Characterization of the peak value behavior of the Hilbert transform of bounded bandlimited signals

Characterization of the Peak Value Behavior of the Hilbert Transform of Bounded Bandlimited Signals

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Abstract

The peak value of a signal is a characteristic that has to be controlled in many applications. In this paper we analyze the peak value of the Hilbert transform for the space $\mathcal{B}_\pi^\infty$ of bounded bandlimited signals. It is known that for this space the Hilbert transform cannot be calculated by the common principal value integral, because there are signals for which it diverges everywhere. Although the classical definition fails for $\mathcal{B}_\pi^\infty$, there is a more general definition of the Hilbert transform, which is based on the abstract $\mathcal{H}^1$-BMO($\mathbb{R}$) duality. It was recently shown in the paper "On the Hilbert Transform of Bounded Bandlimited Signals," Problems of Information Transmission, vol. 48, 2012 [1] that, in addition to this abstract definition, there exists an explicit formula for the calculation of the Hilbert transform. Based on this formula we study the properties of the Hilbert transform for the space $\mathcal{B}_\pi^\infty$ of bounded bandlimited signals. We analyze its asymptotic growth behavior, and thereby solve the peak value problem of the Hilbert transform for this space. Further, we obtain results for the growth behavior of the Hilbert transform for the subspace $\mathcal{B}_{\pi,0}^\infty$ of bounded bandlimited signals that vanish at infinity. By studying the properties of the Hilbert transform, we continue the work of Korzhik "The extended Hilbert transformation and its application in signal theory," Problems of Information Transmission, vol. 5, 1969 [2].

Index Terms

Hilbert transform, peak value, bounded bandlimited signal, growth

I. Introduction

The peak value is a basic characteristic of signals. In many applications it is crucial to control the peak value. For example, in wireless communication systems high peak-to-average power ratios (PAPRs) are problematic because high peak values can overload the power amplifiers, which in turn leads to undesired out-of-band radiation [3], [4], [5]. In this paper we analyze the asymptotic growth behavior of the Hilbert transform for the space of bounded bandlimited signals, and thereby solve the peak value problem of the Hilbert transform for this space.

The Hilbert transform is an important operation in numerous fields, in particular in communication theory and signal processing. For example the "analytic signal" [6], which was used by Dennis Gabor in his "Theory of Communication" [7], is based on the Hilbert transform. Further concepts and theories in which the Hilbert transform is an integral part are the instantaneous amplitude, phase, and frequency of a signal [6], [8], [9], [10], [11], [12], [13] and the theory of modulation [6], [14], [15], [16].

In an analytic signal $\psi(t) = u(t) + iv(t)$ the imaginary part v is the Hilbert transform of the real part u, i.e., $v = Hu$. Based on the analytic signal it is possible to define the instantaneous amplitude and frequency of a signal [8], [9]. The instantaneous amplitude $A_u(t)$ of a signal u is then defined by

$$A_u(t) := \sqrt{u^2(t) + v^2(t)},$$

the instantaneous phase $\phi_u(t)$ by

$$\phi_u(t) = \arctan\left(\frac{v(t)}{u(t)}\right),$$

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and the instantaneous frequency, which is the derivative of the instantaneous phase, by

$$\phi_u'(t) = \arctan\left(\frac{v(t)}{u(t)}\right) = \frac{v'(t)u(t) - v(t)u'(t)}{u^2(t) + v^2(t)}.$$

Although there are other possibilities to define the instantaneous amplitude and frequency [9], [17], it was shown in [9] that the only definition that satisfies certain physical requirements is the definition based on the Hilbert transform and the analytic signal. In [12] and [13] interesting approaches are developed to find generalizations of the amplitude-phase representation to non-smooth functions. In these papers the application of the Hilbert transform and the use of techniques from the theory of analytic functions are central. Our approach in this paper is different, because we consider smooth signals, more precisely, the practically important class of bandlimited signals.

A further interesting application of the Hilbert transform is presented in [18], where the classical Hardy spaces are characterized as $L^p(\mathbb{R})$ functions with non-negative spectrum, and an $L^p(\mathbb{R})$ extension of the Bedrosian theorem is developed.

Classically, the Hilbert transform of a smooth signal f with compact support is defined as the principal value integral

$$\begin{aligned}(Hf)(t) &= \frac{1}{\pi}\,\text{V.P.}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau = \frac{1}{\pi}\lim_{\epsilon\to 0} \int\limits_{\epsilon\le|t-\tau|\le\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau \\ &= \frac{1}{\pi}\lim_{\epsilon\to 0}\left(\int_{t-\frac{1}{\epsilon}}^{t-\epsilon} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau + \int_{t+\epsilon}^{t+\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau}\,\mathrm{d}\tau\right). \qquad (1)\end{aligned}$$

The above integral (1) can be used to define the Hilbert transform for more general spaces only if the integral converges for all signals from this space. There are cases where the integral converges only for almost all t, but where the Hilbert transform can be defined in the L^p-sense. However, the convergence of the integral is delicate and has to be checked from case to case. For bounded bandpass signals, the Hilbert transform exists and is bounded. If f is a bandpass signals, the distributional Fourier transform of which vanishes outside $[-\pi, -\epsilon\pi] \cup [\epsilon\pi, \pi]$, $0 < \epsilon < 1$, then f has a bounded Hilbert transform satisfying

$$\|Hf\|_\infty \le \left(C_1 + \frac{2}{\pi}\log\left(\frac{1}{\epsilon}\right)\right)\|f\|_\infty,$$

where $C_1 < 4/\pi$ is a constant [19], [16]. That is, the upper bound on the peak value of the Hilbert transform diverges as ϵ tends to zero. Probably, observations of this kind led to the conclusion "that an arbitrary bounded bandlimited function does not have a Hilbert transform…" [16]. Such a non-existence of the Hilbert transform for certain bounded bandlimited signals would have far-reaching consequences.

In this paper we use a new representation of the Hilbert transform for bounded bandlimited signals, which was recently found in [1]. With this representation we can explicitly calculate the Hilbert transform of such signals using a mixed signal system. Based on this new mixed signal representation we are able to characterize the peak value behavior of the Hilbert transform and to understand the problems in the evaluation of the standard Hilbert transform integral that probably led to the above cited statement about the non-existence of the Hilbert transform for arbitrary bounded bandlimited functions. Our approach is restricted to bandlimited signals. In the literature other methods for the calculation of the Hilbert transform and the treatment of related applications have been developed for non-smooth functions. For example, the approaches in [18], [12], [13] use Hardy spaces and techniques from complex integration.

The paper is structured as follows. In Section II we introduce some notation. In Sections III and IV we define the Hilbert transform for general bounded bandlimited signals and present the new constructive formula for its calculation. The material in this two sections is a summary of the most important facts from [1], which are necessary for the further understanding of this paper. However, the proofs are omitted because they can be found in [1]. In Section V the peak value problem of the Hilbert transform is solved and in Section VI further results about the peak value of the Hilbert transform for the important subspace of bounded bandlimited signals that vanish at infinity are presented. In Section VII a sufficient condition for the boundedness of the Hilbert transform is derived, and in Section VIII an example of a bounded bandlimited signal that vanishes at infinity with unbounded Hilbert transform is given. Finally, in Section IX we characterize a subset of the bounded bandlimited signals for which the common Hilbert transform integral (1) converges.

II. Notation

Let $\hat{f}$ denote the Fourier transform of a function f. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ is the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. A function that is defined and holomorphic over the whole complex plane is called entire function. For $0 < \sigma < \infty$, let $\mathcal{B}_\sigma$ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leq C(\epsilon)\exp((\sigma+\epsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, consists of all functions in $\mathcal{B}_\sigma$, whose restriction to the real line is in $L^p(\mathbb{R})$. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., $\|\cdot\|_{\mathcal{B}_\sigma^p} = \|\cdot\|_p$. A signal in $\mathcal{B}_\sigma^p$, $1 \leq p \leq \infty$, is called bandlimited to σ, and $\mathcal{B}_\sigma^\infty$ is the space of bandlimited signals that are bounded on the real axis. We call a signal in $\mathcal{B}_\pi^\infty$ bounded bandlimited signal. By the Paley–Wiener–Schwartz theorem [20], the Fourier transform of a signal bandlimited to σ is supported in $[-\sigma, \sigma]$. For $1 \leq p \leq 2$ the Fourier transformation is defined in the classical and for $p > 2$ in the distributional sense.

III. The Operator Q

Consider the linear time-invariant (LTI) system $Q = DH$, which consists of the concatenation of the Hilbert transform H and the differential operator D, as an operator acting on $\mathcal{B}_\pi^2$. Since both operators $H : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ and $D : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ are stable LTI systems, $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$, as the concatenation of two stable LTI systems, is a stable LTI system. The system $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ has the frequency domain representation

$$(Qf)(t) = (DHf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_Q(\omega)\hat{f}(\omega)\, \mathrm{e}^{i\omega t}\, \mathrm{d}\omega, \tag{2}$$

where

$$\hat{h}_Q(\omega) = \begin{cases} |\omega|, & |\omega| \leq \pi \\ 0, & |\omega| > \pi. \end{cases}$$

It is easy to show (for details see [1]) that the system $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ has also the mixed signal representation

$$(Qf)(t) = \sum_{k=-\infty}^{\infty} a_{-k} f(t-k), \tag{3}$$

where the coefficients a_k, $k \in \mathbb{Z}$, are given by

$$a_k = \begin{cases} \frac{\pi}{2}, & k = 0, \\ \frac{(-1)^k - 1}{\pi k^2}, & k \neq 0. \end{cases} \tag{4}$$

We call this representation mixed signal representation, because for a fixed $t \in \mathbb{R}$ we need the signal values on the discrete grid $\{t-k\}_{k\in\mathbb{Z}}$ in order to calculate $(Qf)(t)$. However, for different $t \in \mathbb{R}$ we need other signal values in general. As t ranges over $[0,1]$ we need all the signal values $f(\tau)$, $\tau \in \mathbb{R}$. The mixed signal representation (3) will be important for Section IV, where the Hilbert transform is extended to $\mathcal{B}_\pi^\infty$.

A. Extension of Q to $\mathcal{B}_\pi^\infty$

So far, we have considered the LTI system Q only acting on signals in $\mathcal{B}_\pi^2$. Next, we extend Q to a bounded operator Q^{E} acting on the larger space $\mathcal{B}_\pi^\infty$ of bandlimited signals that are bounded on the real axis. For the operator $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$ we had the representations (2) and (3). However, the frequency domain representations which involves the Fourier transform of the signal makes no sense for signals in $\mathcal{B}_\pi^\infty$. The next theorem shows that the mixed signal representation (3) is still meaningful for signals in $\mathcal{B}_\pi^\infty$, because it is also a valid representation of the extension Q^{E}.

