorphism.

Let A be a local k-algebra with maximal ideal $\mathfrak{m}$. For such a ring, we consider

Standard Hypotheses 1.1.6 *We assume that A has residue field k and that A is a filtered ring with filtration F such that:*

i. A/F^1A is a nonzero finite dimensional vector space.

ii. The Rees ring $\tilde{A} = \oplus(F^pA)t^p$ of A is noetherian.

Proposition 1.1.7 *The standard hypotheses ensure that A and $gr_F A$ are noetherian. Thus A is itself locally finite.*

Proof. Note that $A \simeq \tilde{A}/(t^{-1}-1)$ and $\operatorname{gr}_F A \simeq \tilde{A}/(t^{-1})$ so they are both noetherian. For the last assertion, we need to show that $\operatorname{gr}_F A$ has finite dimensional graded components. Since $\operatorname{gr}_F A$ is noetherian, each graded component is a finitely generated module over the finite dimensional algebra A/F^1A and so we are done.

Given a local ring A satisfying standard hypotheses, it is not necessarily true that the filtration F and the $\mathfrak{m}$-adic filtration are equivalent. However, they are cofinal. We will prove a slightly more general result for which we need some new terminology.

For any n-tuple $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ we consider the following A-filtration F_i on A^n,

$$F_i^p A^n = F^{p+i_1} A \oplus \ldots \oplus F^{p+i_n} A$$

These filtrations are all equivalent. The *standard filtration* is defined to be F_i for $i = (0, \ldots, 0)$. A *good* filtration on a module M is an A-filtration induced from some surjection $A^n \longrightarrow M$ where A^n is filtered by F_i for some n-tuple i. We omit the proof of the next proposition which shows that this definition agrees with the more usual one given in the literature.

Proposition 1.1.8 *A filtration F on a right A-module M is good if and only if there exists generators $x_1, \ldots, x_n$ for M and integers $i_1, \ldots, i_n$ such that,*

$$F^p M = x_1(F^{p+i_1} A) + \ldots + x_n(F^{p+i_n} A)$$

Any two good filtrations are equivalent since the filtrations F_i are equivalent. Also, every good filtration is cocomplete and locally finite since the same is true of the F_i's.

Example 1.1.9 *Let M be a noetherian right A-module. Then $F^pM = M(F^pA)$ is a good filtration on M.*

Proof. Let $A^n \longrightarrow M$ be a surjective map of A-modules. The above filtration coincides with the one induced from the standard filtration on A^n.

We now compare the good and $\mathfrak{m}$-adic filtrations.

Proposition 1.1.10 *Let A be a local ring satisfying standard hypotheses 1.1.6 and let F be a good filtration on a right A-module M. Let G be a filtration on M such that for every $p \in \mathbb{Z}$, G^pM is an A-module and M/G^pM is finite dimensional over k. Then for each $p \in \mathbb{Z}$, there exists a $q \in \mathbb{Z}$ such that $F^qM \subseteq G^pM$. Hence, the filtrations F and $\{M\mathfrak{m}^p\}$ are cofinal.*

Proof. Consider the topology on M induced by the filtration F. All right submodules are closed by [LvO; chapter II, theorem 2.1.2] which means that

$$\bigcap_q (G^pM + F^qM)/G^pM = 0$$

But M/G^pM is finite dimensional so one of the terms in the intersection must actually be zero. Hence, for some $q \in \mathbb{Z}$ we must have $F^qM \subseteq G^pM$.

Now $M/M\mathfrak{m}^p$ is finite dimensional for every integer p. Hence $F^qM \subseteq M\mathfrak{m}^p$ for some $q \in \mathbb{Z}$. On the other hand, since M is locally finite, M/F^pM is finite dimensional for each integer p. Hence, Nakayama's lemma implies that there exists $q \in \mathbb{Z}$ with $M\mathfrak{m}^q \subseteq F^pM$. This proves that the good and $\mathfrak{m}$-adic filtrations are cofinal.

For the rest of this section, we assume that A is a complete local ring satisfying standard hypotheses 1.1.6. Here, complete means with respect to the filtration F or the $\mathfrak{m}$-adic filtration, the two notions being the same by the previous proposition. For complete rings, the condition that $\tilde{A}$ is noetherian is equivalent by [LvO; chapter II, proposition 1.2.3] to the weaker condition that $\operatorname{gr}_F A$ is noetherian. Note that A is an object of fil-A by proposition 1.1.7.

Let $\mathcal{N}(A)$ be the full subcategory of Mod-A consisting of noetherian modules. We will omit the argument A if it is clear. We wish to embed $\mathcal{N}$ in fil-A. We need two lemmas.

Lemma 1.1.11 *Every good filtration is locally finite, cocomplete and complete.*

Proof. We have already noted the first two properties. Since we are assuming that A is complete and the Rees ring $\tilde{A}$ is noetherian, every noetherian module M is complete with respect to the good filtration by [LvO; chapter II theorem 1.2.10 and theorem 2.1.2].

Lemma 1.1.12 *Let M be an A-module with a good filtration and N an A-module with an arbitrary A-filtration. Then every module homomorphism $f : M \longrightarrow N$ is filtered.*

Proof. Let $A^n \longrightarrow M$ be a surjective homomorphism. The standard filtration on A^n induces a good filtration on M so we are reduced to the case $M = A^n$ and thus to the case $M = A$. Every homomorphism from A to N is filtered since the filtration on N is an A-filtration. Hence the lemma holds.

We wish to construct a functor $\iota : \mathcal{N} \longrightarrow \text{fil} - A$. To every noetherian A-module M, we let $\iota(M)$ denote the module M equipped with its equivalence class of good filtrations. For every homomorphism of noetherian modules f, we let $\iota(f) = f$. The previous two lemmas ensure that ι is a functor from $\mathcal{N} \longrightarrow \text{fil} - A$. In fact, ι embeds $\mathcal{N}$ in fil $- A$ so we will also let $\mathcal{N}$ denote the corresponding subcategory of fil-A.

Now good filtrations induce good filtrations on quotients and, since $\tilde{A}$ is noetherian, on submodules as well by [LvO; chapter II, theorem 2.1.2]. Proposition 1.1.1(iv) hence shows that the continuous dual is exact on $\mathcal{N}$. Let $\mathcal{A}$ be the preimage category of $\mathcal{N}$ under $(-)^\vee$. More precisely, the objects of $\mathcal{A}$ are the filtered modules M such that $M^\vee \in \mathcal{N}$ and the morphisms are those whose continuous duals are in $\mathcal{N}$. In particular, duality shows that $N^\vee$ is in Obj $\mathcal{A}$ if $N \in$ Obj $\mathcal{N}$. If we wish to emphasise that the objects we are dealing with are left A-modules, we will write $\mathcal{A}$ as $\mathcal{A}(A^\circ)$.

We wish to show that $\mathcal{A}$ is isomorphic to the category of left artinian A-modules. This proceeds via a series of lemmas.

Lemma 1.1.13 *The equivalence class of filtrations on $M \in Obj\mathcal{A}$ is given by*

$$F^pM = \{m \in M | (F^{-p}A)m = 0\}$$

Proof. Consider a surjection $f : A^n \longrightarrow M^\vee$. The standard filtration on A^n induces a filtration associated to $M^\vee$. Dualising f we see by proposition 1.1.1(iv), that the filtration on M is induced from an embedding into a direct sum of $A^\vee$'s. The lemma now follows from the fact that $F^pA^\vee$ is the submodule of $A^\vee$ consisting of elements annihilated by $F^{-p}A$.

Lemma 1.1.14 *Let $M \in$ Obj $\mathcal{A}$. There is an inclusion reversing bijection between the lattice of submodules of M and the lattice of submodules of $M^\vee$. For N a submodule of M, the corresponding submodule of $M^\vee$ is given by $N' = \ker(M^\vee \longrightarrow N^\vee)$ where the filtration on N is the induced one. Hence, any object M in $\mathcal{A}$ is an artinian A-module.*

Proof. Observe first that by the previous lemma, N is in fil $- A$ so $N^\vee$ is well-defined. The inverse to $N \mapsto N'$ is given by $K \mapsto (M^\vee/K)^\vee$ (once one identifies modules with their biduals).

Lemma 1.1.15 *Let $M, N \in Obj\mathcal{A}$ and $f : M \longrightarrow N$ be a module homomorphism. Then f is also a morphism in $\mathcal{A}$.*

Proof. Lemma 1.1.13 implies that f is filtered so we may consider its continuous dual. Now $f^\vee$ is a morphism in $\mathcal{N}$ so f is a morphism in $\mathcal{A}$.

For the rest of this section, we shall let E denote the bimodule $A^\vee$.

Lemma 1.1.16 *E is the injective hull of k as a left module and as a right module.*

Proof. Let I be a right ideal of A and $f : I \longrightarrow E$ a homomorphism of A-modules. Since I is noetherian, the inclusion $i : I \longrightarrow A$ is filtered by lemma 1.1.12 so we may dualise to obtain by proposition 1.1.1(iv) a surjection $i^\vee : A^\vee \longrightarrow I^\vee$. Similarly, f is filtered and we may thus consider the dual $f^\vee : E^\vee = A \longrightarrow I^\vee$. We lift this to a homomorphism $g : A \longrightarrow A^\vee$ which is automatically filtered. Then $g^\vee$ is a morphism extending f to A. Hence E is injective. To show it is the injective hull of k_A we use the lattice anti-isomorphism between left submodules of A and right submodules of E. Now $k \subset E$ corresponds to $\mathfrak{m} \subset A$. Since $\mathfrak{m}$ contains every left ideal of A except A itself, k must be contained in every right submodule of E except 0. This shows that k_A is an essential submodule of E. Similarly, E is also the injective hull of ${}_Ak$.

Lemma 1.1.17 *Every artinian A-module can be embedded in a finite direct sum of copies of E.*

Proof. Let M be an artinian module. We consider the set of all morphisms from M to finite direct sums of E. From the descending chain condition, we can choose such a morphism $\phi : M \longrightarrow E^n$ so that its kernel is minimal amongst all such morphisms. Then ϕ is injective, for suppose not. Let $x \neq 0$ be in the socle of $\ker \phi$. Then since E is the injective hull of k, we can find a morphism $\psi : M \longrightarrow E$ with $\psi(x) \neq 0$. Then $M \xrightarrow{(\phi,\psi)} E^{n+1}$ has a smaller kernel than ϕ, a contradiction. Hence, M embeds in some finite direct sum E^n.

Putting all these lemmas together yields,

Proposition 1.1.18 *To each left artinian A-module M, let F be the filtration*

$$F^pM = \{m \in M | (F^{-p}A)m = 0\}$$

Then the functor $\eta : M \mapsto (M, F)$ is an embedding of the category of left artinian A-modules onto the category $\mathcal{A}$.

Proof. Consider an embedding $M \longrightarrow E^n$ as in the previous lemma. The above filtration coincides with the one induced from E^n and is complete, cocomplete and locally finite. The dual surjection $A^n \longrightarrow M^\vee$ is in $\mathcal{N}$ so $M \in \text{Obj } \mathcal{A}$. Lemmas 1.1.13 and 1.1.14 ensure that every object of $\mathcal{A}$ is of this form while lemma 1.1.15 ensures that η is a bijection on Hom-sets.

Hence, we will also let $\mathcal{A}(A^\circ)$ denote the category of artinian left A-modules. We call the filtration given in the previous proposition the *cogood* filtration.

Lemma 1.1.19 *The continuous dual is exact on $\mathcal{A}$.*

Proof. Let $L \longrightarrow M \longrightarrow N$ be a sequence of maps in $\mathcal{A}$. If the sequence $N^\vee \longrightarrow M^\vee \longrightarrow L^\vee$ in $\mathcal{N}$ is not exact then neither is $L^{\vee\vee} \longrightarrow M^{\vee\vee} \longrightarrow N^{\vee\vee}$ since $(-)^\vee : \mathcal{N} \longrightarrow \mathcal{A}$ preserves homology. By duality (proposition 1.1.5), we are done.

Lemma 1.1.20 *$A \simeq End_A(E, E)$ with the isomorphism given by $a \in A$ maps to left multiplication by a. Similarly, $A \simeq End_{A^\circ}(E, E)$.*

Proof. Proposition 1.1.18 and duality show that the endomorphisms of E are in 1-1 correspondence with the endomorphisms of A which are right multiplications by elements of A. The lemma follows.

Proposition 1.1.21 *On $\mathcal{N}$ and $\mathcal{A}$, the continuous dual is represented by E i.e. there exists a natural isomorphism of functors $(-)^\vee \longrightarrow Hom_A(-, E)$ from $\mathcal{N} \longrightarrow \mathcal{A}$ and from $\mathcal{A} \longrightarrow \mathcal{N}$.*

Proof. We have to show that for each artinian left A-module M, there is an isomorphism η_M : $M^\vee \simeq \text{Hom}_{A^\circ}(M, E)$ such that for any homomorphism $f : M \longrightarrow N$ of artinian left A-modules, the diagram

$$\begin{array}{ccc} N^\vee & \xrightarrow{f^\vee} & M^\vee \\ \downarrow{\scriptstyle \eta_N} & & \downarrow{\scriptstyle \eta_M} \\ \text{Hom}_{A^\circ}(N, E) & \xrightarrow{\text{Hom}_{A^\circ}(f,E)} & \text{Hom}_{A^\circ}(M, E) \end{array}$$

commutes. We have seen that every artinian module embeds in a finite direct sum of E's. Hence we have an exact sequence $0 \longrightarrow M \longrightarrow E^m \longrightarrow E^n$ for some integers m, n. There is a similar resolution of N. Both the continuous dual and $\mathrm{Hom}_A(-,E)$ are exact functors so it clearly suffices to define the isomorphism η_E and prove commutativity of the square when M, N are finite direct sums of E. This is an immediate consequence of the previous lemma. The case for $\mathcal{N}$ is dual (and easier).

Definition 1.1.22 *We define the* Matlis dual *on modules to be the functor* $M \mapsto M^\vee = Hom_A(M,E)$.

Proposition 1.1.21 shows that the Matlis dual can be computed for noetherian or artinian modules by endowing the module with the good or cogood filtration and then taking the continuous dual.

If M is a bimodule, then there may be some ambiguity as to whether the Matlis dual refers to the left or right module structure. In this case, we will write out the functor in full, as $\mathrm{Hom}_A(-,E)$ or $\mathrm{Hom}_{A^\circ}(-,E)$ accordingly. Fortunately, most of the time, this will not be necessary thanks to

Lemma 1.1.23 *If M is a bimodule which is $\mathfrak{m}$-torsion on both sides, then $Hom_A(M,E) \simeq M^* \simeq Hom_{A^\circ}(M,E)$. If M is a bimodule which is noetherian on both sides then $Hom_A(M,E) \simeq Hom_{A^\circ}(M,E)$. In particular, the left and right Matlis duals coincide for extensions of such bimodules.*

Proof. Suppose first that M is $\mathfrak{m}$-torsion on both sides. The Matlis dual and the k-linear dual both take direct limits to inverse limits, so it suffices to prove $\mathrm{Hom}_A(M,E) = M^*$ for $M \in \mathcal{A}(A)$. Now the right continuous dual of M is M^* since the filtration on M is discrete by lemma 1.1.13. Hence $\mathrm{Hom}_A(M,E) = M^*$ by the previous proposition. Suppose now that M is noetherian on both sides. We show the left and right continuous duals are equal. This amounts to showing that the good filtrations $\{M(F^pA)\}$ and $\{(F^pA)M\}$ of example 1.1.9 define the same topology on M. This is a consequence of proposition 1.1.10 since the $M(F^pA)$ and $(F^pA)M$ are bimodules. The last assertion now follows from exactness of the Matlis dual (lemma 1.1.16).

Let $\mathcal{C}$ be the full subcategory of Mod $-A$ consisting of objects which are extensions of noetherian modules by artinian ones i.e. modules M for which there exists an exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow N' \longrightarrow 0$ where $N \in \mathcal{N}$ and $N' \in \mathcal{A}$. If we wish to emphasise that we are dealing with right modules, we shall write the category as $\mathcal{C}(A)$.

Proposition 1.1.24 *$\mathcal{C}$ is an abelian subcategory of Mod$-A$ which is closed under extensions.*

Proof. We first show that $\mathcal{C}$ is abelian. $\mathcal{C}$ is clearly closed under finite direct sums. We need to show it contains kernels and cokernels. Consider a morphism $f : M \longrightarrow N$ in $\mathcal{C}$. We express M, N as extensions,

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0$$

where $M_1, N_1 \in \mathrm{Obj}\ \mathcal{N}$ and $M_2, N_2 \in \mathrm{Obj}\ \mathcal{A}$. Then we can fit f into the diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N & \longrightarrow & N_2 & \longrightarrow & 0 \end{array}$$

by adding the noetherian module $f(M_1)$ to N_1 if necessary. The snake lemma shows that both the kernel and the cokernel of f are in $\mathcal{C}$.

Consider now an extension

$$0 \longrightarrow M' \longrightarrow M \stackrel{h}{\longrightarrow} M'' \longrightarrow 0$$

where $M', M'' \in \mathrm{Obj}\ \mathcal{C}$. We wish to show that M is an object in $\mathcal{C}$. Since $\mathcal{A}$ and $\mathcal{N}$ are closed under extensions, we may as well assume that M' is artinian and M'' is noetherian. Pick a finitely generated submodule N of M with $h(N) = M''$. Now M/N is isomorphic to a quotient of M' and so is artinian. Hence $M \in \mathrm{Obj}\ \mathcal{C}$.

Let $B \longrightarrow C$ be a ring homomorphism and $\mathcal{K}$ be a full abelian subcategory of Mod-B. Suppose that the full subcategory of C-modules which are in Obj $\mathcal{K}$ when considered as B-modules is closed under extensions. Then we let $D^*_{\mathcal{K}}(B)$ denote the full subcategory of the derived category $D^*(B)$ consisting of complexes with cohomology in $\mathcal{K}$ (here * denotes +, -, b or can be omitted). We will also use $K^*_{\mathcal{K}}(B)$ to denote the corresponding category of chain complexes. The unadorned tensor symbol $\otimes$ will always mean $\otimes_k$. We obtain the following version of Matlis duality.

Theorem 1.1.25 *(Matlis duality) The Matlis dual defines a duality between $\mathcal{N}$ and $\mathcal{A}$ which extends to a duality between $\mathcal{C}(A)$ and $\mathcal{C}(A^\circ)$. It hence induces a duality between $D_{\mathcal{C}}(A)$ and $D_{\mathcal{C}}(A^\circ)$. Let B and C be k-algebras and $M \in D^-_{\mathcal{C}}(B^\circ \otimes A)$ and $N \in D^+_{\mathcal{C}}(C^\circ \otimes A)$. Then there is a natural isomorphism*

$$RHom_A(M,N) \simeq RHom_{A^\circ}(N^\vee, M^\vee)$$

in $D^+(C^\circ \otimes B)$.

Proof. Propositions 1.1.5 and 1.1.21 show that Matlis duality gives a duality between $\mathcal{N}$ and $\mathcal{A}$. This extends to a duality of $\mathcal{C}(A)$ and $\mathcal{C}(A^\circ)$ by the 5-lemma. This in turn induces a duality between $D_{\mathcal{C}}(A)$ and $D_{\mathcal{C}}(A^\circ)$ by [H1; chapter 1, proposition 7.1]. The final assertion follows from the well-known general principle

Proposition 1.1.26 *(derived duality) Let $\mathcal{B}$ and $\mathcal{D}$ be abelian categories with enough projectives and injectives. Let $F : \mathcal{B} \longrightarrow \mathcal{D}$ be a contravariant functor. Let $\mathcal{B}'$ and $\mathcal{D}'$ be full subcategories of $\mathcal{B}$ and $\mathcal{D}$ respectively which are closed under extensions. Suppose that $F(\mathcal{B}') \subseteq \mathcal{D}'$ and that the right derived functor $RF : D_{\mathcal{B}'}(\mathcal{B}) \longrightarrow D_{\mathcal{D}'}(\mathcal{D})$ is a category anti-equivalence. Then for $M \in D^-_{\mathcal{B}'}(\mathcal{B})$ and $N \in D^+_{\mathcal{B}'}(\mathcal{B})$, there is a natural isomorphism*

$$RHom_{\mathcal{B}}(M,N) \simeq RHom_{\mathcal{D}}(RF(N), RF(M))$$

Proof. First note that we may identify $D^-(\mathcal{B})^\circ \times D^+(\mathcal{B})$ with $D^+(\mathcal{B}^\circ \times \mathcal{B})$. If we let Ab denote the category of abelian groups then $\mathrm{RHom}_{\mathcal{B}} : D^+(\mathcal{B}^\circ \times \mathcal{B}) \longrightarrow D(\mathrm{Ab})$ is the right derived functor of the covariant functor $\mathrm{Hom}_{\mathcal{B}} : \mathcal{B}^\circ \times \mathcal{B} \longrightarrow \mathrm{Ab}$. Also, the universal property of derived functors ensures that $R(F \times F) = RF \times RF$. Let $T : \mathcal{B}^\circ \times \mathcal{B} \longrightarrow \mathcal{B} \times \mathcal{B}^\circ$ be the canonical twist functor $(M,N) \mapsto (N,M)$. Then F induces a natural transformation $\mathrm{Hom}_{\mathcal{B}} \longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F \times F) \circ T$. Hence there is a natural transformation in the derived category

$$\mathrm{RHom}_{\mathcal{B}} \longrightarrow R(\mathrm{Hom}_{\mathcal{D}} \circ (F \times F) \circ T) \longrightarrow \mathrm{RHom}_{\mathcal{D}} \circ (RF \times RF) \circ T$$

We need to show this composite is an isomorphism. Now

$$R^i \,\mathrm{Hom}_{\mathcal{B}}(M,N) \simeq \mathrm{Hom}_D(M, N[i]) \simeq \mathrm{Hom}_D(RF(N)[-i], RF(M)) \simeq R^i \,\mathrm{Hom}_{\mathcal{D}}(RF(N), RF(M))$$

where the second isomorphism follows from the fact that RF is a category anti-equivalence. The natural transformation is thus an isomorphism.

1.2 Local Duality and Dualising Complexes

As before, let $(A, \mathfrak{m})$ be a complete local ring satisfying standard hypotheses 1.1.6. In this section we present a noncommutative local version of local duality. The proofs are similar to the ones in the graded case given in [Y1] and [VdB1]. Since A is always noetherian, some of the subtleties in [Y1] and [VdB1] have been avoided.

We denote the $\mathfrak{m}$-torsion functor for right modules by $\Gamma_{\mathfrak{m}}$ and the $\mathfrak{m}$-torsion functor for left modules by $\Gamma_{\mathfrak{m}^\circ}$. The right derived functors are denoted by $H^i_{\mathfrak{m}}$ and $H^i_{\mathfrak{m}^\circ}$. To state the main result, we need some definitions. Following [AZ; definition 3.7], we define

Definition 1.2.1 *A satisfies χ if $R\Gamma_{\mathfrak{m}}(M)$ has artinian cohomology for every noetherian right A-module M. By [H1; chapter 1, proposition 7.3], this is equivalent to the fact that $R\Gamma_{\mathfrak{m}}$ is a functor from $D^+_{\mathcal{N}}$ to $D^+_{\mathcal{A}}$.*

Let $A \longrightarrow B$ be a morphism of k-algebras. We denote the restriction of scalars functor by $\text{res}^B_A : \text{Mod}-B \longrightarrow \text{Mod}-A$. We will omit the subscript or superscript when there is no confusion. The restriction functor is exact and so extends to a functor from $D(B) \longrightarrow D(A)$. In particular, given $M \in D(A^e)$ it makes sense to consider ${}_AM$ and M_A.

Mimicking [Y1; definition 3.3] we make the

Definition 1.2.2 *An element ω in $D^b(A^e)$ is said to be a* dualising complex *for A if*

i. *${}_A\omega$ and ω_A have finite injective dimension i.e. $Ext^i_A(-,\omega) = 0 = Ext^i_{A^\circ}(-,\omega)$ for all $i \ll 0$ and $i \gg 0$.*

ii. *${}_A\omega$ and ω_A have noetherian cohomology.*

iii. *The canonical morphisms $A \longrightarrow RHom_A(\omega,\omega)$ and $A \longrightarrow RHom_{A^\circ}(\omega,\omega)$ are isomorphisms in $D(A^e)$.*

Let $F : \text{Mod}-A \longrightarrow \text{Mod}-k$ be a functor. Every k-algebra C induces a functor $F_C : \text{Mod}-(A\otimes C) \longrightarrow \text{Mod}-C$. With this notation we have

Lemma 1.2.3 *The restriction functor $res^{A\otimes C}_A$ preserves projectives and injectives. Hence $R^*F \circ res^{A\otimes C}_A = res^C_k \circ R^*F_C$ where * is + if F is covariant, - if F is contravariant and can be omitted if F has finite cohomological dimension.*

Proof. ([Y1; lemma 2.1]) The restriction functor $\text{res}^{A\otimes C}_A$ has an exact right adjoint $\text{Hom}_k(C,-)$ and an exact left adjoint $-\otimes C$. Consequently it preserves projectives and injectives. Grothendieck's theorem on derived functors of composites gives the second assertion.

Dualising complexes induce a duality in the following sense,

Proposition 1.2.4 *Let C be a k-algebra. If ω is a dualising complex then the functors $RHom_A(-,\omega)$ and $RHom_{A^\circ}(-,\omega)$ define a duality between $D^b_{\mathcal{N}(A)}(C^\circ\otimes A)$ and $D^b_{\mathcal{N}(A^\circ)}(A^\circ\otimes C)$.*

Proof. ([Y1; proposition 3.5]) The finite injective dimension condition on ω ensures the functors map D^b to D^b while the noetherian cohomology condition ensures $D^b_{\mathcal{N}}$ maps to $D^b_{\mathcal{N}}$. Consider the natural transformations

$$\text{id} \longrightarrow \text{Hom}_{A^\circ}(\text{Hom}_A(-,\omega),\omega) \quad , \quad \text{id} \longrightarrow \text{Hom}_A(\text{Hom}_{A^\circ}(-,\omega),\omega)$$

of endofunctors on $\text{Mod}-(C^\circ\otimes A)$ and $\text{Mod}-(A^\circ\otimes C)$ respectively. We must show that the associated derived functors

$$\text{id} \longrightarrow \text{RHom}_{A^\circ}(\text{RHom}_A(-,\omega),\omega) \quad , \quad \text{id} \longrightarrow \text{RHom}_A(\text{RHom}_{A^\circ}(-,\omega),\omega)$$

are natural isomorphisms of endofunctors on $D^b_{\mathcal{N}(A)}(C^\circ\otimes A)$ and $D^b_{\mathcal{N}(A^\circ)}(A^\circ\otimes C)$ respectively. By the previous lemma, we may restrict scalars and assume $C = k$. Now noetherian modules have finitely generated free resolutions. Hence by [H1; chapter I, proposition 7.1(iv)] we need only check that the natural maps are isomorphisms on finitely generated free modules. This follows from condition (iii) for a complex to be dualising.

Finally, as in [Y1; definition 4.1] we define,

Definition 1.2.5 *Suppose A is complete. A dualising complex ω is said to be* balanced *if $R\Gamma_{\mathfrak{m}}(\omega) \simeq A^\vee \simeq R\Gamma_{\mathfrak{m}^\circ}(\omega)$ in $D(A^e)$.*

We let cd denote the cohomological dimension of a functor. The key result is,

Theorem 1.2.6 *Let A be a complete local ring satisfying the standard hypotheses 1.1.6. Then the following are equivalent,* *i.* *A and* A° *satisfy* χ *and* $cd\ \Gamma_{\mathfrak{m}}$, $cd\ \Gamma_{\mathfrak{m}^\circ}$ *are finite.*

ii. *(Local Duality) There exists* $\omega \in D^b(A^e)$ *satisfying conditions (i),(ii) of 1.2.2 and the following: given any k-algebra C and complexes* $M \in D(C^\circ \otimes A)$ *and* $N \in D(A^\circ \otimes C)$,

$$Hom_A(R\Gamma_{\mathfrak{m}}(M), A^\vee) = RHom_A(M,\omega) \quad , \quad Hom_{A^\circ}(R\Gamma_{\mathfrak{m}^\circ}(N), A^\vee) = RHom_{A^\circ}(N,\omega)$$

hold in $D(A^\circ \otimes C)$ *and* $D(C^\circ \otimes A)$.

iii. *A has a balanced dualising complex.*

If these conditions hold then ω *is the balanced dualising complex and equals*

$$Hom_A(R\Gamma_{\mathfrak{m}}(A), A^\vee) = Hom_{A^\circ}(R\Gamma_{\mathfrak{m}}(A), A^\vee) = Hom_A(R\Gamma_{\mathfrak{m}^\circ}(A), A^\vee) = Hom_{A^\circ}(R\Gamma_{\mathfrak{m}^\circ}(A), A^\vee)$$

Definition 1.2.7 A complete local ring with balanced dualising complex *is a complete local ring satisfying 1.1.6 and the equivalent conditions of theorem 1.2.6.*

The proof of this theorem will occupy the rest of this section.

We start by proving $(i) \Rightarrow (ii)$. As in [VdB1], the first step will be to show that for a bimodule, it does not matter on which side one takes local cohomology. We need several lemmas. Throughout these, we continue to assume that A is complete local and satisfies standard hypotheses.

Lemma 1.2.8 $H^i_{\mathfrak{m}}$ *commutes with direct limits. Consequently, direct limits preserve* $\Gamma_{\mathfrak{m}}$*-acyclics.*

Proof. ([VdB1; lemma 4.3]) Since $H^i_{\mathfrak{m}} = \varinjlim \mathrm{Ext}^i_A(A/\mathfrak{m}^p, -)$ it suffices to show $\mathrm{Ext}^i_A(A/\mathfrak{m}^p, -)$ commutes with direct limits. Let $P \longrightarrow A/\mathfrak{m}^p \longrightarrow 0$ be a finitely generated free resolution so that $\mathrm{Ext}^i_A(A/\mathfrak{m}^p, -) = H^i \,\mathrm{Hom}_A(P, -)$. The lemma now follows from the fact that cohomology and the $\mathrm{Hom}_A(P^j, -)$ commute with direct limits.

Let $\mathcal{T}(A)$ be the category of $\mathfrak{m}$-torsion modules. We will omit the argument A if it is clear.

Lemma 1.2.9 *([VdB1; lemma 4.4]) The canonical natural transformation* $R\Gamma_{\mathfrak{m}} \longrightarrow id$ *is an isomorphism when restricted to the category* $D^+_{\mathcal{T}}$. *If* $cd\, \Gamma_{\mathfrak{m}}$ *is finite then* $R\Gamma_{\mathfrak{m}} \simeq id$ *as endofunctors on* $D_{\mathcal{T}}$.

Proof. By [H1; chapter 1, proposition 7.1(ii) and (iii)] we need only check $R\Gamma_{\mathfrak{m}}(M) \simeq M$ for $M \in \mathrm{Obj}\ \mathcal{T}$. Every torsion module is the direct limit of artinian modules so by the previous lemma we may assume $M \in \mathrm{Obj}\ \mathcal{A}$. Lemma 1.1.17 guarantees that M has an injective resolution consisting of finite direct sums of $A^\vee$. The lemma follows since $\Gamma_{\mathfrak{m}}(A^\vee) = A^\vee$.

