Aspects of Riemannian Geometry in Quantum Field Theories

by

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Abstract

In this thesis we study in detail several situations where the areas of Riemannian geometry and quantum field theory come together. This study is carried out in three distinct situations. In the first part we show how to introduce new local gauge invariant variables for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, explicitly parameterizing the physical Hilbert space of the theory. We show that these gauge invariant variables have a geometrical interpretation, and that they can be constructed such that the emergent geometry is that of $\mathcal{N} = 1$ supergravity: a Riemannian geometry with vector-spinor generated torsion. In the second part we study bosonic and supersymmetric sigma models, investigating to what extent their geometrical target space properties are encoded in the T-duality symmetry they possess. Starting from the consistency requirement between T-duality symmetry and renormalization group flows, we find the two-loop metric beta function for a $d = 2$ bosonic sigma model on a generic, torsionless background. We then consider target space duality transformations for heterotic sigma models and strings away from renormalization group fixed points. By imposing the consistency requirements between the T-duality symmetry and renormalization group flows, the one loop gauge beta function is uniquely determined. The issue of heterotic anomalies and their cancelation is addressed from this duality constraining viewpoint, providing new insight and mechanisms of anomaly cancelation. In the third part we compute a radiative contribution to an anomalous correlation function of one axial current and two energy-momentum tensors, $\langle A_\alpha(z) T_{\mu\nu}(y) T_{\rho\sigma}(x) \rangle$, corresponding to a contribution to the gravitational axial anomaly in the massless Abelian Higgs model. In all three situations there is a rich interplay between geometry and field theory.

Thesis Supervisor: Jeffrey Goldstone
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TO THE MEMORY OF KEN JOHNSON

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Chapter 1

Introduction

Ever since the dawn of modern science, the fields of physics and mathematics have been unequivocally associated to each other in a multitude of situations and areas. One may possibly claim that they also share the same roots, and therefore advances in one field must always reflect on the other and vice-versa, no matter how trivial or fundamental such a reflection might take shape. Examples of such situations are quite often met in the research which is nowadays performed (independently) in both fields.

Having followed on somewhat distinct paths perhaps ever since late in the last century, there is still a very strong interest by many researchers in the boundary of the two fields, exploring the interface science that has come to be known as mathematical physics. One such aspect that we wish to explore in this thesis is what lies in this interface at the point where Riemannian geometry and quantum field theory meet. We shall see, through the three distinct problems that build this thesis, that many interesting results are there to be explored and investigated.

We shall begin by looking at a problem in $3+1$ dimensional supersymmetric gauge theory, to be specific, $\mathcal{N}=1$ supersymmetric $SU(2)$ Yang-Mills theory. In here we develop a new tool to study the strong coupling limit of this theory, in the form of introducing new variables for the Yang-Mills theory, which have the property of being gauge invariant. Indeed gauge invariance is an important constraint on the states of the gauge theory, in the form of Gauss' law. The fact that the Yang-Mills system is constrained has been a difficult drawback to solve in order to fully explore the quantization of this theory. The question of whether one can construct gauge invariant variables then becomes of relevance as one realizes that such variables would allow for a trivial implementation of the Gauss' law constraint. If moreover one can construct these variables such that they are local, they would then seem to be the most appropriate ones to describe the moduli space of the theory. One more point in favor of such a programme is the fact that in temporal gauge the remaining gauge invariance of the Yang-Mills theory is restricted to space-dependent transformations at a fixed time. This is in fact the true quantum mechanical symmetry of the theory. Working with local gauge invariant variables this symmetry of the Hamiltonian can be maintained exactly, even under approximations to the dynamics.

All this said, we strongly believe that this is indeed an interesting problem to explore in quantum field theory, but one would not seem to realize where the connection to Riemannian geometry would come into the game. What we shall see later is that such a connection arrives from the way we will choose to define the new variables: we shall replace the gauge connection of $SU(2)$ by a covariant variable under the gauge group, which shall enjoy the fact that it can be also interpreted as a dreibein, *i.e.*, a square root of a metric. We shall see that this metric lives in a 3 dimensional manifold, and that it can be used as a local gauge invariant variable for Yang-Mills theory. However, our interest in here is, as we mentioned before, on supersymmetric Yang-Mills theory. Therefore, we must not forget to include the fermionic partners of our bosonic variables. In chapter 2 we shall see in detail how this can be accomplished. We shall learn that the local gauge invariant variables we will construct for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory have a Riemannian geometrical interpretation in the sense that they can be constructed such that the emergent geometry is that of $\mathcal{N} = 1$ supergravity: a Riemannian geometry with vector-spinor generated torsion.

After studying this problem, we leave gauge theory behind and move into the domain of sigma models, where we shall study both bosonic and supersymmetric sigma models. In these models, describing maps from a given two dimensional surface into a general target manifold, Riemannian geometry makes its appearance from the very beginning, in the action for the models we shall consider. Indeed, having emerged from string theory – a possibly quantum theory of gravity – it is but to expect that a metric should somehow be incorporated in these models from scratch. Indeed, Riemannian metrics in the target manifold are nothing but infinite-dimensional coupling parameters of the two dimensional quantum field theory. The Riemannian geometry of the target is therefore constrained by the quantum field theoretic properties of the two dimensional theory and, in the particular case of string theory, the condition that the beta functions for the diverse couplings of the sigma model vanish is equivalent to saying that the geometrical structures in the target manifold obey the [generalized] Einstein's equations.

But again because these models are string theory inspired, we can look at all the nice properties of strings and ask which, if any, of such properties are still valid once we move away from the conformal fixed points where the sigma models describe strings – and in particular whether such properties have any chance of being valid throughout all of the parameter space of the sigma model. One such property we shall be interested about, and which we shall study in detail in chapters 3 and 4 of this thesis, is target space duality, henceforth T-duality. This is a perturbative symmetry of string theory which basically relates target manifold compactifications in circles of radius R with compactifications in circles of radius ℓ^2_{string}/R, with ℓ_{string} being the characteristic string length. We will learn that by exploring a consistency requirement between T-duality and the renormalization group flows of the sigma model, we shall be able to find the beta functions of these models for all the coupling parameters. From a string theory point of view this simply means that geometrical target space properties are encoded in this T-duality symmetry. Moreover, in the case of heterotic sigma models, we will also learn that this duality symmetry provides new insight and mechanisms for cancelation of a certain class of anomalies.

Once we are done with sigma models, we shall return to the realm of the 3+1 world, and study the problem of gravitational axial anomalies. If the anomaly is beyond doubt a quantum field theoretic phenomena, it is also not less clear that gravitation is a Riemannian geometrical phenomena (at least in the domain of energy we shall be looking at). We are therefore in a situation where we encounter quantum fields in a curved background spacetime. In this last chapter of the present thesis, chapter 5, we shall compute a radiative contribution to an anomalous correlation function involving one axial current and two energy-momentum tensors, corresponding to a contribution to the gravitational axial anomaly in the massless Abelian Higgs model. We shall learn new techniques to perform such a complicated calculation, and we shall see that the two loop contribution is found not to vanish, due to the presence of two independent tensor structures in the anomalous correlator.

The three problems dealt with in this thesis are clearly quite distinct, but they all share the property of presenting yet some new examples of interactions between Riemannian geometry and quantum field theory. These problems appeared in the literature as four distinct publications. Chapter 2 was published in Nuclear Physics, [65]. Chapter 3 was published in Physical Review Letters, [47], and chapter 4 is to be published in International Journal of Modern Physics, [58]. Finally, chapter 5 was published in Physical Review, [60]. During the process of five years of study at MIT, I also enjoyed the opportunity of doing other research, not directly related to this thesis. In particular, other matters and problems were studied, and I believe they should be mentioned in here. These research projects were not included in this thesis as they did not share the same theme studied in here, the one of interactions between geometry and quantum theories. These projects were (a) studies on classical configurations of string theory in $3+1$ dimensional target manifolds, where the strings under consideration had an initial knotted topology. These investigations were published in Physics Letters, [24, 66]. The other research project was (b) a study of the quantum cosmology of an S-duality invariant $\mathcal{N}=1$ supergravity model in a closed homogeneous and isotropic Friedmann-Robertson-Walker spacetime, and which is to appear in Classical and Quantum Gravity, [16].

Chapter 2

Super-Yang-Mills Theory

2.1 Introduction

For quite sometime now there exists a nice geometrical setting for Yang-Mills theory. That is based on fiber bundle differential geometry, where the configuration space is obtained by factoring out time independent gauge transformations, and is then seen as the base space of a principal fiber bundle, where the structure group is the gauge group [57]. There are many concepts of Riemannian geometry that can then came into the game, as there is the possibility of defining a Riemannian metric on the space of non-equivalent gauge connections [8].

However, this setting must be cast into a more workable form when we want to study the strong coupling regime of Yang-Mills theory. In here, gauge invariance becomes an important constraint on states of the theory in the form of Gauss' law. This constraint amounts to a reduction of the number of degrees of freedom present in the gauge connection: if one starts with a gauge group G, in the canonical formalism and in temporal gauge $A_0^a = 0$, the number of variables is $3 \dim G$, when in fact we only have $2 \dim G$ physical gauge invariant degrees of freedom. The question of whether one can construct local gauge invariant variables is then an important one, as it would allow us to easily implement the Gauss' law constraint. These variables would then seem the most appropriate ones to describe the physical space of the theory. Moreover, observe that in temporal gauge the remaining local gauge invariance is now restricted to space-dependent transformations at a fixed time. This is the true quantum mechanical symmetry of the theory. Working with local gauge invariant variables, this symmetry of the Hamiltonian can be maintained exactly, even under approximations to the dynamics.

This idea first appeared in [48, 36], and has recently gained new momentum with the work in [30, 11, 54, 43, 55, 40, 33, 45, 44], and references therein. In [43], one constructs a change of variables that will allow replacing the coordinates A_i^a by new coordinates u_i^a which have the property of transforming covariantly under the gauge group, as opposed to as a gauge connection. Then, in these new coordinates, the generator of gauge transformations becomes a (color) rotation generator, and by contracting in color we can obtain gauge invariant variables to our theory, $g_{ij} \equiv u_i^a u_j^a$. States $\Psi[g_{ij}]$ depending only on these gauge invariant variables manifestly satisfy Gauss' law. One must be careful, however. Not any choice of gauge covariant variables is adequate: an appropriate set of variables should describe the correct number of gauge invariant degrees of freedom at each point of space, and should also be free of ambiguities such as Wu-Yang ambiguities [79]. In this case, several gauge unrelated vector potentials may lead to the same color magnetic field. Variables that are Wu-Yang insensitive are of no use, as in the functional integral formulation Wu-Yang related potentials must be integrated over – since they are not gauge related –, while functional integration over Wu-Yang insensitive variables always misses these configurations. The absence of Wu-Yang ambiguities will be clear if we are able to invert the variable transformation, *i.e.*, if when transforming $A \rightarrow u$ one can also have an explicit expression for $A[u]$.

In [43], the set of gauge covariant variables $\{u_i^a\}$ that replace the $SU(2)$ gauge connection was defined by the differential equations:

$$\epsilon^{ijk} D_j u_k^a \equiv \epsilon^{ijk}(\partial_j u_k^a + \epsilon^{abc} A_j^b u_k^c) = 0, \tag{2.1.1}$$

which is equivalent to writing,

$$\partial_j u_k^a + \epsilon^{abc} A_j^b u_k^c - \Gamma_{jk}^s u_s^a = 0, \tag{2.1.2}$$

as the $\{u_i^a\}$ have det $u \neq 0$, and so form a complete basis. Observe that we have $\Gamma_{jk}^s = \Gamma_{kj}^s$, and these quantities can be written as,

$$\Gamma_{jk}^s = \frac{1}{2} g^{sn} (\partial_j g_{nk} + \partial_k g_{jn} - \partial_n g_{jk}), \tag{2.1.3}$$

where

$$g_{ij} \equiv u_i^a u_j^a. \tag{2.1.4}$$

So, a "metric" tensor was implicitly introduced by the defining equations for the new variables, (2.1.1). Observe that equation (2.1.2) is simply the so-called dreibein postulate, where the $\{u_i^a\}$ plays the role of a dreibein, $\omega_i^{ac} \equiv \epsilon^{abc} A_i^b$ is a spin-connection, and Γ_{jk}^i is the affine metric connection. A torsion-free Riemannian geometry in a three manifold was then introduced by the definition of the new variables. The metric g_{ij} contains in itself the six local gauge invariant degrees of freedom of the $SU(2)$ gauge theory. Moreover, any gauge invariant wave-functional of A_i^a can be written as a function of g_{ij} only, and any wave-functional of g_{ij} is gauge invariant [43]. This implements gauge invariance exactly. Finally, the dreibein postulate can be inverted so that one obtains,

$$A_i^a = -\frac{1}{2} \epsilon^{abc} u^{bj} \nabla_i u_j^c, \tag{2.1.5}$$

where we use the notation $\nabla_j u_k^a \equiv \partial_j u_k^a - \Gamma_{jk}^s u_s^a$ for the purely geometric covariant derivative (as opposed to the gauge covariant derivative). Therefore, the new variables avoid Wu-Yang ambiguities.

Full geometrization of Yang-Mills theory in this formulation was then carried out in [43, 40]. The electric energy involves the inversion of a differential operator that can generically have zero modes. By deforming equations (2.1.1) it was then shown how one could proceed to compute the electric tensor [45]. Instanton and monopole configurations have been identified as the S^3 and $S^2 \times \mathbf{R}$ geometries [45], and, more recently, the form that the wave-functional for two heavy color sources should take has been calculated [44]. The computations are carried out in the Schrödinger representation of gauge theory, see [51] for a review.

Supersymmetric Yang-Mills theory has also been well established for quite sometime now. It allows for many simplifications in quantum computations, and with an appropriate choice of matter content and/or number of supersymmetry generators, one can obtain finite quantum field theories. Textbook references are [9, 77, 78]. Moreover, recently there has been a lot of progress and activity in the field due to the possibility of actually solving for the low-energy effective action of certain cases of supersymmetric Yang-Mills theories, starting with the work in [68]. It is then natural to extend the work on gauge invariant geometrical variables to the supersymmetric case. That is what we shall do in here.

We shall see that it is possible to define variables that also have a geometrical interpretation, namely, as the variables present in supergravity. We should point out, however, that no coupling to gravity is ever considered. Still, we need a motivation to construct the new variables. As in the pure Yang-Mills case the new variables and geometry have an interpretation as the variables and geometry of three dimensional gravity, it is natural to assume that in the supersymmetric case the new variables and geometry could likewise have an interpretation as the variables and geometry of three dimensional supergravity.

This shall be a guiding principle throughout our work. More geometrical intuition on how to construct the new variables will come from an extra symmetry enjoyed by both the canonical variables and Gauss' law generator. That is a symmetry under $GL(3)$ transformations, a diffeomorphism symmetry. This will allow us to naturally assign tensorial properties to diverse local quantities of the theory. Obviously the Hamiltonian (or any other global operator) will not possess this symmetry. After all, supersymmetric Yang-Mills theory is not diffeomorphism invariant.

The plan of this chapter is as follows. In section 2 we start by reviewing the conventions of $\mathcal{N} = 1$ supersymmetry, and also outline the geometry of supergravity.

In section 3 we will then explore the $GL(3)$ symmetry, assigning tensorial properties to local (composite) operators. With this in hand, we then proceed to define gauge invariant geometrical variables for supersymmetric Yang-Mills theory in section 4, carrying out the full geometrization of the theory in section 5. Section 6 presents a concluding outline.

2.2 Review and Conventions

The conventions in [9, 77, 78] are basically the same. We will follow [9] with minor changes, as we take $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. The $\mathcal{N} = 1$ supersymmetry algebra is obtained by introducing one spinor generator, Q, which is a Majorana spinor, to supplement the usual (bosonic) generators of the Poincaré group. The $\mathcal{N} = 1$ supersymmetry algebra is then the Poincaré algebra plus:

$$[P_\mu, Q] = 0,$$

$$[M_{\mu\nu}, Q] = -i\sigma_{\mu\nu} Q,$$

$$\{Q, \bar{Q}\} = 2\gamma^\mu P_\mu, \tag{2.2.1}$$

where $\bar{Q} \equiv Q\gamma^0$.

Supersymmetric gauge theory, based on gauge group G with gauge algebra $\mathcal{G}$, has as component fields the gluons, or gauge connection, A^a_μ; the gluinos, super-partners of the gauge fields and Majorana spinors, λ^a; and the scalar auxiliary fields D^a. All these fields are in the adjoint representation of $\mathcal{G}$. In Wess-Zumino gauge, the action is,

$$S = \int d^4x \, \{-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{i}{2} \bar{\lambda}^a \gamma^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a\}, \tag{2.2.2}$$

where we can see that the auxiliary fields have no dynamics. The supersymmetry transformation laws of the fields, that leave the action invariant are:

$$\delta A^a_\mu = i\bar{\varepsilon}\gamma_\mu \lambda^a,$$

$$\delta\lambda^a = (\sigma^{\mu\nu} F^a_{\mu\nu} - i\gamma_5 D^a)\varepsilon,$$

$$\delta D^a = \bar{\varepsilon}\gamma_5\gamma^\mu D_\mu \lambda^a, \tag{2.2.3}$$

where ε is a Majorana spinor, which is the parameter of the infinitesimal supersymmetry transformation. These transformation laws implement a representation of the $\mathcal{N} = 1$ supersymmetry algebra in the quantum gauge field theory. The Noether conserved current of supersymmetry is a vector-spinor,

$$J^\mu = i\gamma^\mu \sigma^{\alpha\beta} F^a_{\alpha\beta} \lambda^a, \tag{2.2.4}$$

and so the quantum field theoretic representation of the supersymmetry generator is given by the Majorana spinor,

$$Q = i \int d^3x \, \gamma^0 \sigma^{\mu\nu} F^a_{\mu\nu} \lambda^a. \tag{2.2.5}$$

This outlines our usage of notation for supersymmetric gauge theory. We still have to outline notation for the supergravity geometry. In here, one has a graviton, $g_{\mu\nu}$, and a gravitino which is described by a Rarita-Schwinger field, ψ_μ. So, we need to start by reviewing notation for inserting spinors in curved manifolds. Having a metric, one can define orthonormal frames and so insert a tetrad base at the tangent space to a given point, which will allow one to translate between curved and flat indices. In particular, this allows us to introduce gamma matrices in the manifold, and so introduce spinors. If we consider a manifold M, and pick a point $p \in M$, we can introduce a tetrad base $\{u^a_\mu\}$ at p via,

$$g_{\mu\nu} = u^a_\mu u^b_\nu \eta_{ab}, \tag{2.2.6}$$

defining an orthonormal frame at each point on M. One can now insert gamma matrices as $\gamma^\mu(x) u^a_\mu(x) = \gamma^a$, where the γ^a are numerical matrices. Local Lorentz transformations in the tangent space T_pM are $\Lambda^a{}_b(p)$ and (Dirac) spinors at $p \in M$

rotate as,

$$\psi_\alpha(p) \to S_{\alpha\beta}(\Lambda^a{}_b(p))\psi_\beta(p). \tag{2.2.7}$$

Next, one constructs a covariant derivative, $\mathbf{D}_a\psi_\alpha$, which is a local Lorentz vector, and transforms as a spinor,

$$\mathbf{D}_a\psi_\alpha \to S_{\alpha\beta}(\Lambda)\Lambda_a{}^b\mathbf{D}_b\psi_\beta. \tag{2.2.8}$$

That is done via a connection Ω_μ such that, $\mathbf{D}_a\psi = u^\mu_a(\partial_\mu + \Omega_\mu)\psi$, and,

$$\Omega_\mu = \frac{1}{2}\omega^{ab}_\mu\sigma_{ab} = \frac{1}{2}u^a_\nu\nabla_\mu u^{b\nu}\sigma_{ab}, \tag{2.2.9}$$

where ω^{ab}_μ is the spin-connection.

Now that we have spinors defined on curved manifolds, we can proceed with supergravity. In here, the Riemannian connection $\Gamma^\rho_{\mu\nu}$ is *not* torsion-free. It is still metric compatible, so that one can write,

$$\Gamma^\rho_{\mu\nu} = \hat{\Gamma}^\rho_{\mu\nu} - K_{\mu\nu}{}^\rho, \tag{2.2.10}$$

where $\hat{\Gamma}^\rho_{\mu\nu}$ is the affine metric connection, and $K_{\mu\nu}{}^\rho$ is the contorsion tensor. Hatted symbols will always stand for quantities computed via the affine metric connection. The torsion tensor is,

$$T_{\mu\nu}{}^\rho \equiv \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}, \tag{2.2.11}$$

and so,

$$K_{\mu\nu}{}^\rho = -\frac{1}{2}(T_{\mu\nu}{}^\rho - g_{\nu\lambda}g^{\sigma\rho}T_{\mu\sigma}{}^\lambda - g_{\mu\lambda}g^{\sigma\rho}T_{\nu\sigma}{}^\lambda). \tag{2.2.12}$$

In $\mathcal{N} = 1$ supergravity, the torsion is defined by the Rarita-Schwinger field ψ_μ as,

$$T_{\mu\nu}{}^a = \frac{i}{2}k^2\bar{\psi}_\mu\gamma^a\psi_\nu, \tag{2.2.13}$$

where a is a flat index, and k is the gravitational dimensionfull constant. The tetrad postulate is,

$$\mathcal{D}_\mu u^a_\nu \equiv \partial_\mu u^a_\nu + \omega^{ab}_\mu u_{b\nu} - \Gamma^\rho_{\mu\nu} u^a_\rho = 0, \tag{2.2.14}$$

and the covariant derivative acting on spinor indices is,

$$(\mathbf{D}_\mu)_{\alpha\beta} \equiv \delta_{\alpha\beta}\partial_\mu + \frac{1}{2}\omega_{\mu ab}(\sigma^{ab})_{\alpha\beta}. \tag{2.2.15}$$

Finally, the supersymmetry transformations that leave the $\mathcal{N} = 1$ supergravity action (Einstein-Hilbert plus Rarita-Schwinger) invariant are:

$$\delta u^a_\mu(x) = ik\bar{\xi}(x)\gamma^a\psi_\mu(x),$$

$$\delta\psi_\mu(x) = \frac{2}{k}\mathbf{D}_\mu\xi(x), \tag{2.2.16}$$

where $\xi(x)$ is the infinitesimal parameter of the transformation (now a space-time dependent Majorana spinor), and where we have not included the auxiliary fields. This ends our review and outline of conventions. We can now start analyzing the gauge invariant variables geometrization of supersymmetric Yang-Mills theory.

2.3 Canonical Formulation and $GL(3)$ Properties

In the Lagrangian formulation of the theory, the $\mathcal{N} = 1$ supersymmetry algebra closes only up to the field equations. In order to obtain manifest supersymmetry, and off-shell closure of the algebra, one needs to introduce auxiliary fields. In contrast to this situation, it is known that in the canonical formalism the $\mathcal{N} = 1$ super Lie algebra closes without the introduction of auxiliary fields (in terms of Dirac brackets the algebra closes strongly; otherwise it closes weakly, *i.e.*, up to the first-class constraints) [73, 72]. So, we drop the auxiliary fields.

The Hamiltonian for supersymmetric gauge theory is therefore,

$$H = \int d^3x\{\frac{1}{2}e^2(E^{ai})^2 + \frac{1}{2e^2}(B^{ai}[A^b_j])^2 - \frac{i}{2}\bar{\lambda}^a\gamma^i D_i\lambda^a\}, \tag{2.3.1}$$

where e is the coupling constant. The gauge covariant derivative is,

$$D_i\lambda^a = \partial_i\lambda^a + f^{abc}A_i^b\lambda^c, \tag{2.3.2}$$

and the magnetic field potential energy,

$$B^{ai}[A_n^b] \equiv \frac{1}{2}\epsilon^{ijk}F_{jk}^a[A_n^b] = \epsilon^{ijk}(\partial_j A_k^a + \frac{1}{2}f^{abc}A_j^b A_k^c). \tag{2.3.3}$$

We still have to impose the Gauss' law constraint on the physical states of the theory,

$$\mathcal{G}^a(x) \equiv D_i E^{ai}(x) - \frac{i}{2}f^{abc}\lambda^b(x)\lambda^c(x) \quad , \quad \mathcal{G}^a(x)\Psi[A_i^b, \lambda^c] = 0. \tag{2.3.4}$$

This local composite operator is the generator of local gauge transformations.

There is one more element in the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, and that is the Majorana spinor Q, the generator of supersymmetry. Using the definition,

$$Q \equiv \int d^3x\, \mathcal{Q}(x), \tag{2.3.5}$$

one can then write,

$$\mathcal{Q}(x) = i(-e\gamma_i E^{ai}(x) + \frac{1}{e}\epsilon_{ijk}\gamma^0\sigma^{ij}B^{ak}[A_n^b(x)])\lambda^a(x), \tag{2.3.6}$$

or, using the explicit Weyl representation of the gamma matrices, we can equivalently write this local composite operator in a more compact form,

$$\mathcal{Q}(x) = i\begin{pmatrix} 0 & (eE^{ai}(x) + \frac{i}{e}B^{ai}[A_n^b(x)])\sigma_i \\ (-eE^{ai}(x) + \frac{i}{e}B^{ai}[A_n^b(x)])\sigma_i & 0 \end{pmatrix}\lambda^a(x). \tag{2.3.7}$$

In the bosonic half of the theory, the canonical variables are $A_i^a(x)$ and $E^{ai}(x)$. Canonical quantization is carried out by the commutator,

$$[A_i^a(x), E^{bj}(y)] = i\delta^{ab}\delta_j^i\delta(x-y). \tag{2.3.8}$$

The momentum $E^{ai}(x)$ will be implemented as a functional derivative acting on wave-functionals as,

$$E^{ai}(x)\Psi[A_n^b, \lambda^c] \to -i\frac{\delta}{\delta A_i^a(x)}\Psi[A_n^b, \lambda^c]. \tag{2.3.9}$$

In the fermionic half of the theory, one has a Majorana spinor $\lambda^a(x)$. Canonical quantization is carried out by establishing the anti-commutation relations for the spinorial field,

$$\{\lambda_\alpha^a(x), \lambda_\beta^b(y)\} = \delta^{ab}\delta_{\alpha\beta}\delta(x-y). \tag{2.3.10}$$

Both the commutator and the anti-commutator are to be evaluated at equal times. We can now compute the commutators and anti-commutators of this theory, which involve the composite operators H, $\mathcal{G}^a(x)$ and $\mathcal{Q}(x)$. Clearly, these (anti)-commutators are related to the symmetry transformations generated by these operators.

The commutators involving the generator of local gauge transformations of the canonical variables can be computed to be,

$$[A_i^a(x), \mathcal{G}^b(y)] = i(\delta^{ab}\partial_i - f^{acb}A_i^c(x))\delta(x-y), \tag{2.3.11}$$

$$[E^{ai}(x), \mathcal{G}^b(y)] = if^{abc}E^{ci}(x)\delta(x-y), \tag{2.3.12}$$

$$[\lambda^a(x), \mathcal{G}^b(y)] = if^{abc}\lambda^c(x)\delta(x-y), \tag{2.3.13}$$

and the (anti)-commutators involving the local composite operator associated to the supersymmetry generator can similarly be found to be,

$$[A_i^a(x), \mathcal{Q}(y)] = -\gamma_i\lambda^a(x)\delta(x-y), \tag{2.3.14}$$

$$[E^{ai}(x), \mathcal{Q}(y)] = \epsilon^{ijk}(\epsilon_{knm}\gamma^0\sigma^{nm})D_j\lambda^a\delta(x-y), \tag{2.3.15}$$

$$\{\lambda^a(x), \mathcal{Q}(y)\} = i(-e\gamma_iE^{ai}(x) + \frac{1}{e}\epsilon_{ijk}\gamma^0\sigma^{ij}B^{ak}[A_n^b(x)])\delta(x-y). \tag{2.3.16}$$

Moreover, one will also have that the Hamiltonian and the supersymmetry generator are both gauge invariant composite operators, as,

$$[H, \mathcal{G}^a(x)] = 0, \tag{2.3.17}$$

$$[\mathcal{Q}(x), \mathcal{G}^a(y)] = 0. \tag{2.3.18}$$

As expected, the generators $\mathcal{G}^a(x)$ define the local gauge algebra,

$$[\mathcal{G}^a(x), \mathcal{G}^b(y)] = i f^{abc} \mathcal{G}^c(x) \delta(x - y). \tag{2.3.19}$$

The supersymmetry generator Q defines, along with the generators of the Poincaré algebra, the $\mathcal{N} = 1$ supersymmetry algebra [73]. However, the defined local composite operator $\mathcal{Q}(x)$ does not define a local algebra. That is to be expected as we do not have local supersymmetry in the theory. This local operator was only introduced in order to facilitate the following tensorial analysis based on diffeomorphism transformations of the presented (anti)-commutators.