Theorem 1. *The mapping*

$$Q^E f = \sum_{k=-\infty}^{\infty} a_{-k} f(\,\cdot - k), \tag{5}$$

where the coefficients a_k are defined as in (4), defines a bounded linear operator $Q^E : \mathcal{B}_\pi^\infty \to \mathcal{B}_\pi^\infty$ with norm $\|Q^E\| = \pi$ that coincides with Q on $\mathcal{B}_\pi^2$, i.e., that satisfies $Q^E f = Qf$ for all $f \in \mathcal{B}_\pi^2$.

The proof of Theorem 1 can be found in [1].

IV. The Hilbert Transform for $\mathcal{B}_\pi^\infty$

Despite the convergence problems of the principal value integral, there is a way to define the Hilbert transform for signals in $\mathcal{B}_\pi^\infty$. This definition uses Fefferman's duality theorem, which states that the dual space of $\mathcal{H}^1$ is $\mathrm{BMO}(\mathbb{R})$ [21]. In addition to this rather abstract definition, we will also give a constructive procedure for the calculation of the Hilbert transform. We briefly review some definitions.

Definition 1. The space $\mathcal{H}^1$ denotes the Hardy space of all signals $f \in L^1(\mathbb{R})$ for which $Hf \in L^1(\mathbb{R})$. It is a Banach space endowed with the norm $\|f\|_{\mathcal{H}^1} := \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}$.

Definition 2. A function $f : \mathbb{R} \to \mathbb{C}$ is said to belong to $\mathrm{BMO}(\mathbb{R})$, provided that it is locally in $L^1(\mathbb{R})$ and $\frac{1}{\mu(I)} \int_I |f(t) - m_I(f)| \mathrm{d}t \leq C_2$ for all bounded intervals I, where $m_I(f) := \frac{1}{\mu(I)} \int_I f(t)\mathrm{d}t$ and the constant C_2 is independent of I. μ denotes the Lebesgue measure.

For our further examinations, we need the important fact that the dual space of $\mathcal{H}^1$ is $\mathrm{BMO}(\mathbb{R})$ [22, p. 245]. In order to state this duality, we use the space $\mathcal{H}^1_{\mathrm{D}} = \mathcal{H}^1 \cap \mathcal{S}$, which is dense in $\mathcal{H}^1$. By $\mathcal{S}$ we denote the usual Schwartz space of functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and fulfill $\sup_{t\in\mathbb{R}} |t^a \phi^{(b)}(t)| < \infty$ for all $a, b \in \mathbb{N} \cup \{0\}$.

Theorem 2 (Fefferman)**.** *Suppose $f \in BMO(\mathbb{R})$. Then the linear functional $\mathcal{H}^1_D \to \mathbb{C}$, $\phi \mapsto \int_{-\infty}^{\infty} f(t)\phi(t)\mathrm{d}t$ has a bounded extension to $\mathcal{H}^1$. Conversely, every continuous linear functional L on $\mathcal{H}^1$ is created in this way by a function $f \in BMO(\mathbb{R})$, which is unique up to an additive constant.*

The function $f \in \mathrm{BMO}(\mathbb{R})$ in Theorem 2 is only unique up to an additive constant, because $\phi \in \mathcal{H}^1$ implies $\int_{-\infty}^{\infty} \phi(t)\mathrm{d}t = 0$. Therefore, it will be beneficial to identify two functions in $\mathrm{BMO}(\mathbb{R})$ that differ only by a constant. We do this by introducing the equivalence relation $\sim$ on $\mathrm{BMO}(\mathbb{R})$. We write $f \sim g$ if and only if $f(t) = g(t) + C_{\mathrm{BMO}}$ for almost all $t \in \mathbb{R}$, where C_{BMO} is a constant. By $[f]$ we denote the equivalence class $[f] = \{g \in \mathrm{BMO}(\mathbb{R}) : g \sim f\}$, and $\mathrm{BMO}(\mathbb{R})/\mathbb{C}$ is the set of all equivalence classes in $\mathrm{BMO}(\mathbb{R})$.

A possible extension of the Hilbert transform, which is based on the $\mathcal{H}^1$-$\mathrm{BMO}(\mathbb{R})$ duality is given in the next definition [23].

Definition 3. We define the Hilbert transform $\mathfrak{H}f$ of $f \in L^\infty(\mathbb{R})$ to be the function in $\mathrm{BMO}(\mathbb{R})/\mathbb{C}$ that generates the linear continuous functional

$$\langle \mathfrak{H}f, \phi \rangle = \int_{-\infty}^{\infty} f(t)(H\phi)(t)\mathrm{d}t, \quad \phi \in \mathcal{H}^1.$$

Note that this definition is very abstract, because it gives no information how to calculate the Hilbert transform $\mathfrak{H}f$. However, in [24] it was shown that for bounded signals that are additionally bandlimited, i.e. for signals $f \in \mathcal{B}_\pi^\infty$, it is possible to explicitly calculate the Hilbert transform $\mathfrak{H}f$. Next, we will give this formula, which is based on the Q^{E}-operator from Section III.

Since $Q^{\mathrm{E}} f$ is continuous, for every $f \in \mathcal{B}_\pi^\infty$, the operator $\mathfrak{I}$ given by

$$(\mathfrak{I}f)(t) = \int_0^t (Q^{\mathrm{E}} f)(\tau)\mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{6}$$

is well defined. Since the operator $Q : \mathcal{B}_\pi^2 \to \mathcal{B}_\pi^2$, as an operator on $\mathcal{B}_\pi^2$, was defined to be the concatenation of the Hilbert transform H and the differential operator D, it is clear that, for $g \in \mathcal{B}_\pi^2$, the integral of Qg as in (6) gives—up to a constant—the Hilbert transform Hg of g. Note that for $g \in \mathcal{B}_\pi^2$ we have $Hg \in \mathcal{B}_\pi^2$, which implies that Hg is continuously differentiable. Hence, the fundamental theorem of calculus can be applied in the next equation without problems. For $g \in \mathcal{B}_\pi^2$ we have

$$\begin{aligned}(\mathfrak{I}g)(t) &= \int_0^t (Q^{\mathrm{E}} g)(\tau)\mathrm{d}\tau = \int_0^t (Qg)(\tau)\mathrm{d}\tau \\ &= \int_0^t (DHg)(\tau)\mathrm{d}\tau = (Hg)(t) - (Hg)(0),\end{aligned} \tag{7}$$

i.e., for every signal $g \in \mathcal{B}_\pi^2$, we have $(Hg)(t) = (\mathfrak{I}g)(t) + C_3(g)$, $t \in \mathbb{R}$, where $C_3(g)$ is a constant that depends on g.

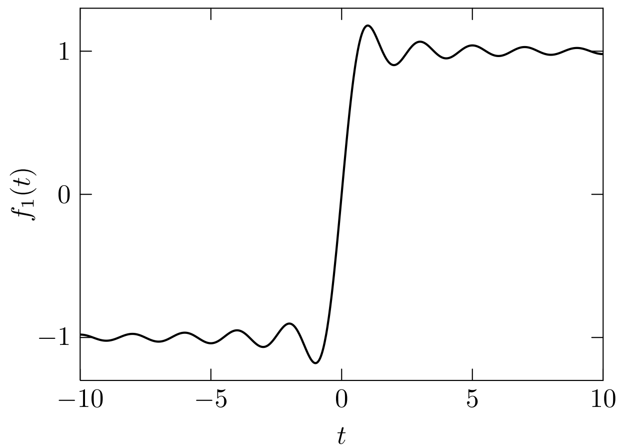

Fig. 1. Plot of the signal f_1.

Based on this observation one could conjecture that, for signals $f \in \mathcal{B}_\pi^\infty$, the integral $\mathfrak{I}f$ is somehow connected to the Hilbert transform $\mathfrak{H}f$ of f. In [1] it was shown that such a connection exists in the sense that $\mathfrak{I}f$ is a representative of the equivalence class $\mathfrak{H}f$.

Theorem 3. *Let $f \in \mathcal{B}_\pi^\infty$. Then we have $\mathfrak{H}f = [\mathfrak{I}f]$.*

Note that according to Definition 3, the Hilbert transform $\mathfrak{H}f$ of a signal $f \in \mathcal{B}_\pi^\infty$ is only defined up to an arbitrary additive constant. This is a consequence of the $\mathcal{H}^1$-BMO($\mathbb{R}$) duality, which was employed for the definition. However, the mapping $\mathfrak{I}$ does not have this ambiguity, it maps every input signal $f \in \mathcal{B}_\pi^\infty$ uniquely to an output signal $\mathfrak{I}f \in \text{BMO}(\mathbb{R})$.

Theorem 3 is very useful, because it enables us to compute the Hilbert transform of bounded bandlimited signals in $\mathcal{B}_\pi^\infty$ by using the constructive formula (6), instead of using the abstract Definition 3. This result is also the key for solving the peak vale problem of the Hilbert transform.

Remark 1. The formula (6) for the calculation of the Hilbert transform is based on the mixed signal representation of the operator Q^{E}. It is an interesting question whether the Hilbert transform of a signal in $\mathcal{B}_\pi^\infty$ can be calculated by using only the samples of the signal. In [25] we have shown that a Nyquist rate sampling based representation of the Hilbert that is based on the Shannon sampling series is not possible even for the subspace $\mathcal{PW}_\pi^1$ of $\mathcal{B}_\pi^\infty$. We conjecture that this negative result holds even in more generality, as long as no oversampling is used. However, if oversampling is used, then a sampling based representation of the Hilbert transform is possible for $\mathcal{B}_\pi^\infty$.

An important fact about the Hilbert transform of bounded bandlimited signals is stated in the next theorem.

Theorem 4. *Let $f \in \mathcal{B}_\pi^\infty$. Then we have $\mathfrak{I}f \in \mathcal{B}_\pi$.*

Theorem 4, the proof of which can be found in [1], shows that the Hilbert transform of a bounded bandlimited signal is again bandlimited.

V. Peak Value Problem

The peak value of signals is important for many applications, e.g., for the hardware design in mobile communications [3], [4]. In the peak value problem we are interested in $\sup_{|t|\leq T}|f(t)|$, i.e., in the peak value of a signal f on the interval $[-T,T]$. Next, we study the Hilbert transform of signals in $\mathcal{B}_\pi^\infty$, in particular its growth behavior on the real axis, and thereby solve the peak value problem for the Hilbert transform.

For all $f \in \mathcal{B}_\pi^\infty$, we have the upper bound

$$|(\mathfrak{I}f)(t)| \leq \int_0^t |(Q^{\text{E}}f)(\tau)| \mathrm{d}\tau \leq \|Q^{\text{E}}f\|_\infty |t| \leq \pi \|f\|_\infty |t|, \quad t \in \mathbb{R}, \tag{8}$$

which shows that the asymptotic growth of the Hilbert transform $\mathfrak{H}f$ of signals $f \in \mathcal{B}_\pi^\infty$ is at most linear. More precisely, for all $f \in \mathcal{B}_\pi^\infty$ there exists a signal $g \in \text{BMO}(\mathbb{R})$ such that $\mathfrak{H}f = [g]$ and $g(t) = O(t)$.