Let $(B, \mathfrak{n})$ be another complete local ring satisfying the standard hypotheses 1.1.6. Let $\mathfrak{p}$ be the maximal ideal $\mathfrak{n} \otimes A + B \otimes \mathfrak{m}$ of $B \otimes A$. The case we will be interested in is $B = A^\circ$.

Lemma 1.2.10 $R\Gamma_{\mathfrak{p}} = R\Gamma_{\mathfrak{n}} \circ R\Gamma_{\mathfrak{m}}$ *holds in* $D^+(B \otimes A)$. *The equality extends to* $D(B \otimes A)$ *if* $cd\ \Gamma_{\mathfrak{m}}$ *and* $cd\ \Gamma_{\mathfrak{n}}$ *are finite.*

Proof. (See [VdB1; lemma 4.5].) An elementary computation shows that $\Gamma_{\mathfrak{p}} = \Gamma_{\mathfrak{n}} \circ \Gamma_{\mathfrak{m}}$. To show $R\Gamma_{\mathfrak{p}}(M) = R\Gamma_{\mathfrak{n}} \circ R\Gamma_{\mathfrak{m}}(M)$, we may replace M with a complex of injectives in $\mathrm{Mod}-(B \otimes A)$ by [BN; application 2.4]. It thus suffices to show for any $(B \otimes A)$-injective M, that $\Gamma_{\mathfrak{m}}(M)$ is $\Gamma_{\mathfrak{n}}$-acyclic. By lemma 1.2.8, we are done if the functor $\mathrm{Hom}_A(A/\mathfrak{m}^p, -) : \mathrm{Mod}-(B \otimes A) \longrightarrow \mathrm{Mod}-B$ maps M to a B-injective. This follows from exactness of the left adjoint $A/\mathfrak{m}^p \otimes -$.

Lemma 1.2.11 *Suppose that A satisfies* χ. *Then* $R\Gamma_{\mathfrak{m}}$ *maps* $D^+_{\mathcal{N}(A)}(B \otimes A)$ *into* $D^+_{\mathcal{T}(B)}(B \otimes A)$. *Furthermore, if* $cd\, \Gamma_{\mathfrak{m}}$ *is finite, then* $R\Gamma_{\mathfrak{m}}$ *maps* $D_{\mathcal{N}(A)}(B \otimes A) \longrightarrow D_{\mathcal{T}(B)}(B \otimes A)$.

Proof. By [H1; chapter 1, proposition 7.3] we need to show that for every $(B \otimes A)$-module M which is noetherian over A, $H^i_{\mathfrak{m}}(M)$ is $\mathfrak{n}$-torsion. This follows from the χ condition and Lenagan's theorem [GW; theorem 7.10] which is easily proved in this case as follows. Let $x \in H^i_{\mathfrak{m}}(M)$. Then xA is a noetherian A-submodule of the artinian A-module $H^i_{\mathfrak{m}}(M)$. Hence, xA is finite dimensional over k

so $x(B \otimes A)$ is a finitely generated right B-module. Nakayama's lemma shows that $x(\mathfrak{n}^i \otimes A)$ is a strictly decreasing sequence of right A-modules unless the sequence terminates at 0. The descending chain condition shows the latter must occur so x is indeed annihilated by a power of $\mathfrak{n}$.

We let $\mathcal{T}(B, A)$ denote the full subcategory of Mod-$(B \otimes A)$ consisting of modules which are $\mathfrak{m}$-torsion and $\mathfrak{n}$-torsion. Similarly, we define subcategories $\mathcal{N}(B, A) = \mathcal{N}(A) \cap \mathcal{N}(B)$ and $\mathcal{A}(B, A) = \mathcal{A}(A) \cap \mathcal{A}(B)$.

Proposition 1.2.12 *There are natural isomorphisms $Hom_A(-, A^\vee) \simeq (-)^*$ and $Hom_{A^\circ}(-, A^\vee) \simeq (-)^*$ of functors on $D_\mathcal{T}(A)$ and $D_\mathcal{T}(A^\circ)$ respectively. Hence, the two Matlis duals $Hom_A(-, A^\vee)$ and $Hom_{A^\circ}(-, A^\vee)$ are naturally isomorphic as functors on $D_{\mathcal{T}(A^\circ, A)}(A^e)$*

Proof. There is a natural transformation $\mathrm{Hom}_A(-, A^\vee) \longrightarrow \mathrm{Hom}_A(-, A^*) = (-)^*$. Hence, the first assertion is a derived version of lemma 1.1.23. By lemma 1.2.3, restriction of scalars commutes with derived functors so the second assertion follows from the first.

Corollary 1.2.13 *If A satisfies χ then $Hom_A(R\Gamma_\mathfrak{m}(-), A^\vee) \simeq Hom_{A^\circ}(R\Gamma_\mathfrak{m}(-), A^\vee)$ as functors on $D_{\mathcal{N}(A)}(A^e)$.*

The corollary allows us to write $R\Gamma_\mathfrak{m}(M)^\vee$ unambiguously whenever M has noetherian cohomology on the right.

Putting lemmas 1.2.9, 1.2.10 and 1.2.11 together we find,

Theorem 1.2.14 *([VdB1; corollary 4.8]) Suppose that A, B satisfy χ. Then $R\Gamma_\mathfrak{m} = R\Gamma_\mathfrak{n}$ as functors on $D^+_{\mathcal{N}(B,A)}(B \otimes A)$. The equality extends to $D_{\mathcal{N}(B,A)}(B \otimes A)$ if cd $\Gamma_\mathfrak{m}$, cd $\Gamma_\mathfrak{n}$ are finite.*

Proof. The above lemmas show in fact that both $R\Gamma_\mathfrak{m}$ and $R\Gamma_\mathfrak{n}$ are naturally isomorphic to $R\Gamma_\mathfrak{p}$.

We are now ready for the

Proof of $(i) \Rightarrow (ii)$ of Theorem 1.2.6 [VdB1; theorem 5.1] and [J2; theorem 2.3].

Let $\omega = R\Gamma_\mathfrak{m}(A)^\vee$ which we note by theorem 1.2.14 equals $R\Gamma_{\mathfrak{m}^\circ}(A)^\vee$. Now $\omega \in D^b_{\mathcal{N}(A^\circ)}$ since $R\Gamma_\mathfrak{m}$ takes $D^b_{\mathcal{N}(A)}$ to $D^b_{\mathcal{A}(A)}$ and the Matlis dual swaps the artinian and noetherian categories by theorem 1.1.25. Similarly, ω has noetherian cohomology on the right.

Let I be an A^e-injective resolution of A. The I^p are A-injective by lemma 1.2.3. Since cd $\Gamma_\mathfrak{m}$ is finite, we may truncate I to a bounded resolution E of A^e-modules which are $\Gamma_\mathfrak{m}$-acyclic. Let $M \in D(C^\circ \otimes A)$ and using [BN; application 2.4], choose a $(C^\circ \otimes A)$-free resolution K of M. Then

$$\mathrm{RHom}_A(M, \omega) = \mathrm{Hom}_A(K, \mathrm{Hom}_A(\Gamma_\mathfrak{m}(E), A^\vee)) = \mathrm{Hom}_A(K \otimes_A \Gamma_\mathfrak{m}(E), A^\vee) = \mathrm{Hom}_A(\Gamma_\mathfrak{m}(K \otimes_A E), A^\vee)$$

where the last equality follows from the fact that the K^p are A-free and $\Gamma_\mathfrak{m}$ commutes with direct sums. Now the spectral sequence for a double complex shows that $K \otimes E$ is quasi-isomorphic to M. Furthermore, it is a complex of $\Gamma_\mathfrak{m}$-acyclics since the E^p are $\Gamma_\mathfrak{m}$-acyclic and $H^i_\mathfrak{m}$ commutes with direct sums by lemma 1.2.8. Hence, the last term above is $\mathrm{Hom}_A(R\Gamma_\mathfrak{m}(M), A^\vee)$ as was to be shown. A symmetric argument shows that ω represents $\mathrm{Hom}_{A^\circ}(R\Gamma_{\mathfrak{m}^\circ}(-), A^\vee)$. Finiteness of the injective dimension of ω now follows from the finiteness of the cohomological dimension of $\Gamma_\mathfrak{m}$.

We move on now to the

Proof of $(ii) \Rightarrow (iii)$ of Theorem 1.2.6 [VdB1; theorem 6.3].

We must have $\omega = \mathrm{Hom}_A(R\Gamma_\mathfrak{m}(A), A^\vee) = \mathrm{Hom}_{A^\circ}(R\Gamma_{\mathfrak{m}^\circ}(A), A^\vee)$. We wish to show that ω is also a balanced dualising complex for A. Since by assumption, ω satisfies conditions (i) and (ii) of definition 1.2.2, we need only verify the third condition and show also that $R\Gamma_\mathfrak{m}(\omega) \simeq A^\vee \simeq R\Gamma_{\mathfrak{m}^\circ}(\omega)$. Using theorem 1.1.25 we see

$$\mathrm{RHom}_A(\omega, \omega) = \mathrm{RHom}_A(\mathrm{Hom}_{A^\circ}(R\Gamma_{\mathfrak{m}^\circ}(A), A^\vee), \omega) = \mathrm{RHom}_{A^\circ}(\mathrm{Hom}_A(\omega, A^\vee), R\Gamma_{\mathfrak{m}^\circ}(A))$$

Recall the well-known

Lemma 1.2.15 *([Y1; (4.7)]) Let B and C be k-algebras. There is a natural isomorphism*

$$RHom_A(-, R\Gamma_{\mathfrak{m}}(-)) \xrightarrow{\sim} RHom_A(-,-)$$

of functors from $D^-_{\mathcal{T}(A)}(B^\circ \otimes A) \times D^+(C^\circ \otimes A) \longrightarrow D^+(C^\circ \otimes B)$.

Proof. The natural map is induced by the natural map $R\Gamma_{\mathfrak{m}} \longrightarrow \mathrm{id}$ in the second variable. By lemma 1.2.3, we may as well assume that $B = k = C$. By [H1; chapter I, proposition 7.1] it suffices to prove the map is an isomorphism of functors from $\mathcal{T} \times \mathrm{Mod}-A \longrightarrow D(k)$. Now

$$\mathrm{Hom}_A(-, \Gamma_{\mathfrak{m}}(-)) = \mathrm{Hom}_A(-,-)$$

on $\mathcal{T} \times \mathrm{Mod}-A$ so the lemma follows from Grothendieck's theorem and the fact that $\Gamma_{\mathfrak{m}}$ preserves injectives (see [Y1; (4.7)] or proposition 1.3.5).

Now $\omega \in D^b_{\mathcal{N}(A)}$ by assumption so Matlis duality ensures $\mathrm{Hom}_A(\omega, A^\vee) \in D^b_{\mathcal{A}(A^\circ)}$. Hence,

$$\mathrm{RHom}_A(\omega,\omega) = \mathrm{RHom}_{A^\circ}(\mathrm{Hom}_A(\omega, A^\vee), A) = \mathrm{RHom}_A(A^\vee, \omega)$$

By local duality and lemma 1.1.20, this last term is just

$$\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}(A^\vee), A^\vee) = \mathrm{Hom}_A(A^\vee, A^\vee) = A$$

yielding half of condition (iii) of definition 1.2.2. The other half is symmetric so ω is indeed a dualising complex. To show it is balanced, note that by local duality

$$\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}(\omega), A^\vee) = \mathrm{RHom}_A(\omega,\omega) = A$$

Hence $R\Gamma_{\mathfrak{m}}(\omega) = A^\vee$. Similarly, one can show $R\Gamma_{\mathfrak{m}^\circ}(\omega) = A^\vee$.

Finally we give the
Proof of $(iii) \Rightarrow (i)$ of Theorem 1.2.6.

We first prove a weak version of local duality in

Claim 1.2.16 *Let C be a k-algebra and ω a dualising complex such that $R\Gamma_{\mathfrak{m}}(\omega) = A^\vee$. Then there is a natural isomorphism $R\Gamma_{\mathfrak{m}} \simeq Hom_{A^\circ}(RHom_A(-,\omega), A^\vee)$ of endofunctors on $D^b_{\mathcal{N}(A)}(C^\circ \otimes A)$.*

Proof. ([Y1; theorem 4.18]) Since $\Gamma_{\mathfrak{m}}$ preserves injectives, there is a natural map

$$\theta : R\Gamma_{\mathfrak{m}} \longrightarrow \mathrm{RHom}_{A^\circ}(\mathrm{RHom}_A(-,\omega), R\Gamma_{\mathfrak{m}}(\omega))$$

which we must show is an isomorphism. By lemma 1.2.3, we may suppose that $C = k$. Let $M \in D^b_{\mathcal{N}}(A)$. Consider the diagram

$$\begin{array}{ccc} \mathrm{RHom}_A(A/\mathfrak{m}^p, M) & \longrightarrow & \mathrm{RHom}_{A^\circ}(\mathrm{RHom}_A(M,\omega), \mathrm{RHom}_A(A/\mathfrak{m}^p,\omega)) \\ \phi\downarrow & & \downarrow\psi \\ R\Gamma_{\mathfrak{m}}(M) & \xrightarrow{\theta} & \mathrm{RHom}_{A^\circ}(\mathrm{RHom}_A(M,\omega), R\Gamma_{\mathfrak{m}}(\omega)) \end{array}$$

where the vertical arrows are induced from the natural map $\mathrm{Hom}_A(A/\mathfrak{m}^p, -) \longrightarrow \Gamma_{\mathfrak{m}}$. The top map in the diagram comes from the duality induced by $\mathrm{RHom}_A(-,\omega) : D^b_{\mathcal{N}}(A) \longrightarrow D^b_{\mathcal{N}}(A^\circ)$ and is thus an isomorphism by proposition 1.1.26. To see that the diagram commutes, let $P \longrightarrow A/\mathfrak{m}^p$ be a free resolution and assume that M is a bounded below complex of injectives and ω is a bounded complex of injectives. We have the following commutative diagram of chain complexes

$$\begin{array}{ccc} \mathrm{Hom}_A(P,M) & \longrightarrow & \mathrm{Hom}_{A^\circ}(\mathrm{Hom}_A(M,\omega),\mathrm{Hom}_A(P,\omega)) \\ {\scriptstyle qis}\uparrow & & \uparrow{\scriptstyle qis} \\ \mathrm{Hom}_A(A/\mathfrak{m}^p,M) & \longrightarrow & \mathrm{Hom}_{A^\circ}(\mathrm{Hom}_A(M,\omega),\mathrm{Hom}_A(A/\mathfrak{m}^p,\omega)) \\ \downarrow & & \downarrow \\ \Gamma_{\mathfrak{m}}(M) & \longrightarrow & \mathrm{Hom}_{A^\circ}(\mathrm{Hom}_A(M,\omega),\Gamma_{\mathfrak{m}}(\omega)) \end{array}$$

where qis denotes that the morphisms are quasi-isomorphisms. Localising yields the previous diagram which thus also commutes.

Thus to show that θ is an isomorphism, it suffices to show ϕ and ψ induce isomorphisms on cohomology in the limit as $p \to \infty$ i.e. $\varinjlim_p H^i(\phi)$ and $\varinjlim_p H^i(\psi)$ are isomorphisms for all i. This is clear for ϕ. For ψ, note first that $\mathrm{RHom}_A(M,\omega) \in D^b_{\mathcal{N}}(A^\circ)$ and so may be replaced by a finitely generated free resolution F. We must show that

$$\varinjlim_p H^i\,\mathrm{Hom}_{A^\circ}(F,\mathrm{Hom}_A(A/\mathfrak{m}^p,\omega)) \simeq H^i\,\mathrm{Hom}_{A^\circ}(F,\Gamma_{\mathfrak{m}}(\omega))$$

This holds since $\mathrm{Hom}_{A^\circ}(F^j,-)$ and cohomology commute with direct limits. The claim is thus proved.

Returning to the proof that $(iii) \Rightarrow (i)$, the claim shows that the cohomological dimension of $\Gamma_{\mathfrak{m}}$ is finite since the right injective dimension of ω is. It also shows that $R\Gamma_{\mathfrak{m}}$ maps $D_{\mathcal{N}}(A)$ to $D_{\mathcal{A}}(A)$ since $\mathrm{RHom}_A(-,\omega) : D^b_{\mathcal{N}}(A) \longrightarrow D^b_{\mathcal{N}}(A^\circ)$ and the Matlis dual maps $D^b_{\mathcal{N}}(A^\circ) \longrightarrow D^b_{\mathcal{A}}(A)$. Hence the χ condition holds. This proves the right-handed conditions in (i). The left-handed conditions are proved symmetrically.

This concludes the proof of theorem 1.2.6.

Definition 1.2.17 *Let $M \in D(A)$. A* minimal injective resolution *of M is a quasi-isomorphism $M \longrightarrow I$ where I is a complex of injectives with the property that the inclusion of cocycles $Z^j \hookrightarrow I^j$ is essential.*

The existence of such a resolution for $M \in D^+(A)$ is guaranteed by [Y1; lemma 4.2]. We wish to determine the shape of a minimal injective resolution of a balanced dualising complex ω, say as a right module. Since ω_A has finite injective dimension, we know it must have the form

$$0 \longrightarrow \omega_{-d} \longrightarrow \omega_{-d+1} \longrightarrow \ldots \longrightarrow \omega_e \longrightarrow 0$$

for some $d, e \in \mathbb{Z}$. We need to introduce some notation.

Definition 1.2.18 *For $M \in D(A)$, we let* $\sup(M) = \sup\{i | H^i(M) \neq 0\}$ *and* $\inf(M) = \inf\{i | H^i(M) \neq 0\}$.

We define the *injective dimension of* ω to be $\mathrm{id}\,\omega := \inf(\omega)$. By the proof of [H1; chapter I, proposition 7.6] and local duality, we see that

$$e = \sup_{M \in \mathrm{Mod}-A} \{\sup \mathrm{RHom}_A(M,\omega)\} = \sup_{M \in \mathrm{Mod}-A} \{\sup R\Gamma_{\mathfrak{m}}(M)^\vee\} = 0$$

Minimality of the resolution ensures that there is non-zero cohomology at the $-d$ term so we have in fact $d = \mathrm{id}\,\omega$. Furthermore, local duality shows that

$$\mathrm{cd}\,\Gamma_{\mathfrak{m}} = \sup_{M \in \mathrm{Mod}-A} \{-\inf R\Gamma_{\mathfrak{m}}(M)^\vee\} = \sup_{M \in \mathrm{Mod}-A} \{-\inf \mathrm{RHom}_A(M,\omega)\} = d$$

The above computation works equally well for a minimal left injective resolution of ω so we obtain,

Proposition 1.2.19 *A minimal injective resolution of a balanced dualising complex ω for A has the form*

$$0 \longrightarrow \omega_{-d} \longrightarrow \omega_{-d+1} \longrightarrow \dots \longrightarrow \omega_0 \longrightarrow 0$$

where $d = id\ \omega = cd\ \Gamma_{\mathfrak{m}} = cd\ \Gamma_{\mathfrak{m}^\circ}$.

1.3 Regular Local Rings

Throughout this section, let A be a noetherian local k-algebra with residue field k. We assume that A is complete. If no mention is made of a filtration on A, complete will mean with respect to the $\mathfrak{m}$-adic filtration.

Regular local rings of dimension two were studied in [AS]. We wish to show they are examples of rings with balanced dualising complexes. First recall,

Definition 1.3.1 *([AS; definition 1.12]) A complete local ring A is* regular of dimension two *if $gr_{\mathfrak{m}} A \simeq k\langle x, y\rangle/(Q)$ where Q is a quadratic form in x, y which is not the product of two linear forms.*

Recall also,

Definition 1.3.2 *Suppose the local ring A has finite global dimension d. Then A satisfies the* Gorenstein condition *if $RHom_A(k, A) = k[-d]$ where $[-d]$ denotes the shift by d places in the complex.*

The Gorenstein condition is equivalent to the following: given a minimal projective resolution $P \longrightarrow k_A \longrightarrow 0$ of k_A, the dual $0 \longleftarrow {}_A k \longleftarrow \mathrm{Hom}_A(P, A)$ is a minimal projective resolution of ${}_A k$. This characterisation is sometimes referred to as *Gorenstein symmetry.*

For any element $x \in A$, we will let $\overline{x}$ denote the image of x in the associated graded ring $\mathrm{gr}_{\mathfrak{m}} A$.

Proposition 1.3.3 *(Artin) A complete local ring A is regular of dimension two if and only if it is a domain which has global dimension two and satisfies the Gorenstein condition. All such rings satisfy the standard hypotheses 1.1.6 with respect to the $\mathfrak{m}$-adic filtration.*

Proof. The forward implication and the last assertion have been proved in [AS; lemma 1.4 and proposition 1.14]. We prove the reverse direction. Consider a minimal projective resolution of k_A which must be of the form

$$0 \longrightarrow A^r \longrightarrow A^s \longrightarrow A \longrightarrow k_A \longrightarrow 0$$

Since A is a noetherian domain, the quotient field $Q(A)$ exists so we may tensor the resolution with $Q(A)$. Additivity of dimension then forces $s = r + 1$. Furthermore, Gorenstein symmetry implies that $r = 1$. Hence the resolution is actually of the form

$$0 \longrightarrow A \xrightarrow{\begin{bmatrix} a \\ b \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} A \longrightarrow k_A \longrightarrow 0 \qquad (*)$$

From this we deduce that $xA + yA = \mathfrak{m}$ and thus x, y are topological generators for A. Gorenstein symmetry shows that a, b are also topological generators for A.

We wish to show that $\mathrm{gr}_{\mathfrak{m}} A$ has no zero divisors of degree one. Suppose this is not the case. Since the above resolution (*) can be built from arbitrary generators x, y of $\mathfrak{m}$, we can assume that $\overline{x}$ is a zero divisor in $\mathrm{gr}_{\mathfrak{m}} A$. Suppose $\overline{x}\,\overline{g} = 0$ and let n be the degree of $\overline{g}$. We may assume that n is minimal among all such pairs x, g. Now $xg \in \mathfrak{m}^{n+2}$ so since x, y are topological generators for A we obtain, by altering g if necessary, a relation $xg + yh = 0$ where $h \in \mathfrak{m}^{n+1}$. Exactness of the resolution (*) at the A^2 term yields an element $r \in A$ such that $\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] r$. Now $\deg r \leq \deg g - \deg a = n - 1$.

But $br = h \in \mathfrak{m}^{n+1}$ so $\overline{br} = 0$, contradicting the minimality of n. Thus there are no zero divisors of degree one in $\mathrm{gr}_{\mathfrak{m}} A$.

Now the monomials $m(x, y)$ in x, y span A topologically so their images in $\mathrm{gr}_{\mathfrak{m}} A$ span $\mathrm{gr}_{\mathfrak{m}} A$. We conclude that $\overline{x}, \overline{y}$ are generators for $\mathrm{gr}_{\mathfrak{m}} A$ since now we know that $m(\overline{x}, \overline{y}) = \overline{m(x, y)}$. Furthermore, $Q := \overline{x}\,\overline{a} + \overline{y}\overline{b} = 0$ is a quadratic relation in $\mathrm{gr}_{\mathfrak{m}} A$. If this is a defining relation, then we are done, for if Q were a product of linear factors, they would give rise to degree one zero divisors in $\mathrm{gr}_{\mathfrak{m}} A$. Consider another relation which we can write out in the form $P := \overline{x}\,\overline{g} + \overline{y}\overline{h} = 0$. We may suppose that P is a relation of minimal degree amongst those which are not in the ideal of $k\langle \overline{x}, \overline{y} \rangle$ generated by Q. By changing g and h if necessary, we may assume that in fact $xg + yh = 0$. As before, exactness of (*) at the A^2 term yields an element $r \in A$ such that $\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] r$. Since $\overline{a}, \overline{b}$ are regular, $\overline{g} = \overline{a}\,\overline{r}$ and $\overline{h} = \overline{b}\overline{r}$. Furthermore, by minimality of the degree of P, these equalities hold in $k\langle \overline{x}, \overline{y} \rangle/(Q)$. Hence in $k\langle \overline{x}, \overline{y} \rangle/(Q)$ we have $P = \overline{x}\,\overline{a}\,\overline{r} + \overline{y}\overline{b}\overline{r} = Q\overline{r} = 0$. This completes the proof of the proposition.

Following [ASch; §0] we define

Definition 1.3.4 *A complete local domain A, satisfying standard hypotheses with respect to the $\mathfrak{m}$-adic filtration is said to be* AS-regular *if it has finite global dimension and satisfies the Gorenstein condition.*

If A has finite global dimension, then A is a dualising complex for A. We wish to show that if A also satisfies the Gorenstein condition then we can modify A into a balanced dualising complex. For this we need a little more theory.

Recall that an invertible bimodule is a bimodule L for which there exists an inverse bimodule L^{-1}, i.e. $L \otimes_A L^{-1} \simeq A \simeq L^{-1} \otimes_A L$. Morita theory ensures that L is finitely generated projective on the left and right. Since A is local noetherian, rank considerations force $L_A \simeq A_A$ and ${}_A L \simeq {}_A A$. Now the only left A-module structure we can impose on A_A which is compatible with the right module structure is $a.x = \phi(a)x$ for $a, x \in A$ where on the left we have scalar multiplication, on the right we have multiplication in A and ϕ is a ring endomorphism. We denote such a bimodule by $A(\phi)$. If ϕ is an automorphism, then ${}_A A(\phi) \simeq {}_A A$ and in fact $A(\phi)$ is invertible for $A(\phi^{-1})$ is an inverse. The converse is true too for suppose we have an isomorphism ${}_A A \xrightarrow{\sim} {}_A A(\phi)$ which maps 1 to e. Then as a runs through the elements of A, $\phi(a)e$ runs through the elements of A once and once only. Hence ϕ must be an automorphism.

We assume for the rest of the section that A is a complete local ring satisfying standard hypotheses 1.1.6 with respect to some filtration.

The category $\mathcal{T}(A)$ of $\mathfrak{m}$-torsion modules is closed under direct limits and so by [G; chapitre III, proposition 8] is a localising subcategory of $\mathrm{Mod}-A$. Hence the quotient functor $\pi : \mathrm{Mod}-A \longrightarrow (\mathrm{Mod}-A)/\mathcal{T}(A)$ has a right adjoint which we shall denote by ρ. Recall that for every $M \in \mathrm{Obj}\ (\mathrm{Mod}-A)/\mathcal{T}(A)$, $\rho(M)$ is $\mathfrak{m}$-torsionfree. The following result is well-known.

Proposition 1.3.5 *The injective hull of an $\mathfrak{m}$- torsion module M is $Hom_A(k, M) \otimes A^{\vee}$. Consequently, every injective I in $Mod-A$ has the form $Hom_A(k, I) \otimes A^{\vee} \oplus \rho(E)$ where E is an injective in $(Mod-A)/\mathcal{T}(A)$.*

Proof. Since A is noetherian, direct limits of injectives are injective. Hence we may extend the natural essential extension $\mathrm{Hom}_A(k, M) \longrightarrow \mathrm{Hom}_A(k, M) \otimes A^{\vee}$ to an essential morphism $f : M \longrightarrow \mathrm{Hom}_A(k, M) \otimes A^{\vee}$. By construction, the socle of $\ker f$ is $\mathrm{socle}(M) \cap \ker f = 0$. Hence $\ker f = 0$ and f is indeed an essential injection. The last statement now follows from [G; chapitre III, corollaire 2 of proposition 6].

The following result of Yekutieli's shows, amongst other things, a relationship between the Gorenstein condition and the condition of being balanced.

Proposition 1.3.6 *Let ω be a dualising complex for A. Then the following are equivalent:*

i. $RHom_A(k, \omega) \simeq k$ *in* $D(A^e)$.

ii. $RHom_{A^\circ}(k, \omega) \simeq k$ *in* $D(A^e)$.

iii. $R\Gamma_{\mathfrak{m}}(\omega) \simeq A^\vee \otimes_A L$ *in* $D(A^e)$ *for some invertible bimodule* L.

iv. $R\Gamma_{\mathfrak{m}^\circ}(\omega) \simeq A^\vee \otimes_A L$ *in* $D(A^e)$ *for some invertible bimodule* L.

Proof. (adapted from [Y1; proposition 4.4]) (i) and (ii) are equivalent for assuming (i), duality gives $\mathrm{RHom}_{A^\circ}(k,\omega) \simeq \mathrm{RHom}_{A^\circ}(\mathrm{RHom}_A(k,\omega),\omega) \simeq k$. We now show that *(i)*, *(ii)* $\Rightarrow$ *(iii)*. Let I be a minimal injective resolution of ω. Now proposition 1.3.5 shows that $\Gamma_{\mathfrak{m}}(I^j)_A \simeq \mathrm{Hom}_A(k,I^j) \otimes A^\vee$. Minimality ensures that the differentials in $\mathrm{Hom}_A(k,I)$ are zero so condition (i) tells us that $\Gamma_{\mathfrak{m}}(I^0) \simeq A^\vee$ and all other $\Gamma_{\mathfrak{m}}(I^j)$ are zero. Hence $R\Gamma_{\mathfrak{m}}(\omega) \simeq A^\vee$ as a right module. If we can show that $R\Gamma_{\mathfrak{m}}(\omega) \simeq A^\vee$ as a left module, then we are done, for then its Matlis dual must be isomorphic to A as a left module and a right module, and hence invertible as a bimodule. To this end, observe first that induction on conditions (i) and (ii) imply that $\mathrm{RHom}_A(-,\omega)$ and $\mathrm{RHom}_{A^\circ}(-,\omega)$ restrict to a duality on the subcategory of finite length modules. We denote these restrictions more precisely by $R^0\,\mathrm{Hom}_A(-,\omega)$ and $R^0\,\mathrm{Hom}_{A^\circ}(-,\omega)$. Then

$$\mathrm{Hom}_{A^\circ}(k, R\Gamma_{\mathfrak{m}}(\omega)) = \mathrm{Hom}_{A^\circ}(k, R^0\Gamma_{\mathfrak{m}}(\omega)) \simeq \varinjlim_q \mathrm{Hom}_{A^\circ}(k, R^0\,\mathrm{Hom}_A(A/\mathfrak{m}^q,\omega))$$

$$\simeq \varinjlim_q \mathrm{Hom}_A(A/\mathfrak{m}^q, R^0\,\mathrm{Hom}_{A^\circ}(k,\omega)) \simeq \varinjlim_q \mathrm{Hom}_A(A/\mathfrak{m}^q, k) = k$$

From lemma 1.1.20 we have $\mathrm{Hom}_A(A^\vee, A^\vee) = A$ so $R\Gamma_{\mathfrak{m}}(\omega)$ must be of the form $A(\phi) \otimes_A A^\vee$ where ϕ is a ring endomorphism of A. We see in particular that $R\Gamma_{\mathfrak{m}}(\omega)$ is torsion on the left. Proposition 1.3.5 and the above socle computation show that $R\Gamma_{\mathfrak{m}}(\omega)$ embeds in $A^\vee$ as a left A-module so it remains to see that the embedding is surjective. Observe that the dimension over k of the right annihilator of $\mathfrak{m}^p$ in $R\Gamma_{\mathfrak{m}}(\omega) \simeq A(\phi) \otimes_A A^\vee$ is at least the dimension of the right annihilator of $\mathfrak{m}^p$ in $A^\vee$ since ϕ preserves powers of $\mathfrak{m}$. Hence ${}_A R\Gamma_{\mathfrak{m}}(\omega)$ maps onto the whole of $A^\vee$ so (iii) holds.

