On quaternionic line bundles

by

Gustavo Granja

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of
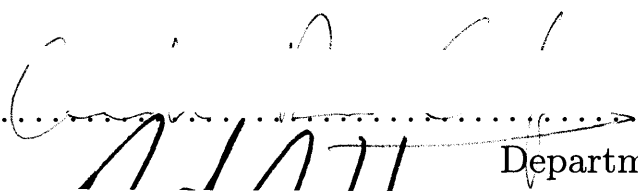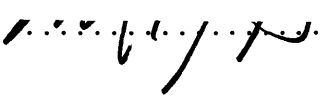
Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1999

Author . . . .
Department of Mathematics
May 7, 1999

Certified by . . . .
Michael Hopkins
Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
Richard Melrose
Chairman, Departmental Committee on Graduate Students

On quaternionic line bundles

by

Gustavo Granja

Submitted to the Department of Mathematics on May 7, 1999
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

Abstract

We use obstruction theory based on the unstable Adams spectral sequence for S^3 to study the classification of quaternionic line bundles over finite dimensional complexes. We pay special attention to universal examples, namely finite dimensional quaternionic projective spaces.

Thesis Supervisor: Michael Hopkins
Title: Professor of Mathematics

Acknowledgements

1. INTRODUCTION.

Let $\mathbb{H}$ denote the division algebra of quaternions. A quaternionic line bundle is a fibre bundle with fibre $\mathbb{H}$ and structure group $\mathbb{H}^\times$ acting on $\mathbb{H}$ by left multiplication. The group of quaternions of unit length is canonically isomorphic to $SU(2)$ so there is a 1-1 correspondence between quaternionic line bundles and principal $SU(2)$-bundles.

Unlike real or complex line bundles which are classified by their characteristic classes, quaternionic line bundles are not. As the group of unit quaternions is nonabelian, the classifying space for quaternionic line bundles $\mathbb{H}P^\infty$ is not an Eilenberg-Maclane space. This makes the problem of classifying quaternionic line bundles over a space X very hard. Consider for example the case when X is a sphere. The homotopy long exact sequence of the universal bundle

$$S^3 \longrightarrow ES^3 \longrightarrow \mathbb{H}P^\infty$$

shows that $\pi_i \mathbb{H}P^\infty = \pi_{i-1}S^3$ so classifying quaternionic line bundles over spheres amounts to describing the underlying sets of the homotopy groups of S^3, a problem whose solution is nowhere in sight.

The previous example illustrates the close relationship between the classification of quaternionic line bundles and the structure of the homotopy groups of S^3. Although the structure of $\pi_* S^3$ is far from being completely understood, much information about it is known. This is best expressed in terms of the unstable Adams spectral sequence. What we will do in this thesis is to apply the available information on $\pi_* S^3$ to the classification of quaternionic line bundles.

The first result which we will prove is a sufficient condition for a cohomology class in $H^4(X;\mathbb{Z})$ to be the Pontryagin class of a quaternionic line bundle over a finite dimensional CW-complex X. We will find $N(d)$ such that if X is a CW-complex of dimension $\leq d$ and $\alpha \in H^4(X;\mathbb{Z})$ is divisible by $N(d)$ then α is the Pontryagin class of a quaternionic line bundle over X. See Proposition 3.6 for a precise statement.

Most of this thesis deals with classification of quaternionic line bundles over $\mathbb{H}P^n$. A quaternionic line bundle over $\mathbb{H}P^n$ is a homotopy class of maps $\mathbb{H}P^n \longrightarrow \mathbb{H}P^\infty$ or equivalently, by cellular approximation, a homotopy class of self maps of $\mathbb{H}P^n$. In this case, the classification is of special interest for two reasons. The first is that finite dimensional quaternionic projective spaces play a universal role: a self map of $\mathbb{H}P^n$ will give an operation on quaternionic line bundles over complexes of dimensions $\leq 4n$. The second, which was the original motivation for this work, is that the study of self maps of $\mathbb{H}P^n$ gives information about the homotopy properties of the standard multiplication of the Lie group $SU(2)$.

Stasheff [St] shows that there is a hierarchy on the set of homotopy classes of self maps of a loop space X

$$[X,X] = A_1(X) \supset A_2(X) \supset \ldots \supset A_\infty(X)$$

where denoting the loop multiplication on X by μ,

$$A_2(X) = \{f \in [X,X] : f \circ \mu \simeq \mu \circ (f \times f)\}$$

is the set of H-self maps,

$$A_3(X) = \{f \in A_2(X) : \mu \circ (\mu \circ (f \times f) \times f) \simeq \mu \circ (f \times \mu \circ (f \times f))\}$$

and in general $A_n(X)$ is the set of H-self maps of X which "preserve associativity up to $(n-2)$-th higher order homotopy" (see [St]). $A_\infty(X)$ is the set of loop self maps. When $X = SU(2)$, $A_1(X) = \mathbb{Z}$ and results of Stasheff [St] imply that there is a self map of $\mathbb{H}P^n$ of degree k (i.e. whose restriction to $\mathbb{H}P^1$ has degree k) if and only if $k \in A_n(SU(2))$. Thus the homological classification of self maps of $\mathbb{H}P^n$ is the same as the determination of the sets $A_n(SU(2))$.

Finally, knowledge of the sets $A_n(SU(2))$ would be of interest because it would give an indication of which power maps of a finite loop space can be expected to admit an A_n-structure (cf. the work of McGibbon [MG2, MG3] for $n = 2, 3$).

Our main result is the construction of a number of new self maps of $\mathbb{H}P^n$ (see Theorem 5.10 for a detailed statement). The main method used in the proof will be obstruction theory based on the unstable Adams spectral sequence for S^3. As a corollary, a conjecture of Feder and Gitler [FG] regarding the homological classification of self maps of $\mathbb{H}P^n$ is verified for $n \le 5$.

Organization of the paper. In section 2 we record the information about the unstable Adams spectral sequence for $\mathbb{H}P^\infty$ that we will need in the rest of the paper. In section 3 we use this to prove the result on the existence of quaternionic line bundles over finite dimensional CW-complexes.

The remaining sections are concerned with self maps of $\mathbb{H}P^n$. In section 4 we define the obstruction to extension of a self map of $\mathbb{H}P^n$ and prove some of its basic properties. We also discuss the homotopy classification of self maps and obtain it in low dimensions in terms of the homological classification. The latter is the subject of section 5 where we construct quaternionic line bundles over $\mathbb{H}P^n$ with specified Pontryagin classes and as a corollary prove the Feder-Gitler conjecture for $n \le 5$. Section 6 is devoted to some remarks on certain interesting spherical fibrations over $\mathbb{H}P^n$.

2. THE UNSTABLE ADAMS SPECTRAL SEQUENCE FOR $\mathbb{H}P^\infty$.

In this section we record some facts about the unstable Adams spectral sequence for $\mathbb{H}P^\infty$ and explain how to use the spectral sequence as the basis of an obstruction theory for constructing maps into $\mathbb{H}P^\infty$. A good general reference for the undefined terms we will use concerning the Steenrod algebra is [Sc].

The Massey-Peterson spectral sequence. Let $\mathcal{U}$ denote the category of unstable modules over the mod p Steenrod algebra. If $M \in \mathcal{U}$, $U(M)$ denotes the free unstable algebra generated by M and for $P \in \mathcal{U}$ a free module, we write $K(P)$ for the mod p generalized Eilenberg-Maclane space such that

$$H^*(K(P); \mathbb{Z}/p) = U(P)$$

Recall from [HM2] that if X is a simply connected space such that

$$H^*(X; \mathbb{Z}/p) = U(M)$$

for some $M \in \mathcal{U}$, then the Massey-Peterson spectral sequence is defined. It is the spectral sequence of a tower of principal fibrations under X

(2.1)

where

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \ldots$$

is a free resolution of M. The diagram (2.1) is called an unstable **Adams resolution** for X. For a simply connected space of finite type this spectral sequence converges strongly and

$$E_2^{s,t}(X) = Ext_{\mathcal{U}}^{s,t}(M, \mathbb{F}_2) \Rightarrow \pi_{t-s}X \otimes \widehat{\mathbb{Z}}_2$$

Let $I(\mathcal{A})$ denote the augmentation ideal of the Steenrod algebra. Recall that if $M, N \in \mathcal{U}$, a map $f : M \longrightarrow N$ is minimal if $\ker(f) \subset I(\mathcal{A})M$. A resolution is minimal if each of its maps is minimal.

Lemma 2.1. *If $H^*(X) = U(M)$ there is an Adams resolution of X so that in the corresponding homotopy spectral sequence*

$$E_2^{(s,t)}(X) = E_1^{(s,t)}(X)$$

Proof. Pick a minimal resolution P_* for M. This has the property that all the differentials in the complex $Hom_{\mathcal{U}}(P_*, \Sigma^t \mathbb{Z}/2)$ are trivial (see for example [MT]). Since this complex is the E_1-term of the Massey-Peterson spectral sequence we conclude that, as bigraded vector spaces, $E_2(X) = E_1(X)$. $\square$

We will consider the associated tower of principal fibrations over X obtained by setting X^{s+1} to be the homotopy fiber of $X \to E_s$.

(2.2)

$$
\begin{array}{c}
\vdots \\
\downarrow{\scriptstyle p_2} \\
X^2 \xrightarrow{\;j_2\;} K(\Omega^2 P_2) \\
\downarrow{\scriptstyle p_1} \\
X^1 \xrightarrow{\;j_1\;} K(\Omega P_1) \\
\downarrow{\scriptstyle p_0} \\
X^0 = X \xrightarrow{\;j_0\;} K(P_0)
\end{array}
$$

The homotopy spectral sequence of this tower is clearly isomorphic to the Massey-Peterson spectral sequence.

Definition 2.2. *The **Adams filtration** of a map $f : W \to X$ is the largest s such that f factors through X^s.*

We can use the tower (2.2) to construct maps $W \to X$ as follows. Assuming by induction that we have constructed a map on the k-skeleton $f_k : W^{(k)} \to X$ of Adams filtration s, it follows that the obstructions to extending the map to the $(k+1)$-skeleton $W^{(k+1)}$ have filtration $\geq s$. This restricts the set of obstructions we have to rule out in order to guarantee that f_k extends to $f_{k+1} : W^{(k+1)} \longrightarrow X$. In order to proceed by induction we will need an estimate of the Adams filtration of the extension f_{k+1}. The following lemma will be useful for this purpose. It gives a criterion for the vanishing of the obstruction to extension to imply the vanishing of the obstruction on an Adams cover. If $s > t$, let $p_t^s = p_t \circ \cdots \circ p_{s-1}$.

Lemma 2.3. *Suppose that in the homotopy spectral sequence of (2.2) the image of d_r in $E_r^{s+i,t+i}$ is 0 for each $r \geq q$ and $i \geq 0$. Then p_{0*}^s is injective on $p_{s*}^{s+q-1}(\pi_{t-s}X^{s+q-1}) \subset \pi_{t-s}X^s$.*

Proof. Write $F_s = K(\Omega^s P_s)$ and consider the $(q-1)$-th derived exact couple of the homotopy exact couple of (2.2):

$$
\begin{array}{c}
\vdots \\
\downarrow{\scriptstyle p_s} \\
\pi_{t-s+1}F_{s-1}^{(q-1)} \xrightarrow{\;\delta\;} \pi_{t-s}X^{s\,(q-1)} \xrightarrow{\;j_s\;} \pi_{t-s}F_s^{(q-1)} \\
\downarrow{\scriptstyle p_{s-1}} \\
\pi_{t-s+1}F_{s-2}^{(q-1)} \xrightarrow{\;\delta\;} \pi_{t-s}X^{s-1\,(q-1)} \xrightarrow{\;j_{s-1}\;} \pi_{t-s}F_{s-1}^{(q-1)} \\
\downarrow{\scriptstyle p_{s-2}} \\
\pi_{t-s+1}F_{s-3}^{(q-1)} \xrightarrow{\;\delta\;} \pi_{t-s}X^{s-2\,(q-1)} \xrightarrow{\;j_{s-2}\;} \pi_{t-s}F_{s-2}^{(q-1)} \\
\downarrow{\scriptstyle p_{s-3}} \\
\vdots
\end{array}
$$

By definition $\pi_{t-s}X^{s\,(q-1)} = p_{s*}^{s+q-1}(\pi_{t-s}X^{s+q-1}) \subset \pi_{t-s}X^s$. Suppose $\alpha \in \ker(p_{0*}^s : \pi_{t-s}X^{s\,(q-1)} \longrightarrow \pi_{t-s}X^{0\,(q-1)})$ is nonzero.