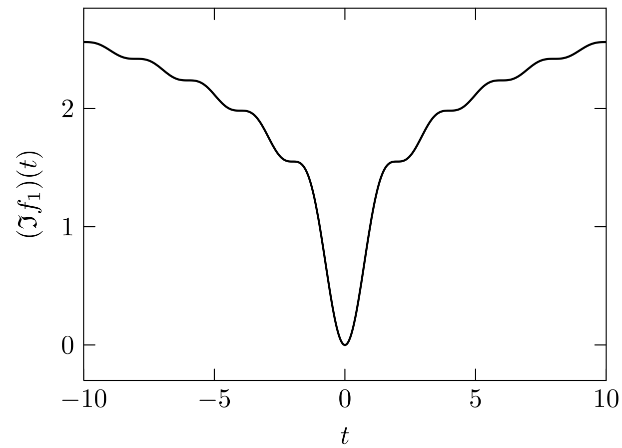

Fig. 2. Plot of the signal $\mathfrak{I}f_1$.

On the other hand, using the identity (6), it has been shown in [1] that for the $\mathcal{B}_\pi^\infty$-signal

$$f_1(t) = \frac{2}{\pi}\int_0^\pi \frac{\sin(\omega t)}{\omega}\mathrm{d}\omega \tag{9}$$

we have

$$|(\mathfrak{I}f_1)(t)| \geq \frac{2}{\pi}\left(\log(|t|) - \frac{\pi^2}{4} - 1 - \frac{1}{\pi}\right) \tag{10}$$

for all $t \in \mathbb{R}$ with $|t| \geq 1$. The signals f_1 and $\mathfrak{I}f_1$ are visualized in Fig. 1 and Fig. 2, respectively. Thus, there are signals $f \in \mathcal{B}_\pi^\infty$, such that the growth of the Hilbert transform $\mathfrak{H}f$ is logarithmic, in the sense that there exists a signal $g \in \mathrm{BMO}(\mathbb{R})$ such that $\mathfrak{H}f = [g]$ and $g(t) = \Omega(\log(t))$.

From this the question arises whether the asymptotically logarithmic growth is actually the maximum possible growth, i.e., whether the upper bound (8) can be improved. The next theorem gives a positive answer.

Theorem 5. *There exist two positive constants C_4 and C_5 such that for all $f \in \mathcal{B}_\pi^\infty$ and all $t \in \mathbb{R}$ we have*

$$|(\mathfrak{I}f)(t)| \leq C_4 \log(1 + |t|)\|f\|_\infty + C_5\|f\|_\infty.$$

For the proof we need the following lemma, the proof of which can be found in [1].

Lemma 1. *Let $f \in \mathcal{B}_\pi^\infty$ and, for $0 < \epsilon < 1$,*

$$f_\epsilon(t) = f((1-\epsilon)t)\frac{\sin(\epsilon\pi t)}{\epsilon\pi t}, \qquad t \in \mathbb{R}. \tag{11}$$

Then we have $(\mathfrak{I}f)(t) = \lim_{\epsilon\to 0}(\mathfrak{I}f_\epsilon)(t)$ for all $t \in \mathbb{R}$.

Now, we are in the position to proof Theorem 5.

Proof of Theorem 5: Let $f \in \mathcal{B}_\pi^\infty$ be arbitrary but fixed. For $0 < \epsilon < 1$ consider the functions f_ϵ that were defined in (11). We have $f_\epsilon \in \mathcal{B}_\pi^2$ and $\|f_\epsilon\|_\infty \leq \|f\|_\infty$ for all $1 < \epsilon < 1$, as well as $\lim_{\epsilon\to 0} f_\epsilon(t) = f(t)$ for all $t \in \mathbb{R}$, where the convergence is locally uniform. Lemma 1 is a key observation. Due to the representation (6) and the properties of the operator Q, we can work with $\mathcal{B}_\pi^2$-functions in the following. Next, we analyze

$$(\mathfrak{I}f_\epsilon)(t) = \int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau.$$

We have to distinguish two cases: $|t| < 2$ and $|t| \geq 2$. For $|t| < 2$ we have

$$\left|\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau\right| \leq \|Qf_\epsilon\|_\infty |t| \leq 2\pi\|f\|_\infty, \tag{12}$$

where we used $\|Qf_\epsilon\|_\infty = \|Q^{\mathrm{E}} f_\epsilon\|_\infty \leq \|Q^{\mathrm{E}}\| \, \|f_\epsilon\|_\infty \leq \pi \|f\|_\infty$ in the second inequality. Now, we come to the second case $|t| \geq 2$. We can restrict ourselves to the case $t \geq 2$, because the case $t \leq 2$ is treated analogously. Let $t \geq 2$ be arbitrary but fixed. Using (2), i.e., the frequency domain representation of Q, we obtain

$$\begin{aligned}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau &= \int_0^t \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega| \hat{f}_\epsilon(\omega)\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega|\hat{f}_\epsilon(\omega)\int_0^t \mathrm{e}^{i\omega\tau}\,\mathrm{d}\tau\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega|\hat{f}_\epsilon(\omega)\frac{\mathrm{e}^{i\omega t}-1}{i\omega}\mathrm{d}\omega. \end{aligned} \tag{13}$$

The order of integration was exchanged according to Fubini's theorem, which can be applied because

$$\int_0^t \frac{1}{2\pi}\int_{-\pi}^{\pi} |\omega||\hat{f}_\epsilon(\omega)|\mathrm{d}\omega\mathrm{d}\tau \leq |t|\pi\|f\|_{\mathcal{B}_\pi^2} < \infty.$$

Furthermore, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\omega|\hat{f}_\epsilon(\omega)\frac{\mathrm{e}^{i\omega t}-1}{i\omega}\mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{-i\,\mathrm{sgn}(\omega)\hat{\phi}(\omega)}_{=\hat{u}(\omega)}\hat{f}_\epsilon(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \\ &\quad - \frac{1}{2\pi}\int_{-\pi}^{\pi} -i\,\mathrm{sgn}(\omega)\hat{\phi}(\omega)\hat{f}_\epsilon(\omega)\mathrm{d}\omega, \end{aligned} \tag{14}$$

where the function

$$\hat{\phi}(\omega) = \begin{cases} 1, & |\omega| \leq \pi, \\ 2 - |\omega|/\pi, & \pi < |\omega| < 2\pi, \\ 0, & |\omega| \geq 2\pi, \end{cases}$$

was inserted without altering the integrals, because $\hat{\phi}(\omega) = 1$ for $\omega \in [-\pi, \pi]$. Using the abbreviation $\hat{u}(\omega) = -i\,\mathrm{sgn}(\omega)\hat{\phi}(\omega)$ and applying the generalized Parseval equality, we obtain from (13) and (14) that

$$\begin{aligned}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau &= \int_{-\infty}^{\infty} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau - \int_{-\infty}^{\infty} f_\epsilon(\tau)u(-\tau)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau + \int_{-\infty}^{\infty} f_\epsilon(\tau)u(\tau)\mathrm{d}\tau, \end{aligned} \tag{15}$$

where u is given by

$$u(\tau) = \frac{1}{\pi\tau} + \frac{\sin(\pi\tau) - \sin(2\pi\tau)}{(\pi\tau)^2}, \quad \tau \in \mathbb{R}.$$

Dividing the integration range of the first and the second integral in (15) into three parts gives

$$\begin{aligned}\int_0^t (Qf_\epsilon)(\tau)\mathrm{d}\tau = &\overbrace{\int_{|\tau|\leq 1} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau}^{=(A_1)} + \overbrace{\int_{|\tau-t|\leq 1} \ldots\,\mathrm{d}\tau}^{=(A_2)} + \overbrace{\int_{\substack{|\tau-t|\geq 1\\|\tau|\geq 1}} \ldots\,\mathrm{d}\tau}^{=(A_3)} \\ &+ \underbrace{\int_{|\tau|\leq 1} f_\epsilon(\tau)u(\tau)\mathrm{d}\tau}_{=(B_1)} + \underbrace{\int_{|\tau-t|\leq 1} \ldots\,\mathrm{d}\tau}_{=(B_2)} + \underbrace{\int_{\substack{|\tau-t|\geq 1\\|\tau|\geq 1}} \ldots\,\mathrm{d}\tau}_{=(B_3)}. \end{aligned}$$

For (A_1) we have

$$|(A_1)| = \left|\int_{|\tau|\leq 1} f_\epsilon(\tau)u(t-\tau)\mathrm{d}\tau\right| \leq \int_{|\tau|\leq 1} |f_\epsilon(\tau)|\,|u(t-\tau)|\mathrm{d}\tau \leq 2\|f_\epsilon\|_\infty\|u\|_\infty.$$

The same calculation shows that $|(A_2)| \leq 2\|f_\epsilon\|_\infty \|u\|_\infty$, $|(B_1)| \leq 2\|f_\epsilon\|_\infty \|u\|_\infty$, and $|(B_2)| \leq 2\|f_\epsilon\|_\infty \|u\|_\infty$. It remains to analyze $(A_3) + (B_3)$. We have

$$\begin{aligned} |(A_3) + (B_3)| &= \left| \int_{\substack{|\tau - t| \geq 1 \\ |\tau| \geq 1}} f_\epsilon(\tau)(u(t-\tau) + u(\tau)) \mathrm{d}\tau \right| \\ &\leq \|f_\epsilon\|_\infty \left(\int_{\substack{|\tau - t| \geq 1 \\ |\tau| \geq 1}} \left| \frac{1}{\pi(t-\tau)} + \frac{1}{\pi\tau} \right| \mathrm{d}\tau + \int_{\substack{|\tau - t| \geq 1 \\ |\tau| \geq 1}} \left(\frac{2}{\pi(t-\tau)^2} + \frac{2}{\pi\tau^2} \right) \mathrm{d}\tau \right) \\ &\leq \|f_\epsilon\|_\infty \left(\frac{1}{\pi} \int_{\substack{|\tau - t| \geq 1 \\ |\tau| \geq 1}} \frac{|t|}{|t-\tau|\,|\tau|} \mathrm{d}\tau + \frac{8}{\pi} \right), \end{aligned}$$

because

$$\int_{\substack{|\tau - t| \geq 1 \\ |\tau| \geq 1}} \left(\frac{2}{\pi(t-\tau)^2} + \frac{2}{\pi\tau^2} \right) \mathrm{d}\tau \leq \frac{8}{\pi}.$$