Finally, by symmetry, it suffices to show that *(iii)* $\Rightarrow$ *(i)*. We have by lemma 1.2.15,

$$\mathrm{RHom}_A(k,\omega) \simeq \mathrm{RHom}_A(k, R\Gamma_{\mathfrak{m}}(\omega)) \simeq \mathrm{RHom}_A(k, A^\vee \otimes_A L) \simeq \mathrm{RHom}_A(k \otimes_A L^{-1}, A^\vee) \simeq k^\vee = k$$

We show finally that the condition of being balanced depends only on the right or left local cohomology.

Theorem 1.3.7 *([Y1; theorem 4.8]) Let* ω *be a dualising complex with* $R\Gamma_{\mathfrak{m}}(\omega) \simeq A^\vee$ *as bimodules. Then* ω *is balanced.*

Proof. By the previous proposition, $R\Gamma_{\mathfrak{m}^\circ}(\omega) \simeq A^\vee \otimes_A L$ for some invertible bimodule L. We need to prove that $L = A$. The weak version of local duality as expressed in claim 1.2.16 shows that $\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}(A), A^\vee) \simeq \omega$ in $D(A^e)$. Making repeated use of lemma 1.2.15 yields

$$L \simeq \mathrm{RHom}_{A^\circ}(A^\vee, R\Gamma_{\mathfrak{m}^\circ}(\omega)) \simeq \mathrm{RHom}_{A^\circ}(A^\vee, \omega) \simeq \mathrm{RHom}_A(\mathrm{Hom}_{A^\circ}(\omega, A^\vee), A) \simeq$$

$$\mathrm{RHom}_A(\mathrm{Hom}_{A^\circ}(\omega, A^\vee), R\Gamma_{\mathfrak{m}}(A)) \simeq \mathrm{RHom}_{A^\circ}(\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}(A), A^\vee), \omega) \simeq \mathrm{RHom}_{A^\circ}(\omega,\omega) \simeq A$$

as desired.

Corollary 1.3.8 *Suppose that* A *has finite injective dimension as a left module and as a right module and that* $RHom_A(k,A) \simeq k[-d]$ *for some* $d \in \mathbb{Z}$. *Then* A *has a balanced dualising complex of the form* $L[d]$ *where* L *is an invertible bimodule. Furthermore,* d *is the injective dimension of* A *as a right module or a left module.*

Proof. (taken from [Y1; corollary 4.10]) Note first that $A[d]$ is a dualising complex for A. Since it satisfies condition (i) of proposition 1.3.6 we see that $R\Gamma_{\mathfrak{m}}(A)[d] = L^{-1} \otimes_A A^\vee$ for some invertible bimodule L. Then $R\Gamma_{\mathfrak{m}}(L[d]) = L[d] \otimes_A L^{-1} \otimes_A A^\vee[-d] = A^\vee$. Furthermore, $L[d]$ is a dualising complex which, by the previous theorem, is balanced. From the shape of a minimal injective resolution of ω given in proposition 1.2.19, we see that $d = \text{id } A_A = \text{id}_A A$.

Finally, we prove as promised,

Corollary 1.3.9 *Let A be a complete local domain satisfying standard hypotheses with respect to the $\mathfrak{m}$-adic filtration. Then A is AS-regular if and only if it has finite global dimension and a balanced dualising complex.*

Proof. The forward direction follows from the previous corollary. For the reverse direction, first recall Yekutieli's

Theorem 1.3.10 *([Y1; theorem 3.9]) Let ω be a dualising complex for A. Then ω' is a dualising complex for A if and only if $\omega' \simeq \omega \otimes_A L[d]$ in $D(A^e)$ for some invertible bimodule L and $d \in \mathbb{Z}$.*

Comment on Proof. The proof in the graded case given in [Y1] carries over to the local case.

Since, A is a dualising complex, the above theorem shows that the balanced dualising complex is $\omega = L[d]$ for some invertible bimodule L. Then proposition 1.3.6 shows that $\text{RHom}_A(k, A[d]) = k$ so the projective dimension of k is pd $k = d$. Now pd $k =$ gl.dim A by [McR; corollary 7.1.14] so the Gorenstein condition does indeed hold.

1.4 Existence Theorems for Balanced Dualising Complexes

In this section, we use theorem 1.2.6 to show how the existence of a balanced dualising complex for some particular ring, guarantees the existence of a balanced dualising complex of various related rings. Throughout this section, let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be complete local rings satisfying standard hypotheses 1.1.6.

A graded version of the following result has already appeared in [J2; lemma 3.1].

Proposition 1.4.1 *Suppose there is a local homomorphism $A \longrightarrow B$ such that B is a finite A-module on the left and on the right. Let res denote the restriction functor from B-modules to A-modules. Then there is a natural isomorphism*

$$res \circ R\Gamma_{\mathfrak{n}} \simeq R\Gamma_{\mathfrak{m}} \circ res$$

of functors from $D^+(B) \longrightarrow D^+(A)$ or from $D^+(B^e) \longrightarrow D^+(B^\circ \otimes A)$.

Proof. Note first that $\text{res} \circ \Gamma_{\mathfrak{n}} \simeq \Gamma_{\mathfrak{m}} \circ \text{res}$ since B is finitely generated over A. Because restriction is an exact functor, it suffices by Grothendieck's theorem to show that restriction maps any B-injective I to a $\Gamma_{\mathfrak{m}}$-acyclic. Now by proposition 1.3.5, we are reduced to two cases: i) I is $\mathfrak{n}$-torsionfree and ii) I is a direct sum of $B^\vee$'s. In the first case we consider the spectral sequence

$$\text{Ext}^i_B(\text{Tor}^A_j(A/\mathfrak{m}^p, B), I) \Longrightarrow \text{Ext}^{i+j}_A(A/\mathfrak{m}^p, I)$$

Since I is B-injective, the sequence collapses to

$$\text{Hom}_B(\text{Tor}^A_j(A/\mathfrak{m}^p, B), I) \simeq \text{Ext}^j_A(A/\mathfrak{m}^p, I)$$

Now $\text{Tor}^A_j(A/\mathfrak{m}^p, B)$ is finite dimensional over k so since we are assuming that I is $\mathfrak{n}$-torsionfree, the left hand side must be zero. Taking direct limits, we see that $R\Gamma_{\mathfrak{m}}(I) = 0$. In the second case $R\Gamma_{\mathfrak{m}}(I) = I$ by lemma 1.2.9 so I is again $\Gamma_{\mathfrak{m}}$-acyclic and the proposition follows.

I wish to thank Amnon Yekutieli for suggesting the following proof of the adjunction formula.

Corollary 1.4.2 *(Adjunction Formula for Finite Morphisms) Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be as in the previous proposition. If A has a balanced dualising complex ω_A, then B has a balanced dualising complex and it is given by*

$$\omega_B \simeq RHom_A(B, \omega_A) \quad in \quad D(A^\circ \otimes B)$$

$$\omega_B \simeq RHom_{A^\circ}(B,\omega_A) \quad in \quad D(B^\circ \otimes A)$$

Proof. By theorem 1.2.6, existence of the dualising complex is guaranteed once we show that cd $\Gamma_{\mathfrak{m}}$, cd $\Gamma_{\mathfrak{m}^\circ}$ are finite and A, A° satisfy χ. Every noetherian B-module is noetherian as an A-module while every B-module which is artinian as an A-module is artinian as a B-module. Hence, existence of the balanced dualising complex follows from the previous proposition and its left-handed counterpart.

Now by local duality (theorem 1.2.6) and the previous proposition,

$$\mathrm{RHom}_A(B,\omega_A) \simeq R\Gamma_{\mathfrak{m}}(B)^\vee \simeq R\Gamma_{\mathfrak{n}}(B)^\vee \simeq \omega_B \quad \text{in } D(A^\circ \otimes B)$$

where the second isomorphism holds since the two Matlis duals are just the k-linear duals by proposition 1.2.12. The other adjunction formula is similar.

Lemma 1.4.3 *Let $(A,\mathfrak{m})$ and $(B,\mathfrak{n})$ be as in proposition 1.4.1. For every noetherian B-module M, $Hom_A(M, A^\vee) \simeq Hom_B(M, B^\vee)$.*

Proof. Since M is noetherian as an A-module too, we need only show that the continuous dual of M as an A-module and as a B-module agree. Equivalently, we must show that the good filtrations of M as an A-module and as a B-module are cofinal. This follows from lemma 1.1.23 since the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic filtrations are cofinal.

The next result was communicated to me by James Zhang,

Corollary 1.4.4 *Let $(A,\mathfrak{m})$ and $(B,\mathfrak{n})$ be as in proposition 1.4.1 and suppose further that the map $A \longrightarrow B$ is split as a morphism of right A-modules and of left A-modules. If B has a balanced dualising complex ω_B, then A has a balanced dualising complex ω_A which is a direct summand of ω_B when considered as an object in $D(A)$ or $D(A^\circ)$.*

Proof. We wish to verify the χ condition and bound the cohomological dimension of $\Gamma_{\mathfrak{m}}$. Let $M \in$ Obj $\mathcal{N}(A)$ and let $N = M \otimes_A B \in$ Obj $\mathcal{N}(B)$. Since $A \longrightarrow B$ splits, there exists a right A-module M' such that $N \simeq M \oplus M'$. The proposition gives an isomorphism $R\Gamma_{\mathfrak{n}}(N) \simeq R\Gamma_{\mathfrak{m}}(M) \oplus R\Gamma_{\mathfrak{m}}(M')$ in $D(A)$. This shows that cd $\Gamma_{\mathfrak{m}} \leq$ cd $\Gamma_{\mathfrak{n}}$ since local cohomology commutes with direct limits. For the χ condition, it suffices to show that every artinian B-module is artinian as an A-module. By lemmas 1.1.23 and 1.4.3, the Matlis dual of an $\mathfrak{n}$-torsion or noetherian B-module M, is the same as the Matlis dual of M regarded as an A-module. Suppose that M is an artinian B-module. Then $M^\vee$ is a noetherian B-module and hence a noetherian A-module. Thus by duality, M is an artinian A-module. The left-handed conditions are proved similarly.

It remains now to prove the last assertion. Let $B \simeq A \oplus A'$ be a direct sum decomposition of right A-modules. Then we have the following isomorphisms in $D(A)$

$$\omega_B \simeq R\Gamma_{\mathfrak{n}}(B)^\vee \simeq R\Gamma_{\mathfrak{m}}(B)^\vee \simeq R\Gamma_{\mathfrak{m}}(A)^\vee \oplus R\Gamma_{\mathfrak{m}}(A')^\vee \simeq \omega_A \oplus R\Gamma_{\mathfrak{m}}(A')^\vee$$

as desired.

We next give another characterisation of the χ condition which is closer to Zhang's original definition. It is a mild extension of [AZ; proposition 7.7].

Proposition 1.4.5 *Let M be a noetherian A-module. Then $R\Gamma_{\mathfrak{m}}(M)$ has artinian cohomology if and only if $Ext^i_A(k, M)$ is finite dimensional for all i. Hence A satisfies χ if and only if $RHom_A(k, -)$ is a functor from $D^+_{\mathcal{N}}(A) \longrightarrow D^+_{\mathcal{N}}(k)$.*

Proof. Let $0 \longrightarrow M \longrightarrow I$ be a minimal injective resolution. By proposition 1.3.5, $\Gamma_{\mathfrak{m}}(I^i) = \mathrm{Hom}_A(k, I^i) \otimes_A A^\vee$. But since the resolution is minimal, the differential in $\mathrm{Hom}_A(k, I)$ is zero so $\mathrm{Hom}_A(k, I^i) = \mathrm{Ext}^i_A(k, M)$. The reverse implication follows immediately.

We prove the forward implication by induction on i. The case $i = 0$ follows from the fact that $\mathrm{Hom}_A(k, M)$ is $\mathfrak{m}$-torsion noetherian and thus finite dimensional. We now prove the inductive step. Consider the exact sequence

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^p \longrightarrow A/\mathfrak{m}^p \longrightarrow k \longrightarrow 0$$

Taking direct limits of long exact sequences we obtain the exact sequence

$$\varinjlim_{p} \operatorname{Ext}_A^{i-1}(\mathfrak{m}/\mathfrak{m}^p, M) \longrightarrow \operatorname{Ext}_A^i(k, M) \longrightarrow H_{\mathfrak{m}}^i(M)$$

To show that $\operatorname{Ext}_A^i(k, M)$ is finite dimensional, it suffices to prove that it is artinian since it is annihilated by $\mathfrak{m}$. Since we are assuming that $H_{\mathfrak{m}}^i(M)$ is artinian we must prove that $\varinjlim \operatorname{Ext}_A^{i-1}(\mathfrak{m}/\mathfrak{m}^p, M)$ is artinian. Now $\varinjlim \operatorname{Ext}_A^{i-1}(\mathfrak{m}/\mathfrak{m}^p, M)$ is the $(i-1)$-th cohomology of the complex $\varinjlim \operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^p, I)$ which in turn is a subcomplex of $\operatorname{Hom}_A(\mathfrak{m}, \Gamma_{\mathfrak{m}}(I))$. Hence it suffices to show that $\operatorname{Hom}_A(\mathfrak{m}, \Gamma_{\mathfrak{m}}(I^{i-1}))$ is artinian. This module is $\mathfrak{m}$-torsion as $\mathfrak{m}$ is finitely generated so by proposition 1.3.5, we need only show that its socle is finite dimensional. Choose n so that $\mathfrak{m}/\mathfrak{m}^2 \simeq k^n$ as right A-modules. Then

$$\operatorname{Hom}_A(k, \operatorname{Hom}_A(\mathfrak{m}, \Gamma_{\mathfrak{m}}(I^{i-1}))) = \operatorname{Hom}_A(k \otimes_A \mathfrak{m}, \Gamma_{\mathfrak{m}}(I^{i-1})) =$$

$$\operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, I^{i-1}) = \operatorname{Hom}_A(k, I^{i-1})^n = \operatorname{Ext}_A^{i-1}(k, M)^n$$

which is finite dimensional by the induction hypothesis. This completes the proof of the proposition.

Theorem 1.4.6 *([AZ; theorem 8.8]) Let $x \in \mathfrak{m}$ be a regular normal element. Then A has a balanced dualising complex if and only if A/xA does.*

Proof. Let $B = A/xA$ and $\mathfrak{n}$ be its maximal ideal. Since $A \longrightarrow B$ is a finite local homomorphism of rings, B has a balanced dualising complex if A does by corollary 1.4.2.

To show the converse, we check the χ condition and the cohomological dimension of the torsion functors. As was noted in [AZ; proposition 3.12(3)], to show the χ condition holds for a ring A, it suffices to show that for every noetherian module M, there exists a non-zero submodule M' such that $R\Gamma_{\mathfrak{m}}(M')$ has artinian cohomology. Indeed, we may choose M' maximal with respect to the above property, and observe that we must have $M' = M$ or else we would be able to find a submodule M'' properly containing M', such that $R\Gamma_{\mathfrak{m}}(M''/M')$ and hence $R\Gamma_{\mathfrak{m}}(M'')$ have artinian cohomology.

Let L be the invertible bimodule xA. We consider the exact sequence

$$0 \longrightarrow K \longrightarrow M \otimes_A L \xrightarrow{\mu} M \longrightarrow C \longrightarrow 0 \qquad (*)$$

where μ is the restriction of the module multiplication map and K and C are the kernel and cokernel respectively. Note that if one identifies M with $M \otimes_A L$ via $m \mapsto m \otimes_A x$ then μ is just the multiplication by x map. Now $- \otimes_A L$ is a category equivalence which commutes with $\Gamma_{\mathfrak{m}}$ so $R\Gamma_{\mathfrak{m}}(M \otimes_A L) \simeq R\Gamma_{\mathfrak{m}}(M) \otimes_A L$.

Note that K is the kernel of a multiplication by x map and so is a noetherian B-module. Hence, by proposition 1.4.1, $R\Gamma_{\mathfrak{m}}(K) = R\Gamma_{\mathfrak{n}}(K)$ which has artinian cohomology since B satisfies χ by theorem 1.2.6. Thus to verify the χ condition we may assume that $K = 0$. We will use the Ext condition of the previous proposition. First note that the map $\operatorname{Ext}_A^i(k, M \otimes_A L) \longrightarrow \operatorname{Ext}_A^i(k, M)$ is zero since we have assumed that x annihilates k. Hence the long exact sequence stemming from (*) breaks up to give the short exact sequences

$$0 \longrightarrow \operatorname{Ext}_A^i(k, M) \longrightarrow \operatorname{Ext}_A^i(k, C) \longrightarrow \operatorname{Ext}_A^{i+1}(k, M \otimes_A L) \longrightarrow 0$$

It suffices to show that the middle term is finite dimensional. This follows from the previous proposition since C is a noetherian B-module. We have thus verified that A satisfies the χ condition. The χ condition for A° follows similarly.

It remains to bound the cohomological dimensions of $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}^{\circ}}$. Let $d = \operatorname{cd} \Gamma_{\mathfrak{n}}$. We break up (*) into the short exact sequences

$$0 \longrightarrow K \longrightarrow M \otimes_A L \longrightarrow N \longrightarrow 0 \quad , \quad 0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0$$

Let $i > d+1$. Then the long exact sequences in cohomology show that

$$H^i_{\mathfrak{m}}(M) \otimes_A L \xrightarrow{\sim} H^i_{\mathfrak{m}}(N) \xrightarrow{\sim} H^i_{\mathfrak{m}}(M)$$

Since $H^i_{\mathfrak{m}}(M)$ is artinian, multiplication by x must annihilate the socle. The above isomorphisms show that $H^i_{\mathfrak{m}}(M)$ has zero socle and thus $H^i_{\mathfrak{m}}(M)$ is itself zero. Hence cd $\Gamma_{\mathfrak{m}} \leq$ cd $\Gamma_{\mathfrak{n}}+1$. Similarly, cd $\Gamma_{\mathfrak{m}^\circ}$ is bounded.

1.5 Reflexive and Cohen-Macaulay Modules

In this section, we collect some well-known results about reflexive modules and introduce Cohen-Macaulay modules. Throughout, let A denote a noetherian ring and D denote the A-linear dual $\mathrm{Hom}_A(-, A)$ or $\mathrm{Hom}_{A^\circ}(-, A)$ as the case may be.

Definition 1.5.1 *A finitely generated A-module is said to be* reflexive *if the natural morphism $M \longrightarrow DD(M)$ is an isomorphism.*

Recall that if A is semiprime Goldie and $Q(A)$ is its ring of fractions, then a module M is said to be *torsionfree* if the natural morphism $M \longrightarrow M \otimes_A Q(A)$ is an injection, and *torsion* if the map is zero. There are many ways to characterise torsionfree modules.

Proposition 1.5.2 *Let A be a noetherian domain and M be a finitely generated right A-module. Then*

$$\ker(M \longrightarrow DD(M)) = \ker(M \longrightarrow M \otimes_A Q(A))$$

Hence, the following are equivalent:

i. M is torsionfree.

ii. The natural morphism $M \longrightarrow DD(M)$ is an injection.

iii. M can be embedded in a finitely generated free module F.

In (iii), the free module F may be chosen so that F/M is torsion.

Proof. We first prove equality of the kernels which will give the equivalence of (i) and (ii).

Let $m \in \ker(M \longrightarrow M \otimes_A Q(A))$ and $f \in \mathrm{Hom}_A(M, A)$. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ M \otimes_A Q(A) & \xrightarrow{f \otimes_A Q(A)} & Q(A) \end{array}$$

Since the vertical morphism on the right is injective, $f(m) = 0$ so $\ker(M \longrightarrow M \otimes_A Q(A)) \subseteq \ker(M \longrightarrow DD(M))$. Suppose now that $m \notin \ker(M \longrightarrow M \otimes_A Q(A))$. Since $Q(A)$ is a skew-field, we have an isomorphism $M \otimes_A Q(A) \simeq Q(A)^n$ for some n. Further, as M is finitely generated, we have a commutative diagram of the form

$$\begin{array}{ccc} M & \longrightarrow & M \otimes_A Q(A) \\ f\downarrow & & \downarrow\wr \\ A^n & \xrightarrow{\lambda_x} & Q(A)^n \end{array}$$

in Mod$-A$ where λ_x is left multiplication by some $x \in Q(A)$. We must have $f(m) \neq 0$ so we can find a homomorphism $g : A^n \longrightarrow A$ such that the composite $(g \circ f)(m) \neq 0$. Hence $m \notin \ker(M \longrightarrow$

$DD(M)$) and the kernels must coincide. If (i) holds then f above gives the desired embedding of (iii) with $A^n/f(M)$ torsion. Conversely, (iii) implies (i) since free modules are torsionfree.

The next proposition gives alternative characterisations of reflexivity.

Proposition 1.5.3 *Let A be a noetherian domain and M be a finitely generated right A-module. The following are equivalent:*

i. M is reflexive.

ii. There exists a finitely generated free module F and a torsionfree module G such that M fits into an exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$$

iii. There exists a reflexive module F and a torsionfree module G such that M fits into an exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$$

Proof. (adapted from [H2; proposition 1.1]) First assume that M is reflexive. Consider a finitely generated presentation of D(M)

$$F_1 \longrightarrow F_0 \longrightarrow D(M) \longrightarrow 0$$

Taking the A-linear dual gives an exact sequence

$$0 \longrightarrow M \longrightarrow D(F_0) \longrightarrow D(F_1)$$

This implies condition (ii) by the previous proposition. Free modules are reflexive so *(ii)* $\Rightarrow$ *(iii)*. It only remains to prove (i) assuming (iii). Note that M is torsionfree. Consider the dual exact sequence

$$0 \longrightarrow D(G) \longrightarrow D(F) \longrightarrow D(M) \longrightarrow N \longrightarrow 0$$

where N is the appropriate submodule of $\mathrm{Ext}^1_A(G, A)$. Dualising again we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow & & \\ D(N) & \longrightarrow & DD(M) & \longrightarrow & DD(F) & \longrightarrow & DD(G) & & \end{array}$$

where the top row is exact but the bottom row is just a complex which is exact at the $DD(M)$ term.

We first observe that N is torsion. Indeed, by the previous proposition, we can find a finitely generated free module P containing G with P/G torsion. Since $Q(A)$ is flat over A we have

$$Q(A) \otimes_A \mathrm{Ext}^1_A(G, A) \simeq Q(A) \otimes_A \mathrm{Ext}^2_A(P/G, A) \simeq \mathrm{Ext}^2_{Q(A)}(P/G \otimes_A Q(A), Q(A)) = 0$$

N being a submodule of $\mathrm{Ext}^1_A(G, A)$ must be torsion. Hence proposition 1.5.2 shows that $D(N) = 0$ and so the map $DD(M) \longrightarrow DD(F)$ is injective. The previous proposition also shows that $M \longrightarrow DD(M)$ and $G \longrightarrow DD(G)$ are injections. A diagram chase now completes the proof.

Proposition 1.5.4 *Let A, B be noetherian domains, M, a finitely generated right A-module and L, a (B, A)-bimodule such that ${}_BL$ is reflexive. Then $Hom_A(M, L)$ is also a reflexive B-module.*

Proof. Consider a presentation

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

Taking homomorphisms into L gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, L) \longrightarrow L^n \longrightarrow L^m$$

Since L^n is reflexive by assumption and every submodule of L^m is torsionfree, $\mathrm{Hom}_A(M, L)$ is reflexive by proposition 1.5.3.

Following [Y2; definition 1.4], we consider the

Definition 1.5.5 *Let A be a local ring satisfying standard hypotheses 1.1.6. Given a finitely generated A-module M, we define the* canonical dimension *of M to be $\delta(M) = \sup(R\Gamma_{\mathfrak{m}}(M))$ and the* depth *of M to be $depth(M) = \inf(R\Gamma_{\mathfrak{m}}(M))$.*

Definition 1.5.6 *Let A be a local ring satisfying standard hypotheses 1.1.6 such that cd $\Gamma_{\mathfrak{m}}$ is finite. Let M be a finitely generated A-module. M is said to be* Cohen-Macaulay *if $\delta(M) = depth(M)$. If $depth(M) = cd\, \Gamma_{\mathfrak{m}}$ then M is said to be* maximal Cohen-Macaulay.

If A has a balanced dualising complex ω, then by local duality we may use $\mathrm{RHom}_A(-, \omega)$ instead of $R\Gamma_{\mathfrak{m}}$ to test if a module is Cohen-Macaulay. A module M is Cohen-Macaulay if and only if its dual $\mathrm{RHom}_A(M, \omega)$ has only one non-zero cohomology group.

Definition 1.5.7 *Let A be a complete local ring with balanced dualising complex ω. We say that A is* AS-Cohen-Macaulay *if ω is isomorphic to a shift of a bimodule. If this bimodule is invertible then we say that A is* AS-Gorenstein.

Remark: If A is AS-Cohen-Macaulay then $\delta(A) = \mathrm{depth}(A)$ and we call the common number the *dimension* of A. As an A-module, A is maximal Cohen-Macaulay since a minimal injective resolution for its dual ω has the form

$$0 \longrightarrow \omega_{-d} \longrightarrow \omega_{-d+1} \longrightarrow \ldots \longrightarrow \omega_0 \longrightarrow 0$$

where $d = \mathrm{cd}\, \Gamma_{\mathfrak{m}}$ by proposition 1.2.19.

Remark: By corollary 1.3.8, a complete local ring A satisfying standard hypotheses 1.1.6 is AS-Gorenstein if and only if A has finite injective dimension on the left and right and $\mathrm{RHom}_A(k, A) \simeq k[-d]$ for some $d \in \mathbb{Z}$. In this case, the dimension of A is d which is also the injective dimension of A as a left or right module. If A is AS-regular, then the dimension of A is also the global dimension of A by corollary 1.3.9.

Here is another classical result from commutative algebra.

Proposition 1.5.8 *Let A be an AS-Cohen-Macaulay domain of dimension two. If M is reflexive then it is maximal Cohen-Macaulay.*

Proof. As per proposition 1.5.3, we consider an exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$$

where F is finitely generated free and G is torsionfree. We examine the associated long exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(F) \longrightarrow H^0_{\mathfrak{m}}(G) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow H^1_{\mathfrak{m}}(F)$$

Now G is $\mathfrak{m}$-torsionfree so $H^0_{\mathfrak{m}}(G) = 0$ while F is maximal Cohen-Macaulay by the above remark so $H^0_{\mathfrak{m}}(F) = H^1_{\mathfrak{m}}(F) = 0$. Hence $H^0_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(M) = 0$ as was to be shown.

There is a natural noncommutative analogue of the Auslander-Buchsbaum formula. In the graded case, this has already been established by Jørgensen in [J1] and presumably the same proof works. We present a different one.

Theorem 1.5.9 *(Auslander-Buchsbaum formula) Let A be a complete local ring with balanced dualising complex and M be a finitely generated module of finite projective dimension. Then the projective dimension is given by*

$$pd\ M = depth(A) - depth(M)$$

Proof. Consider a minimal free resolution $P \longrightarrow M \longrightarrow 0$ of M so that P has the form

$$\ldots 0 \longrightarrow P^s \longrightarrow P^{s-1} \longrightarrow \ldots \longrightarrow P^0 \longrightarrow 0 \ldots$$

where $s = \mathrm{pd}(M)$. Let X be a complex of bimodules representing ω of the form

$$\ldots \longrightarrow X^{-r-1} \longrightarrow X^{-r} \longrightarrow 0 \longrightarrow \ldots$$

where $r = \mathrm{depth}(A)$. We use $\mathrm{Hom}_A(P, X)$ to compute $\sup(\mathrm{RHom}_A(M, \omega))$. The top right corner of the double complex is

$$\begin{array}{ccccc} 0 & & 0 & & \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}_A(P^s, X^{-r-1}) & \longrightarrow & \mathrm{Hom}_A(P^s, X^{-r}) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \\ \mathrm{Hom}_A(P^{s-1}, X^{-r-1}) & \longrightarrow & \mathrm{Hom}_A(P^{s-1}, X^{-r}) & \longrightarrow & 0 \end{array}$$

If the natural morphism

$$\phi : \mathrm{Hom}_A(P^{s-1}, X^{-r}) \longrightarrow \mathrm{Hom}_A(P^s, X^{-r})/d\,\mathrm{Hom}_A(P^s, X^{-r-1}) \simeq \mathrm{Hom}_A(P^s, X^{-r}/dX^{-r-1})$$

is not surjective then the double complex has non-zero cohomology in degree $s - r$. Hence by local duality, $\mathrm{depth}(M) = -(s - r)$ and the proof will be complete. Now the morphism ϕ factors via

$$\mathrm{Hom}_A(P^{s-1}, X^{-r}) \longrightarrow \mathrm{Hom}_A(P^{s-1}, X^{-r}/dX^{-r-1}) \longrightarrow \mathrm{Hom}_A(P^s, X^{-r}/dX^{-r-1})$$

The second morphism in the composition is not surjective since P is minimal and X^{-r}/dX^{-r-1} is finitely generated on the right. Consequently, ϕ is not surjective either and we are done.

Corollary 1.5.10 *Let A be an AS-regular ring (definition 1.3.4). Then every maximal Cohen-Macaulay module is free.*

1.6 Krull-Schmidt Theorem

In this section, we prove a version of the Krull-Schmidt theorem which applies to the rings we will study in the following chapters. Throughout, A will denote a complete local ring satisfying the standard hypotheses 1.1.6.

Lemma 1.6.1 *Let M be an A-module with a good filtration F. Suppose further that for every endomorphism $f \in End(M)$, $f(F^pM) \subseteq F^pM$ for any $p \in \mathbb{Z}$. Then $End(M)$ is a local ring if M is indecomposable.*

Proof. Let E denote the endomorphism ring $\mathrm{End}(M)$. We filter E by declaring $f \in F^pE$ if $f(F^qM) \subseteq F^{p+q}M$ for all q. This turns E into a filtered ring with $F^0E = E$. It is complete since M is.

We wish to show that any idempotent $\bar{e}$ of E/F^pE can be lifted to $E/F^{2p}E$. Let $e \in E$ represent $\bar{e}$ so that $e^2 - e \in F^pE$. Then setting $x = (1-2e)(e^2-e) \in F^pE$, we see that $e + x$ is an idempotent modulo $F^{2p}E$ lifting $\bar{e}$ for

$$(e + x)^2 - (e + x) = (e^2 - e) + (2e - 1)x + x^2 = -4(e^2 - e)^2 + x^2$$

which lies in $F^{2p}E$. Since E is complete, we can by induction lift any idempotent of $\overline{E} = E/F^1E$ to E. E has no non-trivial idempotents as M is indecomposable so neither does $\overline{E}$.

