CFT correlation functions from AdS/CFT correspondence

by

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Abstract

In this thesis we discuss correlation functions of $\mathcal{N}=4$, $d=4$ Super-Yang–Mills theory in the strong coupling regime. Namely, the recent conjecture of the equivalence of the string theory in $AdS_5 \times S^5$ background to the $\mathcal{N}=4$, $d=4$ SYM theory with $SU(N)$ gauge group allows to find correlation functions of the CFT in the limit of large t'Hooft coupling and at large N by evaluating relatively simple tree–level supergravity amplitudes. We discuss the basic ideas of the AdS supergravity computations, and establish the techniques for evaluating tree-level AdS supergravity scattering amplitudes with fixed rates of fall–offs of the fields as they approach AdS boundary. We translate these supergravity results into field theory language and learn several interesting things. First, at the level of the two–point correlation functions we learn about the necessity for the introduction of a cut–off in seemingly convergent AdS supergravity computations. Next, we find a non–renormalization property of certain 3–point functions. Finally, we find an explicit expression for certain 4–point functions, which deviate from free–field approximation in perturbation theory, thus providing some new non–perturbative information about SYM. We study various limits of these 4–point functions, with intention to give them an OPE interpretation. We find logarithmic singularities in all limits, and discuss their compatibility with existence of an OPE at strong coupling.

Thesis Supervisor: Samir Mathur
Title: Associate Professor of Physics

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Contents

Chapter 1

Introduction

The fact that the near horizon geometry [5]–[12] of typical brane configurations in string/M theory is the product space $AdS_{d+1} \times S_p$ with $d+1+p = 10/11$ has suggested an intriguing conjecture [1] relating string theory theory on AdS_{d+1} space–time background with a superconformal theory on its d–dimensional boundary [1]. The concept of this correspondence emerged in an earlier work on black holes [13]–[17] and has been further elaborated in [32]–[75].

Precise forms of the conjecture [1] have been stated and investigated in [2, 3] (see also [4]) for the $AdS_5 \times S_5$ geometry of N 3–branes in Type–IIB string theory. The superconformal theory on the world–volume of the N branes is $\mathcal{N} = 4$ SUSY Yang–Mills with gauge group $SU(N)$. The conjecture holds in the limit of a large number N of branes with $g_{st}N \sim g_{YM}^2 N$ fixed but large. As $N \to \infty$ the string theory becomes weakly coupled and one can neglect string loop corrections; Ng_{st} large ensures that the AdS curvature is small so one can trust the supergravity approximation to string theory. In this limit one finds the maximally supersymmetric 5–dimensional supergravity with gauged $SU(4)$ symmetry [18]–[20] together with the Kaluza–Klein modes for the "internal" S_5. There is a map [3] between elementary fields in the supergravity theory and gauge invariant composite operators of the boundary $\mathcal{N} = 4$ $SU(N)$ SYM theory. More precisely, in the maximally symmetric $\mathcal{N} = 8$ AdS supergravity in $d = 5$ there are long, short and ultra–sort (singleton) supermultiplets, containing 2^{16}, 2^8 and 2^4 states respectively. It can be shown that the singleton supermultiplet action can be rewritten as a boundary action of a free $U(1)$ gauge theory, and since we are interested in $SU(n)$ rather than $U(N)$ boundary theory we will ignore this multiplet in the further discussion. Short multiplets, which contain the lowest supergravity multiplet as well as all Kaluza–Klein modes of S^5 directly correspond to the short multiplets of chiral operators of $N = 4$ superconformal theory. This is based on the fact that the (super) isometry group of the AdS supergravity is the same as (super) conformal group of the dual gauge theory. Finally, the long multiplets of the supergravity, which come from the string modes (their mass is of the order of Plank scale) correspond to long multiplets of non–chiral operators on the SCFT. Another class of long multiplets in the CTF can be constructed by simply multiplying short multiplets. The members of these long multiplets are thus normal ordered products of chiral operators, which are non–chiral. They naturally correspond to the multi–particle states of the supergravity. At this point one should mention an important non–renormalization theorem of $\mathcal{N} = 4$ SYM theory, which states that the dimension of chiral operators, i.e. ones belonging to the short multiplets are protected from perturbative corrections. There is no such theorem for long–multiplet operators. Thanks to the equivalence of AdS (super) isometry group to (super) conformal group one can derive a simple relation between mass of the field in the supergravity and the conformal dimension of the corresponding operator in the CFT. At large mass the relation basically tells us that dimension is proportional to mass. Due to this, one can assume that the dimension of an operator corresponding to a string state is typically $\Delta_{\text{string}} \sim (g_{\text{YM}}^2 N)^{\frac{1}{4}}\ \Delta_{\text{chiral}}$, which allows us to ignore the string states in this (large t'Hooft coupling) limit. The non–chiral products of chiral operators, however, do not acquire such large dimensions, and their importance will be discussed in the last chapter. The OPE of chiral operators will generally contain their products with non–protected dimensions. From the study of 4–point functions one can deduce the mixing of order $1/N^2$ for this non–protected operators.

The calculation of correlation functions is one useful way to test and explore the

AdS/CFT correspondence. We consider the simplest example of the correspondence which is the duality between $\mathcal{N}=4,\ d=4\ SU(N)$ SYM theory and type IIB string theory on $AdS_5\times S^5$ with N units of 5–form flux and compactification radius $R^2=\alpha'\left(g_{YM}^2N\right)^{\frac{1}{2}}$. In the large N limit with $\lambda=g_{YM}^2N$ fixed and large the supergravity approximation is valid. Correlators of gauge invariant local operators in the CFT at large N and strong t'Hooft coupling λ are related to supergravity amplitudes according to the prescription of [2, 3]. Namely, the precise relation between the boundary CFT and AdSsupergravity is

$$\left\langle \exp\left(\int_{\partial AdS}\phi_0\mathcal{O}\right)\right\rangle_{\mathrm{CFT}}=Z\left[\phi_0\right] \tag{1.0.1}$$

where in this schematic notation a CFT operator $\mathcal{O}(x)$ is a boundary source for the corresponding supergravity field $\phi(x)$ and the supergravity partition function $Z\left[\phi_0\right]$ is calculated with the value ϕ_0 on the boundary. To compute a correlation function in the CFT by the correspondence, one has to implement a perturbation theory in the supergravity, with fixed values of fields on the boundary that correspond to the operators of interest in the CFT. The 5–dimensional Newton constant $G_5\sim R^3/N^2$, so that the perturbative expansion in supergravity, if ultraviolet convergent, corresponds to the $1/N$ expansion in the CFT. In the next chapter we give a more precise meaning to this scheme by working out a particular example.

Chapter 2

Two– and three–point functions

2.1 Introduction

To describe the conjecture for correlators in more detail, we note that correlators of the $\mathcal{N}=4$ $SU(N)$ SYM theory are conformally related to those on the 4–sphere which is the boundary of (Euclidean) AdS_5. Consider an operator $\mathcal{O}(\vec{x})$ of the boundary theory, coupled to a source $\phi_0(\vec{x})$ ($\vec{x}$ is a point on the boundary S_4), and let $e^{-W[\phi_0]}$ denote the generating functional for correlators of $\mathcal{O}(x)$. Suppose $\phi(z)$ is the field of the interior supergravity theory which corresponds to $\mathcal{O}(\vec{x})$ in the operator map. Propagators $K(z,\vec{x})$ between the bulk point z and the boundary point $\vec{x}$ can be defined and used to construct a perturbative solution of the classical supergravity field equation for $\phi(z)$ which is determined by the boundary data $\phi_0(\vec{x})$. Let $S_{cl}[\phi]$ denote the value of the supergravity action for the field configuration $\phi(z)$. Then the conjecture [2, 3] is precisely that $W[\phi_0]=S_{cl}[\phi]$. This leads to a graphical algorithm, see Fig.1, involving AdS_5 propagators and interaction vertices determined by the classical supergravity Lagrangian. Each vertex entails a 5–dimensional integral over AdS_5.

Actually, the prescriptions of [2] and [3] are somewhat different. In the first [2], solutions $\phi(z)$ of the supergravity theory satisfy a Dirichlet condition with boundary data $\phi_0(\vec{x})$ on a sphere of radius R equal to the AdS length scale. In the second method [3], it is the infinite boundary of (Euclidean) AdS space which is relevant. Massless scalar and gauge fields satisfy Dirichlet boundary conditions at infinity, but fields with AdS mass different from zero scale near the boundary like $\phi(z) \rightarrow z_0^{d-\Delta}\phi_0(\vec{x})$ where z_0 is a coordinate in the direction perpendicular to the boundary and Δ is the dimension of the corresponding operator $O(\vec{x})$. This is explained in detail below. Our methods apply readily only to the prescription of [3], although for 2–point functions we will be led to consider a prescription similar to [2].

The purpose of the present chapter is to present a method to calculate multi–point correlators and present specific applications to 3–point correlators of various scalar composite operators and the flavor currents J_i^a of the boundary gauge theory. Our calculations provide explicit formulas for AdS_{d+1} integrals needed to evaluate generic supergravity 3–point amplitudes involving gauge fields and scalar fields of arbitrary mass. Integrals are evaluated for AdS_{d+1}, for general dimension, to facilitate future applications of our results. The method uses conformal symmetry to simplify the integrand, so that the internal $(d+1)$–dimensional integral can be simply done. This technique, which uses a simultaneous inversion of external coordinates and external points, has been applied to many two–loop Feynman integrals of flat four–dimensional theories [21, 22, 26]. The method works well in four flat dimensions, although there are difficulties for gauge fields, which arise because the invariant action $F_{\mu\nu}^2$ is inversion symmetric but the gauge–fixing term is not [21]. It is a nice surprise that it works even better in AdS because the inversion is an isometry, and not merely a conformal isometry as in flat space. Thus the method works perfectly for massive fields and for gauge interactions in AdS_{d+1} for any dimension d.

It is well–known that conformal symmetry severely restricts the tensor form of 2– and 3–point correlation functions and frequently determines these tensors uniquely up to a constant multiple. (For a recent discussion, see [27]). This simplifies the study of the 3–point functions.

One of the issues we are concerned with are Ward identities that relate 3–point correlators with one or more currents to 2–point functions. It was a surprise to us this

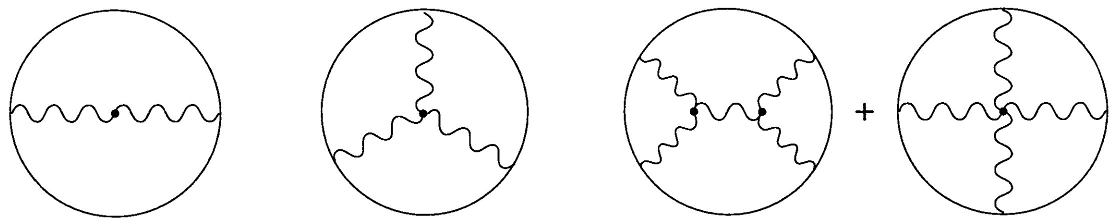

Figure 2-1: Witten diagrams.

requires a minor modification of the prescription of [3] for the computation of $\langle \mathcal{O}^I \mathcal{O}^J \rangle$ for gauge–invariant composite scalar operators.

It is also the case that some of the correlators we study obey superconformal non–renormalization theorems, so that the coefficients of the conformal tensors are determined by the free–field content of the $\mathcal{N} = 4$ theory and are not corrected by interactions. The evaluation of n–point correlators, for $n \geq 4$, contains more information about large N dynamics, and they are given by more difficult integrals in the supergravity construction. We hope, but cannot promise, that our conformal techniques will be helpful here. The integrals encountered also appear well–suited to Feynman parameter techniques, so traditional methods may also apply. In practice, the inversion method reduces the number of denominators in an amplitude, and we do apply standard Feynman parameter techniques to the "reduced amplitude" which appears after inversion of coordinates.

2.2 Scalar amplitudes

It is simplest to work [3] in the Euclidean continuation of AdS_{d+1} which is the $Y_{-1} > 0$ sheet of the hyperboloid:

$$-(Y_{-1})^2 + (Y_0)^2 + \sum_{i=1}^{d} (Y_i)^2 = -\frac{1}{a^2} \tag{2.2.1}$$

which has curvature $R = -d(d+1)a^2$. The change of coordinates:

$$
\begin{aligned}
z_i &= \frac{Y_i}{a(Y_0 + Y_{-1})} \\
z_0 &= \frac{1}{a^2(Y_0 + Y_{-1})}
\end{aligned}
\tag{2.2.2}
$$

brings the induced metric to the form of the Lobaschevsky upper half–space:

$$
ds^2 = \frac{1}{a^2 z_0^2} \left(\sum_{\mu=0}^{d} dz_\mu^2 \right) = \frac{1}{a^2 z_0^2} \left(dz_0^2 + \sum_{i=1}^{d} dz_i^2 \right) = \frac{1}{a^2 z_0^2} \left(dz_0^2 + d\vec{z}^2 \right) \tag{2.2.3}
$$

We henceforth set $a \equiv 1$. One can verify that the inversion:

$$
z'_\mu = \frac{z_\mu}{z^2} \tag{2.2.4}
$$

is an isometry of (2.2.3). Its Jacobian:

$$
\begin{aligned}
\frac{\partial z'_\mu}{\partial z_\nu} &= (z')^2 \left(\delta_{\mu\nu} - 2\frac{z'_\mu z'_\nu}{(z')^2} \right) \\
&\equiv (z')^2 J_{\mu\nu}(z') = (z')^2 J_{\mu\nu}(z)
\end{aligned}
\tag{2.2.5}
$$

has negative determinant showing that it is a discrete isometry which is not a proper element of the $SO(d+1,1)$ group of (2.2.1) and (2.2.3). Note that we define contractions such as $(z')^2$ using the Euclidean metric $\delta_{\mu\nu}$, and we are usually indifferent to the question of raised or lowered coordinate indices, *i.e.* $z^\mu = z_\mu$. When we need to contract indices using the AdS metric we do so explicitly, *e.g.*, $g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$, with $g^{\mu\nu} = z_0^2\delta_{\mu\nu}$.

The Jacobian tensor $J_{\mu\nu}$ obeys a number of identities that will be very useful below. These include the pretty inversion property

$$
J_{\mu\nu}(x-y) = J_{\mu\rho}(x')J_{\rho\sigma}(x'-y')J_{\sigma\nu}(y') \tag{2.2.6}
$$

and the orthogonality relation

$$J_{\mu\nu}(x)J_{\nu\rho}(x) = \delta_{\mu\rho} \tag{2.2.7}$$

The (Euclidean) action of any massive scalar field

$$S[\phi] = \frac{1}{2}\int d^dz dz_0 \sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] \tag{2.2.8}$$

is inversion invariant if $\phi(z)$ transforms as a scalar, *i.e.* $\phi(z) \to \phi'(z) = \phi(z')$. The wave equation is:

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0 \tag{2.2.9}$$

$$z_0^{d+1}\frac{\partial}{\partial z_0}\left[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{z})\right] + z_0^2\frac{\partial}{\partial \vec{z}^2}\phi(z_0,\vec{z}) - m^2\phi(z_0,\vec{z}) = 0 \tag{2.2.10}$$

A generic solution which vanishes as $z_0 \to \infty$ behaves like $\phi(z_0,\vec{z}) \to z_0^{d-\Delta}\phi_0(\vec{z})$ as $z_0 \to 0$, where $\Delta = \Delta_+$ is the largest root of the indicial equation of (2.2.10), namely $\Delta_\pm = \frac{1}{2}(d \pm \sqrt{d^2+4m^2})$. Witten [3] has constructed a Green's function solution which explicitly realizes the relation between the field $\phi(z_0,\vec{z})$ in the bulk and the boundary configuration $\phi_0(\vec{x})$. The normalized bulk–to–boundary Green's function*, for $\Delta > \frac{d}{2}$:

$$K_\Delta(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x})^2}\right)^\Delta \tag{2.2.11}$$

is a solution of (2.2.10) with the necessary singular behavior as $z_0 \to 0$, namely:

$$z_0^{\Delta-d}K_\Delta(z_0,\vec{z},\vec{x}) \to 1\cdot\delta(\vec{z}-\vec{x}) \tag{2.2.12}$$

---

*The special case $\Delta = \frac{d}{2}$ corresponds to the lowest AdS mass allowed by unitarity, *i.e.* $m^2 = -\frac{d^2}{4}$. In this case $\phi(z_0,\vec{z}) \to -z_0^{\frac{d}{2}}\ln z_0\,\phi_0(\vec{z})$ as $z_0 \to 0$ and the Green's function which gives this asymptotic behavior is $K_{\frac{d}{2}}(z_0,\vec{z},\vec{x}) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x})^2}\right)^{\frac{d}{2}}$. All the formulas in the text assume the generic normalization (2.2.11) valid for $\Delta > \frac{d}{2}$, obvious modifications are needed for $\Delta = \frac{d}{2}$.

The solution of (2.2.10) is then related to the boundary data by:

$$\phi(z_0, \vec{z}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} \int d^d x \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^{\Delta} \phi_0(\vec{x}) \tag{2.2.13}$$

Note that the choice of K_Δ that we have taken is invariant under translations in $\vec{x}$. This choice corresponds to working with a metric on the boundary of the AdS space that is flat R^d with all curvature at infinity. Thus our correlation functions will be for CFT$_d$ on R^d. Correlation function for other boundary metrics can be obtained by multiplying by the corresponding conformal factors.

It is vital to the CFT$_d$/AdS_{d+1} correspondence, and to our method, that isometries in AdS_{d+1} correspond to conformal isometries in CFT$_d$. In particular the inversion isometry of AdS_{d+1} is realized by the well–known conformal inversion in CFT$_d$. A scalar field (a scalar source from the point of view of the boundary theory) $\phi_0(\vec{x})$ of scale dimension α transforms under the inversion as $x_i \to x'_i/|\vec{x}'|^2$ as $\phi_0(\vec{x}) \to \phi'_0(\vec{x}) = |\vec{x}'|^{2\alpha}\phi_0(\vec{x}')$. The construction (2.2.13) can be used to show that a bulk scalar of mass m^2 is related to boundary data $\phi_0(\vec{x})$ with scale dimension $d - \Delta$. To see this one uses the equalities:

$$\begin{aligned} d^d x &= \frac{d^d x'}{|\vec{x}'|^{2d}} \\ \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^{\Delta} &= \left(\frac{z'_0}{(z'_0)^2 + (\vec{z}' - \vec{x}')^2}\right)^{\Delta} |\vec{x}'|^{2\Delta} \end{aligned} \tag{2.2.14}$$

and $\phi'_0(\vec{x}) = |\vec{x}'|^{2(d-\Delta)}\phi_0(\vec{x}')$. We then find directly that:

$$\frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})} \int d^d x \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2}\right)^{\Delta} \phi'_0(\vec{x}) = \phi(z') \tag{2.2.15}$$

Thus conformal inversion of boundary data with scale dimension $d - \Delta$ produces the inversion isometry in AdS_{d+1}. In the CFT$_d$/AdS_{d+1} correspondence, $\phi_0(\vec{x})$ is viewed as the source for a scalar operator $\mathcal{O}(\vec{x})$ of the CFT$_d$. From $\int d^d x \mathcal{O}(\vec{x})\phi_0(\vec{x})$ one sees that $\mathcal{O}(\vec{x}) \to \mathcal{O}'(\vec{x}) = |\vec{x}'|^{2\Delta}\mathcal{O}(\vec{x}')$ so that $\mathcal{O}(\vec{x})$ has scale dimension Δ.

Let us first review the computation of the 2–point correlator $\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})$ for a CFT$_d$ scalar operator of dimension Δ [3]. We assume that the kinetic term (2.2.8) of the corresponding field ϕ of AdS_{d+1} supergravity is multiplied by a constant η determined from the parent 10–dimensional theory. We have, accounting for the 2 Wick contractions:

$$\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = -2\cdot\frac{\eta}{2}\int\frac{d^dzdz_0}{z_0^{d+1}}\left(\partial_\mu K_\Delta(z,\vec{x})z_0^2\partial_\mu K_\Delta(z,\vec{y}) + m^2K_\Delta(z,\vec{x})K_\Delta(z,\vec{y})\right) \tag{2.2.16}$$

We integrate by parts; the bulk term vanishes by the free equation of motion for K, and we get:

$$\begin{aligned}\langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= +\eta\lim_{\epsilon\to 0}\int d^dz\epsilon^{1-d}K_\Delta(\epsilon,\vec{z},\vec{x})\left[\frac{\partial}{\partial z_0}K_\Delta(z_0,\vec{z},\vec{y})\right]_{z_0=\epsilon} \qquad (2.2.17)\\ &= \eta\frac{\Gamma[\Delta+1]}{\pi^{\frac{d}{2}}\Gamma[\Delta-\frac{d}{2}]}\frac{1}{|\vec{x}-\vec{y}|^{2\Delta}}\end{aligned}$$

where (2.2.12) has been used. We warn readers that considerations of Ward identities will suggest a modification of this result for $\Delta\neq d$. One indication that the procedure above is delicate is that the $\partial_\mu K\partial_\mu K$ and m^2KK integrals in (2.2.16) are separately divergent as $\epsilon\to 0$.

We are now ready to apply conformal methods to simplify the integrals in AdS_{d+1} which give 3–point scalar correlators in CFT$_d$. We consider 3 scalar fields $\phi_I(z)$, $I=1,2,3$, in the supergravity theory with masses m_I and interaction vertices of the form $\mathcal{L}_1=\phi_1\phi_2\phi_3$ and $\mathcal{L}_2=\phi_1 g^{\mu\nu}\partial_\mu\phi_2\partial_\nu\phi_3$. The corresponding 3–point amplitudes are:

$$A_1(\vec{x},\vec{y},\vec{z}) = -\int\frac{d^dwdw_0}{w_0^{d+1}}K_{\Delta_1}(w,\vec{x})K_{\Delta_2}(w,\vec{y})K_{\Delta_3}(w,\vec{z}) \tag{2.2.18}$$

$$A_2(\vec{x},\vec{y},\vec{z}) = -\int\frac{d^dwdw_0}{w_0^{d+1}}K_{\Delta_1}(w,\vec{x})\partial_\mu K_{\Delta_2}(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,\vec{z}) \tag{2.2.19}$$

where $K_{\Delta_I}(w,\vec{x})$ is the Green function (2.2.11). These correlators are conformally covariant and are of the form required by conformal symmetry:

$$A_i(\vec{x},\vec{y},\vec{z}) = \frac{a_i}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{z}-\vec{x}|^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.2.20}$$

so the only issue is how to obtain the coefficients a_1, a_2.

The basic idea of our method is to use the inversion $w_\mu = \frac{w'_\mu}{w'^2}$ as a change of variables. In order to use the simple inversion property (2.2.14) of the propagator, we must also refer boundary points to their inverses, *e.g.* $x_i = \frac{x'_i}{x'^2}$. If this is done for a generic configuration of $\vec{x}$, $\vec{y}$, $\vec{z}$, there is nothing to be gained because the same integral is obtained in the w' variable. However, if we use translation symmetry to place one boundary point at 0, say $\vec{z}=0$, it turns out that the denominator of the propagator attached to this point drops out of the integral, essentially because the inverted point is at ∞, and the integral simplifies.

Applied to $A_1(\vec{x},\vec{y},0)$, using (2.2.14), these steps immediately give:

$$A_1(\vec{x},\vec{y},0) = -\frac{1}{|\vec{x}|^{2\Delta_1}}\frac{1}{|\vec{y}|^{2\Delta_2}}\frac{\Gamma(\Delta_3)}{\pi^{\frac{d}{2}}\Gamma(\Delta_3-\frac{d}{2})}\int\frac{d^dw'\,dw'_0}{(w'_0)^{d+1}}K_{\Delta_1}(w',\vec{x}')K_{\Delta_2}(w',\vec{y}')\,(w'_0)^{\Delta_3} \tag{2.2.21}$$

The remaining integral has two denominators, and it is easily done by conventional Feynman parameter methods. We will encounter similar integrals below so we record the general form:

$$\int_0^\infty dz_0\int d^d\vec{z}\frac{z_0^a}{[z_0^2+(\vec{z}-\vec{x})^2]^b[z_0^2+(\vec{z}-\vec{y})^2]^c} \equiv I[a,b,c,d]|\vec{x}-\vec{y}|^{1+a+d-2b-2c} \tag{2.2.22}$$

$$\begin{aligned} I[a,b,c,d] &= \frac{\pi^{d/2}}{2}\frac{\Gamma[\frac{a}{2}+\frac{1}{2}]\Gamma[b+c-\frac{d}{2}-\frac{a}{2}-\frac{1}{2}]}{\Gamma[b]\Gamma[c]} \\ &\quad \frac{\Gamma[\frac{1}{2}+\frac{a}{2}+\frac{d}{2}-b]\Gamma[\frac{1}{2}+\frac{a}{2}+\frac{d}{2}-c]}{\Gamma[1+a+d-b-c]} \end{aligned} \tag{2.2.23}$$

We thus find that $A_1(\vec{x},\vec{y},0)$ has the spatial dependence:

$$\frac{1}{|\vec{x}|^{2\Delta_1}|\vec{y}|^{2\Delta_2}|\vec{x'}-\vec{y'}|^{(\Delta_1+\Delta_2-\Delta_3)}}=\frac{1}{|\vec{x}|^{\Delta_1+\Delta_3-\Delta_2}|\vec{y}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{x}-\vec{y}|^{(\Delta_1+\Delta_2-\Delta_3)}} \tag{2.2.24}$$

which agrees with (2.2.20) after the translation $\vec{x}\to(\vec{x}-\vec{z})$, $\vec{y}\to(\vec{y}-\vec{z})$. The coefficient a_1 is then:

$$a_1=-\frac{\Gamma[\frac{1}{2}(\Delta_1+\Delta_2-\Delta_3)]\Gamma[\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)]\Gamma[\frac{1}{2}(\Delta_3+\Delta_1-\Delta_2)]}{2\pi^d\Gamma[\Delta_1-\frac{d}{2}]\Gamma[\Delta_2-\frac{d}{2}]\Gamma[\Delta_3-\frac{d}{2}]}\Gamma[\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3-d)] \tag{2.2.25}$$

We now turn to the integral $A_2(\vec{x},\vec{y},\vec{z})$ in (2.2.19). It is convenient to set $\vec{z}=0$. Since the structure $\partial_\mu K_2 w_0^2\partial_\mu K_3$ is an invariant contraction and the inversion of the bulk point a is diffeomorphism, we have, using (2.2.14):

$$\partial_\mu K_2(w,\vec{y})w_0^2\partial_\mu K_{\Delta_3}(w,0)=|\vec{y'}|^{2\Delta_2}\partial'_\mu K_{\Delta_2}(w',\vec{y'})(w'_0)^2\partial_\mu K_{\Delta_3}(w',0) \tag{2.2.26}$$

$$\sim|\vec{y'}|^{2\Delta_2}\frac{\partial}{\partial w'_0}\left(\frac{w'_0}{(w'_0)^2+(\vec{w'}-\vec{y'})^2}\right)^{\Delta_2}(w'_0)^2\frac{\partial}{\partial w'_0}(w'_0)^{\Delta_3} \tag{2.2.27}$$

$$=\Delta_2\Delta_3|\vec{y'}|^{2\Delta_2}(w'_0)^{(\Delta_2+\Delta_3)}\left[\frac{1}{((w'_0)^2+(\vec{w'}-\vec{y'})^2)^{\Delta_2}}-\frac{2(w'_0)^2}{((w'_0)^2+(\vec{w'}-\vec{y'})^2)^{\Delta_2+1}}\right] \tag{2.2.28}$$

where the normalization constants are temporarily omitted. We then find two integrals of the form $I(a,b,c,d)$ with different parameters. The result is:

$$a_2=a_1\left[\Delta_2\Delta_3+\frac{1}{2}\left(d-\Delta_1-\Delta_2-\Delta_3\right)\left(\Delta_2+\Delta_3-\Delta_1\right)\right] \tag{2.2.29}$$

As described by Witten [3], massive AdS_5 scalars are sources of various composite gauge–invariant scalar operators of the $\mathcal{N}=4$ SYM theory. The values of the 3–point correlators of these operators can be obtained by combining our amplitudes $A_1(\vec{x},\vec{y},\vec{z})$ and $A_2(\vec{x},\vec{y},\vec{z})$ weighted by appropriate couplings from the gauged supergravity Lagrangian.

2.3 Flavor current correlators

2.3.1 Review of field theory results

We first review the conformal structure of the correlators $\langle J_i^a(x) J_j^b(y)\rangle$ and $\langle J_i^a(x) J_j^b(y) J_k^c(z)\rangle$ and their non-renormalization theorems.† The situation is best understood in 4-dimensions, so we mostly limit our discussion to this physically relevant case. The needed information probably appears in many places, but we shall use the reference best known to us [29]. Conserved currents $J_i^a(x)$ have dimension $d-1$, and transform under the inversion as $J_i^a(x) \to (x'^2)^{(d-1)} J_{ij}(x') J_j^a(x')$. The two-point function must take the inversion covariant, gauge–invariant form

$$\begin{aligned} \langle J_i^a(x) J_j^b(y)\rangle &= B\,\delta^{ab}\,\frac{2(d-1)(d-2)}{(2\pi)^d}\frac{J_{ij}(x-y)}{(x-y)^{2(d-1)}} \\ &= B\,\frac{\delta^{ab}}{(2\pi)^d}(\Box\delta_{ij}-\partial_i\partial_j)\frac{1}{(x-y)^{2(d-2)}} \end{aligned} \tag{2.3.30}$$

where B is a positive constant, the central charge of the $J(x)J(y)$ OPE.