As for the remaining integral, we have

$$\begin{aligned} \int_{\substack{|\tau - t| \geq 1 \\ |\tau| \geq 1}} \frac{|t|}{|t-\tau|\,|\tau|} \mathrm{d}\tau &= \int_{-\infty}^{-1} \frac{t}{(\tau - t)\tau} \mathrm{d}\tau + \int_{1}^{t-1} \frac{t}{(t-\tau)\tau} \mathrm{d}\tau + \int_{t+1}^{\infty} \frac{t}{(\tau - t)\tau} \mathrm{d}\tau \\ &= \int_{1}^{t-1} \frac{t}{(t-\tau)\tau} \mathrm{d}\tau + 2 \int_{t+1}^{\infty} \frac{t}{(\tau - t)\tau} \mathrm{d}\tau. \end{aligned}$$

Since

$$\begin{aligned} \int_{t+1}^{\infty} \frac{t}{(\tau - t)\tau} \mathrm{d}\tau &= \lim_{M \to \infty} \int_{t+1}^{M} \left(\frac{1}{\tau - t} - \frac{1}{\tau} \right) \mathrm{d}\tau \\ &= \lim_{M \to \infty} \left(\log\left(\frac{M - t}{M} \right) + \log(t+1) \right) \\ &= \log(t+1) \end{aligned}$$

and

$$\int_{1}^{t-1} \frac{t}{(t-\tau)\tau} \mathrm{d}\tau = \int_{1}^{t-1} \frac{1}{t-\tau} + \frac{1}{\tau} \mathrm{d}\tau = 2\log(t-1),$$

we obtain

$$|(A_3) + (B_3)| \leq \|f_\epsilon\|_\infty \frac{4}{\pi} (\log(1+t) + 2).$$

Combining the partial results gives

$$\begin{aligned} \left| \int_0^t (Qf_\epsilon)(\tau) \mathrm{d}\tau \right| &\leq 8\|f_\epsilon\|_\infty \|u\|_\infty + \|f_\epsilon\|_\infty \frac{4}{\pi} (\log(1+t) + 2) \\ &= C_4 \log(1+t) \|f_\epsilon\|_\infty + C_6 \|f_\epsilon\|_\infty \\ &\leq C_4 \log(1+t) \|f\|_\infty + C_6 \|f\|_\infty. \end{aligned} \tag{16}$$

Finally, the assertion follows from Lemma 1, (12), and (16). ■

Remark 2. The growth result in Theorem 5 implies that

$$\int_{-\infty}^{\infty} \frac{|(\mathfrak{I}f)(t)|^2}{(1+t^2)^\alpha} \mathrm{d}t < \infty$$

for all $\alpha > 1/2$. This shows that, for all $f \in \mathcal{B}_\pi^\infty$, the Hilbert transform $\mathfrak{H}f$ is in the Zakai class, in the sense that there exists a signal g in the Zakai class, satisfying $\mathfrak{H}f = [g]$.

A direct consequence of Theorem 5 is the following corollary concerning the peak value problem of the Hilbert transform.

Corollary 1. *For all $f \in \mathcal{B}_\pi^\infty$ and all $g \in BMO(\mathbb{R})$ satisfying $\mathfrak{H}f = [g]$ there exists a constant $C_7 = C_7(g)$ such that for all $T > 2$ we have*

$$\max_{|t|\leq T} |g(t)| \leq C_7 \log(1+T).$$

Remark 3. The signal f_1 is also a good example where commonly used formal substitution rules fail. The Hilbert transform of f_1 cannot be obtained by replacing $\sin$ with $-\cos$ because the resulting integral

$$\frac{2}{\pi}\int_0^\pi \frac{-\cos(\omega t)}{\omega} \mathrm{d}\omega$$

diverges for all $t \in \mathbb{R}$.

VI. Asymptotic Behavior for $\mathcal{B}_{\pi,0}^\infty$

Next, we present two further results about the peak value of the Hilbert transform for signals in the space $\mathcal{B}_{\pi,0}^\infty$, which is the subspace consisting of $\mathcal{B}_\pi^\infty$-signals f that vanish on the real axis at infinity, i.e., satisfy $\lim_{|t|\to\infty} |f(t)| = 0$.

Theorem 6. *For all $f \in \mathcal{B}_{\pi,0}^\infty$ we have*

$$\lim_{T\to\infty} \frac{1}{\log(1+T)} \max_{|t|\leq T} |(\mathfrak{I}f)(t)| = 0.$$

Proof: Let $f \in \mathcal{B}_{\pi,0}^\infty$ and $\epsilon > 0$ be arbitrary but fixed. Since $\mathcal{B}_\pi^2$ is dense in $\mathcal{B}_{\pi,0}^\infty$, there exists a function $g \in \mathcal{B}_\pi^2$ such that $\|f-g\|_\infty < \epsilon$. Thus, for all $t \in \mathbb{R}$, we have

$$\begin{aligned}
|(\mathfrak{I}f)(t)| &= |(\mathfrak{I}f)(t) - (\mathfrak{I}g)(t) + (\mathfrak{I}g)(t)| \\
&\leq |(\mathfrak{I}(f-g))(t)| + |(\mathfrak{I}g)(t)| \\
&\leq C_4 \log(1+|t|)\|f-g\|_\infty + C_5\|f-g\|_\infty + |(Hg)(t)| + |(Hg)(0)| \\
&\leq C_4 \log(1+|t|)\epsilon + C_5\epsilon + 2\|Hg\|_\infty,
\end{aligned}$$

where we used Theorem 5 and equation (7) in the second to last line. It follows, for $T > 0$, that

$$\frac{1}{\log(1+T)} \max_{|t|\leq T} |(\mathfrak{I}f)(t)| \leq C_4\epsilon + \frac{C_5\epsilon}{\log(1+T)} + \frac{2\|Hg\|_\infty}{\log(1+T)}.$$

Choosing $T_0 = \exp(\max\{2\|Hg\|_\infty, 1\}/\epsilon) - 1$, we obtain

$$\frac{1}{\log(1+T)} \max_{|t|\leq T} |(\mathfrak{I}f)(t)| \leq (C_4 + C_5 + 1)\epsilon$$

for all $T \geq T_0$. Since $\epsilon > 0$ was arbitrary, the proof is complete. ■

Due to Theorem 3, we immediately obtain the following corollary about the asymptotic growth behavior of the Hilbert transform $\mathfrak{H}f$ for $f \in \mathcal{B}_{\pi,0}^\infty$.

Corollary 2. *For all $f \in \mathcal{B}_{\pi,0}^\infty$ we have*

$$\lim_{T\to\infty} \frac{1}{\log(1+T)} \max_{|t|\leq T} |(\mathfrak{H}f)(t)| = 0.$$

A further result about the asymptotic growth is the following.

Theorem 7. *Let ϕ be an arbitrary positive function with $\lim_{t\to\infty} \phi(t) = 0$. Then there exists a signal $f_2 \in \mathcal{B}_{\pi,0}^\infty$ such that*

$$\limsup_{T\to\infty} \frac{1}{\phi(T)\log(1+T)} \max_{|t|\leq T} |(\mathfrak{I}f_2)(t)| = \infty. \tag{17}$$

Proof: For $t \geq 1$ consider the family of bounded linear functionals $U_t \colon \mathcal{B}_{\pi,0}^\infty \to \mathbb{C}$, defined by

$$U_t f = \frac{(\mathfrak{I}f)(t)}{\phi(t)\log(1+t)}.$$

Further, for $0 < \epsilon < 1$, let

$$f_{1,\epsilon}(t) = \frac{f_1((1-\epsilon)t)}{\|f_1\|_\infty} \frac{\sin(\epsilon\pi t)}{\epsilon\pi t},$$

where f_1 is the function defined in (9). Then, we have $\|f_{1,\epsilon}\|_\infty \leq 1$ and it follows that

$$\|U_t\| = \sup_{\substack{f\in\mathcal{B}_{\pi,0}^\infty \\ \|f\|_\infty\leq 1}} |U_t f| \geq |U_t f_{1,\epsilon}|.$$

Since

$$\lim_{\epsilon\to 0}|U_t f_{1,\epsilon}| = \frac{\lim_{\epsilon\to 0}|(\mathfrak{I} f_{1,\epsilon})(t)|}{\phi(t)\log(1+t)} = \frac{|(\mathfrak{I} f_1)(t)|}{\|f_1\|_\infty \phi(t)\log(1+t)} \geq \frac{\frac{2}{\pi}\left(\log(t) - \frac{\pi^2}{4} - 1 - \frac{1}{\pi}\right)}{\|f_1\|_\infty \phi(t)\log(1+t)},$$

where we used Lemma 1 in the second equality and (10) in the last inequality, we obtain

$$\|U_t\| \geq \frac{2}{\pi\|f\|_\infty \phi(t)}\left(\frac{\log(t)}{\log(1+t)} - \left(\frac{\pi^2}{4} + 1 + \frac{1}{\pi}\right)\frac{1}{\log(1+t)}\right).$$

From this we see that $\lim_{t\to\infty}\|U_t\|_\infty = \infty$. Thus, the Banach–Steinhaus Theorem [26, p. 68] implies that there exists a signal $f_2 \in \mathcal{B}_{\pi,0}^\infty$ such that

$$\limsup_{t\to\infty}|U_t f_2| = \limsup_{t\to\infty} \frac{|(\mathfrak{I} f)(t)|}{\phi(t)\log(1+t)} = \infty,$$

which completes the proof. ∎

Again, we obtain, as a corollary, the analogous result for the asymptotic growth of the Hilbert transform.

Corollary 3. *Let ϕ be an arbitrary positive function with $\lim_{t\to\infty}\phi(t) = 0$. Then there exists a signal $f_2 \in \mathcal{B}_{\pi,0}^\infty$ such that*

$$\limsup_{T\to\infty} \frac{1}{\phi(T)\log(1+T)} \max_{|t|\leq T}|(\mathfrak{H} f_2)(t)| = \infty. \tag{18}$$

Corollaries 2 and 3 show that, for signals in the space $\mathcal{B}_{\pi,0}^\infty$, the peak value of the Hilbert transform grows not as fast as $\log(1+T)$ but not "substantially" slower.

VII. A Condition for the Boundedness of the Hilbert Transform

Thanks to Theorem 3, we can use the simple formula (6) to compute the Hilbert transform of bounded bandlimited signal. In Section V we have seen that there exists a signal $f_1 \in \mathcal{B}_\pi^\infty$ such that $\mathfrak{I} f_1$ is unbounded on the real axis. Thus, the Hilbert transform of a bounded bandlimited signal is again a bandlimited but not necessarily a bounded signal.

Remark 4. It is well-known that there exist discontinuous signals the Hilbert transforms of which have singularities [27], [28]. However, those signals are not bandlimited, and the divergence effects are consequences of the non-smoothness of the signals. This is in contrast to this paper where we treat bandlimited, smooth signals.

For practical applications is important to know when the Hilbert transform is bounded. Theorem 8 gives a necessary and sufficient condition for the boundedness of the Hilbert transform. The proof of Theorem 8 is given in Appendix A.