Now $\overline{E}$ embeds in the finite dimensional space $\mathrm{Hom}_{\mathrm{Gr}-A}(\mathrm{gr}\, M, \mathrm{gr}\, M)$ where Gr-A denotes the category of graded A-modules (and degree zero morphisms). Hence $\overline{E}$ is artinian and $\overline{E}/\mathrm{rad}\ \overline{E}$ cannot have any non-trivial idempotents either. Being semisimple, $\overline{E}/\mathrm{rad}\ \overline{E}$ must then be a skew field. Now $F^1E \subseteq \mathrm{rad}\ E$ as E is complete so $E/\mathrm{rad}\ E = \overline{E}/\mathrm{rad}\ \overline{E}$ and E is indeed local.

Corollary 1.6.2 *The category* mod $- A$ *of noetherian A-modules is Krull-Schmidt.*

Proof. By [Jac; theorem 3.6] it suffices to show that the endomorphism ring of any finitely generated indecomposable A-module M is local. Consider a surjection of the form $A^n \longrightarrow M$. The standard filtration on A^n induces a filtration on M for which the lemma applies. The corollary follows.

Chapter 2

Quotient Surface Singularities

Throughout this chapter we fix a base field k. The aim of this chapter is to study finite group actions on regular complete local rings of dimension two and their invariant rings.

2.1 The Diamond Lemma for Power Series

We will need Bergman's diamond lemma [B; theorem 1.2] both in its usual algebra form, and in its power series form. For more information about the algebra form, the reader may wish to look at theorem 4.2.2 where the result is generalised to monoids in a category. To state the power series form we need some definitions. Throughout this section, let A be a complete filtered k-algebra with filtration F. First note the following fact,

Proposition 2.1.1 *Let $\Lambda = \{m_\alpha\} \subset A$. Then the following are equivalent:*

i. For $p \in \mathbb{Z}$, the set $\Lambda_p = \{m_\alpha + F^pA | m_\alpha \notin F^pA\} \subseteq A/F^pA$ is a k-basis of A/F^pA.

ii. The images of the $m_\alpha \in \Lambda$ in $gr_F A$ form a basis of $gr_F A$.

If these hold then every element $a \in A$ has a unique representation as a convergent series of the form $a = \sum_{m_\alpha \in \Lambda} c_\alpha m_\alpha$ where $c_\alpha \in k$.

Definition 2.1.2 *A subset $\Lambda \subset A$ is said to be a* strict topological basis for *A if it satisfies the equivalent conditions of the previous proposition.*

Let $X_1, \dots, X_n$ be a set of indeterminates of degrees $d_i > 0$ and let Γ be the semigroup generated by the X_i's. We extend the degree map to a semigroup homomorphism $\deg : \Gamma \longrightarrow \mathbb{N}$ and set $\Gamma_d \subset \Gamma$ to be the subset of degree d elements. Let $P = k\langle\langle X_i \rangle\rangle$ denote the power series ring in the indeterminates $X_1, \dots, X_n$. We filter P by setting F^pP to be the set of power series of degree at least p, i.e. every term of the power series has degree at least p. In general, we shall say that an element α in a filtered object (M, F) has *degree* r if $x \in F^rM - F^{r+1}M$. This agrees with the terminology in the previous case where $M = P$.

Let $r_j, s_j \in P$. We let $(r_j - s_j)$ denote the closed ideal generated by the $r_j - s_j$ and define the *quotient of P with defining relations $r_j = s_j$* to be $P/(r_j - s_j)$. This notation should not cause confusion since we will only be interested in complete quotients of P. Using the language of [B; p.180-181] we may now state the power series version of the diamond lemma:

Theorem 2.1.3 *(Diamond Lemma for Power Series) Let P be the noncommutative power series ring in n indeterminates as above and $r_j \in \Gamma$, $s_j \in P$ for j in some index set be such that r_j does not occur as a term in s_j. Let $A = P/(r_j - s_j)$. Suppose there is a semigroup partial ordering $<$ on Γ satisfying $m_1 < m_2$ whenever $\deg m_1 > \deg m_2$ and such that the restriction of $<$ to each Γ_d satisfies the descending chain condition. Suppose that each relation $r_j = s_j$ is compatible with $<$ in* *the sense that every monomial $m \in \Gamma$ occurring in s_j is less than r_j. If all the overlap and inclusion ambiguities can be resolved then A has a strict topological basis of the form,*

$$\Lambda = \{\tilde{m} | m \in \Gamma \text{ has no subword of the form } r_j\}$$

where $\tilde{m}$ denotes the image of m in A.

Proof. Let F be the filtration on A induced by the degree filtration on P. The theorem follows from applying the usual diamond lemma on A/F^pA by adding relations of the form $m = 0$ where $m \in \Gamma$ has degree at least p.

The strict topological basis provided by this version of the diamond lemma is called the *topological Gröbner basis.*

The two diamond lemmas in tandem provide an effective way of passing between a complete filtered ring and its associated graded ring as the following proposition shows. To keep notation straight, given $r \in P$ let $r(x)$ be the element obtained by substituting the elements $x = x_1, \dots, x_n$ for $X_1, \dots, X_n$.

Proposition 2.1.4 *Suppose $P, A, <, r_j, s_j$ are as hypothesised in the previous theorem. Let F be the filtration on A induced from the degree filtration on P. Consider the degree deg r_j part, t_j of $r_j - s_j$ and let $\overline{r_j - s_j} = t_j(\overline{X})$ for some new indeterminates $\overline{X}_i$ of degree deg X_i. If all overlap and inclusion ambiguities can be resolved then the associated graded ring of A is*

$$gr_F\, A \simeq k\langle \overline{X}_i \rangle / (\overline{r_j - s_j})$$

Conversely, let A be a complete filtered ring with filtration F. Suppose the associated graded ring $gr_F\, A \simeq k\langle \overline{X}_i \rangle / (\overline{r_j} - \overline{s_j})$. Suppose also that there is a semigroup partial ordering $<$ on monomials in $\overline{X}_i$ satisfying the descending chain condition and compatible with the relations $\overline{r_j} = \overline{s_j}$. Let X_i be new indeterminates of degree deg $\overline{X}_i$ and $r_j = \overline{r_j}(X), s_j = \overline{s_j}(X)$. If all overlap and inclusion ambiguities can be resolved, then there exist power series t_j in the X_i of degree greater than deg r_j such that

$$A \simeq k\langle\langle X_i \rangle\rangle / (r_j - s_j - t_j)$$

Furthermore, if x_i are arbitrary lifts of $\overline{X}_i$ to A, then we may assume by changing the t_j if necessary, that the isomorphism maps x_i to $X_i + (r_j - s_j - t_j)$.

Proof. We attack the forward implication first. Let x_i denote the image of X_i in A and $\overline{x}_i$ denote the image of x_i in $\mathrm{gr}_F\, A$. Choose $\Lambda \subset \Gamma$ so that $\{m(x) | m \in \Lambda\}$ is the topological Gröbner basis for A obtained from the power series version of the diamond lemma. Note firstly that for $m \in \Lambda$, deg m = deg $m(x)$ for otherwise we can write $m(x)$ as a power series consisting of terms of degree greater than deg m and then rewrite these as a power series with terms in Λ of degree greater than deg m. This would contradict the uniqueness of power series representations given in proposition 2.1.1. We wish to show that $\overline{X}_i \mapsto \overline{x}_i$ induces the desired isomorphism. We first show that the $\overline{x}_i$ satisfy the relations $\overline{r_j - s_j} = 0$. By reducing monomials, we may assume that no term of the s_j's has r_i as a subword since reduction does not alter $\mathrm{gr}_F\, A$ nor the ring $k\langle \overline{X}_i \rangle / (\overline{r_j - s_j})$. This implies in particular that each term of s_j lies in Λ. There are two cases. If deg r_j = deg $r_j(x) = p$ say, then $(\overline{r_j - s_j})(\overline{x})$ is just the image of $r_j(x) - s_j(x)$ in $F^pA/F^{p+1}A$ and so is zero. If deg $r_j <$ deg $r_j(x)$ then $(\overline{r_j - s_j})(\overline{x}) = r_j(\overline{x})$ which is zero by the assumption on the degree of $r_j(x)$. We thus conclude that $\overline{X}_i \mapsto \overline{x}_i$ induces a map $\phi : k\langle \overline{X}_i \rangle / (\overline{r_j - s_j}) \longrightarrow \mathrm{gr}_F\, A$.

The hypotheses ensure that you can apply the diamond lemma to $k\langle \overline{X}_i \rangle / (\overline{r_j - s_j})$ to obtain the basis $\{m(\overline{X}) | m \in \Lambda\}$. On the other hand, deg m = deg $m(x)$ for $m \in \Lambda$ shows that $\mathrm{gr}_F\, m(x) = m(\overline{x})$ so the $m(\overline{x})$ form a basis for $\mathrm{gr}_F\, A$ by proposition 2.1.1. Thus ϕ restricts to a bijection on Gröbner bases and so must be an isomorphism as was to be shown.

For the converse, let x_i denote arbitrary lifts of $\overline{X}_i$ to A. Then $\overline{r_j}(x) - \overline{s_j}(x) = t_j(x)$ for some power series t_j of degree greater than deg r_j. Since A is complete, we obtain a natural map $\phi : k\langle\langle X_i \rangle\rangle / (r_j - s_j - t_j) \longrightarrow A$. Now ϕ maps the strict topological basis for $k\langle\langle X_i \rangle\rangle / (r_j - s_j - t_j)$

obtained in the first half of the proof to the strict topological basis for A obtained using the power series version of the diamond lemma. Hence ϕ is an isomorphism.

2.2 Group Actions on Regular Rings

In this section we classify certain group actions on regular complete local rings of dimension two. Throughout, let k denote an algebraically closed field of characteristic zero.

Let $x_1, \ldots, x_n$ be a set of indeterminates of degree one which is totally ordered by a relation $<$. Let Γ be the semigroup generated by the x_i's. We need a partial ordering on Γ in order to apply the diamond lemmas of the previous section. We will use a right to left version of the lexicographic ordering on Γ defined as follows: Let $a, b \in \Gamma$ be monomials of the same degree. Write a and b as a product of monomials $a = a'x_ic$, $b = b'x_jc$ where c has as large a degree as possible. We write $a \prec b$ if $x_i < x_j$. As usual, we use the language of [B; p.180-181]. Recall,

Lemma 2.2.1 *The relation $\prec$ is a semigroup partial ordering on Γ which satisfies the descending chain condition.*

This is the partial order we use when applying the usual algebra form of the diamond lemma. When using the power series version of the diamond lemma, we refine the order $\prec$ by insisting also that $a \prec b$ if deg $a >$ deg b. There should be no confusion in using the same notation to denote these two orders since one will only be used for graded algebras and the other for complete local rings.

A monomial $x_{i_1} \ldots x_{i_m}$ is said to be *in order* or *ordered* if $x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_m}$. Otherwise, the monomial is said to be *out of order.*

For the rest of this section, $(B, \mathfrak{n})$ will denote a regular complete local ring of dimension two. We recall the following result of Artin-Stafford and Cartan. As usual, given r an element of a power series ring, we let (r) denote the closed ideal generated by r.

Proposition 2.2.2 *([AS; lemma 1.4]) $B \simeq k\langle\langle u, v\rangle\rangle/(r)$ where r is one of the following,*

i. ($\overline{q}$-Plane) $r = vu - \overline{q}uv - c$ for some $\overline{q} \in k - \{0\}$.

ii. (Jordan Plane) $r = vu - uv - v^2 - c$

and c is a noncommutative power series of degree at least three. Furthermore, if there is a finite group G acting on B, then one can assume that G acts linearly on u, v i.e. G restricts to an action on $V = ku + kv$. Conversely, every ring of the form $B \simeq k\langle\langle u, v\rangle\rangle/(r)$ where r is given as in (i) or (ii) is regular complete local of dimension two.

Proof. Suppose there is a finite group action on B. Then by Maschke's theorem one can lift the action of G on $\mathfrak{n}/\mathfrak{n}^2$ to a vector subspace V of $\mathfrak{n}$ so that the natural map $V \longrightarrow \mathfrak{n}/\mathfrak{n}^2$ is a G-module isomorphism. Since B is regular of dimension two, $\mathrm{gr}_{\mathfrak{n}} B \simeq k\langle\overline{u}, \overline{v}\rangle/(Q)$ for some quadratic form Q which is not the product of two linear factors. By changing variables, we may assume that Q is $\overline{v}\,\overline{u} - \overline{q}\,\overline{u}\,\overline{v}$ or $\overline{v}\,\overline{u} - \overline{u}\,\overline{v} - \overline{v}^2$. Let u, v be the unique lifts of $\overline{u}, \overline{v}$ to V. We wish to apply the converse half of proposition 2.1.4. Order the variables by $\overline{v} < \overline{u}$ and consider the order $\prec$ of lemma 2.2.1. We express the relations of $\mathrm{gr}_{\mathfrak{n}} B$ in the form $\overline{v}\,\overline{u} = \overline{q}\,\overline{u}\,\overline{v}$ or $\overline{u}\,\overline{v} + \overline{v}^2$. In this form, the relations are compatible with $\prec$ and there are no overlap or inclusion ambiguities to resolve. Hence, proposition 2.1.4 shows that B has the desired presentation.

To see that every ring of the form $k\langle\langle u, v\rangle\rangle/(r)$ with r as above is regular of dimension two, we need only apply the first half of proposition 2.1.4.

The power series version of the diamond lemma allows us to read off a strict topological basis for B.

Lemma 2.2.3 *B has a strict topological basis of the form $\{u^iv^j | i, j \geq 0\}$.*

Note 1: Every automorphism of B must preserve the $\mathfrak{n}$-adic filtration and so induces an automorphism of the associated graded ring $\mathrm{gr}_{\mathfrak{n}} B$. Hence, every automorphism of B is continuous.

Fix $V \subset \mathfrak{n}$ such that the natural map $V \longrightarrow \mathfrak{n}/\mathfrak{n}^2$ is an isomorphism. We say that the group G acts *linearly on* B (with respect to V) if the action preserves V.
Note 2: Suppose that G acts linearly on B. Then the action of G on B is determined by the action of G on $\mathrm{gr}_{\mathfrak{n}} B$.
Remark: The possible associated graded rings of B namely, $\overline{B}_{\overline{q}} = k\langle u,v\rangle/(vu - \overline{q}uv)$ and $\overline{B}_J = k\langle u,v\rangle/(vu - uv - v^2)$ can be nicely interpreted using Van den Bergh's notion of a twisted homogeneous coordinate ring (see [AV] or chapter 4 for definitions). In fact, $\overline{B}_{\overline{q}} \simeq B(\mathbb{P}^1, \mathcal{O}(1)_\tau)$ where $\tau \in \mathrm{Aut}(\mathbb{P}^1)$ is multiplication by $\overline{q}$ and $\overline{B}_J \simeq B(\mathbb{P}^1, \mathcal{O}(1)_\tau)$ where τ is translation by 1.

Since we are interested in invariant rings, we will only study faithful actions of a finite group G on B. If G acts linearly on B, then it will be convenient to identify G with a finite subgroup of $GL(V)$. If furthermore, we are given a basis u, v for V, then we will also identify $GL(V)$ with $GL_2(k)$ via the basis u, v.

Lemma 2.2.4 *We continue the notation of proposition 2.2.2. Suppose an automorphism σ of B acts linearly and has order d. Then one of the following must hold:*

i. B is a $\overline{q}$-plane and $\sigma = \left(\begin{smallmatrix}\omega^r & 0\\ 0 & \omega^s\end{smallmatrix}\right)$ where ω is a primitive d-th root of unity and $r, s \in \mathbb{Z}$.

ii. B is a $\overline{q}$-plane where $\overline{q} = -1$, d is even and $\sigma = \left(\begin{smallmatrix}0 & a\\ b & 0\end{smallmatrix}\right)$ where ab is a $\frac{d}{2}$-th root of unity.

iii. B is a $\overline{q}$-plane with $\overline{q} = 1$.

iv. B is a Jordan plane and $\sigma = \omega\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ where ω is a d-th root of unity.

Proof. By note 2 above, we need only consider the induced automorphism of σ on $gr_{\mathfrak{n}}B$. Let $\overline{u}, \overline{v}$ be the images of u, v in $gr_{\mathfrak{n}}B$. For the $\overline{q}$-plane where $\overline{q} \neq 1$, the set of normal degree 1 elements (together with 0) is the union of $k\overline{u}$ and $k\overline{v}$ so these must be σ-stable. If σ stabilises each of these lines then we are in case (i). If σ swaps these lines then $\sigma = \left(\begin{smallmatrix}0 & a\\ b & 0\end{smallmatrix}\right)$ where ab is a $\frac{d}{2}$-th root of unity. For σ to preserve the skew commutation relation $\overline{v}\,\overline{u} = \overline{q}\,\overline{u}\,\overline{v}$ of $\mathrm{gr}_{\mathfrak{n}} B$ we must have $\overline{q} = -1$. In the Jordan plane case, $k\overline{v}$ is the set of all normal elements of degree 1 (plus 0) so must be stable under G. By Maschke's theorem, there is another σ-stable vector space of the form $k(\overline{u} + \alpha\overline{v})$, $\alpha \in k$. We may replace $\overline{u}$ with $\overline{u} + \alpha\overline{v}$ since this will not alter the relation $\overline{v}\,\overline{u} = \overline{u}\,\overline{v} + \overline{v}^2$. For σ to preserve this relation, the eigenvectors $\overline{u}\,\overline{v}$ and $\overline{v}^2$ must have the same eigenvalue. This implies that σ is scalar on $k\overline{u} + k\overline{v}$ and we are done.
Remark: Perhaps the most illuminating way of seeing the classification of actions on $\mathrm{gr}_{\mathfrak{n}} B$ is to view $\mathrm{gr}_{\mathfrak{n}} B$ as a twisted homogeneous coordinate ring $B(\mathbb{P}^1, \mathcal{O}(1)_\tau)$. Now σ induces an automorphism of $\mathbb{P}^1$ which commutes with τ. If τ is multiplication by $\overline{q} \neq 1$ then it has two fixed points 0 and ∞. If σ fixes these then we are in case (i). If τ swaps these, then up to multiplication, τ is inversion so $\overline{q} = \overline{q}^{-1}$ and we are in case (ii) or (iii). In the Jordan case, ∞ is a fixed point of τ and thus also of σ. Hence σ induces an automorphism of $\mathbb{A}^1$ which commutes with translation. The only such maps are translations of which the only one of finite order is the identity.

If G is any finite group acting on B, then the $\mathfrak{n}$-adic filtration induces a filtration F on the invariant ring B^G. There is a natural graded ring homomorphism $\mathrm{gr}_F B^G \longrightarrow (\mathrm{gr}_{\mathfrak{n}} B)^G$. In fact we have

Proposition 2.2.5 *The natural map $gr_F B^G \longrightarrow (gr_{\mathfrak{n}} B)^G$ is an isomorphism.*

Proof. The morphism is injective because the filtration is induced. To see surjectivity, consider the G-module epimorphism $\mathfrak{n}^p \longrightarrow \mathfrak{n}^p/\mathfrak{n}^{p+1}$. By Maschke's theorem this map splits so fixed elements lift as desired.

In commutative algebra, the rational double points are the quotients of Spec $k[[u,v]]$ by a finite subgroup of SL_2. We classify noncommutative analogues in

Proposition 2.2.6 *Let u, v be indeterminates and $V = ku \oplus kv$. Let $\overline{B} = k\langle u,v\rangle/(vu - \overline{q}uv)$ and suppose $G \subset SL(V)$ is a finite group whose action extends to $\overline{B}$. Then one of the following holds,* *i. G is a cyclic group of order d generated by* $\sigma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ *where* ξ *is a primitive d-th root of unity. Furthermore,* $\overline{B}^G$ *is the k-algebra on 3 generators* x, y, z *subject to the relations*

$$yx = qxy \quad , \quad zx = q^d xz \quad , \quad zy = qyz \quad , \quad y^d = xz \tag{2.1}$$

where $q = \overline{q}^d$.

ii. $\overline{q} = -1$ *and G is the binary dihedral group* $< \sigma, \tau | \sigma^d = \tau^2, \tau^4 = 1, \tau\sigma = \sigma^{-1}\tau >$. *Up to scaling u and v we have* $\sigma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, *(*$\xi$ *a 2d-th root of unity) and* $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. *In this case,* $\overline{B}^G \simeq k[x, y]$.

iii. $\overline{q} = 1$ *and G is either binary dihedral, tetrahedral, octahedral or icosahedral.*

Let $\overline{B} = k\langle u, v \rangle / (vu - uv - v^2)$ *and* $1 \neq G \subset SL(V)$ *extend to an action of* $\overline{B}$ *as before. Then G is* ± 1 *and* $\overline{B}^G$ *is the k-algebra on 3 generators* x, y, z *subject to the relations*

$$yx = xy + xz \; , \; zx = xz + 2yz + 2z^2 \; , \; zy = yz + z^2 \; , \; y^2 = xz \tag{2.2}$$

Proof. We suppose first that $\overline{B} = k\langle u, v \rangle / (vu - \overline{q}uv)$.

Case (i) G is diagonal, $\overline{q} \neq 1$: This is always the case if $\overline{q} \neq \pm 1$ by lemma 2.2.4. Then G must be as in (i) for the diagonal subgroup of SL_2 is isomorphic to k^* and the only finite subgroups of k^* are cyclic. The invariant ring $\overline{B}^G$ is spanned by the $u^i v^j$ where $d | j - i$. Since u and v are normal, we see immediately that $\overline{B}^G$ is generated by u^d, uv, v^d. Let C be the k-algebra on three generators x, y, z with defining relations (2.1). We set the degree of x and z to be d and the degree of y to be 2 so that C is a graded algebra. Then there is a surjective graded ring homomorphism $\phi : C \longrightarrow \overline{B}^G$ defined by $x \mapsto u^d, y \mapsto cuv, z \mapsto v^d$ where $c = \overline{q}^{\frac{1-d}{2}}$. Now the relations (2.1) ensure that the $x^i y^r z^j$ for $0 \leq r < d$ span C. Furthermore, the $\phi(x^i y^r z^j) \in k u^{id+r} v^{jd+r}$ are linearly independent by lemma 2.2.3. Hence ϕ is injective and $C \simeq \overline{B}^G$.

Case (ii) G not diagonal, $\overline{q} = -1$: Let H be the subgroup of diagonal elements. As was seen in the previous case, H must be cyclic generated by some automorphism $\sigma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$, where ξ an e-th root of unity and e is the order of H. Let τ be a non-diagonal element of G which by the lemma has the form $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. Its determinant $-ab = 1$ so by scaling u and v if necessary, we may assume $a = 1, b = -1$. The product of any two off-diagonal elements in G must be diagonal and so lies in H. Hence G is generated by σ and τ. Now $\tau^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ so $e = 2d$ for some d. Computing the relations satisfied by σ and τ, we see that G is the binary dihedral group.

To determine $\overline{B}^G$, first recall from the previous case that $\overline{B}^H$ has a basis consisting of monomials $u^i v^j$ where $2d | j - i$. Since τ has order two modulo H, $\overline{B}^G = (\mathrm{id} + \tau)(\overline{B}^H)$. To compute this latter we consider monomials $u^i v^j$ with $2d | j - i$ so that in particular, i and j have the same parity. Then the

$$u^i v^j + \tau(u^i v^j) = u^i v^j + (-1)^i v^i u^j = u^i v^j + (-1)^{i+ij} u^j v^i = u^i v^j + u^j v^i$$

form a basis for $\overline{B}^G$. We wish to show this element lies in the algebra generated by $x = uv$ and $y = u^{2d} + v^{2d}$. First note that u^2 and v^2 are central in $\overline{B}$. In particular, x and y commute. They are also algebraically independent. Now if $j \geq i$ then since u^{j-i} is central we see $u^i v^j + u^j v^i = u^i v^i (v^{j-i} + u^{j-i})$. We show by induction on $j - i$ that this lies in $k[x, y]$. If $j = i$ then $u^i v^i = \pm x^i$ so we may assume $i = 0$ and $j = 2ld$. From the binomial theorem, we see that $(u^{2ld} + v^{2ld}) - y^l$ is a sum of binomials of the form $u^{2md} v^{2(l-m)d} + u^{2(l-m)d} v^{2md}$ where $0 < m < l$. Hence by the inductive hypothesis, $u^{2ld} + v^{2ld}$ also lies in $k[x, y]$. The case when $i > j$ is similar so $\overline{B}^G \simeq k[x, y]$.

Case (iii), $\overline{q} = 1$: This is the classical commutative case for which we refer the reader to [Pink].

Now assume that $\overline{B} = k\langle u, v \rangle / (vu - uv - v^2)$. Lemma 2.2.4 shows that the only nontrivial group $G \subset SL(V)$ acting on B is $\{\pm 1\}$. The invariant ring $\overline{B}^G$ consists of the even degree elements in

$\overline{B}$ and thus is generated by u^2, uv, v^2. Let C be the algebra defined by equation (2.2). Checking relations, we see that there is a graded surjective ring homomorphism $C \longrightarrow \overline{B}^G$ defined by

$$x \mapsto \frac{1}{2}(u^2 + uv)\ ,\ y \mapsto uv\ ,\ z \mapsto 2v^2$$

Injectivity is proved as in (i).

Case (ii) of the proposition is somewhat anomalous. There is no commutative analogue.

We have a converse to the previous proposition:

Proposition 2.2.7 *The action of G on V in proposition 2.2.6 extends to an action of G on B.*

which is proved by verifying that G preserves the defining relation $vu = \overline{q}uv$ or $vu = uv + v^2$.

Whether or not the actions of proposition 2.2.6 will induce actions on a regular complete local ring B will depend on what the cubic term c of proposition 2.2.2 is. If $c = 0$ then there will be an induced action on B.

Definition 2.2.8 *A* quotient surface singularity *is a ring of the form $A = B^G$ where G is a finite group acting on B. If $G \subset SL(V)$ then A is said to be a* special quotient surface singularity.

2.3 The Associated Graded Ring

Throughout this section, let k denote an algebraically closed field of characteristic zero.

Many properties of filtered rings may be deduced from the associated graded ring. This latter is usually easier to work with. Let $\overline{B} = k\langle \overline{u}, \overline{v}\rangle/(\overline{v}\,\overline{u} - \overline{q}\,\overline{u}\,\overline{v})$ or $k\langle \overline{u}, \overline{v}\rangle/(\overline{v}\,\overline{u} - \overline{u}\,\overline{v} - \overline{v}^2)$. Proposition 2.2.5 states that the associated graded ring of a special quotient surface singularity has the form $\overline{B}^G$ (where G is a finite subgroup of $SL(ku + kv)$). These graded rings were computed in proposition 2.2.6. There were two cases when the ring was not commutative: let $\overline{A}_q$ denote the ring defined by the relations (2.1) of the proposition and $\overline{A}_J$ denote the ring defined by the relations (2.2). We study these two rings in this section.

All rational double points have embedding dimension three. This shows that they are Gorenstein. We would like a noncommutative version of this result, at least in the associated graded case. As in previous section, let $q = \overline{q}^d$. The rôle of the coordinate ring of ambient 3-space will be played by the ring

$$S := k\langle x, y, z\rangle/(yx - qxy, zx - q^d xz, zy - qyz)$$

for $\overline{A}_q$ and by

$$T := k\langle x, y, z\rangle/(yx - xy - xz, zx - xz - 2yz - 2z^2, zy - yz - z^2)$$

for $\overline{A}_J$. We study these two rings first.

Lemma 2.3.1 *The ordered monomials $\{x^i y^j z^k | i, j, k \geq 0\}$ form a k-basis for S and for T.*

Proof. We wish to apply Bergman's diamond lemma. Order the variables by $z < y < x$. The relations in S and T yield reductions which replace the out of order monomials yx, zx, zy with sums of ordered monomials. This reduction is compatible with the order $\prec$ of lemma 2.2.1 so we may apply Bergman's diamond lemma. This will show that the ordered monomials form a basis once we resolve the overlap ambiguity for the monomial zyx. Using $\mapsto$ to denote reductions, we have in $\overline{A}_q$,

$$z(yx) \mapsto qzxy \mapsto q^{d+1}xzy \mapsto q^{d+2}xyz \ \text{ and }\ (zy)x \mapsto qyzx \mapsto q^{d+1}yxz \mapsto q^{d+2}xyz$$

which checks out. In $\overline{A}_J$ we have

$$z(yx) \mapsto zxy + zxz \mapsto xzy + 2yzy + 2z^2y + zxz \mapsto xyz + xz^2 + 2y^2z + 2yz^2 + 2zyz + 2z^3 + zxz$$

$$(zy)x \mapsto yzx + z^2x \mapsto yxz + 2y^2z + 2yz^2 + zxz + 2zyz + 2z^3$$

$$\mapsto xyz + xz^2 + 2y^2z + 2yz^2 + zxz + 2zyz + 2z^3$$

which agree as well so we are done.

Let R be a ring, α an automorphism of R and δ an α-derivation of R i.e. δ satisfies the skew-Leibniz rule $\delta(rs) = r\delta(s) + \delta(r)\alpha(s), r, s \in R$. Recall that associated to this data, there exists a ring called the Ore extension $R[x; \alpha, \delta]$. As an R-module, this ring is isomorphic to the free module $\oplus_{i\geq 0} x^i R$. Multiplication is determined by the rule $rx = x\alpha(r) + \delta(r)$ for all $r \in R$. There is a converse result whose proof we omit.

Lemma 2.3.2 *Let $R \longrightarrow R'$ be a ring extension such that as an R-module, $R' \simeq \oplus_{i\geq 0} x^i R$ for some $x \in R'$. Suppose there exists an automorphism α of R such that $rx - x\alpha(r) \in R$ for all $r \in R$. Then R' is an Ore extension of R.*

Proposition 2.3.3 *Let $R_0 = S$ or T. Then there exists a chain of subrings $R_0 \supset R_1 \supset R_2 \supset R_3 = k$ where R_i is an Ore extension of R_{i+1} for $i = 0, 1, 2$.*

Proof. Let $R_2 = k[z]$ which is an Ore extension of k. The subring R_1 of R_0 generated by y and z (over k) is $\oplus_{i\geq 0} y^i R_2$ as an R_2-module by lemma 2.3.1. Consider first $R_0 = S$. Let α in the previous lemma be the automorphism of R_2 which sends $z \mapsto qz$. The above lemma shows that R_1 is an Ore extension of R_2. Let α' be the automorphism of R_1 mapping $y \mapsto qy, z \mapsto q^d z$. We conclude similarly, using lemmas 2.3.1 and 2.3.2 that S is an Ore extension of R_1.

Now consider the case $R_0 = T$. As before, we see that R_1 is an Ore extension of R_2. Let α in lemma 2.3.2 be the automorphism of R_1 which maps $y \mapsto y + z, z \mapsto z$. This is an automorphism since $R_1 \simeq k\langle y, z\rangle/(zy - yz - z^2)$. Applying lemma 2.3.2 we conclude that T is an Ore extension of R_1.

There are surjective ring homomorphisms $S \longrightarrow \overline{A}_q$ and $T \longrightarrow \overline{A}_J$. The kernels of these maps is given by the two-sided ideal generated by $y^d - xz$ where d is as in (2.1) of proposition 2.2.6 for $\overline{A}_q$ and equals 2 for $\overline{A}_J$. We wish to show that $y^d - xz$ is in fact normal. We need a different ordering of the monomials. The following is elementary.