In 4 dimensions the 3–point function has normal and abnormal parity parts which we denote by $\langle J_i^a(x) J_j^b(y) J_k^c(z)\rangle_\pm$. It is an old result [28] that the normal parity part is a superposition of two possible conformal tensors (extensively studied in [29]), namely

$$\langle J_i^a(x) J_j^b(y) J_k^c(z)\rangle_+ = f^{abc}(k_1 D_{ijk}^{\mathrm{sym}}(x,y,z) + k_2 C_{ijk}^{\mathrm{sym}}(x,y,z)), \tag{2.3.31}$$

where $D_{ijk}^{\mathrm{sym}}(x,y,z)$ and $C_{ijk}^{\mathrm{sym}}(x,y,z)$ are permutation–odd tensor functions, obtained from the specific tensors

$$D_{ijk}(x,y,z) = \frac{1}{(x-y)^2(z-y)^2(x-z)^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}\log(x-y)^2\frac{\partial}{\partial z_k}\log\left(\frac{(x-z)^2}{(y-z)^2}\right) \tag{2.3.32}$$

†In this subsection, x, y, z always indicate d–dimensional vectors in flat d–dimensional Euclidean space–time.

$$C_{ijk}(x,y,z) \;=\; \frac{1}{(x-y)^4}\frac{\partial}{\partial x_i}\frac{\partial}{\partial z_l}\log{(x-z)^2}\frac{\partial}{\partial y_j}\frac{\partial}{\partial z_l}\log{(y-z)^2}\frac{\partial}{\partial z_k}\log\left(\frac{(x-z)^2}{(y-z)^2}\right)$$

by adding cyclic permutations

$$\begin{aligned} D^{\rm sym}_{ijk}(x,y,z) &= D_{ijk}(x,y,z)+D_{jki}(y,z,x)+D_{kij}(z,x,y) \\ C^{\rm sym}_{ijk}(x,y,z) &= C_{ijk}(x,y,z)+C_{jki}(y,z,x)+C_{kij}(z,x,y). \end{aligned} \tag{2.3.33}$$

Both symmetrized tensors are conserved for separated points (but the individual permutations are not); $\frac{\partial}{\partial z_k}D^{\rm sym}_{ijk}(x,y,z)$ has the local $\delta^4(x-z)$ and $\delta^4(y-z)$ terms expected from the standard Ward identity relating 2- and 3-point correlators, while $\frac{\partial}{\partial z_k}C^{\rm sym}_{ijk}(x,y,z)=0$ even locally. Thus the Ward identity implies $k_1=\frac{B}{16\pi^6}$, while k_2 is an independent constant. The symmetrized tensors are characterized by relatively simple forms in the limit that one coordinate, say y, tends to infinity:

$$\begin{aligned} D^{\rm sym}_{ijk}(x,y,0) &\xrightarrow[y\to\infty]{} \frac{-4}{y^6x^4}J_{jl}(y)\left\{\delta_{ik}x_l-\delta_{il}x_k-\delta_{kl}x_i-2\frac{x_ix_jx_l}{x^2}\right\} \\ C^{\rm sym}_{ijk}(x,y,0) &\xrightarrow[y\to\infty]{} \frac{8}{y^6x^4}J_{jl}(y)\left\{\delta_{ik}x_l-\delta_{il}x_k-\delta_{kl}x_i+4\frac{x_ix_jx_l}{x^2}\right\} \end{aligned} \tag{2.3.34}$$

In a superconformal–invariant theory with a fixed line parametrized by the gauge coupling, such as $\mathcal{N}=4$ SYM theory, the constant B is exactly determined by the free field content of the theory, *i.e.* 1–loop graphs. This is the non-renormalization theorem for flavor central charges proved in [25]. The argument is quite simple. The fixed point value of the central charge is equal to the external trace anomaly of the theory with source for the currents [23, 22]. Global $\mathcal{N}=1$ supersymmetry relates the trace anomaly to the R-current anomaly, specifically to the $U(1)_RF^2$ (F is for flavor) which is one-loop exact in a conformal theory. Its value depends on the r–charges and the flavour quantum numbers of the fermions of the theory, and it is independent of the couplings. For an $\mathcal{N}=1$ theory with chiral superfields Φ^i with (anomaly–free) r–charges r_i in irreducible representations R_i of the gauge group, the fixed point value of the central charge was given in (2.28) of [24] as

$$B\delta^{ab} = 3\sum_i (\mathrm{dim}R_i)(1-r_i)\mathrm{Tr}_i(T^aT^b). \tag{2.3.35}$$

For $\mathcal{N}=4$ SYM we can restrict to the $SU(3)$ subgroup of the full $SU(4)$ flavour group that is manifest in an $\mathcal{N}=1$ description. There is a triplet of $SU(N)$ adjoint Φ^i with $r=\frac{2}{3}$. We thus obtain

$$B = 3(N^2-1)\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}(N^2-1). \tag{2.3.36}$$

We might now look forward to the AdS_5 calculation with the expectation that the value found for k_1 will be determined by the non–renormalization theorem, but k_2 will depend on the large N dynamics and differ from the free field value. Actual results will force us to revise this intuition. We now discuss the 1–loop contributions in the field theory and obtain the values of k_1 and k_2 for later comparison with AdS_5.

Spinor and scalar 1-loop graphs were expressed as linear combinations of D^{sym} and C^{sym} in [29]. For a single $SU(3)$ triplet of left handed fermions and a single triplet of complex bosons one finds

$$\begin{aligned}\langle J_i^a(x)J_j^b(y)J_k^c(z)\rangle_+^{\mathrm{fermi}} &= \frac{4}{3}\frac{f^{abc}}{(4\pi^2)^3}(D_{ijk}^{\mathrm{sym}}(x,y,z) - \frac{1}{4}C_{ijk}^{\mathrm{sym}}(x,y,z)) \\ \langle J_i^a(x)J_j^b(y)J_k^c(z)\rangle^{\mathrm{bose}} &= \frac{2}{3}\frac{f^{abc}}{(4\pi^2)^3}(D_{ijk}^{\mathrm{sym}}(x,y,z) + \frac{1}{8}C_{ijk}^{\mathrm{sym}}(x,y,z))\end{aligned} \tag{2.3.37}$$

The sum of these, multiplied by N^2-1 is the total 1-loop result in the $\mathcal{N}=4$ theory:

$$\langle J_i^a(x)J_j^b(y)J_k^c(z)\rangle_+^{\mathcal{N}=4} = \frac{(N^2-1)f^{abc}}{32\pi^6}(D_{ijk}^{\mathrm{sym}}(x,y,z) - \frac{1}{8}C_{ijk}^{\mathrm{sym}}(x,y,z)). \tag{2.3.38}$$

We observe the agreement with the value of B in (2.3.36) and the fact that the free field ratio of C^{sym} and D^{sym} tensors is $-\frac{1}{8}$.

Since the $SU(4)$ flavor symmetry is chiral, the 3–point current correlator also has an abnormal parity part $\langle J_i^aJ_j^bJ_k^c\rangle_-$. It is well–known that there is a unique conformal tensor–amplitude [28] in this section, which is a constant multiple of the fermion triangle amplitude, namely

$$\langle J_i^a(x)J_j^b(y)J_k^c(z)\rangle_- = -\frac{N^2-1}{32\pi^6}id^{abc}\frac{\mathrm{Tr}\,[\gamma_5\gamma_i(\not{x}-\not{y})\gamma_j(\not{y}-\not{z})\gamma_k(\not{z}-\not{x})]}{(x-y)^4(y-z)^4(z-x)^4} \tag{2.3.39}$$

where the $SU(N)$ f and d symbols are defined by $\mathrm{Tr}(T^aT^bT^c) \equiv \frac{1}{4}(if^{abc}+d^{abc})$ with T^a hermitian generators normalized as $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta^{ab}$. The coefficient is again "protected" by a non–renormalization theorem, namely the Adler–Bardeen theorem (which is independent of SUSY and conformal symmetry). After bose–symmetric regularization [26] of the short distance singularity, one finds the anomaly

$$\frac{\partial}{\partial z_k}\langle J_i^a(x)J_j^b(y)J_k^c(z)\rangle_- = -\frac{N^2-1}{48\pi^2}id^{abc}\epsilon^{ijlm}\frac{\partial}{\partial x_l}\frac{\partial}{\partial y_m}\delta(x-z)\delta(y-z) \tag{2.3.40}$$

If we minimally couple the currents $J_i^a(x)$ to background sources $A_i^a(x)$ by adding to the action a term $\int d^4x J_i^a(x)A_i^a(x)$, this information can be presented as the operator equation:

$$(D_iJ_i(z))^a = \frac{\partial}{\partial z_i}J_i^a(z) + f^{abc}A_i^b(z)J_i^c(z) = \frac{N^2-1}{96\pi^2}id^{abc}\epsilon_{jklm}\partial_j(A_k^b\partial_l A_m^c + \frac{1}{4}f^{cde}A_k^bA_l^dA_m^e) \tag{2.3.41}$$

where the cubic term in A_i^a is determined by the Wess–Zumino consistency conditions (see e.g. [30]).

The CFT_4/AdS_5 correspondence can also be used to calculate the large N limit of correlators $\langle J_i^a(x)\mathcal{O}^I(y)\mathcal{O}^J(z)\rangle$ and $\langle J_i^a(x)J_j^b(y)\mathcal{O}^I(z)\rangle$ where $\mathcal{O}^I$ is a gauge–invariant composite scalar operator of the $\mathcal{N}=4$ SYM theory. For example, one can take $\mathcal{O}^I$ to be a k–th rank traceless symmetric tensor $\mathrm{Tr}\,X^{\alpha_1}\cdots X^{\alpha_k}$ (the explicit subtraction of traces is not indicated) formed from the real scalars X^α, $\alpha=1,...,6$, in the 6–dimensional representation of $SU(4)\cong SO(6)$, and there are other possibilities in the operator map discussed by Witten [3]. We will compute the corresponding supergravity amplitudes in the next section, and we record here the tensor form required by conformal symmetry.

For $\langle J_i^a \mathcal{O}^I \mathcal{O}^J \rangle$ there is a unique conformal tensor for every dimension d given by

$$\langle J_i^a(z)\mathcal{O}^I(x)\mathcal{O}^J(y)\rangle = \xi \overset{a}{S_i^{IJ}} (z,x,y) \tag{2.3.42}$$

$$\equiv -\xi(d-2) \overset{a}{T^{IJ}} \frac{1}{(x-y)^{2\Delta-d+2}} \frac{1}{(x-z)^{d-2}(y-z)^{d-2}} \left[\frac{(x-z)_i}{(x-z)^2} - \frac{(y-z)_i}{(y-z)^2}\right] \tag{2.3.43}$$

where ξ is a constant and $\overset{a}{T^{IJ}}$ are the Lie algebra generators. This correlator satisfies a Ward identity which relates it to the 2–point function $\langle \mathcal{O}^I(x)\mathcal{O}^J(y)\rangle$. Specifically:

$$\begin{aligned} \xi \frac{\partial}{\partial z_i} \overset{a}{S_i^{IJ}} (z,x,y) &= \xi \frac{(d-2)2\pi^{\frac{d}{2}}}{\Gamma[\frac{d}{2}]} \overset{a}{T^{IJ}} \left(\delta^d(x-z) - \delta^d(y-z)\right) \frac{1}{(x-y)^{2\Delta}} \\ &= \delta^d(x-z) \overset{a}{T^{IK}} \langle \mathcal{O}^K(x)\mathcal{O}^J(y)\rangle + \delta^d(y-z) \overset{a}{T^{JK}} \langle \mathcal{O}^I(x)\mathcal{O}^K(y)\rangle \end{aligned} \tag{2.3.44}$$

There is also a unique tensor form for $\langle J_i J_j \mathcal{O}\rangle$ (we suppress group theory labels) which is given in [22]:

$$\langle J_i(x)J_j(y)\mathcal{O}(z)\rangle = \zeta R_{ij}(x,y,z) \equiv \zeta \frac{(6-\Delta)J_{ij}(x-y) - \Delta J_{ik}(x-z)J_{kj}(z-y)}{(x-y)^{6-\Delta}(x-z)^{\Delta}(y-z)^{\Delta}} \tag{2.3.45}$$

where ζ is a constant.

2.3.2 Calculations in AdS supergravity

The boundary values $A_i^a(\vec{x})$ of the gauge potentials $A_\mu^a(x)$ of gauged supergravity are the sources for the conserved flavor currents $J_i^a(\vec{x})$ of the boundary SCFT$_4$. It is sufficient for our purposes to ignore non-renormalizable $\phi^n F_{\mu\nu}^2$ interactions and represent the gauge sector of the supergravity by the Yang–Mills and Chern–Simons terms (the latter for $d+1=5$)

$$S_{cl}[A] = \int d^d z dz_0 \left[\sqrt{g} \frac{F_{\mu\nu}^a F^{\mu\nu a}}{4g_{SG}^2} + \frac{ik}{96\pi^2}\left(d^{abc}\epsilon^{\mu\nu\lambda\rho\sigma} A_\mu^a \partial_\nu A_\lambda^b \partial_\rho A_\sigma^c + \cdots\right)\right] \tag{2.3.46}$$

The coefficient $\frac{k}{96\pi^2}$, where k is an integer, is the correct normalization factor for the 5–dimensional Chern–Simons term ensuring that under a large gauge transformation the action changes by an unobservable phase $2\pi i n$ (see e.g. [30]). The couplings g_{SG} and k could in principle be determined from dimensional reduction of the parent 10 dimensional theory, but we shall ignore this here. Instead, they will be fixed in terms of current correlators of the boundary theory which are exactly known because they satisfy non-renormalization theorems.

To obtain flavor–current correlators in the boundary CFT from AdS supergravity, we need a Green's function $G_{\mu i}(z, \vec{x})$ to construct the gauge potential $A^a_\mu(z)$ in the bulk from its boundary values $A^a_i(\vec{x})$. We will work in d dimensions. There is the gauge freedom to redefine $G_{\mu i}(z, \vec{x}) \to G_{\mu i}(z, \vec{x}) + \frac{\partial}{\partial z_\mu}\Lambda_i(z, \vec{x})$ which leaves boundary amplitudes obtained from the action (2.3.46) invariant. Our method requires a conformal–covariant propagator, namely

$$G_{\mu i}(z, \vec{x}) = C^d \frac{z_0^{d-2}}{[z_0^2 + (\vec{z} - \vec{x})^2]^{d-1}} J_{\mu i}(z - \vec{x}) \tag{2.3.47}$$

$$= C^d \left(\frac{z_0}{(z - \vec{x})^2}\right)^{d-2} \partial_\mu \left(\frac{(z - \vec{x})_i}{(z - \vec{x})^2}\right) \tag{2.3.48}$$

which satisfies the gauge field equations of motion in the bulk variable z. The normalization constant C^d is determined by requiring that as $z_0 \to 0$, $G_{ji}(z, \vec{x}) \to 1 \cdot \delta_{ji}\delta(\vec{x})$:

$$C^d = \frac{\Gamma(d)}{2\,\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})} \tag{2.3.49}$$

This Green's function does not satisfy boundary transversality ($i.e.$ $\frac{\partial}{\partial x_i}G_{\mu i}(z, \vec{x}) = 0$), but the following gauge–related propagator does‡:

$$\bar{G}_{\mu i}(z,\vec{x}) = G_{\mu i}(z,\vec{x}) + \frac{\partial}{\partial z_\mu}\left\{\frac{C^d z_0^{2-d}}{(d-2)(d-1)(\Gamma[\frac{d}{2}])^2}\frac{\partial}{\partial z_i}F\left[d-1,\frac{d}{2}-1,\frac{d}{2};-\frac{(\vec{z}-\vec{x})^2}{z_0^2}\right]\right\} \tag{2.3.50}$$

(Both $G_{\mu i}(z,\vec{x})$ and $\bar{G}_{\mu i}(z,\vec{x})$ differ by gauge terms from the Green's function used by Witten [3]). The gauge equivalence of inversion–covariant and transverse propagators ensures that the method produces boundary current correlators which are conserved.

Notice that in terms of the conformal tensors $J_{\mu i}$ the abelian field strength made from the Green's function takes a remarkably simple form:

$$\partial_{[\mu}G_{\nu]i}(z,\vec{x}) = (d-2)C^d\frac{z_0^{d-3}}{[z_0^2+(\vec{z}-\vec{x})^2]^{d-1}}J_{0[\mu}(z-\vec{x})J_{\nu]i}(z-\vec{x}) \tag{2.3.51}$$

as easily checked by using for $G_{\mu i}$ the representation (2.3.48).

We stress again that the inversion $z_\mu = z'_\mu/(z')^2$ is a coordinate transformation which is an isometry of AdS_{d+1}. It acts as a diffeomorphism on the internal indices $\mu,\nu,\ldots$ of $G_{\mu i}, G_{\nu j},\ldots$. Since these indices are covariantly contracted at an internal point z, much of the algebra required to change integration variables can be avoided. The inversion $\vec{x} = \vec{x'}/(\vec{x'})^2$ of boundary points is a conformal isometry which acts on the external index i and also changes the Green's function by a conformal factor. Thus the change of variables amounts to the replacement:

$$\begin{aligned}
G_{\mu i}(z,\vec{x}) &= z'^2 J_{\mu\nu}(z')\cdot(\vec{x'})^2 J_{ki}(\vec{x'})\cdot(\vec{x'})^{2(d-2)}\,C^d\,\frac{(z'_0)^{d-2}J_{\nu k}(z'-\vec{x'})}{[(z'_0)^2+(\vec{z'}-\vec{x'})^2]^{d-1}} \qquad (2.3.52)\\
&= \frac{\partial z'_\nu}{\partial z_\mu}\cdot\frac{\partial x'_k}{\partial x_i}\cdot(\vec{x'})^{2(d-2)}G_{\nu k}(z',\vec{x'})\\
&= \frac{\partial z'_\nu}{\partial z_\mu}\cdot\frac{\partial x'_k}{\partial x_i}\cdot G'_{\nu k}(z',\vec{x'})
\end{aligned}$$

‡For even d, the hypergeometric function in (2.3.50) is actually a rational function. For instance for $d=4$, $\bar{G}_{\mu i}(z,\vec{x}) = G_{\mu i}(z,\vec{x}) + \frac{\partial}{\partial z_\mu}\left\{\frac{C^d}{12}\frac{\partial}{\partial z_i}\left(\frac{2z_0^2+(\vec{z}-\vec{x})^2}{[z_0^2+(\vec{z}-\vec{x})^2]^2}\right)\right\}$.

$\partial_{[\mu}G_{\nu]i}(z,\vec{x})$ will also transform conformal–covariantly under inversion (compare equ.(2.3.52)):

$$\partial_{[\mu}G_{\nu]i}(\vec{x},z) = (z')^2 J_{\mu\rho}(z') \cdot (z')^2 J_{\nu\sigma}(z') \cdot (\vec{x}')^2 J_{ki}(\vec{x}') \cdot (\vec{x}')^{2(d-2)} \partial'_{[\rho}G_{\sigma]k}(\vec{x}',z') \qquad (2.3.53)$$

as one can directly check from (2.3.51) using the identity (2.2.6).

$\langle \mathbf{J^a_i J^b_j} \rangle$: To obtain the current–current correlator we follow the same procedure [3] as for the scalar 2–point function, eq.(2.2.16–2.2.17):

$$\begin{aligned}
\langle J^a_i(\vec{x}) J^b_j(\vec{y}) \rangle &= -\delta^{ab}\, 2 \cdot \frac{1}{4g^2_{SG}} \int \frac{d^d z dz_0}{z_0^{d+1}}\, \partial_{[\mu}G_{\nu]i}(z,\vec{x})\, z_0^4\, \partial_{[\mu}G_{\nu]j}(z,\vec{y}) \\
&= +\frac{\delta^{ab}}{2g^2_{SG}} \lim_{\epsilon\to 0} \int d^d z\, \epsilon^{3-d}\; 2G_{\nu i}(\epsilon,\vec{z},\vec{x})\, \left[\partial_{[0}G_{\nu]j}(z_0,\vec{z},\vec{y})\right]_{z_0=\epsilon} \\
&= \delta^{ab} \frac{C^d(d-2)}{g^2_{SG}} \frac{J_{ij}(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^{2(d-1)}} \qquad (2.3.54)
\end{aligned}$$

which is of the form (2.3.30) with $B = \frac{1}{g^2_{SG}} \frac{2^{d-2}\pi^{\frac{d}{2}}\Gamma[d]}{(d-1)\Gamma[\frac{d}{2}]}$. According to the conjecture [1, 2, 3], (2.3.54) represents the large–N value of the 2–point function for $g^2_{YM}N$ fixed but large. Let us now consider the case $d=4$. By the non–renormalization theorem proven in [25], the coefficient in (2.3.30) is protected against quantum corrections. Hence, at leading order in N, the strong–coupling result (2.3.54) has to match the 1–loop computation (2.3.36). We thus learn:

$$g_{SG}^{d+1=5} = \frac{4\pi}{N} \qquad (2.3.55)$$

$\langle \mathbf{J^a_i J^b_j J^c_k} \rangle_+$: The vertex relevant to the computation of the normal parity part of $< J^a_i(\vec{x}) J^b_j(\vec{y}) J^c_k(\vec{z}) >$ comes from the Yang–Mills term of the action (2.3.46), namely

$$\frac{1}{2g^2_{SG}} \int \frac{d^d w dw_0}{w_0^{d+1}}\, i f^{abc}\, \partial_{[\mu}A^a_{\nu]}(w)\, w_0^4\, A^b_\mu(w) A^c_\nu(w) \qquad (2.3.56)$$

We then have

$$\langle J^a_i(\vec{x}) J^b_j(\vec{y}) J^c_k(\vec{z}) \rangle_+ = -\frac{i f^{abc}}{2g^2_{SG}}\, 2 \cdot F^{\mathrm{sym}}_{ijk}(\vec{x},\vec{y},\vec{z}) \qquad (2.3.57)$$

$$\equiv\ -\frac{if^{abc}}{2g_{SG}^2}\,2\left[F_{ijk}(\vec{x},\vec{y},\vec{z})+F_{jki}(\vec{y},\vec{z},\vec{x})+F_{kij}(\vec{z},\vec{x},\vec{y})\right]$$

where

$$F_{ijk}(\vec{x},\vec{y},\vec{z})=\int\frac{d^dw dw_0}{w_0^{d+1}}\,\partial_{[\mu}G_{\nu]i}(w,\vec{x})\,w_0^4\,G_{\mu j}(w,\vec{y})G_{\nu k}(w,\vec{z}) \qquad (2.3.58)$$

(The extra factor of 2 in (2.3.57) correctly accounts for the 3! Wick contractions). To apply the method of inversion, it is convenient to set $\vec{x}=0$. Then, changing integration variable $w_\mu=\frac{w'_\mu}{(w')^2}$ and inverting the external points, $y_i=\frac{y'_i}{|\vec{y'}|^2}$, $z_i=\frac{z'_i}{|\vec{z'}|^2}$, we achieve the simplification (using (2.3.52),(2.3.53),(2.2.7)):

$$\begin{aligned}
&F_{ijk}(0,\vec{y},\vec{z})=\\
&=|\vec{y'}|^{2(d-1)}|\vec{z'}|^{2(d-1)}J_{jl}(\vec{y'})J_{km}(\vec{z'})\\
&\cdot\int\frac{d^dw'dw'_0}{(w'_0)^{d+1}}\,\partial'_{[\mu}G_{\nu]i}(w',0)\,(w'_0)^4\,G_{\mu l}(w,\vec{y'})G_{\nu m}(w',\vec{z'}) && (2.3.59)\\
&=(C^d)^3\frac{J_{jl}(\vec{y})}{|\vec{y}|^{2(d-1)}}\frac{J_{km}(\vec{z})}{|\vec{z}|^{2(d-1)}}\int\frac{d^dw'dw'_0}{(w'_0)^{d+1}}\Big[\partial'_{[\mu}(w'_0)^{d-2}\partial'_{\nu]}(w'_i)\,(w'_0)^4\\
&\qquad\frac{(w'_0)^{d-2}}{(w'-\vec{y'})^{2(d-2)}}\frac{(w'_0)^{d-2}}{(w'-\vec{z'})^{2(d-2)}}J_{\mu l}(w,\vec{y'})J_{\nu m}(w',\vec{z'})\Big] && (2.3.60)\\
&=(C^d)^3\frac{J_{jl}(\vec{y})}{|\vec{y}|^{2(d-1)}}\frac{J_{km}(\vec{z})}{|\vec{z}|^{2(d-1)}}\int d^dw'dw'_0\,\frac{(d-2)(w'_0)^{2d-4}J_{l[0}(w'-\vec{t})J_{i]m}(w')}{[(w'_0)^2+(\vec{w'}-\vec{t})^2]^{d-1}[(w'_0)^2+(\vec{w'})^2]^{d-1}}
\end{aligned}$$

where in the last step we have defined $\vec{t}\equiv\vec{y'}-\vec{z'}$. Observe that in going from (2.3.58) to (2.3.59) we just had to replace the original variables with primed ones and pick conformal Jacobians for the external (Latin) indices: the internal Jacobians nicely collapsed with each other (recall the contraction rule (2.2.7) for $J_{\mu i}$ tensors) and with the factors of w' coming from the inverse metric. The integrals in (2.3.60) now have two denominators and through straightforward manipulations can be rewritten as derivatives with respect to the external coordinate $\vec{t}$ of standard integrals of the form (2.2.23). We thus obtain:

$$F_{ijk}(\vec{x},\vec{y},\vec{z})\ =\ -\frac{J_{jl}(\vec{y}-\vec{x})}{|\vec{y}-\vec{x}|^{2(d-1)}}\frac{J_{km}(\vec{z}-\vec{x})}{|\vec{z}-\vec{x}|^{2(d-1)}}(C^d)^3\pi^{\frac{d+2}{2}}2^{3-2d}\left(\frac{d-2}{d-1}\right)\frac{\Gamma\left[\frac{d}{2}\right]}{\left[\Gamma\left[\frac{d+1}{2}\right]\right]^2}$$

$$\cdot \frac{1}{|\vec{t}|^d}\left[\delta_{lm}t_i + (d-1)\delta_{il}t_m + (d-1)\delta_{im}t_l - d\frac{t_i t_l t_m}{|\vec{t}|^2}\right] \tag{2.3.61}$$

where we have restored the $\vec{x}$ dependence, so that now $\vec{t} \equiv (\vec{y}-\vec{x})' - (\vec{z}-\vec{x})'$. We now add permutations to obtain $F^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z})$ in (2.3.57). The final step is to express $F^{\rm sym}_{ijk}$ as a linear combination of the conformal tensors $D^{\rm sym}_{ijk}$ and $C^{\rm sym}_{ijk}$ of Section 3.1. It is simplest, and by conformal invariance not less general, to work in the special configuration $\vec{z}=0$ and $|\vec{y}| \to \infty$. After careful algebra we obtain

$$\begin{aligned} F^{\rm sym}_{ijk}(\vec{x},|\vec{y}|\to\infty,0) &= -(C^d)^3\pi^{\frac{d+2}{2}}2^{2-2d}(2d-3)\left(\frac{d-2}{d-1}\right)\frac{\Gamma\left[\frac{d}{2}\right]}{\left[\Gamma\left[\frac{d+1}{2}\right]\right]^2} \\ &\cdot \frac{J_{jl}(\vec{y})}{|\vec{y}|^{2(d-1)}|\vec{x}|^d}\left\{\delta_{ik}x_l - \delta_{il}x_k - \delta_{kl}x_i - \frac{d}{2d-3}\frac{x_i x_j x_l}{x^2}\right\} \end{aligned} \tag{2.3.62}$$

Now take $d=4$; comparison with (2.3.34) gives

$$F^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z}) = \frac{1}{\pi^4}\left(D^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z}) - \frac{1}{8}C^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z})\right) \tag{2.3.63}$$

and finally, from (2.3.57) and (2.3.55):

$$\begin{aligned} \langle J^a_i(\vec{x})J^b_j(\vec{y})J^c_k(\vec{z})\rangle_+ &= \frac{f^{abc}}{2\pi^4 g^2_{SG}}\left(D^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z}) - \frac{1}{8}C^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z})\right) \\ &= \frac{N^2 f^{abc}}{32\pi^6}\left(D^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z}) - \frac{1}{8}C^{\rm sym}_{ijk}(\vec{x},\vec{y},\vec{z})\right) \end{aligned} \tag{2.3.64}$$

which, at leading order in N, precisely agrees with the 1–loop result (2.3.38).

The correlator (2.3.64) calculated from AdS_5 supergravity is supposed to reflect the strong–coupling dynamics of the $\mathcal{N}=4$ SYM theory at large N. The exact agreement found with the free–field result therefore requires some comment. As discussed in Section 3.1, the coefficient of the D tensor is fixed by the Ward identity that relates it to the constant B in the 2–point function, and we matched the latter to the 1–loop result by a non–renormalization theorem. So agreement here is just a check that we have done the integral correctly. However, the fact that the ratio of the C and D tensors coefficients also agrees with the free field value was initially a surprise. Upon further thought, we see that our argument that the value of k_2 was a free parameter used only $\mathcal{N}=0$ conformal symmetry, and superconformal symmetry may impose some constraint. Indeed, in an $\mathcal{N}=1$ description of the $\mathcal{N}=4$ SYM theory, we have the flavor $SU(3)$ triplet Φ^i of ($SU(N)$ adjoint) chiral superfields, together with their adjoints $\bar{\Phi}^i$. The $SU(3)$ flavor currents are the $\bar{\theta}\theta$ components of composite scalar superfields $K^a(\vec{x},\theta,\bar{\theta}) = \mathrm{Tr}\,\bar{\Phi}T^a\Phi$, where T^a is a fundamental $SU(3)$ matrix. Just as $\mathcal{N}=0$ conformal invariance constrains the tensor form of 2– and 3–point correlators, $\mathcal{N}=1$ superconformal symmetry will constrain the superfield correlators $\langle K^aK^b\rangle$ and $\langle K^aK^bK^c\rangle$. We are not aware of a specific analysis, but it seems likely [31] that there are only two possible superconformal amplitudes for $\langle K^aK^bK^c\rangle$, one proportional to f^{abc} and the other to d^{abc}. The f^{abc} amplitude contains the normal parity $\langle J_i^aJ_j^bJ_k^c\rangle_+$ in its θ–expansion, and this would imply that the ratio $-\frac{1}{8}$ of the coefficients of the C and D tensors must hold in any $\mathcal{N}=1$ superconformal theory.

$\langle \mathbf{J_i^aJ_j^bJ_k^c}\rangle_-$ Witten [3] has sketched an elegant argument that allows to read the value of the abnormal parity part of the 3–current correlator directly from the supergravity action (2.3.46), with no integral to compute. Under an infinitesimal gauge transformation of the bulk gauge potentials, $\delta_\Lambda A_\mu^a = (D_\mu\Lambda)^a$, the variation of the the action is purely a boundary term coming from the Chern–Simons 5–form:

$$\delta_\Lambda\, S_{cl} = \int d^4z\,\Lambda^a(\vec{z})\left(-\frac{ik}{96\pi^2}\right) d^{abc}\epsilon^{ijkl}\partial_i(A_j^b\partial_kA_l^c + \frac{1}{4}f^{cde}A_j^bA_k^dA_l^e) \tag{2.3.65}$$

By the conjecture [1, 2, 3], $S_{cl}[A_\mu^a(z)] = W[A_i^a(\vec{z})]$, the generating functional for current correlators in the boundary theory. Since by construction $J_i^a(\vec{x}) = \frac{\delta W[A]}{\delta A_i^a(\vec{x})}$, one has:

$$\delta_\Lambda\, S_{cl}[A_\mu^a(z)] = \delta_\Lambda\, W[A_\mu^a(\vec{z})] = \int d^4z[D_i\Lambda(\vec{z})]^aJ_i^a(\vec{z}) = -\int d^4z\Lambda^a(\vec{z})[D_iJ_i(\vec{x})]^a \tag{2.3.66}$$

and comparison with (2.3.65) gives

$$(D_i J_i(\vec{z}))^a = \frac{ik}{96\pi^2} d^{abc} \epsilon^{jklm} \partial_j (A_k^b \partial_l A_m^c + \frac{1}{4} f^{cde} A_j^b A_k^d A_l^e) \tag{2.3.67}$$

which has precisely the structure (2.3.41). Thus the CFT_4/AdS_5 correspondence gives a very concrete physical realization of the well–known mathematical relation between the gauge anomaly in d dimensions and the gauge variation of a $(d+1)$–dimensional Chern–Simons form. Witten [3] has argued that (2.3.67) is an exact statement even at finite N (string–loop effects) and for finite 't Hooft coupling $g_{YM}^2 N$ (string corrections to the classical supergravity action), which is of course what one expects from the Adler–Bardeen theorem. Matching (2.3.67) with the 1–loop result (2.3.41) we are thus led to identify $k = N^2 - 1$.