Theorem 8. *Let $f \in \mathcal{B}_\pi^\infty$ be real-valued. We have $\mathfrak{I} f \in \mathcal{B}_\pi^\infty$ if and only if there exists a constant C_8 such that*

$$\left|\frac{1}{\pi}\int_{\epsilon\leq|t-\tau|\leq\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau}\mathrm{d}\tau\right| \leq C_8 \tag{19}$$

for all $0 < \epsilon < 1$ and all $t \in \mathbb{R}$.

Thanks to Theorem 8 we have a complete characterization of the signals in $\mathcal{B}_\pi^\infty$ that have a bounded Hilbert transform.

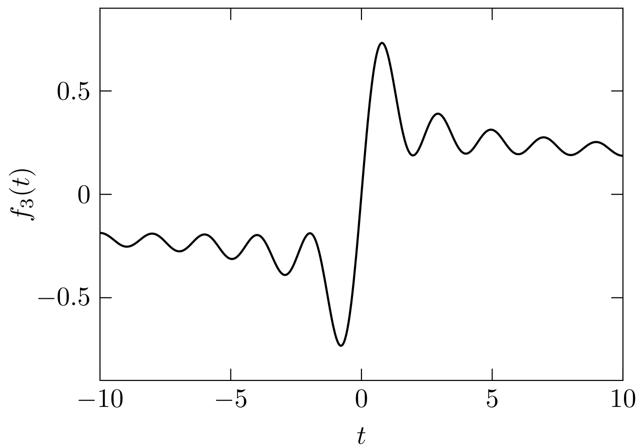

Fig. 3. Plot of the signal f_3.

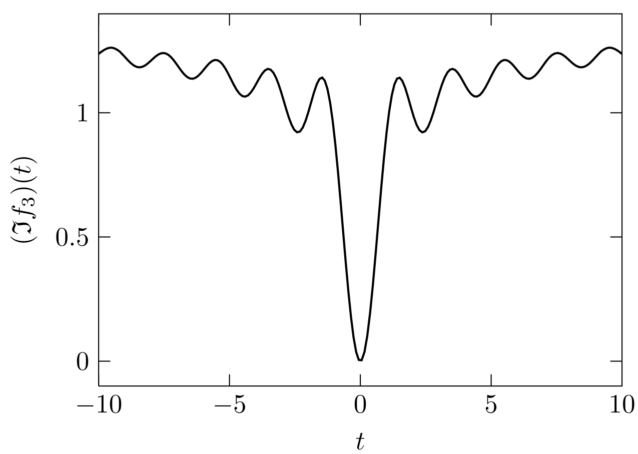

Fig. 4. Plot of the signal $\mathfrak{I} f_3$.

Remark 5. Note that condition (19) in Theorem 8 does not imply that the principal value integral of the Hilbert transform converges, i.e., that the limit in (1) exists. The convergence of the principal value integral is the content of Theorem 10.

Remark 6. Theorem 8 shows that the unbounded divergence of the principal value integral for a signal $f \in \mathcal{B}_\pi^\infty$ implies that the Hilbert transform of f (which is nevertheless well-defined) is not a signal in $\mathcal{B}_\pi^\infty$.

VIII. A Signal in $\mathcal{B}_{\pi,0}^\infty$ with Unbounded Hilbert Transform

In Section V we have seen that there exists a signal $f_1 \in \mathcal{B}_\pi^\infty$, the Hilbert transform of which is unbounded. Next, we strengthen this result by showing that there even exists a signal $f_3 \in \mathcal{B}_{\pi,0}^\infty$ with unbounded Hilbert transform.

Theorem 9. *There exists a signal $f_3 \in \mathcal{B}_{\pi,0}^\infty$ such that $\|\mathfrak{H} f_3\|_\infty = \infty$.*

We do not prove Theorem 9 here, but sketch the proof idea only. Theorem 9 can be proved by showing that the Hilbert transform of the signal

$$f_3(t) = \frac{2}{\pi} \int_0^\pi \frac{1}{\log\left(\frac{2\pi}{\omega}\right)} \frac{\sin(\omega t)}{\omega} \mathrm{d}\omega,$$

which is plotted in Fig. 3, is unbounded. Using integration by parts it is easy to show that f_3 satisfies $\lim_{|t|\to\infty} f_3(t) = 0$, i.e., that f_3 is a signal in $\mathcal{B}_{\pi,0}^\infty$. The plot of $\mathfrak{I} f_3$ in Fig. 4 indicated the unboundedness of the Hilbert transform of f_3. The actual proof of Theorem 9 can be done indirectly by using Theorem 8 and showing that

$$\lim_{\epsilon \to 0} \int\limits_{\epsilon \le |\tau| \le \frac{1}{\epsilon}} \frac{f_3(\tau)}{\tau} \mathrm{d}\tau = \infty.$$

IX. Convergence of the Hilbert Transform Integral

Theorem 8 characterizes when $\mathfrak{I}f$ is bounded. It links the boundedness of $\mathfrak{I}f$ to the boundedness of the principal value integral (1). In Theorem 10 we characterize a subset of the bounded bandlimited signals, for which the Hilbert transform integral (1) converges, and thus give a sufficient condition for being able to calculate the Hilbert transformation by the integral (1).

Theorem 10. *Let $f \in \mathcal{B}_{\pi,0}^{\infty}$ be real-valued. If $\mathfrak{I}f - C_I \in \mathcal{B}_{\pi,0}^{\infty}$ for some constant C_I, then we have*

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \int\limits_{\epsilon \le |t-\tau| \le \frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau = (\mathfrak{I}f)(t) - C_I$$

and

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \int\limits_{\epsilon \le |t-\tau| \le \frac{1}{\epsilon}} \frac{(\mathfrak{I}f)(\tau)}{t-\tau} \mathrm{d}\tau = -f(t)$$

for all $t \in \mathbb{R}$.

The proof of Theorem 10 is given in Appendix B.

X. Conclusion

In this paper we solved the peak value problem of the Hilbert transform for bounded bandlimited signals. By analyzing the problem we also clarified the causes which, in the classical literature, led to the misbelief that general bounded bandlimited signals do not have a Hilbert transform. For general bounded bandlimited signals, the Hilbert transform cannot be calculated by the principal value integral (1), and formal substitution rules, like the one where, in certain expressions, "sin" is simply replaced by "$-\cos$" can no longer be used. The necessary theory to define to Hilbert transform for bounded signals is build on the abstract and nonconstructive $\mathcal{H}^1$-BMO($\mathbb{R}$) duality. Because the duality approach gives no constructive procedure for the calculation of the Hilbert transform, its usefulness for practical applications is limited. However, in this paper we considered bounded signals that are additionally bandlimited and thus could use a simple formula, which was recently found, for calculating the Hilbert transform, avoiding the abstract duality theory. Based on this novel formula, we were able to solve the peak value problem of the Hilbert transform and to provide growth estimates.

Appendix

A. Proof of Theorem 8

For the proof of Theorem 8 we need Lemma 2.

Lemma 2. *Let $f \in \mathcal{B}_{\pi}^{\infty}$ and $\mathfrak{I}f \in \mathcal{B}_{\pi}^{\infty}$. Then, for $F = f + i\mathfrak{I}f$, we have*

$$|F(t+iy)| \le \|F\|_\infty$$

for all $t \in \mathbb{R}$ and $y \ge 0$.

Proof: Let $t \in \mathbb{R}$ and $y > 0$ be arbitrary but fixed. Since $f \in \mathcal{B}_{\pi}^{\infty}$ and $\mathfrak{I}f \in \mathcal{B}_{\pi}^{\infty}$, we have $F \in \mathcal{B}_{\pi}^{\infty}$ and therefore the integral

$$G(t,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\tau) \frac{y}{y^2 + (t-\tau)^2} \mathrm{d}\tau$$

is absolutely convergent.

Next, we show that

$$G(t,y) = F(t+iy). \tag{20}$$

For $0 < \epsilon < 1$ consider the functions

$$f_\epsilon(z) = f((1-\epsilon)z)\frac{\sin(\epsilon\pi z)}{\epsilon\pi z}, \quad z \in \mathbb{C},$$

which are in $\mathcal{B}_\pi^2$ and set

$$F_\epsilon(z) = f_\epsilon(z) + i(\Im f_\epsilon)(z), \quad z \in \mathbb{C}.$$

Thus, according to (7), we have

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty} F_\epsilon(\tau)\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} (f_\epsilon(\tau)+i(Hf_\epsilon)(\tau))\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau - \frac{1}{\pi}\int_{-\infty}^{\infty} i(Hf_\epsilon)(0)\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau. \end{aligned} \tag{21}$$

Since $f_\epsilon \in \mathcal{B}_\pi^2$, we obtain

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} (f_\epsilon(\tau)+i(Hf_\epsilon)(\tau))\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\hat{f}_\epsilon(\omega) + i(-i\operatorname{sgn}(\omega))\hat{f}_\epsilon(\omega))\,\mathrm{e}^{-y|\omega|}\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{0}^{\pi} 2\hat{f}_\epsilon(\omega)\,\mathrm{e}^{i\omega(t+iy)}\,\mathrm{d}\omega \\ &= f_\epsilon(t+iy) + i(Hf_\epsilon)(t+iy)\end{aligned}$$

for the first integral on the right hand side of (21). For the second integral we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} i(Hf_\epsilon)(0)\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau = i(Hf_\epsilon)(0),$$

because $(Hf_\epsilon)(0)$ is a constant and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau = 1.$$

Thus, it follows that

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} F_\epsilon(\tau)\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau &= f_\epsilon(t+iy) + i((Hf_\epsilon)(t+iy) - (Hf_\epsilon)(0)) \\ &= f_\epsilon(t+iy) + i(\Im f_\epsilon)(t+iy) \\ &= F_\epsilon(t+iy). \end{aligned} \tag{22}$$

Let $\delta > 0$ be arbitrary but fixed. Then there exists a $\tau_0 = \tau_0(\delta) > 0$ such that

$$\int_{-\infty}^{-\tau_0} \log(1+|\tau|)\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau < \delta \tag{23}$$

and

$$\int_{\tau_0}^{\infty} \log(1+\tau)\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau < \delta.$$

Further, it can be shown, that there exists a $\epsilon_0 = \epsilon_0(\tau) > 0$ such that

$$\max_{|\tau|\le\tau_0} |F_\epsilon(\tau) - F(\tau)| < \delta \tag{24}$$

for all $0 < \epsilon \le \epsilon_0$. Using (22), we have

$$\begin{aligned}|F_\epsilon(t+iy) - G(t,y)| &= \left|\frac{1}{\pi}\int_{-\infty}^{\infty} (F_\epsilon(\tau) - F(\tau))\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau\right| \\ &\le \frac{1}{\pi}\int_{-\infty}^{-\tau_0} |F_\epsilon(\tau) - F(\tau)|\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau \\ &\quad + \frac{1}{\pi}\int_{-\tau_0}^{\tau_0} |F_\epsilon(\tau) - F(\tau)|\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau \\ &\quad + \frac{1}{\pi}\int_{\tau_0}^{\infty} |F_\epsilon(\tau) - F(\tau)|\frac{y}{y^2+(t-\tau)^2}\mathrm{d}\tau. \end{aligned} \tag{25}$$