Lemma 2.3.4 *Let Γ and Γ' be semigroups and $<$ a semigroup partial ordering on Γ' which satisfies the descending chain condition. Let $f : \Gamma \longrightarrow \Gamma'$ be a semigroup homomorphism. Then the induced order on Γ defined by $a < b$ whenever $f(a) < f(b)$ is a semigroup partial ordering satisfying the descending chain condition.*

Proposition 2.3.5 *The monomials $\{x^i y^j z^k | i, k \geq 0, 0 \leq j < d\}$ form a k-basis for $\overline{A}_q$ and $\overline{A}_J$.*

Proof. Let Γ the free semigroup generated by x, y, z and Γ' be the free semigroup generated by u, v. Let f be the semigroup homomorphism from $\Gamma \longrightarrow \Gamma'$ which maps $x \mapsto u^d$, $y \mapsto uv$, $z \mapsto v^d$. Let $\prec$ be the order on Γ' considered in the previous section which comes from setting $v < u$. The partial order $<$ on Γ we wish to use is that induced from Γ' via f.

When we apply Bergman's diamond lemma in this case, we have all the reductions that we used for S and T plus the additional reduction $y^d \mapsto xz$. These reductions are all compatible with the new partial order $<$. The monomials above are precisely the irreducible ones. We need only resolve overlap ambiguities for the monomials $zyx, zy^d, y^d x$ and y^{d+1}. We have already resolved the ambiguity for the monomial zyx in S and T so only the other three need to be resolved.

Consider first $\overline{A}_q$.

$$(zy)y^{d-1} \mapsto qyzy^{d-1} \mapsto \ldots \mapsto q^d y^d z \mapsto q^d xz^2 \quad \text{and} \quad z(y^d) \mapsto zxz \mapsto q^d xz^2$$

which agree while

$$y^{d-1}(yx) \mapsto qy^{d-1}xy \mapsto \ldots \mapsto q^d xy^d \mapsto q^d x^2 z \quad \text{and} \quad (y^d)x \mapsto xzx \mapsto q^d x^2 z$$

which verifies the second monomial. Finally

$$y(y^d) \mapsto yxz \mapsto qxyz \quad \text{and} \quad (y^d)y \mapsto xzy \mapsto qxyz$$

resolving the last ambiguity.

Now consider $\overline{A}_J$. Here $d = 2$. We check the monomial zy^2.

$$(zy)y \mapsto yzy + z^2y \mapsto y^2z + yz^2 + zyz + z^3 \mapsto xz^2 + yz^2 + yz^2 + 2z^3 \mapsto xz^2 + 2yz^2 + 2z^3$$

$$z(y^2) \mapsto zxz \mapsto xz^2 + 2yz^2 + 2z^3$$

For the monomial y^2x we have

$$y(yx) \mapsto yxy + yxz \mapsto xy^2 + xzy + xyz + xz^2 \mapsto x^2z + xyz + xz^2 + xyz + xz^2 \mapsto x^2z + 2xyz + 2xz^2$$

$$(y^2)x \mapsto xzx \mapsto x^2z + 2xyz + 2xz^2$$

which agree as well so it remains to consider y^3:

$$y(y^2) \mapsto yxz \mapsto xyz + xz^2 \quad , \quad (y^2)y \mapsto xzy \mapsto xyz + xz^2$$

which verifies the last overlap check.

Recall that for any graded k-module $M = \oplus M_n$, the Hilbert function of M is $h_M(n) = \dim_k M_n$.

Proposition 2.3.6 *The element $h = y^d - xz$ is normal in S and T. Hence $\overline{A}_q \simeq S/hS$ and $\overline{A}_J \simeq T/hT$.*

Proof. In S, h skew commutes with x, y and z so the proposition follows. We give a proof for T which works equally well for S. Lemma 2.3.1 shows that $h_T(n) = \frac{1}{2}(n+2)(n+1)$ while the previous proposition shows that $h_{\overline{A}_J}(n) = 2n + 1$. Now h is a regular element since T is an iterated Ore extension of a domain. This gives an exact sequence,

$$0 \longrightarrow T[-2] \xrightarrow{h} T \longrightarrow T/hT \longrightarrow 0$$

where the [-2] denotes the shift in grading by two. We deduce that

$$h_{T/hT}(n) = h_T(n) - h_T(n-2) = \frac{1}{2}[(n+2)(n+1) - n(n-1)] = 2n + 1$$

Since $\overline{A}_J \simeq T/ThT$ has the same Hilbert function, we must have $hT = ThT$ which proves the proposition.

2.4 Finiteness of Representation Type

Throughout this section let k denote an algebraically closed field of characteristic zero.

Our first goal is to show that quotient surface singularities have balanced dualising complexes so we may apply the results of chapter 1.

Proposition 2.4.1 *Every quotient surface singularity is AS-Cohen-Macaulay of dimension two.*

Proof. Let $(B, \mathfrak{n})$ be a regular complete local ring of dimension two and let A be the quotient surface singularity B^G (where G is a finite group). Let F be the filtration on A induced from the $\mathfrak{n}$-adic filtration on B. We first show that A is a complete local ring satisfying standard hypotheses 1.1.6 with respect to F. The filtration F is complete since the $\mathfrak{n}$-adic filtration is complete on B and the action of G is continuous. Proposition 2.2.5 shows that $\operatorname{gr}_F A \simeq (\operatorname{gr}_{\mathfrak{n}} B)^G$ so $A/F^1A \simeq k$ and

$\mathrm{gr}_F A$ is noetherian by [Mont; corollary 1.12]. This latter implies that the Rees ring $\oplus(F^p A)t^p$ is noetherian. Hence A satisfies standard hypotheses 1.1.6. We wish to apply corollary 1.4.4. By [Mont; corollary 5.9], B is a finitely generated left and right A-module. The inclusion $A \longrightarrow B$ is split as a morphism of left A-modules and of right A-modules since we are assuming that the characteristic of k is zero. Hence corollary 1.4.4 implies that the dualising complex ω_A of A exists and is a direct summand of the dualising complex ω_B of B. Since B is regular of dimension two, corollary 1.3.8 shows that ω_B is isomorphic to $L[2]$ where L is an invertible bimodule. Hence, ω_A has cohomology in the -2 spot only and A is AS-Cohen-Macaulay of dimension two.

We next establish a noncommutative analogue of the fact that rational double points are Gorenstein. For this, we will need to introduce the Auslander condition. Below we use the inf function defined in definition 1.2.18.

Definition 2.4.2 *A noetherian ring A satisfies the* Auslander condition *if for every finitely generated left or right module M, every $i \in \mathbb{N}$ and every submodule $N \subseteq Ext^i_A(M, A)$ we have*

$$inf(RHom_A(N, A)) \geq i$$

Definition 2.4.3 *A noetherian ring A is said to be* Auslander-Gorenstein *if it has finite injective dimension and satisfies the Auslander condition. If furthermore, A has finite global dimension then A is said to be* Auslander regular.

For us, the relevance of this notion is given by Levasseur's

Theorem 2.4.4 *([L; theorem 6.3]) Let A be a complete local ring satisfying standard hypotheses 1.1.6. If A is Auslander-Gorenstein then it is AS-Gorenstein.*

Proof. Levasseur proves the graded version of this theorem but the same proof works in the complete local case.

Much is known about Auslander-Gorenstein rings. We will need some results concerning such rings but first, a

Definition 2.4.5 *([Bj; definition 2.2]) A filtered ring (A, F) satisfies the* strong closure condition *if for every finite set of elements $x_1, \ldots, x_n \in A$ and integers $p_1, \ldots, p_n \in \mathbb{Z}$, the ideals $x_1(F^{p_1}A) + \ldots + x_n(F^{p_n}A)$ and $(F^{p_1}A)x_1 + \ldots + (F^{p_n}A)x_n$ are closed.*

Theorem 2.4.6 *([Bj; theorem 3.9]) Let (A, F) be a filtered ring which satisfies the strong closure condition and such that $gr_F A$ is noetherian. Suppose that $gr_F A$ is Auslander-Gorenstein of injective dimension d. Then A is Auslander-Gorenstein of injective dimension $\leq d$.*

Theorem 2.4.7 *([L; theorem 3.6(2)]) Let A be a noetherian $\mathbb{N}$-graded algebra. Let $x \in A$ be a homogeneous regular element of positive degree. Then A/xA is Auslander-Gorenstein of injective dimension d if and only if A is Auslander-Gorenstein of injective dimension $d + 1$.*

Theorem 2.4.8 *([Ba: §1 theorem and definition]) Every commutative Gorenstein ring is Auslander-Gorenstein.*

Let S and T be the rings defined in the previous section.

Corollary 2.4.9 *The rings S and T are Auslander regular of global dimension three.*

Proof. Since S and T are 3-fold iterated Ore extensions (lemma 2.3.3), Hilbert's syzygies theorem guarantees that S and T have global dimension three. We need to show they are Auslander-Gorenstein. Now x, y and z are normal in S so applying theorem 2.4.7 three times shows that S is Auslander-Gorenstein. Also, z is normal in T and $T/zT \simeq k[x, y]$ which is commutative Gorenstein so Levasseur's theorem again shows that T is Auslander-Gorenstein.

Let $\overline{B} = k\langle \overline{u}, \overline{v}\rangle/(\overline{v}\,\overline{u} - \overline{q}\,\overline{u}\,\overline{v})$ or $k\langle \overline{u}, \overline{v}\rangle/(\overline{v}\,\overline{u} - \overline{u}\,\overline{v} - \overline{v}^2)$. Let $\overline{A} = \overline{B}^G$ (where G is a finite subgroup of $SL(ku + kv)$).

Proposition 2.4.10 *Let (A, F) be a filtered ring whose associated graded ring is isomorphic to $\overline{A}$. Then A is Auslander-Gorenstein of injective dimension ≤ 2 and hence AS-Gorenstein. If A is a special quotient surface singularity then the injective dimension actually equals two.*

Proof. By proposition 2.2.6 $\overline{A}$ is either commutative or equals $\overline{A}_q$ or $\overline{A}_J$ of the previous section. In the commutative case, it is well known that $\overline{A}$ is Gorenstein of injective dimension two and hence Auslander-Gorenstein. In the other cases, proposition 2.3.6 shows that $\overline{A}$ is a quotient of the domain S or T by a normal homogeneous element so by theorem 2.4.7, $\overline{A}$ is Auslander-Gorenstein of injective dimension two.

We verify the hypotheses of theorem 2.4.6. We know that $\mathrm{gr}_F A$ is noetherian by proposition 2.2.5. The strong closure condition follows from [LvO; chapter II, theorem 2.1.2] which states that every ideal is closed. Hence Björk's theorem shows that A must be Auslander-Gorenstein of injective dimension at most two. Suppose now that A is a special quotient surface singularity. By proposition 2.4.1, A has dimension two so the injective dimension of A is two (see the remarks following definition 1.5.7).

Definition 2.4.11 *A complete local ring A with balanced dualising complex is said to have* finite representation type *if the number of isomorphism classes of indecomposable maximal Cohen-Macaulay modules is finite.*

Theorem 2.4.12 *Every special quotient surface singularity has finite representation type.*

Proof. Let B be a regular complete local ring of dimension two and $A = B^G$ (where as usual $G \subset SL(V)$). First note that mod-A is a Krull-Schmidt category by corollary 1.6.2 so it suffices to show that any indecomposable maximal Cohen-Macaulay module M is a direct summand of B. Let $M' = \mathrm{Hom}_A(M, A)$. Since A is AS-Gorenstein, A is a dualising complex for A. Furthermore, since M is maximal Cohen-Macaulay, $\mathrm{RHom}_A(M, A) \simeq \mathrm{Hom}_A(M, A)$ so M is reflexive. Now $0 \longrightarrow A \longrightarrow B$ is split as a sequence of left A-modules so we have a split sequence of right A-modules:

$$0 \longrightarrow \mathrm{Hom}_{A^\circ}(M', A) \longrightarrow \mathrm{Hom}_{A^\circ}(M', B)$$

Proposition 1.5.4 shows that $\mathrm{Hom}_{A^\circ}(M', B)$ is a reflexive B-module so by proposition 1.5.8, it is maximal Cohen-Macaulay and hence free over B (corollary 1.5.10). The theorem follows.

2.5 Regularity in Codimension One

Every commutative quotient singularities is regular in codimension one. In this section we prove an analogue of this fact for certain special quotient surface singularities.

Let A be a ring. If A is commutative, then regularity in codimension one is traditionally defined by looking at the localisations at height one primes of A. Unfortunately, noncommutative geometry tends not to be local so this definition fails for noncommutative rings. An alternative way to define regularity in codimension one in the commutative case is to consider the scheme X consisting of Spec A with all the points of codimension two or greater removed. Then A is regular in codimension one if X is smooth. This suggests,

Definition 2.5.1 *Let $(A, \mathfrak{m})$ be a two dimensional complete local ring with balanced dualising complex and let $\mathcal{T}$ denote the category of $\mathfrak{m}$-torsion A-modules. Then A is said to be* regular in codimension one *if the quotient category Mod-A / $\mathcal{T}$ has finite injective dimension.*

The category Mod-A / $\mathcal{T}$ is sometimes called the punctured spectrum since when A is commutative, it corresponds to the category of quasi-coherent sheaves on Spec A with the closed point removed.

Let B be a ring and G a finite group which acts on B. Let $A = B^G$. There is a standard technique for studying the invariant ring A inaugurated by Montgomery in [Mont]. We review some of the theory, details of which can be found in [Mont] or [McR; chapter 7, section 8].

Let S be the skew group ring $G * B$. As a right B-module it is $\oplus_{\sigma \in G} \sigma B$ while multiplication is given by the commutation rule $b\sigma = \sigma(\sigma(b))$. The definition is such that B is naturally a right S-module. Furthermore, left multiplication by elements of A commute with both the G-action and right multiplication by elements of B so B is an (A, S)-bimodule.

We wish to set up a Morita context involving S and A. Let e be the idempotent $\frac{1}{|G|}\sum_\sigma \sigma$ in S so that eS is a projective S-module. Note that e commutes with elements of A. The Pierce decomposition shows that $\mathrm{End}_S(eS) \simeq eSe$. A simple computation [McR; 7.8.7] shows that there is an isomorphism $A \longrightarrow eSe$ given by $a \mapsto eae = ae = ea$. Now eS is an (A, S)-bimodule which by [McR; proposition 7.8.5] is isomorphic to ${}_AB_S$, the isomorphism being given by $e \mapsto 1$. This shows in particular that $eS = eB$. The Pierce decomposition also shows that $\mathrm{Hom}_S(eS, S) = Se$. From this data, we obtain a Morita context

$$\phi : eS \otimes_S Se \to eSe \simeq A \ , \ \psi : Se \otimes_A eS \to S$$

where the maps are appropriate restrictions of the multiplication map in S. The map ϕ is always an isomorphism. Morita theory states that if ψ is surjective then the functors $- \otimes_A eS$ and $- \otimes_S Se$ define a natural equivalence between Mod-A and Mod-S. This will never occur if G is non-trivial and B is local with residue field k. We wish to obtain a Morita theory for quotients of Mod-A and Mod-S.

Given an ideal J of some ring, let J-tors denote the category of J-torsion modules.

Proposition 2.5.2 *Let J be an ideal of the ring B and I be an ideal of the ring $A = B^G$. Suppose that (*) $I^n \subseteq J$ and $J^n \subseteq IB$ for some $n \in \mathbb{N}$. If S/SeS is J-torsion then $- \otimes_A eS$ and $- \otimes_S Se$ induce inverse equivalences between the quotient categories (Mod-A)/(I-tors) and (Mod-S)/(J-tors). The condition (*) holds if B is a noetherian algebra over a field k, G acts by k-automorphisms, J is the Jacobson radical of B, $\dim_k B/J$ is finite and $I = J \cap A$.*

Proof. We first show that the map ψ of the Morita context is an isomorphism modulo J-torsion. By hypothesis, it is surjective so we need only show that $K := \ker \psi$ is J-torsion. Now Se is a projective left S-module so we have an exact sequence

$$0 \longrightarrow K \otimes_S Se \longrightarrow Se \otimes_A eS \otimes_S Se \longrightarrow Se$$

The last map is an isomorphism since $\phi : eS \otimes_S Se \longrightarrow eSe$ is. Hence $K \otimes_S Se = 0$. This shows that $K \simeq K/(K \otimes_S Se \otimes_A eS) \simeq K \otimes_S (S/SeS)$ which is J-torsion so ψ is indeed an isomorphism modulo J-torsion.

We now show that $- \otimes_S Se$ induces a functor from $(\mathrm{Mod} - S)/(J - \mathrm{tors}) \longrightarrow (\mathrm{Mod} - A)/(I - \mathrm{tors})$. By [G; chapitre III, §1, corollaire 2], it suffices to show that the composite

$$\mathrm{Mod} - S \xrightarrow{- \otimes_S Se} \mathrm{Mod} - A \longrightarrow (\mathrm{Mod} - A)/(I - \mathrm{tors})$$

is exact and maps J-torsion modules to 0. Exactness holds since Se is projective so we need to verify that $M \otimes_S Se$ is I-torsion whenever M is a J-torsion S-module. This follows from condition (*) which guarantees that

$$M \otimes_S SeI^n = M \otimes_S BeI^n = M \otimes_S BI^n e \subseteq M \otimes_S Je = MJ \otimes_S Se$$

Now we show that $- \otimes_A eS$ induces a functor on quotient categories by showing that the composite

$$\alpha : \mathrm{Mod} - A \xrightarrow{- \otimes_A eS} \mathrm{Mod} - S \longrightarrow (\mathrm{Mod} - S)/(J - \mathrm{tors})$$

is exact and annihilates I-torsion modules. We know that α is right exact so consider an injection $M \longrightarrow N$ of A-modules. Let $K := \ker(M \otimes_A eS \longrightarrow N \otimes_A eS)$. Since, Se is a projective S-module we have an exact sequence,

$$0 \longrightarrow K \otimes_S Se \longrightarrow M \otimes_A eS \otimes_S Se \longrightarrow N \otimes_A eS \otimes_S Se$$

Since $\phi : eS \otimes_A Se \longrightarrow A$ is an isomorphism, the map on the right is injective and $K \otimes_S Se = 0$. By the argument in the first paragraph, this shows that K is J-torsion. Hence α is exact. Now let M be an I-torsion module. Then

$$M \otimes_A eJ^n \subseteq M \otimes_A eIB = M \otimes_A IeB = MI \otimes_A eB$$

so $M \otimes_A eS$ is J-torsion so $- \otimes_A eS$ induces a functor on quotient categories as desired.

To see that the functors $- \otimes_A eS$ and $- \otimes_S Se$ induce inverse equivalences between quotient categories, we need to show that the composites $- \otimes_A eS \otimes_S Se$ and $- \otimes_S Se \otimes_A eS$ are naturally equivalent to the identity on the quotient categories. This follows from the fact that ϕ and ψ are isomorphisms modulo torsion.

Finally, we check the last assertion. We have $I \subset J$ so it remains to show that $J^n \subseteq IB$ for some integer n. The hypotheses ensure that all objects are defined over k. Now B is a finitely generated right A-module while A/I is finite dimensional over k since it embeds in B/J. Hence B/IB is finite dimensional and consequently annihilated by a power of J. This completes the proof of the proposition.

Now let k denote an algebraically closed field of characteristic zero. For the rest of the section, $(B, \mathfrak{n})$ will denote a regular complete local ring of dimension two. The first step towards proving regularity in codimension one is

Proposition 2.5.3 *Let $(B, \mathfrak{n})$ be a regular complete local ring of dimension two and S the corresponding skew group ring. The quotient category $(Mod-S)/(\mathfrak{n} - tors)$ has injective dimension at most two.*

Proof. Using the averaging trick (see [McR; theorem 7.5.6]), we see that gl. dim S = gl. dim $B = 2$. It thus suffices to show that the quotient functor $\pi : \text{Mod}-S \longrightarrow (\text{Mod}-S)/(\mathfrak{n} - \text{tors})$ maps injectives to injectives. By [G; chapitre III, §3, corollaire 1], π has a right adjoint which we shall denote by ρ. Furthermore, [G; chapitre III, §3, corollaire 2] states that every injective S-module has the form $I \oplus \rho(J)$ where I is the injective hull of an $\mathfrak{n}$-torsion S-module and J is an injective in $(\text{Mod}-S)/(\mathfrak{n} - \text{tors})$. Since $\pi\rho = \text{id}$, we are reduced to showing that injective hulls of $\mathfrak{n}$-torsion S-modules are $\mathfrak{n}$-torsion.

Let M be an $\mathfrak{n}$-torsion S-module and E be the injective hull of the underlying B-module M_B. We know already by proposition 1.3.5 that E is $\mathfrak{n}$-torsion. Now $F = \text{Hom}_B(S, E)$ is an injective S-module since $\text{Hom}_S(-, F)$ is naturally isomorphic to the exact functor $\text{Hom}_B(-, E)$. Consider the sequence of S-module morphisms,

$$M \to \text{Hom}_S(S, M) \longrightarrow \text{Hom}_B(S, M) \longrightarrow \text{Hom}_B(S, E) = F$$

All the maps above are injective so the injective hull of M embeds in F. Since S is finitely generated and E is $\mathfrak{n}$-torsion, F is $\mathfrak{n}$-torsion. Hence so is the injective hull of M. This completes the proof of the proposition.

One can actually show that the injective dimension of $(\text{Mod}-S)/(\mathfrak{n}-\text{tors})$ is at most the injective dimension of $(\text{Mod}-B)/\mathcal{T}(B)$. Presumably, this latter is one since when B is commutative, the corresponding punctured spectrum is a smooth scheme of dimension one.

Let u, v be topological generators for B as in proposition 2.2.2. As in §2.1, we let $V = ku + kv$ and identify $GL(V)$ with GL_2 using the basis u, v. Let ξ, η be primitive d-th roots of unity for some $d \in \mathbb{N}$ and $\sigma = \left(\begin{smallmatrix} \xi & 0 \\ 0 & \eta \end{smallmatrix}\right)$. We first prove regularity in codimension one for the following quotient singularities,

Proposition 2.5.4 *Suppose that the action of σ on V extends to B and let G be the group generated by σ. Then SeS contains $S\mathfrak{n}^{2d}$. Hence the quotient singularity $A = B^G$ is regular in codimension one in this case.*

Proof. To show $SeS \supseteq S\mathfrak{n}^{2d}$ it suffices to show that $u^i v^{2d-i} \in SeS$ for $0 \leq i \leq 2d$. We will prove this in the case where $i \geq d$. The case $i \leq d$ is similar. For any $0 \leq l < d$, SeS contains

$$z_l := u^l \sum_{0 \leq r < d} \sigma^r u^{i-l} v^{2d-i} = \sum_{0 \leq r < d} \sigma^r \xi^{rl} u^i v^{2d-i}$$

Since ξ is a primitive d-th root of unity, we see that SeS also contains $\frac{1}{d}\sum_{l=0}^{d-1} z_l = u^i v^{2d-i}$ as was to be shown. Hence $SeS \supseteq S\mathfrak{n}^{2d}$ and we may invoke proposition 2.5.2 to conclude that $\mathrm{Mod}-A/\mathcal{T}$ is naturally equivalent to $(\mathrm{Mod}-S)/(\mathfrak{n}-\mathrm{tors})$. This has finite injective dimension by the previous proposition so A is regular in codimension one.

For other examples, we need to reduce the question to the graded case. Let $\overline{B} = \mathrm{gr}_{\mathfrak{n}} B$ and $\overline{\mathfrak{n}} = \mathrm{gr}_{\mathfrak{n}} \mathfrak{n}$. Note that there is an $\mathfrak{n}$-adic filtration on S arising from the right B-module structure. Let $\overline{S}$ denote the associated graded ring $\mathrm{gr}_{\mathfrak{n}} S$ which is isomorphic to $G * \overline{B}$.

Lemma 2.5.5 *If $\overline{S}/\overline{S}e\overline{S}$ is $\overline{\mathfrak{n}}$-torsion then S/SeS is $\mathfrak{n}$-torsion.*

Proof. The lemma follows from the fact that the ideal $\mathrm{gr}_{\mathfrak{n}} SeS$ of $\overline{S}$ contains $\overline{S}e\overline{S}$.

Recall that $\sigma \in GL_2$ is a *pseudo-reflection* if $\sigma \neq 1$ and 1 is an eigenvalue for σ.

Proposition 2.5.6 *Suppose G is a finite subgroup of GL_2 without pseudo-reflections which acts linearly on B and $\overline{B} = gr_{\mathfrak{n}} B$ is commutative. Then B^G is regular in codimension one.*

Proof. By the previous lemma, it suffices to show that $\overline{S}/\overline{S}e\overline{S}$ is $\overline{\mathfrak{n}}$-torsion. This is well-known. We may choose homogeneous elements $f_1, f_2 \in \overline{A} := \overline{B}^G$ such that $\overline{B}/(f_1, f_2)$ is $\overline{\mathfrak{n}}$-torsion. By [Yos; lemma 10.7.2], the hypotheses ensure that the map Spec $\overline{B} \longrightarrow$ Spec $\overline{A}$ is étale away from the vertex $\overline{\mathfrak{n}}$. This implies in particular that $A' = \overline{A}[f_i^{-1}] \longrightarrow B' = \overline{B}[f_i^{-1}]$ for $i = 1, 2$ is separable. This means that the multiplication map $\mu : B' \otimes_{A'} B' \longrightarrow B'$ has a section ϕ which is also a (B', B')-bimodule homomorphism. Let $\phi(1) = \sum x_j \otimes y_j$. Note that for $\sigma \in G$, $\sigma \otimes 1$ induces a map $B' \otimes_{A'} B' \longrightarrow B'$. Hence for any $b \in B'$ we have

$$\sigma(b) \sum_j \sigma(x_j) y_j = (\sigma \otimes 1)(\sum_j b x_j \otimes y_j) = (\sigma \otimes 1)(\phi(b)) = (\sigma \otimes 1)(\sum_j x_j \otimes y_j b) = b \sum_j \sigma(x_j) y_j$$

If $\sigma \neq 1$, then choosing b such that $\sigma(b) \neq b$ shows that $\sum_j \sigma(x_j) y_j = 0$. Let $n \in \mathbb{N}$ be large enough that $f_i^n x_j, f_i^n y_j$ are in $\overline{B}$ for all j. Then $\overline{S}e\overline{S}$ contains

$$\sum_{\sigma, j} f_i^n x_j \sigma f_i^n y_j = f_i^{2n} \sum_{\sigma, j} \sigma \sigma(x_j) y_j = f_i^{2n} \sum_j x_j y_j = f_i^{2n} (\mu\phi)(1) = f_i^{2n}$$

Since $\overline{B}/(f_1^n, f_2^n)$ is $\overline{\mathfrak{n}}$-torsion, $\overline{S}/\overline{S}e\overline{S}$ must also be $\overline{\mathfrak{n}}$-torsion.

We have thus shown that every special quotient surface singularity is regular in codimension one with the possible exception of the rings B^G where G is non-diagonal and $\mathrm{gr}_{\mathfrak{n}} B$ is not commutative. The result is not known in this case but the method of proof above certainly fails as we shall see presently. Let $B = k\langle\langle u, v\rangle\rangle/(vu + uv)$ and G be the cyclic group of order four generated by $\tau = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The invariant ring $A = B^G$ is regular in codimension one since it is just the commutative power series ring in two variables $u^2 + v^2, uv$. However,

Claim 2.5.7 *The bimodule S/SeS is not $\mathfrak{n}$-torsion.*

Proof. We make a change of variables $u' = u + iv, v' = u - iv$. The action of τ is given by $\tau(u') = iu', \tau(v') = -iv'$. The defining relation in the new variables is $u'u' = v'v'$. We use Bergman's diamond lemma to compute a Gröbner basis with respect to these variables. Checking the monomial $u'u'u'$ yields a new relation $v'v'u' = u'v'v'$ but otherwise, all overlap ambiguities are resolved. We hence obtain a Gröbner basis consisting of monomials of the form $u'v'u'v' \dots u'v'v' \dots v'$ or $v'u'v'u'v' \dots u'v'v' \dots v'$. We wish to show that $(u'v')^n$ does not lie in SeS for any $n \in \mathbb{N}$. Let $x_i = (u'v')^i, y_i = x_i u', z_{n-i} = v' x_{n-i-1}$. Now given any two monomials, m_1, m_2 in the Gröbner basis,

$\tau^j(m_1)m_2$ is a monomial which can be rewritten as a scalar multiple of a monomial in the Gröbner basis. It will be a scalar multiple of x_n if and only if $m_1 = x_l, m_2 = x_{n-l}$ or $m_1 = y_l, m_2 = z_{n-l}$. Now given $a, b \in B$, $aeb = \frac{1}{4}\sum_{j=0}^{3} \tau^j\tau^j(a)b$. Hence if $x_n \in SeS = BeB$, then there exist $a_l \in k, 0 \leq l \leq n$ and $b_l \in k, 0 \leq l < n$ such that

$$\begin{aligned} x_n &= \sum_{j=0}^{3}\sum_{l=0}^{n} a_l\tau^j\tau^j(x_l)x_{n-l} + \sum_{j=0}^{3}\sum_{l=0}^{n-1} b_l\tau^j\tau^j(y_l)z_{n-l} \\ &= [(\sum a_l + \sum b_l) + (\sum a_l + i\sum b_l)\tau + (\sum a_l - \sum b_l)\tau^2 + (\sum a_l - i\sum b_l)\tau^3]x_n \end{aligned}$$

Comparing coefficients of τ^j we see that we cannot solve for the a_l, b_l. This contradiction shows that $(u'v')^n = x_n$ does not lie in SeS for any n.

Chapter 3

Singularities of Type A_{d-1}

Throughout this chapter, we fix an algebraically closed field k of characteristic 0.

The approach in the previous chapter for finding noncommutative analogues of rational double points was to examine invariant rings of noncommutative analogues of $k[[x,y]]$. One problem with this method is that it is difficult to describe such rings in terms of (topological) generators and relations.

Properties of filtered rings are often determined by their associated graded rings. This suggests that nice classes of rings can be defined by imposing a condition on the associated graded ring. For example, the class of regular complete local rings of dimension two was defined this way (see definition 1.3.1).

In this chapter, we adopt this approach to define a class of noncommutative analogues of rational double points. To this end, let $\overline{B}$ be the associated graded ring of a regular local ring of dimension two i.e. $k\langle u,v\rangle/(vu - \overline{q}uv)$ or $k\langle u,v\rangle/(vu - uv - v^2)$. Let G be as usual, a subgroup of $SL(\overline{B}_1)$ whose action extends to $\overline{B}$. We will say that G is *diagonalisable* if by changing variables u, v in a manner which preserves the relation $vu = \overline{q}uv$ or $vu = uv + v^2$, G can be made into diagonal subgroup of $SL(\overline{B})$. By lemma 2.2.4, if $\overline{B}$ is not commutative, then G is diagonalisable if and only if it is diagonal.