Then there exists $u < s$ and $0 \neq \gamma \in im(\delta) \subset \pi_{t-s} X^{u(q-1)}$ such that $p^s_{u*}(\alpha) = \gamma$. Since $holim_{\leftarrow} X^s \simeq *$ there also is $v \geq s$ and $\beta \in \pi_{t-s} X^{v(q-1)}$ such that $p^v_{s*}(\beta) = \alpha$ and $\beta \notin im\, p_{v*}$. This gives a nonzero differential $d_r : E_r^{u,t+u-s+1} \to E_r^{v,t+v-s}$ and concludes the proof. $\square$

The spectral sequence for $\mathbb{H}P^\infty$. Let M denote the following unstable module

$$\begin{cases} \mathbb{Z}/p < x, \mathrm{P}^2\, x, \mathrm{P}^{2p}\, \mathrm{P}^2\, x, \mathrm{P}^{2p^2}\, \mathrm{P}^{2p}\, \mathrm{P}^2\, x, \ldots > & \text{if } p > 2 \\ \mathbb{Z}/p < x, \mathrm{Sq}^4\, x, \mathrm{Sq}^8\, \mathrm{Sq}^4\, x, \mathrm{Sq}^{16}\, \mathrm{Sq}^8\, \mathrm{Sq}^4\, x, \ldots > & \text{if } p = 2 \end{cases}$$

where x denotes an element of degree 4. Then $H^*(\mathbb{H}P^\infty; \mathbb{Z}/p) = U(M)$. In general, Adams resolutions are not preserved by looping and so the unstable Adams spectral sequences of X and ΩX are quite different. In our special case, however, we have the following

Lemma 2.4. *The Massey-Peterson spectral sequences for S^3 and $\mathbb{H}P^\infty$ are naturally isomorphic. For each $r \geq 2, s, t$*

$$E_r^{s,t}(S^3) = E_r^{s,t+1}(\mathbb{H}P^\infty)$$

Proof. First note that $H^*(S^3; \mathbb{Z}/p) = U(\Sigma^3 \mathbb{Z}/p)$ and $\Omega M = \Sigma^3 \mathbb{Z}/p$. Since $\Omega_1 M = 0$, applying Ω to a free resolution of M yields a free resolution of $\Sigma^3 \mathbb{Z}/p$. Thus an Adams resolution for S^3 can be obtained by looping an Adams resolution for $\mathbb{H}P^\infty$ which implies the result $\square$

The previous lemma makes things easier for us since the Adams spectral sequence for S^3 has been studied extensively. As usual, if p is an odd prime, we set $q = 2(p-1)$.

Figures 1 and 2 describe the portion of the E_2 term of the unstable Adams spectral sequence for $\mathbb{H}P^\infty$ along the vanishing line. As usual, vertical lines represent multiplication by p and the slanted lines composition on the right with η if the prime is 2 and with α_1 if the prime is odd.

Theorem 2.5 (Mahowald, Miller, Harper-Miller, Thompson). *Consider Figures 1 and 2.*

(a) Above the classes shown and the dotted lines in the columns where no classes appear, the E_2 term vanishes.

(b) The classes in dimensions $qk + 3$ for p odd and the circled classes in dimensions $8k + 7$ and filtration $\geq 4k + 2$ for $p = 2$ correspond to elements on $\pi_* S^3$ which are stable and detected by the real e-invariant.

(c) The classes in dimensions $qk + 2$ for p odd are not boundaries.

Proof. Recall from [Ma1] and [HM1] that for each prime there is a bigraded complex $(\Lambda(3), d)$ such that

$$E_2^{s,t}(S^3) = H^{s,t-3} \Lambda(3)$$

There is a short exact sequence of complexes

$$0 \longrightarrow \Lambda(1) \longrightarrow \Lambda(3) \longrightarrow W(1) \longrightarrow 0$$

which induces a split short exact sequence on homology. $\Lambda(1)$ is a complex with 0 differential which corresponds to the $\mathbb{Z}$-tower in $t - s = 3$.

Let $M \otimes \Lambda$ denote the E_1-term of the stable Adams spectral sequence for the mod$p$ Moore spectrum given by the Λ-algebra. The main results of [Ma1] and [HM1] state that there is a map of complexes $W(1) \longrightarrow M \otimes \Lambda$ inducing an isomorphism on homology for $t - s < 5s - 16$ for $p = 2$ and $t - s < (p+1)qs - (p+2)q$ for p odd. This reduces (a) to a statement about the stable E_2-term of the Moore spectrum except in low dimensions where it is easily checked directly by computing the homology of the complex $\Lambda(3)$.

The E_2-term of the Moore spectrum is computed in the required range in [Mi] for p odd and in [Ma3] for $p = 2$. It agrees with the description given in Figures 1 and 2 shifted according to Lemma 2.4. This proves (a).
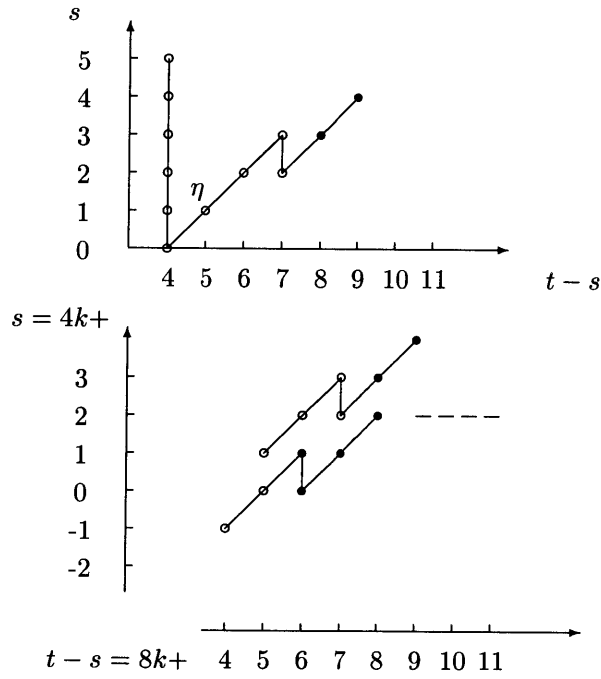
12

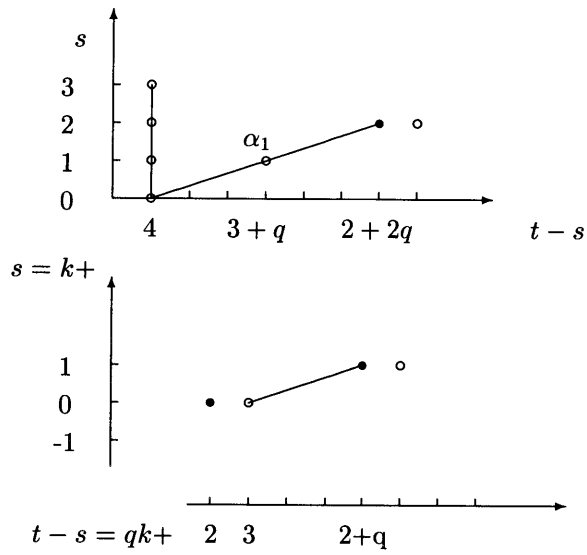FIGURE 1. $E_2^{(s,t)}(\mathbb{H}P^\infty)$ for $p = 2$.

FIGURE 2. $E_2^{(s,t)}(\mathbb{H}P^\infty)$ for p odd.

Let M^n denote the mod p Moore space with top cell in dimension n and $i : S^{n-1} \longrightarrow M^n$ denote the inclusion of the bottom cell. Let $A : M^{n+r} \longrightarrow M^n$ with $r = 8$ if $p = 2$ and $r = q$ if $p > 2$ be a map inducing an isomorphism in K-theory (see [Ad]).

Suppose first that p is odd. Let $\tilde{\alpha}$ denote an extension of a generator $\alpha \in \pi_{2p} S^3$ to M^{2p+1}. Then by [Ad, Proposition 12.7], $\alpha_k = \tilde{\alpha} \circ A^{k-1} \circ i \in \pi_{qk+2} S^3$ survives to a stable class which is detected by the e-invariant. Since α_k has Adams filtration at least k it must be represented in $E_2(S^3)$ by the class in bidegree $(t - s, s) = (qk + 2, k)$.

Next let $p = 2$. Let $\tilde{\mu}$ denote an extension to M^{13} of the generator μ of $\pi_{12} S^3$. Then $\mu_k = \tilde{\mu} \circ A^{k-1} \circ i \in \pi_{8k+4} S^3$ has Adams filtration at least $4k + 1$ (the Adams filtration of the Adams map is 4 since we are in the stable range). It follows that the Toda bracket $< \mu_k, 2, \eta > \in \pi_{8k+7} S^3$ has Adams filtration at least $4k + 2$. Proposition 12.18 of [Ad] shows that this bracket survives to a stable class of order 4 detected by the e-invariant so the bracket is represented on $E_2(S^3)$ by the class in bidegree $(t - s, s) = (8k + 6, 4k + 2)$. This completes the proof of (b).

Finally, let $\tilde{\beta}$ denote an extension of $\alpha^2 \in \pi_{4p+3} S^3$ to M^{4p+4}. Then $\beta_k = \tilde{\beta} \circ A^{k-1} \circ i \in \pi_{q(k+1)+1} S^3$ has Adams filtration at least $k + 1$ and is not null by Theorem 1.3 of [Th]. This proves (c). □

Remark 2.6. In the previous theorem we have only stated the results we will need in the sequel. The classes shown in Figures 1 and 2 are all permanent cycles and represent the v_1-periodic homotopy of S^3. The circled classes are stable and the filled ones are killed by double suspension. For more details, see [Ma2] and [Th].

The previous theorem gives partial information on all the homotopy groups of $\mathbb{H}P^\infty$. We will also require complete information in low dimensions. Given Lemma 2.4, the following theorem is a small part of the main result of [CM].

Theorem 2.7 (Curtis and Mahowald). *The Adams spectral sequence for* $\mathbb{H}P^\infty$ *at the prime 2 for* $t - s \leq 25$ *is completely described by Figure 3.*

For an odd prime, we have the following result

Proposition 2.8. *Let* p *be an odd prime. Then* $E_2^{s,t}(\mathbb{H}P^\infty)$ *is described by Figure 4 for* $t - s \leq (2p + 1)q - 1$.

Proof. By Lemma 2.4 we can consider $E_2^{s,t}(S^3)$ instead. As was already mentioned in the proof of Theorem 2.5, it follows from [HM1] that if M denotes the mod p Moore spectrum (with top cell in dimension 1), there is a map

$$E_2^{s,t}(S^3) \longrightarrow E_2^{s-1,t-q-3}(M)$$

defined for $t > s > 0$ which is an isomorphism for $s > 3$ and $0 < t - s \leq (2p + 1)q - 2$ and an epimorphism for $s > 2$ and $t - s \leq (2p + 1)q - 2$.

By the Localization Theorem [Mi], for $s \geq 3$ and $t - s \leq 2pq - 4$, $E_2^{s,t}(M)$ is isomorphic to

$$\mathbb{F}_p[v_1, b_{1,0}] \otimes E(h_{1,0}, h_{2,0})$$

where the bidegrees $(t - s, s)$ are given by $|v_1| = (q, 1), |h_{i,0}| = (2(p^i - 1) - 1, 1)$, and $|b_{(1,0)}| = (2p(p - 1) - 2, 2)$.

