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VARIABLE METRIC METHODS AND FILTERING THEORY

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ABSTRACT

In this paper we show that there is a close relationship between variable metric methods of function minimization and filtering of linear stochastic systems with disturbances which are modelled as unknown but bounded functions. We develop new variable metric algorithms for function minimization.

1. INTRODUCTION

The objective of this paper is to show that there is a close relationship between variable metric methods of function minimization and filtering of linear stochastic systems with disturbances which are modelled as unknown but bounded functions.

It is well known that Newton's method for function minimization exhibits quadratic convergence in the neighborhood of the minimum. This rapid convergence rate however is obtained at the expense of requiring second derivative computations and solution of a linear equation at each iteration stage. On the other hand, variable metric methods do not require second derivative computations nor matrix inversion (solution of a linear equation) and versions of this algorithm are known to exhibit reasonably rapid convergence. Intuitively, one may consider a variable metric method as one where an estimate of the Hessian (or inverse of a Hessian) is obtained on the basis of information on function values and gradient values in past iterations and the next step is determined on the basis of this estimate. In this paper, we attempt to make this intuitive notion precise.

The work closest in spirit to this work is the doctoral dissertation of THOMAS [4]. The stochastic models we derive are however, somewhat different and we exploit linear filtering theory to the fullest extent possible. We obtain algorithms which do not require accurate line search algorithms as was also done by Thomas.

2. FILTERING MODEL FOR THE ALGORITHM

Consider the problem of minimizing

$$\{f(x) \mid x \in \mathbb{R}^n\} \quad , \text{ where} \tag{2.1}$$

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f is assumed to be thrice continuously differentiable on $\mathbb{R}^n$.

Let

$$\nabla f(x) = g(x) \quad \text{and} \quad D^2 f(x) = G(x) . \tag{2.2}$$

Let x^* be a local minimum of f and in some open, convex neighborhood D of x^*, let us assume

$$||G(x_k + \theta_1 s_k) - G(x_k + \theta_2 s_k)|| \leq L|\theta_1 - \theta_2| \cdot ||s_k|| , \text{ where } L > 0 \tag{2.3}$$

for all $x_k, x_k + s_k \varepsilon D$, all $\theta_1, \theta_2 \varepsilon [0,1]$.

We wish to discuss iterative algorithms for minimizing $f(x)$ and the algorithm proceeds as $x_{k+1} = x_k + s_k$, $k = 0,1,2,\ldots$.

Let us use the notation

$$\left.\begin{aligned} G_k(\theta) &= G(x_k + \theta s_k) \\ g_k(\theta) &= g(x_k + \theta s_k) , \quad k = 0,1,2,\ldots \end{aligned}\right. \tag{2.4}$$

It is easy to see that there exists $U_k \varepsilon L^1(0,1;\mathcal{L}(\mathbb{R}^n))$ such that

$$\left.\begin{aligned} &G_k(\theta) - G_k(0) = \int_0^\theta U_k(t)\,dt , \text{ with} \\ &||U_k(\theta)|| \leq L||s_k|| , \ k = 0,1,2,\ldots,\theta\varepsilon[0,1] . \end{aligned}\right\} \tag{2.5}$$

Also

$$g_k(\theta) = g_k(0) + \int_0^\theta G_k(t) s_k\,dt . \tag{2.6}$$

Evaluating (2.5) and (2.6) at $\theta = 1$, and using the natural notation $G_k(1) = G_{k+1}$, $G_k(0) = G_k$, $g_k(1) = g_{k+1}$, etc., we get

$$\left.\begin{aligned} G_{k+1} &= G_k + \int_0^1 U_k(t)\,dt \\ g_{k+1} &= g_k + G_k s_k + \int_0^1 [G_k(t) - G_k(0)] s_k\,dt . \end{aligned}\right\} \tag{2.8}$$

Let

$$V_k = \int_0^1 U_k(t)dt$$

$$w_k = \int_0^1 [G_k(t) - G_k(0)]s_k dt \quad .$$

Then, we may rewrite (2.8) as

$$\left.\begin{aligned} G_{k+1} &= G_k + V_k \\ g_{k+1} &= g_k + G_k s_k + w_k \quad . \end{aligned}\right\} \tag{2.9}$$

It is natural to think of V_k and w_k as process and observation noise respectively. They are obviously correlated. We now attempt to bound the noise.