Definition 3.1.8 *Let A be a complete local ring satisfying standard hypotheses 1.1.6, whose associated graded ring $gr_F A \simeq \overline{B}^G$. We say that A is a* singularity of type*:*

- A_{d-1} *if G is a diagonalisable subgroup which is cyclic of order d.*
- D_d *if G is a binary dihedral group of order $4(d-2)$ which is not diagonalisable.*
- E_6, E_7 *or* E_8 *if G is binary tetrahedral, octahedral or icosahedral respectively.*

Often, we will omit the subscript and speak of singularities of type A, D or E accordingly. Note that by the proof of proposition 2.2.6, if G is diagonal then it is automatically cyclic. Note also that the binary dihedral group of order four is just the cyclic group of order four. This D_3 case is the only instance where G is cyclic but not diagonalisable.

Note that by proposition 2.2.5 we have,

Remark 3.1.9 *Every special quotient surface singularity is a singularity of type A, D or E.*

Note: Recall from proposition 2.4.10, that any singularity A of type A,D or E is Auslander-Gorenstein of dimension two. Furthermore, since $\overline{B}$ is a domain, $\mathrm{gr}_F A$ is a domain and hence so is A.

We will concentrate on singularities of type A_{d-1} so let A be such a ring. Let $\overline{A}_q$ be the graded ring defined by the relations (2.1) of proposition 2.2.6 and $\overline{A}_J$ the graded ring defined by the relations (2.2). We call these relations the *standard relations* for $\overline{A}_q$ and $\overline{A}_J$. Proposition 2.2.6 shows that the associated graded ring of A is either $\overline{A}_q$ or $\overline{A}_J$. In the first case, we will say that A is a *q-singularity* and in the second case we will say that A is a *Jordan singularity.*

We wish to determine all singularities of type A_{d-1}. Let P be the noncommutative power series ring $k\langle\langle x,y,z\rangle\rangle$. We set the degree of x and z to be d and the degree of y to be 2. We filter P using the degree as in §2.1. Our first step will be construct a family of singularities of type A_{d-1} depending on a parameter $g \in kz + F^3P$. Note that when $d \geq 3$ then $kz \subset F^3P$ so it plays no role. The inclusion of kz is to account for Jordan singularities in the $d = 2$ case.

Theorem 3.1.10 *Suppose A is a quotient of $P = k\langle\langle x,y,z\rangle\rangle$ with the induced filtration and defining relations,*

$$yx = qxy + xg \quad , \quad zx = (qy+g)^d \quad , \quad zy = qyz + gz \quad , \quad y^d = xz \tag{3.1}$$

where $g \in kz + F^3P$. Then A is a singularity of type A_{d-1}. It is a Jordan singularity when $g \notin F^3P$ and $q = 1$ and a q-singularity otherwise.

N.B. The origin of these relations will be clear from the proof of the next theorem.

Proof. We wish to apply proposition 2.1.4 to determine the associated graded ring of A. We order monomials as in proposition 2.3.5 and note that the relations 3.1 are compatible with this partial order. We need to check overlaps for the monomials y^dx, zyx, zy^d and y^{d+1}.

$$y^{d-1}(yx) \mapsto y^{d-1}x(qy+g) \mapsto y^{d-2}x(qy+g)^2 \mapsto \ldots \mapsto x(qy+g)^d$$

$$(y^d)x \mapsto xzx \mapsto x(qy+g)^d$$

verifying the overlap check for y^dx. The overlap check for zy^d is similar so we consider zyx.

$$z(yx) \mapsto zx(qy+g) \mapsto (qy+g)^{d+1} \quad \text{and} \quad (zy)x \mapsto (qy+g)zx \mapsto (qy+g)^{d+1}$$

which also agree. Finally, we have

$$(y^{d+1})y \mapsto xzy \mapsto qxyz + xgz \quad \text{and} \quad y(y^{d+1}) \mapsto yxz \mapsto qxyz + xgz$$

This verifies the hypotheses for proposition 2.1.4. Thus if $\overline{x},\overline{y},\overline{z}$ are the images of x,y,z in $\mathrm{gr}_F A$ and $\overline{g}$ is the image of g in F^2P/F^3P, then $\mathrm{gr}_F A$ is the k-algebra with generators $\overline{x}\,,\overline{y}\,,\overline{z}$ and defining relations

$$\overline{y}\,\overline{x} = q\overline{x}\,\overline{y} + \overline{x}\,\overline{g}\ ,\ \overline{z}\,\overline{x} = (q\overline{y} + \overline{g})^d\ ,\ \overline{z}\,\overline{y} = q\overline{y}\,\overline{z} + \overline{g}\,\overline{z}\ ,\ \overline{y}^{\,d} = \overline{x}\,\overline{z}$$

These are in fact defining relations for $\overline{A}_q$ or $\overline{A}_J$ too. To see this, observe first that when $g \in F^3P$ so that $\overline{g} = 0$, then the above equations reduce to the standard relations ((2.1) of proposition 2.2.6) for $\overline{A}_q$. Suppose now that $\overline{g} = a\overline{z}$ for some nonzero $a \in k$. If $q = 1$ then the linear change of variables $\overline{x} \mapsto a^{-2}\overline{x}\ ,\overline{y} \mapsto a^{-1}\overline{y}\ ,\overline{z} \mapsto \overline{z}$ gives the standard relations ((2.2) of proposition 2.2.6) for $\overline{A}_J$. Similarly, when $q \neq 1$ the linear change of variables

$$\overline{x} \mapsto \overline{x} + a(q+1)(q-1)^{-1}\overline{y} + q(q+1)^2(q-1)^{-2}\overline{z}\ ,\ \overline{y} \mapsto \overline{y} + a(q-1)^{-1}\overline{x}\ ,\ \overline{z} \mapsto \overline{z}$$

gives the standard relations ((2.1) of proposition 2.2.6) for $\overline{A}_q$. This finishes the proof of the theorem.

We have a converse result.

Theorem 3.1.11 *Any singularity A of type A_{d-1} is isomorphic to the quotient of $P = k\langle\langle x,y,z\rangle\rangle$ with defining relations (3.1) of theorem 3.1.10. More precisely, let $\overline{x},\overline{y},\overline{z}$ be generators of $gr_F A$ satisfying the standard relations for $\overline{A}_q$ or $\overline{A}_J$ and x,y,z be lifts of $\overline{x},\overline{y},\overline{z}$ to A. Suppose the last relation $y^d = xz$ holds. If A is a q-singularity then there exists $g \in F^3A$ such that the other relations in (3.1) also hold. If A is a Jordan singularity then there exists $h \in F^3A$ such that the relations in (3.1) hold with $g = z + h$.*

Proof. Suppose first that A is a q-singularity. We saw in proposition 2.3.5 that the hypotheses for Bergman's diamond lemma were satisfied for $gr_F A$. Hence, we may apply proposition 2.1.4 which shows that A is a quotient of P with defining relations:

$$(*) \qquad yx = qxy + z' \ , \ zx = q^d y^d + y' \ , \ zy = qyz + x' \ , \ y^d = xz + w'$$

where $x', z' \in F^{d+3}A$, $y', w' \in F^{2d+1}A$. Observe that in the last relation we can eliminate the higher order term as follows. First note that propositions 2.3.5 and 2.1.1 yield the topological Gröbner basis $\{x^i y^j z^k | i, k \geq 0, 0 \leq j < d\}$ for A. Hence, since the degree r of w' is at least $2d+1$, we can write $w' = x\tilde{z} + \tilde{x}z$ with $\tilde{x}, \tilde{z} \in F^{r-d}A$. Replacing x with $x_1 = x + \tilde{x}$ and z with $z_1 = z + \tilde{z}$ we find $y^d = x_1 z_1 - w''$ where $w'' = \tilde{x}\tilde{z} \in F^{2r-2d}A$ which is deeper in the filtration than w'. Iterating this procedure produces the desired elimination so henceforth we shall assume that $w' = 0$.

We now determine some consequences of associativity.

$$(y^d)x = xzx = q^d xy^d + xy' = q^d x^2 z + xy'$$

This also equals,

$$(y \dots (yx) \dots) = q^d x^2 z + \sum_{i=0}^{d-1} q^i y^{d-1-i} z' y^i$$

Similarly,

$$qxyz + z'z = yxz = y(y^d) = (y^d)y = xzy = qxyz + xx'$$

This yields two "diamond lemma" equations,

$$xy' = \sum_{i=0}^{d-1} q^i y^{d-1-i} z' y^i \qquad (3.2)$$

$$z'z = xx' \qquad (3.3)$$

Writing out x' and z' in terms of the Gröbner basis and comparing coefficients in equation (3.3), we see we must have $x' = gz$ and $z' = xg$ for some $g \in F^3A$. Note that by induction, $y^i x = x(qy+g)^i$ so (3.2) gives,

$$xy' = \sum_{i=0}^{d-1} q^i x(qy+g)^{d-1-i} g y^i$$

Since A is a domain, we may left cancel by x and then use the noncommutative binomial theorem to deduce

$$y' = (qy+g)^d - q^d y^d$$

proving the theorem when A is a q-singularity.

Suppose now that A is a Jordan singularity. As in the previous case, we may apply proposition 2.1.4 to deduce that A is a quotient of P with defining relations

$$yx = xy + xz + z' \ , \ zx = y^2 + 2yz + 2z^2 + y' \ , \ zy = yz + z^2 + x' \ , \ y^d = xz + w'$$

As before, we may change variables to eliminate w'. We consider some consequences of associativity.

$$xyz + xz^2 + z'z = yxz = y^3 = xzy = xyz + xz^2 + xx'$$

Hence we get $xx' = z'z$ as in equation (3.3). It is thus possible to find $h \in F^3P$ such that $x' = hz, z' = xh$. This gives all the relations in (3.1) except the second one. We determine y' by considering the monomial zy^2.

$$z(y^2) = zxz = y^2z + 2yz^2 + 2z^3 + y'z$$

$$(zy)y = yzy + z^2y + hzy = y^2z + yz^2 + yhz + zyz + z^3 + zhz + hyz + hz^2 + h^2z$$

$$= y^2z + yz^2 + yhz + yz^2 + z^3 + hz^2 + z^3 + zhz + hyz + hz^2 + h^2z$$

On comparing these two equations we see that

$$y' = yh + 2hz + zh + hy + h^2$$

Substituting back in gives

$$zx = y^2 + 2yz + 2z^2 + yh + 2hz + zh + hy + h^2$$

We compare this with the right hand side of the corresponding relation in (3.1):

$$(y + z + h)^2 = y^2 + yz + zy + z^2 + yh + hz + zh + hy + h^2$$

$$= y^2 + 2yz + 2z^2 + yh + 2hz + zh + hy + h^2$$

Since these coincide, we are done.

We now consider the question of when a q-singularity of type A_{d-1} is a special quotient surface singularity, that is, we wish to find smooth d-fold covers of "Spec A". Recall that $q \in k$ is said to be *generic with respect to d-th roots of unity* if 1 is the only power of q which is a d-th root of unity. This is equivalent to the fact that either q is not a root of unity or q is a primitive n-th root of unity where n is coprime to d. Note for such q, the geometric sums $\sum_{i=0}^{d-1} q^{li}$ are non-zero for every $l \in \mathbb{Z}$. To simplify notation, we will suppress the P in F^pP and denote it by just F^p.

Theorem 3.1.12 *Let $q \in k$ be generic with respect to d-th roots of unity and A be a q-singularity of type A_{d-1}. Let $\overline{q}$ be any d-th root of q which is also generic with respect to d-th roots of unity. (N.B. Such a $\overline{q}$ always exists). Then there exists a complete regular local ring of dimension two, $B = k\langle\langle u, v\rangle\rangle/(vu - \overline{q}uv - c)$ such that $A \simeq B^G$ where G is the cyclic group $< \sigma >$ of order d which acts on B via, $\sigma : u \mapsto \epsilon u\,,\, v \mapsto \epsilon^{-1}v$ for any primitive d-th root of unity ϵ.*

Proof. By theorem 3.1.11 we may assume A has the relations (3.1). We consider g in the relations as a noncommutative power series in x, y, z. There are many choices for g and we pick one at random. We set $P = k\langle\langle u\,,\, v\rangle\rangle$ where u, v have degree one and let G act by $\sigma : u \mapsto \epsilon u\,,\, v \mapsto \epsilon^{-1}v$. Let F be the natural filtration on P induced by degree. We need to find $x, y, z, c \in P^G$ such that x, y, z satisfy relations (3.1) modulo the relation $vu = \overline{q}uv + c$. It turns out that we may choose $y = \overline{q}^{(1-d)/2}uv$ where $\overline{q}^{(1-d)/2}$ denotes either of the two square roots of $\overline{q}^{(1-d)}$. The key reduction in the proof of the theorem is the following lemma.

Lemma 3.1.13 *Let $y = \overline{q}^{(1-d)/2}uv, x_0 = u^d, z_0 = v^d, c_0 = 0$. It suffices to find inductively $x_i, z_i, c_i \in P^G$ such that the following equations hold,*

$$\left.\begin{array}{rcll} yx_i & \equiv & qx_iy + x_ig_i & \mod (r_i) + ku^dv^{2+i} + F^{d+3+i} \\ z_iy & \equiv & qyz_i + g_iz_i & \mod (r_i) + uF^{1+i}v^d + F^{d+3+i} \\ y^d & \equiv & x_iz_i & \mod (r_i) \end{array}\right\} (\dagger)$$

where $r_i = vu - \overline{q}uv - c_i$, $g_i = g(x_i, y, z_i)$ and also $x_{i+1} \equiv x_i, z_{i+1} \equiv z_i \mod F^{d+1+i}$ and $c_{i+1} \equiv c_i \mod F^{3+i}$. The congruences hold when $i = 0$.

Proof. We need to deal simultaneously with all the regular complete local rings $P/(r_i)$ so it is convenient to work in P as follows. Recall from lemma 2.2.3 that the diamond lemma on $P/(r_i)$ yields a topological Gröbner basis consisting of the monomials $\{u^j v^l\}$. The diamond lemma actually gives more. It gives a projection $\rho_i \in \mathrm{End}_k(P)$ onto the completion of the linear span of the Gröbner basis such that, for $a, b \in P$, $a \equiv b \bmod (r_i)$ if and only if $\rho_i(a) = \rho_i(b)$. This projection comes from the reduction system "replace the submonomial vu with $\bar{q}uv + c_i$" (see [B; p.180] or theorem 4.2.2). Also, for any $p \in \mathbb{N}$, the congruence $a \equiv b \bmod (r_i) + F^p$ is equivalent to $\rho_i(a) \equiv \rho_i(b) \bmod F^p$ since the ρ_i's do not decrease degree. In general, the computation of $\rho_i(a)$ involves an infinite number of replacements $vu \mapsto \bar{q}uv + c_i$ but the computation of $\rho_i(a)$ modulo F^p stops after a finite number of replacements.

Let x, z, c be the limits of $\{x_i\}, \{z_i\}, \{c_i\}$ and $\rho \in \mathrm{End}_k P$ the projection associated to c. We wish to show that $B = P/(r)$ is the sought for "smooth d-fold cover" of the theorem. Note that since $c_i, c \in P^G$, G acts on $P/(r_i)$ and $P/(r)$.

Theorem 3.1.11 shows there exists $\tilde{g} \in P^G$ such that,

$$\left.\begin{array}{rcll} yx_i & \equiv & qx_iy + x_i\tilde{g} & \mathrm{mod}(r_i) \\ z_iy & \equiv & qyz_i + \tilde{g}z_i & \mathrm{mod}(r_i) \\ y^d & \equiv & x_iz_i & \mathrm{mod}(r_i) \end{array}\right\} (*)$$

We compare the first congruence with the first congruence of (†). Note that the lowest degree term of $\rho_i(x_i(g_i - \tilde{g}))$ is the product of u^d with the lowest degree term of $\rho_i(g_i - \tilde{g})$. Hence

$$\rho_i(g_i - \tilde{g}) \equiv 0 \qquad \mathrm{mod}\ kv^{2+i} + F^{3+i}$$

Similarly, comparing the second congruences of (*) and (†) we find

$$\rho_i(g_i - \tilde{g}) \equiv 0 \qquad \mathrm{mod}\ uF^{1+i} + F^{3+i}$$

Now the intersection of $kv^{2+i} + F^{3+i}$ and $uF^{1+i} + F^{3+i}$ is F^{3+i} so in fact $g_i \equiv \tilde{g} \mod (r_i) + F^{3+i}$ and the first two congruences of (†) become,

$$yx_i \equiv qx_iy + x_ig_i \qquad \mathrm{mod}\ (r_i) + F^{d+3+i} \tag{3.4}$$

$$z_iy \equiv qyz_i + g_iz_i \qquad \mathrm{mod}\ (r_i) + F^{d+3+i} \tag{3.5}$$

We can now show that $A \simeq B^G$. It is clear that B^G is (topologically) generated by the images of x, y, z in B so it suffices to show they satisfy the desired defining relations. Now let g also denote $g(x, y, z)$. Modulo F^{d+3+i} we have,

$$0 \equiv \rho_i(yx_i - qx_iy - x_ig_i) \equiv \rho_i(yx - qxy - xg) \equiv \rho(yx - qxy - xg)$$

The last congruence follows from the fact that if W is of degree $\geq d+2$ then $\rho(W) - \rho_i(W) \in F^{d+3+i}$ since $c \equiv c_i \mod F^{3+i}$. Hence in B we have $yx = qxy + xg$ which is the first of the defining relations in (3.1). Similarly we have $y^d = xz$ so we may apply theorem 3.1.11 to find relations

$$yx = qxy + xg' \quad , \quad zx = (qy + g')^d \quad , \quad zy = qyz + g'z$$

Since B is a domain we must have $g = g'$ verifying the lemma.

We now return to the proof of the theorem for which we need to verify the inductive step in the lemma. We may assume the congruences (3.4), (3.5) and the last congruence of (†) hold for some particular value of i and set $c_{i+1} = c_i + \Delta c, x_{i+1} = x_i + \Delta x, z_{i+1} = z_i + \Delta z$ where $\Delta c \in F^{3+i}P^G$ and $\Delta x, \Delta z \in F^{d+1+i}P^G$ are to be solved for. We may replace the last congruence in (†) with the weaker congruence

$$y^d \equiv x_i z_i \qquad \mod (r_i) + F^{2d+1+i} \tag{3.6}$$

since, by the argument in theorem 3.1.11, we can alter x_i and z_i in degrees $\geq d+1+i$ to obtain $y^d \equiv x_i z_i \bmod (r_i)$.

For the rest of the proof, it may be helpful to keep in mind that we are only interested in Δc modulo F^{4+i} and $\Delta x, \Delta z$ modulo F^{d+2+i} and that the congruences (†) we wish to solve lead to linear congruences.

We consider the first congruence in (*) and determine the effect of replacing c_i with c_{i+1}. As has been noted already, any term W of degree $\geq d+3$ satisfies $\rho_i(W) \equiv \rho_{i+1}(W) \mod F^{d+4+i}$. This accounts for all terms in the congruence except the lowest degree terms of yx_i and $x_i y$. The lowest degree term of $x_i y$ is $u^{d+1}v$ which is already in lexicographic order so we deal with the former,

$$\begin{aligned}
\rho_i((\overline{q}^{(1-d)/2}uv)u^d) &= qu^d(\overline{q}^{(1-d)/2}uv) + \overline{q}^{(1-d)/2}\rho_i(\sum_{j=1}^{d}\overline{q}^{j-1}u^j c_i u^{d-j}) \\
&= qu^d(\overline{q}^{(1-d)/2}uv) + \overline{q}^{(1-d)/2}\rho_i(\sum_{j=1}^{d}\overline{q}^{j-1}u^j c_{i+1} u^{d-j}) - \\
&\qquad\qquad \overline{q}^{(1-d)/2}\rho_i(\sum_{j=1}^{d}\overline{q}^{j-1}u^j (\Delta c) u^{d-j}) \\
&\equiv \rho_{i+1}((\overline{q}^{(1-d)/2}uv)u^d) - \overline{q}^{(1-d)/2}\rho_{i+1}(\sum_{j=1}^{d}\overline{q}^{j-1}u^j (\Delta c) u^{d-j}) \mod F^{d+4+i}
\end{aligned}$$

which yields,

$$yx_i \equiv qx_i y + x_i \tilde{g} + \overline{q}^{(1-d)/2}\sum_{j=1}^{d}\overline{q}^{j-1}u^j(\Delta c)u^{d-j} \qquad \mod (r_{i+1}) + F^{d+4+i}$$

We carry out a similar computation with the other congruences in (*). The second is similar to the first so we consider the last. Set $t(n) = \frac{1}{2}n(n+1)$ to be the n-th triangular number. Again, the key term will be the lowest degree one,

$$\begin{aligned}
\rho_i((\overline{q}^{(1-d)/2}uv)^d) &= \overline{q}^{-t(d-1)}\rho_i(u(\overline{q}uv + c_i)^{d-1}v) \\
&= \overline{q}^{-t(d-1)}\rho_i(u[(\overline{q}uv + c_i)^{d-1} - (\overline{q}uv)^{d-1}]v + \overline{q}^{d-1}u(uv)^{d-1}v) \\
&= \overline{q}^{-t(d-1)}\rho_i(\sum_{j=1}^{d-1}\overline{q}^{t(d-1)-t(j)}u^{d-j}[(\overline{q}uv + c_i)^j - (\overline{q}uv)^j]v^{d-j}) \ + \ u^d v^d
\end{aligned}$$

Now the ρ_i's preserve left multiplication by u and right multiplication by v so we may concentrate on the square bracketed term,

$$\begin{aligned}
\rho_i[(\overline{q}uv + c_i)^j - (\overline{q}uv)^j] &= \rho_i[(\overline{q}uv + c_{i+1} - \Delta c)^j - (\overline{q}uv)^j] \\
&\equiv \rho_i[(\overline{q}uv + c_{i+1})^j - (\overline{q}uv)^j] - \rho_i[\overline{q}^{j-1}\sum_{l=1}^{j}(uv)^{l-1}(\Delta c)(uv)^{j-l}] \\
&\qquad\qquad \mod F^{2j+2+i} \\
&\equiv \rho_{i+1}[(\overline{q}uv + c_{i+1})^j - (\overline{q}uv)^j] - \rho_{i+1}[\overline{q}^{j-1}\sum_{l=1}^{j}(uv)^{l-1}(\Delta c)(uv)^{j-l}] \\
&\qquad\qquad \mod F^{2j+2+i}
\end{aligned}$$

Substituting back in and reversing the above computation for $i+1$ instead of i gives,

$$\rho_i((\overline{q}^{(1-d)/2}uv)^d) \equiv \rho_{i+1}((\overline{q}^{(1-d)/2}uv)^d) - \rho_{i+1}[\sum_{j=1}^{d-1}\sum_{l=1}^{j}\overline{q}^{j-1-t(j)}u^{d-j}(uv)^{l-1}(\Delta c)(uv)^{j-l}v^{d-j}] \mod F^{2d+2+i}$$

Hence,

$$y^d \equiv x_i z_i + \sum_{j=1}^{d-1}\sum_{l=1}^{j}\overline{q}^{j-1-t(j)}u^{d-j}(uv)^{l-1}(\Delta c)(uv)^{j-l}v^{d-j} \qquad \mod (r_{i+1}) + F^{2d+2+i}$$

Incorporating also the effect of changing x_i, z_i to x_{i+1}, z_{i+1} we find,

$$\left.\begin{aligned}
yx_{i+1} &\equiv qx_{i+1}y + x_{i+1}\tilde{g} + \overline{q}^{(1-d)/2}\sum_{j=1}^{d}\overline{q}^{j-1}u^j(\Delta c)u^{d-j} + y(\Delta x) - q(\Delta x)y \\
&\qquad \mod (r_{i+1}) + F^{d+4+i} \\
z_{i+1}y &\equiv qyz_{i+1} + \tilde{g}z_{i+1} + \overline{q}^{(1-d)/2}\sum_{j=1}^{d}\overline{q}^{j-1}v^{d-j}(\Delta c)v^j + (\Delta z)y - qy(\Delta z) \\
&\qquad \mod (r_{i+1}) + F^{d+4+i} \\
y^d &\equiv x_{i+1}z_{i+1} + \sum_{j=1}^{d-1}\sum_{l=1}^{j}\overline{q}^{j-1-t(j)}u^{d-j}(uv)^{l-1}(\Delta c)(uv)^{j-l}v^{d-j} - (\Delta x)z_{i+1} - x_{i+1}(\Delta z) \\
&\qquad \mod (r_{i+1}) + F^{2d+2+i}
\end{aligned}\right\} (\ddagger)$$

Let $\Delta g = \rho_i(g_i - \tilde{g})$ which, as was noted in the proof of the lemma, lies in $F^{3+i}P^G$. Note that since $g \in F^3$ we have $g_{i+1} - g_i \in F^{4+i}$. Also,

$$\rho_{i+1}(g_{i+1} - \tilde{g}) \equiv \rho_i(g_{i+1} - \tilde{g}) \qquad \mod F^{4+i}$$

Hence on comparing the congruences ($\ddagger$) with the desired ones ($\dagger$) and (3.6), we see the theorem amounts to solving,

$$x_{i+1}(\Delta g) \equiv \overline{q}^{(1-d)/2}\sum_{j=1}^{d}\overline{q}^{j-1}u^j(\Delta c)u^{d-j} + y(\Delta x) - q(\Delta x)y \qquad \mod (r_{i+1}) + ku^dv^{3+i} + F^{d+4+i} \tag{3.7}$$

$$(\Delta g)z_{i+1} \equiv \overline{q}^{(1-d)/2}\sum_{j=1}^{d}\overline{q}^{j-1}v^{d-j}(\Delta c)v^j + (\Delta z)y - qy(\Delta z) \qquad \mod (r_{i+1}) + uF^{2+i}v^d + F^{d+4+i} \tag{3.8}$$

$$0 \equiv \sum_{j=1}^{d-1}\sum_{l=1}^{j}\overline{q}^{j-1-t(j)}u^{d-j}(uv)^{l-1}(\Delta c)(uv)^{j-l}v^{d-j} - (\Delta x)z_{i+1} - x_{i+1}(\Delta z) \mod (r_{i+1}) + F^{2d+2+i} \tag{3.9}$$

The above congruences are linear in $\Delta x, \Delta z, \Delta c, \Delta g$ and admit the trivial solution when $\Delta g \in F^{4+i}$. Hence, we may assume $\Delta g = u^\beta v^\gamma$ where $\beta + \gamma = 3 + i$. There are two cases:

Case 1 $\beta > 0$: Let $\Delta x = 0$ and $\Delta c = \alpha u^{\beta} v^{\gamma}$. Note that $\Delta c \in F^{3+i}P^G$ since $\Delta g \in F^{3+i}P^G$. Since,

$$\rho_{i+1}(\sum_{j=1}^{d} \overline{q}^{j-1} u^j (\Delta c) u^{d-j}) \equiv \alpha(\sum_{j=1}^{d} \overline{q}^{j-1+\gamma(d-j)}) u^{d+\beta} v^{\gamma} \bmod F^{d+4+i}$$

solving (3.7) depends on the invertibility of $\sum_{j=1}^{d} \overline{q}^{j-1+\gamma(d-j)}$. But our hypothesis on $\overline{q}$ guarantees this geometric sum is non-zero. Now ρ_{i+1} of the double sum term in (3.9) is a multiple of $u^{d+\beta-1} v^{d+\gamma-1}$ modulo F^{2d+2+i} so we may solve (3.9) by setting Δz to be an appropriate multiple of $u^{\beta-1} v^{d+\gamma-1}$. Note that $\Delta z \in F^{d+1+i}P^G$ as was required. Finally, to check (3.8) we need only observe that ρ_{i+1} of every term lies in uPv^d.
Case 2 $\beta = 0$: Let $\Delta z = 0$. As in the previous case we can solve (3.8) to find $\Delta c = \alpha v^{\gamma}$ and (3.9) to obtain Δx as some multiple of $u^{d-1} v^{\gamma-1}$. The verification of (3.7) is as in the previous case.

This completes the proof of the theorem.

Chapter 4

Twisted Multihomogeneous Coordinate Rings

4.1 Introduction

Given a projective scheme X over a field k and an ample invertible sheaf L on X, one can construct the homogeneous coordinate ring

$$B = \oplus_{i \geq 0} H^0(X, L^{\otimes i})$$

which reflects the geometry of X in the sense of Serre's theorem (see [FAC; chapter III §2] or [EGA; §2.7]). In [AV], Van den Bergh defined a noncommutative version of this ring by using the same formula but replacing the invertible sheaf with its noncommutative analogue, an invertible bimodule (see below for definitions). Furthermore, this twisted homogeneous coordinate ring, denoted by $B(X; L)$, reflects the geometry of X in the same way the commutative coordinate ring does. This tight connection with geometry enabled Artin and Van den Bergh to prove that $B(X; L)$ is noetherian when L is ample.

Let $B = B(X; L)$ and $\mathfrak{m} = B_1 \oplus B_2 \oplus \dots$ denote its augmentation ideal. Then one expects the Rees algebra $B[\mathfrak{m}t]$ to be noetherian since, algebro-geometrically, it corresponds to a blowing up. Similarly, since the tensor product corresponds to the fibre product, one expects the tensor product of twisted homogeneous coordinate rings to be noetherian. The main goal of this paper is to prove these two facts along the lines of [AV] by interpreting these rings from a geometric standpoint.

In both the case of the Rees algebra and the tensor product, the resulting algebra is bigraded, which in commutative algebra corresponds to biprojective geometry. Given two line bundles L, M on the projective scheme X one can form the bihomogeneous coordinate ring,

$$B(X; L, M) = \oplus_{i,j \geq 0} H^0(X, L^{\otimes i} \otimes M^{\otimes j})$$

We mimic this construction replacing L and M with invertible bimodules. The only obstacle is that $M \otimes L$ and $L \otimes M$ are not necessarily isomorphic, let alone canonically isomorphic, so it is unclear how to define multiplication. The construction of the twisted bihomogeneous coordinate ring thus involves the additional data of a commutation relation, which is a bimodule isomorphism $\phi : M \otimes L \longrightarrow L \otimes M$. We in fact construct multi-homogeneous coordinate rings from an arbitrary number of invertible bimodules. It turns out that the Rees algebra and tensor product of twisted homogeneous coordinate rings are indeed examples of such rings. We show that the multi-homogeneous coordinate ring reflects the geometry as per Serre's theorem if the invertible bimodules satisfy some analogue of ampleness. Unfortunately, such rings are not automatically noetherian. However, we do give a criterion for the ascending chain condition (ACC) to hold which includes the case of the Rees algebra and tensor product of twisted homogeneous coordinate rings.

4.2 Bimodule Generators and Relations

We wish to construct a bimodule algebra analogous to the construction of algebras via generators and relations. Before recalling Van den Bergh's notion of a bimodule algebra we consider algebras in a general setting.