Next, we analyze the three integrals on the right hand side of (25). In order to bound the first and third integral from above, we need an auxiliary result. According to Theorem 5 there exist two constants C_4 and C_5 such that

$$|(\mathfrak{I}f)(\tau)| \leq (C_5 + C_4 \log(1+|\tau|))\|f\|_\infty$$

for all $\tau \in \mathbb{R}$. Hence, we have

$$\begin{aligned} |F(\tau)| &= |f(\tau) + i(\mathfrak{I}f)(\tau)| \leq |f(\tau)| + |(\mathfrak{I}f)(\tau)| \\ &\leq \|f\|_\infty + (C_5 + C_4 \log(1+|\tau|))\|f\|_\infty \\ &= (1 + C_5 + C_4 \log(1+|\tau|))\|f\|_\infty \end{aligned}$$

and

$$|F_\epsilon(\tau)| \leq (1 + C_5 + C_4 \log(1+|\tau|))\|f_\epsilon\|_\infty \leq (1 + C_5 + C_4 \log(1+|\tau|))\|f\|_\infty$$

for all $\tau \in \mathbb{R}$. This implies that

$$|F_\epsilon(\tau) - F(\tau)| \leq 2(1 + C_5 + C_4 \log(1+|\tau|))\|f\|_\infty \tag{26}$$

for all $\tau \in \mathbb{R}$. Thus, for the first integral on the right hand side of (25), we obtain

$$\begin{aligned} \frac{1}{\pi}\int_{-\infty}^{-\tau_0} |F_\epsilon(\tau) - F(\tau)| \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau &\leq C_9 \|f\|_\infty \int_{-\infty}^{-\tau_0} \log(1+|\tau|) \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau \\ &< C_9 \|f\|_\infty \delta, \end{aligned} \tag{27}$$

where we used (26) in the first inequality and (23) in the second, and C_9 is a constant. By the same arguments, we obtain

$$\frac{1}{\pi}\int_{\tau_0}^{\infty} |F_\epsilon(\tau) - F(\tau)| \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau < C_9 \|f\|_\infty \delta. \tag{28}$$

For the second integral on the right hand side of (25), we have for all $0 < \epsilon \leq \epsilon_0$ that

$$\begin{aligned} &\frac{1}{\pi}\int_{-\tau_0}^{\tau_0} |F_\epsilon(\tau) - F(\tau)| \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau \\ &\leq \max_{|\tau|\leq\tau_0} |F_\epsilon(\tau) - F(\tau)| \frac{1}{\pi}\int_{-\tau_0}^{\tau_0} \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau \\ &< \delta, \end{aligned} \tag{29}$$

where we used (24) in the last inequality. Combining (25), (27), (28), and (29), it follows that

$$|F_\epsilon(t+iy) - G(t,y)| < (1 + 2C_9\|f\|_\infty)\delta$$

for all $0 < \epsilon \leq \epsilon_0$, which shows that

$$\lim_{\epsilon \to 0} F_\epsilon(t+iy) = G(t,y).$$

Further, a short calculation gives

$$\lim_{\epsilon \to 0} F_\epsilon(t+iy) = F(t+iy).$$

Hence, we have (20), i.e.,

$$F(t+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\tau) \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau.$$

It follows that

$$|F(t+iy)| \leq \frac{1}{\pi}\int_{-\infty}^{\infty} |F(\tau)| \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau \leq \|F\|_\infty \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{y^2+(t-\tau)^2} \mathrm{d}\tau = \|F\|_\infty,$$

which completes the proof. ■

Proof of Theorem 8:

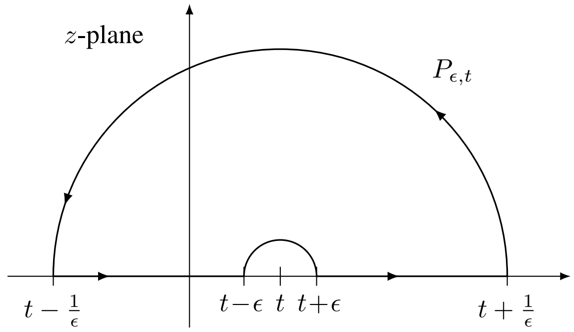

Fig. 5. Integration path $P_{\epsilon,t}$ in the complex plane.

We start with the proof of the "⇒" direction. Let $f \in \mathcal{B}_\pi^\infty$ be real-valued, such that $\mathfrak{I}f \in \mathcal{B}_\pi^\infty$. Further, let ϵ with $0 < \epsilon < 1$ and $t \in \mathbb{R}$ be arbitrary but fixed, and consider the complex contour that is depicted in Fig. 5. Since $F = f + i\mathfrak{I}f$ is an entire function, we have according to Cauchy's integral theorem that

$$\int\limits_{P_{\epsilon,t}} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi = 0.$$

Further, we have

$$\int\limits_{P_{\epsilon,t}} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi = \int\limits_{\curvearrowleft} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi + \int\limits_{\frown} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi + \int\limits_{\to\ \to} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi.$$

Thus, it follows that

$$\int\limits_{\to\ \to} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi = -\int\limits_{\curvearrowleft} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi - \int\limits_{\frown} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi. \tag{30}$$

Next, we analyze the two integrals on the right hand side of (30). For the first integral we have

$$\int\limits_{\curvearrowleft} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi = \int_{-\pi}^{0} \frac{F(t+\epsilon\,\mathrm{e}^{i\phi})}{\epsilon\,\mathrm{e}^{i\phi}} i\epsilon\,\mathrm{e}^{i\phi}\,\mathrm{d}\phi = i\int_{-\pi}^{0} F(t+\epsilon\,\mathrm{e}^{i\phi})\mathrm{d}\phi, \tag{31}$$

and consequently

$$\left| \int\limits_{\curvearrowleft} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi \right| \leq \pi \sup_{\mathrm{Im}(z)\geq 0} F(z) \leq \pi \|F\|_\infty, \tag{32}$$

where we used Lemma 2 in the last inequality. For the second integral, a similar calculation yields

$$\left| \int\limits_{\frown} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi \right| \leq \pi \|F\|_\infty. \tag{33}$$

Combining (30), (32), and (33), we obtain

$$\left| \frac{1}{\pi} \int\limits_{\to\ \to} \frac{F(\xi)}{t-\xi} \mathrm{d}\xi \right| \leq 2\|F\|_\infty.$$

Since $|\mathrm{Re}\, z| \leq |z|$ for all $z \in \mathbb{C}$ and f is real-valued, this implies that

$$\left| \frac{1}{\pi} \int\limits_{\epsilon\leq|t-\tau|\leq\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau \right| = \left| \frac{1}{\pi} \int\limits_{\to\ \to} \frac{f(\xi)}{t-\xi} \mathrm{d}\xi \right| \leq 2\|F\|_\infty,$$

which completes the proof of the "⇒" direction, because $1 < \epsilon < 1$, and $t \in \mathbb{R}$ were arbitrary.

Next, we prove the “$\Leftarrow$” direction. Consider the operator J defined by

$$(Jf)(t) = \lim_{\epsilon \to 0} \left((H_\epsilon f)(t) - (H_\epsilon f)(0) \right), \quad t \in \mathbb{R},$$

where

$$(H_\epsilon f)(t) = \frac{1}{\pi} \int\limits_{\epsilon \leq |t-\tau| \leq \frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau.$$

We first show that J is a well-defined operator on $\mathcal{B}_\pi^\infty$. Let $t>0$ be arbitrary but fixed. Without loss of generality we can assume that $t>0$. The case $t<0$ is treated analogously. Set $t_0 = t/2$. For ϵ, satisfying $0<\epsilon<t_0$ and $1/\epsilon > t + t_0$, we obtain, by splitting and rearranging integrals, that

$$\begin{aligned}
((H_\epsilon f)(t) - (H_\epsilon f)(0))\pi = &\int\limits_{\epsilon \leq |t-\tau| \leq t_0} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau + \int\limits_{\epsilon \leq |\tau| \leq t_0} \frac{f(\tau)}{\tau} \mathrm{d}\tau \\
&+ \int_{\frac{1}{\epsilon}}^{t+\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau + \int_{-\frac{1}{\epsilon}}^{t-\frac{1}{\epsilon}} \frac{f(\tau)}{\tau} \mathrm{d}\tau \\
&+ \int_{-t_0}^{t_0} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau + \int_{t-t_0}^{t+t_0} \frac{f(\tau)}{\tau} \mathrm{d}\tau \\
&+ \int_{t-\frac{1}{\epsilon}}^{-t_0} \frac{f(\tau)t}{(t-\tau)\tau} \mathrm{d}\tau + \int_{t+t_0}^{\frac{1}{\epsilon}} \frac{f(\tau)t}{(t-\tau)\tau} \mathrm{d}\tau. \qquad (34)
\end{aligned}$$

For the first integral in (34) we have

$$\int\limits_{\epsilon \leq |t-\tau| \leq t_0} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau = \int\limits_{\epsilon \leq |t-\tau| \leq t_0} \frac{f(\tau)-f(t)}{t-\tau} \mathrm{d}\tau. \qquad (35)$$

Since $|f(\tau)-f(t)| \leq |t-\tau|\pi\|f\|_\infty$, we see that the integrand of the integral on the right hand side of (35) is continuous on $[t-t_0, t+t_0]$. Thus, it follows that

$$\lim_{\epsilon \to 0} \int\limits_{\epsilon \leq |t-\tau| \leq t_0} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau = \int_{t-t_0}^{t+t_0} \frac{f(\tau)-f(t)}{t-\tau} \mathrm{d}\tau.$$

The same consideration for the second integral in (34) shows that

$$\lim_{\epsilon \to 0} \int\limits_{\epsilon \leq |\tau| \leq t_0} \frac{f(\tau)}{\tau} \mathrm{d}\tau = \int_{-t_0}^{t_0} \frac{f(\tau)-f(0)}{\tau} \mathrm{d}\tau.$$

As for the third integral in (34), we have

$$\int_{\frac{1}{\epsilon}}^{t+\frac{1}{\epsilon}} \left| \frac{f(\tau)}{t-\tau} \right| \mathrm{d}\tau \leq \frac{\|f\|_\infty t}{\frac{1}{\epsilon} - t},$$

and consequently

$$\lim_{\epsilon \to 0} \int_{\frac{1}{\epsilon}}^{t+\frac{1}{\epsilon}} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau = 0.$$

The same holds for the fourth integral

$$\lim_{\epsilon \to 0} \int_{-\frac{1}{\epsilon}}^{t-\frac{1}{\epsilon}} \frac{f(\tau)}{\tau} \mathrm{d}\tau = 0.$$

