ROBUST STABILITY OF LINEAR DYNAMIC SYSTEMS WITH APPLICATION TO SINGULAR PERTURBATION THEORY*

by

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In this paper we will give a simple approach to determining conditions for stability of linear feedback systems subject to additive and multiplicative perturbations in the operators describing these systems. The approach is based on techniques used in functional analysis, and provides an alternative development and generalization of some conditions for the time - invariant case that have appeared in the literature very recently. As an example of the application of the conditions, we consider the determination of finite regions of stability for singularly perturbed systems.

1. Introduction

An important theme in system theory is the preservation of various system theoretic properties in the face of variations in the system model.

It is possible to distinguish two variations on this theme. In the first, attention is restricted to infinitesimal changes in the parameters of the nominal system model. Thus one begins by assuming that the nominal system has a certain property, and then asks if there exists an open set about the nominal system parameters such that all the systems with parameters in this set have the desired property. We will refer to investigations of this first type as sensitivity theory. A second approach requires the explicit delineation of finite regions of models about the nominal model for which the given property is preserved. We will refer to investigations of this second type as robustness theory.

Within the context of sensitivity or robustness theory, there are several properties that have been investigated. For example, it is well known that the controllability property is insensitive to small parameter variations [1,p. 43]. As another example, it is well known that type-1 servomechanisms have zero steady state step tracking error despite large (but not destabilizing) variations in their transfer function matrices [2].

In this paper we will focus on the robustness of the stability property of linear multivariable feedback systems. This subject is of special interest, since stability is the most basic system theoretic issue and since practical feedback systems must remain stable in the face of large parameter variations.

The importance of obtaining robustly stable feedback control systems has long been recognized by designers [3]. Indeed, a principal reason for using feedback rather than open-loop control is the presence of

model uncertainties. Any model is at best an approximation of reality, and the relatively low order, linear time-invariant models most often used for controller synthesis are bound to be rather crude approximations.

In classical frequency domain techniques for single-input, single-output (SISO) control system design, the robustness issue is naturally handled [3]. These techniques employ various graphical means (e.g., Bode, Nyquist, inverse-Nyquist, Nichols plots) of displaying the system model in terms of its frequency response. From these plots, one can determine by inspection the minimum charge in the model frequency response that leads to instability. These changes are often quantified by the gain and phase margins of the feedback system; sometimes the design is required to have certain minimal margins in order to be acceptable [4, p. 43].

In modern time domain techniques (such as the pole placement or linear-quadratic-Gaussian approaches) for multiple-input, multiple-output (MIMO) systems, the robustness issue is not directly dealt with. Instead, it is necessary to transform the resulting design to the frequency domain to examine its robustness properties. For SISO systems, this is accomplished as for classical designs, but the situation is less clear in the MIMO case, where it is necessary to consider simultaneous variations in the frequency responses of all the loops.

Very recently, there has been some important work addressing the multivariable robustness issue. In his thesis [5,6] Safonov gives a powerful approach, based on a multivariable sector stability theorem, that can characterize robustness for very general nonlinear MIMO feedback systems. In a recent paper [7], Doyle develops a robustness characterization for the linear time invariant MIMO case (It can be shown that Doyle's result can also be obtained by Safonov's

approach [8].) Doyle's characterization involves computing the minimum singular value of a certain transfer function matrix, and this computation essentially determines the minimum simultaneous variation of the system frequency responses that leads to instability. Since there is sophisticated and widely accessible software to compute singular values [9], this characterization is of great practical value.

The present paper is prompted by two observations. First, the use of singular values to characterize robustness is suggestive of connections with numerical analysis, but these connections are not clear from Doyle's approach utilizing the multivariable Nyquist theorem. Second, a specific instance of the robustness question arises when a system is approximated by making a singular perturbation to reduce its order. A related motivation, although only briefly discussed in this paper, arises from the desire to use multi-model techniques in the design of decentralized controllers for large scale systems [10,11].