Let $\mathcal{C}$ be a monoidal, abelian category with infinite direct sums. For the definition of a monoidal category we refer the reader to [M1; Chapter 7, Section 1]. Let the bifunctor from the monoidal structure be called tensor product and be denoted by $\otimes$. Assume that the tensor product is right exact in each variable and commutes with infinite direct sums. Hence $\mathcal{C}$ is essentially a tensored category in the sense of [M2; Section 6] except that we do not insist that the tensor product be commutative, but we do require infinite direct sums and the compatibility of such sums with tensor products. We define the free algebra on $L_1, \dots, L_s \in \mathcal{C}$ to be,

$$T = T(L_1, \dots, L_s) := \bigoplus_{i_1, \dots, i_k} L_{i_1} \otimes \dots \otimes L_{i_k} \tag{4.1}$$

where the indices $i_j \in \{1, \dots, s\}$ and $k \in \mathbb{N}$. As usual, when $k = 0$, we get the empty tensor product which we define to be the two-sided identity I, of $\otimes$. In future, we will omit the tensor symbol if its meaning is clear. Now T is naturally graded by the free semigroup Γ on s generators. For $\nu \in \Gamma$ we will denote the ν-th graded component by T_ν.

Recall that an *algebra* or *monoid* in $\mathcal{C}$ is an object A in $\mathcal{C}$ equipped with a multiplication morphism $\mu : A \otimes A \longrightarrow A$ and a unit morphism $I \longrightarrow A$, such that the usual compatibilities hold (see [M1; Chapter 7, Section 3] for details). The object $T(L_1, \dots, L_s)$ is naturally an algebra in $\mathcal{C}$. The unit map is precisely the identification of I with T_e where e is the identity in Γ. Since tensor products commute with infinite direct sums, the multiplication map can be defined by giving morphisms $\mu_{\nu\eta} : T_\nu \otimes T_\eta \longrightarrow T_{\nu\eta}$. We define $\mu_{\nu\eta}$ to be the canonical isomorphism.

Let $\mathcal{C}_{alg}$ denote the category of algebras in $\mathcal{C}$ and let A be an object in $\mathcal{C}_{alg}$. Suppose we are given a set of objects $\{U_\alpha\}_{\alpha \in J}$ in $\mathcal{C}$ and sets of morphisms $\{\phi_{\alpha,i} : U_\alpha \longrightarrow A\}_{i \in J_\alpha}$ in $\mathcal{C}$. Even though the U_α's are only objects in $\mathcal{C}$, we can still speak of the coequaliser in $\mathcal{C}_{alg}$ of the diagram,

$$U_\alpha \xrightarrow{\phi_{\alpha,i}} A$$

where α runs through J and the $\phi_{\alpha,i}$ run through J_α. The *coequaliser in* $\mathcal{C}_{alg}$ is an object B in $\mathcal{C}_{alg}$ equipped with a morphism $\pi : A \longrightarrow B$ in $\mathcal{C}_{alg}$ which firstly; satisfies the property that given any two maps $\phi_{\alpha,i}, \phi_{\alpha,j} : U_\alpha \longrightarrow A$ in the diagram, we have the equality $\pi \circ \phi_{\alpha,i} = \pi \circ \phi_{\alpha,j}$ in $\mathcal{C}$ and secondly; B is the universal object in $\mathcal{C}_{alg}$ with respect to this property.

We define a *relation* on T to be a morphism of the form $\phi : U \longrightarrow T$ where U is a subobject of T (in $\mathcal{C}$).

Definition 4.2.1 *Let* $\Phi = \{\phi_\alpha : U_\alpha \longrightarrow T\}$ *be a set of relations on* $T = T(L_1, \dots, L_s)$. *We define the* algebra with generators L_i and relations Φ *to be the coequaliser in* $\mathcal{C}_{alg}$ *of the diagram*

$$U_\alpha \underset{i_\alpha}{\overset{\phi_\alpha}{\rightrightarrows}} T$$

where i_α *is the canonical inclusion. We denote this algebra by* $T(L)/(\Phi)$.

Hence $T(L)/(\Phi)$ is the "largest" quotient algebra of T which identifies the domains of the ϕ_α's with their images.

Example 1 We show here, how an algebra over a field k defined by generators and relations, can be constructed in this setting. Let $\mathcal{C}$ be the category of vector spaces over k and $\otimes$ be the usual tensor product. The category $\mathcal{C}_{alg}$ of algebras in $\mathcal{C}$ is precisely the category of k-algebras. Let A be the algebra with generators x_i for $i = 1, \dots, n$ and relations $r_\alpha = s_\alpha$ where r_α and s_α are noncommutative polynomials in the x_i. Let $L_i = kx_i$, $U_\alpha = kr_\alpha$ and $\phi_\alpha : U_\alpha \longrightarrow T$ be the map which sends $r_\alpha \mapsto s_\alpha$. Then $A \simeq T(L)/(\Phi)$.

We will only be interested in relations which are of the form $\phi : T_\nu \longrightarrow T$, where $\nu \in \Gamma$ and the image lies in the sum of finitely many graded components of T. We shall call such relations *monic.*

Since $\otimes$ is a bifunctor, we may for each monic relation $\phi : T_\nu \longrightarrow T$, consider the morphism $i_\xi \otimes \phi \otimes i_\eta : T_{\xi\nu\eta} = T_\xi \otimes T_\nu \otimes T_\eta \longrightarrow T \otimes T \otimes T$ where $\xi, \eta \in \Gamma$ and i_ξ, i_η are the natural inclusions. Abusing notation, we will let $i_\xi \otimes \phi \otimes i_\eta$ also denote the composite,

$$T_{\xi\nu\eta} \xrightarrow{i_\xi \otimes \phi \otimes i_\eta} T \otimes T \otimes T \xrightarrow{\mu \circ (\mu \otimes 1)} T$$

where μ is the multiplication map. We extend this is to an endomorphism $1_\xi \otimes \phi \otimes 1_\eta$ of T defined componentwise by,

$$(1_\xi \otimes \phi \otimes 1_\eta)|_{T_{\nu'}} = \begin{cases} i_{\nu'} & \text{if } \nu' \neq \xi\nu\eta \\ \mu \circ (\mu \otimes 1) \circ (i_\xi \otimes \phi \otimes i_\eta) & \text{if } \nu' = \xi\nu\eta \end{cases}$$

Let $\mathcal{D}$ be the smallest subcategory of $\mathcal{C}$ with one object T and containing the morphisms $1_\xi \otimes \phi \otimes 1_\eta$ for all $\xi, \eta \in \Gamma$ and $\phi \in \Phi$.

As usual, it is convenient to have a notion of a Gröbner basis and hence also a version of the diamond lemma. We translate the appropriate notions in [B] to our context. Let $\leq$ be a partial order on Γ. We say that a morphism of the form $\phi : T_\nu \longrightarrow T$ is *decreasing* (with respect to $\leq$) if it has image in $\bigoplus_{\nu' < \nu} T_{\nu'}$. We say that the monic relations Φ are *decreasing* if every relation in Φ is decreasing. Given two morphisms f_1, f_2 in $\mathcal{C}$, we say the *categorical confluence* or *diamond condition* holds for them in $\mathcal{D}$ if there exist morphisms g_1, g_2 in $\mathcal{D}$ such that $g_1 f_1 = g_2 f_2$. In the applications, f_1 and f_2 will always have the correct codomain T, for composition.

Now suppose $\leq$ is a semigroup partial ordering on Γ with the descending chain condition (see [B; p.180] for definitions) and Φ is decreasing with respect to this ordering. This ensures in particular, that each $i_\xi \otimes \phi \otimes i_\eta$ is decreasing for $\xi, \eta \in \Gamma$ and $\phi \in \Phi$. Let $\Lambda \subset \Gamma$ be the subset of elements which are not of the form $\xi\lambda\eta$ for any $\xi, \eta, \lambda \in \Gamma$ with T_λ the domain of some relation. Let $\nu \in \Gamma - \Lambda$. Then there are $\xi, \eta \in \Gamma$ and $\phi \in \Phi$ such that T_ν is the domain of $i_\xi \otimes \phi \otimes i_\eta$. We factor $i_\xi \otimes \phi \otimes i_\eta$ into $T_\nu \longrightarrow \oplus_{\mu \in M} T_\mu \longrightarrow T$ where $M \subset \Gamma$ is chosen to be as small as possible. If the image of $i_\xi \otimes \phi \otimes i_\eta$ does not lie in $\oplus_{\lambda \in \Lambda} T_\lambda$, then there is some $\mu \in M$ which is the domain of some $i_{\xi'} \otimes \phi' \otimes i_{\eta'}$ where $\xi', \eta' \in \Gamma$ and $\phi' \in \Phi$. We can ask again whether or not the image of the composite $(1_{\xi'} \otimes \phi' \otimes 1_{\eta'}) \circ (i_\xi \otimes \phi \otimes i_\eta)$ lies in $\oplus_{\lambda \in \Lambda} T_\lambda$ and if not, compose it with another map of the form $1_{\xi''} \otimes \phi'' \otimes 1_{\eta''}$ with $\xi'', \eta'' \in \Gamma$ and $\phi'' \in \Phi$. Continuing in this fashion, the descending chain condition ensures the process terminates. We shall call the resulting composite map a *terminal* morphism from $T_\nu \longrightarrow T$. By definition, the image of a terminal morphism lies in $\oplus_{\lambda \in \Lambda} T_\lambda$.

Example 2 As an illustration, we shall construct the Jordan affine plane $A = k\langle x, y\rangle/(yx - xy - x^2)$ (k a field) in two different ways. As in example 1, let $\mathcal{C}$ be the category of vector spaces over k. One possibility for constructing A is to use two generators $L_1 = kx$ and $L_2 = ky$. Then Γ can be taken to be monomials in L_1 and L_2. Let Φ consist of a single monic relation given by $\phi : L_2 L_1 \longrightarrow T(L_1, L_2) : yx \mapsto xy + x^2$. Then $A \simeq T(L_1, L_2)/(\Phi)$. Furthermore, if we order the set of monomials Γ lexicographically, then Φ is decreasing. Another way to construct A is to have just the one generator $L = kx \oplus ky$. Let Φ consist of the single monic relation $\phi : LL \longrightarrow T$ defined by

$$x^2 \mapsto x^2 \quad , \quad xy \mapsto xy \quad , \quad yx \mapsto xy + x^2 \quad , \quad y^2 \mapsto y^2$$

We still have $A \simeq T(L)/(\Phi)$, but we cannot order Γ so that ϕ is decreasing.

The desired version of Bergman's diamond lemma is,

Theorem 4.2.2 *Let $\leq$ be a semigroup partial ordering on Γ with the descending chain condition. Let $L_1, \ldots, L_s \in \mathcal{C}$ generate the free algebra T and Φ be a set of decreasing monic relations on T. Let $\mathcal{D}$ denote the category with one object T, defined above. For any $\xi, \eta, \zeta \in \Gamma$, assume further that the following two conditions hold,*

(OV) *If there are two distinct relations $\phi_1, \phi_2 \in \Phi$ whose domains are respectively $T_{\xi\eta}$ and $T_{\eta\zeta}$ then the morphisms, $i_\xi \otimes \phi_2$ and $\phi_1 \otimes i_\zeta$ satisfy the confluence condition in $\mathcal{D}$.*

(IN) *If there are two distinct relations $\phi_1, \phi_2 \in \Phi$ whose domains are respectively $T_{\xi\eta\zeta}$ and T_η then ϕ_1 and $i_\xi \otimes \phi_2 \otimes i_\zeta$ satisfy the confluence condition in $\mathcal{D}$.*

Then we have the following isomorphism in $\mathcal{C}$,

$$T(L)/(\Phi) \cong \bigoplus_{\lambda \in \Lambda} T_\lambda$$

where Λ is the subset of Γ defined above. Furthermore, the universal morphism $\pi : T \longrightarrow T/(\Phi)$ is a projection which, on each graded component T_ν, can be defined by any terminal morphism from $T_\nu \longrightarrow T$.

Conversely, suppose $\phi_1, \phi_2 \in \Phi$ have domains $T_{\xi\eta}$ and $T_{\eta\zeta}$ as in (OV). Choose morphisms $g_1, g_2 \in Mor\ \mathcal{D}$ so that $g_1 \circ (\phi_1 \otimes i_\zeta)$ and $g_2 \circ (i_\xi \otimes \phi_2)$ are terminal. If $g_1 \circ (\phi_1 \otimes i_\zeta) \neq g_2 \circ (i_\xi \otimes \phi_2)$ then $T(L)/(\Phi)$ is a proper quotient of $\oplus_{\lambda\in\Lambda} T_\lambda$. A similar statement can be made for the case (IN).

Comment on Proof. Bergman's diamond lemma [B; theorem 1.2] is the case where $\mathcal{C}$ is the category of k-modules, k a commutative ring, and the generators are free rank one modules. The proof given there is actually set up in the context given above and so generalises to the desired theorem painlessly.

Remark: 1. We have seen that there is an algorithm for finding terminal morphisms. Hence, the converse half of the diamond lemma provides a simple procedure for verifying the hypotheses (OV) and (IN). We will refer to this as checking overlaps or inclusions as the case may be.
2. We will call the set of graded components $\{T_\lambda\}_{\lambda\in\Lambda}$ a Gröbner basis for $T(L)/(\Phi)$.
3. In the example of the Jordan affine plane, the diamond lemma only applies to the first construction. In general, we expect that the more we can decompose the generators L_i, the more information we can extract from the diamond lemma.

We return now to the study of bimodule algebras. Fix a noetherian base scheme S. All morphisms and products will be considered over S unless otherwise stated. Let X be an S-scheme of finite type and $pr_1, pr_2 : X \times X \to X$, and $pr_{ij} : X \times X \times X \to X \times X$ for $1 \le i < j \le 3$ be the canonical projections. Recall,

Definition 4.2.3 *([VdB2]) An ($\mathcal{O}_S -$central) $\mathcal{O}_X$-* bimodule *is a quasi-coherent sheaf M on $X \times X$ where pr_1, pr_2 are* relatively locally finite, *i.e. given any coherent subsheaf L of M, then if Z is the support of L we have $pr_1|_Z, pr_2|_Z$ finite. Given two such bimodules, L and M we define the tensor product to be $L \otimes_{\mathcal{O}_X} M := pr_{13*}(pr_{12}^* L \otimes_{\mathcal{O}_{X^3}} pr_{23}^* M)$. We define the tensor product of an $\mathcal{O}_X$-module M with a bimodule L by $M \otimes_{\mathcal{O}_X} L = pr_{2*}(pr_1^* M \otimes_{\mathcal{O}_X} L)$. An $\mathcal{O}_X$-*bimodule algebra *is an algebra $\mathcal{B}$ in the category of $\mathcal{O}_X$-bimodules. A (right) $\mathcal{B}$-*module *is a quasi-coherent sheaf $\mathcal{M}$ on X equipped with a morphism $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B} \longrightarrow \mathcal{M}$ satisfying the usual module axioms.*

The category of $\mathcal{O}_X$-bimodules is an abelian category with infinite direct sums. Furthermore, the tensor product and the bimodule $\mathcal{O}_X$ furnish the category with a monoidal structure (see [VdB2; proposition 2.5]). Finally, $\otimes$ is right exact [VdB2; proposition 2.6] and it commutes with infinite direct sums. Hence, given a set of bimodule generators $L_1, \dots, L_s$ and relations Φ we may form the corresponding $\mathcal{O}_X$-bimodule algebra with relations, $\mathcal{B} = T(L)/(\Phi)$. Since bimodules are in particular sheaves, we can consider their global sections as well as their higher sheaf cohomology. As was noted in [VdB2; p.452], $H^0(\mathcal{B})$ is an $\mathcal{O}(S)$-algebra for any $\mathcal{O}_X$-bimodule algebra $\mathcal{B}$. The multiplication is given by the composition $H^0(\mathcal{B}) \otimes H^0(\mathcal{B}) \to H^0(\mathcal{B} \otimes \mathcal{B}) \xrightarrow{H^0(\mu)} H^0(\mathcal{B})$. The unit map is given by $H^0(\mathcal{O}_S) \to H^0(\mathcal{O}_X) \xrightarrow{H_0(1)} H^0(\mathcal{B})$. Ring axioms follow from the corresponding axioms for bimodule algebras and functoriality.

4.3 Twisted Multi-Homogeneous Coordinate Rings

We shall assume henceforth that our base scheme S is the spectrum of an algebraically closed field k and that X is projective. Suppose now that the generating bimodules $L_1, \dots, L_s$ are invertible bimodules in the sense of [AV], i.e. they are units in the semigroup of $\mathcal{O}_X$-bimodules. We consider monic relations of the form $\phi_{ij} : L_j L_i \tilde{\to} L_i L_j$ for all $1 \leq i < j \leq s$ which are all ($\mathcal{O}_X$-bimodule) isomorphisms. These shall be referred to as commutation relations for the L_i. They are said to be compatible if in $\mathrm{Hom}(L_k L_j L_i, L_i L_j L_k)$ we have,

$$(\phi_{ij} \otimes 1_{L_k}) \circ (1_{L_j} \otimes \phi_{ik}) \circ (\phi_{jk} \otimes 1_{L_i}) = (1_{L_i} \otimes \phi_{jk}) \circ (\phi_{ik} \otimes 1_{L_j}) \circ (1_{L_k} \otimes \phi_{ij})$$

This is of course nothing more than the overlap hypothesis (OV) in theorem 4.2.2. Case (IN) of theorem 4.2.2 does not arise. We partially order the set Γ lexicographically. This ordering satisfies the hypotheses of theorem 4.2.2 (see for example [B; p.186]).

Definition 4.3.1 *The* twisted multi-homogeneous coordinate ring *on X with respect to the invertible bimodules $L_1, \dots L_s$ and compatible commutation relations $\Phi = \{\phi_{ij}\}$ is defined to be the k-algebra $H^0(T(L_1, \dots, L_s)/(\Phi))$. We denote this algebra by $B(X; L_1, \dots, L_s; \phi_{ij})$ omitting arguments ϕ_{ij}, L_i and X when they are understood.*

One obtains the commutative multi-homogeneous coordinate ring when the L_i's are invertible sheaves and the commutation relations are the canonical morphisms $\phi_{ij} : L_j L_i \longrightarrow L_i L_j$ which map $s \otimes t \mapsto t \otimes s$ for every section s of L_j and t of L_i.

We apply theorem 4.2.2 to $T(L)/(\Phi)$. First note that one can define for each graded component T_ν, the degree in L_i. Hence, we can also define the multi-degree as the s-tuple of degrees in the L_i's. Now, $T(L)/(\Phi) = \oplus_{n_1, \dots, n_s \geq 0} \mathcal{B}_{n_1, \dots, n_s}$ where $\mathcal{B}_{n_1, \dots, n_s} = L_1^{n_1} \dots L_s^{n_s}$. This is a $\mathbb{Z}^s$-graded ring. Note that since the commutation relations ϕ_{ij} are isomorphisms, theorem 4.2.2 gives a canonical isomorphism between all graded components of a given multi-degree. Hence, we can take for $\mathcal{B}_{n_1, \dots, n_s}$ any graded component T_ν with multi-degree $(n_1, \dots, n_s)$. It will often be convenient to do this and we shall do so in future without further comment. Taking global sections we obtain a similar decomposition of $B(X; L_1, \dots, L_s; \phi_{ij})$ into multi-graded components.

As an exercise, we shall unravel the definition of a twisted multi-homogeneous coordinate ring in the case where there are only two invertible bimodules, expressing our result in the pre-[AV] language of say the [ATV] paper. We need some basic facts from [AV; lemma 2.11]. Recall that every invertible bimodule has the form $L_\sigma := \pi_1^* L$ where L is an invertible sheaf on X, σ is an automorphism of X and $\pi_1 : G \to X$ the first projection of the graph $G \subset X \times X$ of σ to X. Clearly $H^0(L_\sigma) = H^0(L)$. Let L_σ, M_τ be two such invertible bimodules. We have a tensor product formula for invertible bimodules (see [AV; lemma 2.14]), namely, $L_\sigma M_\tau = (L \otimes \sigma^* M)_{\tau\sigma}$. Hence for there to be a commutation relation between L_σ and M_τ, we must have a sheaf isomorphism $\phi : M \otimes \tau^* L \tilde{\to} L \otimes \sigma^* M$ and $\tau\sigma = \sigma\tau$. There are no overlap checks so we may construct a twisted bihomogeneous coordinate ring from this data. From the previous paragraph we see using the tensor product formula that,

$$B(X; L_\sigma, M_\tau; \phi) = \bigoplus_{i,j \geq 0} H^0(L \otimes \sigma^* L \otimes \dots \sigma^{i-1*} L \otimes \sigma^{i*} M \otimes (\sigma^i \tau)^* M \otimes \dots \otimes (\sigma^i \tau^{j-1})^* M) \quad (4.2)$$

where by abuse of notation we have used ϕ to represent the bimodule isomorphism induced by the sheaf isomorphism and we have chosen the Gröbner basis. In [AV; equation (1.2)], the multiplication rule was given by the map

$$\mu : H^0(L \otimes \dots \otimes (\sigma^{i_1} \tau^{j_1 - 1})^* M) \otimes H^0(L \otimes \dots \otimes (\sigma^{i_2} \tau^{j_2 - 1})^* M) \longrightarrow$$

$$H^0(L \otimes \dots \otimes (\sigma^{i_1} \tau^{j_1 - 1})^* M \otimes (\sigma^{i_1} \tau^{j_1})^* L \otimes \dots \otimes (\sigma^{i_1 + i_2} \tau^{j_1 + j_2 - 1})^* M)$$

which sends

$$a \otimes b \mapsto a \otimes (\sigma^{i_1} \tau^{j_1})^* b$$

Of course the right hand side must be identified with its image in $H^0(T(L_\sigma, M_\tau)/(\phi))$. Alternatively, if one wishes to convert this back into the Gröbner basis as in equation (4.2) then one needs to compose this with the map,

$$H^0(L \otimes \ldots \otimes (\sigma^{i_1}\tau^{j_1-1})^* M \otimes (\sigma^{i_1}\tau^{j_1})^* L \otimes \ldots \otimes (\sigma^{i_1+i_2}\tau^{j_1+j_2-1})^* M) \xrightarrow{\sim}$$

$$H^0(L_\sigma^{i_1} M_\tau^{j_1} L_\sigma^{i_2} M_\tau^{j_2}) \xrightarrow{H^0(\pi)} H^0(L_\sigma^{i_1+i_2} M_\tau^{j_1+j_2}) \xrightarrow{\sim}$$

$$H^0(L \otimes \ldots \otimes (\sigma^{i_1+i_2-1})^* L \otimes (\sigma^{i_1+i_2})^* M \otimes \ldots \otimes (\sigma^{i_1+i_2}\tau^{j_1+j_2-1})^* M)$$

where the first and last morphisms are the canonical ones and π is the universal morphism of theorem 4.2.2 appropriately restricted. For example, multiplication of the $(2,1)$-graded piece with the $(1,1)$-graded piece is given by $a \otimes b \mapsto (\mathrm{id}_{L\otimes\sigma^* L} \otimes \sigma^{2*}\phi \otimes \mathrm{id}_{(\sigma^3\tau)^* M})(a \otimes (\sigma^2\tau)^* b)$.

One can see from this computation that multiplication in the [ATV] language of invertible sheaves and their twists is considerably more complicated than in the language of invertible bimodules. There are several conceptual advantages in working with invertible bimodules as opposed to invertible sheaves and their twists. For example, the definition of a commutation relation is much more transparent in the bimodule setting and associativity is an immediate consequence of the definition of multiplication.

We now turn our attention to generalising Serre's theorem along the lines of [AV; theorem 3.12]. To this end we define,

Definition 4.3.2 *Let $\geq$ denote the partial order on $\mathbb{Z}^s$ defined by taking the product order of the usual order on each $\mathbb{Z}$. The expression "for $i_1, \ldots, i_s \in \mathbb{Z}^s$* large enough*" will mean for all $(i_1, \ldots, i_s) \geq (m_1, \ldots, m_s)$ for some fixed $m_1, \ldots m_s$. We say that the generating bimodules $L_1, \ldots, L_s$ for a twisted multi-homogeneous ring form a* right ample set *if given any coherent sheaf F on X, we have $H^q(FL^{i_1} \ldots L^{i_s}) = 0$ for $q > 0$ and $i_1, \ldots, i_s$ large enough.*

We leave it to the reader to define left ample. Ample will mean left and right ample.

Definition 4.3.3 *We say that a $\mathbb{Z}^s$-graded object M in an abelian category is* first quadrant bounded *if $M_{i_1 \ldots i_s} = 0$ for $i_1, \ldots, i_s$ large enough. Direct limits of such objects are said to be* torsion. *Let tors denote the category of torsion $\mathbb{Z}^s$-graded right B-modules.*

Let $\mathcal{A}$ be an $\mathcal{O}_X$-bimodule algebra graded by a group G. If e is the identity of G, assume that $\mathcal{A}_e \simeq \mathcal{O}_X$. We say that $\mathcal{A}$ is *strongly G-graded* if the multiplication map restricts to surjections $\mathcal{A}_g \otimes \mathcal{A}_h \longrightarrow \mathcal{A}_{gh}$ for all $g, h \in G$.

Let $\mathrm{Gr} - B$ denote the category of (multi-)graded right B-modules and $\mathcal{O}_X - \mathrm{Mod}$ denote the category of quasi-coherent sheaves on X. Then Serre's theorem in this case is,

Theorem 4.3.4 *Let B be a twisted homogeneous coordinate ring with respect to a right ample set of invertible bimodules $L_1, \ldots, L_s$ and compatible commutation relations $\Phi = \{\phi_{ij}\}$. There exists a strongly $\mathbb{Z}^s$-graded bimodule algebra $\mathcal{B}$ such that $T(L)/(\Phi) = \mathcal{B}_{\geq 0}$. Further, $\mathcal{B}$ has the property that the functors*

$$H^0(\cdot \otimes_{\mathcal{O}_X} \mathcal{B})_{\geq 0} : \mathcal{O}_X - Mod \longrightarrow Gr - B$$

$$(\cdot \otimes_B \mathcal{B})_0 : Gr - B \longrightarrow \mathcal{O}_X - Mod$$

are adjoint and induce inverse category equivalences between $\mathcal{O}_X - Mod$ and $Gr - B/tors$.

N.B. For a right B-module M, $M \otimes_B \mathcal{B}$ is defined to be the sheaf associated to the presheaf $U \mapsto M \otimes_B \mathcal{B}(X \times U)$.

Proof. The proof in the $\mathbb{Z}$-graded case given in [AV] carries over readily. The desired category equivalence comes from composing the following equivalences-

$$\mathcal{O}_X - \text{Mod} \longrightarrow \text{Gr} - \mathcal{B} : \quad F \mapsto F \otimes_{\mathcal{O}_X} \mathcal{B}$$

$$\text{Gr} - \mathcal{B} \longrightarrow \mathcal{O}_X - \text{Mod} : \quad \mathcal{M} \mapsto \mathcal{M}_0$$

and,

$$\text{Gr} - \mathcal{B} \longrightarrow \text{Gr} - B/(\text{tors}) : \quad \mathcal{M} \mapsto H^0(\mathcal{M})_{\geq 0}$$

$$\text{Gr} - B/(\text{tors}) \longrightarrow \text{Gr} - \mathcal{B} : \quad M \mapsto M \otimes_B \mathcal{B}$$

We first construct the ring $\mathcal{B}$. Let M_i for $i = 1, \dots, s$ be the inverses of the L_i. We consider a bimodule algebra with generating bimodules $L_i, M_i,\ i = 1, \dots, s$ and monic relations of the form:

$$\phi_{ij} : L_j L_i \xrightarrow{\sim} L_i L_j \qquad \phi_{ij}^{\vee} : M_j M_i \xrightarrow{\sim} M_i M_j$$

$$\phi_i : L_i M_i \xrightarrow{\sim} \mathcal{O}_X \qquad \phi_i' : M_i L_i \xrightarrow{\sim} \mathcal{O}_X$$

$$\phi_{ij}' : M_j L_i \xrightarrow{\sim} L_i M_j \qquad \phi_{ij}'^{\vee} : L_j M_i \xrightarrow{\sim} M_i L_j$$

to be defined below.

We need first observe that inverting an invertible bimodule L is a functor on the category of invertible bimodules and isomorphisms. This follows from the tensor product formula, which gives the inverse of L as $L^{\vee} = (\sigma^{-1*}(pr_{1*}L)^{\vee})_{\sigma^{-1}}$ where σ is the automorphism whose graph is Supp L. We observe also that for arbitrary invertible bimodules L and M, $(LM)^{\vee}$ is canonically isomorphic to $M^{\vee}L^{\vee}$.

We can now give the defining relations. We define $\phi_{ij}^{\vee}$ by applying the inverse functor to ϕ_{ij} and using the canonical isomorphism above. For any invertible bimodule L and open $U \subseteq X$, let $L(U) = L(U \times X)$. Note that $L^{\vee}(\sigma U) = L(U)^{\vee}$ for U small enough and σ the above automorphism. There is a canonical map $\phi_L : L \otimes L^{\vee} \xrightarrow{\sim} \mathcal{O}_X$ defined by $s \otimes t \in L(U) \otimes_{\mathcal{O}(\sigma U)} L^{\vee}(\sigma U)$ maps to $t(s) \in \mathcal{O}_X(U)$ for all U sufficiently small. Define $\phi_i = \phi_{L_i}$. Similarly, there is a canonical map $\phi_L' : L^{\vee} \otimes L \xrightarrow{\sim} \mathcal{O}_X$ defined by $s \otimes t \in L^{\vee}(U) \otimes_{\mathcal{O}_X(\sigma^{-1}U)} L(\sigma^{-1}U)$ maps to $s(t)^{\sigma^{-1}} \in \mathcal{O}_X(U)$. We set $\phi_i' = \phi_{L_i}'$. We define ϕ_{ij}' to be the composite,

$$\phi_{ij}' : M_j L_i \xrightarrow{(\phi_i)^{-1} \otimes 1} L_i M_i M_j L_i \xrightarrow{1 \otimes (\phi_{ij}^{\vee})^{-1} \otimes 1} L_i M_j M_i L_i \xrightarrow{1 \otimes \phi_i'} L_i M_j$$

We obtain $\phi_{ij}'^{\vee}$ from ϕ_{ij}' by applying the inverse functor.

We wish to show that the diamond lemma holds in this case so that $\mathcal{B} = \bigoplus_{i_1, \dots, i_s \in \mathbb{Z}} L_1^{i_1} \dots L_s^{i_s}$ where if $i < 0$ then L_j^i denotes M_j^{-i}. The case (IN) of theorem 4.2.2 does not arise. There are overlap checks for the monomials $L_k L_j L_i$, $L_k L_j M_i$, $L_k L_j M_j$, $L_k M_j L_i$, $L_k M_j L_j$, $L_k M_k L_j$, $L_k M_k L_k$, $M_k L_k L_j$, $M_k L_j L_i$ for $i < j < k$. The other overlap checks are for monomials obtained from the above ones by swapping M with L. Note that the set of defining relations is self-dual in that applying the inverse functor to any defining relation yields another. Hence, checking overlaps for any monomial verifies the overlap check for the monomial obtained by swapping L with M so we need only consider the 9 monomials above. The overlap check for $L_k L_j L_i$ was hypothesised while that for $L_j M_j L_j$ is easily verified on sections. We carry out the somewhat tedious verification of the others.