$\langle \mathbf{J_i^a J_j^b} \mathcal{O} \rangle$: The next 3–point correlator to be discussed is $\langle J_i^a(\vec{x}) J_j^b(\vec{y}) \mathcal{O}^I(\vec{z}) \rangle$. For this purpose we suppress group indices and consider a supergravity interaction of the form

$$\frac{1}{4} \int d^d w dw_0 \sqrt{g}\, g^{\mu\rho} g^{\nu\sigma}\, \phi\, \partial_{[\mu} A_{\nu]} \partial_{[\rho} A_{\sigma]} \tag{2.3.68}$$

This leads to the boundary amplitude

$$\frac{1}{2} \int \frac{d_d w dw_0}{w_0^{d+1}} K_\Delta(w, \vec{z}) \partial_{[\mu} G_{\nu]}(w, \vec{x}) w_0^2 \partial_{[\mu} G_{\nu]}(w, \vec{y}) \tag{2.3.69}$$

We set $\vec{y} = 0$, apply the method of inversion and obtain the integral

$$\begin{aligned} T_{ij}(\vec{x}, 0, \vec{z}) \quad &= \quad (C^d)^2 \frac{\Gamma[\Delta]}{\pi^{\frac{d}{2}} \Gamma[\Delta - \frac{d}{2}]} \frac{(d-2) J_{ik}(\vec{x})}{|\vec{z}|^{2\Delta} |\vec{x}|^{2(d-1)}} \\ &\cdot \int d^d w' dw_0' \left(\frac{w_0'}{(w' - \vec{z}')^2} \right)^\Delta \frac{\partial}{\partial w_{[0}'} \left(\frac{w_0'}{(w' - \vec{z}')^2} \right)^{d-2} \frac{\partial}{\partial w_{j]}'} \frac{(w' - x')_k}{(w' - \vec{x}')^2} \end{aligned} \tag{2.3.70}$$

This can be evaluated as a fairly standard Feynman integral with two denominators. The result is

$$T_{ij}(\vec{x},\vec{y},\vec{z}) = -\frac{\Delta}{8\pi^2}\frac{\Gamma[\Delta]}{\pi^{\frac{d}{2}}\Gamma[\Delta-\frac{d}{2}]}R_{ij}(\vec{x},\vec{y},\vec{z}) \tag{2.3.71}$$

where R_{ij} is the conformal tensor (2.3.42).

$\langle \mathbf{J_i^a}\mathcal{O}^\mathbf{I}\mathcal{O}^\mathbf{J}\rangle$: It is useful to study the correlator $\langle J_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\rangle$ from the AdS viewpoint because the Ward identity (2.3.44) which relates it to $\langle \mathcal{O}(\vec{y})^I\mathcal{O}^J(\vec{z})\rangle$ is a further check on the CFT/AdS conjecture. We assume that $\mathcal{O}^I(\vec{x})$ is a scalar composite operator, in a real representation of the $SO(6)$ flavor group with generators $\overset{a}{T}{}^{IJ}$ which are imaginary antisymmetric matrices, and that $\mathcal{O}^I(\vec{x})$ corresponds to a real scalar field $\phi^I(\vec{x})$ in AdS_5 supergravity. Actually we will present an AdS_{d+1} calculation based on a gauge–invariant extension of (2.2.8), namely

$$\begin{aligned} S[\phi^I, A_\mu^a] &= \frac{1}{2}\int d^dz dz_0\sqrt{g}\left[g^{\mu\nu}D_\mu\phi^I D_\nu\phi^I + m^2\phi^I\phi^I\right] \\ D_\mu\phi^I &= \partial_\mu\phi^I - iA_\mu^a\,\overset{a}{T}{}^{IJ}\,\phi^J \end{aligned} \tag{2.3.72}$$

The cubic vertex then leads to the AdS integral representation of the gauge theory correlator

$$\langle J_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\rangle = \overset{a}{T}{}^{IJ}\int\frac{d^dw dw_0}{w_0^{d+1}}G_{\mu i}(w,\vec{z})w_0^2K_\Delta(w,\vec{x})\frac{\overleftrightarrow{\partial}}{\partial w_\mu}K_\Delta(w,\vec{y}) \tag{2.3.73}$$

The integral is easily done by setting $\vec{z}=0$ and applying inversion. We have also shown that $\vec{y}=0$ followed by inversion gives the same final result, which is

$$\begin{aligned} \langle J_i^a(\vec{z})\mathcal{O}^I(\vec{x})\mathcal{O}^J(\vec{y})\rangle &= \frac{2C^d\,\overset{a}{T}{}^{IJ}}{|\vec{x}|^{2\Delta}|\vec{y}|^{2\Delta}}\frac{\partial}{\partial x_i'}\int\frac{d^dw'dw_0'}{w_0'}K_\Delta(w',\vec{x}')K_\Delta(w',\vec{y}') \\ &= -\xi\,\overset{a}{S}{}^{IJ}\,(\vec{z},\vec{x},\vec{y}) \\ \xi &= \frac{(\Delta-\frac{d}{2})\Gamma[\frac{d}{2}]\Gamma[\Delta]}{\pi^d(d-2)\Gamma[\Delta-\frac{d}{2}]} \end{aligned} \tag{2.3.74}$$

where $\overset{a}{S}{}^{IJ}\,(\vec{z},\vec{x},\vec{y})$ is the conformal amplitude of (2.3.42). Comparing with (2.3.44) and

(2.2.17), we see that the expected Ward identity is *not* satisfied; there is a mismatch by a factor $\frac{2\Delta-d}{\Delta}$. Although we have checked the integral thoroughly, this is an important point, so we now give a heuristic argument that the answer is correct. We compute the divergence of the correlator (2.3.73) using the following identity inside the integral:

$$\frac{\partial}{\partial z_i} G_{\mu i}(w, \vec{z}) = -\frac{\partial}{\partial w_\mu} K_d(w, \vec{z}) \tag{2.3.75}$$

where $K_d(w, \vec{z})$ is the Green's function of a massless scalar, *i.e.* $\Delta = d$. If we integrate by parts, the bulk term vanishes and we find

$$\frac{\partial}{\partial z_i} \langle J_i^a(\vec{z}) \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \rangle = \lim_{\epsilon \to 0} \int d^d w \epsilon^{1-d} K_d(\epsilon, \vec{w}, \vec{z}) \left[K_\Delta(w, \vec{x}) \frac{\overleftrightarrow{\partial}}{\partial w_0} K_\Delta(w, \vec{y}) \right]_{w_0 = \epsilon} \tag{2.3.76}$$

$$= -\left(\frac{\Gamma[\Delta]}{\pi^{\frac{d}{2}} \Gamma[\Delta - \frac{d}{2}]} \right)^2 2\Delta \lim_{\epsilon \to 0} \int d^d w \delta(\vec{w} - \vec{z}) \left[\frac{w_0^{2\Delta-d+2}}{(w - \vec{y})^{2(\Delta+1)}} \frac{1}{(w - \vec{x})^{2\Delta}} - (\vec{x} \leftrightarrow \vec{y}) \right]_{w_0 = \epsilon} \tag{2.3.77}$$

where we used the property $\lim_{w_0 \to 0} K_d = \delta(\vec{w} - \vec{z})$ (see (2.2.12)). It also follows from (2.2.11–2.2.12) that

$$\lim_{w_0 \to 0} \frac{w_0^{2\Delta-d+2}}{(w - \vec{y})^{2(\Delta+1)}} = \frac{\pi^{\frac{d}{2}} \Gamma[\Delta - \frac{d}{2} + 1]}{\Gamma[\Delta + 1]} \delta^d(\vec{w} - \vec{y}) \tag{2.3.78}$$

This gives

$$\frac{\partial}{\partial z_i} \langle J_i^a(\vec{z}) \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \rangle = \xi \frac{(d-2) 2\pi^{\frac{d}{2}}}{\Gamma[\frac{d}{2}]} \overset{a}{T}{}^{IJ} \left(\delta^d(\vec{x} - \vec{z}) - \delta^d(\vec{y} - \vec{z}) \right) \frac{1}{|\vec{x} - \vec{y}|^{2\Delta}} \tag{2.3.79}$$

which is consistent with (2.3.74) and confirms the previously found mismatch between $\langle J_i^a \mathcal{O}^I \mathcal{O}^J \rangle$ and $\langle \mathcal{O}^I \mathcal{O}^J \rangle$.

Thus the observed phenomenon is that the Ward identity relating the correlators $\langle J_i^a \mathcal{O}^I \mathcal{O}^J \rangle$ and $\langle \mathcal{O}^I \mathcal{O}^J \rangle$, as calculated from AdS_{d+1} supergravity, is satisfied for operators $\mathcal{O}^I$ of scale dimension $\Delta = d$, for which the corresponding AdS_{d+1} scalar is massless, but fails for $\Delta \neq d$.

We suggest the following interpretation of the problem, namely that the prescription of [3] is correct for n–point correlators in the boundary CFT_d for $n \geq 3$, but 2–point correlators are more singular, so a more careful procedure is required. The fact that the kinetic and mass term integrals in (2.2.16) are each divergent has already been noted. In the Appendix we outline an alternate calculation of 2–point functions, very similar to that of [2], in which we Fourier transform in $\vec{x}$ and write a solution $\phi(z_0, \vec{k})$ of the massive scalar field equation which satisfies a Dirichlet boundary–value problem at a small finite value $z_o = \epsilon$, compute the 2–point correlator at this value and then scale to $\epsilon = 0$. This procedure gives a value of $\langle \mathcal{O}^I \mathcal{O}^J \rangle$ which is exactly a factor $\frac{2\Delta - d}{\Delta}$ times that of (2.2.17) and thus agrees with the Ward identity.

Chapter 3

Evidence of logarithms in the short–distance expansions of 4–point functions.

3.1 Introduction

Several interesting physical issues arise when we move to the study of 4-point functions. We will focus on the limit $N \to \infty$, $g_{YM} \to 0$, $g_{YM}^2 N \to \infty$ mentioned above. In the CFT the scaling dimensions of the chiral primary operators (and their superconformal descendents) are protected, while the dimensions of fields corresponding to massive string states are infinite in this limit. Does there exist a 'complete' set of fields and an operator product expansion (OPE) structure that allows us to obtain 4-point functions much the same as in the case of 2-D CFT? If so, do the chiral primaries and their descendents form the complete set, or do we need other fields in the CFT? Is there a connection between supergravity fields propagating in the internal leg of a supergravity graph, and the contribution of a specific chiral primary (plus descendents) in the OPE expansion of the corresponding CFT correlator? Preliminary results on these questions were presented in [88] and [89].

To address such issues we study in this chapter some simple supergravity graphs corresponding to 4-point functions in the CFT. We consider the dilaton (ϕ) and axion (C) sector. (This sector has also been studied in [89], and, while we use similar methods, we arrive at somewhat different conclusions).

3.2 4-point functions in the dilaton-axion sector

The relevant part of the $AdS_5 \times S_5$ supergravity action is

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int_{AdS_5} d^5x\sqrt{g}[-R+\frac{1}{2}(\partial\phi)^2+\frac{1}{2}e^{2\phi}(\partial C)^2] \\ &= \frac{1}{2\kappa^2}\int_{AdS_5} d^5x\sqrt{g}[-R+\frac{1}{2}(\partial\phi)^2+\frac{1}{2}(\partial C)^2+a\phi(\partial C)^2+b\phi^2(\partial C)^2+\ldots] \end{aligned} \quad (3.2.1)$$

where $a=1, b=1$. We use coordinates where the (Euclidean) AdS space appears as the upper half space ($z_0 > 0$) with metric:

$$ds^2 = \frac{1}{z_0^2}[dz_0^2 + \sum_{i=1}^{d} dx_i dx_i] \quad (3.2.2)$$

The AdS space has dimension $d+1$; thus in our present case $d=4$.

First consider the CFT correlator $\langle O_\phi(x_1)O_C(x_2)O_\phi(x_3)O_C(x_4)\rangle$. In the AdS calculation we encounter the supergravity graphs shown in Figure 1. The s-channel amplitude is

$$s = -(4a^2)I^s_{\phi C\phi C}(x_1,x_2,x_3,x_4) \quad (3.2.3)$$

$$I^s_{\phi C\phi C}(x_1,x_2,x_3,x_4) \equiv \int \frac{d^5z}{z_0^5}\frac{d^5w}{w_0^5} z_0^2 w_0^2 K(z,x_1)\partial_{z_\mu}K(z,x_2)\partial_{z_\mu}\partial_{w_\nu}G(z,w)K(w,x_3)\partial_{w_\nu}K(w,x_4) \quad (3.2.4)$$

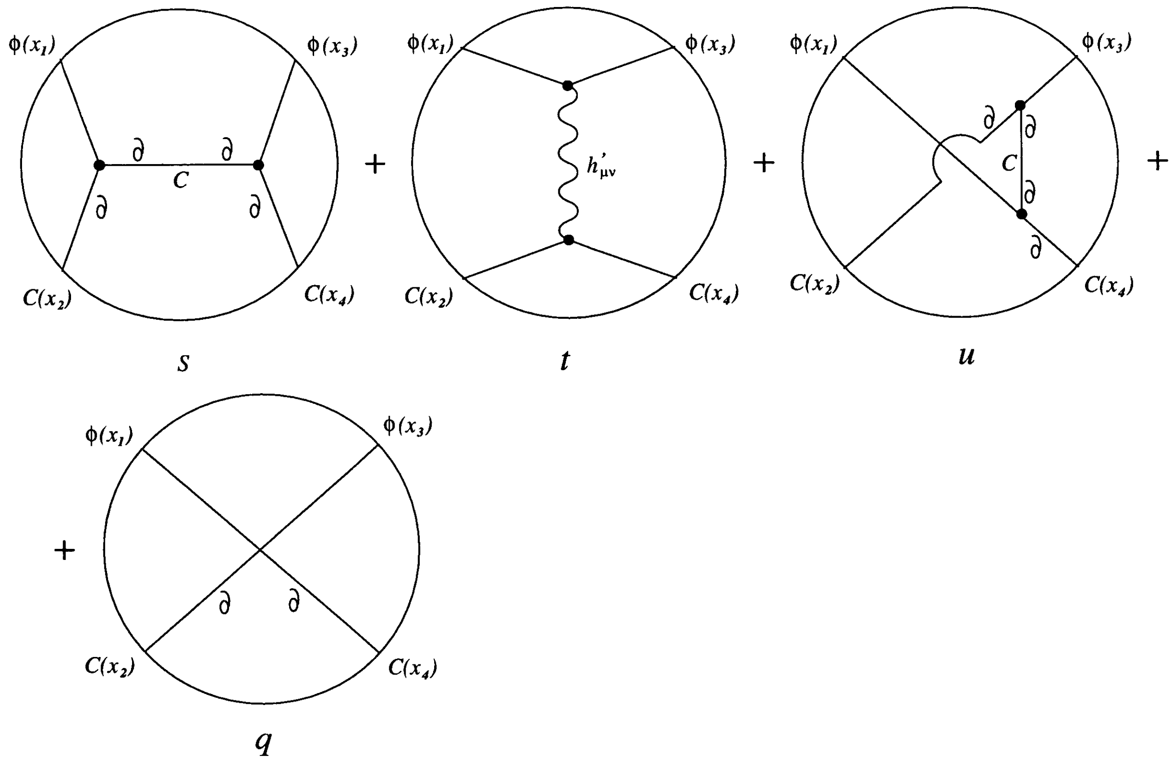

Figure 3-1: Supergravity graphs contributing to $\langle O_\phi(x_1)O_C(x_2)O_\phi(x_3)O_C(x_4)\rangle$.

where*

$$K_\Delta(z,x) = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma[\Delta - d/2]}\left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x})^2}\right)^\Delta \tag{3.2.5}$$

is the normalized boundary to bulk propagator for scalar fields in supergravity corresponding to primary operators in the CFT of scaling dimension Δ [3, 77]. We have $d = 4$ and note that for both ϕ and C we have $\Delta = 4$. For this case we will simply write K without subscript. $G(z,w)$ is the bulk to bulk propagator in the AdS_5 space for massless scalar fields, satisfying †

$$\Delta_z G(z,w) = \delta(z,w) \tag{3.2.6}$$

We will not need the explicit form of $G(z,w)$.

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*We assume $\Delta > d/2$. The case $\Delta = d/2$ saturates the unitarity bound and requires a special normalisation[77].

†In [89] the notation is instead $\Delta_z G(z,w) = -\delta(z,w)$.

The quartic graph is

$$q = -(4b) I^q_{\phi C \phi C}(x_1, x_2, x_3, x_4) \tag{3.2.7}$$

$$\begin{aligned} &I^q_{\phi C \phi C}(x_1, x_2, x_3, x_4) \equiv \\ &\quad \int \frac{d^5 z}{z_0^5} z_0^2 K(z, x_1) \partial_{z_\mu} K(z, x_2) K(z, x_3) \partial_{z_\mu} K(z, x_4) \end{aligned} \tag{3.2.8}$$

The combinatoric factors in (3.2.3), (3.2.7) can be obtained either from Feynman perturbation theory of supergravity or directly from the fourth variation of the supergravity action (3.2.1) with respect to boundary values of the fields.

In [89] a nice manipulation was given which relates I^s to a 4-point contact graph:

$$\begin{aligned} &\int \frac{d^5 z}{z_0^5} \frac{d^5 w}{w_0^5} z_0^2 w_0^2 K(z, x_1) \partial_{z_\mu} K(z, x_2) \partial_{z_\mu} \partial_{w_\nu} G(z, w) K(w, x_3) \partial_{w_\nu} K(w, x_4) \\ &= \int \frac{d^5 z}{z_0^5} \frac{d^5 w}{w_0^5} z_0^2 w_0^2 \partial_{z_\mu} K(z, x_1) K(z, x_2) \partial_{z_\mu} \partial_{w_\nu} G(z, w) K(w, x_3) \partial_{w_\nu} K(w, x_4) \\ &= \frac{1}{2} \int \frac{d^5 z}{z_0^5} \frac{d^5 w}{w_0^5} z_0^2 w_0^2 \partial_{z_\mu} [K(z, x_1) K(z, x_2)] \partial_{z_\mu} \partial_{w_\nu} G(z, w) K(w, x_3) \partial_{w_\nu} K(w, x_4) \\ &= \frac{1}{2} \int \frac{d^5 z}{z_0^5} \frac{d^5 w}{w_0^5} w_0^2 K(z, x_1) K(z, x_2) \delta(z, w) \partial_{w_\nu} K(w, x_3) \partial_{w_\nu} K(w, x_4) \\ &= \frac{1}{2} \int \frac{d^5 z}{z_0^5} z_0^2 K(z, x_1) K(z, x_2) \partial_{z_\nu} K(z, x_3) \partial_{z_\nu} K(z, x_4) \end{aligned} \tag{3.2.9}$$

where we have integrated by parts (noting that surface terms vanish), used the fact that $\triangle_z K(z, x) = 0$, and used (3.2.6). Thus we see that

$$I^s_{\phi C \phi C}(x_1, x_2, x_3, x_4) = \frac{1}{2} I^q_{\phi \phi C C}(x_1, x_2, x_3, x_4) \tag{3.2.10}$$

$$I^u_{\phi C \phi C}(x_1, x_2, x_3, x_4) = \frac{1}{2} I^q_{C \phi \phi C}(x_1, x_2, x_3, x_4) \tag{3.2.11}$$

Note that the RHS of (3.2.10) or (3.2.11) is not the same as the quartic graph in Figure 1(q) since the derivatives act on different variables.

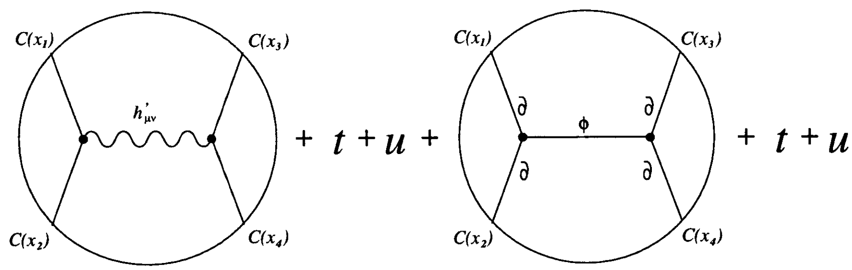

Figure 3-2: Supergravity graphs contributing to $\langle O_C(x_1)O_C(x_2)O_C(x_3)O_C(x_4)\rangle$.

It is easy to see by using integration by parts that

$$I^q_{\phi\phi CC}(x_1,x_2,x_3,x_4) = I^q_{CC\phi\phi}(x_1,x_2,x_3,x_4) \tag{3.2.12}$$

$$I^q_{\phi\phi CC}(x_1,x_2,x_3,x_4) + I^q_{\phi C\phi C}(x_1,x_2,x_3,x_4) + I^q_{C\phi\phi C}(x_1,x_2,x_3,x_4) = 0 \tag{3.2.13}$$

Thus we find that the contributions to $\langle O_\phi(x_1)O_C(x_2)O_\phi(x_3)O_C(x_4)\rangle$ from the s, u and quartic graphs add up to

$$\begin{aligned} &-4a^2\frac{1}{2}I^q_{\phi\phi CC}(x_1,x_2,x_3,x_4) - 4a^2\frac{1}{2}I^q_{C\phi\phi C}(x_1,x_2,x_3,x_4) - 4bI^q_{\phi C\phi C}(x_1,x_2,x_3,x_4) \\ &\quad = (-4b+2a^2)I^q_{\phi C\phi C}(x_1,x_2,x_3,x_4) \end{aligned} \tag{3.2.14}$$

Putting $a = 1, b = 1$ we see that the coefficient on the RHS is not zero. In the next section we show that the function $I^q_{\phi C\phi C}(x_1,x_2,x_3,x_4)$ is nonzero by computing its leading singularities.

The 4-point function of the primary operator corresponding to the axion field $\langle O_C(x_1)O_C(x_2)O_C(x_3)O_C(x_4)\rangle$ is given by the AdS graphs in Figure 2. Using (3.2.13) we see that the sum of the three dilaton exchange graphs sums to zero, though each of these graphs will not separately vanish.

3.3 Singularities in 4-point graphs

We have seen that the s and u graphs of Figure 1 reduce to the form of an I^q integral. In the function $I^q_{\phi\phi CC}(x_1, x_2, x_3, x_4)$ there are two independent short distance limits to be considered:

(a) $x_{12} \equiv |x_1 - x_2| \to 0$.

(b) $x_{13} \equiv |x_1 - x_3| \to 0$.

(From (3.2.12) we see that $x_{34} \to 0$ is similar to $x_{12} \to 0$ etc.).

We first observe the identity

$$\begin{aligned}\int \frac{d^{d+1}z}{z_0^{d+1}} z_0^2 K_{\Delta_1}(z, x_1) K(z, x_2)_{\Delta_2} \partial_{z_\mu} K(z, x_3)_{\Delta_3} \partial_{z_\mu} K(z, x_4)_{\Delta_4} \\ = \Delta_3 \Delta_4 J_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(x_1, x_2, x_3, x_4) \\ - 2(\Delta_3 - \frac{d}{2})(\Delta_4 - \frac{d}{2}) x_{34}^2 J_{\Delta_1,\Delta_2,\Delta_3+1,\Delta_4+1}(x_1, x_2, x_3, x_4)\end{aligned} \tag{3.3.15}$$

where

$$J_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(x_1, x_2, x_3, x_4) \equiv \int \frac{d^{d+1}z}{z_0^{d+1}} K_{\Delta_1}(z, x_1) K(z, x_2)_{\Delta_2} K(z, x_3)_{\Delta_3} K(z, x_4)_{\Delta_4} \tag{3.3.16}$$

This identity can be derived by methods similar to those in [77] (translating x_3 to the origin, performing an inversion $z_\mu = \frac{z'_\mu}{(z')^2}, x_i = \frac{x'_i}{(x')^2}$, evaluating the derivatives and inverting back).

This manipulation reduces the calculation of an integral of the type I^q to computing the quartic graph with no derivatives on any of the legs. A special case of this latter calculation (with all $\Delta_i = \Delta$) was given in [76]; we make a straightforward extension of their calculation to the case with arbitrary Δ_i:

$$\begin{aligned}J_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(x_1, x_2, x_3, x_4) = \\ \frac{1}{2\pi^{3d/2}} \frac{\Gamma[-\frac{d}{2} + \frac{\sum_i \Delta_i}{2}]\Gamma[-\Delta_4 + \frac{\sum_i \Delta_i}{2}]\Gamma[-\Delta_3 + \frac{\sum_i \Delta_i}{2}]\Gamma[\Delta_3]\Gamma[\Delta_4]}{\Gamma[\frac{\sum_i \Delta_i}{2}]\prod_i \Gamma[\Delta_i - \frac{d}{2}]} \\ \int_0^\infty \frac{d\beta_2}{\beta_2}(\beta_2 x_{24}^2 + x_{14}^2)^{-\Delta_4}(\beta_2 x_{12}^2)^{\Delta_4 - \sum_i \frac{\Delta_i}{2}} \left(\frac{x_{34}^2}{(\beta_2 x_{24}^2 + x_{14}^2)}\right)^{-\Delta_3} \beta_2^{\Delta_2}\end{aligned}$$

$$
{}_2F_1[-\Delta_4 + \sum_i \frac{\Delta_i}{2}, \Delta_3, \sum_i \frac{\Delta_i}{2}, 1-\alpha] \tag{3.3.17}
$$

where

$$
\alpha = \frac{(\beta_2 x_{23}^2 + x_{13}^2)(\beta_2 x_{24}^2 + x_{14}^2)}{\beta_2 x_{12}^2 x_{34}^2} \tag{3.3.18}
$$

and ${}_2F_1$ is the hypergeometric function. For the estimates below it is helpful to use the integral representation:

$$
{}_2F_1[\alpha, \beta; \gamma, z] = \frac{1}{B[\beta, \gamma-\beta]} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} dt \tag{3.3.19}
$$

where $B[\alpha, \beta]$ is the Beta function.

From (3.3.17) and (3.3.19) we find that as $x_{12} \to 0$:

$$
I^q_{\phi\phi CC}(x_1, x_2, x_3, x_4) \to \frac{6^4}{\pi^6} \frac{4}{21} \frac{1}{x_{13}^8 x_{14}^8} \ln \frac{x_{13} x_{14}}{x_{12}^2} \tag{3.3.20}
$$

As $x_{13} \to 0$:

$$
I^q_{\phi\phi CC}(x_1, x_2, x_3, x_4) \to -\frac{6^4}{\pi^6} \frac{2}{21} \frac{1}{x_{12}^8 x_{14}^8} \ln \frac{x_{12} x_{14}}{x_{13}^2} \tag{3.3.21}
$$

Note that the strengths of the singularities in (3.3.20) and (3.3.21) are such that they respect the identity (3.2.13).

In [89] it was argued that each of the s,u and quartic graphs given in Figure 1 vanishes separately, while we have reached a somewhat different conclusion.[‡] We have not evaluated the graviton exchange graph, which was speculated to vanish in [89], but we discuss in the next section our expectations for its contribution.

[‡]The resubmitted version (v4) of [89] appears to agree with our conclusions.

3.4 Discussion

We know that the $\mathcal{N} = 4$ SYM theory is exactly conformal. Consider a 4-point function $\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle$ in the limit $x_1 \to x_2$, $x_3 \to x_4$. We might try to expand[§]

$$O_1(x_1)O_2(x_2) = \sum_n \frac{\alpha_n O_n(x_1)}{(x_1 - x_2)^{\Delta_1+\Delta_2-\Delta_n}}\ ,\ O_3(x_3)O_4(x_4) = \sum_m \frac{\beta_m O_m(x_3)}{(x_3 - x_4)^{\Delta_3+\Delta_4-\Delta_m}} \tag{3.4.22}$$

and get

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \sum_{n,m} \frac{\alpha_n \beta_m \langle O_n(x_1)O_m(x_3)\rangle}{(x_1 - x_2)^{\Delta_1+\Delta_2-\Delta_m}(x_3 - x_4)^{\Delta_3+\Delta_4-\Delta_n}} \tag{3.4.23}$$

In a non-conformal theory, where a mass scale m would be available, we could also have, for instance, $O_{\Delta_1}(x_1)O_{\Delta_2}(x_2) \sim \log(m|x_1 - x_2|)O_{\Delta_1+\Delta_2}(x_1)$, but in a conformal theory such a term should not arise. Thus if the sums in (3.4.23) are to converge, we expect that the limit $x_{12} \to 0$ in the correlator would have no term in $\log(x_{12})$. Individual graphs from supergravity, however, are generically expected to have such logarithmic singularities and (3.3.20),(3.3.21) are examples of this fact. Thus either the logs all cancel when the supergravity graphs are summed, or a naive OPE summation of the form (3.4.23) is invalid.

We now proceed to discuss our results for 4-point functions in the dilaton-axion sector in the light of the questions of cancellation of logs and expectations for power singularities. For the correlator $\langle O_\phi O_C O_\phi O_C\rangle$ we found in (14) that the sum of s,u and quartic graphs is proportional to the contact amplitude and contains logarithmic singularities. We have not evaluated the t-channel graviton exchange graph, which is quite difficult, but which could contain logarithms that cancel those in the sum s+u+quartic. Note that if such a cancellation occurs for the $AdS_5 \times S_5$ supergravity theory then it would certainly fail to occur for an arbitrary choice of couplings between the fields. Thus a generic theory in AdS would not give a boundary theory which would possess a convergent local OPE.

[§]See also [90] for discussions of conformal OPEs and the the contribution of a given primary operator and its descendents to the CFT 4-point function.

In the $\langle O_C O_C O_C O_C \rangle$ correlator we found a cancellation among 3 ϕ-exchange graphs which each have a log singularity. The t-channel graviton exchange diagram in this correlator is the same as the t-channel graviton exchange in $\langle O_\phi O_C O_\phi O_C \rangle$. Suppose that this latter graph does contain the cancelling logarithms discussed above. It is then a simple consequence of 3.2.12 and 3.2.13 that the sum of log singularities in the t,s, and u channel graviton exchange diagrams will also cancel in $\langle O_C O_C O_C O_C \rangle$.