Thus, in order to prove the proposition, we must check that for $s \leq 3$ and $t - s \leq (2p + 1)q - 2$, $E_2^{s,t}(S^3) = 0$ except for one class in each of the bidegrees $(t - s, s)$

$$
\begin{aligned}
(kq + 2, k) \quad &: \quad 1 \leq k \leq 3 \\
(kq + 1, k) \quad &: \quad 1 < k \leq 3 \\
(3, (p + 1)q) \quad & \\
(3, (p + 3)q + 2) \quad & \\
(3, (p + 3)q + 1) \quad &
\end{aligned}
$$
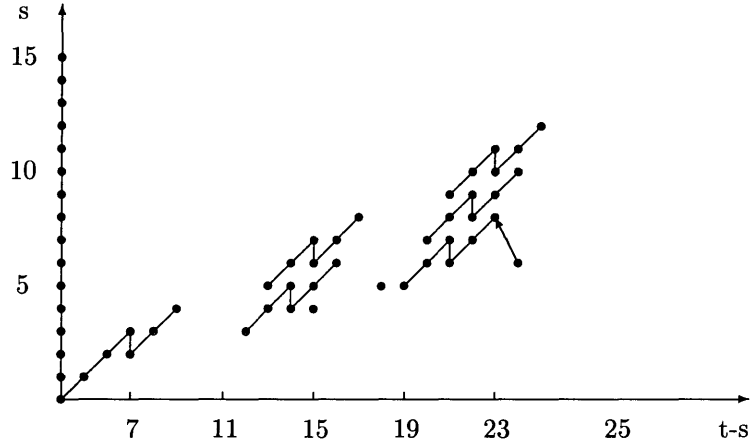
FIGURE 3. $E_2^{s,t}(\mathbb{H}P^\infty)$ at $p = 2$ for $t - s \leq 25$.
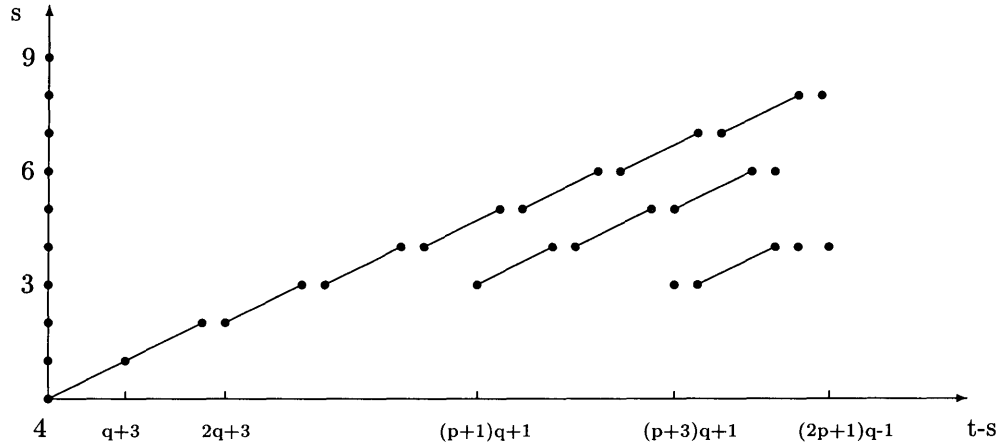
FIGURE 4. $E_2^{s,t}(\mathbb{H}P^\infty)$ for $t - s \leq (2p + 1)q - 1$, for p odd.

We will show this by using the Λ-algebra. Tangora's memoir [Ta] is a good reference for any terms we use which are unfamiliar to the reader. See also [HM1]. Recall that the vector space $\tilde{\Lambda}(3) \subset \Lambda(3)$ generated by the set of admissible monomials with λ-endings is a subcomplex and the inclusion induces an isomorphism in homology for $t - s > 0$.

A basis for $\tilde{\Lambda}(3)$ for $s \leq 3$ is given by

$$
\begin{aligned}
s = 1 \quad &: \quad \lambda_1 \\
s = 2 \quad &: \quad \lambda_1 \lambda_m (1 \leq m < p) \\
&\quad\ \ \mu_1 \lambda_m (1 \leq m \leq p) \\
s = 3 \quad &: \quad \lambda_1 \lambda_m \lambda_k (1 \leq m < p, 1 \leq k < mp) \\
&\quad\ \ \mu_1 \lambda_m \lambda_k (1 \leq m \leq p, 1 \leq k < mp) \\
&\quad\ \ \lambda_1 \mu_m \lambda_k (1 \leq m < p, 1 \leq k \leq mp) \\
&\quad\ \ \mu_1 \mu_m \lambda_k (1 \leq m \leq p, 1 \leq k \leq mp)
\end{aligned}
$$

To compute the homology of $\tilde{\Lambda}(3)$ it is useful to notice that the differential preserves the Cartan degree (i.e. the number of μ's in a monomial).

$\lambda_1, \lambda_1 \lambda_1$ and $\mu_1 \lambda_1$ are cycles. For $m > 1$, the leading term of $d(\lambda_1 \lambda_m)$ is $-m \lambda_1 \lambda_{m-1} \lambda_1$ so the differential is injective on the subspace generated by $\lambda_1 \lambda_m$ with $2 \leq m < p$. Similarly one checks that d is injective in the subspace generated by $\mu_1 \lambda_m$ for $2 < m \leq p$. This computes $E_2^{s,t}(S^3)$ for $s \leq 2$.

For $s = 3$, we need only consider the monomials for which $m + k < 2p$ since the others lie outside the range we are interested in. We have a lower bound on the dimension of $\ker d$ given by the rank of the image of d plus the number of classes in filtration 2 on the Moore spectrum. We can get a lower bound for the rank of d by giving $x_i \in E_1^{3,*}$ such that the leading terms of $\{dx_i\}$ are all distinct. We then must show that the two bounds agree. We will give the details only for the Cartan degree 2 summand in $\tilde{\Lambda}(3)$ since this is the only case of interest for Corollary 2.9.

First note that $\mu_1 \mu_1 \lambda_1$ is a cycle. The leading term of $d(\mu_1 \mu_m \lambda_k)$ is

$$-k \mu_1 \mu_m \lambda_{k-1} \lambda_1 \quad \text{if} \quad 1 < k < p \text{ or } k > p$$

$$(1 - m) \mu_1 \mu_{m-1} \lambda_1 \lambda_1 \quad \text{if} \quad k = 1, 1 < m \leq p$$

$$\mu_1 \lambda_m \mu_{p-1} \lambda_1 \quad \text{if} \quad k = p$$

so we see that the only monomials of Cartan degree 2 whose differentials have the same leading term are $\mu_1 \mu_l \lambda_2$ and $\mu_1 \mu_{l+1} \lambda_1$ for $1 \leq l < p$.

If $p \not\equiv 1 \bmod 4$, replace $\mu_1 \mu_l \lambda_2$ with

$$\mu_1 \mu_l \lambda_2 + \frac{2}{l} \mu_1 \mu_{l+1} \lambda_1$$

The leading term of the differential of this element is now $\mu_1 \lambda_l \mu_1 \lambda_1$ which shows that d is injective on the linear span of $\{\mu_1 \mu_m \lambda_k : m \text{ or } k \neq 1\}$ in the range under consideration.

If $p \equiv 1 \bmod 4$ replace $\mu_1 \mu_l \lambda_2$ with

$$\mu_1 \mu_l \lambda_2 + \frac{2}{l} \mu_1 \mu_{l+1} \lambda_1 + \frac{l-1}{3} \mu_1 \mu_{l-1} \lambda_3$$

The leading term of the differential of this element is $\mu_1 \lambda_{l-1} \mu_2 \lambda_1$ which again shows d is injective. This completes the proof. $\qquad \square$

Corollary 2.9. *Let p be an odd prime. Then $\pi_{4n-1}(S^3) \otimes \mathbb{Z}_{(p)}$ is detected by the e-invariant for $n < (2p+1)q/4$ but not for $n = (2p+1)q/4$.*

Proof. The first part follows immediately from Proposition 2.8 and Theorem 2.5 (b). Let γ denote a generator of $E_2^{4,(2p+1)q-2}(S^3)$. Then γ is a permanent cycle because, for $p > 3$ there are no classes in the Λ-algebra $E_1^{*,(2p+1)q-3}(S^3)$ and for $p = 3$ the only class is the generator of $E_2^{6,25}$ which is not a boundary by Theorem 2.5. Since $E_1^{s,(2p+1)q-1} = 0$ for $s \leq 2$ we conclude that γ detects a nontrivial homotopy class in $\pi_{(2p+1)q-2}S^3$. This class is not detected by the e-invariant. Otherwise it would be v_1-periodic and it follows from Corollary 1.2 of [Th] that there is only one v_1-periodic class in this dimension, namely the one accounted for in Theorem 2.5. $\qquad \square$

3. PONTRYAGIN CLASSES OF QUATERNIONIC LINE BUNDLES.

Let X be a finite dimensional CW complex. In this section we give sufficient conditions for elements of $H^4(X;\mathbb{Z})$ to be Pontryagin classes of quaternionic line bundles.

We will write P for the universal Pontryagin class, which is a generator of $H^4(\mathbb{H}P^\infty;\mathbb{Z})$

$$\mathbb{H}P^\infty \xrightarrow{P} K(\mathbb{Z},4)$$

P is a rational equivalence so we would expect that given $\alpha \in H^4(X;\mathbb{Z})$, some multiple of α can be realized as a Pontryagin class. This is in fact the case as we will now see.

Definition 3.1. *Let $k \in \mathbb{N}$. We say that $d \in \mathbb{Z}$ is a k-divisor if for all CW complexes X of dimension $\leq k$ and $\alpha \in H^4(X;\mathbb{Z})$ such that $d|\alpha$, α is the Pontryagin class of a quaternionic line bundle over X.*

We denote the set of k-divisors by $\mathbf{D}(k)$.

Let ι_4 denote the fundamental class in $H^4(K(\mathbb{Z},4);\mathbb{Z})$. If X is a CW-complex, $X^{(k)}$ denotes the k skeleton of some cellular decomposition. In all that follows, the choice of cellular decomposition will not play a role.

Lemma 3.2. *$d \in \mathbf{D}(k)$ if and only if there is a bundle over $K(\mathbb{Z},4)^{(k)}$ with Pontryagin class $d\iota_4$.*

Proof. The condition is obviously necessary. Conversely, suppose a bundle

$$K(\mathbb{Z},4)^{(k)} \xrightarrow{\xi} \mathbb{H}P^\infty$$

with Pontryagin class $d\iota_4$ exists and X has dimension $\leq k$. By cellular approximation, any class $\beta \in H^4(X;\mathbb{Z})$ factors through $K(\mathbb{Z},4)^{(k)}$

$$
\begin{array}{ccc}
 & & K(\mathbb{Z},4)^{(k)} \\
 & \tilde{\beta} \nearrow & \downarrow \\
X & \xrightarrow{\beta} & K(\mathbb{Z},4)
\end{array}
$$

and clearly $\xi \circ \tilde{\beta}$ classifies a bundle with Pontryagin class $d\beta$. $\qquad\square$

Because the rationalization of $\mathbb{H}P^\infty$ is so simple, it is easy to assemble maps to $\mathbb{H}P^\infty$ from local maps. Let $X_{(p)}$ denote the p-localization of the nilpotent space X and $X_{(0)}$ denote the rationalization. Recall from [Su] that X is the homotopy inverse limit of the diagram

$$(3.1) \qquad
\begin{array}{ccccccc}
X_{(2)} & & X_{(3)} & & X_{(5)} & & \cdots \\
 & \searrow & \downarrow & \swarrow & \swarrow & \\
 & & X_{(0)} & &
\end{array}
$$

where all the maps are rationalization maps. If $X = \mathbb{H}P^\infty$ they classify the fundamental classes

$$\mathbb{H}P^\infty_{(p)} \longrightarrow K(\mathbb{Q},4)$$

We will abuse notation and use the same letter to denote an element in $H^4(X;\mathbb{Z})$ and its image in $H^4(X;\mathbb{Z}_{(p)})$ under extension of coefficients.

Lemma 3.3. *Let X be a space. Then $\alpha \in H^4(X;\mathbb{Z})$ is the Pontryagin class of a quaternionic line bundle over X if and only if α is in the image of*

$$[X,\mathbb{H}P^\infty_{(p)}] \xrightarrow{P_*} [X,K(\mathbb{Z}_{(p)},4)]$$

for every prime p.

Proof. The condition is clearly necessary. If $(\xi_{(p)})$ are local bundles with $P_*(\xi_{(p)}) = \alpha$ then their rationalizations agree up to homotopy. By (3.1), a choice of homotopy yields a bundle ξ with Pontryagin class $\alpha \in H^4(X;\mathbb{Z})$. $\square$

Fix a prime p. Definition 3.1 and Lemma 3.2 have the obvious local formulations: $d \in \mathbb{Z}_{(p)}$ is a p-local k-divisor if for all CW complexes of dimension $\leq k$, $dH^4(X;\mathbb{Z}_{(p)})$ is in the image of P_* and we denote the set of p-local k-divisors by $\mathbf{D}_p(k)$. As before, d is a p-local k-divisor iff there exists

$$K(\mathbb{Z}_{(p)}, 4)^{(k)} \xrightarrow{\xi} \mathbb{H}P^\infty_{(p)}$$

with $P_*(\xi) = d\iota_4$.

Proposition 3.4. *There exists* $\alpha_p(k) \in \mathbb{Z}_{(p)}$ *such that* $\mathbf{D}_p(k) = p^{\alpha_p(k)}\mathbb{Z}_{(p)}$.

Proof. For any $l \in \mathbb{Z}_{(p)}$ there is a self map of $K(\mathbb{Z}_{(p)}, 4)^{(k)}$ inducing multiplication by l on $H_4(-;\mathbb{Z}_{(p)})$. This gives an action of $\mathbb{Z}_{(p)}$ on the set of p-local k-divisors for every k. Therefore we can take

$$\alpha_p(k) = \min_{l \in \mathbf{D}_p(k)} \nu_p(l)$$

where ν_p denotes the p-adic valuation. $\square$

Note that the functions $k \mapsto \alpha_p(k)$ are nondecreasing. Moreover, since $P : \mathbb{H}P^\infty \longrightarrow K(\mathbb{Z}_{(p)}, 4)$ is a p-local equivalence through dimension $2p+1$, $\alpha_p(k) = 0$ for $k \leq 2p+1$. In particular the product appearing in the following statement is finite.