So, we now want to check that there is a $GL(3)$ symmetry at work for the formulae (2.3.8), (2.3.10) and (2.3.11-13), (2.3.19). The bosonic part tensorial assignments will be just like in the pure Yang-Mills case [43], as is to be expected. The mentioned canonical relations are covariant under diffeomorphisms $x^i \to y^n(x^i)$ on the domain $\mathbf{R}^3$, provided $A_i^a(x)$ is a one-form in $\mathbf{R}^3$, transforming as

$$A'^a_n(y^m) = \frac{\partial x^i}{\partial y^n} A_i^a(x^j), \tag{2.3.20}$$

where $[\partial x^i / \partial y^n]$ is a $GL(3)$ matrix. That $\mathbf{A}(x) \equiv A_\mu^a(x) T^a dx^\mu$ is a Lie algebra valued one-form is a well known fact from the fiber bundle geometry of gauge theory; so consistency holds. Also, provided $E^{ai}(x)$ is a vector density (weight -1) in $\mathbf{R}^3$, transforming as

$$E'^{an}(y^m) = \det[\frac{\partial x^i}{\partial y^n}] \frac{\partial y^n}{\partial x^i} E^{ai}(x^j). \tag{2.3.21}$$

The same property holds for $B^{ak}(x)$. This is consistent with the implementation of $E^{ai}(x)$ as a functional derivative (2.3.9), and with the definition of the magnetic field (2.3.3). Commutator (2.3.8) is then clearly diffeomorphic invariant, without the intervention of a space metric. However, to introduce spinors, one does need a metric (more precisely, a dreibein base). We shall assume there is a metric, g_{ij}, and later we will construct it using the bosonic dynamical variables of the theory.

When restricted to three dimensional Euclidean space, Lorentz transformations become rotations in $\mathbf{R}^3$. The spinor representation of a rotation is then, at a point $p \in M$, given by the orthogonal matrix acting on spinor indices,

$$S_{\alpha\beta}(\Lambda(p)) = \exp(\frac{1}{2}\omega_{ab}(\Lambda(p))\sigma^{ab})_{\alpha\beta}, \tag{2.3.22}$$

where ω is the rotation parameter. We can now define the $GL(3)$ properties of $\lambda^a(x)$, in order to maintain the anti-commutator (2.3.10) diffeomorphism invariant. That relation is invariant under diffeomorphisms, provided $\lambda^a(x)$ is a spinorial density (weight $-\frac{1}{2}$) in $\mathbf{R}^3$, transforming as

$$\lambda'^a_\alpha(y^m) = \det[\frac{\partial x^i}{\partial y^n}]^{\frac{1}{2}} S_{\alpha\beta}(\Lambda(p))\lambda^a_\beta(x^j). \tag{2.3.23}$$

Let us see what are the consequences of these $GL(3)$ properties on the composite local operators $\mathcal{G}^a(x)$ and $\mathcal{Q}(x)$. Starting with the generator of local gauge transformations, we observe that the tensorial properties of the canonical variables imply that under diffeomorphisms one will have that $\mathcal{G}^a(x)$ is a scalar density (weight -1) in $\mathbf{R}^3$, transforming as

$$\mathcal{G}'^a(y^m) = \det[\frac{\partial x^i}{\partial y^n}]\mathcal{G}^a(x^j). \tag{2.3.24}$$

This automatically verifies that the canonical commutators (2.3.11-13) and (2.3.19) are invariant under local diffeomorphisms on the domain of the local canonical variables.

Now, look at the other local composite operator, $\mathcal{Q}(x)$, (2.3.6) or (2.3.7). First observe that in (2.3.7) the Pauli matrices $\vec{\sigma}$ are numerical matrices, and not dynamical ones (in which case one would write $\sigma^i(x)u_i^a(x) = \sigma^a$, $\{u_i^a\}$ being a dreibein base).

Write (2.3.7) as:

$$\mathcal{Q}(x) \equiv \sigma_i \otimes \Pi^i(x), \tag{2.3.25}$$

where,

$$\Pi^i(x) \equiv i \begin{pmatrix} 0 & eE^{ai}(x) + \frac{i}{e}B^{ai}[A_n^b(x)] \\ -eE^{ai}(x) + \frac{i}{e}B^{ai}[A_n^b(x)] & 0 \end{pmatrix} \lambda^a(x). \tag{2.3.26}$$

Then, the tensorial properties under $GL(3)$ of the canonical variables imply that, under diffeomorphisms, one will have that $\Pi^i(x)$ is a vector-spinor density (weight $-\frac{3}{2}$) in $\mathbf{R}^3$, transforming as

$$\Pi'^n_\alpha(y^m) = \det[\frac{\partial x^i}{\partial y^n}]^{\frac{3}{2}} \frac{\partial y^n}{\partial x^i} S_{\alpha\beta}(\Lambda(p)) \Pi^i_\beta(x^j). \tag{2.3.27}$$

However, as the σ_i's in (2.3.25) are numerical, they do not transform under the diffeomorphism, and so $\mathcal{Q}(x)$ fails to be covariant. This is to be expected, as we will see below.

The $GL(3)$ symmetry of the (anti)-commutation relations involving local (composite) operators and local variables has been established, given the tensorial properties assigned to the canonical variables. Clearly, the theory itself fails to be $GL(3)$ invariant, and that is to be expected: the Hamiltonian is not covariant under diffeomorphisms (the metric δ_{ij} appears instead of g_{ij}, the measure d^3x appears instead of $\sqrt{g}\, d^3x$, *etc.*). This can be related to the lack of covariance of the supersymmetry generator (2.3.25), (2.3.27). Indeed, one can regard $\mathcal{Q}(x)$ as the square root of the Hamiltonian; and so if the Hamiltonian fails to be covariant, so should the supersymmetry generator. Moreover, observe that when we square (2.3.25) we will obtain a term like $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, and this can be seen as the origin of the "wrong" metric δ_{ij} in the Hamiltonian, which shall destroy the possibility of local covariance. Also, no global (composite) operator can have this $GL(3)$ symmetry, due to the "wrong" choice of integration measure. Now that we assigned tensorial properties to local quantities in the supersymmetric theory, we are ready to proceed in looking for further geometrization in this canonical framework.

2.4 Geometric Variables

We shall limit ourselves to the simplest case of non-Abelian gauge group, namely, $G = SU(2)$. Then, the structure constants are simply $f^{abc} = \epsilon^{abc}$. We will assume knowledge of the previous work done for pure Yang-Mills theory [43, 40, 45].

One wants to have a representation of supersymmetry once we are to transform to the new variables. As we know from the bosonic case [43], the gluon field is transformed into a "metric" field. The supersymmetry representation that includes a metric field is that of supergravity, and it also includes a vector-spinor field. So, one will expect that the gluino field will be transformed into a "gravitino" field. We shall therefore wish to transform the supersymmetric Yang-Mills variables, $\{A_i^a(x), \lambda^a(x)\}$, into the variables of three dimensional supergravity, $\{g_{ij}(x), \psi_k(x)\}$. We shall also expect to obtain a geometry similar to the one of supergravity. After all, the defining equation for the $\{u_i^a(x)\}$ variables identifies them with a dreibein base in a three dimensional manifold.

Recall form section 2 what one is to expect. The geometry will have torsion, defined as,

$$T_{ij}{}^a = \frac{i}{2}\bar{\psi}_i\gamma^a\psi_j. \tag{2.4.1}$$

We can insert a dreibein base through,

$$g_{ij} = u_i^a u_j^b \delta^{ab}, \tag{2.4.2}$$

and also expect that there could be some local supersymmetry transformation in these new variables, which we shall call the "geometrical supersymmetry variation", and which would look like the supersymmetric transformation laws of supergravity,

$$\delta u_i^a(x) = i\bar{\xi}(x)\gamma^a\psi_i(x),$$

$$\delta\psi_k(x) = 2\mathbf{D}_k\xi(x), \tag{2.4.3}$$

where the covariant derivative acting on spinor indices was defined in (2.2.15). With this at hand, the dreibein postulate is now written as,

$$\mathcal{D}_j u_k^a \equiv \partial_j u_k^a + \omega_j^{ab} u_k^b - \Gamma_{jk}^n u_n^a = 0, \tag{2.4.4}$$

also defining the operator $\mathcal{D}_j$. Multiplying this equation by ϵ^{ijk}, and defining the spin-connection via the gauge connection, as in,

$$\omega_j^{ab}(x) \equiv \epsilon^{acb} A_j^c(x), \tag{2.4.5}$$

the dreibein postulate becomes,

$$\epsilon^{ijk} D_j u_k^a = \epsilon^{ijk}(\partial_j u_k^a + \epsilon^{abc} A_j^b u_k^c) = \frac{1}{2}\epsilon^{ijk} T_{jk}{}^a = \frac{i}{4}\epsilon^{ijk}\bar{\psi}_j \gamma^a \psi_k. \tag{2.4.6}$$

We shall take these differential equations to define the change of variables $A_i^a(x) \to u_i^a(x)$. Then, the reverse line of argument holds: the new variables $\{u_i^a(x)\}$ play the role of a dreibein, and from them one can construct a metric $g_{ij} = u_i^a u_j^a$ which is a local gauge invariant variable. The geometry defined by this new variable has torsion, given by (2.4.1). Clearly, for the change of variables to be well defined, we still need to specify what $\psi_j(x)$ is. That is the problem we shall now address.

Let us begin with some dimensional analysis. We know that the gauge field $A_i^a(x)$ has mass dimension one, and the gaugino field $\lambda^a(x)$ has mass dimension three halfs. We also know that the mass dimension of the fermionic generator of the supersymmetry algebra is one half. Through definition (2.3.5) and expression (2.3.25) one observes that if one is ever to modify the gauge theory in order to covariantize it (inserting $\sigma^i(x) u_i^a(x) = \sigma^a$ in (2.3.25) and from then on), one would require the dreibein field to have zero mass dimension, as well as the metric. Though we are not going to modify the gauge theory in this work, we may as well stick to this broader perspective. Then, through the dreibein defining equation (2.4.6), we conclude that the "gravitino" field has mass dimension one half. These dimensional assignments are just like what happens in supergravity.

One can now see that this will have some influence on the construction of the "gravitino" defining equation. In fact, there are some *a priori* requirements for such an equation. It must be geometrical, either in a differential or algebraic way; one needs 12 equations, to change the 12 variables λ^a_α to the 12 variables $\psi_{k\alpha}$; and the gluino field must be present in such an equation. If we moreover require linearity on fermionic variables (like we had linearity on the bosonic variables in (2.4.6)), we see that, by simple dimensional analysis, we can not write such an equation algebraically, but only differentially. Moreover, the equation is constrained to be of the form,

$$\epsilon^{ijk}\mathbf{D}_j\psi_{k\alpha} = \mathcal{M}^{ia}_{\alpha\beta}\lambda^a_\beta, \tag{2.4.7}$$

where the matrix $\mathcal{M}^{ia}_{\alpha\beta}$ must have zero mass dimension, being so far otherwise arbitrary. However, one must be cautious. Not only do we want to have a geometrical way in which to define the vector-spinor field, but we also want to be compatible with the fact that we are studying a supersymmetric theory. In particular, we would like the geometrical supersymmetry variation (2.4.3) to generate the gauge supersymmetry variation (2.2.3). So we shall ask for the geometrical variation (2.4.3) to generate the gauge supersymmetry variation on the bosonic variables $A^a_i(x)$, and in the simplest case where $\xi(x) \equiv \varepsilon$.

Under a generic variation of the fields, one obtains for (2.4.6),

$$\epsilon^{ijk}D_j\delta u^a_k = -\epsilon^{ijk}\epsilon^{abc}u^c_k\delta A^b_j + \frac{1}{2}\epsilon^{ijk}\delta T_{jk}{}^a, \tag{2.4.8}$$

where,

$$\frac{1}{2}\epsilon^{ijk}\delta T_{jk}{}^a = \frac{i}{2}\epsilon^{ijk}\bar{\psi}_j\gamma^a\delta\psi_k. \tag{2.4.9}$$

The supersymmetry transformation laws we shall need are (2.2.3) and (2.4.3). So, a supersymmetry transformation of the dreibein defining equation yields the "gravitino" defining equation. Performing the computations, based on the previous formulae, we are led to,

$$\epsilon^{ijk}\mathbf{D}_j\psi_k = \epsilon^{ijk}(\partial_j\psi_k + \frac{1}{2}\omega^{ab}_j\sigma^{ab}\psi_k) = \frac{1}{3}\epsilon^{ijk}\epsilon^{abc}\gamma^a\gamma_j\lambda^b u^c_k, \tag{2.4.10}$$

where $\gamma_i(x) = u_i^a(x)\gamma^a$.

We shall take these differential equations to define the change of variables $\lambda^a(x) \to \psi_k(x)$. Observe that this equation is precisely of the required form (2.4.7), and the matrix $\mathcal{M}^{ia}_{\alpha\beta}$ has been uniquely defined. Also, this alone guarantees that the geometric variation (2.4.3) will generate the bosonic gauge supersymmetry variation (2.2.3), when $\xi \equiv \varepsilon$. This does not guarantee however that the geometric variation will generate the fermionic gauge supersymmetry variation under the same circumstances. In fact, we can choose $\xi \equiv \xi[\varepsilon]$ through a differential equation (2.4.20) for ξ, such that the geometric variation generates the gauge supersymmetry variation on $\lambda^a(x)$, but we shall not have $\xi \equiv \varepsilon$ in this case. This shows that, even though we can generate the gauge supersymmetry variation via the geometrical supersymmetry variation under special circumstances, the geometric variation is not the original supersymmetry of Yang-Mills theory. The actual expressions for the supersymmetry variations on the new geometrical variables can nevertheless be computed using the usual expression,

$$\delta\Phi = i[Q, \Phi], \tag{2.4.11}$$

where Φ is any of the geometrical variables, and where we should express the supersymmetry generator in this geometric framework (see section 5). The resulting expressions would not be as simple as (2.2.3) or (2.4.3).

All together, one sees that we can now define local gauge invariant geometric variables for supersymmetric Yang-Mills theory via the system of coupled non-linear partial differential equations, (2.4.6) and (2.4.10). These equations define a variable change $\{A_i^a, \lambda^b\} \to \{u_i^a, \psi_k\}$. They also introduce a three dimensional Riemannian geometry with torsion as given by (2.4.1-2) and (2.4.4).

Now that the definition of the new geometrical gauge invariant variables is concluded, one would like to invert the defining equations, in order to express $A_i^a(x)$ and $\lambda^a(x)$ in terms of the geometric variables. This inversion will make it clear that there are no Wu-Yang ambiguities related to these new variables. The defining equation for the dreibein (2.4.6) is equivalent to the dreibein postulate (2.4.4), where the connection is with torsion,

$$\Gamma^n_{jk} = \hat{\Gamma}^n_{jk} - K_{jk}{}^n, \qquad (2.4.12)$$

hatted symbols always denoting affine metric connection quantities. The contorsion tensor is computed from the torsion tensor, through (2.2.12), and one obtains,

$$K_{ijn} = \frac{i}{4}(\bar{\psi}_i\gamma_j\psi_n + \bar{\psi}_j\gamma_i\psi_n - \bar{\psi}_i\gamma_n\psi_j). \qquad (2.4.13)$$

Define a purely geometric derivative through,

$$\nabla_j u^a_k \equiv \partial_j u^a_k - \Gamma^n_{jk} u^a_n, \qquad (2.4.14)$$

and we can find the expression for the inversion,

$$A^a_i(x) = -\frac{1}{2}\epsilon^{abc}u^{bk}(x)\nabla_i u^c_k(x). \qquad (2.4.15)$$

We shall next compute a generic variation of this equation, so that one can later use it to compute the inversion for the gluino field. In order to carry out the calculation, we will need to know what is the generic variation of the connection (2.4.12). Using the fact that it is metric compatible, this can be computed to be,

$$\delta\Gamma^n_{jk} = \frac{1}{2}g^{nm}(\nabla_j\delta g_{mk} + \nabla_k\delta g_{mj} - \nabla_m\delta g_{jk}) - \delta K_{jk}{}^n. \qquad (2.4.16)$$

One can now carry out the variation of the dreibein postulate, and from there obtain the variation of equation (2.4.15). The result is,

$$\delta A^a_i = \frac{\epsilon^{nml}}{2\sqrt{g}}u^a_m(\nabla_i(u^b_l\delta u^b_n) + \nabla_l(\delta g_{ni}) + \frac{i}{2}((\bar{\psi}_i\gamma^b\psi_l)\delta u^b_n + \frac{1}{2}(\bar{\psi}_n\gamma^b\psi_l)\delta u^b_i) +$$

$$+\frac{i}{2}(\bar{\psi}_n\gamma_l\delta\psi_i + \bar{\psi}_i\gamma_n\delta\psi_l + \bar{\psi}_n\gamma_i\delta\psi_l)), \qquad (2.4.17)$$

where $\sqrt{g} = \det u$.

One can use this equation to invert for the gluino field. As we know the geometrical variation (2.4.3) with $\xi \equiv \varepsilon$ generates the bosonic supersymmetry variation. So, we only need to use (2.2.3) and (2.4.3) in (2.4.17), and rearrange, so that we find the expression for the inversion,

$$\lambda^a(x) = -\frac{\epsilon^{nml}}{6\sqrt{g(x)}} u^a_m(x)\gamma^i(x)(\gamma_l(x)\mathcal{D}_i\psi_n(x)+\gamma_n(x)\mathcal{D}_l\psi_i(x)+\gamma_i(x)\mathcal{D}_l\psi_n(x)), \quad (2.4.18)$$

where one uses the vector-spinor full covariant derivative, defined as,

$$\mathcal{D}_i\psi_{k\alpha} \equiv \partial_i\psi_{k\alpha} + \frac{1}{2}\omega_i^{ab}(\sigma^{ab})_{\alpha\beta}\psi_{k\beta} - \Gamma^s_{ik}\psi_{s\alpha}. \quad (2.4.19)$$

Observe that even though the spin-connection is defined via the gauge connection, it is a fully geometric quantity through the dreibein postulate. Later on we shall also require an expression for the generic variation of this equation, so we will address such a problem now. The computation is rather long, and so is the result. One obtains,

$$\delta\lambda^a = -\frac{\epsilon^{nml}}{6\sqrt{g}} u^a_m(u^{bi}\delta u^c_l + u^c_l\delta u^{bi})\gamma^b\gamma^c(\mathcal{D}_i\psi_n - \mathcal{D}_n\psi_i) -$$

$$-\frac{\epsilon^{nml}}{6\sqrt{g}} u^a_m\gamma^i(\gamma_l\mathcal{D}_i(\delta\psi_n) + \gamma_n\mathcal{D}_l(\delta\psi_i) + \gamma_i\mathcal{D}_l(\delta\psi_n) -$$

$$-\frac{1}{2}(\gamma_l\sigma^{jk}\psi_n\nabla_i + \gamma_n\sigma^{jk}\psi_i\nabla_l + \gamma_i\sigma^{jk}\psi_n\nabla_l)(u^b_j\delta u^b_k) -$$

$$-\frac{1}{2}(\gamma_l\sigma^{jk}\psi_n\nabla_j(\delta g_{ki}) + \gamma_n\sigma^{jk}\psi_i\nabla_j(\delta g_{kl}) + \gamma_i\sigma^{jk}\psi_n\nabla_j(\delta g_{kl})) +$$

$$+\frac{1}{2}(\gamma_l\sigma^{jk}\psi_n\delta K_{ijk} + \gamma_n\sigma^{jk}\psi_i\delta K_{ljk} + \gamma_i\sigma^{jk}\psi_n\delta K_{ljk}) + (\gamma_l\delta K_{in}{}^s + \gamma_n\delta K_{li}{}^s + \gamma_i\delta K_{ln}{}^s)\psi_s) -$$

$$-\frac{1}{6\sqrt{g}} u^a_j u^c_k(\epsilon^{njk}\delta u^{cl} - \epsilon^{ljk}\delta u^{cn})\gamma^i(\gamma_l\mathcal{D}_i\psi_n + \gamma_n\mathcal{D}_l\psi_i + \gamma_i\mathcal{D}_l\psi_n), \quad (2.4.20)$$

where the generic variation of the contorsion tensor can be written as,

$$\delta K_{inl} = \frac{i}{4}((\bar{\psi}_i\gamma^a\psi_l)\delta u^a_n + (\bar{\psi}_n\gamma^a\psi_l)\delta u^a_i - (\bar{\psi}_i\gamma^a\psi_n)\delta u^a_l) +$$

$$+\frac{i}{4}((\bar{\psi}_n\gamma_l - \bar{\psi}_l\gamma_n)\delta\psi_i + (\bar{\psi}_i\gamma_n + \bar{\psi}_n\gamma_i)\delta\psi_l - (\bar{\psi}_l\gamma_i + \bar{\psi}_i\gamma_l)\delta\psi_n). \quad (2.4.21)$$

The variations (2.4.17) and (2.4.20) allow us to express a variation of the wave-functional in terms of the variations of the geometric variables. This will be helpful in section 5.

The inversion completed proves the non existence of Wu-Yang ambiguities in the new geometrical variables. Therefore, we have managed to define new gauge invariant variables for supersymmetric Yang-Mills. Moreover, it can be shown that gauge invariant physical wave-functionals of the theory depend only on these geometric variables (see section 5), $\Psi \equiv \Psi[g_{ij}, \psi_k]$, so that we have in these variables an explicit parameterization of the physical Hilbert space (moduli space) of the gauge theory. A final remark on diffeomorphisms is now in order. As said before, only the variables of the theory are diffeomorphism covariant. The Hamiltonian fails to be diffeomorphism covariant. Given that the variables of the theory are now $\{g_{ij}, \psi_k\}$, this has an interesting consequence: a configuration diffeomorphic to the previous one yields a different configuration to the gauge theory. Therefore, we can extend solutions to the gauge theory by action of the group of diffeomorphisms, by simply moving along the orbit of the geometrical configuration.

2.5 Gauge Tensors as Geometric Tensors

We now wish to write the tensors and composite operators of our theory in terms of the new geometric variables, *i.e.*, as geometric tensors and geometric composite operators. We shall first address the electric and magnetic tensors. The Hamiltonian, Gauss' law generator, and the supersymmetry generator composite operators then easily follow from these two tensors and the previous equations for the inversions of the gluon and gluino fields.

Let us start with the gauge Ricci identity,

$$\epsilon^{abc} F^b_{ij} = [D_i, D_j]^{ac}, \tag{2.5.1}$$

and apply it to the dreibein field. We will obtain,

$$\epsilon^{abc} F^b_{ij} u^c_k = R^l{}_{kij} u^a_l, \tag{2.5.2}$$

where $R^l{}_{kij}$ is the Riemann tensor of the connection Γ,

$$R^l{}_{kij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}. \tag{2.5.3}$$

From (2.5.2) one can express the field strength in terms of the Riemann curvature, and so from (2.3.3) we can express the magnetic field vector geometrically, as,

$$B^{am} = -\frac{1}{4\sqrt{g}} \epsilon^{mij} \epsilon^{nlk} u^a_n R_{lkij}. \tag{2.5.4}$$

So, the gauge invariant tensor which gives the Yang-Mills magnetic energy density is,

$$B^{ai} B^{aj} = \frac{1}{16} \epsilon^{imn} \epsilon^{jkl} R^{uv}{}_{mn} (R_{uvkl} - R_{vukl}). \tag{2.5.5}$$

As we can see, this expression gives the gauge invariant tensor in a manifestly gauge invariant form, in terms of the "metric" g_{ij}, and the "gravitino" ψ_k (which is present via the torsion contribution to the Riemann tensor).

The electric field vector is the momentum canonically conjugated to the canonical variable, the gauge connection. In canonical quantization it is represented by a functional derivative (2.3.9). We define a gauge invariant tensor operator e^{ij} by,

$$\frac{\delta}{\delta A^a_i(x)} = i E^{ai}(x) \equiv \sqrt{g(x)} u^a_j(x) e^{ij}(x). \tag{2.5.6}$$

Clearly, $e^{ij}(x)$ is an ordinary $\binom{2}{0}$ tensor under $GL(3)$. From this expression, the electric gauge invariant Yang-Mills tensor, *i.e.*, the manifestly gauge invariant tensor which gives the Yang-Mills electric energy density, now follows as,

$$E^{ai} E^{aj} = -g e^i{}_k e^{jk}. \tag{2.5.7}$$

In order to finally obtain the Hamiltonian in a manifestly gauge invariant form in terms of the geometrical variables, one still needs the expression for the fermionic energy density, as is clear from (2.3.1). The expression for this gauge invariant tensor can be obtained by simply inserting (2.4.18-19) in the required expression. The result we obtain is,

$$\bar{\lambda}^a \gamma^i D_i \lambda^a = \frac{\epsilon^{njk}}{36\sqrt{g}} g_{jm}((\bar{\mathcal{D}}_k \bar{\psi}_n)\gamma_l + (\bar{\mathcal{D}}_l \bar{\psi}_n)\gamma_k +$$

$$+(\bar{\mathcal{D}}_k \bar{\psi}_l)\gamma_n)\gamma^l \gamma^i \nabla_i (\frac{\epsilon^{umv}}{\sqrt{g}} \gamma^s (\gamma_v (\mathcal{D}_s \psi_u) + \gamma_u (\mathcal{D}_v \psi_s) + \gamma_s (\mathcal{D}_v \psi_u))), \tag{2.5.8}$$

where we have defined $\bar{\mathcal{D}}_i \bar{\psi}_j \equiv (\mathcal{D}_i \psi_j)^\dagger \gamma^0$; and where in the contraction $\gamma^i \nabla_i$ the gamma matrices are to be considered as numerical, not as space dependent. The sum of (2.5.5), (2.5.7) and (2.5.8) according to (2.3.1) finally yields the manifestly gauge invariant Hamiltonian.

As was done for the gluon functional derivative, we shall similarly define a gauge invariant vector-spinor density operator χ_i to deal with the gluino functional derivative,

$$\frac{\delta}{\delta \lambda^a(x)} \equiv \sqrt{g(x)} u^{ai}(x) \chi_i(x). \tag{2.5.9}$$

$\chi_i(x)$ is a $\binom{0}{1}$ vector-spinor density (weight $\frac{1}{2}$) under $GL(3)$. With these definitions at hand, one can now express the functional dependence of the wave-functional $\Psi[A_i^a, \lambda^b]$ in terms of the new variables. Under a variation, we have,

$$\delta\Psi = \int d^3x \, \{\frac{\delta\Psi}{\delta A_i^a(x)} \delta A_i^a(x) + \frac{\delta\Psi}{\delta \lambda^a(x)} \delta \lambda^a(x)\} =$$

$$= \int d^3x \, \{\sqrt{g(x)} u_j^a(x) \delta A_i^a(x) [e^{ij}(x)\Psi] + \sqrt{g(x)} u^{ai}(x) [\chi_i(x)\Psi] \delta \lambda^a(x)\}, \tag{2.5.10}$$

where one should use the expression for the variations of the gauge fields in (2.4.17) and (2.4.20-21). Expanding this expression through rather lengthy calculations, it can then be seen that the term in δu_i^a is proportional to the Gauss' law operator (2.3.4), when expressed in geometrical terms, and acting on the wave-functional. Observe that this is $\mathcal{G}^a\Psi[g,\psi]=0$, which in the new variables can be written as,

$$(\nabla_i e^{ij}+\frac{i}{2}\bar{\psi}_i\gamma^k\psi_k e^{ij}+\frac{1}{72g^2}(g^{jn}\epsilon^{ilk}-g^{jl}\epsilon^{ink})((\bar{\mathcal{D}}_s\bar{\psi}_n)\gamma_l+(\bar{\mathcal{D}}_l\bar{\psi}_s)\gamma_n+$$

$$+(\bar{\mathcal{D}}_l\bar{\psi}_n)\gamma_s)\gamma^s\gamma^0\gamma^r(\gamma_k(\mathcal{D}_r\psi_i)+\gamma_i(\mathcal{D}_k\psi_r)+\gamma_r(\mathcal{D}_k\psi_i)))\Psi[g,\psi]=0. \qquad (2.5.11)$$

So, wave-functionals whose dependence is solely on the new gauge invariant variables are gauge invariant, and gauge invariant wave-functionals depend solely on the new gauge invariant variables. It is in these physical gauge invariant wave-functionals that we are mainly interested, and for these the previous expression for $\delta\Psi$ reduces to,

$$\delta\Psi[g_{ij},\psi_k]=\int d^3x\,\{\frac{1}{2}\epsilon^{nml}\nabla_l(\delta g_{ni})[e^i_m\Psi]+$$

$$+\frac{1}{12}\epsilon^{nml}[\chi_m\Psi]\gamma^i(\gamma_l\sigma^{jk}\psi_n\nabla_j(\delta g_{ki})+\gamma_n\sigma^{jk}\psi_i\nabla_j(\delta g_{kl})+\gamma_i\sigma^{jk}\psi_n\nabla_j(\delta g_{kl}))+$$

$$+\frac{i}{4}\epsilon^{nml}(\bar{\psi}_n\gamma_l\delta\psi_i+\bar{\psi}_i\gamma_n\delta\psi_l+\bar{\psi}_n\gamma_i\delta\psi_l)[e^i_m\Psi]-$$

$$-\frac{1}{6}\epsilon^{nml}[\chi_m\Psi]\gamma^i(\gamma_l\mathcal{D}_i(\delta\psi_n)+\gamma_n\mathcal{D}_l(\delta\psi_i)+\gamma_i\mathcal{D}_l(\delta\psi_n))-$$

$$-\frac{i}{24}\epsilon^{nml}[\chi_m\Psi]\gamma^i(\gamma_l\sigma^{jk}\psi_n(\bar{\psi}_j\gamma_k\delta\psi_i+\bar{\psi}_i\gamma_j\delta\psi_k+\bar{\psi}_j\gamma_i\delta\psi_k)+$$

$$+\gamma_n\sigma^{jk}\psi_i(\bar{\psi}_j\gamma_k\delta\psi_l+\bar{\psi}_l\gamma_j\delta\psi_k+\bar{\psi}_j\gamma_l\delta\psi_k)+\gamma_i\sigma^{jk}\psi_n(\bar{\psi}_j\gamma_k\delta\psi_l+\bar{\psi}_l\gamma_j\delta\psi_k+\bar{\psi}_j\gamma_l\delta\psi_k))-$$

$$-\frac{i}{12}\epsilon^{nml}[\chi_m\Psi]\gamma^i(\gamma_l\psi_s(\bar{\psi}_n\gamma^s\delta\psi_i)+\gamma_n\psi_s(\bar{\psi}_i\gamma^s\delta\psi_l)+\gamma_i\psi_s(\bar{\psi}_n\gamma^s\delta\psi_l))\}. \qquad (2.5.12)$$

From here one can now extract expressions for the electric and spinor fields, $e^{ij}\Psi$ and $\chi_i\Psi$, in terms of functional derivatives of gauge invariant wave-functionals, with respect to the gauge invariant variables. Observe that for such, one has to solve a linear system of differential equations, therefore involving the inversion of differential operators. One can then conclude that in general both operators $e^{ij}\Psi$ and $\chi_i\Psi$, will depend non-locally on the functional derivatives $\delta\Psi/\delta g_{ij}$ and $\delta\Psi/\delta\psi_k$. Like in the non-supersymmetric case [43], the Hamiltonian will thus be a non-local composite operator.