Although we have not evaluated the graviton exchange graphs in Figs. 1 and 2, it does appear on physical grounds that they are non-vanishing and have a strong singularity $\sim 1/x^4$ for $x \to 0$, where x is the separation of any two boundary operators connected to the same internal vertex. Part of this physical intuition stems from the fact that the 3-point functions $\langle O_C(x_1) O_C(x_2) T_{ij}(x_3) \rangle$ and $\langle O_\phi(x_1) O_\phi(x_2) T_{ij}(x_3) \rangle$, where T_{ij} is the stress-energy tensor, are different from zero [80], so that we expect from the leading term of the OPE the singularity $\sim 1/x^{\Delta_1+\Delta_2-\Delta_3}$, where all $\Delta_i = 4$. This would imply that the t-channel graph in Fig.1 is more singular as $x_{13} \to 0$ than any of the other graphs, so that the overall sum of all diagrams contributing to $\langle O_\phi O_C O_\phi O_C \rangle$ is not expected to vanish. One can state the same physical expectation in the language of the boundary $\mathcal{N} = 4$ SYM theory, in which $O_\phi = TrF^2$ and $O_C = TrF\tilde{F}$, and the 2- and 3-point functions of these operators are exactly given by their free-field values due to superconformal non-renormalization theorems. It is easy to calculate the free field OPE's and see that $TrF^2(x)TrF^2(y)$ and $TrF\tilde{F}(x)TrF\tilde{F}(y)$ contain the stress tensor with expected $1/(x-y)^4$ singularity. Thus physical considerations within the boundary CFT lead us to expect a non-vanishing t-channel contribution to $\langle O_\phi O_C O_\phi O_C \rangle$.

It is also easy to understand on physical grounds why the naively expected $1/(x_{12})^4$ singularity of the s-channel graph for $\langle O_\phi O_C O_\phi O_C \rangle$ is not present. First, one can use the formulae of [5] to show that $\langle O_\phi O_C O_C \rangle = 0$ (The AdS integral $\int \frac{d^5 z}{z_0^5} z_0^2 K \partial_{z_\mu} K \partial_{z_\mu} K$ vanishes even though the action (3.2.1) contains the vertex $\phi(\partial C)^2$.) Second, one can compute the free field OPE $TrF^2(x)TrF\tilde{F}(y)$ and see that there is no $1/(x-y)^4$ singularity (although we expect a weaker singularity from operators of dimension greater than 4).

We comment on the relation between supergravity graphs and OPE's. Consider a 4-point correlator of chiral primaries, $\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle$. In the expansion (3.4.23), let us consider the sum over chiral primaries and their conformal descendents. The $SO(6)$ symmetry of the $N = 4$ SYM theory allows only a finite number of chiral primaries to appear in this expansion. The same symmetry of the $AdS_5 \times S_5$ supergravity theory allows only a finite number of fields to propagate in the internal lines of the corresponding AdS graphs. It is thus tempting to seek a relation between, say, the s-channel AdS graph whose internal line corresponds to a specific primary operator $O(x)$ and the contribution of $O(x)$ and its descendents (i.e. derivatives) in the double OPE (23). Consider the limit x_{12} small, x_{34} small, x_{13} large. The s–channel supergravity graph has two 3-point vertices in the interior of AdS. Generically, we expect large contributions from two distinct domains of integration in the space of z and w: (a) z is near $\vec{x}_1$,$\vec{x}_2$, while w is near $\vec{x}_3$,$\vec{x}_4$; (b) both z and w are near $\vec{x}_1$,$\vec{x}_2$ (or both near $\vec{x}_3$, $\vec{x}_4$).

In region (a) the bulk supergravity propagator goes from near one pair to near the other pair, so this contribution might correspond to the double OPE (3.4.23). A toy example to study this hypothesis was presented in [88]. The CFT and AdS calculations were compared to fourth order in $\frac{x_{12}}{x_{13}}$ and $\frac{x_{34}}{x_{13}}$, and exact agreement was obtained. Recently, in [89] it was argued that a generic s–channel supergravity graph exactly matches the corresponding OPE contribution. However the argument relied on an implicit assumption of analyticity (in order to separate terms with physical and shadow singularities) which is not satisfied if there are logarithmic singularities. Thus the identification of s-channel graphs and double OPE contributions may not be exact. For example, since the 3-point function $\langle O_\phi O_C O_C\rangle$ vanishes, the double OPE for the correlator $\langle O_\phi O_C O_\phi O_C\rangle$ would also be naively expected to vanish. However, we showed explicitly in Section 3 that the corresponding supergravity s-channel graph (Fig.1,s) has a leading singularity which is logarithmic. It is an important problem for future work to determine the exact circumstances under which logarthmic singularities occur. This will require detailed input from the $AdS_5 \times S_5$ bulk supergravity theory, since s-channel graphs formed from derivative and non-derivative ϕ^3 vertices may have different analyticity properties.

We finally would like to make some comments on the issues of duality both on the supergravity and the CFT side. Supergravity graphs are not expected to be dual, indeed in the $\phi C \phi C$ example we found that the s and u channels are manifestly different since they exhibit different singularities. Operator product expansions are instead dual by definition under the assumption of their convergence. It appears unlikely that $\mathcal{N} = 4$, $d = 4$ $SU(N)$ SYM in the $N \to \infty$, $g_{YM}^2 N \to \infty$ limit possesses a convergent OPE in terms of only chiral primaries and their descendents, if one assumes the validity of the AdS/CFT correspondence. Consider again $\langle O_\phi(x_1) O_C(x_2) O_\phi(x_3) O_C(x_4) \rangle$. The only chiral primary that could enter the double OPE (3.4.23) is O_C, but the coupling is zero since $\langle O_\phi O_C O_C \rangle = 0$. Hence in this way of doing the OPE we expect a zero answer from the chiral sector. However, using the OPE to expand $O_\phi(x_1) O_\phi(x_3)$ and $O_C(x_2) O_C(x_4)$, only the stress-energy tensor T_{ij} can enter as an intermediate chiral operator, and the coupling is this time non-zero since $\langle O_\phi O_\phi T_{ij} \rangle$ and $\langle O_C O_C T_{ij} \rangle$ do not vanish as shown in [80]. We thus see that the assumption of a convergent OPE in terms of only chiral operators appears to lead to a contradiction. It would be interesting to find out the minimum set of operators needed in the theory to allow duality of the OPE expansion for chiral field correlators.

Chapter 4

Complete four point functions and OPE interpretation

4.1 Introduction

Broadly speaking, 2– and 3–point functions (see *e.g.* [91, 85, 92]) have provided evidence that the conjectured correspondence is correct, but 4–point functions are expected to contain more information about the non-perturbative dynamics of the CFT. Previous studies relevant to 4–point correlators include [76]–[103]. 4–point correlators for contact interactions of scalars in the bulk theory were the first to be studied [76, 93, 94] followed by diagrams with exchanged gauge bosons [95] and scalars [88, 96, 97]. (See also [98, 99] for a different approach). α'/R^2 corrections are considered in [100], and there is an extensive literature on instanton contributions, see *e.g.* [101].

The simplest 4–point correlators that can be studied are those involving the marginal operators $\mathcal{O}_\phi \sim \mathrm{Tr}(F^2 + \ldots)$ and $\mathcal{O}_C \sim \mathrm{Tr}(F\tilde{F} + \ldots)$ corresponding to the dilaton and axion supergravity fields, as first stressed in [93]. To leading order in N, the amplitudes $\langle \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \rangle$, $\langle \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \rangle$ and $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$ factorize in products of 2–point functions (see Figures 1a and 3). Thanks to the non–renormalization theorem for the 2–point functions [22, 91], these disconnected contributions do not receive corrections in powers of $\alpha'/R^2 = 1/\lambda^{1/2}$. The next contribution to the 4–point amplitudes is thus a $1/N^2$ effect and involves tree–level, connected supergravity diagrams like the ones in Figure 2. The computation of $\langle \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \rangle$, $\langle \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \rangle$ and $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$ was started in [94] with the evaluation of the relevant quartic and scalar exchange diagrams (Figure 2s,u,q and Figure 4). Here we complete the computation by evaluating the remaining graviton exchange diagram (Figure 2t) and we initiate the analysis of the first realistic 4–point amplitude in the AdS/CFT correspondence.

We also present what we believe is a cross–checked and reliable calculation of the graviton exchange diagram between pairs of external scalars of arbitrary mass in AdS_{d+1} for arbitrary d. The calculation was facilitated by the recently derived covariant form of the graviton propagator [104], but it is still very complex compared to earlier work.

One theoretical framework to analyze results on 4–point functions in the operator product expansion (OPE) [88, 105]. The mere assumption of an OPE is quite restrictive and imposes constraints on the allowed form of the result. Let us assume a double "t–channel" OPE of the schematic form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{n,m} \frac{\alpha_n \, \langle \mathcal{O}_n(x_1)\mathcal{O}_m(x_2)\rangle \, \beta_m}{(x_1-x_3)^{\Delta_1+\Delta_3-\Delta_m}(x_2-x_4)^{\Delta_2+\Delta_4-\Delta_n}} \tag{4.1.1}$$

containing the contribution of various primary operators $\mathcal{O}_p$ and their descendents $\nabla^k \mathcal{O}_p$ in the intermediate state. For simplicity we have assumed that these are scalars, but vector and tensor operators contribute in a similar way, each with a characteristic tensor structure. (For primary operators, $\langle \mathcal{O}_p \mathcal{O}_{p'} \rangle$ vanishes unless $\Delta_p = \Delta_{p'}$).

Recognizing in the supergravity 4–point results a structure of the form (4.1.1) should allow to determine the operator content of the theory and its OPE structure in the large N, large λ limit. Preliminary computations [88, 96] have indicated that the supergravity diagrams contain the expected contributions to (4.1.1) of chiral primary operators and their superconformal descendents. It is however clear that these contributions alone do not reproduce the supergravity result [94]. A natural expectation is that appropriately defined normal–ordered *products* of chiral primaries and descendents also contribute to the OPE and form the full operator content of the theory in this limit. This set of operators has a dual interpretation in terms of multi–particle Kaluza–Klein states in supergravity. Massive string states are expected to decouple in this limit*. The computation of a complete realistic 4–point correlator presented here should allow to put these ideas to test.

An interesting issue raised in the previous chapter is the presence in the 4–point supergravity amplitudes of logarithms of the coordinate separation between two points in the limit when the points come close. Logarithmic singularities appear to be a generic feature of all the AdS processes studied so far [95, 96, 97], and we find the same situation for the graviton exchange. The question then is whether the logarithms cancel when the various contributions to a realistic correlator are assembled. If not, we should ask whether the logarithms can still be incorporated in the OPE framework. Here we find that logarithmic singularities do indeed occur in the complete 4–point functions.

As pointed out by Witten [106], logarithms can generically arise in the perturbative expansion of a CFT 4–point correlator as renormalization effects like mixings and corrections to the dimensions of the exchanged operators. The perturbative parameter is in this case $1/N$, which is mapped by the correspondence to the gravitational coupling constant. The operators $\mathcal{O}_\phi$ and $\mathcal{O}_C$ are chiral and hence their dimensions are protected, but their OPE's contain (besides chiral contributions like the stress–energy tensor) non–chiral composite operators like $:\mathcal{O}_\phi \mathcal{O}_\phi:$ that require a careful definition and can lead to renormalization effects [106]. (A somewhat different viewpoint has been described in a very recent paper [107], see also [108]).

It is an interesting subject for future work to analyze the constraints imposed by this interpretation on the allowed form of the logarithmic singularities and to assess the compatibility of these constraints with the supergravity results.

The chapter is organized as follows.

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*Group–theoretic aspects of multi–particle and string states have been considered in [111].

In Section 2, we present the supergravity graphs that contribute to 4–point functions involving $\mathcal{O}_\phi$, $\mathcal{O}_C$, summarize our results for the amplitudes and make some remarks about their OPE interpretation.

In Section 3, we describe the general set–up for the calculation of the graviton exchange amplitude. We give a few geometric identities, summarize the results for the scalar and graviton propagators and present the integral associated with the graviton exchange graph.

In Section 4 and Section 5 we separately describe two independent computations of the graviton amplitude, for $\Delta = \Delta' = d = 4$ in Section 4 and for general Δ, Δ' and d in Section 5. Both computations reduce the graviton exchange amplitude to finite sums of scalar quartic graphs (see Figure 6). The two results are shown to precisely agree for $\Delta = \Delta' = d = 4$.

In Section 6, we develop integral representations and asymptotic series expansions for the quartic graphs (Figure 5), which are the basic building blocks of the answer. We find asymptotic serieses for the graviton exchange in terms of two conformally invariant variables.

Finally, in the Appendix B we discuss some properties and mathematical identities of the quartic graphs.

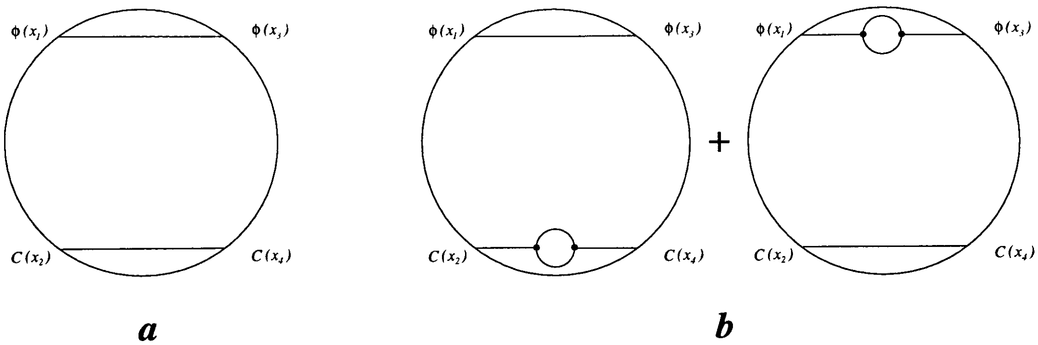

Figure 4-1: Disconnected contribution to $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$. *a*: $O(N^4)$; *b*: $O(N^2)$.

4.2 4–point functions in the dilaton–axion sector

Following [93], we first discuss the dilaton–axion–graviton sector of IIB supergravity, dimensionally reduced on the classical background solution $AdS_5 \times S_5$ keeping only the constant modes on S_5. The relevant part of 5–dimensional action is†

$$S = \frac{1}{2\kappa^2} \int_{AdS_5} d^5 z \sqrt{g} \left(-\mathcal{R} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{2\phi} g^{\mu\nu} \partial_\mu C \partial_\nu C \right) . \tag{4.2.1}$$

Th 5–dimensional gravitational coupling κ is related to the parameters of the compactification by $2\kappa^2 = \frac{15\pi^3 R^3}{N^2}$, where N is the number of units of 5–form flux and R the radius of the 5–sphere (equal to the AdS_5 scale, see equ.(4.3.16) below). We will usually set the AdS_5 scale $R \equiv 1$.

4.2.1 Witten diagrams

We wish to implement the prescription of [2, 3] to compute the CFT correlators $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$, $\langle \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \rangle$, $\langle \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \rangle$, where $\mathcal{O}_\phi \sim \mathrm{Tr}(F^2 + \ldots)$, $\mathcal{O}_C \sim \mathrm{Tr}(F\tilde{F} + \ldots)$ are the exactly

†The metric appearing in (4.2.1) is *not* the restriction of the original 10–dimensional metric to AdS_5, but it is related to it by a Weyl rescaling of the metric fluctuations [109, 93]. The fluctuation $h'_{\mu\nu}$ that gives the massless graviton in AdS_5 is given in terms of the original $h_{\mu\nu}$ by $h_{\mu\nu} = h'_{\mu\nu} - \frac{1}{3}\bar{g}_{\mu\nu} h^\alpha_\alpha$, where α is an index along S_5 and $\bar{g}_{\mu\nu}$ the background metric [109].

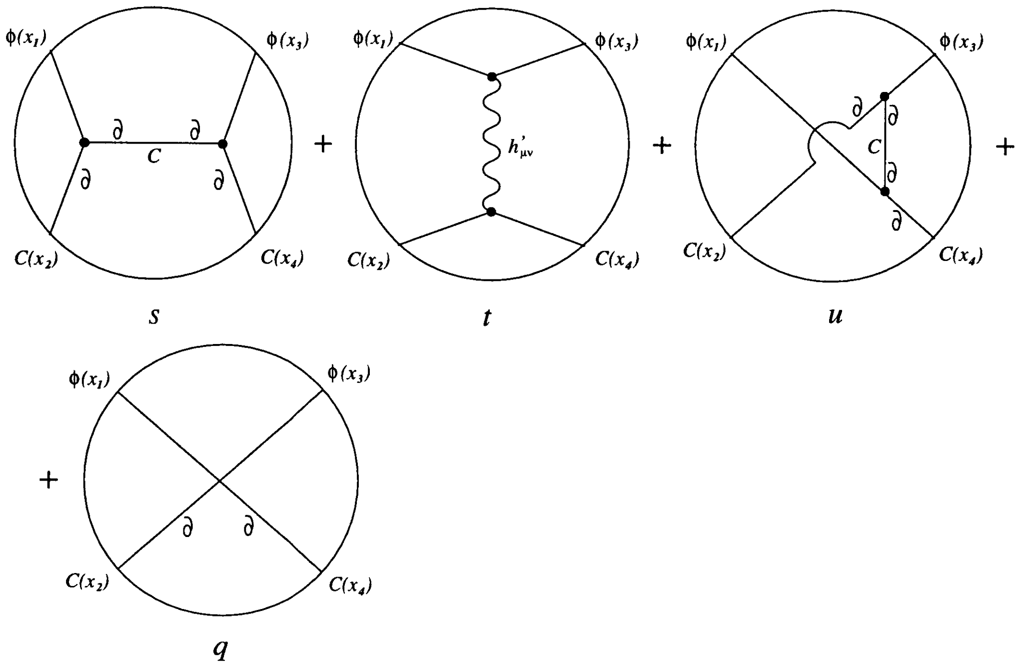

Figure 4-2: Connected $O(N^2)$ contributions to $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$.

marginal ($\Delta = 4$) SYM operators corresponding to the dilaton and axion fields [3][‡].

Let us first consider $\langle \mathcal{O}_\phi(x_1)\mathcal{O}_C(x_2)\mathcal{O}_\phi(x_3)\mathcal{O}_C(x_4)\rangle$. The leading large N contribution is given by the disconnected diargam in Figure 1a. This diagram, being the product of two 2–point functions, is proportional to $N^4/(x_{13}^8 x_{24}^8)$.

The next contribution, of order N^2, comes from the diagrams in Figures 1b and 2. However, the one–loop diagrams in Figure 1b, thanks to the fact that the dimensions of the chiral operators $\mathcal{O}_\phi, \mathcal{O}_C$ are protected, only give a $1/N^2$ correction to the overall coefficient of the amplitude in Figure 1a[§]. Among the diagrams in Figure 2, the sum s+u+q has been computed in [94].

[‡]The precise structure of the composite operators $\mathcal{O}_\phi$ and $\mathcal{O}_C$ in terms of elementary SYM fields is in principle given by the variation of the on–shell $\mathcal{N} = 4$ lagrangian with respect to the marginal couplings g_{YM} and θ, or by supersymmetry transformations starting from the chiral primary $\mathrm{Tr} X^{(i}X^{j)}$.

[§]This correction precisely accounts for the fact the gauge group is $SU(N)$ rather than $U(N)$. Note that validity of the correspondence seems to require that there are no higher loop corrections in the supergravity 2–point functions.

Sections 4 and 5 of the chapter are devoted to evaluation of the remaining graviton exchange diagram t.

Similarly, Figures 3 and 4 reproduce the relevant diagrams for $\langle \mathcal{O}_C(x_1)\mathcal{O}_C(x_2)\mathcal{O}_C(x_3)\mathcal{O}_C(x_4)\rangle$. The connected diagrams for $\langle \mathcal{O}_\phi(x_1)\mathcal{O}_\phi(x_2)\mathcal{O}_\phi(x_3)\mathcal{O}_\phi(x_4)\rangle$ involve only graviton exchanges. As shown in [94] the s,t,u scalar exchange diagrams in Figure 4 add up to zero. Hence, to this order,

$$\langle \mathcal{O}_\phi\mathcal{O}_\phi\mathcal{O}_\phi\mathcal{O}_\phi\rangle = \langle \mathcal{O}_C\mathcal{O}_C\mathcal{O}_C\mathcal{O}_C\rangle. \tag{4.2.2}$$

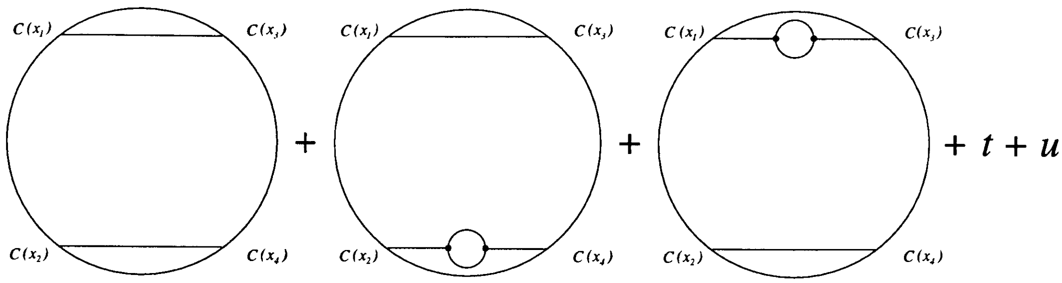

Figure 4-3: Disconnected $O(N^4)$ and $O(N^2)$ contributions to $\langle \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \rangle$.

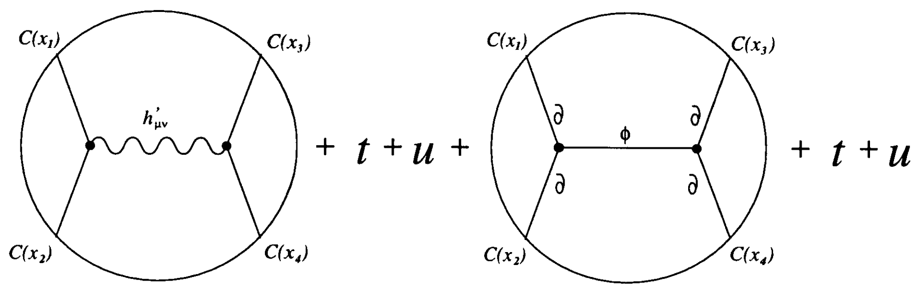

Figure 4-4: Connected $O(N^2)$ contributions to $\langle \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \mathcal{O}_C \rangle$.

4.2.2 Summary of results

It turns out that upon integration over one of the bulk points, all 4–point AdS processes with external scalars, including the graviton exchange, reduce to a finite sum of scalar quartic graphs (see Figure 6). We denote quartic graphs of external conformal dimensions Δ_i with the symbol $D_{\Delta_1\Delta_3\Delta_2\Delta_4}(x_1, x_3, x_2, x_4)$, as in Figure 5 (see equation (B.1) for the precise definition and the Appendix for a discussion of properties of these functions).

The final result for the graviton exchange graph in Figure 2t as sum of quartic graphs (for $\Delta = \Delta' = d = 4$), derived in Sections 4 and 5 below, is

$$\begin{aligned} I_{\text{grav}} \;&=\; \left(\frac{6}{\pi^2}\right)^4 \left[16\, x_{24}^2 \left(\frac{1}{2s} - 1\right) D_{4455} + \frac{64}{9}\frac{x_{24}^2}{x_{13}^2}\frac{1}{s}\, D_{3355} + \frac{16}{3}\frac{x_{24}^2}{x_{13}^4}\frac{1}{s}\, D_{2255} \right. \\ &\left. +18\, D_{4444} - \frac{46}{9\, x_{13}^2}\, D_{3344} - \frac{40}{9\, x_{13}^4}\, D_{2244} - \frac{8}{3\, x_{13}^6}\, D_{1144} \right], \end{aligned} \tag{4.2.3}$$

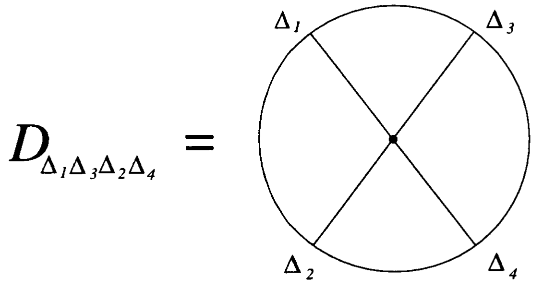

Figure 4-5: Definition of $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$.

where we have introduced the conformally invariant variable

$$s \equiv \frac{1}{2}\frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2}. \tag{4.2.4}$$

See equations (4.5.23, 4.5.57–4.5.58) for the analogous result in the general case of arbitrary Δ, Δ', d.

We also recall the result [94] for the sum of the amplitudes s, q, u in Figure 2

$$I_s + I_u + I_q = \left(\frac{6}{\pi^2}\right)^4 \left[64\, x_{24}^2 D_{4455} - 32\, D_{4444}\right]. \tag{4.2.5}$$

The sum of (4.2.3) and (4.2.5) gives the connected order N^2 contribution to the correlator $\langle \mathcal{O}_\phi(x_1)\mathcal{O}_C(x_2)\mathcal{O}_\phi(x_3)\mathcal{O}_C(x_4)\rangle$. The analogous result for $\langle \mathcal{O}_\phi\mathcal{O}_\phi\mathcal{O}_\phi\mathcal{O}_\phi\rangle = \langle \mathcal{O}_C\mathcal{O}_C\mathcal{O}_C\mathcal{O}_C\rangle$ is obtained by cross–symmetrization of (4.2.3).

The functions $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ admit simple integral representations (see Section 6.1) and can all be obtained as derivatives with respect to x_{ij}^2 of a single function (see Section A.3). In Section 6 we develop asymptotic series expansions for $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ in the conformally invariant variables s and t,

$$t = \frac{x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2}. \tag{4.2.6}$$

We consider the "direct" or t–channel limit $|x_{13}| \ll |x_{12}|$, $|x_{24}| \ll |x_{12}|$ which corresponds to $s, t \to 0$. The singular power terms in this limit are given by

$$I_{\rm grav}\Big|_{\rm sing} = \frac{2^{10}}{35\pi^6}\frac{1}{x_{13}^8 x_{24}^8}\Big[s\,(7t^2+6t^4)+s^2\,(-7-+3t^2)-8\,s^3\Big]. \tag{4.2.7}$$

In addition, as in [94, 95, 96, 97] we find an infinite series of terms logarithmic in s:

$$\begin{aligned} I_{\rm grav}\Big|_{\rm log} &= \frac{3\cdot 2^3}{\pi^6}\frac{\ln s}{x_{13}^8 x_{24}^8}\sum_{k=0}^{\infty} s^{4+k}\frac{\Gamma(k+4)}{\Gamma(k+1)}\Big\{-2(5k^2+20k+16)(3k^2+15k+22)a_{k+3}(t) \\ &+(k+4)^2(15k^2+55k^2+42)a_{k+4}(t)\Big\}, \end{aligned} \tag{4.2.8}$$

where the functions $a_k(t)$ are given by

$$a_k(t) = \int_{-1}^{1} d\lambda \frac{(1-\lambda^2)^k}{(1+\lambda t)^{k+1}} = \sqrt{\pi}\frac{\Gamma(k+1)}{\Gamma\left(k+\frac{3}{2}\right)}\,F\left(\frac{k+1}{2},\frac{k}{2}+1;k+\frac{3}{2};t^2\right). \tag{4.2.9}$$

As clear from the hypergeometric representation, $a_k(t)$ admit power series expansions in t^2 with radius of convergence 1. Here we do not display the non–singular power terms in $I_{\rm grav}$ (see Section 6.2).

The analogous result for the sum of the graphs s+u+q in Figure 2 is

$$I_{\rm s}+I_{\rm u}+I_{\rm q}\Big|_{\rm log} = \frac{2^6\cdot 3\cdot 5}{\pi^6}\frac{\ln s}{x_{13}^8 x_{24}^8}\sum_{k=0}^{\infty} s^{k+4}\left\{(k+1)^2(k+2)^2(k+3)^2(3k+4)\,a_{k+3}(t)\right\}. \tag{4.2.10}$$

The contribution $I_{\rm s}+I_{\rm u}+I_{\rm q}$ has no power singularities.

We now turn to discuss some physical implications of these results.

4.2.3 OPE interpretation

Let us compare the singular power terms of (4.2.7) with those expected form the OPE (4.1.1). In the direct channel limit $|x_{13}| \ll |x_{12}|$, $|x_{24}| \ll |x_{12}|$ the leading terms of the variables s and t are

$$s \sim \frac{x_{13}^2 x_{24}^2}{4\,x_{12}^4} \qquad t \sim -\frac{x_{13}\cdot J(x_{12})\cdot x_{24}}{x_{12}^2} \tag{4.2.11}$$

where $J_{ij} = \delta_{ij} - 2y_iy_j/y^2$ is the well–known Jacobian tensor of the conformal inversion $y_i' = y_i/y^2$. The leading term of (4.2.7) can then be written as

$$I_{\text{grav}}\Big|_{\text{sing}} = \frac{2^6}{5\pi^6}\frac{1}{x_{13}^6x_{24}^6}\frac{4(x_{13}\cdot J(x_{12})\cdot x_{24})^2 - x_{13}^2x_{24}^2}{x_{12}^8} + \ldots \tag{4.2.12}$$

with subleading terms suppressed by powers of $|x_{13}|/|x_{12}|$ and $|x_{24}|/|x_{12}|$. We note from (4.1.1) that (4.2.12) describes the contribution to the OPE of an operator $\mathcal{O}_p$ of dimension $\Delta = 4$. We show below that the tensorial structure agrees with the the expected contribuion of the stress–energy tensor of the boundary theory. It is worth mentioning first that various subcontributions to the amplitude I_{grav} (some of the D functions in (4.2.3)) have leading power $1/(x_{13}^6x_{24}^6x_{12}^4)$ indicative of a scalar operator of dimension $\Delta = 2$, which would not be expected in the graviton exchange process. The fact that this term cancels in the full amplitude is then an important check of the calculation.

Let us consider a scalar operator $\mathcal{O}_\Delta$ of scale–dimension Δ in d–dimensional space–time. The contribution of the conserved traceless stress–tensor T_{ij} to the OPE of $\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_3)$ is

$$\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_3) \sim k\,\frac{x_{13\,i}\,x_{13\,j}}{x_{13}^{2\Delta+2-d}}\,T_{ij}(x_1) \tag{4.2.13}$$

and the 2–point function of the stress tensor is

$$\langle T_{ij}(x_1)T_{kl}(x_2)\rangle = \frac{c}{2}\,\frac{J_{ik}(x_{12})J_{jl}(x_{12}) + J_{il}(x_{12})J_{jk}(x_{12}) - \frac{2}{d}\delta_{ij}\delta_{kl}}{x_{12}^{2d}} \tag{4.2.14}$$

which is conserved and traceless in any dimension. Note that $J_{ik}(y)J_{kj}(y) = \delta_{ij}$. We thus see that the stress tensor contribution to the general scalar double OPE is

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_{\Delta'}(x_2)\mathcal{O}_\Delta(x_3)\mathcal{O}_{\Delta'}(x_4) \sim \frac{kck'}{d}\,\frac{d(x_{13}\cdot J(x_{12})\cdot x_{24})^2 - x_{13}^2x_{24}^2}{x_{13}^{2\Delta+2-d}\;x_{24}^{2\Delta'+2-d}\;x_{12}^8}\,. \tag{4.2.15}$$

This form is in perfect agreement with (4.2.12). Further relevant information on 2– and 3–point functions of the stress–energy tensor can be found in [110].