Corollary 3.5. $\mathbf{D}(k) = \bigcap_p \mathbf{D}_p(k) = \left(\prod_p p^{\alpha_p(k)}\right)\mathbb{Z}$

Proof. It follows from Lemma 3.3 that

$$\mathbf{D}(k) \supset \bigcap_p \mathbf{D}_p(k)$$

On the other hand, if $d \in \mathbf{D}(k)$ let ξ be a bundle over $K(\mathbb{Z}, 4)^{(k)}$ with Pontryagin class $d\iota_4$. Consider the diagram

$$
\left(K(\mathbb{Z}, 4)^{(k)}\right)_{(p)} \xrightarrow{j_{(p)}} K(\mathbb{Z}, 4)_{(p)}
$$

$$
K(\mathbb{Z}_{(p)}, 4)^{(k)}
$$

with maps i into $K(\mathbb{Z},4)_{(p)}$ and the dashed arrow into $\left(K(\mathbb{Z},4)^{(k)}\right)_{(p)}$.

The map $j_{(p)}$ is a k-equivalence, so the inclusion i of the k-skeleton of $K(\mathbb{Z}_{(p)}, 4)$ lifts and the composite

$$K(\mathbb{Z}_{(p)}, 4)^{(k)} \longrightarrow \left(K(\mathbb{Z}, 4)^{(k)}\right)_{(p)} \xrightarrow{\xi_{(p)}} \mathbb{H}P^\infty_{(p)}$$

shows that $d \in \mathbf{D}_p(k)$ which completes the proof. $\square$

We would like to determine the functions $\alpha_p(k)$. It is not hard to find an upper bound for $\alpha_p(k)$ just using elementary obstruction theory. Fix a prime p. Let $\mathbb{H}P^\infty[r]$ denote the homotopy fiber of a generator of $H^4(\mathbb{H}P^\infty_{(p)}; \mathbb{Z}/p^r)$

$$\mathbb{H}P^\infty_{(p)} \longrightarrow K(\mathbb{Z}/p^r, 4)$$

Then, p^r is a k-divisor if and only if the map

$$\mathbb{H}P^\infty[r] \longrightarrow K(\mathbb{Z}_{(p)}, 4)$$

classifying the fundamental class in $H^4(\mathbb{H}P^\infty[r]; \mathbb{Z}_{(p)})$ admits a section over the k-skeleton. The obstructions to extending a section lie in

$$H^n(K(\mathbb{Z}_{(p)}, 4); \pi_{n-1}\mathbb{H}P^\infty < 4 >)$$

At an odd prime, p is an exponent for $\pi_*(\mathbb{H}P^\infty < 4 >)$ and at the prime 2, 4 is an exponent [Se]. Since the obstructions are natural, the pullback square

$$
\begin{array}{ccc}
\mathbb{H}P^\infty[r+1] & \longrightarrow & \mathbb{H}P^\infty[r] \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_{(p)}, 4) & \xrightarrow{\ p\ } & K(\mathbb{Z}_{(p)}, 4)
\end{array}
$$

shows that $\alpha_p(k+1) \le \alpha_p(k) + 1$ for p odd and $\alpha_2(k+1) \le \alpha_2(k) + 2$.

However, obstruction theory based on the Adams spectral sequence provides much better bounds for $\alpha_p(k)$. Let

$$
\beta_2(k) = \begin{cases}
0 & \text{if } k \le 5 \\
4i + j + 1 & \text{if } k = 8i + 5 + j \text{ with } 1 \le j \le 3 \text{ or } j = 0 \text{ and } i > 0 \\
4i + 4 & \text{if } k = 8i + 9 \\
4i + 5 & \text{if } k = 8i + 9 + j \text{ with } 1 \le j \le 3
\end{cases}
$$

and for p odd,

$$
\beta_p(k) = \begin{cases}
0 & \text{if } k \le 3 + q \\
i + 1 & \text{if } k = iq + j \text{ with } 4 \le j \le 2 + q \text{ or } j = 3 \text{ and } i > 1
\end{cases}
$$

These functions describe the vanishing line in the E_2 term of the unstable Adams spectral sequence for $\mathbb{H}P^\infty$.

Proposition 3.6. $\alpha_p(k) \le \beta_p(k)$.

Proof. We may assume $k > 4$. We need to show the existence of a map $K(\mathbb{Z}_{(p)}, 4)^{(k)} \longrightarrow \mathbb{H}P^\infty_{(p)}$ which is multiplication by $p^{\beta_p(k)}$ on $\pi_4(-)$. Consider an Adams resolution of $\mathbb{H}P^\infty_{(p)}$ as in Lemma 2.1. Then, using the notation of (2.2), $\pi_4(X^m) = \mathbb{Z}_{(p)}$ and the maps

$$
X^{m+1} \xrightarrow{\ p_m\ } X^m
$$

induce multiplication by p on $\pi_4(-)$.

Hence it is enough to produce a map

$$
K(\mathbb{Z}_{(p)}, 4)^{(k)} \longrightarrow X^{\beta_p(k)}
$$

inducing an isomorphism on $\pi_4(-)$. We certainly have such a map on $S^4_{(p)} \subset K(\mathbb{Z}_{(p)}, 4)^{(5)}$.

Since in the homotopy spectral sequence of (2.2) $E_1 = E_2$, the vanishing line for the E_2 term of the Adams spectral sequence described in Theorem 2.5 implies that the space $X^{\beta_p(k)}$ is $(k-1)$-connected. Therefore the obstructions to extending the map from $S^4_{(p)}$ to $K(\mathbb{Z}_{(p)}, 4)^{(k)}$ vanish. $\square$

A lower bound for $\alpha_p(k)$ can be obtained by computing the image of the Pontryagin class map on particular spaces. For instance the results of section 5 (see Proposition 5.5) imply logarithmic lower bounds for $\alpha_p(k)$. However, low dimensional calculations indicate that the upper bound given in Proposition 3.6 is close to the actual value of $\alpha_p(k)$.

4. Homotopy classification over $\mathbb{H}P^n$.

In this section we discuss the classification of quaternionic line bundles over $\mathbb{H}P^n$ and we obtain the classification for $n \leq 3$ in terms of the homological classification.

First note that by cellular approximation we have

$$[\mathbb{H}P^n, \mathbb{H}P^n] \xrightarrow{\sim} [\mathbb{H}P^n, \mathbb{H}P^\infty]$$

so the classification of line bundles is the same as the classification of self maps of $\mathbb{H}P^n$. It is more convenient to take the latter point of view to describe the homotopy classification. Again by cellular approximation, we have a chain of restriction maps

$$\mathbb{Z} = [\mathbb{H}P^1, \mathbb{H}P^1] \longleftarrow [\mathbb{H}P^2, \mathbb{H}P^2] \longleftarrow \ldots \longleftarrow [\mathbb{H}P^\infty, \mathbb{H}P^\infty]$$

Definition 4.1. *Let f be a self map of $\mathbb{H}P^n$. The degree of f is the degree of the restriction of f to $\mathbb{H}P^1$.*

Thus we see that the classification problem breaks up into two steps:

(i) Determine which self maps of $\mathbb{H}P^n$ extend.

(ii) For each map which extends, classify the possible extensions.

We'll begin by dealing with (i). Recall that we have the Hopf fibration

$$S^3 \xrightarrow{i} S^{4n+3} \xrightarrow{h} \mathbb{H}P^n$$

Since i is null

$$\Omega\mathbb{H}P^n \simeq S^3 \times \Omega S^{4n+3}$$

and an explicit homotopy equivalence is given by the product of the two maps

$$S^3 \xrightarrow{E} \Omega S^4 \longrightarrow \Omega\mathbb{H}P^n$$

and

$$\Omega S^{4n+3} \xrightarrow{\Omega h} \Omega\mathbb{H}P^n$$

so that we have an isomorphism

$$\pi_i(S^{4n+3}) \oplus \pi_{i-1}S^3 \to \pi_i(\mathbb{H}P^n)$$

$$(\alpha, \beta) \mapsto h_*(\alpha) + E_*(\beta)$$

Definition 4.2. *Let $f : \mathbb{H}P^n \longrightarrow \mathbb{H}P^n$ be a map of degree k. Then $o(f) \in \pi_{4n+3}\mathbb{H}P^n$ is defined by $o(f) := f_*(h) - k^{n+1}h$.*

Proposition 4.3. *Let f be a self map of $\mathbb{H}P^n$. Then f extends to $\mathbb{H}P^{n+1}$ if and only if $o(f) = 0$. Moreover $o(f) \in E_*(\pi_{4n+2}S^3)$.*

Proof. Consider the diagram where the lines are cofiber sequences

$$
\begin{array}{ccccc}
S^{4n+3} & \xrightarrow{h} & \mathbb{H}P^n & \longrightarrow & \mathbb{H}P^{n+1} \\
\downarrow{\scriptstyle l} & & \downarrow{\scriptstyle f} & & \vdots \\
S^{4n+3} & \xrightarrow{h} & \mathbb{H}P^n & \longrightarrow & \mathbb{H}P^{n+1}
\end{array}
$$

If the dotted arrow exists, it follows that $l = k^{n+1}$. Since the lines are also fiber sequences through dimension $4n + 5$, the extension exists if and only if $f_*(h) - k^{n+1}h \in \pi_{4n+3}(\mathbb{H}P^n)$ vanishes. It remains to see that $f_*(h) - k^{n+1}h$ is in the torsion summand of $\pi_{4n+3}(\mathbb{H}P^n)$. This follows from the fact that $\mathbb{H}P^\infty_{(0)} = K(\mathbb{Q}, 4)$ and so the obstruction is 0 rationally. $\square$

Remark 4.4. Note that equivalently we can define the obstruction to extension as the composite

$$S^{4n+3} \xrightarrow{h} \mathbb{H}P^n \longrightarrow \mathbb{H}P^n \longrightarrow \mathbb{H}P^\infty$$

22

Now, to handle (ii), let f be a self map of $\mathbb{H}P^n$ such that $o(f) = 0$. An extension of f is determined by a choice of nulhomotopy of the composite

$$S^{4n+3} \xrightarrow{h} \mathbb{H}P^n \xrightarrow{f} \mathbb{H}P^n \longrightarrow \mathbb{H}P^{n+1}$$

These nulhomotopies form a set on which $\pi_{4n+4}(\mathbb{H}P^{n+1}) = \pi_{4n+3}(S^3)$ acts transitively. To describe the action of this group more explicitly, let

$$\nabla : \mathbb{H}P^n \longrightarrow \mathbb{H}P^n \vee S^{4n}$$

be a cellular approximation of the map

$$\mathbb{H}P^n \xrightarrow{id \times c} \mathbb{H}P^n \times S^{4n}$$

where $c : \mathbb{H}P^n \longrightarrow S^{4n}$ is the map collapsing the $4(n-1)$-skeleton. Then ∇ sends a generator of $H_{4n}(\mathbb{H}P^n; \mathbb{Z})$ to the sum of the generators of $H_{4n}(\mathbb{H}P^n; \mathbb{Z})$ and $H_{4n}(S^{4n}; \mathbb{Z})$.

Definition 4.5. *Given* $\alpha \in \pi_{4n+4}\mathbb{H}P^{n+1}$ *and* g *a self map of* $\mathbb{H}P^{n+1}$ *the* **coaction** *of* α *on* g *is the composite*

$$g \dashv \alpha : \mathbb{H}P^{n+1} \xrightarrow{\nabla} \mathbb{H}P^{n+1} \vee S^{4n+4} \xrightarrow{g \vee \alpha} \mathbb{H}P^{n+1}$$

The action of $\pi_{4n+4}(\mathbb{H}P^{n+1})$ on the set of extensions of f parametrizes the set of homotopy classes of extensions *relative* to the $4n$-skeleton. $\pi_1(\text{Map}(\mathbb{H}P^n, \mathbb{H}P^n)_f)$ acts on this set and the orbits of this action form the actual homotopy classes of extensions. Letting $\text{Ext}(f)$ denote the set of extensions of f and picking an identification of the set of relative homotopy classes of extensions with $\pi_{4n+3}(S^3)$ we have an exact sequence of sets

$$(4.1) \qquad \pi_1 \text{Map}(\mathbb{H}P^n, \mathbb{H}P^n)_f \xrightarrow{d} \pi_{4n+3}(S^3) \longrightarrow \text{Ext}(f)$$

To calculate these exact sequences for $n \leq 3$ we will need some formulas for the obstruction to extension.