Before proceeding with the study of these non-local operators, there is one more composite operator that we still would like to express in a manifestly gauge invariant way, *i.e.*, that we would like to geometrize. Such an operator is the supersymmetry generator, (2.3.5-6). In particular, we will look at its structure as depicted in equations (2.3.5), (2.3.25-26), and geometrize the tensor $\Pi^i(x)$. For that, one simply has to make use of the previous formulae into equation (2.3.26), and obtain,

$$\Pi^i = \begin{pmatrix} 0 & -\frac{e}{6}\epsilon^{nml}e^i{}_m + \frac{1}{24eg}\epsilon^{ijk}(R^{nl}{}_{jk} - R^{ln}{}_{jk}) \\ \frac{e}{6}\epsilon^{nml}e^i{}_m + \frac{1}{24eg}\epsilon^{ijk}(R^{nl}{}_{jk} - R^{ln}{}_{jk}) & 0 \end{pmatrix} \cdot$$

$$\cdot\gamma^r(\gamma_l\mathcal{D}_r\psi_n + \gamma_n\mathcal{D}_l\psi_r + \gamma_r\mathcal{D}_l\psi_n), \tag{2.5.13}$$

from where the supersymmetry generator then follows, according to (2.3.25) and (2.3.5).

Some words are now in order, concerning supersymmetry and its quantum field theoretic representation on the geometrized fields. One of the elements that is present in Π^i is the non-local operator e^{ij}, thus turning the supersymmetry generator into a non-local composite operator, when expressed in the geometrical variables. As we shall see in the following, information about the Green's functions present in this operator can be obtained, albeit in a formal way. By this, we mean that an explicit construction of these Green's functions can only be obtained given a particular geometrical configuration (see [45] for this same situation in the non-supersymmetric case). Moreover, the geometric supersymmetry generator includes the Riemann tensor which is non-linear in the metric and "gravitino" fields, and their derivatives; one would therefore also prefer to have a geometrical configuration with a high degree of symmetry (a maximal number of Killing vectors), in order to simplify it. An example involving spherical geometries, generalizing the one in [45] to this supersymmetric case, shows how this situation could be handled [64].

In the pure Yang-Mills case [43], the calculation of the electric field tensor involved the inversion of a differential operator that could generically have zero modes. Subtleties associated to the inversion of such an operator were later handled with the insertion of a deformation into the dreibein defining equation [45]. We shall now see that in this supersymmetric case those problems can be better handled, by computing the bosonic Green's function for the electric field tensor e^{ij}. We shall learn that one will not need to deform our equations in order to obtain a well-defined result. We start by inverting the defining equation for the electric tensor (2.5.6), to obtain,

$$e^{ij} = \frac{1}{\sqrt{g}} u^{aj} \frac{\delta}{\delta A_i^a}. \tag{2.5.14}$$

Recall that through the dreibein defining equation (2.4.6), the gauge connection depends on both the dreibein and the "gravitino" field. Therefore, one can further expand the geometric tensor e^{ij} as,

$$e^{ij}(x) = \frac{1}{\sqrt{g(x)}} u^{aj}(x) \int d^3y \, (\frac{\delta u_k^b(y)}{\delta A_i^a(x)} \frac{\delta}{\delta u_k^b(y)} + \frac{\delta \psi_k(y)}{\delta A_i^a(x)} \frac{\delta}{\delta \psi_k(y)}). \tag{2.5.15}$$

Variations of the dreibein can be further separated into variations of the six gauge invariant degrees of freedom g_{ij} and of the three gauge degrees of freedom. As we are considering operators that act on gauge invariant wave-functionals only, we simply obtain,

$$e^{ij}(x) = \int d^3y \, \frac{1}{\sqrt{g(x)}} \{ (u^{aj}(x) \frac{\delta u_k^b(y)}{\delta A_i^a(x)} u_m^b(y)) 2 \frac{\delta}{\delta g_{km}(y)} + u^{aj}(x) \frac{\delta \psi_k(y)}{\delta A_i^a(x)} \frac{\delta}{\delta \psi_k(y)} \}. \tag{2.5.16}$$

We next want to study the bosonic Jacobian matrix $\delta u / \delta A$, and see that it has a better behavior in here, than in the non-supersymmetric Yang-Mills case. For that, one needs to start by geometrizing such a matrix. Let us re-write (2.4.6) as,

$$\epsilon^{ijk} (\delta^{ac} \delta_k^n \partial_j + \delta_k^n \epsilon^{abc} A_j^b - \frac{i}{4} \delta^{ac} (\bar{\psi}_j \gamma^n \psi_k)) u_n^c = 0. \tag{2.5.17}$$

Variation of this equation is (2.4.8-9),

$$\epsilon^{ijk} D_j \delta u_k^a = -\epsilon^{ijk} \epsilon^{abc} u_k^c \delta A_j^b + \frac{i}{2} \epsilon^{ijk} \bar{\psi}_j \gamma^a \delta \psi_k, \tag{2.5.18}$$

and to obtain δu_i^a in terms of δA_i^a, the operator acting on δu_i^a must be inverted. In order to do so, let us consider the associated eigenvalue problem,

$$\epsilon^{ijk}(\delta^{ac}\delta^n_k\partial_j + \delta^n_k\epsilon^{abc}A^b_j - \frac{i}{4}\delta^{ac}(\bar{\psi}_j\gamma^n\psi_k))w_{A}{}^c_n = \sqrt{g}\Lambda_A w_A{}^{ia}. \tag{2.5.19}$$

By definition, one solution to this equation with $\Lambda_A = 0$ is u^a_i itself. In our notation, A labels all the eigenfunctions, except the particular one given by u^a_i. Moreover, it will be assumed that $\{u^a_i, w_{A}{}^a_i\}$ forms a complete orthonormal spectrum of real eigenfunctions for the considered operator. By orthonormality, we mean,

$$\int \sqrt{g}\, d^3x\, g^{ij}(u^a_i w_{A}{}^a_j) = 0,$$

$$\int \sqrt{g}\, d^3x\, g^{ij}(w_{A}{}^a_i w_{B}{}^a_j) = 3V\delta_{AB},$$

$$\int \sqrt{g}\, d^3x\, g^{ij}(u^a_i u^a_j) = 3V, \tag{2.5.20}$$

where V is the volume of the space described by g_{ij} (*i.e.*, V is a "dynamical" volume), and δ_{AB} is a Kronecker or Dirac delta, depending on whether the spectrum is discrete or continuous. Now, expand a generic variation of the dreibein in this complete set,

$$\delta u^a_i = \eta u^a_i + \sum_A \eta_A w_{A}{}^a_i, \tag{2.5.21}$$

and substitute this in (2.5.18). If we dot on the left (meaning inner product with the required measure (2.5.20)) with the same complete set $\{u^a_i, w_{A}{}^a_i\}$, we shall obtain a non-homogeneous linear system of equations for the expansion parameters, η and η_A. Solving that system, and inserting the result in (2.5.21) yields an expansion of the variation δu^a_i in terms of the variations δA^a_i and $\delta\psi_k$. It is then easy to compute the Jacobian matrix $\delta u/\delta A$. However, such a result will not be naturally geometric, as it involves the eigenfunctions $w_{A}{}^a_i$, which are gauge vectors. To solve this problem, we introduce the geometric modes $z_{Ai}{}^j$, associated to the gauge modes $w_{A}{}^a_i$, and defined via,

$$w_{A}{}^a_i \equiv z_{Ai}{}^j u^a_j. \tag{2.5.22}$$

It can then be shown that these geometric modes obey,

$$\epsilon^{ijk}\nabla_j z_{Ak}{}^m = \sqrt{g}\Lambda_A z_A{}^{im}. \tag{2.5.23}$$

So, the $z_A{}^{ij}$ are the eigenmodes of the geometric curl operator, with the same eigenvalues as the gauge modes w_{Ai}^a, Λ_A. Full geometrization of the Green's function,

$$u^{aj}(x)\frac{\delta u_k^b(y)}{\delta A_i^a(x)}u_m^b(y), \tag{2.5.24}$$

is now at hand. The result is,

$$u^{aj}(x)\frac{\delta u_k^b(y)}{\delta A_i^a(x)}u_m^b(y) = \sqrt{g(x)}g^{ij}(x)\Big\{\mathcal{H}_{kms}{}^s(y,x) -$$

$$-\frac{3}{\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma_i\psi_k)}\int d^3u\, \epsilon^{ijk}(\bar{\psi}_j\gamma_n\psi_k)(u)\mathcal{H}_{kmi}{}^n(y,u) +$$

$$+g_{km}(y)\frac{8i}{\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma_i\psi_k)}\{3 + \frac{i}{8}\int d^3u\, \epsilon^{ijk}(\bar{\psi}_j\gamma^n\psi_k)(u)\mathcal{H}_{nis}{}^s(u,x) -$$

$$-\frac{3i}{8\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma_i\psi_k)}\int\int d^3u\, d^3v\, \epsilon^{ijk}(\bar{\psi}_j\gamma^n\psi_k)(u)\epsilon^{mrs}(\bar{\psi}_r\gamma_l\psi_s)(v)\mathcal{H}_{nim}{}^l(u,v)\}\Big\}, \tag{2.5.25}$$

where we have defined the Green's functions,

$$\mathcal{H}_{ijmn}(x,y) \equiv \sum_{AB} z_{Aij}(x)I_{AB}^{-1}z_{Bmn}(y), \tag{2.5.26}$$

and the matrix,

$$I_{AB} \equiv -\frac{3}{2}V\Lambda_A\delta_{AB} - \frac{i}{8}\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma^n\psi_k)z_{Aim}z_{Bn}{}^m +$$

$$+\frac{i}{8}\frac{1}{\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma_i\psi_k)}(\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma_l\psi_k)z_{Ai}{}^l)(\int d^3x\, \epsilon^{ijk}(\bar{\psi}_j\gamma^n\psi_k)z_{Bni}). \tag{2.5.27}$$

One sees that we have obtained a well-defined result, unlike it would have happen in the non-supersymmetric case [43], where there were divergences in the electric energy that were independent of the geometry. Clearly, the Green's functions (2.5.26) may still have geometry dependent divergences associated with the degree of symmetry of a given geometrical configuration (as determined by its Killing vectors). Also, this may seem a somewhat formal result, but observe that now we have a constructive definition of the Green's function (2.5.24): given a geometrical configuration on the domain manifold, we start by solving the eigenvalue equation in order to obtain the geometric eigenmodes. Once we have such eigenmodes, first construct the matrix I_{AB}, then invert it (most likely through a symbolic manipulation program), and finally compute the Green's functions $\mathcal{H}_{ijmn}(x,y)$.

We have geometrized all the tensors appearing in the Hamiltonian formulation of $\mathcal{N}=1$ supersymmetric Yang-Mills theory. All composite operators should now follow in a straightforward fashion.

2.6 Conclusions

We have defined new local gauge invariant variables for supersymmetric gauge theory. These variables have moreover a geometrical interpretation, as they are a three dimensional "metric" and "gravitino". The geometry associated to the theory is then just like the geometry of supergravity.

We have also shown that these new variables are free of Wu-Yang ambiguities; so they seem to be quite appropriate for the study of nonperturbative phenomena in supersymmetric gauge theories, as they explicitly parameterize the physical Hilbert space of the theory. We have also seen that these variables have a better behavior here than in the non-supersymmetric case. Namely, there are no geometry independent divergences in the bosonic half of the electric energy tensor operator.

The treatment presented here was rather formal, and the issue of renormalization was not addressed. Further work on this formalism should focus on this issue. We could think of using the known beta functions of supersymmetric Yang-Mills theory to perform the renormalization of the (geometric) composite operators that we have presented. These renormalized operators could then be used to extract information on the ground state wave-functional of the gauge theory. In this supersymmetric case, the functional differential equation for the ground state is $Q\Psi[g, \psi] = 0$, which is first-order in the non-local functional derivatives. There would seem to be hope that one could then extract some information about the solution to the theory.

It would also be interesting to study special solutions to supersymmetric Yang-Mills theory in this framework. Namely, we could try to extend to this supersymmetric case the example of spherical geometries that was introduced in [45]. In particular, in order to define a vector-spinor on a three manifold, the manifold must be parallelisable, and its second Stiefel-Whitney cohomology class must be trivial. Such is the case for S^3, so that the example in [45] could indeed be generalizable to this framework [64]. There is also the possibility of extending this formalism to higher $\mathcal{N}$ supersymmetric gauge theory. This could be interesting, specially if some connection to the work in [68] could then be established.

All these lines of work are quite interesting to follow as there is good knowledge about some properties of supersymmetric Yang-Mills theories (see [70] for a modern review, and references therein). In the example of spherical geometries, a bridge between our formalism and the well known instanton solutions of gauge theory can be established; while in the case of extended $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, one can observe the interesting fact that BPS states obey – in the geometrical formulation – a three dimensional "Einstein field equation" where the "stress tensor" is the one associated to a gauge vector field [64].

Chapter 3

Bosonic σ–Models

3.1 Introduction

Since the time of its discovery [52, 63], target space duality has been studied mostly as a symmetry of string backgrounds. That is to say, it is realized as a transformation taking one set of fields $\{g_{\mu\nu}, b_{\mu\nu}, \phi\}$ (respectively metric, antisymmetric tensor and dilaton) satisfying background field equations of motion, into another set $\{\tilde{g}_{\mu\nu}, \tilde{b}_{\mu\nu}, \tilde{\phi}\}$ satisfying the same equations of motion. As such, it represents a parameter space symmetry of the associated sigma model at its conformal points only. It was recently observed, however, that it is also natural to impose it as a symmetry of the sigma model *away* from the conformal points, throughout the entire parameter space [41]. This is expressed as the requirement (to be made precise below) that duality flows "covariantly" with the renormalization group (RG) evolution of the background fields. Because information about the RG flow is typically difficult to obtain, while a T-duality symmetry is considerably easier to identify, such an interplay between duality and RG flows can be of more than academic interest if it yields restrictions on the renormalization patterns of the theory.

At one-loop order ($\mathcal{O}(\alpha')$), it was shown in [41] that indeed the requirement of duality symmetry away from conformal points of the two dimensional bosonic sigma model led to highly restrictive consistency conditions on the RG beta functions of the model. It was found that these conditions uniquely determine all beta functions at $\mathcal{O}(\alpha')$. This is a particularly striking fact, in that essentially the only condition imposed is that of duality, a symmetry which is *prima facie* entirely unaware of the renormalization structure of the model. Similar (albeit weaker) restrictions have also been seen to follow from analogous consistency conditions in altogether different contexts, such as the 2d Ising and Potts models [21], and the quantum Hall system [17].

Naturally, for sigma models, this would probably be an inconsequential curiosity if such conditions only operated at $\mathcal{O}(\alpha')$. This motivated further investigations of the consistency conditions at two-loop order [46]. For a restricted, purely metric background, it was found that while both the beta functions and the duality transformations are modified by perturbative corrections, the ensuing consistency conditions (also modified) continue nonetheless to be satisfied. This indicates that, at least to $\mathcal{O}(\alpha'^2)$, duality transformations mysteriously remain informed of the renormalization properties of the theory.

If this is so, one is led to inquire whether consistency conditions at $\mathcal{O}(\alpha'^2)$ again allow for a determination of the beta functions at that order. The purpose of the investigation in this chapter is to show that indeed such a determination is possible.

After briefly reviewing the first nontrivial order, we will consider, as in [46], a restricted class of backgrounds in order to probe the consistency conditions at $\mathcal{O}(\alpha'^2)$. In order to be self-contained we begin by deriving, from basic principles, the corrected duality transformations at $\mathcal{O}(\alpha'^2)$ first presented in [76]. From these follow the $\mathcal{O}(\alpha'^2)$ consistency conditions on the beta functions of the theory. We will then show that, out of the ten different tensor structures possibly appearing in the two-loop beta function, only the known, correct structure satisfies the consistency conditions. This represents a completely independent and diagram-free determination of the two-loop beta function of the purely metric 2d bosonic sigma model.

To be precise, with the restricted class of backgrounds we consider, this $\mathcal{O}(\alpha'^2)$ beta function is only determined up to a global constant. However, it should be noted firstly that the beta function determined is valid for entirely generic metric backgrounds and, secondly, that the mechanism at work at $\mathcal{O}(\alpha')$ indicates that, had we considered a more generic background at $\mathcal{O}(\alpha'^2)$, even this global constant would have been determined.

3.2 Order α'

We consider a $d=2$ bosonic sigma model with a target Abelian isometry ($\theta \to \theta+$ constant):

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \Big[g_{00}(X)\partial_\alpha\theta\partial^\alpha\theta + 2g_{0i}(X)\partial_\alpha\theta\partial^\alpha X^i + g_{ij}(X)\partial_\alpha X^i\partial^\alpha X^j + \\ + i\varepsilon^{\alpha\beta}\Big(2b_{0i}(X)\partial_\alpha\theta\partial_\beta X^i + b_{ij}(X)\partial_\alpha X^i\partial_\beta X^j\Big)\Big]. \tag{3.2.1}$$

The adapted target space coordinates are $X^\mu = (\theta, X^i)$, $i = 1, \ldots, D$, and the isometry is made manifest through the independence of background tensors on θ. "Classical" duality transformations [18, 19] take a background $\{g_{\mu\nu}, b_{\mu\nu}\}$ into,

$$\tilde{g}_{00} = \frac{1}{g_{00}}, \quad \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}}, \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}},$$

$$\tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}},$$

$$\tilde{b}_{ij} = b_{ij} - \frac{g_{0i}b_{0j} - b_{0i}g_{0j}}{g_{00}}. \tag{3.2.2}$$

On a curved world-sheet, another background coupling must be introduced, that of the dilaton $\phi(X)$. The RG flow of background couplings is given by their respective beta functions:

$$\beta^g_{\mu\nu} \equiv \mu\frac{d}{d\mu}g_{\mu\nu}\ , \ \beta^b_{\mu\nu} \equiv \mu\frac{d}{d\mu}b_{\mu\nu}\ , \ \beta^\phi \equiv \mu\frac{d}{d\mu}\phi\ , \tag{3.2.3}$$

while the trace of the stress energy tensor is found from the Weyl anomaly coefficients [74],

$$\bar{\beta}^g_{\mu\nu} = \beta^g_{\mu\nu} + 2\alpha'\nabla_\mu\partial_\nu\phi,$$

$$\bar{\beta}^b_{\mu\nu} = \beta^b_{\mu\nu} + \alpha' H_{\mu\nu}{}^\lambda\partial_\lambda\phi,$$

$$\bar{\beta}^\phi = \beta^\phi + \alpha'(\partial_\mu\phi)^2. \tag{3.2.4}$$

Both the beta functions and the Weyl anomaly coefficients will satisfy the consistency conditions to be presented below. However, while the latter satisfy them exactly, the former satisfy them up to a target reparameterization [41, 46]. Since both encode essentially the same RG information, for simplicity we shall consider RG motions as generated by the Weyl anomaly coefficients in what follows. We define (at any order) an operation R on a generic functional $F[g,b,\phi]$ to be,

$$RF[g,b,\phi] = \frac{\delta F}{\delta g_{\mu\nu}} \cdot \bar{\beta}^g_{\mu\nu} + \frac{\delta F}{\delta b_{\mu\nu}} \cdot \bar{\beta}^b_{\mu\nu} + \frac{\delta F}{\delta \phi} \cdot \bar{\beta}^\phi \ , \tag{3.2.5}$$

and an operation T affecting (at lowest order) the transformations (3.2.2) through,

$$TF[g,b,\phi] = F[\tilde{g},\tilde{b},\tilde{\phi}], \tag{3.2.6}$$

(where $\tilde{\phi}$ will be defined shortly). The requirement that duality flows "covariantly" with the RG is expressed as:

$$[T,R] = 0 \ . \tag{3.2.7}$$

When applied to (3.2.2) this leads to the consistency conditions first presented in [41] for the Weyl anomaly coefficients,

$$\bar{\beta}^{\tilde{g}}_{00} = -\frac{1}{g_{00}^2}\bar{\beta}^g_{00} \ ,$$

$$\bar{\beta}^{\tilde{g}}_{0i} = -\frac{1}{g_{00}^2}\Big(b_{0i}\bar{\beta}^g_{00} - \bar{\beta}^b_{0i}g_{00}\Big) \ ,$$

$$\bar{\beta}^{\tilde{b}}_{0i} = -\frac{1}{g_{00}^2}\Big(g_{0i}\bar{\beta}^g_{00} - \bar{\beta}^g_{0i}g_{00}\Big) \ ,$$

$$\bar{\beta}^{\tilde{g}}_{ij} = \bar{\beta}^g_{ij} - \frac{1}{g_{00}}\Big(\bar{\beta}^g_{0i}g_{0j} + \bar{\beta}^g_{0j}g_{0i} - \bar{\beta}^b_{0i}b_{0j} - \bar{\beta}^b_{0j}b_{0i}\Big) + \frac{1}{g_{00}^2}\Big(g_{0i}g_{0j} - b_{0i}b_{0j}\Big)\bar{\beta}^g_{00} \ ,$$

$$\bar{\beta}^{\tilde{b}}_{ij} = \bar{\beta}^b_{ij} - \frac{1}{g_{00}}\Big(\bar{\beta}^g_{0i}b_{0j} + \bar{\beta}^b_{0j}g_{0i} - \bar{\beta}^g_{0j}b_{0i} - \bar{\beta}^b_{0i}g_{0j}\Big) + \frac{1}{g_{00}^2}\Big(g_{0i}b_{0j} - b_{0i}g_{0j}\Big)\bar{\beta}^g_{00} \ . \tag{3.2.8}$$

At loop order ℓ, the possible tensor structures $T_{\mu\nu}$ appearing in the beta function must scale as $T_{\mu\nu}(\Lambda g, \Lambda b) = \Lambda^{1-\ell}T_{\mu\nu}(g,b)$ under global scalings of the background fields [5]. At $\mathcal{O}(\alpha')$ one may then have,

$$\beta^g_{\mu\nu} = \alpha'\Big(A\,R_{\mu\nu} + B\,H_{\mu\lambda\rho}H_\nu^{\ \lambda\rho} + C\,g_{\mu\nu}R + D\,g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}\Big)\ ,$$

$$\beta^b_{\mu\nu} = \alpha'\Big(E\,\nabla^\lambda H_{\mu\nu\lambda}\Big)\ , \tag{3.2.9}$$

with A, B, C, D, E being determined from one-loop Feynman diagrams. As found in [41], requiring (3.2.8) to be satisfied, and choosing $A = 1$ determines $B = -1/4$, $E = -1/2$, and $C = D = 0$, independently of any diagram calculations. As it turns out, the consistency conditions (3.2.8) on $g_{\mu\nu}$ and $b_{\mu\nu}$ alone also allows for an independent determination of the dilaton transformation (or "shift") $\tilde{\phi} = \phi - \frac{1}{2}\ln g_{00}$. Applying (3.2.7) to this then yields the dilaton beta function [46].

3.3 Order α'^2

At the next order R is modified by the two-loop beta functions, and one must determine the appropriate modifications in T such that $[T, R] = 0$ continues to hold. We work at this order with restricted backgrounds of the form,

$$g_{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & \bar{g}_{ij} \end{pmatrix}\ , \tag{3.3.1}$$

and $b_{\mu\nu} = 0$, so that no torsion appears in the dual background either. It is useful to define at this point the following two quantities: $a_i \equiv \partial_i \ln a$, and $q_{ij} \equiv \bar{\nabla}_i a_j + \frac{1}{2}a_i a_j$, where barred quantities here and below refer to the metric $\bar{g}_{ij}$ (also, indices $i, j, \ldots$, are contracted with the metric $\bar{g}_{ij}$). Within this class of backgrounds classical duality transformations reduce to the operation $a \to 1/a$, and it is simple to determine the possible corrections to T from a few basic requirements: *i)* $\tilde{g}_{ij} = g_{ij} = \bar{g}_{ij}$ does not get modified, as it corresponds to sigma model couplings entirely disconnected from the path integral dualization procedure (cf. [18, 19]); *ii)* corrections should be D-dimensional generally covariant; *iii)* corrections to $\tilde{a} = 1/a$ must be proportional to

a_i:

$$\ln \tilde{a} = - \ln a + \alpha' m_i a^i , \;\; m_i = m_i(a, \bar{g}_{ij}) \;, \tag{3.3.2}$$

as it is simple to see that classical consistency conditions would be satisfied for $a =$ constant; *iv)* dimensional analysis: $[\alpha'] = L^2$ and $[a_i] = 1/L$, where L is a target length, so that $[m_i] = 1/L$; *v)* m_i should not contain nontrivial denominators, as the corrections should be finite for finite geometries; *vi)* because the duality group should still be $\mathbb{Z}_2$, by applying the transformations (3.3.2) twice one should re-obtain the original model. This constrains m_i to be odd under classical duality:

$$\tilde{m}_i \equiv m_i(1/a, \bar{g}_{ij}) = -m_i(a, \bar{g}_{ij}) \;. \tag{3.3.3}$$

All of the above then yields,

$$m_i = \lambda \, a_i \;, \tag{3.3.4}$$

with λ an undetermined real constant. As discussed in [46], moreover, we shall also require the measure factor $\sqrt{g} \exp{(-2\phi)}$ to be invariant (so that $[T, R] = 0$ implies invariance of the string background effective action), thus fixing also the correction on the dilaton transformation to be 1/4 that of g_{00}. Altogether, for the backgrounds (3.3.1) the corrected duality transformations are:

$$\ln \tilde{a} = - \ln a + \lambda \alpha' a_i a^i \;,$$

$$\tilde{g}_{ij} = g_{ij} = \bar{g}_{ij} \;,$$

$$\tilde{\phi} = \phi - \frac{1}{2} \ln a + \frac{\lambda}{4} \alpha' a_i a^i \;. \tag{3.3.5}$$

The consistency conditions again follow by applying R to the above and using $[T, R] = 0$ on the LHS:

$$\frac{1}{\tilde{a}} \tilde{\bar{\beta}}_{00} = - \frac{1}{a} \bar{\beta}_{00} + 2\lambda\alpha' \Big[a^i \partial_i \Big(\frac{1}{a} \bar{\beta}_{00} \Big) - \frac{1}{2} a^i a^j \bar{\beta}_{ij} \Big] \;,$$

$$\tilde{\bar{\beta}}_{ij} = \bar{\beta}_{ij} \;,$$

$$\tilde{\bar{\beta}}^{\phi} = \bar{\beta}^{\phi} - \frac{1}{2a} \bar{\beta}_{00} + \frac{\lambda}{2} \alpha' \Big[a^i \partial_i \Big(\frac{1}{a} \bar{\beta}_{00} \Big) - \frac{1}{2} a^i a^j \bar{\beta}_{ij} \Big] \;. \tag{3.3.6}$$

The terms scaling correctly under $g \to \Lambda g$ at this order, and thus possibly present in the beta function, are

$$\beta^{(2)}_{\mu\nu} = A_1 \nabla_\mu \nabla_\nu R + A_2 \nabla^2 R_{\mu\nu} + A_3 R_{\mu\alpha\nu\beta} R^{\alpha\beta} + A_4 R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} + A_5 R_{\mu\alpha} R_\nu{}^\alpha +$$

$$+ A_6 R_{\mu\nu} R + A_7 g_{\mu\nu} \nabla^2 R + A_8 g_{\mu\nu} R^2 + A_9 g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + A_{10} g_{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \quad (3.3.7)$$

(we have used Bianchi identities to reduce from a larger set of tensor structures). It will suffice in fact to study the consistency conditions for the (ij) components, $\tilde{\bar{\beta}}_{ij} = \bar{\beta}_{ij}$, in order to determine the only structure satisfying all the consistency conditions.