The structure of this paper is as follows. In Section 2 we consider the robust stability of MIMO linear feedback systems using a generalized numerical-analytic approach. When specialized to the time invariant case, with rational transfer function matrix perturbations, Doyle's characterization results. In Section 3 we will apply the results of Section 2 to a specific robustness question arising in singular perturbation theory. Section 4 contains the summary and conclusions.

Notation

We will use the standard notation of input-output stability theory: see [12, pp. 13-14], or [13, pp. 38-39].

X = some Banach space of functions $x:T \to X$

T = subset of the real numbers

X = finite dimensional vector space

X_e = $\{x : P_\tau x \in X$ for all $\tau \in T\}$

$$(P_\tau x)(t) = \begin{cases} x(t) & t \leq \tau \\ 0 & t > \tau \end{cases}$$

$\mathscr{L}_2^m$ = space of m-vector functions on T
with integrable Euclidean norm

$I : X_e \to X_e$ = identity operator

$G : X_e \to X_e$ is causal if $P_\tau G P_\tau = P_\tau G$
for all τ

A^* = conjugate transpose of a complex
matrix A.

2.   Robust Stability of Linear Systems

We consider the feedback system depicted in Figure 1. Here

the causal linear operator $G : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$ represents the plant

plus any compensation that is used. The basic feedback equation

is

$$(I + G)e = u \qquad\qquad (2.1)$$

and the basic stability question is whether $(I + G)^{-1} : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^m$

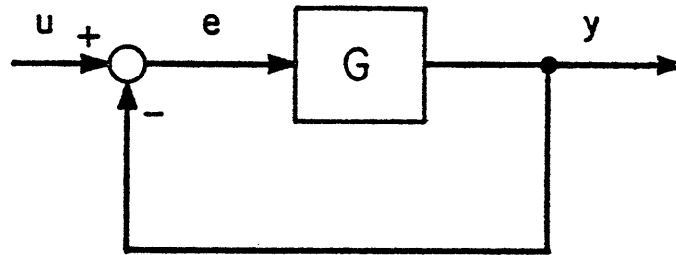exists, is causal, and is a bounded linear operator when restricted to the

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Figure 1. Basic MIMO Linear Feedback System

subspace $\mathscr{L}_2^m$ of $\mathscr{L}_{2e}^m$. We will assume that the nominal system is stable in this sense throughout this section, i.e. for u ε $\mathscr{L}_{2e}^m$ there exists a unique causally related e ε $\mathscr{L}_{2e}^m$ satisfying (2.1), and that u $\varepsilon \mathscr{L}_2^m$ implies that the corresponding e $\varepsilon \mathscr{L}_2^m$ and consequently y = u-e $\varepsilon \mathscr{L}_2^m$. We are interested in whether the closed loop system retains these properties when subject to additive (Figure 2) or multiplicative (Figure 3) perturbations representing uncertainty in the dynamical behavior of the system.

The following theorem provides the basis for our analysis.

THEOREM 1

Let $A: X_e \rightarrow X_e$ be a causal linear operator, and suppose A^{-1} exists, is causal, and is bounded when restricted to X. Then, if $\Delta A : X_e \rightarrow X_e$ is a causal linear operator that is bounded when restricted to X, and if

$$||A^{-1}\Delta A||_X < 1, \qquad (2.2)$$

it follows that $(A + \Delta A)^{-1} : X_e \rightarrow X_e$ exists, is causal and is bounded when restricted to X.