Proposition 4.6. *Let* f, g *be self maps of* $\mathbb{H}P^n$ *of degree* k *and* l *respectively. Let* $\alpha \in \pi_{4n}(\mathbb{H}P^n)$. *Then*

(a) $o(f \dashv \alpha) = o(f) \pm \alpha \circ (nh) \pm k[\iota_4, \alpha]$
(b) $o(f \circ g) = l^n o(f) + k o(g)$

Proof. Consider the diagram

$$
\begin{array}{ccc}
 & \mathbb{H}P^n \vee S^{4n} & \\
 \nabla \nearrow & & \searrow f \vee \alpha \\
S^{4n+3} \xrightarrow{h} \mathbb{H}P^n & \xrightarrow{\quad f \dashv \alpha \quad} & \mathbb{H}P^n
\end{array}
$$

Recall from [Wh] that

$$(4.2) \qquad \pi_{4n+3}(\mathbb{H}P^n \vee S^{4n}) = \pi_{4n+3}(\mathbb{H}P^n) \oplus \pi_{4n+3}(S^{4n}) \oplus \pi_4(\mathbb{H}P^n) \otimes \pi_{4n}(S^{4n})$$

with the last summand embedded by the Whitehead product of the fundamental classes. The projections onto the first two summands in (4.2) are given by composition with the collapse maps on to the wedge summands. Let X denote the cofiber of

$$S^{4n+3} \xrightarrow{\beta} \mathbb{H}P^n \vee S^{4n}$$

Let x be a generator of $H^4(X; \mathbb{Z})$, z a generator of $H^{4n+4}(X; \mathbb{Z})$ and y an element in $H^{4n}(X; \mathbb{Z})$ restricting to the generator of S^{4n}. Then if $xy = kz$, the component of β along the third summand is $\pm k[\iota_4, \iota_{4n}]$.

Since by [Ja], $c \circ h = n\nu_{4n}$ we conclude that in terms of the decomposition (4.2), we have $\nabla \circ h = h \pm n\nu_{4n} \pm [\iota_4, \iota_{4n}]$. Hence we get

$$o(f \dashv \alpha) = o(f) \pm \alpha \circ (nh) \pm k[\iota_4, \alpha]$$

which proves (a).

As for (b)

$$
\begin{aligned}
o(f \circ g) &= f_*(g_*(h)) - (kl)^n h \\
&= f_*(l^n h + o(g)) - (kl)^n h \\
&= l^n(k^n h + o(f)) + ko(g) - (kl)^n h
\end{aligned}
$$

as required. $\square$

We can also obtain a formula relating composition with the coaction

Proposition 4.7. *Let f, g be self maps of $\mathbb{H}P^n$ of degrees k, l respectively and $\alpha, \beta \in \pi_{4n}\mathbb{H}P^n$. Then*

$$
(g \dashv \beta) \circ (f \dashv \alpha) = (g \circ f) \dashv (k^n \beta + l\alpha)
$$

Proof. Note that we have a $(4n + 3)$-equivalence

$$
\mathbb{H}P^n \vee S^{4n} \simeq_{4n+3} \mathbb{H}P^n \times S^{4n}
$$

so denoting by $k^n : \mathbb{H}P^n \to S^{4n}$ the composition of the degree k^n map with the collapse of the $4(n-1)$-skeleton of $\mathbb{H}P^n$, we have a commutative diagram

hence

$$
(g \vee \beta) \circ (f \times k^n) = (g \circ f) \dashv k^n \beta
$$

By restricting to each of the wedge summands we see that the following diagram commutes

We conclude that

$$
\begin{aligned}
(g \vee \beta) \circ ((f \times k^n) \vee (\alpha \times 0)) \circ \nabla &= (((g \circ f) \dashv k^n \beta) \vee l\alpha) \circ \nabla \\
&= ((g \circ f) \dashv k^n \beta) \dashv l\alpha \\
&= (g \circ f) \dashv (k^n \beta + l\alpha)
\end{aligned}
$$

as required. $\square$

We can now describe the homotopy classification of self maps of $\mathbb{H}P^n$ in terms of the homological classification for $n \le 3$. Let $S(n, k)$ be the set of self maps of $\mathbb{H}P^n$ of degree k.

The order of the sets $S(n, k)$ for k odd and $n \le 3$ has also been obtained by Iwase, Maruyama and Oka in [IMO] using similar methods.

Proposition 4.8. *Let $S(n, k)$ be the set of self maps of $\mathbb{H}P^n$ of degree k. Suppose $S(n, k)$ is nonempty. Then*

(a) $\; |S(2, k)| = \begin{cases} 2 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}$

(b) $\; |S(3, k)| = \begin{cases} 4 & \text{if } k \text{ is odd} \\ 2 & \text{if } k \text{ is even} \end{cases}$

Proof. Consider the sequence (4.1). Since $\pi_7(S^3) = \mathbb{Z}/2$, $\mid S(2,k) \mid$ is at most 2. Also, since S^4 is a co-H-space we have for any k, $\pi_1(map(S^4, S^4), k) = \pi_5(S^4) = \mathbb{Z}/2$. A generator of this group is $k + \eta_5$ where k denotes a degree k self map of S^4 and $+$ is induced by the comultiplication on S^4.

Let f be a self map of $\mathbb{H}P^2$ of degree k and let $o(f, f, H) \in \pi_8(\mathbb{H}P^2)$ denote the obstruction to homotopy between f and itself relative to the homotopy H on the 4 skeleton. Then one can check that,

$$o(f, f, k + \eta_5) = \eta_4 \circ \nu_5 + k[\eta_4, \iota_4]$$

Since

$$
\begin{aligned}
[\eta_4, \iota_4] &= [\iota_4 \circ \Sigma\eta_3, \iota_4 \circ \Sigma\iota_3] \\
&= [\iota_4, \iota_4] \circ \Sigma(\eta_3 \wedge \iota_3) \\
&= [\iota_4, \iota_4] \circ \eta_7 = (2\nu_4 + \Sigma\nu') \circ \eta_7 \\
&= \Sigma\nu' \circ \eta_7 = \eta_4 \circ \nu_5
\end{aligned}
$$

is a generator of $\pi_8(\mathbb{H}P^2)$, we see that the homomorphism

$$\pi_1 map(S^7, \mathbb{H}P^2)_{koH} \xrightarrow{d} \pi_8(\mathbb{H}P^2)$$

is surjective if and only if the degree is even, which proves (a).

Since $\pi_{12}\mathbb{H}P^2 = \pi_{11}S^3 = \mathbb{Z}/2$ there are at most 2 distinct extensions to $\mathbb{H}P^3$ of a self map f of $\mathbb{H}P^2$, which differ precisely by the coaction of the nonzero element $\epsilon_4 \in \pi_{12}\mathbb{H}P^2$. By Proposition 4.6 we have

$$o(\epsilon_4 \dashv f) = o(f) \pm \epsilon_4 \circ 3\nu_{12} \pm k[\iota_4, \epsilon_4]$$

where k is the degree of f. Now $\epsilon_4 \circ 3\nu_{12} = \epsilon_4 \circ \nu_{12}$, while $[\iota_4, \epsilon_4] = P(\epsilon_9) = \Sigma\nu' \circ \epsilon_7$ (see Toda [To, p. 69]).

Since these two elements generate two distinct $\mathbb{Z}/2$ summands in $\pi_{15}\mathbb{H}P^3$ we see that the obstruction to extension is always changed by the coaction and hence that d is trivial which proves (b). $\qquad\square$

We have seen examples in the proof above of how coaction on a map can alter the obstruction to extension. Therefore we have the following

Remark 4.9. The vanishing of $o(f)$ does not depend exclusively on the degree of f.

I would like to thank McGibbon for pointing out that this remark also follows from a Theorem of Mislin [Ms] which states that any self map of $\mathbb{H}P^\infty$ of degree 0 is null and the existence of essential self maps of $\mathbb{H}P^n$ of degree 0 for $n = 3, 4$ and 5 proved by Marcum and Randall [MR].

In the proof of Proposition 4.8 we saw that $o(f)$ may have a nonzero component along $\epsilon_4 \circ \nu_{12}$. This element of $\pi_{15}\mathbb{H}P^\infty$ is detected in the unstable Adams spectral sequence in filtration 4 so we have the following

Remark 4.10. o(f) is not necessarily v_1-periodic.

Notice however that, in this case, given a self map of $\mathbb{H}P^3$ we can always alter it by coaction so as to make the obstruction to extension v_1-periodic.

Let SHE(X) denote the group of homotopy classes of self homotopy equivalences of the space X. For $n \geq 2$ any element of SHE$(\mathbb{H}P^n)$ must have degree 1 since there is no self map of $\mathbb{H}P^2$ of degree -1 (see Theorem 5.2). Hence it follows from the previous proposition that SHE$(\mathbb{H}P^2) = \mathbb{Z}/2$ and SHE$(\mathbb{H}P^3)$ is a group with 4 elements. In fact, Iwase, Maruyama and Oka have shown in [IMO] that SHE$(\mathbb{H}P^3) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In the same paper the authors also show that SHE$(\mathbb{H}P^4)$ is either 0 or $\mathbb{Z}/2$ and conjecture it is 0.

5. HOMOLOGICAL CLASSIFICATION OVER HP^n.

In this section we discuss the image of the Pontryagin class of quaternionic line bundles over $\mathbb{H}P^n$ or, equivalently, the homological classification of self maps of $\mathbb{H}P^n$.

We begin by describing a conjecture of Feder and Gitler. We then formulate local versions of the conjecture. We rephrase these in terms of the obstruction to extension of a self map and then use obstruction theory based on the Adams spectral sequence to construct maps and prove the conjecture for $n \leq 5$. We conclude by formulating the conjecture in terms of geometric dimension of certain elements in K-theory of $\mathbb{H}P^n$.

The Feder-Gitler conjecture. We begin with some preliminaries on the K-theory of $\mathbb{H}P^n$. Recall that the coefficient rings for real and complex K-theories are

$$KO^*(pt) = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(\eta^3, 2\eta, \alpha^2 - 4\beta)$$

$$K^*(pt) = \mathbb{Z}[v^{\pm 1}]$$

where $|\eta| = -1, |\alpha| = -4, |\beta| = -8$ and $|v| = -2$. Recall that there is a natural ring map c induced by complexification of bundles and a natural KO^*-module map r induced by forgetting the complex structure

$$c : KO^*(X) \longrightarrow K^*(X)$$

$$r : K^*(X) \longrightarrow KO^*(X)$$

which on coefficients are determined by $c(\alpha) = 2v^2$, $r(v^2) = \alpha$.

As usual, we set $KO(X) := KO^0(X), KSp(X) := KO^4(X), K(X) = K^0(X)$. The natural forgetful map

$$KSp(X) \xrightarrow{p} K(X)$$

is given by $p(x) = v^2 c(x)$.

Recall also that there are Adams operations ψ^k acting on $KO(X)$ and $K(X)$ which are ring homomorphisms and are conjugated by the maps c and r.

Using the Atiyah-Hirzebruch spectral sequence it is easy to see that for $1 \leq n \leq \infty$ we have

$$KO^*(\mathbb{H}P^n) = KO^*[[x_4]]/(x_4^{n+1})$$

$$K^*(\mathbb{H}P^n) = K^*[[x_4]]/(x_4^{n+1})$$

We have denoted both generators by x_4 because they are identified by the map c. Note that c is an injection of the torsion free part of $KO^*(\mathbb{H}P^n)$ in $K^*(\mathbb{H}P^n)$ and similarly for r.

Since the natural projection map $\mathbb{C}P^{2n+1} \longrightarrow \mathbb{H}P^n$ determines an injection in complex K-theory, it is easy to find a formula for the action of the Adams operations on $K(\mathbb{H}P^n) = \mathbb{Z}[x]/x^{n+1}$ where we have written x for the ring generator $v^2 x_4$. In particular we have

$$\psi^2 x = 4x + x^2$$

A map $f : \mathbb{H}P^n \longrightarrow \mathbb{H}P^n$ determines a ring endomorphism of $K(\mathbb{H}P^n)$ commuting with Adams operations. If we write

$$f^*(x) = a_1 x + a_2 x^2 + \dots$$

then one easily checks that a_1 is the degree of f and the relation $\psi^2 f^* x = f^* \psi^2(x)$ recursively determines the coefficients a_n in terms of a_1. Actually, it is not easy to find an explicit formula for the coefficients from the recursive relations, but it is proved in [FG] that writing $k = a_1$,

$$(5.1) \qquad a_n = \frac{2}{(2n)!} \prod_{i=0}^{n-1} (k - i^2)$$

Note that since the Adams operations ψ^l are ring endomorphisms, this gives a formula for the Adams operations on $K(\mathbb{H}P^n)$

$$\psi^l(x) = l^2 x + \ldots + x^l$$

If we write H for the Hopf bundle over $\mathbb{H}P^n$, then $x = H - 2\mathbb{C}$ hence for any map f, f^*x is in the image of $p : KSp(\mathbb{H}P^n) \longrightarrow K(\mathbb{H}P^n)$. The image of p is additively generated by the elements $x, 2x^2, x^3, \ldots$. This together with (5.1) gives us conditions which must be satisfied by the degrees of self maps of $\mathbb{H}P^n$.