We write

$$\bar{\beta}_{ij} = \alpha' \Big(\beta^{(1)}_{ij} + 2 \bar{\nabla}_i \partial_j \phi \Big) + \alpha'^2 \beta^{(2)}_{ij} \ , \quad (3.3.8)$$

where $\beta^{(1)}_{ij} = R_{ij} = \bar{R}_{ij} - \frac{1}{2} q_{ij}$ is the one-loop beta function, and perform the duality transformation (3.3.5), keeping terms to order $\mathcal{O}(\alpha'^2)$. Using the fact that the one-loop Weyl anomaly coefficient satisfies the one-loop consistency conditions (3.2.8), we arrive at,

$$\tilde{\beta}^{(2)}_{ij} = \beta^{(2)}_{ij} - \frac{1}{4} \lambda a_{(i} \partial_{j)} (a^k a_k) \ , \quad (3.3.9)$$

where the duality transformation of $\beta^{(2)}_{ij}$ is given simply by $a \to 1/a$ without α' corrections, since this is already $\mathcal{O}(\alpha'^2)$. Separating the possible tensor structures into even and odd tensors under $a \to 1/a$,

$$\beta^{(2)}_{ij} = E_{ij} + O_{ij} \ , \quad \tilde{E}_{ij} = E_{ij} \ , \quad \tilde{O}_{ij} = -O_{ij} \ , \quad (3.3.10)$$

the even structures drop out and we are left with

$$O_{ij} = \frac{1}{8} \lambda a_{(i} \partial_{j)} (a^k a_k) \ . \quad (3.3.11)$$

We now perform a standard Kaluza-Klein reduction on the ten terms in (3.3.7) to identify which if any satisfy this condition. The results can be obtained using the formulas in the Appendix of [46], and are as follows:

$$(1) \ : \ \nabla_i\nabla_j R = \bar{\nabla}_i\bar{\nabla}_j(\bar{R} - q_n{}^n) \ ,$$

$$(2) \ : \ \nabla^2 R_{ij} = (\bar{\nabla}^2 + \frac{1}{2}a_k\bar{\nabla}^k)(\bar{R}_{ij} - \frac{1}{2}q_{ij}) - \frac{1}{4}a_i a_j q_n{}^n - \frac{1}{4}a^k a_{(i}\Big(\bar{R}_{j)k} - \frac{1}{2}q_{j)k}\Big) \ ,$$

$$(3) \ : \ R_{i\alpha j\beta}R^{\alpha\beta} = \frac{1}{4}q_{ij}q_n{}^n + \bar{R}_{injm}(\bar{R}^{nm} - \frac{1}{2}q^{nm}) \ ,$$

$$(4) \ : \ R_{i\alpha\beta\gamma}R_j{}^{\alpha\beta\gamma} = \frac{1}{2}q_{ik}q_j{}^k + \bar{R}_{iknm}\bar{R}_j{}^{knm} \ ,$$

$$(5) \ : \ R_{i\alpha}R_j{}^{\alpha} = \bar{R}_{ik}\bar{R}_j{}^k - \frac{1}{2}\bar{R}_{k(i}q_{j)}{}^k + \frac{1}{4}q_{ik}q_j{}^k \ ,$$

$$(6) \ : \ R_{ij}R = (\bar{R}_{ij} - \frac{1}{2}q_{ij})(\bar{R} - q_n{}^n) \ ,$$

$$(7) \ : \ g_{ij}\nabla^2 R = \bar{g}_{ij}\Big[\frac{1}{2}a^k\partial_k(\bar{R} - q_m{}^m) + \bar{\nabla}^k\partial_k(\bar{R} - q_m{}^m)\Big] \ ,$$

$$(8) \ : \ g_{ij}R^2 = \bar{g}_{ij}\left(\bar{R} - q_m{}^m\right)^2 \ ,$$

$$(9) \ : \ g_{ij}R_{\alpha\beta}R^{\alpha\beta} = \bar{g}_{ij}\Big[\frac{1}{4}(q_m{}^m)^2 + (\bar{R}_{km} - \frac{1}{2}q_{km})^2\Big] \ ,$$

$$(10) \ : \ g_{ij}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \bar{g}_{ij}\Big[q_{km}q^{km} + \bar{R}_{k\ell mn}\bar{R}^{k\ell mn}\Big] \ . \qquad (3.3.12)$$

The respective odd parts are

$$O_{ij}^{(1)} = -\bar{\nabla}_i\bar{\nabla}_j\bar{\nabla}_n a^n \ ,$$

$$O_{ij}^{(2)} = \frac{1}{2}a_k\bar{\nabla}^k\bar{R}_{ij} - \frac{1}{2}\bar{\nabla}^2\bar{\nabla}_i a_j - \frac{1}{4}a_i a_j\bar{\nabla}_k a^k \ ,$$

$$O_{ij}^{(3)} = -\frac{1}{2}\bar{R}_{injm}\bar{\nabla}^n a^m + \frac{1}{8}a_n a^n\bar{\nabla}_i a_j + \frac{1}{8}a_i a_j\bar{\nabla}_n a^n \ ,$$

$$O_{ij}^{(4)} = \frac{1}{4}a_k a_{(i}\bar{\nabla}_{j)}a^k \ ,$$

$$O_{ij}^{(5)} = -\frac{1}{2}\bar{R}_{k(i}\bar{\nabla}_{j)}a^k + \frac{1}{8}a_k a_{(i}\bar{\nabla}_{j)}a^k \ ,$$

$$O_{ij}^{(6)} = -\frac{1}{2}\bar{R}\bar{\nabla}_i a_j - \bar{R}_{ij}\bar{\nabla}_n a^n + \frac{1}{4}a_i a_j\bar{\nabla}_n a^n + \frac{1}{4}a_n a^n\bar{\nabla}_i a_j \ ,$$

$$O_{ij}^{(7)} = \bar{g}_{ij}\Big[\frac{1}{2}a^k\partial_k(\bar{R} - \frac{1}{2}a_m a^m) - \bar{\nabla}^k\partial_k(\bar{\nabla}_m a^m)\Big] \ ,$$

$$O_{ij}^{(8)} = \bar{g}_{ij}\Big[-2(\bar{\nabla}^k a_k)\bar{R} + (\bar{\nabla}^k a_k)a^m a_m\Big] \ ,$$

$$O_{ij}^{(9)} = \bar{g}_{ij}\Big[\frac{1}{4}(\bar{\nabla}^k a_k)a^m a_m - (\bar{\nabla}_k a_m)\bar{R}^{km} + \frac{1}{4}(\bar{\nabla}_k a_m)a^k a^m\Big] \ ,$$

$$O_{ij}^{(10)} = \bar{g}_{ij}(\bar{\nabla}_k a_m)a^k a^m \ . \tag{3.3.13}$$

It is fortunate that none of these tensors contain purely even structures, since such structures are left unconstrained (and thus undetermined) by duality. The only odd term of the form (3.3.11) comes from $A_4 R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}$, and a detailed inspection shows that no linear combination of the other terms gives rise to odd tensors generically of the form (3.3.11). This determines that, with the requirement of covariance of duality under the RG, the $\mathcal{O}(\alpha'^2)$ term in the beta function is

$$\beta_{\mu\nu}^{(2)} = \lambda R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} \ . \tag{3.3.14}$$

One should now check that the corresponding (00) component also satisfies its consistency condition. A straightforward computation shows that it does, and the determination of the two-loop beta function is thus complete.

Although we treated a restricted class of metric backgrounds, our result is valid for a generic metric, since none of the possible tensor structures are built out of the off-block-diagonal g_{0i} elements alone (in which case our consistency conditions would be blind to them, just as they are to the even terms E_{ij}).

Some final comments on scheme dependence are also in order: for a purely metric background, it is well-known that the two-loop beta function is scheme independent within the standard set of subtraction schemes determined by minimal and non-minimal subtractions of the one-loop divergent structure $R_{\mu\nu}$. Under a broader definition of subtraction scheme, however, when other terms may also be subtracted, *e.g.* of the type $g_{\mu\nu}R$, then the beta function becomes scheme dependent and differs from (3.3.14). Our duality constraints have determined a beta function falling into the first (and standard) class of schemes, *i.e.*, those in which one-loop subtractions are of the form (constant $+ 1/\epsilon)R_{\mu\nu}$. This is natural to expect, as these represent the subtraction of the inherent divergence of the theory. However, it raises the question of whether the duality constraints clash against the possibility of making more general subtractions. It has recently been found [42] that in fact there is no clash, since it is possible to explicitly determine the modification in the duality transformations themselves under a field redefinition, and they will be such as to preserve the consistency conditions with respect to the redefined beta functions. The statement $[T, R] = 0$ thus acquires a meaning beyond and independent of any field redefinition ambiguity.

Simply using the requirements that duality and the RG commute as motions in the parameter space of the sigma model, we have been able to determine the two-loop beta function to be

$$\beta_{\mu\nu} = \alpha' R_{\mu\nu} + \alpha'^2 \lambda R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} \,, \tag{3.3.15}$$

for an entirely generic metric background, without any Feynman diagram calculations. Because we used an extremely restrictive class of backgrounds, it was not possible to determine the value of λ (the correct value is $\lambda = \frac{1}{2}$). However, we expect that, similarly to what happens at $\mathcal{O}(\alpha')$, once a more generic background is used in the consistency conditions, even this constant should be determined[1].

That duality symmetry should yield information on the renormalization structure of a theory is to us a striking fact, and one which we intend to further explore in the next chapter.

[1] In the supersymmetric case, a further restriction is available: when the target is restricted to a Kähler manifold, the beta function must be a Kähler tensor. Because the structure in (3.3.15) is not such a tensor, λ must vanish. Thus, in the supersymmetric case our considerations do determine that there are no $\mathcal{O}(\alpha')$ corrections to duality and that the two-loop beta function vanishes.

Chapter 4

Heterotic σ–Models

4.1 Introduction

Symmetry is a central concept in quantum field theory. Usually, one thinks of symmetries as transformations acting on the fields of a theory, leaving its partition function invariant. More fashionable these days is a different concept. This is the idea of duality symmetries, transformations on the parameter space of a theory which leave its partition function invariant. One such example is the well known target space duality (T-duality henceforth), see [34] for a complete set of references. Another important action on the parameter space of a quantum field theory is that of the renormalization group (RG henceforth), as RG transformations also leave the partition function invariant. The study of the interplay between the RG and duality symmetries then seems quite natural [56].

The idea of T-duality symmetry first came about in the context of string theory [34], but it was soon realized that a proof of its existence could be given directly from sigma model path integral considerations [18, 19]. On the other hand, sigma models are well defined two dimensional quantum field theories away from the conformal backgrounds that are of interest for string theory [32, 6]. A study was then initiated concerning the possibility of having T-duality as a symmetry of the quantum sigma model away from the (conformal) RG fixed points, when the target manifold admits an Abelian isometry. Central to this study was the aforementioned interplay between the duality symmetry and the RG . It was observed that this interplay translates to consistency conditions to be verified by the RG flows of the model; and that indeed they were verified by, and only by, the correct RG flows of the bosonic sigma model. Such a study was carried out up to two loops, order $\mathcal{O}(\alpha'^2)$, in references [41, 46, 47, 42], and partially described in the previous chapter.

Such symmetry being verified in the bosonic sigma model – where the target fields are a metric, an antisymmetric field and a dilaton – one then wonders what happens for the supersymmetric extensions of such models. With relation to the $\mathcal{N} = 2$ supersymmetric sigma model [6], where the target fields are similar, the bosonic results do have something to say. This is due to the fact that this model asks for target Kähler geometry [80, 5], if it is to be supersymmetric. Including this extra constraint in the analysis of [41, 46, 47] one then sees that when restricted to background Kähler tensor structures, the results obtained in there translate to the well known results for this supersymmetric sigma model, as we also remarked in the last chapter. Corresponding results for the $\mathcal{N} = 1$ supersymmetric sigma model can also be obtained.

Another interesting supersymmetric extension of the bosonic sigma model is the heterotic sigma model [50]. One extra feature is that one now has a target gauge field. It is this new coupling that we shall study in here, following the point of view in [41, 46, 47, 42]. We shall work to one loop, order $\mathcal{O}(\alpha')$, and we will see that T-duality is again a good quantum symmetry of this sigma model. This shall be done by deriving consistency conditions for the RG flows of the model under T-duality and observing that they are satisfied by, and only by, the correct RG flows of the heterotic sigma model. However, yet another extra feature arises. In these models the measure of integration over the quantum fields involves chiral fermions. Such fermions produce potential anomalies, and we therefore have a first example where we can analyze the interplay of T-duality and the RG flow in the presence of anomalies. It is then reasonable to expect that the consistency conditions may have something to say about these anomalies, as they need to cancel in order to define an RG flow.

One should finally remark that it is indeed interesting that duality, a symmetry which is apparently entirely unaware of the renormalization structure of the model, should yield such strong constraints as to uniquely determine the sigma model beta functions. Work similar in spirit to the one we perform here has also been carried out in condensed matter systems [21, 17], and more recently in systems that have a strong-weak coupling duality [61, 53].

Following [41, 46, 47, 42], let us begin with a theory with an arbitrary number of couplings, g^i, $i = 1, \ldots, n$, and consider a duality symmetry, T, acting as a map between equivalent points in the parameter space, such that,

$$Tg^i \equiv \tilde{g}^i = \tilde{g}^i(g). \tag{4.1.1}$$

Let us also assume that our system has a renormalization group flow, R, encoded by a set of beta functions, and acting on the parameter space by,

$$Rg^i \equiv \beta^i(g) = \mu \frac{dg^i}{d\mu}, \tag{4.1.2}$$

where μ is some appropriate subtraction scale. Given any function on the parameter space of the theory, $F(g)$, the previous operations act as follows:

$$TF(g) = F(\tilde{g}(g)) \qquad , \qquad RF(g) = \frac{\delta F}{\delta g^i}(g) \cdot \beta^i(g). \tag{4.1.3}$$

For a finite number of couplings the derivatives above should be understood as ordinary derivatives, whereas in the case of the sigma model these will be functional derivatives, and the dot will imply an integration over the target manifold.

The consistency requirements governing the interplay of the duality symmetry and the RG can now be stated simply as,

$$[T, R] = 0, \tag{4.1.4}$$

or in words: duality transformations and RG flows commute as motions in the parameter space of the theory. This amounts to a set of consistency conditions on the beta functions of our system:

$$\beta^i(\tilde{g}) = \frac{\delta \tilde{g}^i}{\delta g^j} \cdot \beta^j(g). \qquad (4.1.5)$$

As we shall see, this is a very strict set of requirements in our model.

The organization of this chapter is as follows. In section 2 we will give a brief review of the heterotic sigma model, and on how to construct the T-duality transformation acting on the target space. Then, in section 3, we shall see how these transformations translate to a set of consistency conditions to be satisfied by the beta functions of the model. In section 4, we study such conditions in the heterotic sigma model; for the simpler case of torsionless backgrounds and paying special attention to the cancelation of anomalies. The results obtained in this section are then extended to torsionfull backgrounds in section 5, where the calculations are more involved. Finally, in section 6, we present a concluding outline.

4.2 Duality in the Heterotic Sigma Model

We shall start by reviewing the construction of the heterotic sigma model in $(1,0)$ superspace, and the standard procedure of dualizing such model. We will closely follow the main references on the subject, [50, 49, 3, 4], and refer to them for further details.

Superspace will have two bosonic coordinates, z_0 and z_1, and a single fermionic coordinate of positive chirality, θ. The supersymmetry is $\mathcal{N} = \frac{1}{2}$ Majorana-Weyl, as only the left moving bosons have fermionic partners. We will consider two types of superfields, one scalar coordinate superfield and one spinor gauge superfield,

$$\Phi^\mu(z,\theta) = X^\mu(z) + \theta\lambda^\mu(z) \quad , \quad \Psi^I(z,\theta) = \psi^I(z) + \theta F^I(z). \qquad (4.2.1)$$

In here the Φ^μ are coordinates in a $(d+1)$-dimensional target manifold $\mathcal{M}$, so that $\mu = 0, 1, ..., d$, while the Ψ^I are sections of a G-bundle over $\mathcal{M}$ with n-dimensional fibers, so that $I = 1, ..., n$. These spinor superfields transform under a representation R of the gauge group G, with $n = \dim R$. We will consider arbitrary n, d, even though for the heterotic superstring $d+1 = 10$, $n = 32$ and $G = \mathrm{Spin}(32)/\mathbf{Z}_2$ or $G = E_8 \otimes E_8$ [39]. Using light-cone coordinates, $z^{\pm} = \frac{1}{\sqrt{2}}(z^0 \pm z^1)$ and $\partial_{\pm} = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$, the superspace $(1,0)$ covariant derivative is written as:

$$D = \partial_\theta + i\theta\partial_+ \qquad , \qquad D^2 = i\partial_+. \tag{4.2.2}$$

We consider the target manifold endowed with a metric $g_{\mu\nu}$, antisymmetric tensor field $b_{\mu\nu}$ and a gauge connection $A_{\mu IJ}$ associated to the gauge group G. The Lagrangian density of such model is given by [50, 3]:

$$\mathcal{L} = -i \int d\theta \{ (g_{\mu\nu}(\Phi) + b_{\mu\nu}(\Phi))D\Phi^\mu \partial_- \Phi^\nu - i\delta_{IJ}\Psi^I(D\Psi^J + A_\mu{}^J{}_K(\Phi)D\Phi^\mu\Psi^K) \}. \tag{4.2.3}$$

One should keep in mind that the action has an overall coefficient of $\frac{1}{4\pi\alpha'}$, as usual. A good exercise is to do the θ integration and eliminate the auxiliary fields. One should find:

$$\mathcal{L} = (g_{\mu\nu} + b_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu + i g_{\mu\nu}\lambda^\mu(\partial_-\lambda^\nu + (\Gamma^\nu_{\rho\sigma} + \frac{1}{2}H^\nu{}_{\rho\sigma})\partial_- X^\rho \lambda^\sigma) +$$

$$+ i\psi^I(\partial_+\psi^I + A_\mu{}^I{}_J \partial_+ X^\mu \psi^J) + \frac{1}{2}F_{\mu\nu IJ}\lambda^\mu\lambda^\nu\psi^I\psi^J, \tag{4.2.4}$$

where,

$$H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu} \qquad \text{and} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{4.2.5}$$

We need to assume that the sigma model has an Abelian isometry in the target manifold, which will enable duality transformations [49, 3, 4]. Let ξ be the Killing vector that generates the Abelian isometry. The diffeomorphism generated by ξ transforms the scalar superfields, and the total action is invariant under the isometry only if we can compensate this transformation in the scalar superfields by a gauge transformation in the spinor superfields [49, 3]. This introduces a target gauge transformation parameter κ, such that $\delta_\xi A_\mu \equiv \mathcal{L}_\xi A_\mu = \mathcal{D}_\mu \kappa$.

Choose adapted coordinates to the Killing vector, $\xi^\mu \partial_\mu \equiv \partial_0$, and split the coordinates as $\mu, \nu = 0, 1, ..., d = 0, i$, so that the $\mu = 0$ component is singled out. In these adapted coordinates the isometry is manifest through independence of the background fields on the coordinate X^0. Moreover, in these coordinates the target gauge transformation parameter will satisfy [49, 3],

$$\mathcal{D}_\mu \kappa \equiv \partial_\mu \kappa + [A_\mu, \kappa] = 0. \tag{4.2.6}$$

The duality transformations are then [3, 4]:

$$\tilde{g}_{00} = \frac{1}{g_{00}} \quad , \quad \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}} \quad , \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}},$$

$$\tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}} \quad , \quad \tilde{b}_{ij} = b_{ij} - \frac{g_{0i}b_{0j} - g_{0j}b_{0i}}{g_{00}}, \tag{4.2.7}$$

$$\tilde{A}_{0IJ} = -\frac{1}{g_{00}} \mu_{IJ}, \tag{4.2.8}$$

$$\tilde{A}_{iIJ} = A_{iIJ} - \frac{g_{0i} + b_{0i}}{g_{00}} \mu_{IJ}. \tag{4.2.9}$$

where we have used $\mu_{IJ} \equiv (\kappa - \xi^\alpha A_\alpha)_{IJ}$ following [49, 3], and which in adapted coordinates becomes $\mu_{IJ} \equiv (\kappa - A_0)_{IJ}$. Observe that the gauge transformation properties of κ are such that μ_{IJ} will transform covariantly under gauge transformations [49, 3]. Equations (4.2.7) are well known since [18, 19], and their interplay with the RG has been studied in [41, 46, 47, 42], and also partially described in the previous chapter. They shall not be dealt with in here, as to our one loop order there is nothing new to be found relative to the work in [41]. We shall rather concentrate on the new additions (4.2.8) and (4.2.9) yielding the duality transformations for the gauge connection.

There is one more duality transformation one needs to pay attention to, the one for the dilaton field. As is well known, in a curved world-sheet we have to include one further coupling in our action,

$$\frac{1}{4\pi}\int d^2z\,\sqrt{h}R^{(2)}\phi(X), \tag{4.2.10}$$

where $h = \det h_{ab}$, h_{ab} being the two dimensional world-sheet metric, and $R^{(2)}$ its scalar curvature. $\phi(X)$ is the background dilaton field in $\mathcal{M}$. This term is required in order to construct the Weyl anomaly coefficients (see section 3). We should however point out that the addition of such coupling to the heterotic string is not entirely trivial as it is not invariant under the so-called kappa-symmetry [14, 13]. Taking into account the one loop Jacobian from integrating out auxiliary fields in the dualization procedure, one finds as usual the dilaton shift [19, 15]:

$$\tilde{\phi} = \phi - \frac{1}{2}\ln g_{00}. \tag{4.2.11}$$

Formulas (4.2.7-9) were obtained using classical manipulations alone. Only (4.2.11) involves quantum considerations. So, for this heterotic sigma model, we need to be careful in the following as there will be anomalies generated by the chiral fermion rotations in the quantum measure, and if so the original and dual action will not be equivalent. If we want these two theories to be equivalent one must find certain conditions on the background fields in order to cancel the anomalies. We shall see in the following that the consistency conditions (4.1.5) do have something to say on this matter.

4.3 Renormalization and Consistency Conditions

The renormalization of the heterotic sigma model has been studied in many references. Of particular interest to our investigations are the one loop beta functions [38, 69, 37, 20]. However, there are some subtleties we should point out before proceeding, as the one loop effective action is not gauge or Lorentz invariant. It happens that this non-invariance is of a very special kind, organizing itself into the well known gauge and Lorentz Chern-Simmons (order $\mathcal{O}(\alpha')$) completion of the torsion

[50]. Then, starting at two loops, there are non-trivial anomalous contributions to the primitive divergences of the theory, and things get more complicated [37]. None of these problems will be of concern to us to the order $\mathcal{O}(\alpha')$ we shall be working to, appearing only at order $\mathcal{O}(\alpha'^2)$. The one loop, order $\mathcal{O}(\alpha')$, beta functions can be computed to be [37, 20]:

$$\beta^g_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\lambda\rho\nu} + \mathcal{O}(\alpha'), \tag{4.3.1}$$

$$\beta^b_{\mu\nu} = -\frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + \mathcal{O}(\alpha'), \tag{4.3.2}$$

$$\beta^A_\mu = \frac{1}{2}(\mathbf{D}^\lambda F_{\lambda\mu} + \frac{1}{2} H_\mu{}^{\lambda\rho} F_{\lambda\rho}) + \mathcal{O}(\alpha'), \tag{4.3.3}$$

where $R_{\mu\nu}$ is the Ricci tensor of the target manifold, ∇_μ is the metric covariant derivative, and $\mathbf{D}_\mu$ is the covariant derivative involving both the gauge and the metric connections.

Of special interest to us are the Weyl anomaly coefficients [74, 75, 35, 14], which are in general different from the RG beta functions. Their importance comes from the fact that while the definition of the sigma model beta functions (β) is ambiguous due to the freedom of target reparameterization, there is no such ambiguity for the Weyl anomaly coefficients ($\bar{\beta}$) which are invariant under such transformations. This, of course, is related to the fact that the $\bar{\beta}$-functions are used to compute the Weyl anomaly, while the β-functions are used to compute the scale anomaly [74, 75].

The advantage of using Weyl anomaly coefficients in our studies is then due to the fact that while both $\bar{\beta}$ and β satisfy the consistency conditions (4.1.5), the $\bar{\beta}$-functions satisfy them exactly, while the β-functions satisfy them up to a target reparameterization [41, 46]. Since both encode essentially the same RG information, in the following we shall simply consider RG motions as generated by the $\bar{\beta}$-functions. For the heterotic sigma model [35, 14], and for the loop orders considered in this work:

$$\bar{\beta}^g_{\mu\nu} = \beta^g_{\mu\nu} + 2\nabla_\mu \partial_\nu \phi + \mathcal{O}(\alpha'), \tag{4.3.4}$$

$$\bar{\beta}^b_{\mu\nu} = \beta^b_{\mu\nu} + H_{\mu\nu}{}^{\lambda}\partial_\lambda\phi + \mathcal{O}(\alpha'), \tag{4.3.5}$$

$$\bar{\beta}^A_\mu = \beta^A_\mu + F_\mu{}^{\lambda}\partial_\lambda\phi + \mathcal{O}(\alpha'). \tag{4.3.6}$$

The consistency conditions (4.1.5) can now be derived. The couplings are $g^i \equiv \{g_{\mu\nu}, b_{\mu\nu}, A_\mu, \phi\}$, and the duality operation (4.1.1) is defined through (4.2.7-9) and (4.2.11). The RG flow operation is defined in (4.1.2), for our couplings, with the only difference that we shall consider $\bar{\beta}$-generated RG motions as previously explained. It is then a straightforward exercise to write down the consistency conditions (4.1.5) for the heterotic sigma model. The consistency conditions associated to (4.2.7) and (4.2.11) have in fact been studied before [41, 46, 47] and are known to be satisfied by, and only by, (4.3.1-2) or (4.3.4-5). So, we shall not deal with them in here. The consistency conditions associated to the gauge field coupling are:

$$\bar{\beta}^{\tilde{A}}_0 = \frac{1}{g_{00}}\bar{\beta}^A_0 + \frac{1}{g_{00}^2}(\kappa - A_0)\bar{\beta}^g_{00}, \tag{4.3.7}$$

$$\bar{\beta}^{\tilde{A}}_i = \bar{\beta}^A_i - \frac{1}{g_{00}}((\kappa - A_0)(\bar{\beta}^g_{0i} + \bar{\beta}^b_{0i}) - (g_{0i} + b_{0i})\bar{\beta}^A_0) + \frac{1}{g_{00}^2}(g_{0i} + b_{0i})(\kappa - A_0)\bar{\beta}^g_{00}, \tag{4.3.8}$$

where we have used the notation $\bar{\beta}^{\tilde{A}}_\mu \equiv \bar{\beta}^A_\mu[\tilde{g}, \tilde{b}, \tilde{A}, \tilde{\phi}]$. These are the main equations to be studied in this paper. The task now at hand is to see if these two conditions on the gauge field $\bar{\beta}$-functions are satisfied by – and only by – expressions (4.3.3), (4.3.6); and if so under what conditions are they satisfied. For that we need to perform a standard Kaluza-Klein decomposition of the target tensors. This procedure is familiar from previous work [41, 46, 47], and in particular we will use the formulas in the Appendix of [46], supplemented with the ones in the Appendix of this thesis.