It remains to analyze the seventh and eighth integral in (34). Since

$$\begin{aligned}\int_{t-\frac{1}{\epsilon}}^{-t_0} \left| \frac{f(\tau)t}{(t-\tau)\tau} \right| \mathrm{d}\tau &\leq \|f\|_\infty \int_{t_0}^{\frac{1}{\epsilon}-t} \frac{t}{(t+\tau)\tau} \mathrm{d}\tau \\ &= \|f\|_\infty \int_{t_0}^{\frac{1}{\epsilon}-t} \frac{1}{\tau} - \frac{1}{(t+\tau)} \mathrm{d}\tau \\ &= \|f\|_\infty \left(\log(1-\epsilon t) + \log\left(\frac{t+t_0}{t_0}\right)\right) \\ &\leq \|f\|_\infty \log\left(\frac{t+t_0}{t_0}\right) \\ &\leq C_{10},\end{aligned}$$

with a constant $C_{10} < \infty$ that is independent of ϵ, we see that the limit

$$\lim_{\epsilon\to 0} \int_{t-\frac{1}{\epsilon}}^{-t_0} \frac{f(\tau)t}{(t-\tau)\tau} \mathrm{d}\tau$$

exists and is finite. By a similar calculation we see that

$$\lim_{\epsilon\to 0} \int_{t+t_0}^{\frac{1}{\epsilon}} \frac{f(\tau)t}{(t-\tau)\tau} \mathrm{d}\tau$$

exists and is finite. Thus, the operator J is well-defined on $\mathcal{B}_\pi^\infty$ and we have

$$\begin{aligned}(Jf)(t) =& \frac{1}{\pi} \int_{t-t_0}^{t+t_0} \frac{f(\tau)-f(t)}{t-\tau} \mathrm{d}\tau + \frac{1}{\pi} \int_{-t_0}^{t_0} \frac{f(\tau)-f(0)}{\tau} \mathrm{d}\tau \\ &+ \frac{1}{\pi} \int_{-t_0}^{t_0} \frac{f(\tau)}{t-\tau} \mathrm{d}\tau + \frac{1}{\pi} \int_{t-t_0}^{t+t_0} \frac{f(\tau)}{\tau} \mathrm{d}\tau \\ &+ \frac{1}{\pi} \int_{-\infty}^{-t_0} \frac{f(\tau)t}{(t-\tau)\tau} \mathrm{d}\tau + \frac{1}{\pi} \int_{t+t_0}^{\infty} \frac{f(\tau)t}{(t-\tau)\tau} \mathrm{d}\tau \end{aligned} \tag{36}$$

for all $f \in \mathcal{B}_\pi^\infty$ and $t \in \mathbb{R}$.

Next, let $f \in \mathcal{B}_\pi^\infty$ be real-valued and arbitrary but fixed. For $n \in \mathbb{N}$, consider the $\mathcal{B}_\pi^2$-functions

$$f_n(\tau) = f\left((1-\tfrac{1}{n})\tau\right) \frac{\sin\left(\frac{1}{n}\pi\tau\right)}{\frac{1}{n}\pi\tau}. \quad \tau \in \mathbb{R},$$

We will show that $(Jf)(t) = \lim_{n\to\infty}(Jf_n)(t)$ for all $t \in \mathbb{R}$. Again, we can restrict ourselves to the case $t > 0$, because the case $t < 0$ is treated analogously. Therefore, let $t > 0$ be arbitrary but fixed. We need some additional preliminary considerations. First, note that $\|f_n\|_\infty \leq \|f\|_\infty$, $n \in \mathbb{N}$. Moreover, f_n converges locally uniformly to f, and f_n' converges locally uniformly to f'. Let $\delta > 0$ be arbitrary but fixed. Then, there exists a $\tau_0 = \tau_0(\delta) \geq t + t_0$ such that

$$\int_{-\infty}^{-\tau_0} \frac{t}{(\tau-t)\tau} \mathrm{d}\tau < \delta \tag{37}$$

and

$$\int_{\tau_0}^{\infty} \frac{t}{(\tau-t)\tau} \mathrm{d}\tau < \delta. \tag{38}$$

Due to the local uniform convergence of f_n, there exists a natural number $n_0 = n_0(\delta)$, such that

$$\max_{\tau\in[-\tau_0,\tau_0]} |f(\tau) - f_n(\tau)| < \delta \tag{39}$$

for all $n \geq n_0$. Since

$$\frac{f(\tau)-f(\tau_1)-f_n(\tau)-f_n(\tau_1)}{\tau_1-\tau} = \frac{1}{\tau_1-\tau} \int_{\tau_1}^{\tau} f'(u) - f_n'(u) \mathrm{d}u,$$

it follows that

$$\left|\frac{f(\tau)-f(\tau_1)-f_n(\tau)-f_n(\tau_1)}{\tau_1-\tau}\right| \leq \frac{1}{|\tau_1-\tau|}\int_{\tau_1}^{\tau}|f'(u)-f_n'(u)|\mathrm{d}u \\ \leq \max_{u\in[\min\{\tau_1,\tau\},\max\{\tau_1,\tau\}]}|f'(u)-f_n'(u)|$$

for all $\tau,\tau_1\in\mathbb{R}$. Hence, by the local uniform convergence of f_n', it follows that there exists a natural number $n_1=n_1(\delta)$, such that

$$\left|\frac{f(\tau)-f(t)-f_n(\tau)-f_n(t)}{t-\tau}\right| < \delta \tag{40}$$

and

$$\left|\frac{f(\tau)-f(0)-f_n(\tau)-f_n(0)}{\tau}\right| < \delta \tag{41}$$

for all $\tau\in[-t_0,t+t_0]$ and all $n\geq n_1$. Now, we have all preliminary considerations and can return to the proof. From (36) we obtain

$$\begin{aligned}|(J(f-f_n))(t)| \leq &\frac{1}{\pi}\int_{t-t_0}^{t+t_0}\left|\frac{f(\tau)-f_n(\tau)-f(t)+f_n(t)}{t-\tau}\right|\mathrm{d}\tau+\frac{1}{\pi}\int_{-t_0}^{t_0}\left|\frac{f(\tau)-f_n(\tau)-f(0)+f_n(0)}{\tau}\right|\mathrm{d}\tau\\ &+\frac{1}{\pi}\int_{-t_0}^{t_0}\frac{|f(\tau)-f_n(\tau)|}{t-\tau}\mathrm{d}\tau+\frac{1}{\pi}\int_{t-t_0}^{t+t_0}\frac{|f(\tau)-f_n(\tau)|}{\tau}\mathrm{d}\tau\\ &+\frac{1}{\pi}\int_{-\infty}^{-t_0}\frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau+\frac{1}{\pi}\int_{t+t_0}^{\infty}\frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau.\end{aligned} \tag{42}$$

We treat the integrals in (42) separately. For the first and second integral we easily obtain

$$\frac{1}{\pi}\int_{t-t_0}^{t+t_0}\left|\frac{f(\tau)-f_n(\tau)-f(t)+f_n(t)}{t-\tau}\right|\mathrm{d}\tau<\delta 2t_0$$

and

$$\frac{1}{\pi}\int_{-t_0}^{t_0}\left|\frac{f(\tau)-f_n(\tau)-f(0)+f_n(0)}{\tau}\right|\mathrm{d}\tau<\delta 2t_0$$

for all $n\geq n_1$, by using (40) and (41), respectively. For the third integral we have

$$\int_{-t_0}^{t_0}\frac{|f(\tau)-f_n(\tau)|}{t-\tau}\mathrm{d}\tau\leq\max_{\tau\in[-t_0,t_0]}|f(\tau)-f_n(\tau)|\underbrace{\int_{-t_0}^{t_0}\frac{1}{t-\tau}\mathrm{d}\tau}_{=:C_{11}}<\delta C_{11}$$

for all $n\geq n_0$, where we used (39) in the last inequality. Equally, we obtain

$$\int_{t-t_0}^{t+t_0}\frac{|f(\tau)-f_n(\tau)|}{t-\tau}\mathrm{d}\tau\leq\max_{\tau\in[t-t_0,t+t_0]}|f(\tau)-f_n(\tau)|\underbrace{\int_{t-t_0}^{t+t_0}\frac{1}{t-\tau}\mathrm{d}\tau}_{=:C_{12}}<\delta C_{12}$$

for all $n\geq n_0$. The fifth integral can be upper bounded according to

$$\begin{aligned}\int_{-\infty}^{-t_0}\frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau&=\int_{-\infty}^{-\tau_0}\frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau+\int_{-\tau_0}^{-t_0}\frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau\\ &\leq 2\|f\|_\infty\int_{-\infty}^{-\tau_0}\frac{t}{(\tau-t)\tau}\mathrm{d}\tau+\max_{\tau\in[-\tau_0,-t_0]}|f(\tau)-f_n(\tau)|\underbrace{\int_{-\infty}^{-t_0}\frac{t}{(\tau-t)\tau}\mathrm{d}\tau}_{=:C_{13}}\\ &<\delta(2\|f\|_\infty+C_{13})\end{aligned}$$

for all $n \geq n_0$, where we used (37) and (39) in the last inequality. For the last integral in (42), the same considerations together with (38) and (39) give

$$\begin{aligned}\int_{t+t_0}^{\infty} \frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau &= \int_{t+t_0}^{\tau_0} \frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau + \int_{\tau_0}^{\infty} \frac{|f(\tau)-f_n(\tau)|t}{(\tau-t)\tau}\mathrm{d}\tau \\ &\leq \max_{\tau\in[t+t_0,\tau_0]}|f(\tau)-f_n(\tau)| \underbrace{\int_{-\infty}^{-t_0} \frac{t}{(\tau-t)\tau}\mathrm{d}\tau}_{=:C_{14}} + 2\|f\|_\infty \int_{\tau_0}^{\infty} \frac{t}{(\tau-t)\tau}\mathrm{d}\tau \\ &< \delta(2\|f\|_\infty + C_{14})\end{aligned}$$

for all $n \geq n_0$. Combining all partial results yields

$$|(Jf)(t)-(Jf_n)(t)| = |(J(f-f_n))(t)| < \delta(4t_0 + 4\|f\|_\infty + C_{11} + C_{12} + C_{13} + C_{14})$$

for all for all $n \geq \max\{n_0, n_1\}$. Since $\delta > 0$ was arbitrary, this shows that

$$(Jf)(t) = \lim_{n\to\infty}(Jf_n)(t). \tag{43}$$

The equality (43) is true for all $t \in \mathbb{R}$, because $t > 0$ was arbitrary, and the case $t < 0$ is treated analogously. Thus, we have

$$\begin{aligned}(\mathfrak{I}f)(t) &= \lim_{n\to\infty}(\mathfrak{I}f_n)(t) = \lim_{n\to\infty}((Hf_n)(t)-(Hf_n)(0)) \\ &= \lim_{n\to\infty}\lim_{\epsilon\to 0}((H_\epsilon f_n)(t)-(H_\epsilon f_n)(0)) = \lim_{n\to\infty}(Jf_n)(t) \\ &= (Jf)(t) = \lim_{\epsilon\to 0}((H_\epsilon f)(t)-(H_\epsilon f)(0))\end{aligned}$$

for all $t \in \mathbb{R}$, where we used Lemma 1 in the first equality and (43) in the fifth. It follows from the assumption of the theorem that

$$|(\mathfrak{I}f)(t)| = |\lim_{\epsilon\to 0}((H_\epsilon f)(t)-(H_\epsilon f)(0))| \leq \limsup_{\epsilon\to 0}(|(H_\epsilon f)(t)| + |(H_\epsilon f)(0)|) \leq 2C_8,$$

for all $t \in \mathbb{R}$, which implies that $\mathfrak{I}f \in \mathcal{B}_\pi^\infty$, because of Theorem 4. ■

B. Proof of Theorem 10

For the proof of Theorem 10 we need the following lemma.