Proof

The operator $A^{-1}\Delta A$ is well defined, causal, and bounded on X by assumption. Since $||A^{-1}\Delta A||_X < 1$, the contraction mapping theorem implies that the sequence x_k, k = 0,1..., defined by

$$x_{k+1} = -A^{-1}\Delta A x_k + b; \quad x_o \equiv 0 \qquad (2.3)$$

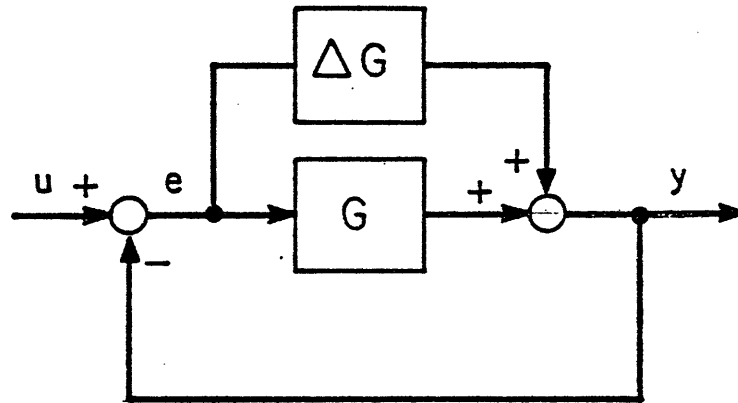converges to a unique solution $x \varepsilon X$ of

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Figure 2. System Subject to Additive Perturbations
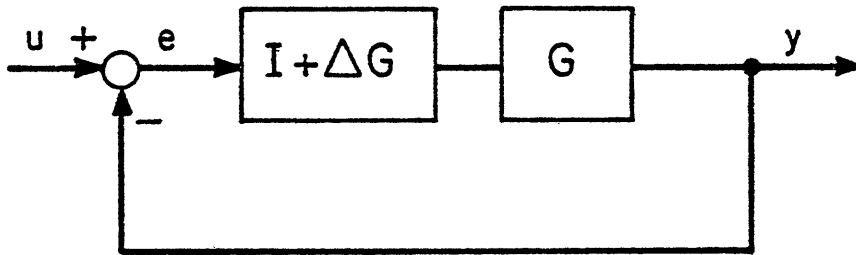
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Figure 3. System Subject to Multiplicative Perturbations

$$(I + A^{-1}\Delta A)x = b \qquad (2.4)$$

for any $b \in X$. Thus $(I + A^{-1}\Delta A)^{-1} : X \to X$ exists, and is therefore bounded since $I + A^{-1}\Delta A$ is. Causality follows since each iterate x_k depends causally on b and consequently the limit x of x_k depends causally on b. Since $(I + A^{-1}\Delta A)^{-1}$ is causal, it can be uniquely extended to X_e by requiring

$$P_\tau (I + A^{-1}\Delta A)^{-1} x = P_\tau (I + A^{-1}\Delta A)^{-1} P_\tau x \qquad (2.5)$$

for $x \in X_e$. Finally, defining

$$(A + \Delta A)^{-1} = (I + A^{-1}\Delta A)^{-1} A^{-1}, \qquad (2.6)$$

gives the required inverse of $A + \Delta A$. Q.E.D.

Remarks

1. Since

$$||A^{-1}\Delta A||_X \leq ||A^{-1}||_X \ ||\Delta A||_X \qquad (2.7)$$

a sufficient condition for (2.1) is

$$||A^{-1}||_X \ ||\Delta A||_X < 1. \qquad (2.8)$$

2. A basic result in numerical analysis is that if an nxm matrix A is invertible, then $A + \Delta A$ is invertible for all ΔA satisfying

$$\overline{\sigma}(\Delta A) < \underline{\sigma}(A) \qquad (2.9)$$

where $\overline{\sigma}(\Delta A) = ||\Delta A||_2$, $\underline{\sigma}(A) = (||A^{-1}||_2)^{-1}$

are respectively the smallest and largest singular values of the matrix A.

Theorem 1 is thus a generalization of this classical finite dimensional result where, of course, boundedness and causality are not at issue. In the finite dimensional case there exists ΔA such that $\overline{\sigma}(\Delta A) = \underline{\sigma}(A)$ and $A + \Delta A$ is singular; this is easily proved by the singular value decomposition[1].

3. The contraction mapping argument in Theorem 1 is a standard technique of applied mathematics; the fact that X_e is not a Banach space complicates the argument. The causality argument has been used by Willems [12, p. 98] in a slightly different context. The linearity of the perturbation operator is not essential.