3. BOUNDS ON THE NOISE

To do the bounding, we use the following device: Let γ_i denote the $i^{\underline{th}}$ row of G . We then use the isomorphism

$$i : \mathcal{L}(\mathbb{R}^n) \to R^{n^2} : G \mapsto \begin{pmatrix} \gamma_i \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{pmatrix} .$$

We can then rewrite equation (2.8) in differential form:

$$\left.\begin{aligned} \frac{d}{d\theta}(iG_k) &= iU_k(\theta) \\ \frac{d}{d\theta} g_k(\theta) &= (I_n \otimes s_k')(iG_i(\theta)) \quad . \end{aligned}\right\} \tag{2.10}$$

In the above ' denotes transpose and $\otimes$ denotes tensor product. Writing (2.10) in vector-matrix form:

$$\frac{d}{d\theta}\begin{bmatrix} iG_k(\theta) \\ g_k(\theta) \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ I_n \otimes s_k' & 0 \end{pmatrix}\begin{pmatrix} iG_k(\theta) \\ g_k(\theta) \end{pmatrix} + \begin{pmatrix} iU_k(\theta) \\ 0 \end{pmatrix} \quad . \tag{2.11}$$

We are interested in bounding V_k and w_k as $U_k(\cdot)$ varies over the class of all mappings given by (2.5). Clearly, the set of all (iV_x, w_k) as $U_k(.)$ varies is a convex set in $\mathbb{R}^{n^2} + \mathbb{R}^n$. Let Ω_k denote the set. We can compute the support function of this set and estimate that the support function $\eta_k(G^*, g^*)$, $G^* \varepsilon \mathcal{L}(R^n)^*$,

$g^* \varepsilon (\mathbb{R}^n)^*$ (* denotes the dual space) satisfies:

$$\eta_k(G^*,g^*) \le L||s_k||\{\frac{1}{3}\, ||s_k||^2 ||g^*||^2 + (g^*,G^*s_k) + ||G^*||^2\}^{1/2} \quad . \tag{2.12}$$

It is easy to see that an appropriate choice of $U_k(\cdot)$ in the class defined by (2.5) attains this bound and hence the support function can be computed as:

$$\eta_k(G^*,g^*) = L||s_k||\{<\begin{pmatrix}G^*\\ g^*\end{pmatrix},Q_k\begin{pmatrix}G^*\\ g^*\end{pmatrix}>\}^{1/2} \tag{2.13}$$

where $<\cdot,\cdot>$ is the obvious inner product in $\mathcal{L}(\mathbb{R}^n) \times \mathbb{R}^n$ and Q_k in the matrix defined from the right hand side of (2.12). We can check that $Q_k > 0$ (unless $s_k = 0$).

The above discussions may be combined in the following:

Consider the problem of estimating G_k from

$$G_{k+1} = G_k + V_k \tag{2.14}$$

$$z_k = G_k s_k + w_k \text{ , where } z_k = g_{k+1} - g_k \quad . \tag{2.15}$$

Let $G_o \varepsilon \Omega_o$ where

$$\Omega_o = \{G\varepsilon\mathcal{L}(\mathbb{R}^n)\,|<G - \hat{G}_o, {}_o^{-1}(G - \hat{G}_o)>_{\mathbb{R}^{n^2}} \le 1 \tag{2.16}$$

and $\hat{G}_o, \pi_o > 0$ are given .

Then

<u>Proposition 1</u>

$$\begin{pmatrix}V_k\\ w_k\end{pmatrix}\varepsilon\Omega_k \text{ , where}$$

$$\Omega_k = \{(\nu)\,\varepsilon\mathcal{L}(\mathbb{R}^n) \times \mathbb{R}^n \left| \frac{1}{L||s_k||} <\nu,Q_k^{-1}\nu>_{\mathcal{L}(\mathbb{R}^n)\times\mathbb{R}^n} \le 1\} \right. \quad . \quad \square \tag{2.17}$$

4. <u>SOLUTION OF THE ESTIMATION PROBLEM</u>

The estimation problem can now be solved using the work of BERTSEKAS [1]. It consists of recursively estimating the sets Ω_k , which are ellipsoids. The centre of the ellipsoid is the desired estimate. These results are summarised in

Proposition 2

$$\hat{\Omega}_{K+1} = \left\{ G \middle| <G-\hat{G}_{K+1},\ \Pi^{-1}_{K+1}(G-\hat{G}_{K+1})>_{\mathbb{R}^{n^2}+\mathbb{R}^n} \leq 1-\gamma_K \right\} \tag{2.18}$$

where Π_{K+1} satisfies

$$\Pi_{K+1}G = GP_{K+1},\ k=0,1,2,\ldots.. \tag{2.19}$$

and P_K is given by

$$P_{K+1} = (1+||s_k||)\left\{ P_k + L^2||s_k||I_n - \frac{[P_k + \frac{L^2||s_k||}{2}I_n]s_k s'_k [P_k + \frac{L^2||s_k||}{2}I_n]}{(s_k,\ [P_k + \frac{L^2||s_k||}{2}I_n]s_k)} \right\} \tag{2.20}$$

$$\hat{G}_{K+1} = \hat{G}_K + \frac{[z_k - \hat{G}_k s_k]s'_k[P_k + \frac{L^2||s_k||}{2} I]}{(s_k, [P_k + \frac{L^2||s_k||}{2} I]s_k)} \tag{2.21}$$

and

$$\nu_k = \frac{||z_k - \hat{G}_k s_k||^2}{L^2||s_k||\ [1+||s_k||]s'_k[P_k + \frac{L^2||s_k||}{3} I_n]s_k} \tag{2.22}$$