A final ingredient to such an investigation is the following [6, 69]. At loop order ℓ, the possible (target) tensor structures $T_{\mu\nu\ldots}$ appearing in the sigma model beta functions must scale as $T_{\mu\nu\ldots}(\Lambda g^i) = \Lambda^{1-\ell}T_{\mu\nu\ldots}(g^i)$ under global scalings of the background fields. In our case at one loop, order $\mathcal{O}(\alpha')$, we have $\ell = 1$. These tensor structures must obviously also share the tensor properties of the beta functions. In our case the gauge beta function is gauge covariant Lie algebra valued, with one lower tensor index.

4.4 Duality, the Gauge Beta Function and Heterotic Anomalies

Let us now start analyzing our main equations, (4.3.7) and (4.3.8), for the case of the heterotic sigma model, as described in section 2. As previously mentioned this model has chiral fermions that, when rotated, introduce potential anomalies into the theory. These anomalies need to be canceled if the dualization is to be consistent at the quantum level. However, our strategy in here is to see if we can get any information on this anomaly cancelation from our consistency conditions (4.3.7-8). So, we will set this question aside for a moment and directly ask: are the consistency conditions (4.3.7-8) verified by (4.3.3), (4.3.6)?

We choose to start with torsionless backgrounds. Such choice can be seen to extremely simplify equation (4.3.8), as the metric is parameterized by:

$$g_{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & \bar{g}_{ij} \end{pmatrix}, \tag{4.4.1}$$

and we take $b_{\mu\nu} = 0$. Therefore, there is also no torsion in the dual background [46, 47]. In this simpler set up it shall be clearer how to deal with anomalies before addressing the case of torsionfull backgrounds (see section 5). All this said, equations (4.3.7-8) become,

$$\bar{\beta}_0^{\tilde{A}} = \frac{1}{a}\bar{\beta}_0^A + \frac{1}{a^2}(\kappa - A_0)\,\bar{\beta}^g_{00}, \tag{4.4.2}$$

$$\bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^A. \tag{4.4.3}$$

Now, use the Kaluza-Klein tensor decomposition of (4.3.3), (4.3.6), under (4.4.1), and compute $\bar{\beta}_0^{\tilde{A}}$ and $\bar{\beta}_i^{\tilde{A}}$ (also see the Appendix). By this we mean the following. One should start with (4.3.3), (4.3.6), and decompose it according to the parameterization (4.4.1). We will obtain expressions for $\bar{\beta}_0^A$ and $\bar{\beta}_i^A$. Then, dualize these two expressions by dualizing the fields according to the rules (4.2.7-9) and (4.2.11). This yields expressions for $\bar{\beta}_0^{\tilde{A}}$ and $\bar{\beta}_i^{\tilde{A}}$. Finally, one should manipulate the obtained expressions so that the result looks as much as possible as a "covariant vector transformation" (4.1.5). Hopefully one would obtain (4.4.2-3), if the gauge beta functions are to satisfy the consistency conditions. However, the result obtained is:

$$\bar{\beta}_0^{\tilde{A}} = \frac{1}{a}\bar{\beta}_0^{A} + \frac{1}{a^2}(\kappa - A_0)(-\bar{\beta}_{00}^{g}), \tag{4.4.4}$$

$$\bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^{A}. \tag{4.4.5}$$

The first thing we observe is that even though (4.4.5) is correct as we compare it to (4.4.3), (4.4.4) is not as we compare it to (4.4.2). There is an extra minus sign that should not be there. Could anything be wrong? A possibility that comes to mind is that nothing is wrong, and indeed (4.4.4) and (4.4.5) are the correct consistency conditions, implying that the duality transformations were incorrect to start with. In that case the duality transformation (4.2.8) would need to be modified in order to yield the correct consistency condition upon differentiation. Let us regard this consistency condition (4.4.4) as a differential relation: a one-form $\bar{\beta}_0^{\tilde{A}}$ which is expressed in the one-form coordinate basis of a "two-manifold" with local coordinates $\{A_0, a\}$. But then, as,

$$\frac{\partial}{\partial a}[\frac{1}{a}] = -\frac{1}{a^2} \qquad \not\equiv \qquad \frac{1}{a^2} = \frac{\partial}{\partial A_0}[-\frac{1}{a^2}(\kappa - A_0)], \tag{4.4.6}$$

we see that the consistency condition (4.4.4) is *not* integrable. Therefore we cannot modify the duality transformation rules.

Let us look at this situation from another perspective. We can make the consistency conditions (4.4.4-5) match (4.4.2-3) if we realize that what (4.4.4) is saying is that, in order for duality to survive as a quantum symmetry of the heterotic sigma model, we need to have,

$$(\kappa - A_0)\,\bar{\beta}_{00}^{g} = 0. \tag{4.4.7}$$

We shall see that this is just the requirement of anomaly cancelation, in a somewhat disguised form – it is the way duality finds to say that these anomalies must be canceled, if the dualization is to be consistent at the quantum level.

As was mentioned before, equations (4.2.8-9) were obtained using classical manipulations alone. In general, however, there will be anomalies and in this case the original theory and its dual will not be equivalent. If we want the two theories to be equivalent one must find the required conditions on the target fields that make these anomalies cancel. The simplest way to do so is to assume that the spin and gauge connections match in the original theory, *i.e.*, $\omega = A$ [50, 3, 4, 39]. Under this assumption, the duality transformation then guarantees that in the dual theory spin and gauge connections also match, $\tilde{\omega} = \tilde{A}$. In the following we choose to cancel the anomalies according to such prescription.

There are two outcomes of such choice [3, 4]. The first one is that if the original theory is conformally invariant to $\mathcal{O}(\alpha')$, so is the dual theory. For the sigma model this means that flowing to a fixed point will be equivalent to dual flowing to the dual fixed point (observe that the duality operation (4.1.1) does map fixed points to fixed points). The second is that we are now required to have $\mu = \Omega$, where we define:

$$\Omega_{\mu\nu} \equiv \frac{1}{2}(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu), \tag{4.4.8}$$

with ξ the Killing vector generating the Abelian isometry and ∇_μ the metric covariant derivative. In particular for our adapted coordinates $\xi_\mu = g_{\mu 0}$, and as the affine connection is metric compatible, $\Omega = 0$. But then,

$$\mu_{IJ} = (\kappa - A_0)_{IJ} = 0, \tag{4.4.9}$$

and we are back to (4.4.7). Then, the consistency conditions are satisfied as long as the anomalies are canceled.

Putting together the information in (4.4.7) and (4.4.9), let us address a few questions. The first thing we notice is that $\bar{\beta}_0^A = 0$ as $\kappa = A_0$ (recall that in adapted coordinates κ satisfies (4.2.6), and so $F_{0i} = 0$), which is consistent with the fact that the target gauge transformation parameter is not renormalized. Then, the consistency conditions become,

$$\bar{\beta}_0^{\tilde{A}} = 0 \qquad , \qquad \bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^A, \tag{4.4.10}$$

stating that the gauge beta function is self-dual under (4.2.8-9). But so, by (4.4.4-

5) with (4.4.7) satisfied, this proves that (4.3.6) explicitly satisfies the consistency conditions (4.4.10) – to the one loop, order $\mathcal{O}(\alpha')$, we are working to.

Given that the gauge field $\bar{\beta}$-function satisfies the consistency conditions, the question that follows is whether the scaling arguments mentioned in section 3 joined with the consistency conditions (4.4.10) are enough information to uniquely determine (4.3.3). This would mean that (4.4.10) is verified by, and only by, the correct gauge RG flows of the heterotic sigma model. Replacing (4.3.6) in (4.4.10) and using the duality transformations, we obtain the beta function constraint:

$$\beta_i^{\tilde{A}} = \beta_i^A + \frac{1}{2} F_i{}^k \partial_k \ln a. \tag{4.4.11}$$

On the other hand, according to scaling arguments the possible tensor structures appearing in the one loop, order $\mathcal{O}(\alpha')$, gauge beta function are:

$$\beta_\mu^A = c_1 \, \mathbf{D}^\lambda F_{\lambda\mu} + c_2 \, H_\mu{}^{\lambda\rho} F_{\lambda\rho}, \tag{4.4.12}$$

where the notation is as in (4.3.3). Dealing with torsionless backgrounds (4.4.1) we set $c_2 = 0$, and are left with c_1 alone. Inserting (4.4.12) in (4.4.11) then yields,

$$(c_1 - \frac{1}{2}) F_i{}^k \partial_k \ln a = 0, \tag{4.4.13}$$

and as the background is general (though torsionless), we obtain $c_1 = \frac{1}{2}$ which is the correct result (4.3.3). Therefore, our consistency conditions were able to uniquely determine the one loop gauge field beta function, in this particular case of vanishing torsion. We shall later see that the same situation happens when one deals with torsionfull backgrounds.

A final point to observe is that the proof of $\mu_{IJ} = 0$ through (4.4.7) (and so, also the proof of validity of the consistency conditions) is telling us that only if the sigma model is consistent at the quantum level (no anomalies) can the duality symmetry be consistent at the quantum level (by having the consistency conditions verified). Still, one could argue that strictly speaking (4.4.7) requires either $\mu_{IJ} = 0$ *or* $\bar{\beta}^g_{00} = 0$. But we also need to cancel all anomalies in order to have an RG flow. So, if one wants to flow away from the fixed point along all directions in the parameter space, one needs to cancel the anomalies in such a way that $\mu_{IJ} = 0$ in the adapted coordinates to the Abelian isometry. Otherwise, if we were to choose an anomaly cancelation procedure yielding non-vanishing μ_{IJ}, it would seem that in order to preserve T-duality at the quantum level away from criticality, expression (4.4.7) would require that one could only flow away from the fixed point along specific regions of the parameter space (*i.e.*, regions with $\bar{\beta}^g_{00} = 0$). As we shall see next when we deal with torsionfull backgrounds, this is actually not a good option: the only reasonable choice one can make is $\mu_{IJ} = 0$.

4.5 Torsionfull Backgrounds

To complete our analysis, we are left with the inclusion of torsion to the previous results. We shall see that even though the calculations are rather involved, the results are basically the same. Let us consider the same situation as in the last section, with the added flavor of torsion. As in [41, 46], we decompose the generic metric $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \begin{pmatrix} a & av_i \\ av_i & \bar{g}_{ij} + av_iv_j \end{pmatrix}, \tag{4.5.1}$$

so that $g_{00} = a$, $g_{0i} = av_i$ and $g_{ij} = \bar{g}_{ij} + av_iv_j$. The components of the antisymmetric tensor are written as $b_{0i} \equiv w_i$ and b_{ij}. We will also find convenient to define the following quantities, $a_i \equiv \partial_i \ln a$, $f_{ij} \equiv \partial_i v_j - \partial_j v_i$ and $G_{ij} \equiv \partial_i w_j - \partial_j w_i$. From (4.2.7) one finds that in terms of the mentioned decomposition, the dual metric and antisymmetric tensor are given by the substitutions $a \to 1/a$, $v_i \leftrightarrow w_i$, and $\tilde{b}_{ij} = b_{ij} + w_iv_j - w_jv_i$.

With all these definitions at hand, we proceed with the Kaluza-Klein decomposition of (4.3.3), (4.3.6), and compute $\bar{\beta}^{\tilde{A}}_0$ (also see the Appendix). From the discussion in section 4 it should be clear what we mean by this, and which are the several steps required to carry out such calculation. Again, one hopes to find (4.3.7) if the gauge beta function is to satisfy the consistency conditions in this torsionfull case. Yet again, this does not happen. Instead we obtain,

$$\bar{\beta}_0^{\tilde{A}} = \frac{1}{a}\bar{\beta}_0^A + \frac{1}{a^2}(\kappa - A_0)(-\bar{\beta}_{00}^g) +$$

$$+\frac{1}{2}(\kappa - A_0)(f^{ij} + \frac{1}{a}G^{ij})f_{ij} + (f^{ij} + \frac{1}{a}G^{ij})v_i F_{j0} + \frac{1}{a}v^i[A_0, F_{i0}]. \tag{4.5.2}$$

At first this looks like a complicated result. However, we already have the experience from the torsionless case, and that should be enough information to guide our way. Indeed, recall the discussion on anomaly cancelation from section 4, and proceed to cancel the anomalies according to $\mu_{IJ} = 0$. Then one has $\kappa = A_0$, and as κ satisfies (4.2.6) in these adapted coordinates we are working in, we also have $F_{i0} = 0$. Looking again at (4.5.2), one sees that the anomaly cancelation condition – just like in the torsionless case – makes (4.5.2) match the consistency condition (4.3.7). Moreover, we also see from (4.5.2) that, unless we are to severely restrict the background fields, the only choice one can make in order to have the consistency conditions verified is to cancel the anomalies through $\mu_{IJ} = (\kappa - A_0)_{IJ} = 0$. Finally, observe that as the target gauge transformation parameter does not get renormalized, and $\kappa = A_0$, we will have in adapted coordinates $\bar{\beta}_0^A = 0$.

One is now left with the analysis of $\bar{\beta}_i^{\tilde{A}}$. Making use of all that has been said in the last paragraph this turns out to be a reasonable calculation as the consistency conditions (4.3.7-8) have once again become,

$$\bar{\beta}_0^{\tilde{A}} = 0 \qquad , \qquad \bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^A, \tag{4.5.3}$$

the same as (4.4.10). Computing $\bar{\beta}_i^{\tilde{A}}$ by the usual procedure (also see the Appendix), one then finds that it indeed satisfies the consistency conditions, modulo gauge transformations. This is reminiscent of the fact that the β-functions only satisfy the consistency conditions modulo target reparameterizations, as they are not invariant under such transformations. In here, the $\bar{\beta}$-functions themselves are not gauge invariant, but gauge covariant. In particular, we can choose a gauge where the consistency conditions are explicitly verified, the gauge $A_0 = 0$.

So the gauge field $\bar{\beta}$-function satisfies the consistency conditions in the torsionfull case as well as it does in the torsionless case. One final question remains: are these consistency conditions enough information to compute the coefficients c_1 and c_2 in (4.4.12)? The constraint these conditions impose on the beta function is obviously the same as (4.4.11). So, when we insert (4.4.12) in (4.4.11) we obtain on one hand, (4.4.13). This is to be expected and allows us to determine $c_1 = \frac{1}{2}$. On the other hand we get the new relation,

$$\frac{1}{2}(c_1 - 2c_2)(av_i + w_i)f^{jk}F_{jk} = 0, \tag{4.5.4}$$

and as the background is general, we obtain $c_2 = \frac{1}{4}$ which is the correct result (4.3.3). Therefore, our consistency conditions were able to uniquely determine the one loop gauge field beta function. Thus, the consistency conditions (4.5.3) are verified by, and only by, the correct RG flows of the heterotic sigma model. In other words, classical target space duality symmetry survives as a valid quantum symmetry of the heterotic sigma model.

4.6 Conclusions

We have studied in this chapter the consistency between RG flows and T-duality in the $d = 2$ heterotic sigma model. The basic statement $[T, R] = 0$ that had been previously studied in bosonic sigma models was shown to keep its full validity in this new situation, with the added bonus of giving us extra information on how one should cancel the anomalies (arising from chiral fermion rotations) of the heterotic sigma model. Moreover, contrary to previously considered cases [41, 46, 47], the requirement $[T, R] = 0$ enabled us to uniquely determine the (gauge field) beta function at one loop order, without any overall global constant left to be determined.

Having considered the cases of closed bosonic, heterotic and (to a certain extent) Type II strings/sigma models, a question that comes to mind is the following. What happens in the open string case? In this case, the duality transformations are [26, 25],

$$\tilde{A}_0 = 0 \qquad , \qquad \tilde{A}_i = A_i. \tag{4.6.1}$$

The consistency conditions associated to (4.6.1) are,

$$\bar{\beta}_0^{\tilde{A}} = 0 \qquad , \qquad \bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^{A}, \tag{4.6.2}$$

the same as (4.4.10). Again, using scaling arguments the only possible form of the gauge field beta function is (4.4.12). If actually the Weyl anomaly for this situation is the same as in (4.3.6), we conclude that also in here $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{4}$. Then, by the same line of arguments as in section 5, we also conclude that for the open string the statement $[T, R] = 0$ is true and determines the beta function exactly, ensuring that duality is a quantum symmetry of the sigma model.

One last sigma model to mention is a truncated version of the heterotic sigma model [15], where one gets rid of the λ^μ fermions in the Lagrangian (4.2.4). The consequences of such truncation are the loss of fermionic partners for the left moving bosons (thus destroying the (1,0) supersymmetry), and the fact that one no longer needs to rotate the ψ^I fermions in the dualization procedure (thus removing the κ parameter from expressions (4.2.8-9) and (4.3.7-8)). Considering the simpler case of torsionless backgrounds, one finds that the known $\bar{\beta}$-functions (4.3.6) satisfy the consistency conditions, modulo gauge transformations. Choosing a gauge where these conditions are explicitly verified ($A_0 = 0$), and following the standard dualization procedure [15] one obtains that the gauge fixed duality transformations are the same as (4.6.1), and so the consistency conditions are the same as (4.4.10) or (4.6.2). Then, by the familiar line of arguments, $[T, R] = 0$ is true and determines the beta function exactly ensuring that duality is a quantum symmetry of this sigma model.

Such a basic statement $[T, R] = 0$ has now been shown to be alive and well in a wide variety of situations, possibly validating the claim in [46, 42] that it should be a more fundamental feature of the models in question than the invariance of the string background effective action.

Chapter 5

The Gravitational Axial Anomaly

5.1 Introduction and Discussion

Perturbation theory anomalies have been known for a long time, starting with the work by Bell and Jackiw [12] and by Adler [1] concerning anomalies in gauge theories. The fact that there are no radiative corrections to the one loop result for the anomaly has been countlessly proven or brought into question since the early work of Adler and Bardeen [2]. For this reason, explicit calculations of possible radiative corrections to the one loop anomaly are of particular interest.

Using a method proposed by Baker and Johnson [10], Erlich and Freedman recently performed such an explicit calculation for the two loop contribution of the anomalous correlation function $\langle A_\mu(x)A_\nu(y)A_\rho(z)\rangle$ of three chiral currents, in the Abelian Higgs model and in the Standard Model [29]. In here, we wish to extend such calculation to the case of a gravitational background.

The Adler-Bell-Jackiw (ABJ) anomaly concerns the divergence of the axial current in a gauge field background. The calculation of the divergence of the axial current in a gravitational field background was later performed by Delbourgo and Salam [23], Eguchi and Freund [27] and Delbourgo [22]. As in the ABJ case, these authors found an anomaly associated to the conservation of the axial current, the gravitational axial anomaly. Later, Alvarez-Gaumé and Witten showed the significance of gravitational anomalies for a wide variety of physical applications [7].

The question of absence of radiative corrections to the one loop result obtained for the gravitational axial anomaly is an issue not as well established as it is in the gauge theory case. This is the reason why we proceed to perform an explicit two loop calculation, adopting the spirit in [29]. However, calculating the two loop contribution to the gravitational axial anomaly is a much longer task than to do so for the gauge axial anomaly. In this chapter we shall address the first part of the computation, by calculating the abnormal parity part of the three point function involving one axial vector and two energy-momentum tensors at a specific two loop order in the Abelian Higgs model. The reason we choose to work in this model is due to the recent interest arising from the gauge anomaly case in [29], and also due to the fact that this model is a simplified version of the Standard Model. In order to set notation, the anomalous correlator we shall be dealing with is:

$$\langle A_{\alpha}(z)T_{\mu\nu}(y)T_{\rho\sigma}(x)\rangle, \tag{5.1.1}$$

where A_{α} is the axial current and $T_{\mu\nu}$ the energy-momentum tensor.

The method of calculation [10, 29] is based on conformal properties of massless field theories, and also involves ideas from the coordinate space method of differential regularization due to Freedman, Johnson and Latorre [31]. In particular, the correlator (5.1.1) will be directly calculated in Euclidean position space and a change of variables suggested by the conformal properties of the correlator will be used in order to simplify the internal integrations. The order in two loops we shall be working involves no internal photons, but only internal matter fields (the scalar and spinor fields in the Abelian Higgs model). However, in this case diagrams containing vertex and self-energy corrections will require a regularization scale. To handle this technicality we shall introduce photons in our calculation, as there is a unique choice of gauge fixing parameter (in the photon propagator) which makes both the self-energies and vertex corrections finite. These "finite gauge photons" are merely a technical tool employed in the calculation.

The use of conformal symmetry to construct three point functions is well established. Of particular interest to us is the work by Schreier [67], where three point functions invariant under conformal transformations were constructed. For the case of one axial and two vector currents, it was shown that there is a unique conformal tensor present in the three point function. More recently, Osborn and Petkos [59] and Erdmenger and Osborn [28] have used conformal invariance to compute several three point functions involving the energy-momentum tensor. However, the case of one axial current and two energy-momentum tensors was not considered.

What we find in here is that, even though at one loop there is only one conformal tensor present in the correlator (5.1.1) – the one that leads to the contraction of the Riemann tensor with its dual in the expression for the anomaly –, at two loops there are two independent conformal tensors present in the correlator. This is unlike the gauge axial anomaly case where the only possible tensor is the one that leads to the field strength contracted with its dual in the anomaly equation. Precisely because of the presence of these two tensors in the two loop result for the three point function, this correlator does not vanish. Again, this is unlike the gauge axial anomaly case [29].

The two linearly independent conformal tensors present in the anomalous correlator are the ones in expressions (5.3.6) and (5.3.7) below (where the notation is explained in the paragraphs leading up to these formulas). One thing we would like to stress is that *every* diagram relevant for our calculation is either a multiple of one of these tensors, or a linear combination of them both.

Two comments are in order. First, the existence of two independent tensors in the two loop correlator could seem to indicate the existence of a radiative correction to the anomaly. On the other hand, the fact that the correlator does not vanish at two loops does not mean that its divergence (the anomaly) does not vanish at two loops.

Another point of interest is to follow [31] and study the differential regularization of the one loop triangle diagram associated to the gravitational axial anomaly, Figure 1(a). This is done in the Appendix. What one finds is that differential regulation entails the introduction of several different mass scales. Renormalization or symmetry conditions may then be used to determine the ratios of these mass scales. In the gauge axial anomaly case it was found that there is only one mass ratio [31]. In this gravitational axial anomaly case, we have shown in the Appendix that there is more than one mass ratio. This multiplicity of the mass ratios introduces new parameters that could be able to cancel all potential (new) anomalies. Apart from presenting part of these different scales we shall not proceed with their study. Here, we shall only restrict to the calculation of the correlation function, which by itself consists a lengthy project. Extracting the two loop contribution to the gravitational axial anomaly from our three point function is a question for the future.

The structure of this chapter is as follows. In section 2 we present the massless Abelian Higgs model, as well as a review of the basic ideas involved in the method of calculation we use. This includes the calculation of the one loop triangle diagram. Then, in section 3 we perform our two loop calculation, with emphasis on rigorous details. The many contributing diagrams are organized into separate groups, and then analyzed one at a time.

5.2 The Abelian Higgs Model and Conformal Symmetry

We shall start by presenting the massless Abelian Higgs model. In four dimensional Euclidean space, its action is given by:

$$S = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^\dagger D_\mu \phi + \bar{\psi} \gamma_\mu D_\mu \psi - f \bar{\psi} (L\phi + R\phi^\dagger) \psi - \frac{\lambda}{4} (\phi^\dagger \phi)^2 \right\}, \quad (5.2.1)$$

where we have used $L = \frac{1}{2}(1 - \gamma_5)$ and $R = \frac{1}{2}(1 + \gamma_5)$, with $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. The covariant derivatives are:

$$D_\mu \phi = (\partial_\mu + ig\mathcal{A}_\mu)\phi,$$

$$D_\mu \psi = (\partial_\mu + \frac{1}{2} ig\mathcal{A}_\mu \gamma_5)\psi, \quad (5.2.2)$$

so that the theory is parity conserving with pure axial gauge coupling.

Next we introduce a background (external) gravitational field, in order to properly define the energy-momentum tensors associated to the scalar and spinor matter degrees of freedom. A simple way to do this is to couple our model to gravity, so that a spacetime metric $g_{\mu\nu}(x)$ is naturally introduced in the Lagrangian as a field variable. Then we can obtain the energy-momentum tensor by varying the Lagrangian with respect to the metric $g_{\mu\nu}(x)$ as $T_{\mu\nu}(x) = 2\frac{\delta}{\delta g^{\mu\nu}(x)} \int d^4x \sqrt{-g}\, \mathcal{L}$, where $T_{\mu\nu}(x)$ is manifestly symmetric. In addition we have to ensure that it is conserved and traceless, obtaining finally for the fermion field,

$$T^{\mathbf{F}}_{\mu\nu} = \bar{\psi}\, \gamma_{(\mu}\partial_{\nu)}\, \psi, \tag{5.2.3}$$

and for the boson field,

$$T^{\mathbf{B}}_{\mu\nu} = \frac{2}{3}\Big\{\partial_\mu\phi^\dagger\, \partial_\nu\phi + \partial_\nu\phi^\dagger\, \partial_\mu\phi - \frac{1}{2}\delta_{\mu\nu}\partial_\alpha\phi^\dagger\, \partial_\alpha\phi - \frac{1}{2}(\phi\, \partial_\mu\partial_\nu\phi^\dagger + \phi^\dagger\, \partial_\mu\partial_\nu\phi)\Big\}, \tag{5.2.4}$$

where $(\mu\nu) \equiv \mu\nu + \nu\mu$.

One should observe that in the two loop calculation we are interested in computing the order $\mathcal{O}(gf^2k^2)$ correction to the correlator, where g is the gauge coupling, f the scalar-spinor coupling, and k the gravitational coupling. This means that there are no internal photons in the associated diagrams, as these would be of order $\mathcal{O}(g^3k^2)$ – we shall only need photons as the external axial current, and in order to handle some of the potential divergences in the calculation (see section 3). This is why in (5.2.3) and (5.2.4) the scalar and spinor matter degrees of freedom are decoupled from the gauge field.

Conformal symmetry plays a central role in our calculations, as it motivates a change of variables that simplifies the two loop integrations. Due to the absence of any scale, our model is conformal invariant. The conformal group of Euclidean field theory is $O(5,1)$ [59]. All transformations which are continuously connected to the identity are obtained via a combination of rotations and translations with the basic conformal inversion,

$$x_\mu = \frac{x'_\mu}{x'^2},$$

$$\frac{\partial x_\mu}{\partial x'_\nu} = x^2(\delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}) \equiv x^2 J_{\mu\nu}(x). \tag{5.2.5}$$

The Jacobian tensor, $J_{\mu\nu}(x)$, which is an improper orthogonal matrix satisfying $J_{\mu\nu}(x) = J_{\mu\nu}(x')$, will play a useful role in the calculation of the coordinate space Feynman diagrams.

The action (5.2.1) is invariant under conformal inversions, as [29]:

$$\phi(x) \to \phi'(x) = x'^2 \phi(x'),$$

$$\psi(x) \to \psi'(x) = x'^2 \gamma_5 \not{x}' \psi(x'),$$

$$\mathcal{A}_\mu(x) \to \mathcal{A}'_\mu(x) = -x'^2 J_{\mu\nu}(x') \mathcal{A}_\nu(x'), \tag{5.2.6}$$

while also the following relations hold,

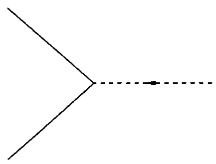
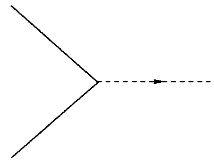
$$d^4x = \frac{d^4x'}{x'^8} \qquad \text{and} \qquad \not{x}'\gamma_\mu\not{x}' = -x'^2 J_{\mu\nu}(x')\gamma_\nu. \tag{5.2.7}$$

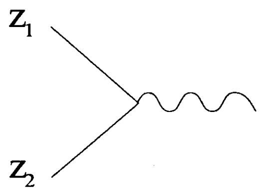
In order to use conformal properties to simplify the two loop Feynman integrals, one should expect that the relevant Feynman rules will consist of vertex factors and propagators with simple inversion properties. In particular for the scalar and spinor propagators we have,

$$\Delta(x-y) = \frac{1}{4\pi^2}\frac{1}{(x-y)^2} = \frac{1}{4\pi^2}\frac{x'^2 y'^2}{(x'-y')^2},$$

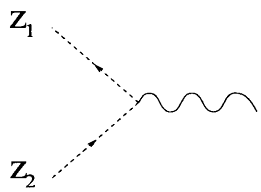
$$S(x-y) = -\not{\partial}\Delta(x-y) = \frac{1}{2\pi^2}\frac{\not{x}-\not{y}}{(x-y)^4} = -\frac{1}{2\pi^2} x'^2 y'^2 \not{x}' \frac{\not{x}'-\not{y}'}{(x'-y')^4}\not{y}'. \tag{5.2.8}$$

The vertex rules, read from the action (5.2.1), are:

 $= -fL,$  $= -fR,$ (5.2.9)



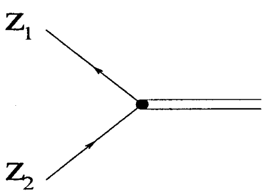
$$\mathbf{z},\alpha \quad = -\,\tfrac{1}{2}\, ig\,\gamma_\alpha\gamma_5\,\delta^4(z-z_1)\,\delta^4(z-z_2), \qquad (5.2.10)$$



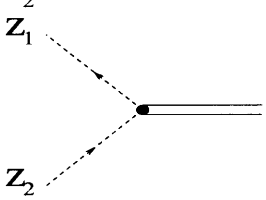
$$\mathbf{z},\alpha \quad = ig\,\left(\tfrac{\partial}{\partial z_2^\alpha} - \tfrac{\partial}{\partial z_1^\alpha}\right)\,\delta^4(z-z_1)\,\delta^4(z-z_2), \qquad (5.2.11)$$

where solid lines are fermions, dashed lines are scalars and wavy lines are gauge fields.