Definition 5.1. *Let $\mathbf{R}_n$ be the set of integers satisfying the congruences:*

$$(5.2) \qquad C_m : \qquad \prod_{i=0}^{m-1} (k - i^2) \equiv 0 \bmod \left\{ \begin{array}{ll} (2m)! & \text{if } m \text{ is even} \\ (2m)!/2 & \text{if } m \text{ is odd} \end{array} \right.$$

for $m = 1, \ldots, n$. Let $\mathbf{R}_{n,p}$ the set of elements in $\mathbb{Z}_{(p)}$ satisfying $C_1, \ldots, C_n$.

The discussion above is now summarized by the following result of [FG]

Theorem 5.2 (Feder and Gitler). *If f is a self map of $\mathbb{H}P^n$ of degree k then $k \in \mathbf{R}_n$.*

In the same paper, Feder and Gitler made the following

Conjecture 5.3 (Feder and Gitler). If $k \in \mathbf{R}_n$ there is a self map of $\mathbb{H}P^n$ of degree k.

The conjecture holds trivially for $n = 1$. For $n = 2$ it follows from a result of Arkowitz and Curjel [AC] and McGibbon [MG2] has given a proof for $n = 3$. It is also true for $n = \infty$. It is proved in [FG] that $\mathbf{R}_\infty$ is the set of odd square integers and 0. Sullivan [Su] constructed self maps of these degrees, the unstable Adams operations, later proved to be unique up to homotopy by Mislin [Ms].

McGibbon [MG1] has given further evidence for the conjecture by proving a stable version which we will now describe. There is a Hurewicz homomorphism

$$[\Sigma^\infty \mathbb{H}P^n, \Sigma^\infty \mathbb{H}P^n] \longrightarrow End(\tilde{H}_*(\mathbb{H}P^n)) \simeq \mathbb{Z}^n$$

Theorem 5.4 (McGibbon). *$(k, k^2, \ldots, k^n)$ is in the image of the stable Hurewicz homomorphism iff $k \in \mathbf{R}_n$.*

Thus the maps of Conjecture 5.3 all exist after appropriate suspension. We should note also that McGibbon obtains in [MG1] the homological classification of stable self maps.

Note that the content of Conjecture 5.3 is that KO-theory together with its *primary* operations is sufficiently powerful to detect the homological classification of self maps of $\mathbb{H}P^n$. This should be made clear by Proposition 5.8.

In the rest of this section we will gather further evidence for Conjecture 5.3. Our basic strategy will be to construct maps inductively by showing that the obstruction to extension vanishes. This will be done using the knowledge of the unstable Adams spectral sequence for $\mathbb{H}P^\infty$ from section 2. We will conclude by rephrasing the conjecture in terms of the geometric dimension of certain elements in $K(\mathbb{H}P^n)$.

The local problem. It follows from Lemma 3.3 that a self map of degree k exists if and only if one exists after localization at each prime. The local formulations of Theorem 5.2 and Conjecture 5.3 are obtained by replacing $\mathbf{R}_n$ with $\mathbf{R}_{n,p}$. It is therefore helpful to have a manageable description of $\mathbf{R}_{n,p}$.

Recall (see [Ei] for example) that k is a 2-adic square and a unit iff it $k \equiv 1 \bmod 8$ and, for p an odd prime, k is a p-adic square iff $k = p^{2l}u$ where u reduces to a non-zero square in $\mathbb{Z}/p$ and $l \in \mathbb{N}_0$.

Proposition 5.5. $R_{\infty,2} = \{k \in \mathbb{Z}_{(2)} : k \equiv 1 \bmod 8$ *or* $k = 0\}$ *and for* p *odd,* $R_{\infty,p} = \{k \in \mathbb{Z}_{(p)} : k = l^2$ *for some* $l \in \mathbb{Z}_p\}$. *If* $2 \le n \le \infty$ *then* $R_{n,p} = R_{\infty,p} \bigcup p^{e_p(n)}\mathbb{Z}_{(p)}$ *where*

(i) $e_2(n) = 1 + 2[\log_2 n]$

(ii) $e_p(n) = \#\{k \le n : k = p^j$ *or* $k = (\frac{p+1}{2})p^{j-1}$ *for some* $j \in \mathbb{N}\}$ *for* $p > 2$.

Proof. Let $\tilde{R}_{\infty,2} = \{k \in \mathbb{Z}_{(2)} : k \equiv 1 \bmod 8$ or $k = 0\}$ and for p odd $\tilde{R}_{\infty,p} = \{k \in \mathbb{Z}_{(p)} : k = l^2$ for some $l \in \mathbb{Z}_p\}$. It is easy to check that these sets are contained in the closure of $R_\infty = \{0, 1, 9, 25, \dots\}$ in $\mathbb{Z}_{(p)}$ with the p-adic topology. Since the solution set of each congruence is closed we conclude that $R_{\infty,p} \supset \tilde{R}_{\infty,p}$ for every p.

We will now show that $R_{n,p} = \tilde{R}_{\infty,p} \cup p^{e_p(n)}\mathbb{Z}_{(p)}$ which will immediately imply the desired result, since $e_p(n)$ tends to ∞ with n.

First take $p = 2$. If $k \in \mathbb{Z}_{(2)}$ satisfies C_2 then either $k \equiv 1 \bmod 8$, that is $k \in \tilde{R}_{\infty,2}$ or $k = 2^3 l$ for some $l \in \mathbb{Z}_{(2)}$ so it suffices to consider solutions of the form $k = 2^e u$ with u a unit and $e \ge 3$. Note also that because $\nu_2((4m+2)!/2) = \nu_2((4m)!)$ we can consider only the congruences C_m for m even.

Now assume the result is true for all $n < 2^d$. Let $2^d \le 2m < 2^{d+1}$ and let $k = 2^e u \in R_{2m,2}$ with $e \ge 3$ and $u \in \mathbb{Z}_{(2)}^\times$. Factoring out units in $\mathbb{Z}_{(2)}$ we see that k satisfies C_{2m} if and only if it satisfies

$$\prod_{i=0}^{m-1} (k - (2i)^2) \equiv 0 \bmod 2^{\nu_2((4m)!)}$$

which, writing $k = 4l$ and noting that $\nu_2((4m)!) = 2m + \nu_2((2m)!)$, is equivalent to

$$\prod_{i=0}^{m-1} (l - i^2) \equiv 0 \bmod 2^{\nu_2((2m)!)}$$

So if k satisfies $C_1, \dots, C_{2m}$ then l satisfies $C_1, \dots, C_m$. Since l is not a unit in $\mathbb{Z}_{(2)}$ it follows by induction that $R_{2m,2} - \tilde{R}_{\infty,2} \subset 2^{2d+1}\mathbb{Z}_{(2)}$.

Now we must check that $k = 2^{2d+1}l$ satisfies $C_1, \dots, C_{2m}$ for any $l \in \mathbb{Z}_{(2)}$. That is we must check that for each $j \le m$

$$\nu_2\left(\prod_{i=0}^{2j-1}(k - i^2)\right) \ge \nu_2((4j)!)$$

$$2d + 1 + \sum_{i=0}^{2j-1} 2\nu_2(i) \ge 2j + j + \dots + [j/2^d]$$

$$2(j + \dots + [j/2^d]) + 2d + 1 - 2\nu_2(2j) \ge 2j + j + \dots + [j/2^d]$$

$$j + \dots + [j/2^d] + 2d + 1 \ge 2j + 2\nu_2(2j)$$

The last inequality is easy to check if we write $2j = 2^r v$ with v a unit in $\mathbb{Z}_{(2)}$ and notice that $r \le d$.

The case when p is odd is similar but simpler. One easily checks that

$$R_{n,p} = \mathbb{Z}_{(p)} \qquad\qquad \text{if } n < (p+1)/2$$
$$R_{n,p} = p\mathbb{Z}_{(p)} \bigcup R_{\infty,p} \qquad \text{if } (p+1)/2 \le n < p$$
$$R_{p,p} = p^2\mathbb{Z}_{(p)} \bigcup R_{\infty,p}$$

Notice that $k \in \mathbb{Z}_{(p)}$ satisfies $C_1, C_2, \dots, C_n$ if and only if it satisfies $C_{(p+1)/2}, C_p, C_{2p}, \dots, C_{[n/p]p}$. If $n > p$ and $k \in R_{n,p} - \tilde{R}_{\infty,p}$ then we can write $k = p^2 v$ for some $v \in \mathbb{Z}_{(p)}$. The result now follows by induction noticing that k satisfies $C_1, \dots C_n$ if and only if v satisfies $C_1, \dots, C_{[n/p]}$. $\square$

Just as in the integral case, self maps of $\mathbb{H}P^\infty$ when localized at a prime are well understood. The following theorem of Rector (see [Rc]) verifies the local Feder-Gitler conjecture for $n = \infty$.

Theorem 5.6 (Rector). *If* $k \in R_{\infty,p}$ *there is a self map of* $\mathbb{H}P^\infty_{(p)}$ *of degree* $k \in \mathbb{Z}_{(p)}$

Actually only the maps with unit degrees are constructed in [Rc] but the same argument gives the statement above.

Since self maps of $\mathbb{H}P^\infty_{(p)}$ give us self maps of $\mathbb{H}P^n_{(p)}$ for each n, Proposition 5.5 has the following

Corollary 5.7. *The Feder-Gitler conjecture holds if there is a self map of $\mathbb{H}P^n_{(p)}$ of degree k for each k such that $p^{e_p(n)} | k$.*

The e-invariant of the obstruction. We will now see what restrictions are imposed on the obstruction to extension of a self map of $\mathbb{H}P^n$ if its degree is in $\mathbf{R}_{n+1}$. Not very surprisingly it turns out that all one can guarantee is that the component of the obstruction which is detected by KO-theory and primary operations vanishes.

Let $\mathcal{A}$ be the abelian category of abelian groups with Adams operations (see [Ad]). Write $KO(X)$ for the reduced KO-theory of X and recall that the e-invariant is a group homomorphism

$$Z \xrightarrow{e} \mathrm{Ext}_\mathcal{A}(KO(X), KO(S^{j+1}))$$

where $Z = \{\alpha \in \pi_j(X) | KO(\alpha) = 0\}$ and

$$e(\alpha) = (0 \to KO(S^{j+1}) \to KO(X \cup_\alpha e^{j+1}) \to KO(X) \to 0)$$

If X is a sphere there is a natural identification of the target of e with a subgroup of $\mathbb{Q}/\mathbb{Z}$. The stable e-invariant of $\beta \in \pi_l(S^0)$ is defined by representing β as a class $\beta' \in \pi_{8m+l}S^{8m}$ for some m and then setting $e^s(\beta) = e(\beta') \in \mathbb{Q}/\mathbb{Z}$. This is independent of the choice of m (for all this see [Ad]).

Proposition 5.8. *Let f be a self map of $\mathbb{H}P^n$ of degree k and $\alpha \in \pi_{4n+2}S^3$ be such that $E_*(\alpha) = o(f)$. Then $e^s(\alpha) = 0$ if and only if $k \in \mathbf{R}_{n+1}$.*

Proof. Recall from the discussion at the beginning of this section that $k \in \mathbf{R}_{n+1}$ if and only if there is a ring endomorphism f^* of $K(\mathbb{H}P^{n+1})$ commuting with Adams operations and such that f^*x is in the image of the forgetful map p. Clearly this still holds if we replace K by KO and p by $r \circ p$.

In turn, this is equivalent to the existence of a map of extensions in the category $\mathcal{A}$ of *groups* with Adams operations

$$
\begin{array}{ccccccccc}
(5.3) & 0 \longrightarrow & KO(S^{4n+4}) & \longrightarrow & KO(\mathbb{H}P^{n+1}) & \longrightarrow & KO(\mathbb{H}P^n) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle k^{n+1}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f^*} & & \\
& 0 \longrightarrow & KO(S^{4n+4}) & \longrightarrow & KO(\mathbb{H}P^{n+1}) & \longrightarrow & KO(\mathbb{H}P^n) & \longrightarrow & 0
\end{array}
$$

Indeed, the elements $\psi^k(r(x))$ generate $KO(\mathbb{H}P^n)$ over the rationals so ϕ is determined by its value on $r(x)$. In turn, this is determined by $f^*(r(x))$ and the fact that $\phi(r(x)^{n+1}) = k^{n+1}r(x)^{n+1}$.

Consider the extension E in $\mathcal{A}$ which is (5.3) as an extension of groups but with Adams operations $\tilde\psi^k = k^2\psi^k$. Because $KO(\mathbb{H}P^n)$ is torsion free, the existence of the map of extensions (5.3) is equivalent to the existence of the corresponding self map of E.