Lemma 3. *Let $f \in \mathcal{B}_{\pi,0}^\infty$ such that $\mathfrak{I}f - C_I \in \mathcal{B}_{\pi,0}^\infty$ for some constant C_I, and let*

$$F^{C_I}(t+iy) = f(t+iy) + i((\mathfrak{I}f)(t+iy) - C_I).$$

Then, for all $\epsilon > 0$ there exists a natural number $R_0 = R_0(\epsilon)$ such that

$$|F^{C_I}(t+iy)| < \epsilon$$

for all $t \in \mathbb{R}$ and $y \geq 0$, satisfying $\sqrt{t^2+y^2} \geq R_0$.

Proof: Consider the Möbius transformation

$$\phi(z) = \frac{z-i}{z+i},$$

which maps the upper half plane to the unit disk. The inverse mapping is given by

$$\phi^{-1}(z) = i\frac{1+z}{1-z}.$$

Since F^{C_I} is analytic in $\mathbb{C}$ and $|F^{C_I}(t+iy)| \leq \|F^{C_I}\|_\infty$ for all $t \in \mathbb{R}$ and $y \geq 0$, according to Lemma 2, it follows that

$$G(z) = F^{C_I}(\phi^{-1}(z)) = F^{C_I}\left(i\frac{1+z}{1-z}\right)$$

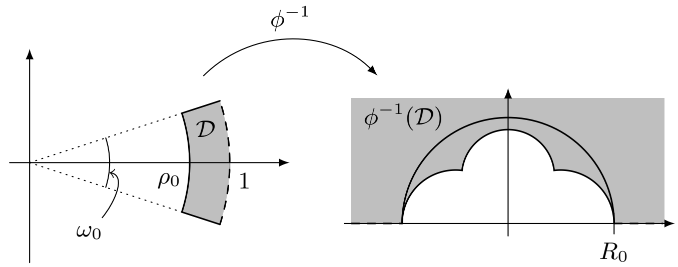

Fig. 6. Visualization of the set $\phi^{-1}(\mathcal{D})$.

is analytic for $|z| < 1$ and that

$$\sup_{|z|<1} |G(z)| < \infty.$$

Further, G is continuous on the unit circle, because F^{C_I} is continuous on the real axis,

$$\lim_{\omega \searrow 0} G(\mathrm{e}^{i\omega}) = \lim_{t\to -\infty} F^{C_\mathrm{I}}(t) = 0, \tag{44}$$

and

$$\lim_{\omega \nearrow 0} G(\mathrm{e}^{i\omega}) = \lim_{t\to \infty} F^{C_\mathrm{I}}(t) = 0. \tag{45}$$

Hence, by [29, p. 340, Theorem 17.11], we have

$$G(\rho\,\mathrm{e}^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\mathrm{e}^{i\omega}) \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega$$

for all $0 \leq \rho < 1$ and $-\pi < \theta < \pi$.

Let $\epsilon > 0$ be arbitrary but fixed. Equations (44) and (45) imply that there exits a $\omega_0 = \omega_0(\epsilon)$, $0 < \omega_0 < \pi$, such that

$$|G(\mathrm{e}^{i\omega})| < \frac{\epsilon}{2} \tag{46}$$

for all $|\omega| \leq \omega_0$. Further, there exists a $\rho_0 = \rho_0(\epsilon)$, $0 < \rho_0 < 1$, such that

$$\frac{\|F^{C_\mathrm{I}}\|_\infty (1-\rho)}{\rho\left(1-\cos\left(\frac{\omega_0}{2}\right)\right)} < \frac{\epsilon}{2} \tag{47}$$

for all $\rho_0 \leq \rho < 1$.

Next, let ρ satisfying $\rho_0 \leq \rho < 1$, and θ satisfying $-\omega_0/2 \leq \theta \leq \omega_0/2$, be arbitrary but fixed. Then, we have

$$\begin{aligned}|G(\rho\,\mathrm{e}^{i\theta})| &\leq \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} |G(\mathrm{e}^{i\omega})| \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega + \frac{1}{2\pi}\int\limits_{\omega_0\leq|\omega|\leq\pi} |G(\mathrm{e}^{i\omega})| \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega \\ &< \frac{\epsilon}{2} + \frac{\|F^{C_\mathrm{I}}\|_\infty}{2\pi}\int\limits_{\omega_0\leq|\omega|\leq\pi} \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega,\end{aligned}$$

where we used (46) and the fact [29, p. 233] that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega = 1.$$

Further, we have

$$\begin{aligned}\frac{\|F^{C_\mathrm{I}}\|_\infty}{2\pi}\int\limits_{\omega_0\leq|\omega|\leq\pi} \frac{1-\rho^2}{1-2\rho\cos(\omega-\theta)+\rho^2}\mathrm{d}\omega &\leq \frac{\|F^{C_\mathrm{I}}\|_\infty}{2\pi}\int\limits_{\omega_0\leq|\omega|\leq\pi} \frac{1-\rho^2}{1-2\rho\cos\left(\frac{\omega_0}{2}\right)+\rho^2}\mathrm{d}\omega \\ < \frac{\|F^{C_\mathrm{I}}\|_\infty(1-\rho^2)}{1-2\rho\cos\left(\frac{\omega_0}{2}\right)+\rho^2} = \frac{\|F^{C_\mathrm{I}}\|_\infty(1-\rho)(1+\rho)}{(1-\rho)^2+2\rho\left(1-\cos\left(\frac{\omega_0}{2}\right)\right)} &< \frac{\|F^{C_\mathrm{I}}\|_\infty(1-\rho)}{\rho\left(1-\cos\left(\frac{\omega_0}{2}\right)\right)} < \frac{\epsilon}{2},\end{aligned}$$

where we used (47) in the last inequality. Hence, it follows that $|G(\rho\,\mathrm{e}^{i\theta})| < \epsilon$ for all $\rho_0 \leq \rho < 1$ and $-\omega_0/2 \leq \theta \leq \omega_0/2$. Let $\mathcal{D} = \{\rho\,\mathrm{e}^{i\theta} : \rho_0 \leq \rho < 1, -\omega_0/2 \leq \theta \leq \omega_0/2\}$. Thus, for $z \in \phi^{-1}(\mathcal{D})$, we have $F^{C_\mathrm{I}}(z) < \epsilon$. The image of $\mathcal{D}$ under the mapping ϕ^{-1} is depicted in Figure 6. Finally, let R_0 be the radius of the smallest circle around the origin, whose restriction to the upper half plane lies completely in $\phi^{-1}(\mathcal{D})$. Then, we have $|F^{C_\mathrm{I}}(t+iy)| < \epsilon$ for all $t \in \mathbb{R}$ and $y \geq 0$, satisfying $\sqrt{t^2+x^2} \geq R_0$. ■

Now we are in the position to prove Theorem 10

Proof of Theorem 10: Let $f \in \mathcal{B}_{\pi,0}^\infty$ be real-valued, such that $\mathfrak{I}f - C_\mathrm{I} \in \mathcal{B}_{\pi,0}^\infty$ for some constant C_I. Further, let $t \in \mathbb{R}$ be arbitrary but fixed. Since $F^{C_\mathrm{I}} = f + i(\mathfrak{I}f - C_\mathrm{I}) \in \mathcal{B}_\pi$ is an entire function, we can use the same argumentation as in the proof of Theorem 8 to obtain

$$\int_{\rightarrow\,\rightarrow} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi = -\int_{\frown} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi - \int_{\frown} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi.$$

From (31) we see that

$$\lim_{\epsilon\to 0}\int_{\frown} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi = \pi i F^{C_\mathrm{I}}(t).$$

Let $\delta > 0$ be arbitrary but fixed. Then, according to Lemma 3, there exists a natural number $R_0 = R_0(\delta)$ such that $|F^{C_\mathrm{I}}(t+iy)| < \delta$ for all $t \in \mathbb{R}$ and $y \geq 0$, satisfying $\sqrt{t^2+y^2} \geq R_0$. Let $\epsilon_0 = 1/(R_0 + |t|)$. Then it follows that $\left|t + \frac{1}{\epsilon}\mathrm{e}^{i\phi}\right| \geq R_0$ for all $0 < \epsilon \leq \epsilon_0$ and consequently that $\left|F^{C_\mathrm{I}}\left(t + \frac{1}{\epsilon}\mathrm{e}^{i\phi}\right)\right| < \delta$ for all $0 < \epsilon \leq \epsilon_0$ and $0 \leq \phi \leq \pi$. Thus, we have

$$\left|\int_{\frown} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi\right| \leq \int_0^\pi \left|F^{C_\mathrm{I}}\left(t + \frac{1}{\epsilon}\mathrm{e}^{i\phi}\right)\right| \mathrm{d}\phi \leq \pi\delta$$

for all $0 < \epsilon \leq \epsilon_0$, which shows that

$$\lim_{\epsilon\to 0}\int_{\frown} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi = 0.$$

Hence, it follows that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\rightarrow\,\rightarrow} \frac{F^{C_\mathrm{I}}(\xi)}{t-\xi}\mathrm{d}\xi = -iF^{C_\mathrm{I}}(t), \tag{48}$$

which in turn implies that the real part of the left hand side of (48) converges to the real part of the right hand side of (48), i.e., that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\rightarrow\,\rightarrow} \frac{f(\xi)}{t-\xi}\mathrm{d}\xi = (\mathfrak{I}f)(t) - C_\mathrm{I},$$

and that the imaginary part of the left hand side of (48) converges to the imaginary part of the right hand side of (48), i.e., that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\rightarrow\,\rightarrow} \frac{(\mathfrak{I}f)(\xi) - C_\mathrm{I}}{t-\xi}\mathrm{d}\xi = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\rightarrow\,\rightarrow} \frac{(\mathfrak{I}f)(\xi)}{t-\xi}\mathrm{d}\xi = -f(t).$$

■

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