In addition the energy-momentum tensor insertions (5.2.3) and (5.2.4) yield the following vertices:



$$\mathbf{z},\mu\nu = k\gamma_{(\mu}\delta_{\nu)\alpha}\,\left(\tfrac{\partial}{\partial z_2^\alpha} - \tfrac{\partial}{\partial z_1^\alpha}\right)\,\delta^4(z-z_1)\,\delta^4(z-z_2), \qquad (5.2.12)$$



$$\mathbf{z},\mu\nu = \tfrac{2}{3}\,k\,\left(\tfrac{\partial}{\partial z_2^{(\mu}}\tfrac{\partial}{\partial z_1^{\nu)}} - \tfrac{1}{2}\,\delta_{\mu\nu}\,\tfrac{\partial}{\partial z_2^\alpha}\tfrac{\partial}{\partial z_1^\alpha} - \tfrac{1}{2}\left(\tfrac{\partial^2}{\partial z_2^\mu\partial z_2^\nu} + \tfrac{\partial^2}{\partial z_1^\mu\partial z_1^\nu}\right)\right)\cdot$$
$$\cdot\delta^4(z-z_1)\,\delta^4(z-z_2), \qquad (5.2.13)$$

where the double solid lines represent gravitons.

Let us analyze the conformal properties of the graviton vertices. In order to do that, we attach the vertices (5.2.12) and (5.2.13) to scalar and spinor legs, and use (5.2.5), (5.2.7) and (5.2.8) to obtain,

$$S(v-x)\,\gamma_{(\mu}\delta_{\nu)\alpha}\,\left(\frac{\overrightarrow{\partial}}{\partial x_\alpha} - \frac{\overleftarrow{\partial}}{\partial x_\alpha}\right)\,S(x-u) =$$

$$= -x'^8\, J_{\bar\mu(\mu}(x')\, J_{\nu)\bar\nu}(x')\left\{ v'^2 \not{v}'\, S(v'-x')\,\gamma_{\bar\mu}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\nu}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar\nu}})\, S(x'-u')\, u'^2 \not{u}' \right\}, \quad (5.2.14)$$

for the fermionic vertex, where the derivatives *only* act inside the curly brackets. Likewise,

$$\Delta(v-x)\,(\frac{\overleftarrow{\partial}}{\partial x_{(\mu}}\frac{\overrightarrow{\partial}}{\partial x_{\nu)}} - \frac{1}{2}\delta_{\mu\nu}\frac{\overleftarrow{\partial}}{\partial x_\alpha}\frac{\overrightarrow{\partial}}{\partial x_\alpha} - \frac{1}{2}(\frac{\overleftarrow{\partial^2}}{\partial x_\mu \partial x_\nu} + \frac{\overrightarrow{\partial^2}}{\partial x_\mu \partial x_\nu}))\,\Delta(x-u) =$$

$$= x'^8\, J_{\bar\mu\mu}(x')\, J_{\nu\bar\nu}(x')\cdot$$

$$\cdot\left\{ v'^2 \Delta(v'-x')\,(\frac{\overleftarrow{\partial}}{\partial x'_{(\bar\mu}}\frac{\overrightarrow{\partial}}{\partial x'_{\bar\nu)}} - \frac{1}{2}\delta_{\bar\mu\bar\nu}\frac{\overleftarrow{\partial}}{\partial x'_\alpha}\frac{\overrightarrow{\partial}}{\partial x'_\alpha} - \frac{1}{2}(\frac{\overleftarrow{\partial^2}}{\partial x'_{\bar\mu}\partial x'_{\bar\nu}} + \frac{\overrightarrow{\partial^2}}{\partial x'_{\bar\mu}\partial x'_{\bar\nu}}))\,\Delta(x'-u')u'^2 \right\}, \quad (5.2.15)$$

for the bosonic vertex, where once again the derivatives only act inside the curly brackets.

As an illustration of these coordinate space propagators and vertex rules, we shall now look at the one loop triangle diagram and perform the conformal inversion on the amplitude's tensor structure. The relevant one loop triangle diagram is depicted in Figure 1(a) and its amplitude, $B_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, can be computed using the previous rules to be:

$$B_{\alpha,\mu\nu,\rho\sigma}(z,y,x) =$$

$$= \frac{1}{2} i g k^2\, \mathbf{Tr}\, \gamma_\alpha \gamma_5\, S(z-y)\, \gamma_{(\mu}\delta_{\nu)\beta}\,(\frac{\overrightarrow{\partial}}{\partial y_\beta} - \frac{\overleftarrow{\partial}}{\partial y_\beta})\, S(y-x)\, \gamma_{(\rho}\delta_{\sigma)\pi}\,(\frac{\overrightarrow{\partial}}{\partial x_\pi} - \frac{\overleftarrow{\partial}}{\partial x_\pi})\, S(x-z). \quad (5.2.16)$$

Due to translation symmetry we are free to set $z = 0$, while we refer the remaining external points x and y to their inverted images (5.2.5). Although this transformation may seem *ad hoc* at this stage, it will later simplify the calculation of the two loop diagrams [29]. The result we obtain is,

$$B_{\alpha,\mu\nu,\rho\sigma}(0,y,x) =$$

$$= -\frac{igk^2}{8\pi^4} y'^8 x'^8\, J_{\bar\mu(\mu}(y') J_{\nu)\bar\nu}(y') J_{\bar\rho(\rho}(x') J_{\sigma)\bar\sigma}(x')\, \mathbf{Tr}\, \gamma_\alpha \gamma_5 \gamma_{\bar\mu}\, \frac{\partial^2}{\partial y'_{\bar\nu} \partial x'_{\bar\sigma}}\, S(y'-x')\, \gamma_{\bar\rho}. \quad (5.2.17)$$

Taking the fermionic trace one finally gets,

$$B_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{igk^2}{4\pi^6}\, y'^8 x'^8\, J_{\bar{\mu}(\mu}(y')\, J_{\nu)\bar{\nu}}(y')\, J_{\bar{\rho}(\rho}(x')\, J_{\sigma)\bar{\sigma}}(x')\, \varepsilon_{\alpha\kappa\bar{\mu}\bar{\rho}}\, \frac{\partial^2}{\partial y'_{\bar{\nu}} \partial x'_{\bar{\sigma}}} \frac{(x'-y')_\kappa}{(x'-y')^4}. \tag{5.2.18}$$

For separated points (5.2.16) is fully Bose symmetric and conserved on all indices. The expected anomaly is a local violation of the conservation Ward identities which arises because the differentiation of singular functions is involved [29]. There are several ways to obtain the anomaly in this coordinate space approach [31, 71, 29]. One way [31, 29] to do this is to recognize that the amplitude (5.2.16) is too singular at short distances to have a well defined Fourier transform. One then regulates, which entails the introduction of several independent mass scales. The regulated amplitude is well defined, and one can check the Ward identities. Also, an important aspect of this coordinate space approach to the axial anomaly is that the well defined amplitude (5.2.16), for separated points, determines the fact that there is an anomaly of specific strength [31].

Some more comments about the role of the conformal symmetry in the calculation of possible radiative corrections to the anomaly are now in order. At first sight this could look as a questionable role, after all the introduction of a scale to handle the divergences of perturbation theory will spoil any expected conformal properties. This is true in general, but our two loop triangle diagrams for this massless Abelian Higgs model are exceptional. Any primitively divergent amplitude is exceptional when studied in coordinate space for separated points, since the internal integrals converge without regularization [29].

As we shall see in the next section, there will be 3 non-planar and 3 planar diagrams, which are primitives. Of course there will be many other diagrams which contain sub-divergent vertex and self-energy corrections, and these require a regularization scale. However, we are dealing with pure axial coupling for the fermion. This means that if we introduce diagrams containing internal photons, then there is a unique choice of gauge-fixing parameter Γ which makes the one loop self-energy finite [29]. Moreover, since the vertex and self-energy corrections are related by a

Ward identity, each vertex correction is also finite in this same gauge. In conclusion, choosing this finite gauge makes it possible to obtain a finite two loop result in our calculation.

5.3 The Three Point Function for the Two Loop Gravitational Axial Anomaly

Let us now proceed to the next loop order in the non-gauge sector, as we are interested in computing possible corrections to the gravitational axial anomaly at order $\mathcal{O}(gf^2k^2)$. At this order we have a total of 36 diagrams that can possibly contribute. Of these diagrams, 3 are non-planar, but they actually only correspond to 2 independent calculations due to reflection symmetry. These are depicted in Figures 1(b) and 1(c). Then, there are 3 scalar self-energy diagrams, and other 3 photon self-energy diagrams (as we shall see, some diagrams involving photons are required in order to choose the finite gauge and compensate some divergences of the non-gauge amplitudes). These 6 self-energy diagrams amount to 2 independent calculations alone, the ones depicted in Figures 1(d) and 1(e). Then, we have 3 axial current insertion vertex corrections, in Figures 1(f), 1(g) and 1(h). At the energy-momentum tensor insertion, we also have vertex corrections. These are 6 diagrams, amounting to the 3 independent calculations in Figures 1(i), 1(j) and 1(k). There are also 6 diagrams that identically vanish due to fermionic traces, the ones in Figures 1(l), 1(m) and 1(n). Associated to the mentioned self-energies there are 3 diagrams corresponding to local self-energy renormalizations. They amount to 1 independent calculation, Figure 1(o). Also, associated to the mentioned vertex corrections at the energy-momentum insertion there are 2 diagrams corresponding to local vertex renormalizations. They amount to 1 independent calculation, Figure 1(p). So, overall, of the 29 initial two loop diagrams in Figure 1, we are left with 12 independent calculations. In Figure 2 we have 7 more diagrams, corresponding to 4 independent calculations. These diagrams are associated to the finite gauge photons and shall be discussed later. We are thus left with an overall number of 16 independent calculations, out of the initial 36 diagrams. Let us see how to perform such calculations, one at a time.

5.3.1 Diagrams in Figures 1(b) and 1(c)

We begin with the non-planar diagram depicted in Figure 1(b), which we shall denote by $N^{(1)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. This amplitude is conformal covariant since no issues of sub-divergences and gauge choice arise. The idea [10] is to use the inversion, $u_\alpha = {u'}_\alpha/{u'}^2$ and $v_\alpha = {v'}_\alpha/{v'}^2$, as a change of variables in the internal integrals. In order to use the simple conformal properties of the propagators (5.2.8) we must also refer the external points to their inverted images (5.2.5), as was done in (5.2.14), (5.2.15), and in (5.2.17), (5.2.18). If in succession we use the translation symmetry to place one point at the origin, say $z=0$, then the propagators attached to that point drop out of the integral, because the inverted point is now at ∞, and the integrals simplify.

After summing over both directions of Higgs field propagation, and setting $z=0$, the amplitude for $N^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ is written as,

$$N^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{8\pi^4}\int d^4u\, d^4v\, (\frac{u_\alpha}{u^4v^2} - \frac{v_\alpha}{v^4u^2})\cdot$$

$$\cdot\mathbf{Tr}\,\gamma_5\, S(v-x)\gamma_{(\rho}\delta_{\sigma)\beta}\,(\frac{\overrightarrow{\partial}}{\partial x_\beta} - \frac{\overleftarrow{\partial}}{\partial x_\beta})\, S(x-u)S(u-y)\gamma_{(\mu}\delta_{\nu)\pi}\,(\frac{\overrightarrow{\partial}}{\partial y_\pi} - \frac{\overleftarrow{\partial}}{\partial y_\pi})\, S(y-v). \tag{5.3.1}$$

The change of variables previously outlined can be performed with the help of (5.2.7), (5.2.8), and the Higgs current transformation,

$$\frac{u_\alpha}{u^4v^2} - \frac{v_\alpha}{v^4u^2} = {v'}^2{u'}^2\,({u'}_\alpha - {v'}_\alpha). \tag{5.3.2}$$

The spinor propagator side factors $\not{x}'$, $\not{y}'$, etc., collapse within the trace, and the Jacobian $(u'v')^{-8}$ cancels with factors in the numerator. Performing the algebra we obtain,

$$N^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{8\pi^4}\,{y'}^8{x'}^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x')\int d^4u'\, d^4v'\,(v'-u')_\alpha\cdot$$

$$\cdot\Big\{\mathbf{Tr}\,\gamma_5\,S(v'-x')\gamma_{\bar\rho}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\,S(x'-u')S(u'-y')\gamma_{\bar\mu}\,(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}})\,S(y'-v')\Big\}, \tag{5.3.3}$$

where the derivatives are acting only inside the curly brackets.

We see that we obtain the expected transformation factors for the energy-momentum tensors at x and y times an integral in which u' and v' each appear in only two denominators. These convolution integrals can be done in several different ways. The final relevant formulas are listed in the Appendix. We begin by using the trace properties to move the $S(y'-v')$ propagator in (5.3.3) close to the $S(v'-x')$ propagator. The differential operators are kept fixed, with the understanding that now the y' derivative that seems to be acting on nothing is actually acting on the propagator $S(y'-v')$ which is now sitting on the left. As usual, all derivatives act only inside curly brackets. We can perform the integrations without the need to make the differentiations first as the integration variables are well separated from the differentiation ones. Expand the product with $(v'-u')_\alpha$, and we are led to the following result:

$$\int d^4u'd^4v'(v'-u')_\alpha\mathbf{Tr}\gamma_5 S(v'-x')\gamma_{\bar\rho}(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})S(x'-u')S(u'-y')\gamma_{\bar\mu}(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}})\cdot$$

$$\cdot S(y'-v')=-\mathbf{Tr}\gamma_5\int d^4v'v'_\alpha S(v'-y')S(v'-x')\,\gamma_{\bar\rho}(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\int d^4u'S(u'-x')\cdot$$

$$\cdot S(u'-y')\,\gamma_{\bar\mu}(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}})+\mathbf{Tr}\gamma_5\int d^4v'S(v'-y')S(v'-x')\,\gamma_{\bar\rho}(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\cdot$$

$$\cdot\int d^4u'u'_\alpha S(u'-x')S(u'-y')\,\gamma_{\bar\mu}(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}}), \tag{5.3.4}$$

where the integrals can be directly read off from the Appendix. When these results are used and substituted within the trace, one finds the final amplitude,

$$N^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)=-\frac{igf^2k^2}{32\pi^8}\,y'^8x'^8\,J_{\bar\mu(\mu}(y')\,J_{\nu)\bar\nu}(y')\,J_{\bar\rho(\rho}(x')\,J_{\sigma)\bar\sigma}(x')\cdot$$

$$\cdot\Big\{\varepsilon_{\alpha\kappa\bar\rho\bar\mu}\,\frac{1}{(x'-y')^2}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\,\frac{(x'-y')_\kappa}{(x'-y')^2}\,(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}})\Big\}, \tag{5.3.5}$$

where we should have the "trace" attitude for taking derivatives: the derivatives *only* act inside the curly brackets, and the y' derivative that seems to be acting on nothing is actually acting on the first $(x'-y')$ term.

We observe that unlike the case in [29], this non-planar amplitude is *not* a numerical multiple of the amplitude for the one loop triangle diagram (5.2.18). This is because the tensor (derivative) structure in (5.3.5) is different from the one in (5.2.18). To see that one just has to explicitly compute both structures, and compare them. For the triangle one has:

$$\frac{\partial^2}{\partial x'_{\bar\sigma}\partial y'_{\bar\nu}}\frac{\Delta_\kappa}{\Delta^4} =$$

$$= \frac{\partial}{\partial x'_{\bar\sigma}}\Big(\frac{\Delta_\kappa}{\Delta^2}\Big)\frac{\partial}{\partial y'_{\bar\nu}}\Big(\frac{1}{\Delta^2}\Big) + \frac{\Delta_\kappa}{\Delta^2}\frac{\partial^2}{\partial x'_{\bar\sigma}\partial y'_{\bar\nu}}\Big(\frac{1}{\Delta^2}\Big) + \frac{1}{\Delta^2}\frac{\partial^2}{\partial x'_{\bar\sigma}\partial y'_{\bar\nu}}\Big(\frac{\Delta_\kappa}{\Delta^2}\Big) + \frac{\partial}{\partial x'_{\bar\sigma}}\Big(\frac{1}{\Delta^2}\Big)\frac{\partial}{\partial y'_{\bar\nu}}\Big(\frac{\Delta_\kappa}{\Delta^2}\Big), \tag{5.3.6}$$

while for the non-planar structure one obtains,

$$\frac{1}{\Delta^2}\Big(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}}\Big)\frac{\Delta_\kappa}{\Delta^2}\Big(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}} - \frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}}\Big) =$$

$$= \frac{\partial}{\partial x'_{\bar\sigma}}\Big(\frac{\Delta_\kappa}{\Delta^2}\Big)\frac{\partial}{\partial y'_{\bar\nu}}\Big(\frac{1}{\Delta^2}\Big) - \frac{\Delta_\kappa}{\Delta^2}\frac{\partial^2}{\partial x'_{\bar\sigma}\partial y'_{\bar\nu}}\Big(\frac{1}{\Delta^2}\Big) - \frac{1}{\Delta^2}\frac{\partial^2}{\partial x'_{\bar\sigma}\partial y'_{\bar\nu}}\Big(\frac{\Delta_\kappa}{\Delta^2}\Big) + \frac{\partial}{\partial x'_{\bar\sigma}}\Big(\frac{1}{\Delta^2}\Big)\frac{\partial}{\partial y'_{\bar\nu}}\Big(\frac{\Delta_\kappa}{\Delta^2}\Big), \tag{5.3.7}$$

where we defined $\Delta \equiv (x'-y')$. The reason such difference can happen is that while in [29] there is a unique conformal tensor structure for the correlator of three axial vector currents, in here we have two conformal tensor structures due to the higher dimensionality of the correlator of the one axial vector current and the two energy-momentum tensors. Also, observe that both these structures (5.3.6) and (5.3.7) are to be understood as always attached to the appropriate factors of $J_{\mu\nu}(y')$, $J_{\rho\sigma}(x')$ and the appropriate powers of y', x'. Moreover the diagrams that give rise to them obey conservation equations for the energy-momentum tensor insertions. (5.3.6) is associated to the one loop diagram in (5.2.18). It can easily be proved that the conservation equation is obeyed, a standard result from [23, 27, 22] (and also from [31] once we are aware of the relation (B.6) from the Appendix). (5.3.7) is associated to the two loop diagram in (5.3.5), and one can also explicitly check the conservation law for this case. This existence of two conformal structures is an extra feature in the discussion of these two loop diagrams, relative to the work in [29].

There are 2 more non-planar diagrams, where the scalar vertex is placed at x and at y. We need to compute them, as they are independent of the previous result (we have a scalar-scalar-tensor vertex instead of a scalar-scalar-vector vertex, among other different vertices), but they amount to 1 independent calculation.

So, we proceed with the non-planar diagram in Figure 1(c), denoted in the following by $N^{(2)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. The method of calculation is very similar to the one for the previous diagram, and so we shall perform it in here with somewhat less details. After summing over both directions of Higgs field propagation, and setting $z=0$, the amplitude for $N^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ is written as,

$$N^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{12\pi^4}\int d^4u\, d^4v\, \mathbf{Tr}\,\gamma_5\, \frac{\not u\gamma_\alpha\not v}{u^4v^4}\, S(v-x)\gamma_{(\rho}\delta_{\sigma)\beta}\,(\frac{\overrightarrow{\partial}}{\partial x_\beta}-\frac{\overleftarrow{\partial}}{\partial x_\beta})\, S(x-u)\cdot$$

$$\cdot\Delta(u-y)\,(\frac{\overleftarrow{\partial}}{\partial y_{(\mu}}\frac{\overrightarrow{\partial}}{\partial y_{\nu)}} - \frac{1}{2}\delta_{\mu\nu}\,\frac{\overleftarrow{\partial}}{\partial y_\pi}\frac{\overrightarrow{\partial}}{\partial y_\pi} - \frac{1}{2}\,(\frac{\overleftarrow{\partial^2}}{\partial y_\mu \partial y_\nu} + \frac{\overrightarrow{\partial^2}}{\partial y_\mu \partial y_\nu}))\,\Delta(y-v). \quad (5.3.8)$$

Performing the conformal inversion is now no harder than it was for the previous diagram. The procedure is essentially the same, and if we carry out the algebra we obtain,

$$N^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{24\pi^4}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x')\cdot$$

$$\cdot\int d^4u'\, d^4v' \Big\{ \mathbf{Tr}\,\gamma_5\gamma_\alpha\, S(v'-x')\gamma_{\bar\rho}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\, S(x'-u')\cdot$$

$$\cdot\Delta(u'-y')\,(\frac{\overleftarrow{\partial}}{\partial y'_{(\bar\mu}}\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu)}} - \frac{1}{2}\delta_{\bar\mu\bar\nu}\,\frac{\overleftarrow{\partial}}{\partial y'_\pi}\frac{\overrightarrow{\partial}}{\partial y'_\pi} - \frac{1}{2}\,(\frac{\overleftarrow{\partial^2}}{\partial y'_{\bar\mu}\partial y'_{\bar\nu}} + \frac{\overrightarrow{\partial^2}}{\partial y'_{\bar\mu}\partial y'_{\bar\nu}}))\,\Delta(y'-v')\Big\}. \quad (5.3.9)$$

Once again the expected structure emerges, and all we have to do is to perform the integrations. Using the relevant formulas from the Appendix, we find the final result as,

$$N^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{igf^2k^2}{384\pi^8}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x') \Big\{ \varepsilon_{\alpha\ell\bar\rho\kappa} \frac{(x'-y')_\ell}{(x'-y')^2} \cdot$$

$$\cdot \Big(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}} \Big) \frac{(x'-y')_\kappa}{(x'-y')^2} \Big(\frac{\overleftarrow{\partial}}{\partial y'_{(\bar\mu}} \frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu)}} - \frac{1}{2}\delta_{\bar\mu\bar\nu} \frac{\overleftarrow{\partial}}{\partial y'_\pi} \frac{\overrightarrow{\partial}}{\partial y'_\pi} - \frac{1}{2} \Big(\frac{\overleftarrow{\partial^2}}{\partial y'_{\bar\mu} \partial y'_{\bar\nu}} + \frac{\overrightarrow{\partial^2}}{\partial y'_{\bar\mu} \partial y'_{\bar\nu}} \Big) \Big) \Big\}. \tag{5.3.10}$$

However, one should note the following. In (5.3.3) both differential operators were first order in the derivatives, but in (5.3.9) the differential operator associated to the vertex (5.2.13) is actually second order. As we expect to have at the end a result similar to (5.3.5) or (5.2.18), we have to perform one of the derivatives in order for both differential operators to become first order. Manipulating this result through a somewhat lengthy calculation, one finds:

$$N^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{128\pi^8}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x') \cdot$$

$$\cdot \Big\{ \varepsilon_{\alpha\kappa\bar\rho\bar\mu} \frac{1}{(x'-y')^2} \Big(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}} \Big) \frac{(x'-y')_\kappa}{(x'-y')^2} \Big(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}} - \frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}} \Big) \Big\}, \tag{5.3.11}$$

where the notation is like in (5.3.5). One realizes that the structure here obtained is the same as in (5.3.5). So, the 3 non-planar diagrams have the same tensor structure, which is different from the one associated to the one loop triangle. From this result we immediately read the last non-planar diagram, the one with the vertex involving the scalar fields and energy-momentum tensor located at x. All we have to do is to exchange x with y and $\mu\nu$ with $\rho\sigma$ in (5.3.11). This actually does not change the amplitude (5.3.11), so that this third diagram contributes with the same amount as its reflection symmetric diagram.

Finally, we can add these 3 diagrams, and obtain the non-planar contribution to the two loop correlator. The overall contribution is simply:

$$N_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \sum_{i=1}^{3} N^{(i)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{3igf^2k^2}{64\pi^8}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y') \cdot$$

$$\cdot J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x')\left\{\varepsilon_{\alpha\kappa\bar\rho\bar\mu}\,\frac{1}{(x'-y')^2}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\,\frac{(x'-y')_\kappa}{(x'-y')^2}\,(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}})\right\}, \quad (5.3.12)$$

5.3.2 Diagrams in Figures 1(d), 1(e) and 1(o)

We now proceed to the self-energy diagrams. These will be the same as in the three gauge current case [29]. We shall see the finite gauge mechanism for the one loop self-energies and vertex corrections coming about, as it handles certain divergences by choosing a gauge where they are zero [10, 29]. For this cancelation of divergences we have introduced the Abelian field which can be decoupled at the end by setting its coupling to zero. Let us see how all that works, by starting with the Higgs self-energy diagram in Figure 1(d), and the photon self-energy diagram in Figure 1(e). These are 3 diagrams in Figure 1(d) (as we can place the self-energy loop at any of the 3 sides of the triangle), which amount to 1 independent calculation, and other 3 diagrams in Figure 1(e) that again amount to 1 independent calculation. If we remove the self-energy leg from the triangle diagram, and add the Higgs and photon contributions we obtain [29],

$$\Sigma(v-u)=\frac{1}{8\pi^4}\,[f^2+\frac{1}{2}g^2(1-\Gamma)]\,\frac{\not v-\not u}{(v-u)^6}+a\,\not\partial\,\delta^4(v-u), \quad (5.3.13)$$

where Γ is the gauge fixing parameter coming from the photon propagator [10, 29]. In this result, the first term is the part of the amplitude which is determined by the Feynman rules read from the diagrams. It has a linearly divergent Fourier transform, but the crucial point is that this amplitude can be made finite by choosing the gauge $\Gamma=1+2f^2/g^2$. It then vanishes for separated points. However, there is a possible local term, the second term in (5.3.13), which is left ambiguous by the Feynman rules, and is represented in Figure 1(o). The constant a will be determined by the Ward identity [29].

In order to proceed with the calculation of this constant using the Ward identity, we first need to look at the following vertex correction diagrams at the axial current insertion: Figure 1(f), $T^{(1)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, Figure 1(g), $T^{(2)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, and Figure 1(h),

$T^{(3)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. Again, we need a diagram involving photons in order to choose the previously introduced finite gauge. Also, these 3 diagrams clearly correspond to 3 distinct calculations.

The amplitudes of the 3 vertex correction subgraphs in these diagrams are the same as in [29]. Therefore we already know that each contribution has a logarithmic divergent Fourier transform, and that the sum of the divergent contributions from these 3 vertex subgraphs is proportional to $-2f^2 - g^2(1-\Gamma)$, therefore vanishing in the same gauge that makes the self-energy finite. Henceforth we shall use this gauge.

Let us then proceed with the Ward identity calculation, by summarizing the result from [29]. From the amplitudes for the vertex subgraphs in the diagrams $T^{(i)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, $i=1,2,3$, we obtain the Ward identity for the theory [29],

$$\frac{\partial}{\partial z_\alpha} T_\alpha(z,u,v) = -i\frac{1}{2} g\gamma_5 \left(\delta^4(z-u) - \delta^4(z-v)\right) \Sigma(u-v), \tag{5.3.14}$$

where $T_\alpha = \sum_{i=1}^{3} T^i_\alpha$, and T^i_α is the vertex subgraph in the diagram $T^{(i)}_{\alpha,\mu\nu,\rho\sigma}$. The constant a in the self-energy (5.3.13) can be calculated as in [29] – where basically one works out the LHS in (5.3.14) (in the finite gauge) in order to find the correct value for (5.3.13) in the RHS –, and the final answer is given by

$$\Sigma(z) = \frac{3}{64\pi^2} (f^2 - \frac{1}{2} g^2)\, \partial\!\!\!/ \,\delta^4(z). \tag{5.3.15}$$

Strictly speaking, one should now proceed to verify that the exact same result is obtained from the Ward identity associated with the vertex correction diagrams at the energy-momentum tensor insertions, Figures 1(i), 1(j) and 1(k). This is in fact true, but for pedagogical reasons we shall postpone such a proof for a couple of pages.