Using the natural identification $KO^{-4}(-) \simeq KO(\Sigma^4(-))$, there is a natural map in $\mathcal{A}$

$$E \xrightarrow{\alpha \times (-)} (0 \to KO(S^{4n+4}) \to KO(\Sigma^4\mathbb{H}P^{n+1}) \to KO(\Sigma^4\mathbb{H}P^n) \to 0)$$

which is termwise an injection of abelian groups. The condition that $f^*(r(x))$ lie in the image of $p \circ r$ is now easily seen to be equivalent to the condition that the self map of E extends over this monomorphism.

We conclude that $k \in \mathbf{R}_{n+1}$ iff the following map of extensions in $\mathcal{A}$ exists

$$
\begin{array}{ccccccccc}
0 \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(\Sigma^4\mathbb{H}P^{n+1}) & \longrightarrow & KO(\Sigma^4\mathbb{H}P^n) & \longrightarrow & 0 \\
& \downarrow{\scriptstyle k^{n+1}} & & \downarrow & & \downarrow{\scriptstyle \Sigma^4 f^*} & & \\
0 \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(\Sigma^4\mathbb{H}P^{n+1}) & \longrightarrow & KO(\Sigma^4\mathbb{H}P^n) & \longrightarrow & 0
\end{array}
$$

The existence of the map is equivalent to the following equality in Ext_A

$$e(\Sigma^4 h) \circ \Sigma^4 f^* = k^{n+1} \circ e(\Sigma^4 h)$$

where $\circ$ denotes the Yoneda product in Ext_A. Since the e-invariant sends compositions of maps to Yoneda products and is a group homomorphism these are equivalent to

$$
\begin{aligned}
e(\Sigma^4(f \circ h)) &= e(k^{n+1}\Sigma^4 h) \\
e(\Sigma^4 o(f)) &= 0
\end{aligned}
$$

By Proposition 4.3 $o(f)$ factors through the inclusion of the bottom cell of $\mathbb{H}P^n$, letting $X = \Sigma^4 \mathbb{H}P^n \cup_{o(f)} e^{4n+8}$, we have a map of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(X) & \longrightarrow & KO(\Sigma^4 \mathbb{H}P^n) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle i^*} & & \\
0 & \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(S^8 \cup_{\Sigma^5 \alpha} e^{4n+8}) & \longrightarrow & KO(S^8) & \longrightarrow & 0
\end{array}
$$

where i denotes the inclusion of the bottom cell. This is equivalent to the equality

$$e^s(\alpha) \circ i^* = e(o(f))$$

The long exact sequence in Ext induced by

$$0 \longrightarrow KO(\Sigma^4 \mathbb{H}P^n / \mathbb{H}P^1) \longrightarrow KO(\Sigma^4 \mathbb{H}P^n) \xrightarrow{i^*} KO(S^8) \longrightarrow 0$$

together with the easily checked fact that $\text{Hom}_A(KO(\Sigma^4 \mathbb{H}P^n / \mathbb{H}P^1), KO(S^{4n+8})) = 0$ show that composition with i^* is injective and this concludes the proof. $\qquad\square$

Remark 5.9. Clearly the proposition also holds locally. That is, for f a self map of $\mathbb{H}P^n_{(p)}$ of degree k, $e(o(f)) = 0$ if and only if $k \in \mathbf{R}_{n+1,p}$.

The previous proposition simultaneously makes Conjecture 5.3 seem unlikely and proves it for $n \leq 3$ since the obstruction groups $\pi_6 S^3$ and $\pi_{10} S^3$ are both cyclic and generated by elements that suspend to the image of the J-homomorphism (see [To]).

For $n \geq 3$ it is no longer true that the obstruction group is detected by the e-invariant so we can't guarantee that the obstruction to extension vanishes. In fact, cf. Remark 4.9, it doesn't always vanish. There are self maps of $\mathbb{H}P^3$ with degree in $\mathbf{R}_4$ whose obstruction to extension doesn't vanish. This might cause us to think that the conjecture should fail already for $n = 4$. That this is not the case (see Corollary 5.11) is evidence for the conjecture of a stronger nature than had previously been obtained.

The main theorem. We will now use the Adams spectral sequence obstruction theory to prove our main theorem. The idea is the same as that of the proof of Proposition 3.6 except that we can use Proposition 5.8 to rule out the first nontrivial obstruction that occurs in the extension process. Define

$$
\gamma_p(n) = \begin{cases}
e_p(n) & \text{if } n < (2p+1)(p-1)/2 \text{ or } n = 3 \\
2n - 3 & \text{if } p = 2 \text{ and } n > 3 \text{ is even} \\
2n - 5 & \text{if } p = 2 \text{ and } n > 3 \text{ is odd} \\
2^{n-(2p+1)q/4+1}3 & \text{if } p > 3 \text{ and } (2p+1)q/4 \leq n < (2p+1)q/4 + [\log_2((2p+1)/3)] \\
[4(n-2)/q] & \text{if } p > 3 \text{ and } n \geq (2p+1)q/4 + [\log_2((2p+1)/3)] \\
& \text{or if } p = 3 \text{ and } n \geq (2p+1)q/4
\end{cases}
$$

Figure 5 shows the graph of $\gamma_p(n)$ at a prime $p > 3$. It bounds the shaded area over the $(t-s)$-axis.
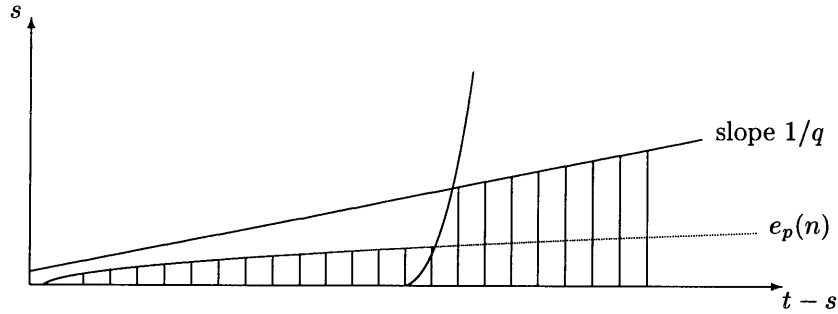
FIGURE 5. $\gamma_p(n)$ for $p > 3$.

Theorem 5.10. *For any $k \in \mathbb{Z}_{(p)}$ there is a map $\mathbb{H}P^n_{(p)} \to \mathbb{H}P^\infty_{(p)}$ of degree $kp^{\gamma_p(n)}$*

Proof. As we have already noted, the e-invariant detects the obstruction group $\pi_6 S^3$ and $\pi_{10} S^3$ at the prime 2. By Corollary 2.9, if p is an odd prime and $n < (2p+1)q/4$ the obstruction group $\pi_{4n-1} S^3_{(p)}$ is detected by the e-invariant. This proves the theorem for $n < (2p+1)q/4$.

Suppose p is odd. Since p is an exponent for $\pi_{4n-2} S^3$, Proposition 4.6(b) implies that the obstruction to extension of the composite of self maps of $\mathbb{H}P^n$ of degree p^k and $p^k l$ with $l \in \mathbb{Z}_{(p)}$ vanishes. Therefore if p^k is a divisor for $\mathbb{H}P^n$ with $k \geq 1$ then p^{2k} is a divisor for $\mathbb{H}P^{n+1}$. Since $e_p((2p+1)q/4) = 3$, this gives the case of the theorem for $(2p+1)q/4 \leq n < (2p+1)q/4 + [\log_2((2p+1)/3)]$ and $p > 5$.

Finally, for the remaining cases, let $m = \gamma_p(n)$ and let $k \in \mathbb{Z}_{(p)}$. It is enough to produce a map $\mathbb{H}P^n \to X^m$ (where X^m is as in (2.2)) of degree k and we have such a map for $n = 1$. We must show that we can extend this map to $\mathbb{H}P^n$. Consider the diagram

$$
\begin{array}{ccccccc}
 & & & & X^{m+1} & & \\
 & & & \nearrow & \downarrow {\scriptstyle p_m} & & \\
S^{4l+3} & \xrightarrow{\ h\ } & \mathbb{H}P^l & \longrightarrow & X^m & \longrightarrow & F_m \\
 & & & & \downarrow {\scriptstyle p_0 \circ \ldots \circ p_{m-1}} & & \\
 & & & & \mathbb{H}P^\infty & &
\end{array}
$$

Suppose that an extension to $\mathbb{H}P^l$ has been constructed and $l < n$. Then, because h has Adams filtration 1, the obstruction to extending the map to $\mathbb{H}P^{k+1}$, $\alpha : S^{4k+3} \to X^m \in \pi_{4k+3} X^m$ lifts to X^{m+1}.

By Theorem 2.5 there are no nonzero differentials with target $E_r^{m+i,4l+3+m+i}$ for $r \geq 2$ and $i \geq 0$. Therefore by Lemma 2.3, $\alpha \in \pi_{4l+3} X^{m\,(1)} \subset \pi_{4l+3} \mathbb{H}P^\infty$ is a class of Adams filtration $> m$ in $\pi_{4l+3} \mathbb{H}P^\infty$.

In the range we are considering, either $E_r^{s,t} = 0$ or $E_r^{s,t} = \mathbb{Z}/p$ corresponds to an element which, by Theorem 2.5 is detected by the e-invariant. Proposition 5.8 now implies that α is 0 so that we get a map $\mathbb{H}P^{l+1} \to X^m$ as required to continue the induction. $\square$

Corollary 5.11. *Conjecture 5.3 holds for $n \leq 5$.*

Proof. This follows immediately from Theorem 5.10 once we note that $e_2(4) = e_2(5) = 5$. $\square$

Note that the *ad hoc* nature of the proof of Theorem 5.10 for $p > 3$ clearly indicates that the value obtained for $\gamma_p(n)$ is not sharp.

We finish this subsection by pointing out the obstructions that must be overcome in order to prove the first unsettled case of the Feder-Gitler conjecture at each prime. At the prime 2, the

first case that remains unsettled is $n = 6$. The obstruction in question is the class in dimension 23 detected in filtration 9 in the Adams spectral sequence. This class is v_1-periodic and hence should be possible to analyze. The difficulty resides in finding a way of detecting the class on $\mathbb{HP}^\infty$ in an effective way. The means given by Thompson in [Th] requires looping three times.

At an odd prime, the first case that remains unsettled is $n = (2p+1)q/4$. The obstruction to be overcome is the class γ in filtration 4 mentioned in the proof of Corollary 2.9. This is not v_1-periodic.

Geometric dimension. Since the inclusion

$$\mathbb{HP}^\infty \longrightarrow BSp$$

classifies an algebra generator x_4 of $\widetilde{KO}^*(\mathbb{HP}^\infty)$, it follows that the map $f : \mathbb{HP}^n \longrightarrow \mathbb{HP}^\infty$ of degree k determines by composition with the inclusion $\mathbb{HP}^\infty \longrightarrow BSp$ the element

$$(5.4) \qquad kx_4 + \frac{k(k-1)}{4!}\alpha x_4^2 + \frac{2k(k-1)(k-4)}{6!}\beta x_4^3 + \ldots \in \widetilde{KO}^4(\mathbb{HP}^n) = \widetilde{KSp}(\mathbb{HP}^n)$$

Thus the problem of deciding whether there is a self map of $\mathbb{HP}^n$ of degree k is equivalent to that of deciding whether the element (5.4) in symplectic K-theory has geometric dimension 1.

Assume that (5.4) has geometric dimension 1 on $\mathbb{HP}^{n-1}$.

Since $[\mathbb{HP}^n, BSp(n)] = KSp(\mathbb{HP}^n)$ and $KSp(\mathbb{HP}^n)$ surjects onto $KSp(\mathbb{HP}^{n-1})$ extensions g of f always exist and we can consider the problem of lifting g up the tower. There is a sequence of obstruction sets $o_k(g) \subset \pi_{4n-1}S^{4(n-k)+3}$ for $1 \le k \le n-1$ defined if the previous obstructions contain 0.

It is not hard to check that under the inclusion map

$$\mathbb{HP}^\infty \longrightarrow BSp(n-k)$$

the obstruction to extension $o(f)$ is sent to an element of $o_k(g)$ (as long as this obstruction is defined). Thus we are simply filtering the obstruction group $\pi_{4n-1}\mathbb{HP}^\infty$ by

$$F_k(\pi_{4n-1}\mathbb{HP}^\infty) = \ker(\pi_{4n-1}\mathbb{HP}^\infty \longrightarrow \pi_{4n-1}BSp(k))$$

and analyzing one filtration quotient at a time. We have

$$0 = F_1 \subset F_2 \subset \ldots \subset F_n = \pi_{4n-1}\mathbb{HP}^\infty$$

The first obstruction to lifting is just the n-th symplectic Pontryagin class of the symplectic vector bundle classified by g. It is possible to compute its value and one finds not surprisingly that g can be chosen so that $o_1(g) = 0$ if and only if the degree of f is in $\mathbf{R}_n$. This gives a different proof of Theorem 5.2.