Let us now consider the logarithmic terms. We see from the sum of (4.2.8) and (4.2.10) that an infinite series of terms logarithmic in s occurs in the direct channel expansion of $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$. Since the serieses (4.2.8) and (4.2.10) have a rather different structure, this conclusion appears quite robust. (In particular, it is insensitive to the relative normalization of $I_{\rm grav}$ and $I_{\rm s} + I_{\rm u} + I_{\rm q}$). We plead exhaustion and excuse ourselves from carrying a similar analysis for the crossed channel limit of $\langle \mathcal{O}_\phi \mathcal{O}_C \mathcal{O}_\phi \mathcal{O}_C \rangle$ and for $\langle \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \mathcal{O}_\phi \rangle$. The reader can find the necessary ingredients in Section 6.2. As mentioned in the Introduction, one should be able to interpret these logarithmic terms as $1/N^2$ renormalization effects related to the contribution of composite operators to the OPE (4.1.1) [106]. For example, the leading logarithmic term in the direct channel limit, $\frac{1}{(x_{12})^{16}} \log(\frac{x_{13}x_{24}}{x_{12}^2})$, could be related to the presence in (4.1.1) of the non–chiral composite operators $: \mathcal{O}_\phi \mathcal{O}_\phi :$ and $: \mathcal{O}_C \mathcal{O}_C :$. It is an interesting topic for future research to precisely identify the contributions of various composite operators, and the patterns of their renormalization and mixing, in the intricate series structures (4.2.8), (4.2.10). A detailed OPE intepretation of these supergravity results should provide us with new non–perturbative information about the $\mathcal{N} = 4$ SYM theory.

4.3 General set–up

As in most previous work on correlation functions, we work on the Euclidean continuation of AdS_{d+1}, viewed as the upper half space in $z_\mu \in \mathbf{R}^{d+1}$, with $z_0 > 0$. The metric $g_{\mu\nu}$ and Christoffel symbols $\Gamma^\kappa_{\mu\nu}$ are given by

$$
\begin{aligned}
ds^2 &= \sum_{\mu,\nu=0}^{d} g_{\mu\nu} dz_\mu dz_\nu = \frac{R^2}{z_0^2}(dz_0^2 + \sum_{i=1}^{d} dz_i^2) && (4.3.16) \\
\Gamma^\kappa_{\mu\nu} &= \frac{1}{R\, z_0}(\delta^\kappa_0 \delta_{\mu\nu} - \delta_{\mu 0}\delta^\kappa_\nu - \delta_{\nu 0}\delta^\kappa_\mu) && (4.3.17)
\end{aligned}
$$

and the curvature scalar is $\mathcal{R} = -d(d+1)/R^2$. We henceforth set the AdS scale $R \equiv 1$. This space is a maximally symmetric solution of the gravitational action

$$S_g = -\frac{1}{2\kappa^2}\int dz\sqrt{g}(\mathcal{R} - \Lambda) \tag{4.3.18}$$

with $\Lambda = -d(d-1)$.

It is well known that invariant bi–scalar functions on AdS_{d+1}, such as scalar field propagators, are most simply expressed in terms of the chordal distance variable u, defined by

$$u \equiv \frac{(z-w)^2}{2z_0 w_0} \tag{4.3.19}$$

where $(z-w)^2 = \delta_{\mu\nu}(z-w)_\mu(z-w)_\nu$ is the "flat Euclidean distance". Invariant tensor functions, such as the gauge or the graviton propagator, may be expanded in terms of bases of invariant bi-tensors, which are derivatives of u. For example, for rank 1, we have ($\partial_\mu = \partial/\partial z^\mu$ and $\partial_{\nu'} = \partial/\partial w^{\nu'}$)

$$\partial_\mu u = \frac{1}{z_0}\left(\frac{(z-w)_\mu}{w_0} - u\delta_{\mu 0}\right) \tag{4.3.20}$$

$$\partial_{\nu'} u = \frac{1}{w_0}\left(\frac{(w-z)_{\nu'}}{z_0} - u\delta_{\nu' 0}\right). \tag{4.3.21}$$

and for rank 2, there is $\partial_\mu u \partial_{\nu'} u$ as well as

$$\partial_\mu\partial_{\nu'} u = -\frac{1}{z_0 w_0}[\delta_{\mu\nu'} + \frac{1}{w_0}(z-w)_\mu\delta_{\nu'0} + \frac{1}{z_0}(w-z)_{\nu'}\delta_{\mu 0} - u\delta_{\mu 0}\delta_{\nu' 0}]. \tag{4.3.22}$$

Throughout this chapter, we shall also make use of differentiation and contraction relations between these basis tensors, which we list here,

$$\Box u = D^\mu\partial_\mu u = (d+1)(1+u) \tag{4.3.23}$$

$$D^\mu u\, \partial_\mu u = u(2+u) \tag{4.3.24}$$

$$D_\mu\partial_\nu u = g_{\mu\nu}(1+u) \tag{4.3.25}$$

$$(D^\mu u)\,(D_\mu \partial_\nu \partial_{\nu'} u) \;=\; \partial_\nu u \partial_{\nu'} u \tag{4.3.26}$$

$$(D^\mu u)\,(\partial_\mu \partial_{\nu'} u) \;=\; (1+u)\partial_{\nu'} u \tag{4.3.27}$$

$$(D^\mu \partial_{\mu'} u)\,(\partial_\mu \partial_{\nu'} u) \;=\; g_{\mu'\nu'} + \partial_{\mu'} u \partial_{\nu'} u\,. \tag{4.3.28}$$

$$\Box F(u) \;=\; u(u+2)F''(u) + (d+1)(1+u)F'(u) \tag{4.3.29}$$

These relations may be derived using (4.3.21), (4.3.22) and the metric and Christoffel symbols of (4.3.17) for AdS_{d+1}.

4.3.1 Scalar and graviton propagators

The bulk–to–boundary propagator (or Poisson kernel) for a scalar field of mass $m^2 = \Delta(\Delta - d)$ is well–known [3, 91] and given by

$$K_\Delta(z,\vec{x}) = C_\Delta \tilde{K}_\Delta(z,\vec{x}) = C_\Delta \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x})^2}\right)^\Delta \tag{4.3.30}$$

with the following normalization

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})}\,. \tag{4.3.31}$$

Bulk–to–bulk propagators for scalar fields of dimension Δ, with mass $m^2 = \Delta(\Delta - d)$, were derived in [112]. They can be expressed as hypergeometric functions in several equivalent ways. The expression which appears best suited for the integrals which occur in exchange diagrams [96, 97] is to use a hypergeometric function whose argument is ξ^2 where

$$\xi \equiv \frac{1}{1+u} = \frac{2z_0 w_0}{(z_0^2 + w_0^2 + (\vec{z}-\vec{w})^2)}\,. \tag{4.3.32}$$

The propagator is then given by

$$G_\Delta(u) = 2^\Delta \tilde{C}_\Delta \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right). \tag{4.3.33}$$

$$\tilde{C}_\Delta = \frac{\Gamma(\Delta)\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{(4\pi)^{(d+1)/2}\Gamma(2\Delta - d + 1)} \tag{4.3.34}$$

The propagator for massless scalars, with $\Delta = d$, is relevant for the graviton. When $\Delta = d$ is an even integer, the hypergeometric expression (4.3.33) can be rewritten [104] in terms of elementary functions. In particular, for $d = 4$, we have

$$G_4(u) = -\frac{1}{8\pi^2}\left\{\frac{2(1+u)}{\sqrt{u(2+u)}} - \frac{1+u}{\sqrt{u(2+u)}^3} - 2\right\} \tag{4.3.35}$$

The graviton propagator [104] can be expressed as a superposition of 5 independent fourth rank bi–tensors, of which 2 are gauge independent and 3 are gauge artifacts. The gauge terms represent pure diffeomorphisms, and their contribution to the integrals in the exchange diagram vanishes because the stress tensor is conserved. The physical part of the propagator involves two scalar functions $G(u)$ and $H(u)$, and is given by

$$G_{\mu\nu\mu'\nu'}(z,w) \;=\; (\partial_\mu\partial_{\mu'}u\;\partial_\nu\partial_{\nu'}u + \partial_\mu\partial_{\nu'}u\;\partial_\nu\partial_{\mu'}u)\;G(u) + g_{\mu\nu}\;g_{\mu'\nu'}\;H(u) \tag{4.3.36}$$

The function $G(u)$ is equal to the massless scalar propagator G_d.

A representation of $H(u)$ as a hypergeometric function was given in [104]. It was also expressed in terms of $G(u)$ and its first integral $\bar{G}(u)$, defined by $\bar{G}(u)' = G(u)$ and the boundary condition $\bar{G}(\infty) = 0$, which is a more useful form, given by

$$-(d-1)H(u) = 2(1+u)^2 G(u) + 2(d-2)(1+u)\bar{G}(u)\,. \tag{4.3.37}$$

Again, when d is even, $H(u)$ admits an elementary expression; in particular, when $d = 4$, we have

$$H(u) = -\frac{1}{12\pi^2}\{-6(1+u)^4 + 9(1+u)^2 - 2\}\frac{1+u}{(u(2+u))^{\frac{3}{2}}} - \frac{1}{2\pi^2}(1+u)^2\,. \tag{4.3.38}$$

4.3.2 Structure of the graviton exchange amplitude

The graviton exchange amplitude associated with the Witten diagram of Figure 2t is given by

$$I_{\text{grav}} = \frac{1}{4} \int dz \sqrt{g} \int dw \sqrt{g}\; T_{13}^{\mu\nu}(z)\; G_{\mu\nu\mu'\nu'}(z,w)\; T_{24}^{\mu'\nu'}(w) \tag{4.3.39}$$

where $G_{\mu\nu\mu'\nu'}$ is the graviton propagator (4.3.36). The vertex factor $T_{13}^{\mu\nu}(z)$ is given by

$$\begin{aligned} T_{13}^{\mu\nu}(z) &= D^\mu K_\Delta(z,x_1) D^\nu K_\Delta(z,x_3) + D^\nu K_\Delta(z,x_1) D^\mu K_\Delta(z,x_3) \\ &\quad - g^{\mu\nu}[\partial_\rho K_\Delta(z,x_1) D^\rho K_\Delta(z,x_3) + m^2 K_\Delta(z,x_1) K_\Delta(z,x_3)] . \end{aligned} \tag{4.3.40}$$

The combination $T_{24}^{\mu'\nu'}(w)$ is obtained from (4.3.40) by replacing $x_1 \to x_2$, $x_3 \to x_4$, $\Delta \to \Delta'$ $z \to w$. The stress–energy tensor $T_{\mu\nu}$ is conserved, $D_\mu T_{13}^{\mu\nu} = D_{\mu'} T_{24}^{\mu'\nu'} = 0$ thanks to the propagator equations $(\Box - m^2)K_\Delta = (\Box - m'^2)K_{\Delta'} = 0$.

It is the high tensorial rank of the propagator and vertex factors that make this amplitude more difficult than previously studied exchanges. The calculation is made tractable by splitting the amplitude into several terms and using partial integration of derivatives. There are several ways to organize this process, and what we have done and will present are complete calculations by two different methods which are then compared and shown to give identical results for the special case $d = \Delta = \Delta' = 4$, *i.e.* axions and dilatons in the type IIB theory. The two methods are separately presented in Sections 4 and 5.

4.4 The graviton exchange graph for $\Delta = \Delta' = d = 4$

The graviton propagator involves non–trivial tensorial structures. Nevertheless, it turns out that it is possible to reduce the graviton exchange graph to the sum of purely scalar amplitudes, with a peculiar pattern of bulk–to–bulk and bulk–to–boundary scalar propagators. We describe this reduction in Section 4.1.

Furthermore, upon integration over one of the two bulk variables, which we carry out in Section 4.2, each effective scalar exchange can be expressed a sum of quartic graphs with

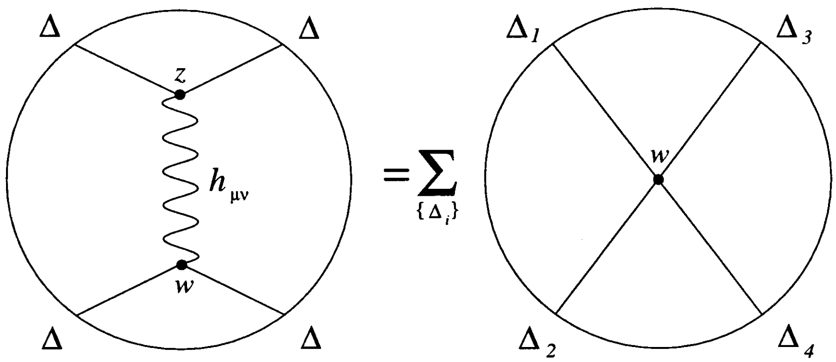

Figure 4-6: Reduction of graviton exchange to quartic graphs.

appropriate external dimensions. The final answer for the graviton exchange in terms of these basic building blocks (see Figure 6) is given in equation (4.4.38). The quartic graphs admit asymptotic series expansion which we describe in Section 6. It is also worth mentioning at this point that each quartic graph can be obtained by taking successive derivatives of a single basic function, see section A.3.

4.4.1 Reduction to scalar exchanges

We need to compute the graviton exchange amplitude (4.3.39) for $m^2 = m'^2 = 0$. Using the form (4.3.36) for the graviton propagator, we have¶ :

$$\begin{aligned}
I_{\rm grav} &= (C_4)^4 (I_H + I_G) && (4.4.1)\\
I_H &= \int [dz][dw] \left[\partial^\mu \tilde{K}_4(z,x_1)\partial^\nu \tilde{K}_4(z,x_3) - \frac{1}{2} g^{\mu\nu}\partial_\lambda \tilde{K}_4(z,x_1)\partial^\lambda \tilde{K}_4(z,x_3)\right] \cdot && (4.4.2)\\
& \quad g_{\mu\nu} g_{\mu'\nu'}\, H(u) \left[\partial^{\mu'} \tilde{K}_4(w,x_2)\partial^{\nu'} \tilde{K}_4(w,x_4) - \frac{1}{2} g^{\mu'\nu'}\partial_{\lambda'} \tilde{K}_4(w,x_2)\partial^{\lambda'} \tilde{K}_4(w,x_4)\right] \\
I_G &= \int [dz][dw] \left[\partial^\mu \tilde{K}_4(z,x_1)\partial^\nu \tilde{K}_4(z,x_3) - \frac{1}{2} g^{\mu\nu}\partial_\lambda \tilde{K}_4(z,x_1)\partial^\lambda \tilde{K}_4(z,x_3)\right] \cdot && (4.4.3)\\
& \quad (\partial_\mu \partial_{\mu'} u\, \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u\, \partial_\nu \partial_{\mu'} u)\, G(u) \cdot \\
& \quad \left[\partial^{\mu'} \tilde{K}_4(w,x_2)\partial^{\nu'} \tilde{K}_4(w,x_4) - \frac{1}{2} g^{\mu'\nu'}\partial_{\lambda'} \tilde{K}_4(w,x_2)\partial^{\lambda'} \tilde{K}_4(w,x_4)\right]
\end{aligned}$$

¶We introduce the notation $[dz] \equiv \sqrt{g}\, d^5 z$.

where $C_4 = \frac{6}{\pi^2}$ is the normalization factor (4.3.31) of the bulk–to–boundary propagator. The tensorial structures in I_H immediately trivialize:

$$I_H = \left(1-\frac{5}{2}\right)^2 \int [dz][dw]\partial_\mu \tilde{K}_4 \partial^\mu \tilde{K}_4 \, H(u) \, \partial_{\mu'} \tilde{K}_4 \partial^{\mu'} \tilde{K}_4 \tag{4.4.4}$$

$$= \left(\frac{9}{4}\right) \int [dz][dw] \tilde{K}_4 \tilde{K}_4 \, \frac{1}{4} \Box^2 H(u) \, \tilde{K}_4 \tilde{K}_4 \tag{4.4.5}$$

where we have used integration by parts and the equation of motion $\Box \tilde{K}_4 = 0$ to eliminate the derivatives on the $\tilde{K}$'s.

Now we consider I_G, and it is useful to split into 4 parts:

$$I_G = I_G^1 + I_G^2 + I_G^3 + I_G^4$$

$$I_G^1 = \int [dz][dw] \, \partial^\mu \tilde{K}_4(z,x_1) \partial^\nu \tilde{K}_4(z,x_3) \cdot \\ (\partial_\mu \partial_{\mu'} u \, \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \, \partial_\nu \partial_{\mu'} u) \, G(u) \, \partial^{\mu'} \tilde{K}_4(w,x_2) \partial^{\nu'} \tilde{K}_4(w,x_4) \tag{4.4.6}$$

$$I_G^2 = \int [dz][dw] \, \partial^\mu \tilde{K}_4(z,x_1) \partial^\nu \tilde{K}_4(z,x_3) \cdot \\ (\partial_\mu \partial_{\mu'} u \, \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \, \partial_\nu \partial_{\mu'} u) \, G(u) \left(-\frac{1}{2} g^{\mu'\nu'} \partial_{\lambda'} \tilde{K}_4(w,x_2) \partial^{\lambda'} \tilde{K}_4(w,x_4)\right) \tag{4.4.7}$$

$$I_G^3 = \int [dz][dw] \left(-\frac{1}{2} g^{\mu\nu} \partial_\lambda \tilde{K}_4(z,x_1) \partial^\lambda \tilde{K}_4(z,x_3)\right) \cdot \\ (\partial_\mu \partial_{\mu'} u \, \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \, \partial_\nu \partial_{\mu'} u) \, G(u) \partial^{\mu'} \tilde{K}_4(w,x_2) \partial^{\nu'} \tilde{K}_4(w,x_4) \tag{4.4.8}$$

$$I_G^4 = \int [dz][dw] \left(-\frac{1}{2} g^{\mu\nu} \partial_\lambda \tilde{K}_4(z,x_1) \partial^\lambda \tilde{K}_4(z,x_3)\right) \cdot \\ (\partial_\mu \partial_{\mu'} u \, \partial_\nu \partial_{\nu'} u + \partial_\mu \partial_{\nu'} u \, \partial_\nu \partial_{\mu'} u) \, G(u) \left(-\frac{1}{2} g^{\mu'\nu'} \partial_{\lambda'} \tilde{K}_4(w,x_2) \partial^{\lambda'} \tilde{K}_4(w,x_4)\right) \tag{4.4.9}$$

We wish to eliminate all the tensor indices and all the derivatives, so that the graviton exchange is reduced to a sum of effective scalar graphs. With this program in mind, we observe a few pretty identities. First:

$$\partial^\mu \tilde{K}_\Delta(z,x_1) \, \partial_\mu \partial_{\nu'} u \, \partial^{\nu'} \tilde{K}_\Delta(w,x_2) = \\ \Delta^2 \Big[-\tilde{K}_\Delta(z,x_1) \tilde{K}_{\Delta+1}(w,x_2) \tilde{K}_{-1}(z,x_2) - \tilde{K}_{\Delta+1}(z,x_1) \tilde{K}_\Delta(w,x_2) \tilde{K}_{-1}(w,x_1) \\ +2x_{12}^2 \tilde{K}_{\Delta+1}(z,x_1) \tilde{K}_{\Delta+1}(w,x_2) + (1+u) \tilde{K}_\Delta(z,x_1) \tilde{K}_\Delta(w,x_2)\Big] \tag{4.4.10}$$

It is simplest to verify this identity by the methods described in [91], where one uses conformal transformations to go to a coordinate system where point x_1 is mapped to infinity and point x_2 to zero. Further:

$$\tilde{K}_{\Delta+1}(z,x_1)\,\tilde{K}_{-1}(w,x_1) = \frac{1}{\Delta}\partial^{\mu}\tilde{K}_{\Delta}(z,x_1)\partial_{\mu}u + (1+u)\tilde{K}_{\Delta}(z,x_1) \tag{4.4.11}$$

Inserting twice (4.4.11) into (4.4.10) we get:

$$\begin{aligned}
&\partial^{\mu}\tilde{K}_{\Delta}(z,x_1)\,\partial_{\mu}\partial_{\nu'}u\,\partial^{\nu'}\tilde{K}_{\Delta}(w,x_2) = \\
&\Delta^2\Big[-\frac{1}{\Delta}\tilde{K}_{\Delta}(z,x_1)\,\partial^{\mu'}\tilde{K}_{\Delta}(w,x_2)\,\partial_{\mu'}u - \frac{1}{\Delta}\partial^{\mu}\tilde{K}_{\Delta}(z,x_1)\,\partial_{\mu}u\,\tilde{K}_{\Delta}(w,x_2) \\
&+2x_{12}^2\,\tilde{K}_{\Delta+1}(z,x_1)\,\tilde{K}_{\Delta+1}(w,x_2) - (1+u)\tilde{K}_{\Delta}(z,x_1)\,\tilde{K}_{\Delta}(w,x_2)\Big]
\end{aligned} \tag{4.4.12}$$

We now evaluate (4.4.6–4.4.9) one by one.

I_G^1

Writing (4.4.6) as

$$\begin{aligned}
I_G^1 = \int [dz][dw]\; &\left(\partial^{\mu}\tilde{K}_4(z,x_1)\,\partial_{\mu}\partial_{\mu'}u\,\partial^{\mu'}\tilde{K}_4(w,x_2)\right)\,G(u)\,\times \\
&\times\left(\partial^{\nu}\tilde{K}_4(z,x_3)\,\partial_{\nu}\partial_{\nu'}u\,\partial^{\nu'}\tilde{K}_4(w,x_4)\right) + \{x_1 \leftrightarrow x_3\}
\end{aligned} \tag{4.4.13}$$

and inserting twice (4.4.12) for $\Delta = 4$ we obtain $16+16$ terms many of which are related by a simple symmetrization. Below we present the manipulations performed on the inequivalent terms. We often suppress the coordinate labels, and give the expressions with the propagators in the following order: $(z,x_1)\,,(z,x_3)\,,(w,x_2)\,,(w,x_1)$ unless stated otherwise. Referring to the terms in (4.4.12) we get:

I × I:

$$\mathrm{I}\times\mathrm{I} \;=\; 4^2\int[dz\,dw]\tilde{K}_4\tilde{K}_4\,G\,\partial^{\mu'}\tilde{K}_4\,\partial_{\mu'}u\,\partial^{\nu'}\tilde{K}_4\,\partial_{\nu'}u = \tag{4.4.14}$$

$$
\begin{aligned}
&= 4^2 \int [dz\,dw]\tilde{K}_4\tilde{K}_4 \left[D_{\mu'}\partial_{\nu'} \int\int^u G - g_{\mu'\nu'}(1+u)\int^u G\right] \times \\
&\quad \times \left(T^{\mu'\nu'} + \frac{1}{2} g^{\mu'\nu'}\partial^{\lambda'}\tilde{K}_4\,\partial_{\lambda'}\tilde{K}_4\right) \\
&= 4^2 \int [dz\,dw]\tilde{K}_4\tilde{K}_4 \left[\frac{1}{4}\Box^2 \int\int^u G - \frac{1}{2}\Box\left\{(1+u)\int^u G\right\}\right]\tilde{K}_4\tilde{K}_4
\end{aligned}
$$

Here we have used[||] $G(u)\,\partial_{\mu'}\partial_{\nu'}u = [D_{\mu'}\partial_{\nu'}\int\int^u G - g_{\mu'\nu'}(1+u)\int^u G]$ thanks to (4.3.25) and we also used the conservation of the stress-energy tensor integrating by parts to get the last equality.

I × II:

$$
\mathrm{I}\times\mathrm{II} = 4^2 \int [dz\,dw]\tilde{K}_4\,\partial^{\mu}\tilde{K}_4\partial_{\mu}u\,G\,\partial^{\mu'}\tilde{K}_4\partial_{\mu'}u\,\tilde{K}_4 \tag{4.4.15}
$$

Using $\partial_\mu u\,G(u) = \partial_\mu \int^u G$ we get by integration by parts:

$$
\begin{aligned}
\mathrm{I}\times\mathrm{II} &= -4^2 \int [dz\,dw]\tilde{K}_4\,\partial^{\mu}\tilde{K}_4\partial_{\mu}u \int^u G\,\partial^{\mu'}\tilde{K}_4\,\partial_{\mu'}\tilde{K}_4 \\
&\quad -4^2 \int [dz\,dw]\tilde{K}_4\,\partial^{\mu}\tilde{K}_4 \int^u G\,\partial_{\mu}\partial_{\mu'}u\,\partial^{\mu'}\tilde{K}_4\,\tilde{K}_4
\end{aligned} \tag{4.4.16}
$$

where we have used $\Box\tilde{K}_4 = 0$ in the bulk. The first term in (4.4.16) can be easily processed to give

$$
4^2 \int [dz\,dw]\tilde{K}_4\tilde{K}_4\,\frac{1}{4}\Box^2 \int\int^u G\,\tilde{K}_4\tilde{K}_4 \tag{4.4.17}
$$

The second term in (4.4.16) is handled by inserting again the identity (4.4.12) with ($x_1 \leftrightarrow x_3$) and going through by now familiar manipulations. It gives

$$
\begin{aligned}
&\int [dz\,dw]\tilde{K}_4\,\tilde{K}_4 \left[-4^3\Box \int\int^u G + 4^4(1+u)\int^u G\right]\tilde{K}_4\,\tilde{K}_4 - \\
&\qquad -2\cdot 4^4 x_{32}^2 \int\int [dz\,dw]\tilde{K}_4\,\tilde{K}_5 \int^u G\,\tilde{K}_5\,\tilde{K}_4
\end{aligned} \tag{4.4.18}
$$

[||]Here our convention is that $\int^u F = \int_a^u F(u)\,dw$, where a is chosen to ensure the fastest possible falloff of $\int^u F$ in the $u \to \infty$ limit.

I × III: Upon integration by parts,

$$\begin{aligned}
\mathrm{I}\times\mathrm{III} &= 2\cdot 4^3 x_{34}^2 \int [dz\,dw]\tilde{K}_4\,\tilde{K}_5 \int^u G\,\partial^{\mu'}\tilde{K}_4\,\partial_{\mu'}\tilde{K}_5 \\
&= 2\cdot 4^3 x_{34}^2 \int [dz\,dw]\tilde{K}_4\,\tilde{K}_5 \int^u G\left[\frac{1}{2}\Box\left(\tilde{K}_4\,\tilde{K}_5\right) - \frac{m_5^2}{2}\tilde{K}_4\,\tilde{K}_5\right] \\
&= 2\cdot 4^3 x_{34}^2 \int [dz\,dw]\tilde{K}_4\,\tilde{K}_5 \left(\frac{1}{2}\Box\int^u G - \frac{5}{2}\int^u G\right)\tilde{K}_4\,\tilde{K}_5
\end{aligned} \tag{4.4.19}$$

I × IV:

$$-4^3\int [dz\,dw]\tilde{K}_4\,\tilde{K}_4\,\frac{1}{2}\Box\int^u ((1+u)G)\,\tilde{K}_4\,\tilde{K}_4 \tag{4.4.20}$$

III × III:

$$4\cdot 4^4 x_{12}^2 x_{34}^2 \int [dz\,dw]\tilde{K}_5\,\tilde{K}_5\,G\tilde{K}_5\,\tilde{K}_5 \tag{4.4.21}$$

III × IV:

$$-2\cdot 4^4 x_{12}^2 \int [dz\,dw]\tilde{K}_5\,\tilde{K}_4\,G(1+u)\tilde{K}_5\,\tilde{K}_4 \tag{4.4.22}$$

IV × IV:

$$4^4\int [dz\,dw]\tilde{K}_4\,\tilde{K}_4\,G(1+u)^2\tilde{K}_4\,\tilde{K}_4 \tag{4.4.23}$$

I_G^2, I_G^3 and I_G^4

Using (4.3.24), after some similar algebra we arrive at

$$I_G^2 = I_G^3 = -\int [dz\,dw]\tilde{K}_4\,\tilde{K}_4\,\frac{1}{4}\Box^2\left[G + \frac{3}{2}(1+u)\int^u G + \frac{1}{2}u(u+2)G\right]\tilde{K}_4\,\tilde{K}_4 \tag{4.4.24}$$

$$I_G^4 = \frac{1}{2}\int [dz\,dw]\tilde{K}_4\,\tilde{K}_4\,\frac{1}{4}\Box^2\left[5G + u(u+2)G\right]\tilde{K}_4\,\tilde{K}_4 \tag{4.4.25}$$

The graviton amplitude in terms of scalar exchanges

Adding all the terms above with the appropriate symmetrizations we get the complete graviton graph in terms of effective scalar exchanges:

$$\frac{I_{\text{grav}}}{(C_4)^4} = \int [dz\,dw]\tilde{K}_4\,\tilde{K}_4\left[\Box^2\left(\frac{9}{16}H + 2\cdot 4^2\int\int^u G + \frac{1}{8}G - \frac{3}{4}(1+u)\int^u G - \frac{1}{8}u(u+2)G\right)\right.$$

$$+\Box\left(-4^4\int^u((1+u)G)-\frac{1}{2}4^3(1+u)\int^u G-4^4\int\int^u G)\right)$$
$$+4^5(1+u)\int^u G+2\cdot 4^4(1+u)^2G\Big]\tilde{K}_4\tilde{K}_4 \qquad (4.4.26)$$
$$+x_{34}^2\int[dz\,dw]\tilde{K}_4\tilde{K}_5\left[-18\cdot 4^3\int^u G+2\cdot 4^3\Box\int^u G-2\cdot 4^4G(1+u)\right]\tilde{K}_4\tilde{K}_5$$
$$+\{3\text{ perms}\}$$
$$+(x_{12}^2x_{34}^2+x_{14}^2x_{23}^2)\int[dz\,dw]\tilde{K}_5\tilde{K}_5\,4\cdot 4^4G\,\tilde{K}_5\tilde{K}_5$$

where the 3 permutations of the second integral are obtained by exchanging $(x_1,x_2)\leftrightarrow(x_3,x_4)$ and $x_1\leftrightarrow x_3$. The formula above can be simplified by explicit application of Laplace operator (4.3.29) and using the equations obeyed by G and H given in Section 3.1. We get

$$\frac{I_{\text{grav}}}{(C_4)^4}=\int[dz\,dw]\tilde{K}_4\tilde{K}_4\left[(-72u^2-144u+168)\,G+168\,(u+1)\int^u G\right]\tilde{K}_4\tilde{K}_4$$
$$+x_{34}^2\int[dz\,dw]\tilde{K}_4\tilde{K}_5\left[-768\int^u G-256\,G(1+u)\right]\tilde{K}_4\tilde{K}_5$$
$$+\{3\text{ perms}\}$$
$$+(x_{12}^2x_{34}^2+x_{14}^2x_{23}^2)\int[dz\,dw]\tilde{K}_5\tilde{K}_5\,1024\,G\,\tilde{K}_5\tilde{K}_5$$
$$+10\int[dw]\tilde{K}_4\tilde{K}_4\tilde{K}_4\tilde{K}_4-16\,x_{24}^2\int[dw]\tilde{K}_4\tilde{K}_4\tilde{K}_5\tilde{K}_5 \qquad (4.4.27)$$

The last two terms in this expression arise from delta functions generated by the application of the Laplace operator**. In particular the last term comes from:

$$\int[dz\,dw]\tilde{K}_4\tilde{K}_4\,\Box\delta(z,w)\,\tilde{K}_4\tilde{K}_4$$
$$=2\int[dw]\tilde{K}_4\tilde{K}_4\,\partial_{\mu'}\tilde{K}_4\partial^{\mu'}\tilde{K}_4=2\int[dw]\tilde{K}_4\tilde{K}_4\left(16\tilde{K}_4\tilde{K}_4-32x_{24}^2\tilde{K}_5\tilde{K}_5\right) \qquad (4.4.28)$$

where in the last equality we have used (B.2.5).