It is this result for $\Sigma(v-u)$ which is to be used to evaluate the local self-energy renormalization, Figure 1(o), therefore yielding the correct value for a in (5.3.13). These again are 3 diagrams that amount to 1 independent calculation as in Figures 1(d) and 1(e). As (5.3.15) is purely local, the integral in u and v required for the previous diagram is trivial, simply yielding a multiple of the one loop triangle amplitude. The final result is that the sum of the self-energy insertion diagrams, Figures

1(d), 1(e) and 1(o), is a multiple of the one loop amplitude,

$$\Sigma'_{\alpha,\mu\nu,\rho\sigma}(z,y,x) = \frac{3}{64\pi^2}\,(f^2 - \frac{1}{2}g^2)\,B_{\alpha,\mu\nu,\rho\sigma}(z,y,x), \tag{5.3.16}$$

exactly like in [29] as the internal fields are the same. Now recall that there is a factor of 3 from the triangular symmetry. There is also a factor of 2 for opposite directions of fermion charge flow (such term was absent in the non-planar diagrams). Finally, we are interested in the $\mathcal{O}(gf^2k^2)$ corrections, so that the term in g^2 in (5.3.16) should be discarded. The overall result for the self-energy contribution to the two loop correlator is finally,

$$\Sigma_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{9f^2}{32\pi^2}\,B_{\alpha,\mu\nu,\rho\sigma}(0,y,x), \tag{5.3.17}$$

where we have set $z = 0$ (for coherence with the other diagram calculations).

5.3.3 Diagrams in Figures 1(f), 1(g) and 1(h)

We can now proceed the calculation of the vertex correction diagrams at the axial current insertion, $T^{(i)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, $i = 1,2,3$. As for the self-energy, the calculation of these 3 diagrams follows from [29]. We shall regard each virtual photon diagram as the sum of two graphs, one with the photon propagator in the Landau gauge $\Gamma = 1$, and the second with inversion covariant pure gauge propagator,

$$\tilde{\Delta}_{\mu\nu}(u-v) = -\frac{1}{4\pi^2}\,\frac{f^2}{g^2}\,\frac{1}{(u-v)^2}\,J_{\mu\nu}(u-v). \tag{5.3.18}$$

The Landau gauge diagrams give order $\mathcal{O}(g^3k^2)$ contributions to the two loop correlator, while the remainder gives an order $\mathcal{O}(gf^2k^2)$ contribution which is what we are interested in. Therefore – and similarly to what was done from (5.3.16) to (5.3.17) – we shall discard the Landau gauge diagrams from our final result, and only use (5.3.18) for the virtual photon propagator in the finite gauge.

With this in mind we turn to the calculation of the diagrams $T^{(i)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. The method of calculation is similar to the one used for the non-planar diagrams, and so we shall follow it here without giving details. After summing over both directions of Higgs field propagation, and setting $z = 0$, the amplitude for $T^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ is written as,

$$T^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{8\pi^4}\int d^4u\, d^4v\, \Delta(v-u)\cdot$$

$$\cdot\mathbf{Tr}\, \frac{\not{u}}{u^4}\gamma_\alpha\gamma_5\frac{\not{v}}{v^4}\, S(v-y)\gamma_{(\mu}\delta_{\nu)\beta}\,(\frac{\overrightarrow{\partial}}{\partial y_\beta}-\frac{\overleftarrow{\partial}}{\partial y_\beta})\, S(y-x)\gamma_{(\rho}\delta_{\sigma)\pi}\,(\frac{\overrightarrow{\partial}}{\partial x_\pi}-\frac{\overleftarrow{\partial}}{\partial x_\pi})\, S(x-u), \tag{5.3.19}$$

and performing the conformal inversion we are led to the result,

$$T^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{8\pi^4}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x')\int d^4u'\, d^4v'\, \Delta(v'-u')\cdot$$

$$\cdot\mathbf{Tr}\,\gamma_5\gamma_\alpha\, S(v'-y')\gamma_{\bar\mu}\,(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}})\, S(y'-x')\gamma_{\bar\rho}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}})\, S(x'-u'). \tag{5.3.20}$$

As usual the expected tensorial structure emerges. We are left with the integrations to be performed. However, as we have seen, this result is divergent; only when we sum the 3 diagrams $T^{(i)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, $i = 1,2,3$ the result will be finite, in the finite gauge. So at this stage we should include in the calculation (5.3.20) the equivalent results coming from the diagrams $T^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ and $T^{(3)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ – where for this last one we should use *only* the inversion covariant pure gauge propagator (5.3.18). The result of including the 3 diagrams all together is to produce an integral of a traceless tensor, which is convergent, and can be read from the formulas in the Appendix. Hence we can write for the net sum of vertex insertions at point z, *i.e.*, $T^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ plus $T^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ plus $T^{(3)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$,

$$T'_{\alpha,\mu\nu,\rho\sigma}(0,y,x) =$$

$$= \frac{igf^2k^2}{256\pi^8}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x')\, \varepsilon_{\alpha\kappa\bar\mu\bar\rho}\,\frac{\partial^2}{\partial y'_{\bar\nu}\partial x'_{\bar\sigma}}\,\frac{(x'-y')_\kappa}{(x'-y')^4}, \tag{5.3.21}$$

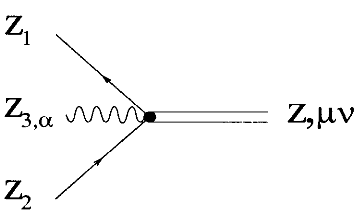
which is a multiple of the triangle one loop amplitude (5.2.18). Recalling that there is a factor of 2 for opposite directions of fermion charge flow, we can finally write for the contribution of the vertex correction diagrams (at the axial current insertion) to the two loop correlator,

$$T_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{f^2}{32\pi^2} B_{\alpha,\mu\nu,\rho\sigma}(0,y,x). \tag{5.3.22}$$

5.3.4 Diagrams in Figures 2(a) and 2(b)

In order to obtain the previous result we had to use finite gauge virtual photons, as given by the propagator (5.3.18). If one has a diagram with an internal virtual photon, one should expect a factor of g from each of the two internal vertices, and so an overall contribution of order $\mathcal{O}(g^3k^2)$. However, if one uses an internal finite gauge virtual photon, there is an extra factor of f^2/g^2 from the propagator (5.3.18), and we therefore obtain an overall contribution of order $\mathcal{O}(gf^2k^2)$, which is the order we are interested in. This means that one now has to include all diagrams with one internal finite gauge photon (5.3.18).

In particular we have to include one more vertex in our rules, that completes (5.2.9-13). This vertex can be read from the action (5.2.1) when coupled to gravity, and is the following,



$$= -\tfrac{1}{2}\, igk\,(\gamma_{(\mu}\delta_{\nu)\alpha} - \tfrac{1}{2}\,\delta_{\mu\nu}\gamma_\alpha)\,\gamma_5\,\delta^4(z-z_1)\,\delta^4(z-z_2)\,\delta^4(z-z_3), \tag{5.3.23}$$

where the notation is as in (5.2.9-13). Observe that there is no similar vertex involving two scalar legs (as opposed to the two fermion legs we have) as such vertex would give diagrams that do not contribute to the abnormal parity part of the correlator we are computing.

When this vertex is considered, one finds that there are 7 new diagrams that must be included in our calculation, the ones presented in Figure 2. There is 1 primitive diagram in Figure 2(a). There are 2 other primitive diagrams which only correspond to 1 independent calculation, the one depicted in Figure 2(b). Then we have energy-momentum insertion vertex corrections. These are Figure 2(c) and Figure 2(d), 2 independent calculations corresponding to 4 diagrams due to reflection symmetry. We shall now proceed to evaluate these diagrams.

We start with the primitive diagram in Figure 2(a), $P'^{(1)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. This diagram is easily evaluated as it involves no integrations. Recall that we have to use (5.3.18) alone, whenever one encounters a virtual photon. Setting $z=0$ and performing the conformal inversion, the amplitude $P'^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ becomes,

$$P'^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) =$$

$$= -\frac{igf^2k^2}{64\pi^8}\, y'^8 x'^8\, J_{\bar{\mu}(\mu}(y')\, J_{\nu)\bar{\nu}}(y')\, J_{\bar{\rho}(\rho}(x')\, J_{\sigma)\bar{\sigma}}(x')\, \varepsilon_{\alpha\kappa\bar{\mu}\bar{\rho}}\, \frac{(x'-y')_\kappa}{(x'-y')^6}\, J_{\bar{\nu}\bar{\sigma}}(x'-y'). \tag{5.3.24}$$

Unlike all the preceeding calculations, this amplitude involves no derivatives. This is certainly to be expected due to the nature of vertex (5.3.23). However, one can manipulate (5.3.24) in order to write it as the second derivative of a tensor involving the structures (5.3.6) and (5.3.7) alone. After some calculations, one can show that (5.3.24) can be re-written as (including the factor of 2 for opposite directions of fermion charge flow):

$$P^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{igf^2k^2}{128\pi^8}\, y'^8 x'^8\, J_{\bar{\mu}(\mu}(y')\, J_{\nu)\bar{\nu}}(y')\, J_{\bar{\rho}(\rho}(x')\, J_{\sigma)\bar{\sigma}}(x') \cdot$$

$$\cdot \Big\{ \varepsilon_{\alpha\kappa\bar{\mu}\bar{\rho}}\, \frac{1}{(x'-y')^2} \big(\frac{\overrightarrow{\partial}}{\partial x'_{\bar{\sigma}}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar{\sigma}}}\big) \frac{(x'-y')_\kappa}{(x'-y')^2} \big(\frac{\overrightarrow{\partial}}{\partial y'_{\bar{\nu}}} - \frac{\overleftarrow{\partial}}{\partial y'_{\bar{\nu}}}\big) \Big\}, \tag{5.3.25}$$

and this tensor structure is precisely the same as the one in the non-planar diagrams (5.3.12). This should not come as a surprise as this diagram – like the non-planar ones – is a primitive.

Let us proceed with the primitive diagram in Figure 2(b), $P'^{(2)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. This diagram involves only one integration, therefore being different from the ones we previously calculated (which either involved two or none integrations). Upon setting

$z=0$ the amplitude for $P'^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ is,

$$P'^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) \;=\; -\frac{igf^2k^2}{128\pi^6}\int\frac{d^4u}{(u-y)^2}\,J_{\tau\beta}(u-y)\cdot$$

$$\cdot\mathbf{Tr}\,\gamma_\beta\,\frac{\not{u}}{u^4}\,\gamma_\alpha\gamma_5\,\frac{\not{y}}{y^4}\,(\gamma_{(\mu}\delta_{\nu)\tau}-\frac{1}{2}\,\delta_{\mu\nu}\gamma_\tau)\,S(y-x)\,\gamma_{(\rho}\delta_{\sigma)\pi}\,(\frac{\overrightarrow{\partial}}{\partial x_\pi}-\frac{\overleftarrow{\partial}}{\partial x_\pi})\,S(x-u), \quad (5.3.26)$$

and performing the conformal inversion one obtains,

$$P'^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) \;=\; \frac{igf^2k^2}{128\pi^6}\,y'^8x'^8\,J_{\bar{\mu}(\mu}(y')\,J_{\nu)\bar{\nu}}(y')\,J_{\bar{\rho}(\rho}(x')\,J_{\sigma)\bar{\sigma}}(x')\cdot$$

$$\cdot\int\frac{d^4u'}{(u'-y')^2}\,J_{\bar{\nu}\beta}(u'-y')\,\mathbf{Tr}\,\gamma_\beta\,\gamma_\alpha\,\gamma_5\,\gamma_{\bar{\mu}}\,S(y'-x')\gamma_{\bar{\rho}}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar{\sigma}}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar{\sigma}}})\,S(x'-u'). \quad (5.3.27)$$

We are left with one integration to perform. However, one should note that there is only one differential operator in (5.3.27), and if we are to obtain a final result for this diagram which involves the tensor structures (5.3.6) and (5.3.7) we shall have to manipulate (5.3.27) in order to re-write it in such a way that it involves two differential operators. This is analogous to the situation we faced from (5.3.24) to (5.3.25). After integrating and performing some calculations, one obtains:

$$P'^{(2)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) \;=\; \frac{igf^2k^2}{128\pi^8}\,y'^8x'^8\,J_{\bar{\mu}(\mu}(y')\,J_{\nu)\bar{\nu}}(y')\,J_{\bar{\rho}(\rho}(x')\,J_{\sigma)\bar{\sigma}}(x')\,\varepsilon_{\alpha\kappa\bar{\mu}\bar{\rho}}\cdot$$

$$\cdot\Big\{\frac{1}{2}\,\frac{\partial^2}{\partial y'_{\bar{\nu}}\partial x'_{\bar{\sigma}}}\,\frac{(x'-y')_\kappa}{(x'-y')^4}+\frac{1}{(x'-y')^2}\,(\frac{\overrightarrow{\partial}}{\partial x'_{\bar{\sigma}}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar{\sigma}}})\,\frac{(x'-y')_\kappa}{(x'-y')^2}\,(\frac{\overrightarrow{\partial}}{\partial y'_{\bar{\nu}}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar{\nu}}})\Big\}. \quad (5.3.28)$$

It is interesting to observe that this tensor structure is a linear combination of both (5.3.6) and (5.3.7). Also, one must now include in this result a factor of 2 for opposite directions of fermion charge flow and another factor of 2 associated to the 2 distinctive diagrams connected through reflection symmetry.

Finally, we can add the 3 diagrams in Figures 2(a) and 2(b), in order to obtain the primitive planar contribution to the two loop correlator,

$$P_{\alpha,\mu\nu,\rho\sigma}(0,y,x) \;=\; \frac{igf^2k^2}{32\pi^8}\,y'^8x'^8\,J_{\bar{\mu}(\mu}(y')\,J_{\nu)\bar{\nu}}(y')\,J_{\bar{\rho}(\rho}(x')\,J_{\sigma)\bar{\sigma}}(x')\,\varepsilon_{\alpha\kappa\bar{\mu}\bar{\rho}}\cdot$$

$$\cdot\Big\{\frac{1}{2}\frac{\partial^2}{\partial y'_{\bar\nu}\partial x'_{\bar\sigma}}\frac{(x'-y')_\kappa}{(x'-y')^4}+\frac{5}{4}\frac{1}{(x'-y')^2}\Big(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}}-\frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}}\Big)\frac{(x'-y')_\kappa}{(x'-y')^2}\Big(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}}\Big)\Big\}. \tag{5.3.29}$$

5.3.5 Diagrams in Figures 1(i), 1(j), 1(k), 1(p), 2(c) and 2(d)

Next, we proceed with the evaluation of the contributions coming from the vertex correction diagrams at the energy-momentum tensor insertions. These amplitudes are presented in Figure 1(i): $V^{(1)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, Figure 1(j): $V^{(2)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, Figure 1(k): $V^{(3)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, Figure 1(p): $V^{(4)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, and also Figure 2(c): $V^{(5)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$, and Figure 2(d): $V^{(6)}_{\alpha,\mu\nu,\rho\sigma}(z,y,x)$. Using the previous treatment with the axial insertion vertex we should insert the photon propagator in our calculation in order to guarantee finiteness of the energy-momentum insertion vertex. The amplitudes $V^{(5)}$ and $V^{(6)}$ have a different structure from the other vertex diagrams as they are semi-local. Moreover, there is a possible local term, $V^{(4)}$, which is left ambiguous by the Feynman rules. This is analogous to the situation we faced when dealing with the self-energy diagrams. As this local term cannot be evaluated by Feynman rules it will be solely determined from the Ward identity. The role it plays is one of regularizing a divergence.

The sum of these amplitudes becomes finite in the finite gauge, which also guaranteed the finiteness of the axial insertion vertex. The finite part of the vertex subgraphs is a traceless tensor with respect to all three included indices, and can be written as,

$$V^{(finite)}_{\mu\nu}(y,v,u)=$$

$$=\frac{3k(f^2-\frac{1}{2}g^2)}{2\pi^4}\frac{1}{(y-u)^6}\gamma_\kappa\Big(\frac{(y-u)_\mu(y-u)_\nu(y-u)_\kappa}{(y-u)^2}-\frac{1}{6}\delta_{(\mu\nu}(y-u)_{\kappa)}\Big). \tag{5.3.30}$$

In addition, it would be thoughtful to check the Ward identity connected to these vertices. For that we express the tensor on the RHS of (5.3.30) in terms of the regularized traceless structure of derivatives,

$$\Big(\frac{(y-u)_\mu\,(y-u)_\nu\,(y-u)_\kappa}{(y-u)^2}-\frac{1}{6}\,\delta_{(\mu\nu}\,(y-u)_{\kappa)}\Big) =$$

$$= -\frac{1}{48}\,\frac{\partial}{\partial y_\kappa}\,\Big(\frac{\partial}{\partial y_\mu}\,\frac{\partial}{\partial y_\nu}-\frac{1}{4}\,\delta_{\mu\nu}\,\Box\Big)\,\frac{1}{(y-u)^2}. \qquad (5.3.31)$$

With this expression it is easy to derive that the following Ward identity is satisfied for the energy-momentum insertion vertex,

$$\frac{\partial}{\partial y^\mu}\int d^4v\sum_{i=1}^{6}V^{(i)}_{\mu\nu}(y,v,u) = -k\,\partial_\nu\,\Sigma(y-u), \qquad (5.3.32)$$

where $V^{(i)}_{\mu\nu}$ is the vertex subgraph in the diagram $V^{(i)}_{\alpha,\mu\nu,\rho\sigma}$.

This Ward identity is essential for the calculation of the amplitude of Figure 1(i). It suggests that it will give a divergent result, and only when we add together the diagrams in Figures 1(i), 1(j), 1(k), 1(p), 2(c) and 2(d) we shall obtain a finite answer. After summing up the two possible directions of the Higgs field and setting $z=0$ due to translation invariance of the amplitude, $V^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ is written as,

$$V^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = -\frac{igf^2k^2}{8\pi^4}\int d^4u\,d^4v\,\Delta(u-v)\cdot$$

$$\cdot\mathbf{Tr}\,S(v-y)\gamma_{(\mu}\delta_{\nu)\beta}\,\big(\frac{\overrightarrow{\partial}}{\partial y_\beta}-\frac{\overleftarrow{\partial}}{\partial y_\beta}\big)\,S(y-u)S(u-x)\gamma_{(\rho}\delta_{\sigma)\pi}\,\big(\frac{\overrightarrow{\partial}}{\partial x_\pi}-\frac{\overleftarrow{\partial}}{\partial x_\pi}\big)\,\frac{\not{x}}{x^4}\,\gamma_\alpha\gamma_5\,\frac{\not{v}}{v^4}. \qquad (5.3.33)$$

Using the conformal properties of the theory we can perform the usual inversion in the spatial variables which results to,

$$V^{(1)}_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{igf^2k^2}{8\pi^4}\,y'^8x'^8\,J_{\bar\mu(\mu}(y')\,J_{\nu)\bar\nu}(y')\,J_{\bar\rho(\rho}(x')\,J_{\sigma)\bar\sigma}(x')\int d^4u'\,d^4v'\,\Delta(v'-u')\cdot$$

$$\cdot\mathbf{Tr}\,\gamma_5\,S(v'-y')\gamma_{\bar\mu}\,\big(\frac{\overrightarrow{\partial}}{\partial {y'}_{\bar\nu}}-\frac{\overleftarrow{\partial}}{\partial {y'}_{\bar\nu}}\big)\,S(y'-u')S(u'-x')\gamma_{\bar\rho}\,\frac{\overleftarrow{\partial}}{\partial {x'}_{\bar\sigma}}\,\gamma_\alpha. \qquad (5.3.34)$$

In order to proceed with the u and v integrations we should also add the diagrams of Figures 1(j), 1(k), 1(p), 2(c) and 2(d) to guarantee finiteness of the contribution from the energy-momentum insertion vertex. Note that all these diagrams do not contribute with a finite part for the order $\mathcal{O}(gf^2k^2)$ we are interested in, but merely make the diagram 1(i) finite. This will make the integrand have a traceless form, as given in the Appendix. After some manipulations we deduce that,

$$V''_{\alpha,\mu\nu,\rho\sigma}(0,y,x) =$$

$$= \frac{igf^2k^2}{256\pi^8}\, y'^8 x'^8\, J_{\bar{\mu}(\mu}(y')\, J_{\nu)\bar{\nu}}(y')\, J_{\bar{\rho}(\rho}(x')\, J_{\sigma)\bar{\sigma}}(x')\, \varepsilon_{\alpha\kappa\bar{\mu}\bar{\rho}}\, \frac{\partial^2}{\partial y'_{\bar{\nu}}\partial x'_{\bar{\sigma}}}\, \frac{(x'-y')_\kappa}{(x'-y')^4}, \quad (5.3.35)$$

which is proportional to the triangle structure. Taking into account the 2 fermionic directions and doubling our answer for the two distinctive diagrams connected with reflection symmetry we obtain finally:

$$V'_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{2f^2}{32\pi^2}\, B_{\alpha,\mu\nu,\rho\sigma}(0,y,x), \quad (5.3.36)$$

which is similar to what we got in (5.3.22). So, we can add all the diagrams that represent vertex corrections (both at axial and energy-momentum insertions). The overall result of the vertex corrections contribution to the two loop correlator is,

$$V_{\alpha,\mu\nu,\rho\sigma}(0,y,x) = \frac{3f^2}{32\pi^2}\, B_{\alpha,\mu\nu,\rho\sigma}(0,y,x). \quad (5.3.37)$$

5.3.6 Diagrams in Figures 1(l), 1(m) and 1(n)

Finally, we would like to mention the diagrams that are zero. That these diagrams vanish can be easily seen either from the fact that the fermion trace vanishes or from arguments of Lorentz symmetry. We mention these diagrams for completeness. They are the following: Figure 1(l), which are 3 diagrams that amount to 1 independent calculation, Figure 1(m), which is only 1 diagram, and Figure 1(n), which are 2 diagrams that amount to 1 independent calculation. We have now completed the calculations for all the 36 diagrams.

5.3.7 The Three Point Function

The next and final step is to add all diagrams together, and find out what is the two loop contribution to the three point function at order $\mathcal{O}(gf^2k^2)$. Adding the results for all our diagrams we obtain the $\mathcal{O}(gf^2k^2)$ two loop contribution to the correlator $\langle A_\alpha(z)T_{\mu\nu}(y)T_{\rho\sigma}(x)\rangle$. There are 4 distinct contributions: the one from the non-planar primitive diagrams, $N_{\alpha,\mu\nu,\rho\sigma}\ (0,y,x)$ in (5.3.12); the one from the self-energy diagrams, $\Sigma_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ in (5.3.17); the one from the planar primitive diagrams, $P_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ in (5.3.29); and the one from the vertex correction diagrams, $V_{\alpha,\mu\nu,\rho\sigma}(0,y,x)$ in (5.3.37). Adding these 4 structures we finally obtain our result: the three point function does not vanish and consists of two independent conformal tensor structures,

$$N_{\alpha,\mu\nu,\rho\sigma}(0,y,x) + \Sigma_{\alpha,\mu\nu,\rho\sigma}(0,y,x) + P_{\alpha,\mu\nu,\rho\sigma}(0,y,x) + V_{\alpha,\mu\nu,\rho\sigma}(0,y,x) =$$

$$= \frac{igf^2k^2}{64\pi^8}\, y'^8 x'^8\, J_{\bar\mu(\mu}(y')\, J_{\nu)\bar\nu}(y')\, J_{\bar\rho(\rho}(x')\, J_{\sigma)\bar\sigma}(x')\, \varepsilon_{\alpha\kappa\bar\mu\bar\rho}\cdot$$

$$\cdot\Big\{7\,\frac{\partial^2}{\partial y'_{\bar\nu}\partial x'_{\bar\sigma}}\frac{(x'-y')_\kappa}{(x'-y')^4} + \frac{11}{2}\,\frac{1}{(x'-y')^2}\big(\frac{\overrightarrow{\partial}}{\partial x'_{\bar\sigma}} - \frac{\overleftarrow{\partial}}{\partial x'_{\bar\sigma}}\big)\frac{(x'-y')_\kappa}{(x'-y')^2}\big(\frac{\overrightarrow{\partial}}{\partial y'_{\bar\nu}} - \frac{\overleftarrow{\partial}}{\partial y'_{\bar\nu}}\big)\Big\}. \tag{5.3.38}$$

There is one consistency check that can be performed on this result. Namely, under the appropriate changes, one can ask: does it reduce to the result obtained in the axial gauge theory case [29]? In order to reduce (5.3.38) to the gauge theory case of [29] we first have to discard all diagrams involving the vertex (5.3.23). Then, in the other diagrams, one has to erase all the "graviton derivatives". Once this is done we are left with a unique conformal tensor, *i.e.*, we are reducing the structure of our theory to the one in [29]. Finally, taking into consideration that the removal of the "graviton derivatives" also includes a factor of -2 in the non-planar contribution (due to the symmetry enhancement of this diagrams), one finds that the overall result vanishes just as it did in [29]. This shows that our result is consistent with the calculations performed for the gauge axial anomaly.

We can trace back the reason why this radiative correction does not vanish (as it *does* vanish in the gauge theory case [29]). This is (5.3.12) and (5.3.29), the contribution of the primitive diagrams, which is *not* a multiple of the one loop amplitude. The existence of two different conformal tensors in our theory is a result of the dimensionality of the correlator $\langle A_\alpha(z)T_{\mu\nu}(y)T_{\rho\sigma}(x)\rangle$.

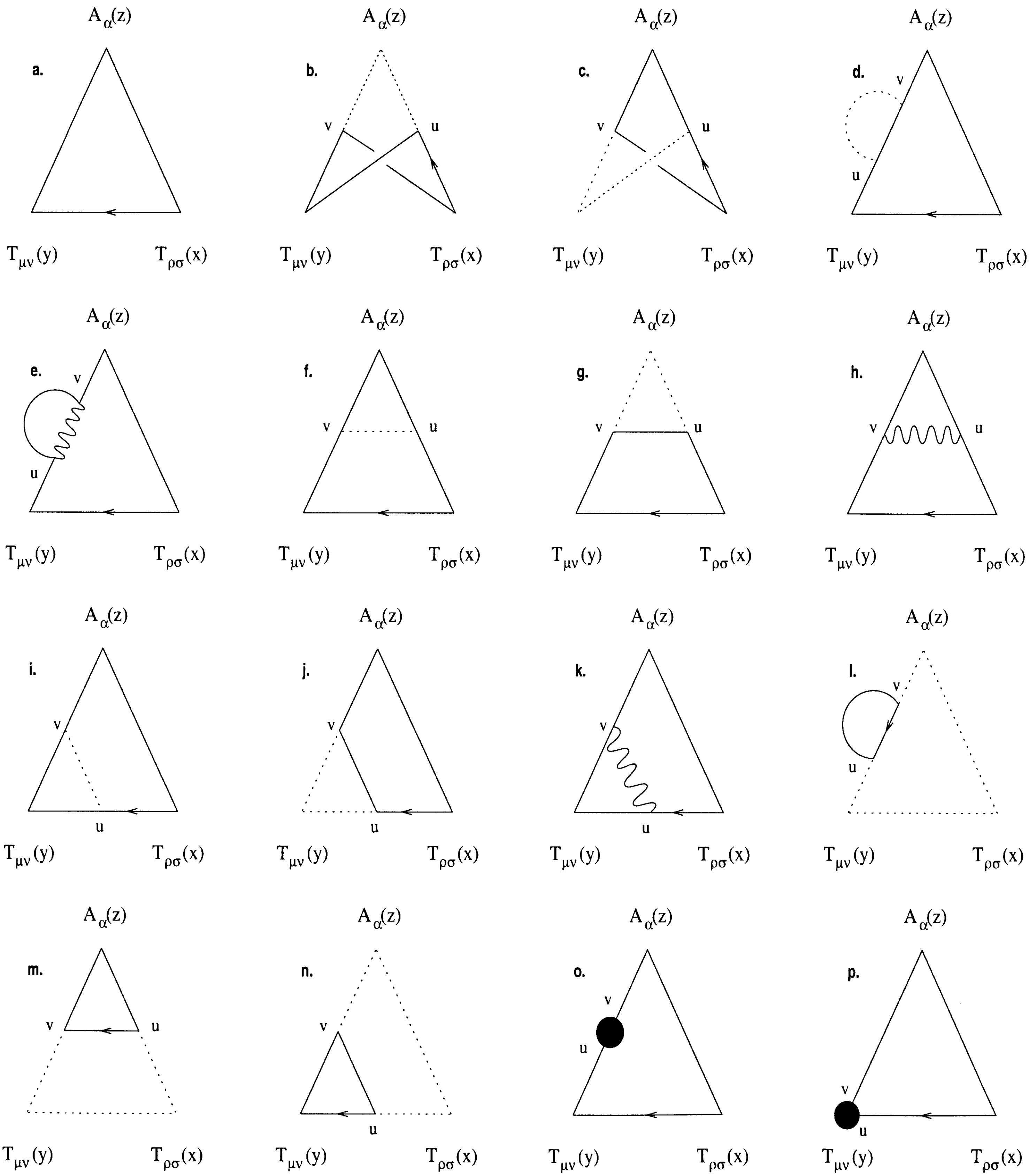


Figure 5-1: One and two loop contributions to the anomaly in the Abelian Higgs theory.