For the monomial $L_kM_kL_j$ we must show the maps,

$$L_kM_kL_j \xrightarrow{\phi_k\otimes 1} L_j \tag{4.3}$$

and

$$L_kM_kL_j \xrightarrow{1\otimes\phi_j^{-1}\otimes 1} L_kL_jM_jM_kL_j \xrightarrow{1\otimes(\phi_{jk}^{\vee})^{-1}\otimes 1} L_kL_jM_kM_jL_j$$

$$\xrightarrow{\phi_{jk}\otimes 1} L_jL_kM_kM_jL_j \xrightarrow{1\otimes\phi_k\otimes\phi_j'} L_j \tag{4.4}$$

are equal. The last map of this composite is equal to,

$$L_jL_kM_kM_jL_j \xrightarrow{1\otimes\phi_k\otimes 1} L_jM_jL_j \xrightarrow{\phi_j\otimes 1} L_j$$

by the overlap check for $L_jM_jL_j$. Tracing the maps in (4.4) we see it suffices to show that the two maps,

$$(L_kL_j)(L_kL_j)^{\vee} \xrightarrow{\phi_{L_kL_j}} \mathcal{O}_X$$

and

$$(L_kL_j)(L_kL_j)^{\vee} \xrightarrow{\phi_{jk}\otimes(\phi_{jk}^{-1})^{\vee}} (L_jL_k)(L_jL_k)^{\vee} \xrightarrow{\phi_{L_jL_k}} \mathcal{O}_X$$

are equal. This can be checked on sections. Checking overlaps for the other monomials where two of the indices are the same is done by a similar (or easier) argument.

We now check the overlap for $M_kL_jL_i$ i.e. equality of the two maps

$$\psi_1 : M_kL_jL_i \xrightarrow{1\otimes\phi_{ij}} M_kL_iL_j \xrightarrow{\phi_{ik}'\otimes 1} L_iM_kL_j \xrightarrow{1\otimes\phi_{jk}'} L_iL_jM_k$$

$$\psi_2 : M_kL_jL_i \xrightarrow{\phi_{jk}'\otimes 1} L_jM_kL_i \xrightarrow{1\otimes\phi_{ik}'} L_jL_iM_k \xrightarrow{\phi_{ij}\otimes 1} L_iL_jM_k$$

It suffices to show that these are equal respectively to,

$$\psi_1' : M_kL_jL_i \longrightarrow M_kL_jL_iL_kM_k \longrightarrow M_kL_iL_jL_kM_k \longrightarrow$$

$$M_kL_iL_kL_jM_k \longrightarrow M_kL_kL_iL_jM_k \longrightarrow L_iL_jM_k$$

and

$$\psi_2' : M_kL_jL_i \longrightarrow M_kL_jL_iL_kM_k \longrightarrow M_kL_jL_kL_iM_k \longrightarrow$$

$$M_kL_kL_jL_iM_k \longrightarrow M_kL_kL_iL_jM_k \longrightarrow L_iL_jM_k$$

since $\psi_1' = \psi_2'$ by the overlap check for $L_kL_jL_i$. We first show that the two maps,

$$M_kL_iL_jL_k \longrightarrow L_iL_jM_kL_k \longrightarrow L_iL_j \tag{4.5}$$

and

$$M_kL_iL_jL_k \longrightarrow M_kL_kL_iL_j \longrightarrow L_iL_j \tag{4.6}$$

are equal. We compare with the two maps

$$M_k L_i L_j L_k \longrightarrow M_k L_i L_k M_k L_j L_k \rightrightarrows L_i L_j \qquad (4.7)$$

where the top map comes from commuting the L_k's through L_i, L_j and then contracting with ϕ'_k, and the bottom from commuting the M_k's. These two maps are equal to (4.5) and (4.6) respectively, by the overlap check for $L_k M_k L_k$ and $M_k L_k M_k$. However, the two maps in (4.7) are equal by the overlap check for $M_k L_k L_i$ and $M_k L_k L_j$.

Now consider the last part of the map ψ'_1 consisting of

$$M_k L_i L_j L_k M_k \longrightarrow M_k L_k L_i L_j M_k \longrightarrow L_i L_j M_k$$

This equals the map

$$M_k L_i L_j L_k M_k \longrightarrow L_i L_j M_k L_k M_k \xrightarrow{1\otimes\phi'_k\otimes 1} L_i L_j M_k$$

by what we just proved. This in turn equals

$$M_k L_i L_j L_k M_k \longrightarrow L_i L_j M_k L_k M_k \xrightarrow{1\otimes\phi_k} L_i L_j M_k$$

by the overlap check for $M_k L_k M_k$. From this we see that $\psi_1 = \psi'_1$. A symmetrical argument shows $\psi_2 = \psi'_2$ as desired. The overlap check for $L_k L_j M_i$ follows by a similar argument comparing the requisite maps with the "two" maps $M_i L_i L_k L_j M_i \rightrightarrows M_i L_j L_k L_i M_i$ and the overlap check for $L_k M_j L_i$ by considering $M_j L_k L_j L_i M_j \rightrightarrows M_j L_i L_j L_k M_j$. Hence, theorem 4.2.2 is applicable and we find that $\mathcal{B}$ is a strongly graded algebra with $\mathcal{B}_{\geq 0} = T(L)/(\Phi)$.

We return now to the two category equivalences mentioned at the beginning of the proof. We prove the first of these in,

Lemma 4.3.5 *We have a natural category equivalence,*

$$\mathcal{O}_X - Mod \leftrightarrow Gr - \mathcal{B}: \quad F \to F \otimes \mathcal{B}, \quad \mathcal{M}_0 \leftarrow \mathcal{M}$$

Proof. This holds true for any $\mathcal{B}$ strongly graded so we shall prove it in this setting. Firstly, it is clear that the composed functor $F \mapsto (F \otimes_{\mathcal{O}_X} \mathcal{B})_0$ is naturally isomorphic to the identity functor on $\mathcal{O}_X$ − Mod. For the reverse composition, we have a natural transformation $\chi_{\mathcal{M}} : \mathcal{M}_0 \otimes \mathcal{B} \longrightarrow \mathcal{M}$ from the multiplication map. Note that $\chi_{\mathcal{M}}$ is surjective since the composite $\mathcal{M}_v \otimes \mathcal{B}_{-v} \otimes \mathcal{B}_v \longrightarrow \mathcal{M}_0 \otimes \mathcal{B}_v \longrightarrow \mathcal{M}_v$ is for any $v \in \mathbb{Z}^s$. Let $\mathcal{K} = \ker \chi_{\mathcal{M}}$. Then $\mathcal{K}_v = \operatorname{im}(\mathcal{K}_v \otimes \mathcal{B}_{-v} \otimes \mathcal{B}_v \longrightarrow \mathcal{K}_v) = 0$ since $\mathcal{K}_v \otimes \mathcal{B}_{-v} \longrightarrow \mathcal{K}_0 = 0$. Hence, $\chi_{\mathcal{M}}$ is an isomorphism.

Finally, we leave it to the reader to verify that the proof of the category equivalence Gr − $\mathcal{B} \leftrightarrow$ Gr − B/tors given in [AV] carries over without change. In the course of the proof, the following fact was established whose multi-graded analogue we will need.

Lemma 4.3.6 *Assume the hypotheses of the theorem hold. If F is a coherent $\mathcal{O}_X$-module then $F \otimes \mathcal{B}_v$ is generated by global sections for $v \in \mathbb{Z}^s$ large enough.*

Proof. The proof is as in [AV; proposition 3.2(ii)].

4.4 Examples

In this section we give some examples of naturally occurring twisted bihomogeneous coordinate rings. Throughout this section we shall assume that our base scheme is the spectrum of the algebraically closed field k.

Example 4.4.1 *Some Ore extensions of twisted homogeneous coordinate rings*

Let X be a projective scheme over k and L a right ample invertible bimodule on X. Suppose τ is an automorphism of X such that there is a bimodule isomorphism $\phi : \mathcal{O}_\tau \otimes L \longrightarrow L \otimes \mathcal{O}_\tau$. We can then consider the twisted bihomogeneous coordinate ring $B = B(X; L, \mathcal{O}_\tau; \phi)$. The (i,j)-th graded component of B is $B_{ij} = H^0(X, L^i \mathcal{O}_\tau^j)$. But every section in $H^0(X, L^i \mathcal{O}_\tau^j)$ has the form $a \otimes 1_{\mathcal{O}_\tau} \otimes \ldots \otimes 1_{\mathcal{O}_\tau}$ where $a \in H^0(X, L^i)$ and $1_{\mathcal{O}_\tau} = pr_1^* 1_{\mathcal{O}_X} = pr_2^* 1_{\mathcal{O}_X} \in \mathcal{O}_\tau(X \times X)$. Hence if we write t for the element $1_{\mathcal{O}_\tau} \in B_{01}$ then right multiplication in B by t^j maps B_{i0} isomorphically onto B_{ij}. We see immediately that B is an Ore extension of the twisted homogeneous coordinate ring $B(X; L)$. Note that $L, \mathcal{O}_\tau$ form a right ample set.

Example 4.4.2 *Rees algebras of twisted homogeneous coordinate rings*

Let X be a projective scheme over k and L a right ample invertible bimodule on X. Let $R = B(X; L)$. We consider $B = B(X; L, L; \phi)$ where $\phi : L \otimes L \longrightarrow L \otimes L$ is the identity map. Then $B_{ij} = H^0(X, L^{i+j})$ so $B = \oplus_{i,j \geq 0} R_{i+j} t^j$ where t is just a place marker to distinguish the different graded components. However, unraveling the definition of multiplication in B we see that the equality above is actually a ring isomorphism if in the right hand side, t is treated as a central indeterminate. Hence B is the Rees algebra $R[\mathfrak{m}t]$ where $\mathfrak{m}$ is the augmentation ideal $\mathfrak{m} = B_1 \oplus B_2 \oplus \ldots$. Furthermore, the pair L, L is right ample.

Example 4.4.3 *Tensor products of twisted homogeneous coordinate rings*

Let X and Y be projective schemes over k and L_σ, M_τ be right ample invertible bimodules on X,Y respectively. Let pr_1, pr_2 be the projections from $X \times Y$ to X and Y respectively and let $\tilde{\sigma} = \sigma \times 1_Y$ and $\tilde{\tau} = 1_X \times \tau$. Observe that $\tilde{\sigma}$ and $\tilde{\tau}$ commute. Also, since $pr_1 = pr_1 \circ \tilde{\tau}$ and $pr_2 = pr_2 \circ \tilde{\sigma}$ there are canonical isomorphisms of sheaves on $X \times Y$,

$$pr_2^* M \otimes \tilde{\tau}^* pr_1^* L \xrightarrow{\sim} pr_2^* M \otimes pr_1^* L \xrightarrow{\sim} pr_1^* L \otimes pr_2^* M \xrightarrow{\sim} pr_1^* L \otimes \tilde{\sigma}^* pr_2^* M \tag{4.8}$$

Thus if we let $pr_1^*(L_\sigma)$ denote $(pr_1^* L)_{\tilde{\sigma}}$ and $pr_2^*(M_\tau)$ denote $(pr_2^* M)_{\tilde{\tau}}$ then we have a bimodule isomorphism $\phi : pr_2^* M_\tau \otimes pr_1^* L_\sigma \longrightarrow pr_2^* M_\tau \otimes pr_1^* L_\sigma$ and so can form the twisted bihomogeneous coordinate ring $B = B(X \times Y; pr_1^* L_\sigma, pr_2^* M_\tau; \phi)$. The (i,j)-th graded component of B in this case is

$$H^0(X \times Y, pr_1^* L_\sigma \otimes \ldots \otimes \tilde{\sigma}^{i-1*} pr_1^* L \otimes pr_2^* M \otimes \ldots \otimes \tilde{\tau}^{j-1*} pr_2^* M)$$

which by the Künneth formula is $B(X; L_\sigma)_i \otimes_k B(Y; M_\tau)_j$. In fact, from (4.8) we see that we have $B = B(X; L_\sigma) \otimes_k B(Y; M_\tau)$ as algebras.

4.5 A Criterion for ACC

Throughout this section, X will be a projective scheme over k.

Serre's category equivalence was used in [AV; theorem 3.14] to give a proof that the twisted homogeneous coordinate ring with respect to a right ample invertible bimodule is right noetherian. It is not surprising that this fails in the multigraded case.

Example 4.5.1 *Non-noetherian $B(X; L, M)$*

We in fact have a commutative example. Let X be a smooth curve of positive genus g. Let L be a line bundle of degree 0 such that no tensor power of L is isomorphic to $\mathcal{O}_X$and M a line bundle of degree greater than $g - 1$. Then L, M form an ample set of line bundles. However, the $(i, 0)$-graded component of $B = B(X; L, M)$ for $i > 0$ is zero while all other graded components are non-zero. Let $I = B_{>0}$ be the augmentation ideal of the associated $\mathbb{Z}$-graded algebra. Then I/I^2 contains a copy of the infinite dimensional vector space $\oplus_{i>0} B_{i1}$. Thus, B is not finitely generated and hence not noetherian.

The problem here is that L does not define a map into projective space. To eliminate this possibility we consider the following condition on an invertible bimodule L_σ on X,

(*) *There exists a projective scheme Y over k with an automorphism σ and a σ-equivariant morphism $f : X \longrightarrow Y$. There also exists a line bundle L' on Y such that $L = f^*L'$ and such that L'_σ is right ample.*

We have the following sufficient criterion for a twisted bihomogeneous coordinate ring to be right noetherian.

Theorem 4.5.2 *Let X be a projective scheme over k and L_σ, M_τ a right ample set of invertible bimodules on X with some commutation relation $\phi : M_\tau L_\sigma \longrightarrow L_\sigma M_\tau$. Suppose that L_σ and M_τ satisfy (*). Then $B(X; L_\sigma, M_\tau; \phi)$ is right noetherian.*

Proof. We first show

Claim 4.5.3 *Under the hypotheses of the theorem we have that $\oplus_{i>i_0} H^q(X, L^i_\sigma M^j_\tau)$ is a noetherian $B(X; L_\sigma)$- module for all j, q, i_0 and similarly $\oplus_{j>j_0} H^q(X, L^i_\sigma M^j_\tau)$ is a noetherian $B(X; M_\tau)$- module for all i, q, j_0.*

Proof. We first observe that there is a natural ring homomorphism $B(Y; L'_\sigma) \longrightarrow B(X; L_\sigma)$. Recall that $B(Y, L'_\sigma)$ is right noetherian by [AV; theorem 3.14]. Now $\oplus_{i>i_0} H^q(X, L^i_\sigma M^j_\tau)$ is a right $B(X; L_\sigma)$-module by functoriality of cohomology. It suffices to show it is finitely generated over $B(Y; L'_\sigma)$. Let $g = f \circ \tau^j$ and consider the Leray spectral sequence,

$$E_2^{pq} = H^p(Y, R^q g_*(\bigoplus_{i>i_0} M \otimes \ldots \otimes \tau^{j-1*} M \otimes \tau^{j*} L \otimes \ldots \otimes \tau^{j*}\sigma^{i-1*} L))$$

$$\Longrightarrow H^{p+q}(X, \bigoplus_{i>i_0} M \otimes \ldots \otimes \tau^{j-1*} M \otimes \tau^{j*} L \otimes \ldots \otimes \tau^{j*}\sigma^{i-1*} L)$$

From the projection formula [EGA 0_{III}; proposition 12.2.3] we see the E_2^{pq} term is,

$$H^p(Y, \oplus_i R^q g_*(M \otimes \ldots \otimes \tau^{j-1*} M) \otimes L' \otimes \ldots \otimes \sigma^{i-1*} L')$$

Since L'_σ is right ample the spectral sequence degenerates in high degree to give isomorphisms $E_2^{0q} \xrightarrow{\sim} H^q(\oplus_i L^i_\sigma M^j_\tau)$ in $(\text{Gr} - B(Y; L'_\sigma))/\text{tors}$ by naturality of the spectral sequence. However, under the category equivalence $(\text{Gr} - B(Y; L'_\sigma))/\text{tors} \leftrightarrow \mathcal{O}_Y - \text{Mod}$, E_2^{0q} corresponds to the coherent sheaf $R^q g_*(M \otimes \ldots \otimes \tau^{j-1*} M)$ and so must be finitely generated as was to be shown. A symmetrical argument fixing i instead of j completes the claim.

We show now how the proof of the ascending chain condition in [AV; p. 261-263] pushes through to the bigraded case under the given hypotheses. Let I be a bigraded right ideal of $B = B(X; L_\sigma, M_\tau)$. It suffices to show that I is finitely generated.

Recall from theorem 4.3.4 that we have two adjoint functors, $\text{Gr} - B \longrightarrow \mathcal{O}_X - \text{Mod} : M \mapsto (M \otimes_B \mathcal{B})_0$ and $\mathcal{O}_X - \text{Mod} \longrightarrow \text{Gr} - B : F \mapsto H^0(F \otimes \mathcal{B})_{\geq 0}$. Composing the two yields an endofunctor $M \mapsto \overline{M}$ of $\text{Gr} - B$. We have a natural transformation $\eta_M : M \longrightarrow \overline{M}$ which is an isomorphism modulo torsion.

Now η_B is an isomorphism so the injection $I \hookrightarrow B$ factors through $\eta_I : I \longrightarrow \overline{I}$. Hence, η_I is an injection. Now, η_I has torsion cokernel so it suffices to prove the following two lemmas,

Lemma 4.5.4 *$\overline{I}$ is finitely generated.*

Lemma 4.5.5 *Given an exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ in $Gr - B$ with M_2 finitely generated and M_3 torsion, then M_1 is also finitely generated.*

Proof.(4.5.4) Since $(\cdot \otimes_B \mathcal{B})_0 : \text{Gr} - B/\text{tors} \longrightarrow \mathcal{O}_X - \text{Mod}$ is one of the inverse equivalences in theorem 4.3.4, $(I \otimes_B \mathcal{B})_0$ is a subsheaf of $\mathcal{O}_X$. It thus suffices to show that for any coherent sheaf

F on X, $H^q(F \otimes \mathcal{B})_{\geq 0}$ for $q \in \mathbb{N}$ is finitely generated. We achieve this by downward induction on q. The case for large q is clear by Grothendieck's vanishing theorem. By lemma 4.3.6, we have a surjective map $\mathcal{O}_X^n \longrightarrow F \otimes \mathcal{B}_v$ for some $n \in \mathbb{N}$ and some $v \in \mathbb{Z}^2$ sufficiently large. Tensoring by a shift of $\mathcal{B}$ yields the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{P} \longrightarrow F \otimes \mathcal{B} \longrightarrow 0$$

where $\mathcal{P}$ is a finite sum of shifts of $\mathcal{B}$. Further, by the equivalence of categories $\mathcal{O}_X$ − Mod $\leftrightarrow$ Gr − $\mathcal{B}$, we see that $\mathcal{K} \simeq \mathcal{K}_0 \otimes \mathcal{B}$ where $\mathcal{K}_0$ is a coherent $\mathcal{O}_X$-module. We have an exact sequence $H^q(\mathcal{P})_{\geq 0} \longrightarrow H^q(F \otimes \mathcal{B})_{\geq 0} \longrightarrow H^{q+1}(\mathcal{K})_{\geq 0}$. By induction, we can assume the last term to be finitely generated so it remains to observe that $H^q(\mathcal{B})_{\geq w}$ is too, for any $w \in \mathbb{Z}^2$. The following fact was proved in [AV; p.261-262] for the case $q = 0, s = 1$:

Lemma 4.5.6 *For each $u \in \mathbb{Z}^s, (s \in \mathbb{N})$ large enough, there exists $v \in \mathbb{Z}^s$ such that the multiplication map $H^q(\mathcal{B})_u \otimes_k H^0(\mathcal{B})_{v'} \longrightarrow H^q(\mathcal{B})_{u+v'}$ is surjective whenever $v' \geq v$.*

Proof. For $q > 0$ the statement is trivial as ampleness ensures $H^q(\mathcal{B})_u = 0$ for u large enough so we assume $q = 0$. By lemma 4.3.6, $\mathcal{B}_u$ is generated by sections for u large enough. This gives an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow H^0(\mathcal{B}_u) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{B}_u \longrightarrow 0$$

Choosing v so that $H^q(\mathcal{G} \otimes \mathcal{B}_{v'}) = 0$ whenever $v' \geq v$ and $q > 0$ gives the lemma.

Returning to the proof of lemma 4.5.4, fix $u \geq w$ large enough that lemma 4.5.6 holds. Then $H^q(\mathcal{B})_{\geq w}$ is generated by the graded components $H^q(\mathcal{B})_{w'}$ where w' is not greater than or equal to $v+u$. But $H^q(\mathcal{B})_{\geq w}/H^q(\mathcal{B})_{\geq v+u}$ is finitely generated since it has a filtration with finitely generated factors by our claim (4.5.3). This shows that $H^q(\mathcal{B})_{\geq w}$ is finitely generated verifying the lemma.

Proof.(4.5.5) Since M_3 is finitely generated torsion, we may choose $v \in \mathbb{Z}^2$ large enough so that $M_1 \supset (M_2)_{\geq v}$. We write M_2 as a quotient of a finite sum of shifts of B, say $\oplus_{i=1}^n B(v_i)$, where $n \in \mathbb{N}, v_i \in \mathbb{Z}^2$. Now, $M_2/(M_2)_{\geq v}$ is a quotient of $\oplus B(v_i)/B(v_i)_{\geq v}$ and so is a noetherian module by our claim. It thus suffices to show that $(M_2)_{\geq v}$ is finitely generated or that $B(v_i)_{\geq v} = B_{\geq v_i+v}$ is finitely generated. This has already been verified above. So ends the proof of the theorem.

On applying the theorem to example 4.4.1, we recover a special case of Hilbert's basis theorem. The theorem also applies to example 4.4.2 giving

Corollary 4.5.7 *Let L be a right ample invertible bimodule on a projective scheme X over k. Then the Rees algebra $B[\mathfrak{m}t]$ of $B = B(X; L)$ is right noetherian.*

Corollary 4.5.8 *Let L_σ, M_τ be right ample invertible bimodules on projective schemes X and Y respectively. Then the tensor product $B(X; L_\sigma) \otimes_k B(Y; M_\tau)$ of the twisted homogeneous coordinate rings is right noetherian.*

Proof. Using the notation of example (4.4.3) we see $pr_1^* L_\sigma, pr_2^* M_\tau$ satisfy (*) so the hypotheses of the theorem hold provided they form a right ample set. We verify this latter now. Let $\overline{L} := pr_1^* L$ and let σ, τ also denote the automorphisms $\sigma \times 1, 1 \times \tau$ of $X \times Y$. Let F be a coherent sheaf on $X \times Y$. As in claim 4.5.3 we consider the Leray spectral sequence,

$$H^p(Y, R^q pr_{2*}(F \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L}) \otimes M \otimes \ldots \otimes \tau^{j-1*}M) \Longrightarrow H^{p+q}(X \times Y, F\overline{L}_\sigma^i \overline{M}_\tau^j)$$

It suffices to show that for i, j large enough, the first term vanishes whenever $p > 0$ or $q > 0$. Pick ample line bundles $\mathcal{O}_X(1), \mathcal{O}_Y(1)$ on X and Y respectively and let $\mathcal{O}(m,n) := pr_1^*\mathcal{O}_X(m) \otimes pr_2^*\mathcal{O}_Y(n)$. We will make use of the exact sequence,

$$0 \longrightarrow K \longrightarrow \oplus\mathcal{O}(n_l, n_l) \longrightarrow F \longrightarrow 0$$

where the sum in the middle is finite. We first show that for any coherent sheaf F on $X \times Y$, $R^q pr_{2*}(F \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L}) = 0$ for $q > 0$ and i large enough, by downward induction on q. Grothendieck's vanishing theorem [EGA III; corollary 4.2.2] again dispenses with the large q case and, assuming inductively the result for $q+1$ and K, we see it suffices to prove the result for $F = \mathcal{O}(n,n)$. But,

$$R^q pr_{2*}(\mathcal{O}(n,n) \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L}) = \mathcal{O}_Y(n) \otimes R^q pr_{2*}(\mathcal{O}(n,0) \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L})$$

$$= \mathcal{O}_Y(n) \otimes H^q(X, L \otimes \ldots \otimes \sigma^{i-1*}L(n))$$

by the projection formula and the commutativity of cohomology with flat base change (see [EGA III; proposition 1.4.15]). Right ampleness of L_σ now ensures the last term is zero for i large enough.

We now show that

$$E_2^{p0} = H^p(Y, pr_{2*}(F \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}L) \otimes M \otimes \ldots \otimes \tau^{j-1*}M) = 0$$

for i, j large enough. We use downward induction on p. By the previous paragraph, for i large enough, we have an exact sequence,

$$0 \longrightarrow pr_{2*}(K \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L}) \longrightarrow pr_{2*}(\oplus_l \mathcal{O}(n_l, n_l) \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L})$$

$$\longrightarrow pr_{2*}(F \otimes \overline{L} \otimes \ldots \otimes \sigma^{i-1*}\overline{L}) \longrightarrow 0$$

Tensoring this sequence by $M \otimes \ldots \otimes \tau^{j-1*}M$ and considering the long exact sequence in cohomology we are again reduced to the case $F = \mathcal{O}(n,n)$. But then

$$E_2^{p0} = H^p(Y, \mathcal{O}_Y(n) \otimes H^0(X, \mathcal{O}_X(n) \otimes L \otimes \ldots \otimes \sigma^{i-1*}L) \otimes M \otimes \ldots \otimes \tau^{j-1*}M)$$

which by ampleness is zero for j large enough independent of i. This completes the proof of the corollary.

Corollary 4.5.9 *The tensor product of 3-dimensional Artin-Schelter regular algebras is noetherian.*

Proof. Let A_1, A_2 be Artin-Schelter regular algebras of dimension 3. From [ATV; proposition 6.7(i) and theorem 6.8(i)] we know there exist normal homogeneous elements $g_i \in A_i$ of positive degree or 0 for $i = 1,2$ such that $A_i/(g_i)$ is the twisted homogeneous coordinate ring of a projective scheme with respect to an ample invertible bimodule. Thus corollary 4.5.8 together with its left-handed companion show that $(A_1 \otimes A_2)/(g_1 \otimes 1, 1 \otimes g_2)$ is noetherian. Since $g_1 \otimes 1$ and $1 \otimes g_2$ are normal, the result follows from two or fewer applications of [ATV; lemma 8.2].

Bibliography

[AS] Artin, Stafford, "Regular Local Rings of Dimension 2", manuscript

[ASch] Artin, Schelter, "Graded Algebras of Global Dimension 3" , *Adv. in Math.*, **66**, (1987), p. 172-216

[ATV] M. Artin, J. Tate, M. Van den Bergh, "Some Algebras Associated to Automorphisms of Elliptic Curves", *Grothendieck Festchrift vol. 1*, Birkhäuser, Boston, (1990) , p. 33-85

[AV] M. Artin, M. Van den Bergh, "Twisted Homogeneous Coordinate Rings",*J. of Algebra*, **133**, (1990) , p. 249-271

[AZ] M. Artin, J, Zhang, "Noncommutative Projective Schemes", *Adv. in Math.*, **109**, (1994), p. 228-287

[B] G. Bergman, "The Diamond Lemma in Ring Theory", *Adv. in Math.*, **29**, (1978) , p. 178-218

[Ba] H. Bass, "On the Ubiquity of Gorenstein Rings", *Math. Zeitschr.*, **82**, (1963), p. 8-28

[Bj] Björk, "The Auslander Condition on Noetherian Rings", *Séminaire Dubreil-Malliavin 1987-88*, Lecture Notes in Math. vol. 1404, Springer-Verlag, Berlin (1989)

[BN] M. Böckstedt, A. Neeman, "Homotopy Limits in Triangulated Categories", *Composition Math.*, **86**, (1993), p. 209-234

[EGA] J. Dieudonné, A. Grothendieck, "Éléments de Géometrie Algébrique , Ch. II & III", *Publ. Inst. Math. Hautes Etudes Sci.*, No. 8 & 11, (1961)

[FAC] J.-P. Serre, "Faisceaux Algébriques Cohérents", *Ann. of Math.*, **61**, (1955), p. 197-278

[G] P. Gabriel, "Des Catégories Abéliennes", *Bull. Soc. Math. France*, **90**, (1962), p. 323-448

[GW] K. Goodearl, R. Warfield, " An Introduction to Noncommutative Noetherian Rings", London Math. Soc. Student Texts vol. 16, Cambridge Univ. Press, Cambridge, 1989

[H1] R. Hartshorne, "Residues and Duality", Lecture Notes in Math. vol. 20, Springer-Verlag, Berlin Heidelberg New York, 1966

[H2] R. Hartshorne, "Stable Reflexive Sheaves", *Math. Ann.*, **254**, (1980), p. 121-176

[J1] P. Jorgensen, "Non-commutative Graded Homological Identities", *J. London Math. Soc.*, to appear

[J2] P. Jorgensen, "Local Cohomology for Non-commutative Graded Algebras", *Comm. Alg.*, **25**, (1997), p. 575-591

[Jac] N. Jacobson, "Basic Algebra II" 2nd edition, Freeman, New York, (1989)

[L] T. Levasseur, "Some Properties of Non-commutative Regular Graded Rings", *Glasgow Math. J.*, **34**, (1992), p. 277-300

[LvO] H. Li, F. van Oystaeyen, "Zariskian Filtrations", Kluwer Academic Publishers, Dordrecht-Boston-London, (1996)

[M1] S. Mac Lane, "Categories for the Working Mathematician", Graduate Texts in Mathematics 5, Springer-Verlag, Berlin Heidelberg New York, (1971)

[M2] S. Mac Lane, "Natural Associativity and Commutativity", *Rice University Studies*, **49**, (1963), p. 28-46

[McR] J, McConnell, Robson, "Noncommutative Noetherian Rings", John Wiley & Sons, Chichester New York Brisbane Toronto Singapore, (1987)

[Mont] S. Montgomery, "Fixed Rings of Finite Automorphism Groups of Associative Rings", Lecture Notes in Math. **818**, Springer-Verlag, Berlin Heidelberg New York, (1980)

[Pink] H. Pinkham, "Singularités de Klein - I,II", *Séminaire sur les Singularités des Surfaces*, Lecture Notes in Math. vol. 777, Springer-Verlag, Berlin Heidelberg New York, (1980)

[VdB1] M. Van den Bergh, "Existence Theorems for Dualizing Complexes over Non-Commutative Graded and Filtered Rings", *to appear in J. of Algebra*

[VdB2] M. Van den Bergh, "A Translation Principle for the Four-dimensional Sklyanin Algebras", *J. of Algebra*, **184**, (1996), p. 435-490

[Y1] A. Yekutieli, "Dualizing Complexes over Noncommutative Graded Algebras", *J. of Algebra*, **153**, (1992), p. 41-84

[Y1] A. Yekutieli, "The Residue Complex of a Noncommutative Graded Algebra", *J. of Algebra*, **186**, (1996), p. 522-543

[Yos] Y. Yoshino, "Cohen-Macaulay Modules over Cohen-Macaulay Rings", London Math. Soc. Lecture Note Series vol. 146, Cambridge University Press, Cambridge New York Port Chester Melbourne Sydney, (1990)