**The coordinate dependence of the K's is: $(w,x_1),(w,x_3),(w,x_2),(w,x_4)$.

4.4.2 Reduction to quartic graphs

We first observe the identity

$$\int^{u} G = -G_3 + (1+u)G\,, \tag{4.4.29}$$

where G_3 is a scalar propagator of $m^2 = -3$, corresponding to a boundary conformal dimension $\Delta = 3$:

$$-(\Box + 3)G_3 = \delta(z,w)\,. \tag{4.4.30}$$

Using (4.4.29), we see that the complete graviton graph (4.4.27) involves effective scalar exchanges of the form

$$I^{\Delta_5,p}_{\Delta_1\Delta_3\Delta_2\Delta_4} \equiv \int [dz\,dw]\tilde{K}_{\Delta_1}\,\tilde{K}_{\Delta_3}\,(1+u)^p G_{\Delta_5}\,\tilde{K}_{\Delta_2}\,\tilde{K}_{\Delta_4}\,, \tag{4.4.31}$$

plus some quartic interactions (last line of (4.4.27)).

We now proceed to derive a general formula to perform the z integration in (4.4.31), following the methods developed in [97]. Quite remarkably, upon integration over z, (4.4.31) reduces to a finite sum of effective quartic graphs, see Figure 6.

Translating $x_1 \to 0$ and performing conformal inversion (see [91] for a detailed account), we can write

$$I^{\Delta_5,p}_{\Delta_1\Delta_3\Delta_2\Delta_4} = |x_{31}|^{-2\Delta_3}|x_{21}|^{-2\Delta_2}|x_{41}|^{-2\Delta_4}\int [dw]R(w - x'_{31})\tilde{K}_{\Delta_2}(w, x'_{21})\,\tilde{K}_{\Delta_4}(w, x'_{41})\,, \tag{4.4.32}$$

where $\vec{x}' \equiv \vec{x}/x^2$ and

$$R^{\Delta_5,p}_{\Delta_1,\Delta_3}(w) = \int [dz] z_0^{\Delta_1}\tilde{K}_{\Delta_3}(z)(1+u)^p G_{\Delta_5}(u)\,. \tag{4.4.33}$$

As compared to [97] we allow the bulk propagator to be multiplied by $(1+u)^p$ (see (3.3–3.4) in [97]). We now use the hypergeometric series expansion (4.3.33). Inserting this series into the expression for $R^{\Delta_5,p}_{\Delta_1,\Delta_3}$, we can perform the z integral term by term by a standard

Feynman parameterization, and resum the resulting series. We get

$$
\begin{aligned}
R^{\Delta_5,p}_{\Delta_1,\Delta_3}(w) &= 2^{\Delta_5-p+1}\tilde{C}_{\Delta_5}\pi^{d/2}\frac{\Gamma[\frac{1}{2}(\Delta_5-p+\Delta_3-\Delta_1)]\Gamma[\frac{1}{2}(\Delta_5-p+\Delta_1+\Delta_3-d)]}{\Gamma[\Delta-p]\Gamma[\Delta_3]} \\
&\times\left(\frac{w_0}{w^2}\right)^{\Delta_3} w_0^{\Delta_1-\Delta_3}\int_0^1 d\gamma\frac{(1-\gamma)^{\Delta_3-1}\gamma^{\frac{1}{2}(\Delta_5-p-\Delta_1-\Delta_3)-1}}{\left(\left(\frac{w_0}{w^2}\right)+\gamma-\left(\frac{w_0}{w^2}\right)\gamma\right)^{\Delta_1}} \\
&\times {}_4F_3\left(\tfrac{\Delta_5}{2},\tfrac{\Delta_5+1}{2},\tfrac{\Delta_5-p+\Delta_3-\Delta_1}{2},\tfrac{\Delta_5-p+\Delta_1+\Delta_3-d}{2};\Delta_5-\tfrac{d}{2}+1,\tfrac{\Delta_5-p}{2},\tfrac{\Delta_5-p+1}{2};\gamma\right)
\end{aligned} \tag{4.4.34}
$$

For $p=0$ we recover equation (3.11) in [97]. It turns out that for the cases relevant to the graviton amplitude, the hypergeometric function ${}_4F_3$ is elementary and the Feynman parameter integral can be explicitly done. The result is always a simple binomial in w_0 and w_0/w^2. The relevant cases are:

$$
\begin{aligned}
R^{4,0}_{4,4} &= \frac{1}{36}w_0^3\left(\frac{w_0}{w^2}\right)^3+\frac{1}{48}w_0^2\left(\frac{w_0}{w^2}\right)^2 \\
R^{4,2}_{4,4} &= \frac{1}{36}w_0^3\left(\frac{w_0}{w^2}\right)^3+\frac{7}{288}w_0^2\left(\frac{w_0}{w^2}\right)^2+\frac{1}{48}w_0\left(\frac{w_0}{w^2}\right) \\
R^{3,1}_{4,4} &= \frac{1}{36}w_0^3\left(\frac{w_0}{w^2}\right)^3+\frac{1}{36}w_0^2\left(\frac{w_0}{w^2}\right)^2+\frac{1}{36}w_0\left(\frac{w_0}{w^2}\right) \\
R^{3,0}_{4,5} &= \frac{1}{48}w_0^3\left(\frac{w_0}{w^2}\right)^4+\frac{1}{48}w_0^2\left(\frac{w_0}{w^2}\right)^3+\frac{1}{48}w_0\left(\frac{w_0}{w^2}\right)^2 \\
R^{4,1}_{4,5} &= \frac{1}{48}w_0^3\left(\frac{w_0}{w^2}\right)^4+\frac{11}{576}w_0^2\left(\frac{w_0}{w^2}\right)^3+\frac{1}{64}w_0\left(\frac{w_0}{w^2}\right)^2 \\
R^{4,0}_{5,5} &= \frac{1}{64}w_0^4\left(\frac{w_0}{w^2}\right)^4+\frac{1}{72}w_0^3\left(\frac{w_0}{w^2}\right)^3+\frac{1}{96}w_0^2\left(\frac{w_0}{w^2}\right)^2
\end{aligned} \tag{4.4.35}
$$

We see that each term in $R^{\Delta_5,p}_{\Delta_1,\Delta_3}(w)$ is a of product of bulk–to–boundary propagators. Indeed, w_0^{Δ} corresponds in this inverted frame to a propagator at $\vec{x}'=\infty$, likewise $(w_0/w^2)^{\tilde{\Delta}}$ corresponds to a propagator at $\vec{x}'=0$. Inserting each such term in the expression for $I^{\Delta_5,p}_{\Delta_1\Delta_3\Delta_2\Delta_4}$ (equ. 4.4.32)), and going back from the inverted variables $\vec{x}'_i$ to the original variables $\vec{x}_i$, we recognize the integral defining a quartic graph. For example

$$
|x_{31}|^{-2\Delta_3}|x_{21}|^{-2\Delta_2}|x_{41}|^{-2\Delta_4}\int[dw]w_0^{2\Delta}\left(\frac{w_0}{(w-x'_{31})^2}\right)^{\tilde{\Delta}}\tilde{K}_{\Delta_2}(w,x'_{21})\,\tilde{K}_{\Delta_4}(w,x'_{41})
$$

$$= \int [dw] \left(\frac{w_0}{(w-x_1)^2}\right)^{\Delta} \left(\frac{w_0}{(w-x_3)^2}\right)^{\tilde{\Delta}} \left(\frac{w_0}{(w-x_2)^2}\right)^{\Delta_2} \left(\frac{w_0}{(w-x_4)^2}\right)^{\Delta_4}$$
$$\equiv D_{\Delta\tilde{\Delta}\Delta_2\Delta_4}(x_1, x_3, x_2, x_4)\,, \tag{4.4.36}$$

where in the last line we have used the important notation for quartic graphs (see Figure 5) introduced in (B.1). We can finally write the full graviton amplitude as a sum of quartic graphs:

$$\begin{aligned} I_{\text{grav}} \;=\; & \left(\frac{6}{\pi^2}\right)^4 \left[16\left(\frac{x_{12}^2\,x_{34}^2}{x_{13}^2} + \frac{x_{14}^2\,x_{23}^2}{x_{13}^2} - x_{24}^2\right) D_{4455} + \frac{128}{9\,x_{13}^4}\left(x_{12}^2\,x_{34}^2 + x_{14}^2\,x_{23}^2\right) D_{3355}\right. \\ & + \frac{32}{3\,x_{13}^6}\left(x_{12}^2\,x_{34}^2 + x_{14}^2\,x_{23}^2\right) D_{2255} + 10\,D_{4444} + \frac{14}{3\,x_{13}^2}\,D_{3344} + \frac{8}{3\,x_{13}^4}\,D_{2244} \\ & - \frac{8}{3\,x_{13}^6}\,D_{1144} - \frac{16}{3\,x_{13}^2}\left(x_{12}^2\,D_{4354} + x_{14}^2\,D_{4345} + x_{34}^2\,D_{3445} + x_{23}^2\,D_{3454}\right) \\ & \left. - \frac{32}{9\,x_{13}^4}\left(x_{12}^2\,D_{3254} + x_{14}^2\,D_{3245} + x_{34}^2\,D_{2345} + x_{23}^2\,D_{2354}\right)\right]. \end{aligned} \tag{4.4.37}$$

The graviton amplitude (4.3.39) is symmetric under $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$. These symmetries are explicit in the final expression for I_{grav}, indeed some of the D functions (of the form $D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}$) are symmetric by themselves, while asymmetric D functions appear in all the symmetric permutations. It turns out that thanks to the remarkable properties of the D functions (see equ. (B.3.11)), the answer can be rewritten in terms of $D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}$'s alone. Introducing the conformal invariant variable $s \equiv \frac{1}{2}\frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2}$, we get

$$\begin{aligned} I_{\text{grav}} \;=\; & \left(\frac{6}{\pi^2}\right)^4 \left[16\,x_{24}^2\left(\frac{1}{2s} - 1\right) D_{4455} + \frac{64}{9}\frac{x_{24}^2}{x_{13}^2}\frac{1}{s}\,D_{3355} + \frac{16}{3}\frac{x_{24}^2}{x_{13}^4}\frac{1}{s}\,D_{2255}\right. \\ & \left. + 18\,D_{4444} - \frac{46}{9\,x_{13}^2}\,D_{3344} - \frac{40}{9\,x_{13}^4}\,D_{2244} - \frac{8}{3\,x_{13}^6}\,D_{1144}\right]. \end{aligned} \tag{4.4.38}$$

The graviton amplitude (4.3.39) is, for the case $\Delta = \Delta' = 4$ that we are considering, also symmetric under $(x_1, x_3) \leftrightarrow (x_2, x_4)$. Although not immediately manifest in the expression above, this symmetry is actually present thanks to the identity (B.3.11) obeyed by the D functions.

4.5 General graviton exchange graph

We expect that the amplitudes for graviton exchange between massive scalars will be useful in general studies of the AdS/CFT correspondence. As in past work [95, 97] we therefore assume initially that d, Δ, and Δ' are arbitrary, constrained only by the unitarity bound $\Delta,\ \Delta' \geq d/2$. We will assume integer values at the point where this step simplifies the calculation, and specialize still later to the case $d = \Delta = \Delta' = 4$ to present detailed asymptotic formulas for dilatons and axions in the type IIB supergravity.

The first step in the evaluation of the amplitude (4.3.39) is to split it into contributions from the terms in $H(u)$ and $G(u)$ in the graviton propagator, and to split the latter into a term proportional to the metric $g^{\mu\nu}$ in $T_{13}^{\mu\nu}(z)$ of (4.3.40) plus the remaining term, viz.

$$I_{\text{grav}} = \frac{1}{4} A_{\text{grav}} = \frac{1}{4}(A^H + A_S^G + A_T^G). \tag{4.5.1}$$

The three contributions are then given by

$$\begin{aligned}
A_S^G &= \int dz\sqrt{g}\int dw\sqrt{g}[\partial_\rho K(1)D^\rho K(3) + m^2 K(1)K(3)](z)\ I_{\mu'\nu'}(z,w)\ T_{24}^{\mu'\nu'}(w) && (4.5.2)\\
A_T^G &= 2\int dz\sqrt{g}\int dw\sqrt{g}\partial_\mu K(1)\partial_\nu K(3)\ D^\mu\partial_{\mu'}uD^\nu\partial_{\nu'}u\ G(u)\ T_{24}^{\mu'\nu'}(w) + (1\leftrightarrow 3) && (4.5.3)\\
A^H &= \int dz\sqrt{g}\int dw\sqrt{g}\ g\cdot T_{13}(z)\ H(u)\ g\cdot T_{24}(w) && (4.5.4)
\end{aligned}$$

where we use the abbreviation $g\cdot T = g_{\mu\nu}T^{\mu\nu}$, and

$$\begin{aligned}
I_{\mu'\nu'} &\equiv -g_{\mu\nu}G(u)[D^\mu\partial_{\mu'}uD^\nu\partial_{\nu'}u + D^\mu\partial_{\nu'}uD^\nu\partial_{\mu'}u] \\
&= -2G(u)(g_{\mu'\nu'} + \partial_{\mu'}u\partial_{\nu'}u)\,. && (4.5.5)
\end{aligned}$$

where (4.3.27) is used to obtain the second line in (4.5.5). The symmetrization in $1\leftrightarrow 3$ in (4.5.3) will be useful for later steps.

The w-integral in A_G^S that involves the tensor $\partial_{\mu'}u\partial_{\nu'}u$ of (4.5.5) may be simplified by using $\partial_{\mu'}uG(u) = \partial_{\mu'}\bar{G}(u)$, integrating by parts in w and using the covariant conservation of T_{24},

$$\int dw\sqrt{g}G(u)\partial_{\mu'}u\partial_{\nu'}uT_{24}^{\mu'\nu'}(w) = -\int dw\sqrt{g}\,\bar{G}(u)\,D_{\mu'}\partial_{\nu'}u\,T_{24}^{\mu'\nu'}(w)$$
$$= -\int dw\sqrt{g}\,(1+u)\bar{G}(u)\,g\cdot T_{24}(w) \qquad (4.5.6)$$

Putting together this rearrangement of the A_S^G part, we have

$$A_S^G = \int dz\sqrt{g}\int dw\sqrt{g}[\partial_\rho K(1)D^\rho K(3)+m^2K(1)K(3)](z)$$
$$\times\{-2G(u)+2(1+u)\bar{G}(u)\}\;g\cdot T_{24}(w) \qquad (4.5.7)$$

Next, we use the propagator equations $(\Box - m^2)K(1) = (\Box - m^2)K(3) = 0$ to obtain the following identity

$$[\partial_\rho K(1)D^\rho K(3)+m^2K(1)K(3)](z) = \frac{1}{2}\Box_z\{K(1)K(3)\}(z) \qquad (4.5.8)$$

Substituting this identity into A_S^G, integrating by parts the operator $\Box_z$, neglecting vanishing boundary terms and using (4.3.29), we find

$$\Box_z\{(1+u)\bar{G}(u)\} = -2G(u)+4(1+u)^2G(u)+2d(1+u)\bar{G}(u) \qquad (4.5.9)$$

which then gives

$$A_S^G = \int dz\sqrt{g}\int dw\sqrt{g}K(1)K(3)\{-\Box_z G(u)-2G(u)+4(1+u)^2G(u)$$
$$+2d(1+u)\bar{G}(u)\}\;g\cdot T_{24}(w) \qquad (4.5.10)$$

Before simplifying the $g\cdot T_{24}$ factor in the integrand, we first treat A_T^G and A^H in a similar manner. For A^H, we use again (4.5.8) to simplify the z-integration and to cast it in the following form

$$A^H = \int dz\sqrt{g}\int dw\sqrt{g}\ K(1)K(3)\{-\frac{1}{2}(d-1)\Box_z H(u) - 2m^2 H(u)\}\ g\cdot T_{24}(w) \quad (4.5.11)$$

To simplify A^G_T, we begin with partial integration of ∂_ν in the z-integral in (4.5.3), and split A^G_T as follows

$$A^G_T = -2A^G_{T1} - 2A^G_{T2} \quad (4.5.12)$$

where

$$A^G_{T1} = \int dz\sqrt{g}\int dw\sqrt{g}\partial_\mu\{K(1)K(3)\}D_\nu[D^\mu\partial_{\mu'}uD^\nu\partial_{\nu'}uG(u)]\ T^{\mu'\nu'}_{24}(w) \quad (4.5.13)$$

$$A^G_{T2} = \int dz\sqrt{g}\int dw\sqrt{g}D_\nu\partial_\mu K(1)K(3)[D^\mu\partial_{\mu'}uD^\nu\partial_{\nu'}uG(u)]T^{\mu'\nu'}_{24}(w) + (1\leftrightarrow 3)\,. \quad (4.5.14)$$

Now, A^G_{T1} may be simplified by working out the tensor algebra using (4.3.23–4.3.27) and again $\partial_{\nu'}uG(u) = \partial_{\nu'}\bar{G}(u)$ to obtain

$$\begin{aligned} D_\nu[D^\mu\partial_{\mu'}uD^\nu\partial_{\nu'}uG(u)] &= D_{\mu'}(\ \ldots) + D_{\nu'}(\ \ldots) - D^\mu u\ g_{\mu'\nu'}J(u) \\ J(u) &= (1+u)G(u) + (d+1)\bar{G}(u) \end{aligned} \quad (4.5.15)$$

The terms with $D_{\mu'}$ and $D_{\nu'}$ cancel by partial integration in (4.5.13) by conservation of T_{24} Finally, integrating by parts once more in ∂_μ and using $D^\mu u\ J(u) = D^\mu \int^u J$, we get the following simple result for A^G_{T1},

$$\begin{aligned} A^G_{T1} &= -\int dz\sqrt{g}\int dw\sqrt{g}\partial_\mu\{K(1)K(3)\}D^\mu u\ J(u)\ g\cdot T_{24}(w) \quad (4.5.16) \\ &= \int dz\sqrt{g}\int dw\sqrt{g}K(1)K(3)\{u(2+u)J'(u) + (d+1)(1+u)J(u)\}\ g\cdot T_{24}(w) \end{aligned}$$

It is more difficult to deal with A^G_{T2} To simplify the integral representation in (4.5.14), it is very convenient to set $x_1 = 0$ in the first term and then perform an inversion transformation of the integral (in z and w) as explained in [91]. The symmetric step in $1 \leftrightarrow 3$ is done later. It is now easy to evaluate the double covariant derivative of the inverted bulk-to-boundary propagator $K(1') = C_\Delta z_0^\Delta$,

$$D_\nu \partial_\mu K(1') = -\Delta K(1') g_{\mu\nu} + \Delta(\Delta+1) K(1') z_0^2 g_{\mu 0} g_{\nu 0} \tag{4.5.17}$$

The contribution of the first term in (4.5.17) is proportional to the metric $g_{\mu\nu}$, and may be treated by the same technique used for A_S^G. It acquires an "effective scalar propagator" proportional to the term in $\{\ldots\}$ in (4.5.7). We thus find for this contribution to A_{T2}^G the term

$$-|x'_{21}|^{2\Delta'}|x'_{31}|^{2\Delta}|x'_{41}|^{2\Delta'}\Delta \int dz\sqrt{g}\int dw\sqrt{g} K(1')K(3')\{G(u)-(1+u)\bar{G}(u)\}\, g\cdot T_{24}(w) + (1\leftrightarrow 3)\,. \tag{4.5.18}$$

Note that the prefactor contains the scale factors from the inversion.

The integral of the second term in (4.5.17) contains the factor.

$$z_0^2 g_{\mu 0} g_{\nu 0} D^\mu \partial_{\mu'} u \; D^\nu \partial_{\nu'} u = (z_0 g_{\mu' 0'} + \partial_{\mu'} u)(z_0 g_{\nu' 0'} + \partial_{\nu'} u) \tag{4.5.19}$$

Integration in w against $G(u) T_{24}^{\mu'\nu'}(w)$ gives rise to three types of terms

$$\begin{aligned}
&\int dw\sqrt{g}(z_0 g_{\mu' 0'} + \partial_{\mu'} u)(z_0 g_{\nu' 0'} + \partial_{\nu'} u) G(u) T_{24}^{\mu'\nu'}(w) \\
&= z_0^2 \int dw\sqrt{g}\; G(u) T_{24}(w)_{0'0'} + 2z_0 \int dw\sqrt{g}\; g_{\mu'0'}\partial_{\nu'} u\; G(u)\; T_{24}^{\mu'\nu'}(w) \\
&\quad - \int dw\sqrt{g}\; (1+u)\bar{G}(u)\; g\cdot T_{24}(w).
\end{aligned} \tag{4.5.20}$$

The second integral on the right hand side may be further simplified by using once more $\partial_{\nu'} u G(u) = \partial_{\nu'} \bar{G}(u)$, integrating by parts, using conservation of T_{24} and being careful to taking into account the fact that the integral is the $0'$ component of a vector instead of a scalar. Thus there is a non-vanishing contribution of Christoffel symbols, which gives

$$\int dw\sqrt{g}\; g_{\mu'0'}\partial_{\nu'} u\; G(u)\; T_{24}^{\mu'\nu'}(w) = \int dw\sqrt{g}\; \frac{1}{w_0}\bar{G}(u)\; g\cdot T_{24}(w) \tag{4.5.21}$$

We now combine (4.5.18),(4.5.20) and (4.5.21) to write an expression for A^G_{T2}, namely

$$\begin{aligned}
A^G_{T2} &= |x'_{21}|^{2\Delta'}|x'_{31}|^{2\Delta}|x'_{41}|^{2\Delta'} \int dz\sqrt{g}\int dw\sqrt{g}K(1')K(3') \\
&\quad \Big\{\Big[-\Delta G(u) - \Delta^2(1+u)\bar{G}(u) + \Delta(\Delta+1)\frac{2z_0}{w_0}\bar{G}(u)\Big]\ g\cdot T_{24}(w) \\
&\quad +\Delta(\Delta+1)\ z_0^2 G(u)T_{24}(w)_{0'0'}\Big\} \\
&\quad +(1\leftrightarrow 3)
\end{aligned} \tag{4.5.22}$$

4.5.1 Final simplified form

We are now in a position to assemble all contributions to the graviton exchange diagram by combining results for A^H, A^G_S, A^G_{T1} and A^G_{T2}. The z-integrals are easiest to carry out after inversion, so we apply inversion to all contributions and rewrite A_{grav} with a universal conformal factor extracted, viz.

$$A_{\mathrm{grav}} = |x'_{21}|^{2\Delta'}|x'_{31}|^{2\Delta}|x'_{41}|^{2\Delta'}(B^{tt}+B^{dd}+B^{00}) + (1\leftrightarrow 3) \tag{4.5.23}$$

where the reduced amplitudes B are given by

$$B^{tt} = \int dz\sqrt{g}\int dw\sqrt{g}K(1')K(3')P(u)\ g\cdot T_{24}(w) \tag{4.5.24}$$

$$B^{dd} = -4\Delta(\Delta+1)\int dz\sqrt{g}\int dw\sqrt{g}\ \frac{z_0}{w_0}K(1')K(3')\bar{G}(u)\ g\cdot T_{24}(w) \tag{4.5.25}$$

$$B^{00} = -2\Delta(\Delta+1)\int dz\sqrt{g}\int dw\sqrt{g}\ z_0^2\ K(1')K(3')\ G(u)\ T_{24}(w)_{0'0'} \tag{4.5.26}$$

The function $P(u)$ is gotten by combining all contributions involving $g\cdot T_{24}$ (except that from A^G_{T2}) and is given by

$$\begin{aligned}
P(u) &= -\frac{1}{2}\Box_z G - \frac{1}{4}(d-1)\Box_z H - G + 2(1+u)^2 G + d(1+u)\bar{G}(u) - m^2 H \\
&\quad -u(2+u)J' - (d+1)(1+u)J + 2\Delta G + 2\Delta^2(1+u)\bar{G}(u)
\end{aligned} \tag{4.5.27}$$

The relation between $H(u)$ and $G(u)$ was given in (4.3.37) and may be used to further simplify the form of $P(u)$. While both $\Box_z G$ and $\Box_z H$ have a term proportional to $\delta(z,w)$, the relative coefficients of both terms are such that this δ-functions cancels out of the full $P(u)$, and we are left with

$$\begin{aligned} P(u) &= 2\Delta G - 2u(2+u)G + 2(\Delta^2 - d - 1)(1+u)\bar{G}(u) - m^2 H(u) \\ &= 2\{\Delta + 1 + \frac{m^2 - d + 1}{d-1}(1+u)^2\}G(u) + 2\{\Delta^2 - d - 1 + \frac{m^2(d-2)}{d-1}\}(1+u)\bar{G}(u) \end{aligned}$$

Finally, the expression $T_{24}(w)_{0'0'}$ may be worked out explicitly,

$$\begin{aligned} T_{24}(w)_{0'0'} &= (\Delta')^2 K_{\Delta'}(2')K_{\Delta'}(4')\Big\{\Big(1 - \frac{(m')^2}{(\Delta')^2}\Big)\frac{1}{w_0^2} - \frac{4}{(w-x_2')^2} \\ &\quad - \frac{4}{(w-x_4')^2} + \frac{8w_0^2 + 2(x_2' - x_4')^2}{(w-x_2')^2(w-x_4')^2}\Big\}, \end{aligned} \qquad (4.5.28)$$

and we can use an identity similar to (4.5.8) to obtain a covariant expression for $g \cdot T_{24}(w)$, namely

$$g \cdot T_{24}(w) = (-\frac{1}{2}(d-1)\Box_w - 2m'^2)\{K(2')K(4')\}. \qquad (4.5.29)$$

4.5.2 General integrals over interaction points

We shall use the following strategy for the calculation of the integrals over the interaction points z and w in the reduced amplitudes of (4.5.24–4.5.26). First, we shift both z and w by x_{31}'; by translation invariance, the integrals depend only upon the new variables $x \equiv x_{41}' - x_{31}'$ and $y \equiv x_{21}' - x_{31}'$. The z-integrations then only depend upon the variable w, and may be carried out explicitly in terms of elementary functions by methods similar to the ones used in [95] and [97]. Only after the z-integrals are carried out are the explicit forms of $g \cdot T_{24}$ and $T_{24}(w)_{0'0'}$ required and used. The remaining w-integrals may be recast as integral representations that admit simple asymptotic expansions.

To prepare for the z–integrations, we note that $P(u)$ in (4.5.24)) and (4.5.28) involves the invariant function $G(u)$ and its first integral $\bar{G}(u)$, and the same functions appear in

(4.5.25,4.5.26). To apply the methods of [95] and [97] we need the series expansions of $G(u)$ and $\bar{G}(u)$ in the variable ξ of (4.3.32). For $G(u)$ this is just the hypergeometric series for $G_d(u)$ in (4.3.33) and we obtain the series for $\bar{G}(u)$ by direct integration. These expansions are given by

$$\begin{aligned} G(u) &= \frac{1}{2} C_G \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{d}{2}+\frac{1}{2})}{\Gamma(\frac{d}{2}+\frac{1}{2})\, k!} \frac{1}{k+\frac{d}{2}} \xi^{2k+d} \\ \bar{G}(u) &= -\frac{1}{4} C_G \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{d}{2}-\frac{1}{2})}{\Gamma(\frac{d}{2}+\frac{1}{2})\, k!} \frac{1}{k+\frac{d}{2}} \xi^{2k+d-1} \end{aligned} \tag{4.5.30}$$

These series expansions are uniformly convergent inside any disc $|\xi| < 1$. The normalization constant may be read off from (4.3.34) and (4.3.31) for $\Delta = d$, and we find $C_G = 2^d\, d\, \tilde{C}_d$.

There are five independent z-integrals required to evaluate the graviton exchange amplitudes. They are as follows,

$$Z_1(w) = \int dz \sqrt{g}\, K(1')K(3')G(u) \tag{4.5.31}$$

$$Z_2(w) = \int dz \sqrt{g}\, K(1')K(3')(1+u)^2 G(u) \tag{4.5.32}$$

$$Z_3(w) = \int dz \sqrt{g}\, K(1')K(3')(1+u)\bar{G}(u) \tag{4.5.33}$$

$$Z_4(w) = \int dz \sqrt{g}\, K(1')K(3') z_0 w_0^{-1} \bar{G}(u) \tag{4.5.34}$$

$$Z_5(w) = \int dz \sqrt{g}\, K(1')K(3') z_0^2 w_0^{-2} G(u) \tag{4.5.35}$$

In terms of these integrals, the original amplitudes are given by

$$\begin{aligned} B^{tt} &= \int dw \sqrt{g}\, \{2(\Delta+1)Z_1(w) + 2\frac{m^2-d+1}{d-1} Z_2(w) \\ &\quad +2(\Delta^2-d-1+\frac{m^2(d-2)}{d-1})Z_3(w)\} g \cdot T_{24}(w) \end{aligned} \tag{4.5.36}$$

$$B^{dd} = \int dw \sqrt{g}\, \{-4\Delta(\Delta+1)Z_4(w)\} g \cdot T_{24}(w) \tag{4.5.37}$$

$$B^{00} = \int dw \sqrt{g}\, \{-2\Delta(\Delta+1)Z_5(w)\} w_0^2 T_{24}(w)_{0'0'} \tag{4.5.38}$$

It remains to evaluate the z-integrals.

Performing the z-integrals

The z-integrations are carried out term by term on the series expansions of (4.5.30), and all the integrals we need in (4.5.24–4.5.26) are of the following form (with $2a,\ 2b = 0, 1$ or 2)

$$\begin{aligned}
&\int_0^\infty dz_0 \int_{\mathbf{R}^d} d^d\vec{z}\ \frac{z_0^{2\Delta+2a-d-1}}{(z_0^2+\vec{z}^2)^\Delta}\Big(\frac{2z_0w_0}{z_0^2+w_0^2+(\vec{z}-\vec{w})^2}\Big)^{2k+d-2b} \\
&= \pi^{\frac{d}{2}} \frac{\Gamma(\frac{1}{2})\Gamma(\Delta+k+a-b)\Gamma(k+\frac{d}{2}-a-b)}{\Gamma(\Delta)\Gamma(k+\frac{d}{2}-b)\Gamma(k+\frac{d}{2}-b+\frac{1}{2})}\ w_0^{2a} \\
&\times \int_0^1 d\alpha\ \alpha^{2a-1}(1-\alpha)^{\Delta-1}\Big(\frac{\alpha w_0^2}{\alpha w_0^2+(1-\alpha)w^2}\Big)^{k+\frac{d}{2}-a-b}
\end{aligned} \tag{4.5.39}$$

In the integrals $Z_j(w)$ of (4.5.31–4.5.35), the values taken by (a,b) are $(0,0)$, $(0,1)$, $(0,1)$, $(\frac{1}{2},\frac{1}{2})$ and $(1,0)$ for $j = 1,2,3,4,5$ respectively. The calculation of the z-integrals is slightly involved, but is essentially the same for each of the Z_j-integrals. Here, we shall present in detail only the calculation for Z_1, and restrict to presenting the final results for the remaining 4 integrals.