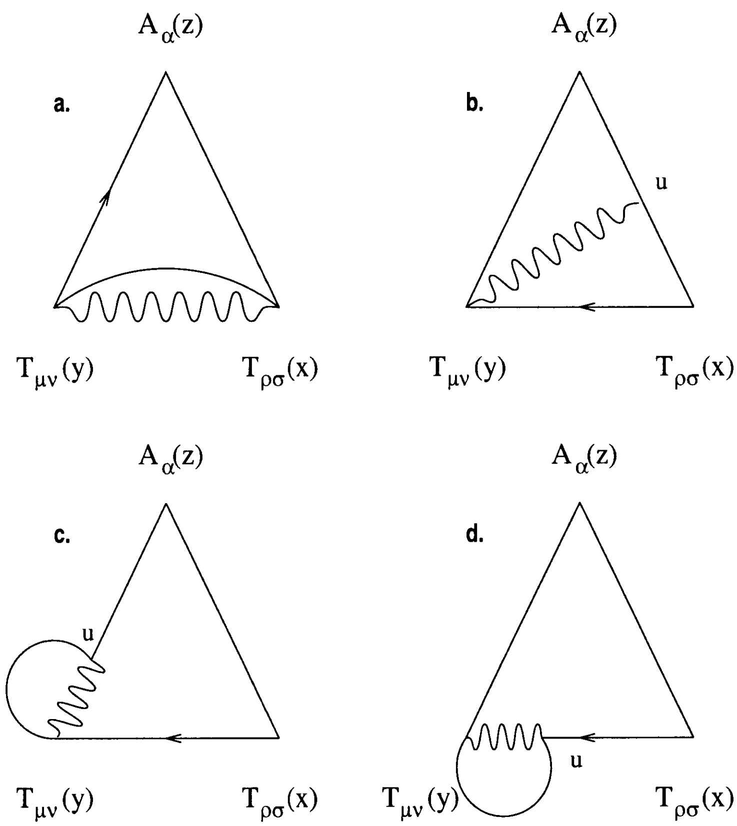


Figure 5-2: Two loop contributions to the anomaly involving the four-point vertex.

Appendix A

Kaluza-Klein Tensor Decompositions

We list below all quantities relevant for our computations in chapter 4, as they were cited upon during the text. We shall consider in here the general torsionfull metric parameterization, as was done in section 4.5 (see expression (4.5.1) and the definitions that follow it in the text). To use these decompositions in section 4.4, all one needs to do is to set $v_i = w_i = b_{ij} = f_{ij} = G_{ij} = 0$ in the following. The tensor decompositions are as follows, for both the gauge beta function, (4.3.3), (4.3.6), and the (00)-component of the metric beta function, (4.3.1), (4.3.4) (where barred quantities will refer to the metric $\bar{g}_{ij}$). Observe that expressions (A.2), (A.4), (A.6) and (A.8) below have $(\kappa - A_0)_{IJ} = 0$.

1. $\nabla^\lambda F_{\lambda\mu}$:

$$\nabla^\lambda F_{\lambda 0} = \bar{g}^{ij}(\partial_i F_{j0} - \bar{\Gamma}^k_{ij} F_{k0}) - \frac{1}{2} a_i F^i{}_0 - \frac{1}{2} a f^{ij} F_{ij} - a v_i f^{ij} F_{j0}, \tag{A.1}$$

$$\nabla^\lambda F_{\lambda i} = \bar{g}^{jk}(\partial_j F_{ki} - \bar{\Gamma}^\ell_{jk} F_{\ell i} - \bar{\Gamma}^\ell_{ji} F_{k\ell}) + \frac{1}{2} a_k F^k{}_i - \frac{1}{2} a v_i f^{jk} F_{jk}, \tag{A.2}$$

2. $[A^\lambda, F_{\lambda\mu}]$:

$$[A^\lambda, F_{\lambda 0}] = -v^i [A_0, F_{i0}] + [A^i, F_{i0}], \tag{A.3}$$

$$[A^\lambda, F_{\lambda i}] = -v^j [A_0, F_{ji}] + [A^j, F_{ji}], \tag{A.4}$$

3. $H_\mu{}^{\lambda\rho}F_{\lambda\rho}$:

$$H_0{}^{\lambda\rho}F_{\lambda\rho} = 2v_i G^{ij}F_{0j} - G^{ij}F_{ij}, \tag{A.5}$$

$$H_i{}^{\lambda\rho}F_{\lambda\rho} = 2v_j G_{ik}F^{kj} + H_{ijk}F^{jk}, \tag{A.6}$$

4. $F_\mu{}^\lambda \partial_\lambda \phi$:

$$F_0{}^\lambda \partial_\lambda \phi = F_0{}^i \partial_i \phi, \tag{A.7}$$

$$F_i{}^\lambda \partial_\lambda \phi = F_i{}^j \partial_j \phi, \tag{A.8}$$

5. $\bar{\beta}^g_{\mu\nu}$:

$$\bar{\beta}^g_{00} = -\frac{a}{2}[\bar{\nabla}^i a_i + \frac{1}{2}a_i a^i - \frac{a}{2}f_{ij}f^{ij}] - \frac{1}{4}G_{ij}G^{ij} + aa^i\partial_i\phi. \tag{A.9}$$

Finally one should also include, for the sake of completeness, the duality transformations acting on the gauge field strength tensor. These are derived directly from expressions (4.2.8-9), with the following results:

6. $(0i)$-component:

$$\tilde{F}_{0i} = \frac{1}{a}(F_{0i} - (\kappa - A_0)a_i), \tag{A.10}$$

7. (ij)-component:

$$\tilde{F}_{ij} = F_{ij} - (v_j + \frac{1}{a}w_j)F_{0i} + (v_i + \frac{1}{a}w_i)F_{0j} - (\kappa - A_0)(f_{ij} + \frac{1}{a}G_{ij} + \frac{1}{a}(w_i a_j - w_j a_i)). \tag{A.11}$$

Appendix B

Differential Regularization

In this Appendix we study the differential regularization of the one loop triangle diagram associated to the gravitational axial anomaly, Figure 1(a), from chapter 5. As we have seen in section 5.2, the amplitude for this diagram is,

$$\frac{1}{2} igk^2 \,\mathbf{Tr}\, \gamma_\alpha \gamma_5 \, S(z-y)\, \gamma_{(\mu}\delta_{\nu)i} \, (\frac{\overrightarrow{\partial}}{\partial y_i} - \frac{\overleftarrow{\partial}}{\partial y_i})\, S(y-x)\, \gamma_{(\rho}\delta_{\sigma)j}\, (\frac{\overrightarrow{\partial}}{\partial x_j} - \frac{\overleftarrow{\partial}}{\partial x_j})\, S(x-z). \quad \text{(B.1)}$$

We can re-write this diagram as a "generic" fermion triangle diagram, for which the bare amplitude takes the form:

$$\mathbf{Tr}\, \gamma_{III}\, S(z-y)\, \gamma_{II}\, (\frac{\overrightarrow{\partial}}{\partial y_i} - \frac{\overleftarrow{\partial}}{\partial y_i})\, S(y-x)\, \gamma_I\, (\frac{\overrightarrow{\partial}}{\partial x_j} - \frac{\overleftarrow{\partial}}{\partial x_j})\, S(x-z). \qquad \text{(B.2)}$$

As compared to (B.1), this is a slightly more general form of the amplitude we want to regulate. For our present purposes we shall only need to concentrate on the regularization of the singular functions present in the amplitude, so we may as well just consider (B.2).

Performing the derivatives, (B.2) can be written as:

$$\mathbf{Tr} \Big\{ -\frac{\partial^2}{\partial y_i \partial x_j} \Big(\gamma_{III}\, S(z-y)\, \gamma_{II}\, S(y-x)\, \gamma_I\, S(x-z) \Big) +$$

$$+ 2\gamma_{II}\, \frac{\partial}{\partial y_i} S(y-x)\, \gamma_I\, \frac{\partial}{\partial x_j} S(x-z)\, \gamma_{III}\, S(z-y) +$$

$$+\,2\gamma_{III}\,\frac{\partial}{\partial y_i}\,S(z-y)\,\gamma_{II}\,\frac{\partial}{\partial x_j}\,S(y-x)\,\gamma_I\,S(x-z)\Big\}\;=$$

$$=\;-\frac{1}{(4\pi^2)^3}\,\mathbf{Tr}\,[\gamma_{III}\,\gamma_a\,\gamma_{II}\,\gamma_b\,\gamma_I\,\gamma_c]\Big\{-\frac{\partial^2}{\partial y_i\partial x_j}\Big(\frac{\partial}{\partial z_a}\,\frac{1}{(z-y)^2}\,\frac{\partial}{\partial y_b}\,\frac{1}{(y-x)^2}\cdot$$

$$\cdot\frac{\partial}{\partial x_c}\,\frac{1}{(x-z)^2}\Big)\Big\}\;-\;\frac{2}{(4\pi^2)^3}\,\mathbf{Tr}\,[\gamma_{II}\,\gamma_a\,\gamma_I\,\gamma_b\,\gamma_{III}\,\gamma_c]\Big\{\frac{\partial}{\partial y_i}\,\frac{\partial}{\partial y_a}\,\frac{1}{(y-x)^2}\cdot$$

$$\cdot\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_b}\,\frac{1}{(x-z)^2}\,\frac{\partial}{\partial z_c}\,\frac{1}{(z-y)^2}\Big\}\;-\;\frac{2}{(4\pi^2)^3}\,\mathbf{Tr}\,[\gamma_{III}\,\gamma_a\,\gamma_{II}\,\gamma_b\,\gamma_I\,\gamma_c]\Big\{\frac{\partial}{\partial y_i}\,\frac{\partial}{\partial z_a}\,\frac{1}{(z-y)^2}\cdot$$

$$\cdot\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial y_b}\,\frac{1}{(y-x)^2}\,\frac{\partial}{\partial x_c}\,\frac{1}{(x-z)^2}\Big\}. \qquad \text{(B.3)}$$

With $x' = x - z$, $y' = y - z$, $\bar{a} = c$, $\bar{b} = a$ and $\bar{c} = b$, we can re-write the first term in the previous result as the singular function,

$$t^{(1)}_{ij\;\bar{a}\bar{b}\bar{c}}(x',y')\;=\;\frac{\partial^2}{\partial y'_i\partial x'_j}\Big(\frac{\partial}{\partial x'_{\bar{a}}}\,\frac{1}{x'^2}\,\frac{\partial}{\partial y'_{\bar{b}}}\,\frac{1}{y'^2}\,\frac{\partial}{\partial x'_{\bar{c}}}\,\frac{1}{(x'-y')^2}\Big), \qquad \text{(B.4)}$$

though in the following we shall drop primes and bars, and simply use the notation (x,y) and $\{a,b,c\}$. By a similar re-naming of variables one can likewise manipulate the second and third terms in (B.3) to obtain – in both cases – the singular function,

$$t^{(2)}_{ij\;abc}(x,y)\;=\;\frac{\partial}{\partial x_j}\,\frac{\partial}{\partial x_a}\,\frac{1}{x^2}\,\frac{\partial}{\partial y_i}\,\frac{\partial}{\partial y_b}\,\frac{1}{y^2}\,\frac{\partial}{\partial x_c}\,\frac{1}{(x-y)^2}. \qquad \text{(B.5)}$$

Observe that (B.5) is independent of (B.4), so that in (B.3) we have two independent singular functions. For particular choices of the vertex gamma matrices, (B.2) describes the anomalous three point function (explicitly studied in chapter 5) as well as other physically interesting amplitudes.

We need to "pull out" derivatives and regularize both $t^{(1)}_{ij\;abc}(x,y)$ and $t^{(2)}_{ij\;abc}(x,y)$. Starting with (B.4), one easily observes that this can be written as,

$$t^{(1)}_{ij\;abc}(x,y)\;=\;\frac{\partial^2}{\partial y_i\partial x_j}\,t_{abc}(x,y), \qquad \text{(B.6)}$$

where $t_{abc}(x,y)$ is the singular function which is present in the bare amplitude for the gauge axial anomaly, and has been previously considered in [45]. In particular, it was shown in this reference how to regularize this singular function. Two derivatives are required to control the linear divergence arising from the singularity at $x \sim y \sim 0$. Making manifest the $x \leftrightarrow y$, $a \leftrightarrow b$ antisymmetry of $t_{abc}(x,y)$ one can write,

$$t_{abc}(x,y) \; = \; F_{abc}(x,y) \, + \, S_{abc}(x,y), \tag{B.7}$$

where the function $F_{abc}(x,y)$ has finite Fourier transform by power counting and trace arguments,

$$\begin{aligned} F_{abc}(x,y) \; = \; & \frac{\partial}{\partial x_a}\frac{\partial}{\partial y_b}\Big[\frac{1}{x^2y^2}\frac{\partial}{\partial x_c}\frac{1}{(x-y)^2}\Big] + \frac{\partial}{\partial x_a}\Big[\frac{1}{x^2y^2}\Big(\frac{\partial^2}{\partial x_b\partial x_c} - \frac{1}{4}\delta_{bc}\Box\Big)\cdot \\ & \cdot\frac{1}{(x-y)^2}\Big] - \frac{\partial}{\partial y_b}\Big[\frac{1}{x^2y^2}\Big(\frac{\partial^2}{\partial x_a\partial x_c} - \frac{1}{4}\delta_{ac}\Box\Big)\frac{1}{(x-y)^2}\Big] - \\ & - \frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_a\partial x_b\partial x_c} - \frac{1}{6}\Big(\delta_{ab}\frac{\partial}{\partial x_c} + \delta_{bc}\frac{\partial}{\partial x_a} + \delta_{ac}\frac{\partial}{\partial x_b}\Big)\Box\Big]\frac{1}{(x-y)^2}. \end{aligned} \tag{B.8}$$

The term $S_{abc}(x,y)$ contains the "traces" subtracted off in (B.8) and thus derivatives of $\delta(x-y)$ times $1/x^4$ factors which are regulated as is standard in differential regularization, yielding:

$$\begin{aligned} S_{abc}(x,y) \; = \; & \frac{1}{4}\pi^2\Big\{\Big[\delta_{bc}\frac{\partial}{\partial x_a} - \delta_{ac}\frac{\partial}{\partial y_b}\Big]\delta(x-y)\Box\frac{\ln M_1^2x^2}{x^2} - \\ - \frac{1}{3}\Big[\delta_{bc}\Big(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial y_a}\Big) & + \delta_{ac}\Big(\frac{\partial}{\partial x_b} - \frac{\partial}{\partial y_b}\Big) + \delta_{ab}\Big(\frac{\partial}{\partial x_c} - \frac{\partial}{\partial y_c}\Big)\Big]\delta(x-y)\Box\frac{\ln M_2^2x^2}{x^2}\Big\}. \end{aligned} \tag{B.9}$$

Two different mass scales were used for the two independent trace terms in $S_{abc}(x,y)$. Renormalization or symmetry conditions may be used to determine the ratio M_1/M_2 in particular cases of the triangle amplitude.

Expressions (B.8) and (B.9) provide the required regularization of (B.4) via expressions (B.6-7). We are thus left with the regularization of (B.5), which can be performed in a similar fashion. Again, one can write,

$$t^{(2)}_{ij\;abc}(x,y) \; = \; F'_{ij\,abc}(x,y) \, + \, S'_{ij\,abc}(x,y). \tag{B.10}$$

The function $F'_{ij_{abc}}(x,y)$ whose Fourier transform is finite by power counting and trace arguments is,

$$F'_{ij_{abc}}(x,y) = \frac{\partial}{\partial y_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_a}\frac{\partial}{\partial y_b}\Big[\frac{1}{x^2y^2}\frac{\partial}{\partial x_c}\frac{1}{(x-y)^2}\Big] + \frac{\partial}{\partial y_i}\frac{\partial}{\partial x_j}\cdot$$

$$\cdot\Big\{\frac{\partial}{\partial x_a}\Big[\frac{1}{x^2y^2}\Big(\frac{\partial^2}{\partial x_b\partial x_c} - \frac{1}{4}\delta_{bc}\Box\Big)\frac{1}{(x-y)^2}\Big] - \frac{\partial}{\partial y_b}\Big[\frac{1}{x^2y^2}\Big(\frac{\partial^2}{\partial x_a\partial x_c} - \frac{1}{4}\delta_{ac}\Box\Big)\cdot$$

$$\cdot\frac{1}{(x-y)^2}\Big] - \frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_a\partial x_b\partial x_c} - \frac{1}{6}\Big(\delta_{ab}\frac{\partial}{\partial x_c} + \delta_{bc}\frac{\partial}{\partial x_a} + \delta_{ac}\frac{\partial}{\partial x_b}\Big)\Box\Big]\frac{1}{(x-y)^2}\Big\} +$$

$$+\frac{\partial}{\partial y_i}\frac{\partial}{\partial x_a}\Big\{\frac{\partial}{\partial y_b}\Big[\frac{1}{x^2y^2}\Big(\frac{\partial^2}{\partial x_j\partial x_c} - \frac{1}{4}\delta_{jc}\Box\Big)\frac{1}{(x-y)^2}\Big] -$$

$$-\frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_b\partial x_j\partial x_c} - \frac{1}{6}\Big(\delta_{bj}\frac{\partial}{\partial x_c} + \delta_{jc}\frac{\partial}{\partial x_b} + \delta_{bc}\frac{\partial}{\partial x_j}\Big)\Box\Big]\frac{1}{(x-y)^2}\Big\} -$$

$$-\frac{\partial}{\partial y_b}\frac{\partial}{\partial x_j}\Big\{\frac{\partial}{\partial x_a}\Big[\frac{1}{x^2y^2}\Big(\frac{\partial^2}{\partial x_i\partial x_c} - \frac{1}{4}\delta_{ic}\Box\Big)\frac{1}{(x-y)^2}\Big] -$$

$$-\frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_a\partial x_i\partial x_c} - \frac{1}{6}\Big(\delta_{ai}\frac{\partial}{\partial x_c} + \delta_{ic}\frac{\partial}{\partial x_a} + \delta_{ac}\frac{\partial}{\partial x_i}\Big)\Box\Big]\frac{1}{(x-y)^2}\Big\} -$$

$$-\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_b}\Big\{\frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_a\partial x_j\partial x_c} - \frac{1}{6}\Big(\delta_{aj}\frac{\partial}{\partial x_c} + \delta_{jc}\frac{\partial}{\partial x_a} + \delta_{ac}\frac{\partial}{\partial x_j}\Big)\Box\Big]\frac{1}{(x-y)^2}\Big\} +$$

$$+\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_a}\Big\{\frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_i\partial x_b\partial x_c} - \frac{1}{6}\Big(\delta_{ib}\frac{\partial}{\partial x_c} + \delta_{bc}\frac{\partial}{\partial x_i} + \delta_{ic}\frac{\partial}{\partial x_b}\Big)\Box\Big]\frac{1}{(x-y)^2}\Big\} -$$

$$-\frac{\partial}{\partial x_a}\frac{\partial}{\partial y_b}\Big\{\frac{1}{x^2y^2}\Big[\frac{\partial^3}{\partial x_i\partial x_j\partial x_c} - \frac{1}{6}\Big(\delta_{ij}\frac{\partial}{\partial x_c} + \delta_{jc}\frac{\partial}{\partial x_i} + \delta_{ic}\frac{\partial}{\partial x_j}\Big)\Box\Big]\frac{1}{(x-y)^2}\Big\} +$$

$$+\,\text{Terms invovling } 4^{th} \text{ and } 5^{th} \text{ order traceless derivatives acting on } \frac{1}{(x-y)^2}. \qquad \text{(B.11)}$$

The fourth and fifth derivatives can be obtained in a straightforward fashion, but their explicit form is not relevant and they would occupy a couple of pages to write down. Therefore we omit these terms.

Again, the term $S'_{ij_{abc}}(x,y)$ contains "traces" subtracted from (B.11) and so its structure is similar to the one of (B.9), containing the usual differential regulated derivatives of $\delta(x-y)$ times $1/x^4$ factors. One obtains,

$$S'_{ij\,abc}(x,y) \;=\; \frac{\partial}{\partial y_i}\frac{\partial}{\partial x_j}\frac{1}{4}\pi^2\Big\{\Big[\delta_{bc}\frac{\partial}{\partial x_a} \;-\; \delta_{ac}\frac{\partial}{\partial y_b}\Big]\,\delta(x-y)\,\Box\,\frac{\ln M_1^2x^2}{x^2} \;-$$

$$-\frac{1}{3}\Big[\delta_{bc}\Big(\frac{\partial}{\partial x_a}-\frac{\partial}{\partial y_a}\Big)+\delta_{ac}\Big(\frac{\partial}{\partial x_b}-\frac{\partial}{\partial y_b}\Big)+\delta_{ab}\Big(\frac{\partial}{\partial x_c}-\frac{\partial}{\partial y_c}\Big)\Big]\,\delta(x-y)\,\Box\,\frac{\ln M_2^2x^2}{x^2}\Big\}-$$

$$-\frac{\partial}{\partial y_i}\frac{\partial}{\partial x_a}\frac{1}{4}\pi^2\Big\{\delta_{jc}\frac{\partial}{\partial y_b}\,\delta(x-y)\,\Box\,\frac{\ln M_3^2x^2}{x^2}+\frac{1}{3}\Big[\delta_{bc}\Big(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial y_j}\Big)+\delta_{jc}\Big(\frac{\partial}{\partial x_b}-\frac{\partial}{\partial y_b}\Big)+$$

$$+\delta_{jb}\Big(\frac{\partial}{\partial x_c}-\frac{\partial}{\partial y_c}\Big)\Big]\,\delta(x-y)\,\Box\,\frac{\ln M_4^2x^2}{x^2}\Big\}-\frac{\partial}{\partial y_b}\frac{\partial}{\partial x_j}\frac{1}{4}\pi^2\Big\{\delta_{ic}\frac{\partial}{\partial x_a}\,\delta(x-y)\,\Box\,\frac{\ln M_3^2x^2}{x^2}-$$

$$-\frac{1}{3}\Big[\delta_{ic}\Big(\frac{\partial}{\partial x_a}-\frac{\partial}{\partial y_a}\Big)+\delta_{ac}\Big(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial y_i}\Big)+\delta_{ai}\Big(\frac{\partial}{\partial x_c}-\frac{\partial}{\partial y_c}\Big)\Big]\,\delta(x-y)\,\Box\,\frac{\ln M_4^2x^2}{x^2}\Big\}-$$

$$-\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_b}\frac{\pi^2}{12}\Big\{\Big[\delta_{jc}\Big(\frac{\partial}{\partial x_a}-\frac{\partial}{\partial y_a}\Big)+\delta_{ac}\Big(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial y_j}\Big)+\delta_{aj}\Big(\frac{\partial}{\partial x_c}-\frac{\partial}{\partial y_c}\Big)\Big]\,\delta(x-y)\cdot$$

$$\cdot\Box\,\frac{\ln M_5^2x^2}{x^2}\Big\}+\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_a}\frac{\pi^2}{12}\Big\{\Big[\delta_{bc}\Big(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial y_i}\Big)+\delta_{ic}\Big(\frac{\partial}{\partial x_b}-\frac{\partial}{\partial y_b}\Big)+\delta_{ib}\Big(\frac{\partial}{\partial x_c}-\frac{\partial}{\partial y_c}\Big)\Big]\cdot$$

$$\cdot\delta(x-y)\,\Box\,\frac{\ln M_5^2x^2}{x^2}\Big\}-\frac{\partial}{\partial x_a}\frac{\partial}{\partial y_b}\frac{\pi^2}{12}\Big\{\Big[\delta_{jc}\Big(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial y_i}\Big)+\delta_{ic}\Big(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial y_j}\Big)+$$

$$+\delta_{ij}\Big(\frac{\partial}{\partial x_c}-\frac{\partial}{\partial y_c}\Big)\Big]\,\delta(x-y)\,\Box\,\frac{\ln M_6^2x^2}{x^2}\Big\}+$$

$$+\text{ Traces subtracted from (A.11) in the } 4^{th} \text{ and } 5^{th} \text{ order derivatives,}$$

$$\text{and their respective mass scales.} \tag{B.12}$$

One can see that several different mass scales were introduced for the several independent trace terms in $S'_{ij\,abc}(x,y)$. Again, as was mentioned for (B.9), renormalization or symmetry conditions may be used to determine the ratios between the mass scales in particular cases of the triangle amplitude.

Appendix C

Convolution Integrals

In this Appendix we list the convolution integrals that are required in order to perform the two loop calculation in chapter 5 [62]. They are inclosed to make this chapter self contained for the reader who wishes to reproduce our result. The table of convolution integrals is (defining $\Delta \equiv x - y$, and using the cutoff Λ):

$$\int \frac{d^4 v}{v^2\,(v-x)^2} \;=\; -\pi^2 \ln \frac{x^2}{\Lambda^2}, \tag{C.1}$$

$$\int \frac{(v-x)_\rho}{v^2\,(v-x)^4}\, d^4 v \;=\; -\pi^2\, \frac{x_\rho}{x^2}, \tag{C.2}$$

$$\int \frac{(v-x)_\rho (v-y)_\sigma}{(v-x)^4\,(v-y)^4}\, d^4 v \;=\; \frac{\pi^2}{2\Delta^2}\,(\delta_{\rho\sigma} - 2\frac{\Delta_\rho \Delta_\sigma}{\Delta^2}), \tag{C.3}$$

$$\int \frac{(v_\rho v_\sigma - \frac{1}{4}\delta_{\rho\sigma} v^2)}{v^4\,(v-x)^2}\, d^4 v \;=\; \frac{\pi^2}{2x^2}\,(x_\rho x_\sigma - \frac{1}{4}\delta_{\rho\sigma} x^2), \tag{C.4}$$

$$\int \frac{((v-x)_\rho (v-x)_\sigma - \frac{1}{4}\delta_{\rho\sigma}(v-x)^2)(v-y)_\lambda}{(v-x)^4\,(v-y)^4}\, d^4 v \;=$$

$$= -\frac{\pi^2}{4\Delta^2}\,(\delta_{\rho\lambda}\Delta_\sigma + \delta_{\sigma\lambda}\Delta_\rho - 2\frac{\Delta_\rho \Delta_\sigma \Delta_\lambda}{\Delta^2}), \tag{C.5}$$

$$\int \frac{(v_\rho v_\sigma - \frac{1}{4}\delta_{\rho\sigma}v^2)}{v^6\,(v-x)^2}\,d^4v \;=\; \frac{\pi^2}{2x^4}\,(x_\rho x_\sigma - \frac{1}{4}\delta_{\rho\sigma}x^2), \qquad \text{(C.6)}$$

$$\int \frac{((v-x)_\rho(v-x)_\sigma - \frac{1}{4}\delta_{\rho\sigma}(v-x)^2)(v-y)_\lambda}{(v-x)^6\,(v-y)^4}\,d^4v \;=$$

$$= \;-\frac{\pi^2}{4\Delta^4}\,(\delta_{\rho\lambda}\Delta_\sigma + \delta_{\sigma\lambda}\Delta_\rho + \frac{1}{2}\delta_{\rho\sigma}\Delta_\lambda - 4\frac{\Delta_\rho\Delta_\sigma\Delta_\lambda}{\Delta^2}), \qquad \text{(C.7)}$$

$$\int \frac{v_\alpha(v-x)_\rho(v-y)_\sigma}{(v-x)^4\,(v-y)^4}\,d^4v \;=\; \frac{\pi^2}{4\Delta^2}\,(\delta_{\alpha\rho}\Delta_\sigma - \delta_{\alpha\sigma}\Delta_\rho + (x+y)_\alpha\,[\delta_{\rho\sigma} - 2\frac{\Delta_\rho\Delta_\sigma}{\Delta^2}]), \quad \text{(C.8)}$$

$$\int \frac{v_\alpha(v-y)_\rho}{(v-x)^2\,(v-y)^4}\,d^4v \;=\; \frac{\pi^2}{2\Delta^2}\,((x+y)_\alpha\Delta_\rho - \frac{1}{4}\delta_{\alpha\rho}\Delta^2) - \frac{\pi^2}{4}\delta_{\alpha\rho}\ln\frac{\Delta^2}{\Lambda^2}. \qquad \text{(C.9)}$$

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