Proposition 3

If $H_k = (\hat{G}_k)^{-1}$ exists and $\hat{G}_{K+1}$ is generated by (2.20) - (2.22) then

$$H_{K+1} = H_k + \frac{[s_k - H_k z_k]d'_k H_k}{\alpha_k + d'_k[s_k - H_k z_k]} \tag{2.23}$$

is the inverse of G_{K+1}, where

$$\alpha_k = \frac{(s_k,\ [P_k + \frac{||s_k||}{3} I_n]\ s_k)}{(s_k,\ [P_k + \frac{||s_k||}{2} I_n]s_k)} < 1 \tag{2.24}$$

$$d_k = \frac{[P_k + \frac{||s_k||}{2} I_n]\ s_k}{(s_k,\ [P_k + \frac{||s_k||}{2} I_n]s_k)} \tag{2.25}$$

5. NEW ALGORITHMS FOR FUNCTION g MINIMIZATION

Since we are looking for an estimate of the Hessian (or inverse of the Hessian)it is desirable that our estimates are symmetric. This suggests the following algorithm:

(i) Propagate H_k and P_k according to (2.23) and (2.20)

(ii) Symmetrize H_k to obtain H_k^s

(iii) Find the closest approximation $\hat{H}_k$ to H_k^s so that the secant equation $s_k = Xz_k$ is satisfied.

(iv) The new step is computed according to Powell's dog-leg strategy (cf. POWELL [3])

We now present a number of convergence results corresponding to the use of different estimates for the Hemian.

Suppose we update P_k and H_k according to (2.20) and (2.23)-(2.25) with $P_o=\sigma^2 I$. The new step is chosen according to the formula

$s_k = -H_k g_k$, and let us update Π_k according to (5.1)

$$\begin{cases} \Pi_{K+1} = (1+||s_k||)\left[\Pi_k+||s_k||I_n - \dfrac{(2\alpha_k-1)}{\alpha_k^2}\;\dfrac{d_k d'_k}{(s'_k d_k)}\right] \\ \Pi_o = \sigma^2 I. \end{cases} \tag{5.2}$$

We then have:

<u>Lemma 5.1</u>

$$\Pi_k \geq P_k \geq 0 \quad \forall k \geq 0 . \tag{5.3}$$

We know, that if $\hat{G}_k$ is non-singular then

$$\hat{G}_{K+1} = \hat{G}_k + \frac{(z_k-\hat{G}_k s_k)d'_k}{\alpha_K}, \quad k=0,1,2,\ldots \tag{5.4}$$

We can then show

<u>Lemma 5.2</u>

There exists a $\mu>0$, such that

$$||G_k-\hat{G}_k||^2 \leq \mu, \quad k \geq 0. \tag{5.5}$$

These ideas enable us to prove the following basic convergence theorem:

Theorem 5.3

Let $g: R^n \to R^n$ be differentiable in an open convex neighborhood D of x^*, where x^* satisfies $g(x^*)=0$ and we also have $Dg(x^*) = G(x^*)$ is non-singular. Let us suppose that $G(\cdot)$ satisfies

$$||G(x)-G(y)|| \leq L||x-y||, \forall x,y \varepsilon D. \tag{5.6}$$

Then for each $\gamma \geq 0$, $r\varepsilon[0,1]$, $\exists \delta=\delta(\gamma,r)$, $\varepsilon=\varepsilon(\gamma,r)$ such that if $||x_o-x^*|| \leq \delta$ and $||\hat{G}_o-G_o|| \leq \gamma\sigma$, $\sigma\varepsilon[0,\varepsilon]$, then, the sequence

$$x_{K+1} = x_k - [\hat{G}_k]^{-1} g_k \tag{5.7}$$

converges to x^*.

Moreover

$$||x_{k+1}-x^*|| \leq \gamma ||x_k-x^*|| \text{ and the sequence} \tag{5.8}$$

$(||\hat{G}_k||)_{k=0,1,\ldots}$ and $(||\hat{H}_k||)_{k=0,1,\ldots}$ are uniformly bounded.

Theorem 5.3 shows that we obtain linear convergence. One can show that the convergence is actually superlinear.

So far we have constructed an algorithm which uses the output of the filter directly. As we have previously remarked it would be desirable to "symmetrize" the estimate and use this as in the algorithm. It can be shown that an algorithm using the symmetrized estimate converges linearly under the same hypotheses as that of Theorem 5.3. However a proof of convergence of the algorithm when the estimates are also chosen to satisfy the secant equation is at present not available. The details of the proof of the various results, presented in this paper will appear elsewhere [cf. MITTER-TOLDALAGI [2]].

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