To compute $Z_1(w)$, we use the expansion of (4.5.30) for the function $G(u)$ and integrate term by term in z using the integral formula of (4.5.39), here with $a = b = 0$. Assembling these results, we notice that the factors $\Gamma(k+\frac{d}{2}+\frac{1}{2})$ and $\Gamma(k+\frac{d}{2})$ cancel between numerators and denominators. Also, interchanging the order of the α-integration of (4.5.39) and the k-sum of (4.5.30), we are left with the following result

$$\begin{aligned}
Z_1(w) &= \frac{\pi^{\frac{d}{2}}\Gamma(\frac{1}{2})}{2\Gamma(\frac{d}{2}+\frac{1}{2})} C_G C_\Delta^2 \int_0^1 \frac{d\alpha}{\alpha}(1-\alpha)^{\Delta-1} f_{\Delta;\frac{d}{2}}\Big(\frac{\alpha w_0^2}{\alpha w_0^2+(1-\alpha)w^2}\Big) \\
f_{\Delta;p}(\zeta) &= \sum_{k=0}^\infty \frac{\Gamma(k+\Delta)}{\Gamma(\Delta)\ k!}\ \frac{\zeta^{k+p}}{k+p}
\end{aligned} \tag{4.5.40}$$

Assuming that d is even and $d \geq 4$ throughout, we have $p > 1$ and the function $f_{\Delta;p}$ may be easily evaluated in terms of elementary functions. We begin by noticing that

$$f_{\Delta;p}(\zeta) = \zeta^p \Big(\frac{d}{d\zeta}\Big)^{p-1} \sum_{k=0}^{\infty} \frac{\Gamma(k+\Delta)}{\Gamma(\Delta)\Gamma(k+p+1)} \zeta^{k+p-1} \tag{4.5.41}$$

In view of the presence of the multiple derivative operation in front, we are free to add into the sum the terms with $k = -p+1, -p+2, \cdots, -1$. Then, we shift $k \to k-p$ and obtain

$$f_{\Delta;p}(\zeta) = \zeta^p \Big(\frac{d}{d\zeta}\Big)^{p-1} \sum_{k=1}^{\infty} \frac{\Gamma(k+\Delta-p)}{\Gamma(\Delta)\Gamma(k+1)} \zeta^{k-1} \tag{4.5.42}$$

The infinite sum is proportional to $\zeta^{-1}[(1-\zeta)^{-\Delta+p} - 1]$ and the multiple differentiations may be carried out explicitly. The final result is

$$f_{\Delta;p}(\zeta) = (-)^p \frac{\Gamma(p)}{\Gamma(\Delta)} \Big[\Gamma(\Delta-p) - \sum_{\ell=0}^{p-1} (-)^\ell \frac{\Gamma(\Delta-p+\ell)}{\ell!} \frac{\zeta^\ell}{(1-\zeta)^{\Delta-p+\ell}}\Big] \tag{4.5.43}$$

Upon substituting the value $\zeta = \alpha w_0^2/(\alpha w_0^2 + (1-\alpha) w^2)$, and using the binomial expansion for the (positive) powers of the combination $\alpha w_0^2 + (1-\alpha)w^2$, we find

$$f_{\Delta;p}(\zeta) = -(-)^p \frac{\Gamma(p)}{\Gamma(\Delta)} \sum_{k=0}^{\Delta-2} \sum_{\ell=0}^{p-1} (-)^\ell \frac{\Gamma(\Delta-p+\ell)\Gamma(\Delta-p+1)}{\ell!\,\Gamma(\Delta-p+\ell-k)\Gamma(k-\ell+2)} \Big(\frac{\alpha w_0^2}{(1-\alpha)w^2}\Big)^{k+1} \tag{4.5.44}$$

Remarkably, upon including the factor of $\alpha^{-1}(1-\alpha)^{\Delta-1}$ of the integral in (4.5.40), the integrand is polynomial in α and may be carried out term by term in (4.5.44). The final result for this calculation as well as for that of the remaining Z_j may be expressed in the following final form

$$Z_j(w) = \sum_{k=0}^{\Delta-2} Z_j^{(k)} \Big(\frac{w_0^2}{w^2}\Big)^{k+1} \qquad j = 1, \cdots, 5 \tag{4.5.45}$$

with the coefficients $Z_j^{(k)}$ dependent only on Δ and d and given as follows

$$Z_j^{(k)} = (-)^{\frac{d}{2}} \frac{\pi^{\frac{d}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{d}{2})\Gamma(\Delta-\frac{d}{2}+1)}{2\Gamma(\frac{d}{2}+\frac{1}{2})\Gamma(\Delta)^2} C_G\, C_\Delta^2\, \hat{Z}_j^{(k)} \tag{4.5.46}$$

$$= \quad (-)^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2})\ (\Delta - \frac{d}{2})^2}{4\pi^d \Gamma(\Delta - \frac{d}{2} + 1)}\ \hat{Z}_j^{(k)}, \tag{4.5.47}$$

with

$$\hat{Z}_1^{(k)} = \sum_{\ell=0}^{\frac{d}{2}-1} -(-)^\ell \frac{\Gamma(\Delta + \ell - \frac{d}{2})\Gamma(\Delta - k - 1)\Gamma(k+1)}{\ell!\ \Gamma(k - \ell + 2)\Gamma(\Delta - k + \ell - \frac{d}{2})} \tag{4.5.48}$$

$$\hat{Z}_2^{(k)} = \hat{Z}_3^{(k)} (-)^{\frac{d}{2}} \frac{\Gamma(\Delta - 1)\Gamma(k+1)}{\Gamma(k - \frac{d}{2} + 3)} \tag{4.5.49}$$

$$\hat{Z}_3^{(k)} = \frac{1}{2} \sum_{\ell=0}^{\frac{d}{2}-1} (-)^\ell \frac{\Gamma(\Delta + \ell - \frac{d}{2} - 1)\Gamma(\Delta - k - 1)\Gamma(k+1)}{\ell!\ \Gamma(k - \ell + 3)\Gamma(\Delta - k + \ell - \frac{d}{2} - 1)} \tag{4.5.50}$$

$$\hat{Z}_4^{(k)} = -\frac{\Delta - \frac{d}{2} + 1}{(d-2)\Delta} \sum_{\ell=0}^{\frac{d}{2}-2} (-)^\ell \frac{(\frac{d}{2} - \ell - 1)\Gamma(\Delta + \ell - \frac{d}{2})\Gamma(\Delta - k - 1)\Gamma(k+2)}{\ell!\ \Gamma(k - \ell + 3)\Gamma(\Delta - k + \ell - \frac{d}{2})} \tag{4.5.51}$$

$$\hat{Z}_5^{(k)} = \frac{2(\Delta - \frac{d}{2} + 2)(\Delta - \frac{d}{2} + 1)}{d\Delta(\Delta+1)} \times \sum_{\ell=0}^{\frac{d}{2}-2} (-)^\ell \frac{(\frac{d}{2} - \ell - 1)\Gamma(\Delta + \ell - \frac{d}{2} + 1)\Gamma(\Delta - k - 1)\Gamma(k+3)}{\ell!\ \Gamma(k - \ell + 3)\Gamma(\Delta - k + \ell - \frac{d}{2} + 1)} \tag{4.5.52}$$

We conclude by noticing that the relation between $Z_j^{(k)}$ and $\hat{Z}_j^{(k)}$ simplifies considerably upon using the explicit forms for C_G and C_Δ, as was done in (4.5.47).

Reduction to w-integrals

Our purpose here is to express the w–integrals in B^{tt}, B^{dd}, and B^{00} of (4.5.36–4.5.38) in terms of the following standard integral

$$W_k^{\Delta'}(a,b) \equiv \int dw \sqrt{g}\ \frac{w_0^{2\Delta'+2a+2k}}{w^{2k}} \frac{1}{(w-x)^{2\Delta'}} \frac{1}{(w-y)^{2\Delta'+2b}}\,. \tag{4.5.53}$$

We also use $\tilde{W}_k^{\Delta'}(a,b)$ which represents $W_k^{\Delta'}(a,b)$ with $x \leftrightarrow y$. Introducing the constants

$$Z^{(k)} = 2(\Delta+1)Z_1^{(k)} + 2\frac{m^2 - d + 1}{d-1} Z_2^{(k)} + 2(\Delta^2 - d - 1 + \frac{m^2(d-2)}{d-1}) Z_3^{(k)} - 4\Delta(\Delta+1) Z_4^{(k)} \tag{4.5.54}$$

we find the following expression for $B^{tt}+B^{dd}$, after partial integration of $\Box_w$,

$$B^{tt}+B^{dd}=\sum_{k=0}^{\Delta-2} Z^{(k)}\int dw\sqrt{g}\left(-\frac{1}{2}(d-1)\Box_w-2m'^2\right)\left(\frac{w_0^2}{w^2}\right)^{k+1}K(2')K(4') \qquad (4.5.55)$$

The action of the Laplace operator on the various powers of w_0^2/w^2 is easily evaluated with the help of the following formula

$$\Box_w\left(\frac{w_0^2}{w^2}\right)^k=2k(2k-d)\left(\frac{w_0^2}{w^2}\right)^k-4k^2\left(\frac{w_0^2}{w^2}\right)^{k+1} \qquad (4.5.56)$$

and we obtain the following expression for the amplitude in terms of W functions

$$\begin{aligned} B^{tt}+B^{dd} &= c_{\Delta'}^2\sum_{k=0}^{\Delta-2} Z^{(k)}\Big[\{-(d-1)(k+1)(2k+2-d)-2m'^2\}W_{k+1}^{\Delta'}(0,0) \\ &\quad +2(d-1)(k+1)^2W_{k+2}^{\Delta'}(0,0)\Big] \end{aligned} \qquad (4.5.57)$$

Proceeding analogously for the contribution of B^{00} with the help of (4.5.28) and (4.5.38), we find

$$\begin{aligned} B^{00} &= -2\Delta(\Delta+1)(\Delta')^2c_{\Delta'}^2\sum_{k=0}^{\Delta-2} Z_5^{(k)}\Big\{(1-\frac{m'^2}{\Delta'^2})W_{k+1}^{\Delta'}(0,0)-4W_{k+1}^{\Delta'}(1,1)-4\tilde{W}_{k+1}^{\Delta'}(1,1) \\ &\quad +8W_{k+1}^{\Delta'+1}(0,0)+2(x-y)^2W_{k+1}^{\Delta'+1}(0,0)\Big\} \end{aligned} \qquad (4.5.58)$$

As in the special case $\Delta=\Delta'=d=4$ already discussed in Section 3, we recognize that the general graviton exchange amplitude is a finite sum of quartic graphs. In fact, each $W_k^{\Delta'}(a,b)$ is the amplitude of a 4–point contact diagram evaluated in the inverted coordinates (with appropriate inversion prefactors omitted). The scale dimension of the external propagators are $\Delta_1=k+2a-b$, $\Delta_3=k$, $\Delta_2=\Delta'+b$ and $\Delta_4=\Delta'$ (see equ.(B.1.3)).

4.5.3 Graviton exchange graph for $d = \Delta = \Delta' = 4$

For $\Delta = \Delta' = d = 4$, the masses of the scalars vanish $m = m' = 0$, and the k and ℓ-sums in the results for the z-integral functions I_j truncate after just a few terms. We need the z-integral functions $Z_j(w)$, $j = 1, \cdots, 5$, which may be read off from (4.5.45) and (4.5.48–4.5.52) with $\Delta = d = 4$,

$$\begin{aligned}
Z_1(w) &= \frac{1}{2\pi^4}(\qquad\quad + \frac{3}{2}\frac{w_0^4}{w^4} + 2\frac{w_0^6}{w^6}) && (4.5.59)\\
Z_2(w) &= \frac{1}{2\pi^4}(\frac{3}{2}\frac{w_0^2}{w^2} + \frac{7}{4}\frac{w_0^4}{w^4} + 2\frac{w_0^6}{w^6}) && (4.5.60)\\
Z_3(w) &= \frac{1}{2\pi^4}(-\frac{1}{2}\frac{w_0^2}{w^2} - \frac{1}{4}\frac{w_0^4}{w^4}) && (4.5.61)\\
Z_4(w) &= \frac{1}{2\pi^4}(-\frac{3}{8}\frac{w_0^2}{w^2} - \frac{1}{8}\frac{w_0^4}{w^4}) && (4.5.62)\\
Z_5(w) &= \frac{3}{10\pi^4}(+\frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6}) && (4.5.63)
\end{aligned}$$

Using these integrals, the expressions for $B^{tt} + B^{dd}$ and B^{00} become quite simple and are given as follows,

$$\begin{aligned}
B^{tt} + B^{dd} &= \frac{8}{\pi^4}\int dw\sqrt{g}\left\{\frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6}\right\} g \cdot T_{24}(w) && (4.5.64)\\
B^{00} &= -\frac{12}{\pi^4}\int dw\sqrt{g}\left\{\frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6}\right\} w_0^2 T_{24}(w)_{0'0'} && (4.5.65)
\end{aligned}$$

When $m' = 0$ and $d = 4$, the combination $g \cdot T_{24}$ in (4.5.29) simplifies. Upon integration by parts, and making use of the differentiation formula (4.5.56), we obtain the following expression

$$\begin{aligned}
B^{tt} + B^{dd} &= \frac{2^6 \cdot 3^3}{\pi^8}\int dw\sqrt{g}\left\{\frac{w_0^2}{w^2} + \frac{w_0^4}{w^4} + \frac{w_0^6}{w^6} + 9\frac{w_0^8}{w^8}\right\}\frac{w_0^8}{(w-x)^8(w-y)^8}\\
&= \frac{2^6 \cdot 3^3}{\pi^8}\Big\{W_1^4(0,0) + W_2^4(0,0) + W_3^4(0,0) + 9W_4^4(0,0)\Big\} \qquad (4.5.66)
\end{aligned}$$

The expression for B^{00} may be obtained in an analogously, using (4.5.28) for $m' = 0$, $\Delta = 4$. This directly gives

$$B^{00} = -\frac{2^9 \cdot 3^3}{\pi^8} \sum_{p=1}^{3} \Big\{ W_p^4(0,0) - 4W_p^4(1,1) - 4\tilde{W}_p^4(1,1) + 8W_p^5(1,0) + 2(x-y)^2 W_p^5(0,0) \Big\} \tag{4.5.67}$$

Using the expression for $W_p^4(1,1) + \tilde{W}_p^4(1,1)$ in terms of $W(0,0)$ to be derived in (4.6.5), this formula may be recast in terms of $W(0,0)$ and $W(1,0)$ only, viz.

$$B^{00} = -\frac{2^9 \cdot 3^3}{\pi^8} \Big[3W_4^4(0,0) + \sum_{p=1}^{3} \{ -2W_p^4(0,0) + 8W_p^5(1,0) + 2(x-y)^2 W_p^5(0,0) \} \Big] \tag{4.5.68}$$

Adding the contributions of $B^{tt} + B^{dd}$ and B^{00}, we finally obtain the expression for the full B in terms of W-functions and we have

$$B = -\frac{2^6 \cdot 3^3}{\pi^8} \Big[15W_4^4(0,0) + \sum_{p=1}^{3} \{ -17W_p^4(0,0) + 64W_p^5(1,0) + 16(x-y)^2 W_p^5(0,0) \} \Big] . \tag{4.5.69}$$

The full graviton amplitude I_{grav} is obtained by multiplying B by the appropriate kinematic factors and symmetrizing under $1 \leftrightarrow 3$ (see (4.5.1), (4.5.23)).

4.5.4 Equivalence with the result in Section 3

We now make contact with the result obtained in Section 4. We recall that $W_k^{\Delta'}(a,b)$ are just scalar quartic graphs in the inverted coordinates (with some kinematic factors omitted), see equ.(B.1.3). One can easily convert (4.5.69) and (4.5.23) into the notations Section 4, and get a sum of D–functions. The representation of the graviton exchange graph that is obtained in this way does not at first appear to coincide with the result (4.4.38). In particular, terms of the form $x_{12}^2 x_{14}^2 D_{p+2p55} + x_{23}^2 x_{34}^2 D_{pp+255}$ arise from $W_p^5(1,0)$ in (4.5.69) and its symmetrization in $1 \leftrightarrow 3$. Thanks to the many identities that connect the D functions (see the Appendix), the two representations of the answer are in fact exactly equal. We first use (B.5.19) to eliminate the "asymmetric" D's in the result of Section 5.

We get

$$
\begin{aligned}
I_{\text{grav}} \;=\; \left(\frac{6}{\pi^2}\right)^4 \Bigg[& 16\, x_{24}^2 \left(\frac{1}{2s}-1\right) D_{4455} + \frac{32}{3}\frac{x_{24}^2}{x_{13}^2}\left(-1+\frac{2}{3s}\right)\frac{x_{24}^2}{x_{13}^2} D_{3355} \qquad (4.5.70)\\
& + \frac{32}{3}\frac{x_{24}^2}{x_{13}^4}\left(\frac{1}{2s}-1\right) D_{2255} - \frac{32}{3}\frac{x_{24}^2}{x_{13}^6} D_{1155} + 24\, D_{4444}\\
& + \frac{8}{9\, x_{13}^2} D_{3344} + \frac{14}{9\, x_{13}^4} D_{2244} + \frac{10}{3\, x_{13}^6} D_{1144} \Bigg] .
\end{aligned}
$$

Now (4.4.38), (4.5.70) are both in terms of D–functions of the form $D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}$. By repeated application of (B.3.9) one can convert one representation into the other. We regard this non–trivial match as a strong check of our result.

4.6 Asymptotic expansions

We have seen that the graviton exchange amplitude (and generically all AdS 4–point processes with external scalars) can be expressed as a finite sum of quartic graphs, see (4.4.38), (4.5.57–4.5.58), (4.5.70). In this Section we develop asymptotic series expansions for the scalar quartic graphs (Figure 5) in terms of conformally invariant variables. This series expansions allow to analyze the supergravity results in terms of the expected double OPE (4.1.1). In Section 3 and 4 we have used slightly different notations for the quartic graphs, namely $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ and $W_p^{\Delta}(a,b)$. The connection between the two is given in (B.1.3). Here the expansions are performed for the $W_p^{\Delta}(a,b)$ representation of the quartic graph.

In Section 6.3 we assemble the series expansions of the W's that appear in the representation (4.5.23,4.5.69) of the graviton exchange for $\Delta = \Delta' = d = 4$. We concentrate on the direct channel and display explicitly the singular terms and all the logarithmic contributions. The complete expansions, in both direct and crossed channels, can be easily obtained from the formulas in Section 6.2.

4.6.1 Integral representations of $W_k^{\Delta'}(a,b)$

To evaluate $W_k^{\Delta'}(a,b)$, we follow the methods of [95] and [97]. We introduce a first Feynman parameter α for the denominators w^2 and $(w-x)^2$ and a second Feynman parameter β for the resulting denominator and $(w-y)^2$. The $\vec{w}$ and w_0 integrals may then be carried out using standard formulas, and we find

$$\begin{aligned} W_k^{\Delta'}(a,b) &= \frac{\pi^{\frac{d}{2}}}{2}\frac{\Gamma(k+\Delta'+a-\frac{d}{2})\Gamma(\Delta'+b-a)}{\Gamma(k)\Gamma(\Delta')\Gamma(\Delta'+b)} \\ &\times \int_0^1 d\alpha \int_0^1 d\beta \frac{\alpha^{\Delta'-1}(1-\alpha)^{k-1}\beta^{\Delta'+b-1}(1-\beta)^{k+a-b-1}}{[\beta(y-\alpha x)^2+\alpha(1-\alpha)x^2]^{\Delta'-a+b}} \end{aligned} \tag{4.6.1}$$

Upon performing the following change of variables familiar from [95] and [97],

$$\alpha = \frac{1}{1+u} \qquad\qquad \beta = \frac{u}{u+v+uv} \tag{4.6.2}$$

we obtain an integral representation similar that of [95] and [97],

$$\begin{aligned} W_k^{\Delta'}(a,b) &= \frac{\pi^{\frac{d}{2}}}{2}\frac{\Gamma(k+\Delta'+a-\frac{d}{2})\Gamma(\Delta'+b-a)}{\Gamma(k)\Gamma(\Delta')\Gamma(\Delta'+b)} \int_0^\infty du \int_0^\infty dv \\ &\times \frac{u^{k+a-1}v^{k+a-b-1}}{(u+v+uv)^{k+2a-b}} \frac{1}{[(x-y)^2+uy^2+vx^2]^{\Delta'-a+b}} \end{aligned} \tag{4.6.3}$$

Now the function W with $b \neq 0$ only enters the calculation of B^{00} (equ.(4.5.67)), and appears there only in the form of the sum $W_k^{\Delta'}(1,1) + \tilde{W}_k^{\Delta'}(1,1)$. This particular combination may be re–expressed in terms of W-functions with $b=0$ only. This would be difficult to see from the w-integral definition (4.5.53), but is manifest from the integral representation (4.6.3), by using the following relation

$$\frac{u}{(u+v+uv)^{k+1}} + \frac{v}{(u+v+uv)^{k+1}} = \frac{1}{(u+v+uv)^k} - \frac{uv}{(u+v+uv)^{k+1}} \tag{4.6.4}$$

Taking normalization factors into account properly, we find

$$W_k^{\Delta'}(1,1) + \tilde{W}_k^{\Delta'}(1,1) = \frac{k+\Delta'-\frac{d}{2}}{\Delta'} W_k^{\Delta'}(0,0) - \frac{k}{\Delta'} W_{k+1}^{\Delta'}(0,0) \tag{4.6.5}$$

As a result of this identity, there will be only two classes of w-integral functions entering into the graviton exchange amplitudes : $W_k^{\Delta'}(0,0)$ and $W_k^{\Delta'}(1,0)$.

Similarly, a relation exists expressing $W_k^{\Delta'}(1,0)$ in terms of $W(0,0)$-functions. This may be established by using the fact that the quantity

$$\Big(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\Big)\Big(\frac{(uv)^{k-1}}{(u+v+uv)^k} \frac{1}{[(x-y)^2 + uy^2 + vx^2]^{\Delta'}}\Big) \tag{4.6.6}$$

has vanishing integral in u and v, and by carrying out the derivatives explicitly and regrouping the result in terms of W-functions. The final result is

$$\begin{aligned} 2(k+1)(\Delta')^2 W_k^{\Delta'}(1,0) \;&=\; k(k+\Delta'-\frac{d}{2})(k+\Delta'-\frac{d}{2}-1)W_k^{\Delta'-1}(0,0) \\ &\quad -k(2k+1)(k+\Delta'-\frac{d}{2})W_{k+1}^{\Delta'-1}(0,0) \\ &\quad +k(k+1)^2 W_{k+2}^{\Delta'-1}(0,0) - k(\Delta')^2(x^2+y^2)W_{k+1}^{\Delta'}(0,0) \end{aligned} \tag{4.6.7}$$

The w-integrals $W_k^{\Delta'}(0,0)$ and $W_k^{\Delta'}(1,0)$ may each be expressed in terms of derivatives on two universal functions. To show this, we proceed as in [95] and [97], where analogous results were obtained for the scalar and gauge exchange graphs. We begin by introducing the conformal invariants

$$s = \frac{1}{2}\frac{(x-y)^2}{x^2+y^2} = \frac{1}{2}\frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \tag{4.6.8}$$

$$t = \frac{x^2-y^2}{x^2+y^2} = \frac{x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2} \tag{4.6.9}$$

whose ranges are $0 \le s \le 1$ and $-1 \le t \le 1$. Next, we perform a change of variables

$$u \;= 2\rho(1-\lambda)$$

$$v = 2\rho(1+\lambda) \tag{4.6.10}$$

under which we have

$$\begin{aligned} W_k^{\Delta'}(a,0) &= \frac{\pi^{\frac{d}{2}}}{2^{\Delta'+a}} \frac{\Gamma(k+\Delta'+a-\frac{d}{2})\Gamma(\Delta'-a)}{\Gamma(k)\Gamma(\Delta')^2(x^2+y^2)^{\Delta'-a}} \int_0^\infty d\rho \int_{-1}^1 d\lambda \\ &\times \frac{\rho^{k-1}(1-\lambda^2)^{k+a-1}}{[1+\rho(1-\lambda^2)]^{k+2a}} \frac{1}{(s+\rho+\rho\lambda t)^{\Delta'-a}} \end{aligned} \tag{4.6.11}$$

It is now possible to write the right hand side as a derivative with respect to s of order $\Delta' - a - 1$ of an integral in which the denominator involving s appears to degree 1, using

$$\frac{1}{(s+\omega)^p} = \frac{(-)^{p+1}}{\Gamma(p)} \left(\frac{\partial}{\partial s}\right)^{p-1} \frac{1}{s+\omega} \tag{4.6.12}$$

Next, we change variables to $\rho = s/\mu$ and recognize that the new integral is a derivative with respect of s of order $k - 1 + 2a$. Putting all together, we obtain

$$\begin{aligned} W_k^{\Delta'}(a,0) &= \pi^{\frac{d}{2}} \frac{(-)^{\Delta'+k+a} 2^{-\Delta'-a} \Gamma(k+\Delta'+a-\frac{d}{2})}{\Gamma(k)\Gamma(k+2a)\Gamma(\Delta')^2(x^2+y^2)^{\Delta'-a}} \\ &\times \left(\frac{\partial}{\partial s}\right)^{\Delta'-a-1} \left\{ s^{k-1} \left(\frac{\partial}{\partial s}\right)^{k-1+2a} I_a(s,t) \right\} \end{aligned} \tag{4.6.13}$$

where the universal functions $I_a(s,t)$ are given by the following integral representations

$$\begin{aligned} I_a(s,t) &= s^{2a} \int_0^\infty d\mu \int_{-1}^1 d\lambda \, \frac{(1-\lambda^2)^a}{\mu + s(1-\lambda^2)} \, \frac{1}{1+\mu+\lambda t} \\ &= s^{2a} \int_{-1}^1 d\lambda \frac{(1-\lambda^2)^a}{1+\lambda t - s(1-\lambda^2)} \ln \frac{1+\lambda t}{s(1-\lambda^2)} \end{aligned} \tag{4.6.14}$$

The integrals $I_a(s,t)$ are perfectly convergent and produce analytic functions in s and t, with logarithmic singularities in s and t.

4.6.2 Series expansions of $W_k^{\Delta'}(a,b)$

Series expansions of the functions $W_k^{\Delta'}(a,0)$ may be obtained easily from the series expansions of the universal functions $I_a(s,t)$. There are two different regions in which the expansion will be needed :

a) The direct channel ("t–channel") limit $|x_{13}| \ll |x_{12}|$, $|x_{24}| \ll |x_{12}|$, which corresponds to $s, t \to 0$.

b) The two crossed channels; one ("s–channel") is the limit $|x_{12}| \ll |x_{13}|$, $|x_{34}| \ll |x_{13}|$, which corresponds to $s \to 1/2$, $t \to -1$, and the other ("u–channel") is $|x_{23}| \ll |x_{34}|$, $|x_{14}| \ll |x_{34}|$ in which $s \to 1/2$, $t \to 1$.

We shall now discuss each limit in turn.

(a) Direct channel series expansion

The direct channel limit is given by $s, t \to 0$, and the expansions of the functions $I_a(s,t)$ are given by

$$I_0(s,t) = \sum_{k=0}^{\infty}\{-\ln s \; a_k(t) + b_k(t)\}s^k \tag{4.6.15}$$

$$I_1(s,t) = \sum_{k=0}^{\infty}\{-\ln s \; \hat{a}_k(t) + \hat{b}_k(t)\}s^{k+2} \tag{4.6.16}$$

where the coefficient functions are given by

$$a_k(t) = \int_{-1}^{1} d\lambda \frac{(1-\lambda^2)^k}{(1+\lambda t)^{k+1}} \qquad b_k(t) = \int_{-1}^{1} d\lambda \frac{(1-\lambda^2)^k}{(1+\lambda t)^{k+1}} \ln\frac{1+\lambda t}{1-\lambda^2} \tag{4.6.17}$$

$$\hat{a}_k(t) = \int_{-1}^{1} d\lambda \frac{(1-\lambda^2)^{k+1}}{(1+\lambda t)^{k+1}} \qquad \hat{b}_k(t) = \int_{-1}^{1} d\lambda \frac{(1-\lambda^2)^{k+1}}{(1+\lambda t)^{k+1}} \ln\frac{1+\lambda t}{1-\lambda^2} \tag{4.6.18}$$

The coefficient functions admit Taylor series expansions in powers of t with radius of convergence 1. Actually, in view of (4.6.7), we have the following relations between these functions

$$(k+2)\hat{a}_k(t) = (k+1)(2a_k(t) - a_{k+1}(t)) \tag{4.6.19}$$

$$(k+2)^2\,\hat{b}_k(t) \;=\; (k+1)(k+2)(2b_k(t)-b_{k+1}(t))-2a_k(t)+a_{k+1}(t) \qquad (4.6.20)$$

From (4.6.13) and (4.6.15, 4.6.16), we obtain the series expansions of $W_k^{\Delta'}(0,0)$ and $W_k^{\Delta'}(1,0)$ using the following differentiation formulas

$$s^p\Big(\frac{\partial}{\partial s}\Big)^p s^k \;=\; \frac{\Gamma(k+1)}{\Gamma(k-p+1)}\,s^k \qquad (4.6.21)$$

$$s^p\Big(\frac{\partial}{\partial s}\Big)^p \{s^k\ln s\} \;=\; \frac{\Gamma(k+1)}{\Gamma(k-p+1)}\,s^k\,\{\ln s+\psi(k+1)-\psi(k-p+1)\} \qquad (4.6.22)$$

We find

$$W_p^{\Delta}(0,0) \;=\; \frac{(-)^{\Delta+p}\pi^{\frac{d}{2}}\Gamma(p+\Delta-\frac{d}{2})}{2^{\Delta}\Gamma(p)^2\Gamma(\Delta)^2(x^2+y^2)^{\Delta}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)^2\,s^{k-\Delta+1}}{\Gamma(k-p+2)\Gamma(k-\Delta+2)}$$
$$\Big\{b_k(t)-a_k(t)[\ln s+2\psi(k+1)-\psi(k-\Delta+2)-\psi(k-p+2)]\Big\} \qquad (4.6.23)$$

and

$$W_p^{\Delta}(1,0) \;=\; \frac{(-)^{\Delta+p+1}\pi^{\frac{d}{2}}\Gamma(p+\Delta-\frac{d}{2}+1)}{2^{\Delta+1}\Gamma(p)\Gamma(p+2)\Gamma(\Delta)^2(x^2+y^2)^{\Delta-1}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1)\Gamma(k+3)\,s^{k-\Delta+2}}{\Gamma(k-p+2)\Gamma(k-\Delta+3)} \qquad (4.6.24)$$
$$\cdot\Big\{\hat{b}_k(t)-\hat{a}_k(t)[\ln s+\psi(k+1)+\psi(k+3)-\psi(k-\Delta+3)-\psi(k-p+2)]\Big\}$$

The presentation of these series expansions is slightly formal in the sense that for $k\le\Delta-2$, the $\Gamma(k-\Delta+2)$ function in the denominator produces a zero, while the $\psi(k-\Delta+2)$ term produces a pole, which together yield a finite result, which amounts to a pole term in s. Its coefficient can be obtained from the formula $\lim_{x\to 0}\psi(x-q)/\Gamma(x-q)=(-)^{q+1}\Gamma(q+1)$ for any non–negative interger q.