Finally note that since $\pi_4 S^0 = 0$ it follows that $F_{n-1} = F_{n-2}$. Therefore the geometric dimension of the element (5.4) is $\le n - 2$ if $k \in \mathbf{R}_n$. In particular, this gives another proof of Conjecture 5.3 for $n \le 3$.

6. Certain spherical fibrations.

In this section we consider certain fibrations over $\mathbb{H}P^n$ with fibre S^3. Homologically these look like quaternionic line bundles so it is tempting to look among them for the bundles whose existence is conjectured in Conjecture 5.3. We will show however, that these fibrations do not contain representatives of all quaternionic line bundles over $\mathbb{H}P^n$.

To simplify the statements we will work locally. It is easy to deduce corresponding integral results. We begin by defining the fibrations mentioned above.

Let l be a prime and $\overline{\mathbb{F}}_l$ denote the algebraic closure of the field $\mathbb{F}_l$. According to Friedlander and Mislin [FM] there exists a map

$$BSL(2; \overline{\mathbb{F}}_l) \longrightarrow BSL(2; \mathbb{C}) \simeq \mathbb{H}P^\infty$$

inducing an isomorphism in homology with $\mathbb{Z}/n$ coefficients for $(n, l) = 1$. In particular, if p is a prime not equal to l then

$$H^*(BSL(2; \overline{\mathbb{F}}_l); \mathbb{Z}/p) = \mathbb{Z}/p[x]$$

with $|x| = 4$. Recall from [FP] for example that

$$H^*(BSL(2; \mathbb{F}_{l^n}); \mathbb{Z}/p) = \mathbb{Z}/p[x] \otimes \mathbb{Z}/p(y)$$

where $|x| = 4, |y| = 3$ and the generators are connected by a higher Bockstein operation $\beta_r(y) = x$. Here r is such that the p-Sylow subgroup of $SL(2; \mathbb{F}_{l^n})$ is cyclic of order p^r for p odd or generalized quaternionic of order 2^r for $p = 2$. By varying l and n we can realize any r for p odd and any $r \geq 3$ for $p = 2$. In cohomology, the map determined by the inclusion

$$BSL(2; \mathbb{F}_{l^n}) \longrightarrow BSL(2; \overline{\mathbb{F}}_l)$$

identifies the classes in degree 4.

Let π denote the composite of the two maps above

$$BSL(2; \mathbb{F}_{l^n}) \xrightarrow{\pi} \mathbb{H}P^\infty$$

Since the p-completion (see [BK]) $BSL(2; \mathbb{F}_{l^n})\widehat{_p}$ is rationally trivial, the p-completion of this map lifts to $\mathbb{H}P^\infty_{(p)}$. We will also use π to denote this lift. A Serre spectral sequence calculation shows that the homotopy fiber of π has the homology of $S^3_{(p)}$ so we have the following fiber sequence

$$S^3\{p^r\} \longrightarrow S^3_{(p)} \xrightarrow{p^r} S^3_{(p)} \longrightarrow BSL(2; \mathbb{F}_{l^n})\widehat{_p} \longrightarrow \mathbb{H}P^\infty_{(p)}$$

where $S^3\{p^r\}$ denotes the homotopy fibre of the degree p^r self map of S^3. In particular, we see (cf. [Co]) that

$$\Omega BSL(2; \mathbb{F}_{l^n})\widehat{_p} \simeq S^3\{p^r\}$$

For this reason we will from now on write

$$BS^3\{p^r\} := BSL(2; \mathbb{F}_{l^n})\widehat{_p}$$

The fibration

(6.1) $$S^3_{(p)} \longrightarrow BS^3\{p^r\} \longrightarrow \mathbb{H}P^\infty_{(p)}$$

is not homologically distinguishable from a (local) quaternionic line bundle over $\mathbb{H}P^\infty_{(p)}$ with Pontryagin class p^r (up to a unit in $\mathbb{Z}_{(p)}$). However we know that for r odd such a bundle can not exist since $up^r \notin \mathbf{R}_{\infty,p}$ for any $u \in \mathbb{Z}_{(p)}^\times$. Now $p^r \in \mathbf{R}_{n,p}$ for certain n so one can hope that the restriction of (6.1) to $\mathbb{H}P^n$ has the fibre homotopy type of a quaternionic line bundle. Let

$$\mathbb{H}P^\infty_{(p)} \xrightarrow{c} B\,\mathrm{SHE}(S^3_{(p)})$$

be the map classifying (6.1). Then we are asking whether a lift exists in the following diagram

$$
\begin{array}{ccc}
 & & \mathbb{H}P^{\infty}_{(p)} \\
 & \nearrow^{\tilde{c}} & \downarrow \\
\mathbb{H}P^{n}_{(p)} \longrightarrow & \mathbb{H}P^{\infty}_{(p)} \xrightarrow{\ c\ } & B\,\mathrm{SHE}(S^3_{(p)})
\end{array}
$$

The following proposition rules this out in general.

Proposition 6.1. *If $\tilde{c}$ exists then $\tilde{c}$ extends to $\mathbb{H}P^{n+1}_{(p)}$.*

Proof. Let F_r be the homotopy fiber of $\tilde{c}$. Equivalently, F_r is the total space of the restriction of (6.1) to $\mathbb{H}P^{n}_{(p)}$. Let $F_{n,r}$ denote the $4n$-skeleton of F_r. Then letting i denote the inclusion, we have

(6.2)
$$
\begin{array}{ccc}
F_{n,r} & \longrightarrow & \mathbb{H}P^{n}_{(p)} \\
\downarrow{\simeq} & & \downarrow{i} \\
BS^3\{p^r\}^{(4n)} & \longrightarrow & \mathbb{H}P^{\infty}_{(p)}
\end{array}
$$

Consider the following diagram

$$
\begin{array}{ccccc}
M^{4n+3}(p^n) & \longrightarrow & S^{4n+3}_{(p)} & \xrightarrow{\ p^n\ } & S^{4n+3}_{(p)} \\
\downarrow & & \downarrow{h} & & \vert \\
BS^3\{p^n\}^{(4n)} & \longrightarrow & \mathbb{H}P^{n}_{(p)} & \xrightarrow{\ \tilde{c}\ } & \mathbb{H}P^{n}_{(p)} \\
\downarrow & & \downarrow & & \\
BS^3\{p^n\}^{(4n+4)} & \longrightarrow & \mathbb{H}P^{n+1}_{(p)} & &
\end{array}
$$

The two left columns are cofiber sequences and the horizontal maps are induced by the map $BS^3\{p^r\} \longrightarrow \mathbb{H}P^{\infty}_{(p)}$. By (6.2), the composition of the two maps in the middle row is null therefore the dotted arrow exists. This implies that $o(\tilde{c}) = \tilde{c} \circ h$ is divisible by p^n. But by the exponent theorem [Se], p^n is an exponent for $\pi_{4n+2}S^3_{(p)}$ so $o(\tilde{c}) = 0$ which completes the proof. $\square$

Thus many of the quaternionic line bundles on $\mathbb{H}P^{n}_{(p)}$ do not arise from the fibrations (6.1). For instance, a self map of degree 8 of $\mathbb{H}P^3_{(2)}$ does not extend to $\mathbb{H}P^4_{(2)}$ and therefore can not be obtained by restricting one of the spherical fibrations above. In fact, one can see that this is already the case for a self map of degree 8 of $\mathbb{H}P^2_{(2)}$.

REFERENCES

[Ad]   J. F. Adams, *On the groups J(X).IV*, Topology 5 (1966) 21-71.

[AC]   M. Arkowitz and C. R. Curjel, *On maps of H-spaces*, Topology 6 (1967) 137-148.

[BK]   A. K. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304, Springer Verlag (1972).

[BL]   C. Broto and R. Levi, *Loop structures on homotopy fibres of self maps of a sphere*, preprint (1998).

[Co]   F. Cohen, *Remarks on the homotopy theory associated to some finite groups*, Lecture Notes in Math. 1509 (1990) 95-103.

[CM]   E. Curtis and M. Mahowald, *The unstable Adams spectral sequence for S^3*, Contemp. Math. 96, AMS (1989) 125-162.

[Ei]   D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Grad. Texts in Math. 150, Springer Verlag (1995).

[FG]   S. Feder and S. Gitler, *Mappings of quaternionic projective spaces*, Bol. Soc. Mat. Mex. 34 (1975) 12-18.

[FP]   Z. Fiedorowicz and S. Priddy, *Homology of classical groups over finite fields and their associated infinite loop spaces*, Lecture Notes in Math. 674, Springer-Verlag (1978).

[FM]   E. Friedlander and G. Mislin, *Cohomology of classifying spaces of complex Lie groups and related discrete groups*, Comment. Math. Helvetici 59 (1984) 347-361.

[Gr]   B. Gray, *On the sphere of origin of infinite families in the homotopy groups of spheres*, Topology 8 (1969) 219-232.

[HM1] J. Harper and H. R. Miller, *On the double suspension homomorphism at odd primes*, Trans. Amer. Math. Soc. 273 (1982) 319-331.

[HM2] J. Harper and H. R. Miller, *Looping Massey-Peterson towers* in: S.M. Salamon, B. Steer, W.A. Sutherland, eds. *Advances in Homotopy Theory*(London Math. Soc. Lec. Notes 139, Cambridge University Press, Cambridge, 1989) 69-86.

[IMO] N. Iwase, K. Maruyama and S. Oka, *A note on $\mathcal{E}(\mathbb{H}P^n)$ for $n \leq 4$*, Math. J. Okayama Univ. 33 (1991) 163-176.

[Ja]   I. M. James, *The topology of Stiefel manifolds*, London Math. Soc. Lec. Notes 24, Cambridge University Press, Cambridge (1976).

[Ma1] M. Mahowald, *On the double suspension homomorphism*, Trans. Amer. Math. Soc. 214 (1975) 169-178.

[Ma2] M. Mahowald, *The image of J in the EHP sequence*, Annals of Math. 116 (1982) 65-112.

[Ma3] M. Mahowald, *The order of the image of the J-homomorphism*, Bull. Amer. Math. Soc. 76 (1970) 1310-1313.

[MR]   H. Marcum and D. Randall, *A note on self-mappings of quaternionic projective spaces*, An. Acad. Brasil. Ci. 48 (1976) 7-9.

[MP]   W. Massey and F. P. Peterson, *The mod 2 cohomology structure of certain fiber spaces*, Mem. Amer. Math. Soc. 74 (1967).

[MG1] C. McGibbon, *Self maps of projective spaces*, Trans. Amer. Math. Soc. 271 (1982) 325-346.

[MG2] C. McGibbon, *Multiplicative properties of power maps. I*, Quart. J. Math. Oxford (2) 31 (1980) 341-350.

[MG3] C. McGibbon, *Multiplicative properties of power maps. II*, Trans. Amer. Math. Soc. 274 (1982) 479-508.

[Mi]   H. Miller, *A localization theorem in homological algebra*, Math. Proc. Camb. Phil. Soc. 84 (1978) 73-84.

[MiR] H. Miller and D. Ravenel, *Mark Mahowald's work on homotopy groups of spheres*, Contemp. Math. 146, AMS (1993) 1-30.

[Ms]   G. Mislin, *The homotopy classification of self-maps of infinite quaternionic projective space*, Quart. J. Math. Oxford (2) 38 (1987) 245-257.

[MT]   R. Mosher and M. Tangora, *Cohomology operations and applications in homotopy theory*, Harper and Row Publishers, New York (1968).

[Rc]   D. Rector, *Loop structures on the homotopy type of S^3*, Lecture Notes in Math. 249 (1971) 99-105.

[Sc]   L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Chicago Lectures in Mathematics, University of Chicago Press (1994).

[Se]   P. Selick, *Odd primary torsion in $\pi_k S^3$*, Topology 17 (1978) 407-412.

[St]   J. Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Math. 161, Springer Verlag (1970).

[Su]   D. Sullivan, *Geometric Topology I. Localization, periodicity and Galois symmetry*, MIT Notes (1970).

[Ta]   M. Tangora, *Computing the homology of the lambda algebra*, Memoirs of the Amer. Math. Soc. 337 (1985).

[Th]   R. Thompson, *The v_1-periodic homotopy groups of an unstable sphere at odd primes*, Trans. Amer. Math. Soc. 319 (1990) 535-559.

[To]   H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies 49, Princeton University Press (1962).

[Wh]   G. Whitehead, *Elements of Homotopy Theory*, Grad. Texts in Math 61, Springer Verlag (1978).