(b) Crossed channel series expansion

The crossed channel asymptotics is given by $s\to\frac{1}{2}$ and $t\to\pm1$, and may also be obtained from the series expansion of the functions $I_a(s,t)$, with $a=0,1$. Actually, it suffices to obtain the expansion of $I_0(s,t)$ and thus of $W_p^{\Delta'}(0,0)$ in this limit and then to compute the series expansion of $W_p^{\Delta'}(1,0)$ by using the relation (4.6.7). This is useful in this case, since the expansion of $I_1(s,t)$ appears more involved than that of $I_0(s,t)$.

We start from the definition of $I_0(s,t)$ in (4.6.14) as a double integral and consecutively perform the following changes of variables $\mu = (1+\lambda t)\sigma$ and $\tau = (1+\sigma)^{-1}$, so that

$$I_0(s,t) = \int_0^1 d\tau \int_{-1}^1 d\lambda \frac{1}{(1-\tau)(1+\lambda t) + \tau s(1-\lambda^2)} \,. \tag{4.6.25}$$

This form of the universal function $I_0(s,t)$ is now precisely of the form studied in [97], and the λ-integral may be performed explicitly in an elementary way. We obtain, as in [97]

$$I_0(s,t) = I_0^{\log}(s,t) + I_0^{\mathrm{reg}}(s,t) \tag{4.6.26}$$

where

$$I_0^{\log}(s,t) = -\ln(1-t^2)\int_0^1 d\tau \frac{1}{\sqrt{\omega^2 - \tau^2(1-t^2)}} \tag{4.6.27}$$

$$I_0^{\mathrm{reg}}(s,t) = 2\int_0^1 d\tau \frac{1}{\sqrt{\omega^2 - \tau^2(1-t^2)}} \ln\Big\{\frac{\omega}{\tau} + \sqrt{\frac{\omega^2}{\tau^2} - (1-t^2)}\Big\} \tag{4.6.28}$$

where the composite variable ω is defined by $\omega = 1-(1-2s)(1-\tau)$. In the neighborhood of $s=\frac{1}{2}$ and $t=\pm 1$, we have $\omega \sim 1$ and $1-t^2 \sim 0$, so that the integrals in (4.6.27,4.6.28) are both uniformly convergent, and may be Taylor expanded in powers of $(2s-1)$ and $(1-t^2)$. Thus, $I_0^{\mathrm{reg}}(s,t)$ is analytic in both s and t in the neighborhood of $s=\frac{1}{2}$ and $t=\pm 1$, and all non-analyticity is contained in the factor $\ln(1-t^2)$ of $I_0^{\log}(s,t)$. The integral admits a double Taylor expansion given by

$$\begin{aligned} I_0^{\log}(s,t) &= -\ln(1-t^2)\sum_{k=0}^{\infty}(1-2s)^k \alpha_k(t) \\ \alpha_k(t) &= \frac{1}{k+1}F\Big(\frac{1}{2},\frac{k+1}{2};\frac{k+3}{2};1-t^2\Big) = \sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\ell!}\,\frac{(1-t^2)^\ell}{2\ell+k+1} \end{aligned} \tag{4.6.29}$$

This expansion may be used to evaluate the logarithmic part of $W_p^{\Delta'}(0,0)$ and we obtain the following result

$$W_p^{\Delta'}(0,0)\Big|_{\log} = -2^{p-2}\pi^{\frac{d}{2}} \ln(1-t^2) \frac{\Gamma(p+\Delta'-\frac{d}{2})}{\Gamma(p)\Gamma(\Delta')(x^2+y^2)^{\Delta'}} \sum_{\ell=0}^{\Delta'-1}\sum_{k=0}^{\infty} \frac{(-2)^{-\ell}\Gamma(k+1)\; s^{p-\ell-1}\;(1-2s)^{k-p+\ell-\Delta'+2}}{\Gamma(\Delta'-\ell)\Gamma(p-\ell)\;\ell!\;\Gamma(k-p+\ell-\Delta'+3)}\,\alpha_k(t) \quad (4.6.30)$$

Notice that in the crossed channel, no power singularities arise.

4.6.3 Asymptotic expansion for the graviton exchange

We now turn to the direct channel asymptotic expansion of the graviton exchange graph for $\Delta = \Delta' = d = 4$. The power singularity terms may be read off directly from the general asymptotic expansion formula (4.6.23) restricted to $d = 4$, and we have

$$W_p^{\Delta}(0,0) \sim \frac{\pi^2\Gamma(p+\Delta-2)}{2^{\Delta}\Gamma(p)^2\Gamma(\Delta)^2}\frac{(-)^{p-1}}{(x^2+y^2)^{\Delta}}\sum_{k=p-1}^{\Delta-2}(-)^k\frac{\Gamma(k+1)^2\Gamma(\Delta-k-1)}{\Gamma(k-p+2)}\frac{a_k(t)}{s^{\Delta-1-k}} \quad (4.6.31)$$

Similarly, we have from (4.6.24)

$$W_p^{\Delta}(1,0) \sim \frac{\pi^2\Gamma(p+\Delta-1)}{2^{\Delta+1}\Gamma(p)\Gamma(p+2)\Gamma(\Delta)^2}\frac{(-)^{p-1}}{(x^2+y^2)^{\Delta-1}} \times\sum_{k=p-1}^{\Delta-3}(-)^k\frac{\Gamma(k+1)\Gamma(k+3)\Gamma(\Delta-k-2)}{\Gamma(k-p+2)}\frac{\hat{a}_k(t)}{s^{\Delta-2-k}} \quad (4.6.32)$$

The full singular power part of the amplitude is now easily obtained by working out the asymptotics above in the cases $W_p^4(0,0)$, $W_p^5(0,0)$ and $W_p^5(1,0)$ with $p = 1,2,3$. The function $W_4^4(0,0)$ has no power singularities and does not contribute here. Putting all together, we have

$$B_{\text{sing}} = -\frac{48}{\pi^6}\frac{1}{(x^2+y^2)^4}\Big[-2\big(\frac{a_0(t)}{s^3}+\frac{a_1(t)}{s^2}+\frac{a_2(t)}{s}\big)+3\big(\frac{\hat{a}_0(t)}{s^3}+\frac{\hat{a}_1(t)}{s^2}+\frac{\hat{a}_2(t)}{s}\big)\Big] \quad (4.6.33)$$

Using the series expansions of the functions $a_k(t)$ and $\hat{a}_k(t)$ to low orders, taking into account that generically, s vanishes like t^2,

$$\begin{aligned}
a_0(t) &= 2+\frac{2}{3}t^2+\frac{2}{5}t^4 & a_1(t) &= \frac{4}{3}+\frac{4}{5}t^2 & a_2(t) &= \frac{16}{15} \\
\hat{a}_0(t) &= \frac{4}{3}+\frac{4}{15}t^2+\frac{4}{35}t^4 & \hat{a}_1(t) &= \frac{16}{15}+\frac{48}{105}t^2 & \hat{a}_2(t) &= \frac{32}{35}
\end{aligned} \tag{4.6.34}$$

The final result for the singular part of B is

$$B_{\text{sing}} = -\frac{2^7}{35\pi^6}\frac{1}{(x^2+y^2)^4}\left[\frac{1}{s^3}(-7t^2-6t^4)+\frac{1}{s^2}(7-3t^2)+\frac{8}{s}\right] \tag{4.6.35}$$

Repristinating the overall kinematic factors we get the final result for the singular terms in the direct channel of the graviton amplitude

$$I_{\text{grav}}\bigg|_{\text{sing}} = \frac{2^{10}}{35\pi^6}\frac{1}{x_{13}^8 x_{24}^8}\left[s\,(7t^2+6t^4)+s^2\,(-7+3t^2)-8\,s^3\right] \tag{4.6.36}$$

Notice that the leading singularity x_{13}^{-6} cancels between the various tensor contributions to the amplitude. The physical interpretation of this singular expansion is discussed in Section 2.3.

The logarithmic singularities may be read off directly from the asymptotic expansion formulas of (4.6.23, 4.6.24), and we have

$$\begin{aligned}
W_p^4(0,0) &= (-)^{p+1}\frac{\pi^2}{2^6\,3^2}\frac{\Gamma(p+2)}{\Gamma(p)^2}\frac{\ln s}{(x^2+y^2)^4}\sum_{k=0}^{\infty}\frac{\Gamma(k+4)^2 s^k}{\Gamma(k+5-p)\Gamma(k+1)}a_{k+3}(t) \\
W_p^5(0,0) &= (-)^{p}\frac{\pi^2}{2^{11}\,3^2}\frac{\Gamma(p+3)}{\Gamma(p)^2}\frac{\ln s}{(x^2+y^2)^5}\sum_{k=0}^{\infty}\frac{\Gamma(k+5)^2 s^k}{\Gamma(k+6-p)\Gamma(k+1)}a_{k+4}(t) \\
W_p^5(1,0) &= (-)^{p+1}\frac{\pi^2}{2^{12}\,3^2}\frac{\Gamma(p+4)}{\Gamma(p)\Gamma(p+2)}\frac{\ln s}{(x^2+y^2)^4}\sum_{k=0}^{\infty}\frac{\Gamma(k+6)\Gamma(k+4) s^k}{\Gamma(k+5-p)\Gamma(k+1)}\hat{a}_{k+3}(t)
\end{aligned} \tag{4.6.37}$$

Assembling these contributions to the logarithmic singularity and expressing the coefficient functions $\hat{a}_k(t)$ in terms of $a_k(t)$ using (4.6.19) we get

$$I_{\rm grav}\Big|_{\log} = \frac{3\cdot 2^3}{\pi^6}\frac{\ln s}{x_{13}^8 x_{24}^8}\sum_{k=0}^{\infty} s^{4+k}\,\frac{\Gamma(k+4)}{\Gamma(k+1)}\Big\{-2(5k^2+20k+16)(3k^2+15k+22)a_{k+3}(t) + (k+4)^2(15k^2+55k^2+42)a_{k+4}(t)\Big\}. \qquad (4.6.38)$$

Appendix A

Normalization of 2–point function.

For scalars with dimension $\Delta = d$ the correlation functions achieve constant limiting values as we approach the boundary of AdS space. If $\Delta \neq d$ then the correlation function goes to zero or infinity as we go towards the boundary, and must be defined with an appropriate scaling. In this case an interesting subtlety is seen to arise in the order in which we take the limits to define various quantities, and we discuss this issue below.

Let us discuss the 2–point function for scalars. We take the metric (2.2.3) on the AdS space, and put the boundary at $z_0 = \epsilon$ with $\epsilon << 1$; at the end of the calculation we take ϵ to zero. We also Fourier transform the variables $\vec{x}$, and follow the discussion of [2].

The wave equation in Fourier space for scalars with mass m is

$$z_0^{d+1}\frac{\partial}{\partial z_0}[z_0^{-d+1}\frac{\partial}{\partial z_0}\phi(z_0,\vec{k})] - (k^2 z_0^2 + m^2)\phi(z_0,\vec{k}) = 0 \tag{A.1}$$

where we have written

$$\phi(z_0,\vec{x}) \;=\; \frac{1}{(2\pi)^{d/2}}\int d\vec{k}\; e^{i\vec{k}\cdot\vec{x}}\;\phi(z_0,\vec{k}) \tag{A.2}$$

The solution to this equation is

$$\phi(z_0,\vec{k}) \;=\; z_0^{\frac{d}{2}} F_\nu[ikz_0] \tag{A.3}$$

where F_ν is a solution of the Bessel equation with index

$$\nu \;=\; \Delta - \frac{d}{2} \;=\; [\frac{d^2}{4} + m^2]^{1/2} \tag{A.4}$$

The action in terms of Fourier components is

$$\begin{gathered} S = \tfrac{1}{2}\int dz_0\, d\vec{k}d\vec{k}'\delta(\vec{k}+\vec{k}')z_0^{-d+1} \\ [\tfrac{\partial}{\partial z_0}\phi(z_0,\vec{k})\tfrac{\partial}{\partial z_0}\phi(z_0,\vec{k}') + (k^2 + \tfrac{m^2}{z_0^2})\phi(z_0,\vec{k})\phi(z_0,\vec{k}')] \end{gathered} \tag{A.5}$$

We have to evaluate this action on a solution of the equation of motion with $\phi(\epsilon,\vec{k}) \equiv \phi_b(\vec{k})$ given. An integration by parts gives

$$S = \frac{1}{2}\int d\vec{k}d\vec{k}'\delta(\vec{k}+\vec{k}') \lim_{z_0\to\epsilon} z_0^{-d+1}[\phi(z_0,\vec{k})\partial_{z_0}\phi(z_0,\vec{k}')] \tag{A.6}$$

If we have a solution to the wave equation $K^\epsilon(z_0,\vec{k})$ such that

$$\lim_{z_0\to\epsilon} K^\epsilon(z_0,\vec{k}) \;=\; 1, \quad \lim_{z_0\to\infty} K(z_0,\vec{k}) = 0 \tag{A.7}$$

then we can write the desired solution to the wave equation as

$$\phi(z_0,\vec{k}) \;=\; K(z_0,\vec{k})\phi_b(\vec{k}) \tag{A.8}$$

Then the 2-point function in Fourier space will be given by

$$\langle\mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\rangle \;=\; -\,\epsilon^{-d+1}\delta(\vec{k}+\vec{k}') \lim_{z_0\to\epsilon}\partial_{z_0}K^\epsilon(z_0,\vec{k}) \tag{A.9}$$

We have

$$K^\epsilon(z_0,\vec{k}) \;=\; (\frac{z_0}{\epsilon})^{d/2}\frac{\mathcal{K}_\nu(kz_0)}{\mathcal{K}_\nu(k\epsilon)} \tag{A.10}$$

where $\mathcal{K}$ is the modified Bessel function which vanishes as $z_0 \to \infty$. For small argument

$\mathcal{K}_\nu$ has the expansion

$$\mathcal{K}_\nu(kz_0) \;=\; 2^{\nu-1}\Gamma(\nu)(kz_0)^{-\nu}[1+\ldots] \;-\; 2^{-\nu-1}\frac{\Gamma(1-\nu)}{\nu}(kz_0)^{\nu}[1+\ldots] \tag{A.11}$$

where the terms represented by '...' are positive integer powers of $(kz_0)^2$. Then (A.9) gives

$$\langle \mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\rangle =$$

$$-\,\epsilon^{-d+1}\delta(\vec{k}+\vec{k}')\,\lim_{z_0\to\epsilon}(\epsilon k)^{-d/2}\partial_{z_0}\frac{(kz_0)^{-\nu+\frac{d}{2}}+\ldots-2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(kz_0)^{\nu+\frac{d}{2}}+\ldots}{(k\epsilon)^{-\nu}+\ldots-2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(k\epsilon)^{\nu}+\ldots}$$

$$= -\epsilon^{2(\Delta-d)}\delta(\vec{k}+\vec{k}')k^{2\nu}2^{-2\nu}\tfrac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(2\nu) \quad + \quad \ldots \tag{A.12}$$

Here in the last line we have written only those terms that correspond to the power law behavior of the correlator in position space, and further only the largest such terms in the limit $\epsilon \to 0$ have been kept. In particular we have dropped terms that are integer powers in k^2, even though some of these terms are multiplied by a smaller power of ϵ than the term that we have kept. The reason for dropping these terms is that they give delta–function contact terms in the correlator after transforming to position space, and we are interested here in the correlation function for separated points.

The result (A.12) is the Fourier transform of the function

$$\frac{1}{\pi^{d/2}}\epsilon^{2(\Delta-d)}\frac{(2\Delta-d)}{\Delta}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-\frac{d}{2})}|\vec{x}-\vec{y}|^{-2\Delta} \tag{A.13}$$

which should therefore be the correctly normalized 2–point function on the boundary $z_0=\epsilon$. It also agrees with the correctly normalized 2–point function required by the Ward identity (2.3.44). The power of ϵ indicates the rate of growth of this correlation function as the boundary of AdS space is moved to infinity, and we can define for convenience a scaled correlator that is the same as above but without this power of ϵ. The correlation functions given in the rest of this paper are in fact written after such a rescaling.

We would however have obtained a different result had we taken the limits in the following way. We first take $\epsilon \to 0$ in the propagator (A.10), obtaining

$$K^{\epsilon}(z_0) \;=\; (\frac{z_0}{\epsilon})^{d/2}\frac{1}{2^{\nu-1}\Gamma(\nu)(k\epsilon)^{-\nu}}\mathcal{K}_{\nu}(kz_0) \tag{A.14}$$

Using (A.14) in (A.9) we get

$$\langle\mathcal{O}(\vec{k})\mathcal{O}(\vec{k}')\rangle = -\epsilon^{-2(\Delta-d)}\delta(\vec{k}+\vec{k}')k^{2\nu}2^{-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}(\nu+\frac{d}{2}) \;+\; \ldots \tag{A.15}$$

which differs from (A.12) by a factor

$$\frac{\Delta}{2\Delta-d} \tag{A.16}$$

The difference between (A.12) and (A.15) can be traced to the fact that the terms in $K^{\epsilon}(z_0)$ which are subleading in ϵ when z_0 is order unity, give a contribution that is not subleading when $z_0 \to \epsilon$, which is the limit that we actually require when computing the 2-point function.

Appendix B

Properties of $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$

We have seen that a basic building block in expressing the 4–point functions is the quantity $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$, defined by

$$D_{\Delta_1\Delta_3\Delta_2\Delta_4}(x_1,x_3,x_2,x_4) = \int \frac{d^{d+1}z}{z_0^{d+1}}\, \tilde{K}_{\Delta_1}(z,x_1)\, \tilde{K}_{\Delta_3}(z,x_3)\, \tilde{K}_{\Delta_2}(z,x_2)\, \tilde{K}_{\Delta_4}(z,x_4) \quad \text{(B.1)}$$

where $\tilde{K}_\Delta(z,x)$ is

$$\tilde{K}_\Delta(z,x) = \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x})^2}\right)^\Delta . \quad \text{(B.2)}$$

(note the different normalization from $K(z,x)$, equ.(4.3.31)). Thus $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ corresponds to a quartic interaction between scalars of dimension Δ_i, with a simple non–derivative interaction vertex, see Figure 5. Note that sometimes we suppress the explicit coordinate dependence of the D functions. Coordinate labels are always understood to be in the order $(x_1 x_3 x_2 x_4)$.

While the result of the computation of the graviton exchange graph gives a sum of many different D functions, in fact all these functions are closely related to each other. We show that one can relate $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ to $D_{\Delta_1-1\Delta_3-1\Delta_2\Delta_4}$ and $D_{\Delta_1\Delta_3\Delta_2+1\Delta_4+1}$ (see for example (B.3.9)). Further, all the D functions can be obtained from differentiating one single expression (which can be obtained in closed form) with respect to the variables x_{ij}^2. This is shown in section A.3. Using this latter fact we show how for example how

$D_{\Delta\Delta+1\tilde{\Delta}\tilde{\Delta}+1}$ (+ symmetrizing permutations) can be related easily to expressions of the form $D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}$ (see (B.5.16)). These relations are useful to arrive at the two simplified forms of the graviton amplitude (4.4.38) , (4.5.70) given in the text and to show their equivalence.

B.1 Relation between $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ and $W_k^{\Delta'}(a,b)$

The standard integral introduced in (4.5.53) is just a quartic graph evaluated in the inverted frame, with some kinematic factors omitted. The precise relation with $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ is

$$W_k^{\Delta'}(a,b) = x_{13}^{2k}\, x_{14}^{2\Delta'}\, x_{12}^{2(\Delta'+b)}\, D_{2a-b+k,\,k,\,\Delta'+b,\,\Delta'} \tag{B.1.3}$$

B.2 Derivative vertices

The first thing we note is that if we have a quartic interaction with derivatives, given by a coupling

$$\phi_{\Delta_1}(z)\,\phi_{\Delta_3}(z)\,\frac{\partial}{\partial z^\mu}\phi_{\Delta_2}(z)\,\frac{\partial}{\partial z^\nu}\phi_{\Delta_1}(z)\,g^{\mu\nu}\,, \tag{B.2.4}$$

then the computation of the 4–point function with such an interaction can again be reduced to a sum of terms of the form (B.1). This is done with the identity [94]

$$\begin{aligned} g^{\mu\nu}\frac{\partial}{\partial z^\mu}\tilde{K}(z,x_1)\frac{\partial}{\partial z^\nu}\tilde{K}(z,x_2) &= \Delta_1\Delta_2\Big[\tilde{K}_{\Delta_1}(z,x_1)\tilde{K}_{\Delta_2}(z,x_2) \\ &\quad -2x_{12}^2\tilde{K}_{\Delta_1+1}(z,x_1)\,\tilde{K}_{\Delta_2+1}(z,x_2)\Big]\,. \end{aligned} \tag{B.2.5}$$

Thus

$$\begin{aligned} D_{\Delta_1\Delta_3\partial\Delta_2\partial\Delta_4} &\equiv \int \frac{d^{d+1}z}{z_0^{d+1}}\tilde{K}_{\Delta_1}(z,x_1)\,\tilde{K}_{\Delta_3}(z,x_3)\,\frac{\partial}{\partial z^\mu}\tilde{K}_{\Delta_2}(z,x_2)\,z_0^2\,\frac{\partial}{\partial z^\mu}\tilde{K}_{\Delta_4}(z,x_4) \\ &= \Delta_2\Delta_4\left(D_{\Delta_1\Delta_3\Delta_2\Delta_4} - 2\,x_{24}^2\,D_{\Delta_1\Delta_3\Delta_2+1\Delta_4+1}\right)\,. \end{aligned} \tag{B.2.6}$$

B.3 Lowering and raising Δ_i

Not only does the identity (B.2.6) allow us to remove derivatives from the quartic vertex, it is also useful to relate various D functions to each other. Let us rewrite the l.h.s. in (B.2.6) as

$$\begin{aligned} D_{\Delta_1\Delta_3\partial\Delta_2\partial\Delta_4} &= \frac{1}{2}\int \frac{d^{d+1}z}{z_0^{d+1}}\tilde{K}_{\Delta_1}(z,x_1)\tilde{K}_{\Delta_3}(z,x_3)\Box_z\left(\tilde{K}_{\Delta_2}(z,x_2)\tilde{K}_{\Delta_4}(z,x_2)\right) \\ &-\frac{1}{2}\left(m_{\Delta_2}^2+m_{\Delta_4}^2\right)D_{\Delta_1\Delta_3\Delta_2\Delta_4} \end{aligned} \tag{B.3.7}$$

where $m_\Delta^2 \equiv \Delta(\Delta-d)$. Upon integrating by parts of the first term in (B.3.7) we get

$$\begin{aligned} \frac{1}{2}\int [dz]\Box_z\left(\tilde{K}_{\Delta_1}\tilde{K}_{\Delta_3}\right)\tilde{K}_{\Delta_2}\tilde{K}_{\Delta_4} &= \int [dz]\frac{\partial}{\partial z^\mu}\tilde{K}_{\Delta_1}z_0^2\frac{\partial}{\partial z^\mu}\tilde{K}_{\Delta_3}\tilde{K}_{\Delta_2}\tilde{K}_{\Delta_4} \\ &+\frac{1}{2}\left(m_{\Delta_1}^2+m_{\Delta_3}^2\right)D_{\Delta_1\Delta_3\Delta_2\Delta_4} \end{aligned} \tag{B.3.8}$$

Putting relations (B.2.6,B.3.7, B.3.8) together, we find in particular, for $\Delta_1 = \Delta_3 = \Delta$, $\Delta_2 = \Delta_4 = \tilde{\Delta}$:

$$\tilde{\Delta}^2 x_{24}^2 D_{\Delta\Delta\tilde{\Delta}+1\tilde{\Delta}+1} = \Delta^2 x_{13}^2 D_{\Delta+1\Delta+1\tilde{\Delta}\tilde{\Delta}} + \frac{1}{2}\left(\tilde{\Delta}^2-\Delta^2+m_{\tilde{\Delta}}^2-m_\Delta\right)D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}} \tag{B.3.9}$$

A special case is $\Delta = \tilde{\Delta}$, which implies:

$$x_{24}^2 D_{\Delta\Delta\Delta+1\Delta+1} = x_{13}^2 D_{\Delta+1\Delta+1\Delta\Delta}\,. \tag{B.3.10}$$

Iteration of (B.3.9) allows one to prove that more generally

$$\left(x_{24}^2\right)^n D_{\Delta\Delta\Delta+n\Delta+n} = \left(x_{13}^2\right)^n D_{\Delta+n\Delta+n\Delta\Delta}\,. \tag{B.3.11}$$

B.4 Obtaining $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ in closed form

By using a Schwinger parameterization and performing the z integrals, one finds [113] (and references therein):

$$D_{\Delta_1\Delta_3\Delta_2\Delta_4}(x_1,x_3,x_2,x_4) = \frac{\pi^{\frac{d}{2}}\Gamma(\frac{\Sigma-d}{2})\Gamma(\frac{\Sigma}{2})}{2\prod_i\Gamma(\Delta_i)}\int\frac{\prod_j d\alpha_j\,\alpha_j^{\Delta_j-1}\delta(\sum_i\alpha_i-1)}{\left(\sum_{k,l}\alpha_k\alpha_l x_{kl}^2\right)^{\frac{\Sigma}{2}}} \tag{B.4.12}$$

where

$$\Sigma \equiv \sum_i \Delta_i\,. \tag{B.4.13}$$

We observe that any $D_{\Delta_1\Delta_3\Delta_2\Delta_4}$ can be obtained by differentiating an appropriate number of times in the variables x_{ij} the basic function

$$B(x_{ij}) = \int\frac{\prod_j d\alpha_j\delta(\sum_i\alpha_i-1)}{\left(\sum_{k,l}\alpha_k\alpha_l x_{kl}^2\right)^2}\,. \tag{B.4.14}$$

$B(x_{ij})$ is given in closed form in [113]. From the integral representation (B.4.12) we immediately find

$$\frac{\partial}{\partial x_{13}^2}D_{\Delta_1\Delta_3\Delta_2\Delta_4} = -\frac{2\Delta_1\Delta_3}{\Sigma-d}\,D_{\Delta_1+1\Delta_3+1\Delta_2\Delta_4} \tag{B.4.15}$$

B.5 Symmetrizing identities

Equation (B.4.15) can be used to show that a sum of D functions which is symmetric under $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$ can always be rewritten in a basis in which each individual term shares this symmetry, *i.e.* each term is of the form $D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}$. For example:

$$x_{12}^2 D_{\Delta+1\Delta\tilde{\Delta}+1\tilde{\Delta}} + x_{14}^2 D_{\Delta+1\Delta\tilde{\Delta}\tilde{\Delta}+1} = \frac{\Sigma-d}{2\tilde{\Delta}}\,D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}} - \frac{\Delta}{\tilde{\Delta}}\,x_{13}^2 D_{\Delta+1\Delta+1\tilde{\Delta}\tilde{\Delta}} \tag{B.5.16}$$

where $\Sigma \equiv 2\Delta + 2\tilde{\Delta}$. Let us see how to derive this identity. It follows from conformal invariance that

$$D_{\Delta_1\Delta_3\Delta_2\Delta_4} = \left(\prod_{i<j}(x_{ij}^2)^{-\frac{\Delta_i+\Delta_j}{2}+\frac{\Sigma}{6}}\right) E_{\Delta_1\Delta_3\Delta_2\Delta_4}(\xi,\eta) \tag{B.5.17}$$

where $\xi \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $\eta \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ are conformal cross ratios. From simple chain rule manipulations we then get

$$\left(x_{12}^2 \frac{\partial}{\partial x_{12}^2} + x_{13}^2 \frac{\partial}{\partial x_{13}^2} + x_{14}^2 \frac{\partial}{\partial x_{14}^2}\right) E_{\Delta_1\Delta_3\Delta_2\Delta_4}(\xi,\eta) = 0\,. \tag{B.5.18}$$

Using (B.4.15), the last equation is tantamount to (B.5.16) for $\Delta_1 = \Delta_3 = \Delta$, $\Delta_2 = \Delta_4 = \tilde{\Delta}$. Similar arguments lead to the more complicated identity

$$\begin{aligned} & x_{12}^2 x_{14}^2 D_{\Delta+2\,\Delta\tilde{\Delta}+1\tilde{\Delta}+1} + x_{23}^2 x_{34}^2 D_{\Delta\Delta+2\,\tilde{\Delta}+1\tilde{\Delta}+1} = \\ & -\frac{\Delta}{\Delta+1}(x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2) D_{\Delta+1\Delta+1\tilde{\Delta}+1\tilde{\Delta}+1} + \frac{\Delta(\Sigma-d)(\Sigma+2-d)}{4(\Delta+1)\tilde{\Delta}^2} D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}} \\ & -\frac{\Delta(2\Delta+1)(\Sigma+2-d)}{2(\Delta+1)\tilde{\Delta}^2} x_{13}^2 D_{\Delta+1\Delta+1\tilde{\Delta}\tilde{\Delta}} + \frac{\Delta(\Delta+1)}{\tilde{\Delta}^2} x_{13}^4 D_{\Delta+2\,\Delta+2\,\tilde{\Delta}\tilde{\Delta}}\,, \end{aligned} \tag{B.5.19}$$

where $\Sigma = 2\Delta + 2\tilde{\Delta}$.

B.6 Series expansion of $D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}$

From (B.1.3) and (4.6.23):

$$\begin{aligned} D_{\Delta\Delta\tilde{\Delta}\tilde{\Delta}}(x_1,x_3,x_2,x_4) = & \frac{(-)^{\Delta+\tilde{\Delta}}\pi^{\frac{d}{2}}\Gamma(\tilde{\Delta}+\Delta-\frac{d}{2})}{\Gamma(\tilde{\Delta})^2\Gamma(\Delta)^2(x_{13}^2)^{\Delta}(x_{24}^2)^{\tilde{\Delta}}} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)^2\, s^{k+1}}{\Gamma(k-\tilde{\Delta}+2)\Gamma(k-\Delta+2)} \\ & \Big\{ b_k(t) - a_k(t)[\ln s + 2\psi(k+1) - \psi(k-\Delta+2) - \psi(k-\tilde{\Delta}+2)] \Big\}\,. \end{aligned} \tag{B.6.20}$$

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