4. Theorem 1 can be used to give $\mathscr{L}_p$ robust stability results, but we will confine ouselves to the case p = 2 in the sequel.

5. Theorem 1 can be used to obtain robust stability results for both continuous and discrete time, but in the sequel we will confine our attention to the case T = $[0,\infty]$.

The robust stability questions posed at the beginning of this section are now answered in terms of Theorem 2.

THEOREM 2

Assume that the basic feedback system of Figure 1 is stable. Then (i) the system remains stable for additive pertubations G (Figure 2) provided

$$||(I + G)^{-1}\Delta G||_{\mathscr{L}_2^m} < 1 \qquad\qquad (2.10)$$

and (ii) the system remains stable for multiplicative perturbations ΔG (Figure 3)

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[1] The singular value decomposition of an nxn nonsingular complex matrix A is $A = U\Sigma V^*$, where U and V are unitary nxn matrices, Σ = diag $(\sigma_1, \ldots\ldots,\sigma_n)$ and the singular values σ_i are the non-negative square roots of the eigenvalues of A^*A. See [14] for references, a more general definition, and an excellent discussion of the fundamental role of the singular value decomposition in linear systems theory.

provided

$$\left\| [I - (I + G)^{-1}]\Delta G \right\|_{\mathscr{L}_2^m} < 1 \qquad (2.11)$$

Proof

For case (i), we apply Theorem 1 to the equation

$$(I + G + \Delta G)e = u \qquad (2.12)$$

while for case (ii) we consider

$$[I+G(I+\Delta G)]e = (I+G+(I+G)\Delta G-\Delta G)e = u. \qquad \text{Q.E.D.} \qquad (2.13)$$

The practical importance of Theorem 2 stems from the fact that the $\mathscr{L}_2^m$ norm of a linear convolution operator can be computed from its transfer function matrix. This fact, which is a consequence of Parseval's Theorem, is well known in the input-output stability theoretic literature; see, e.g., [13, p. 26].

Lemma 1

Let the operator $G: \mathscr{L}_2^m \to \mathscr{L}_2^m$ for $T = [0,\infty]$ be defined by

$$(Gx)(t) = \int_0^\infty G(t-\tau)\, x(\tau)\, d\tau \qquad (2.14)$$

where the elements of the impulse response matrix $G(t)$ are assumed absolutely integrable on T. Then

$$\left\| G \right\|_{\mathscr{L}_2^m} = \sigma_{max} \qquad (2.15)$$

where

$$\sigma_{max} = \max_{\omega \geq 0}\ \max_{1 \leq i \leq m}\ \sigma_i(G(j\omega)) \qquad (2.16)$$

and where $\sigma_i(G(j\omega))$ denotes the ith singular value of the transfer function matrix corresponding to G.

Combining Theorem 2 and Lemma 1, we obtain the following result.

THEOREM 3

Assume that the nominal system in Figure 1 is time invariant, stable, and that the operators $(I+G)^{-1}, \Delta G$ can be represented as convolution operators with impulse response matrices with absolutely integrable elements. Then (i) the system remains stable for additive perturbations ΔG satisfying

$$\overline{\sigma}(\Delta G(j\omega)) < \underline{\sigma}(I+G(j\omega)), \quad \omega > 0 \tag{2.17}$$

and (ii) the system remains stable for multiplicative perturbations satisfying

$$\overline{\sigma}(\Delta G(j\omega)) < \underline{\sigma}(I+G^{-1}(j\omega)), \quad \omega > 0. \tag{2.18}$$

Here $G(j\omega)$ and $\Delta G(j\omega)$ are the transfer functions of G and ΔG, and $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ denote the maximum and minimum singular values of A.

Proof

(i) From Lemma 1 and (2.17) we have

$$||\Delta G|| < ||(I+G)^{-1}||^{-1} \tag{2.19}$$

so that

$$||(I+G)^{-1}|| \; ||\Delta G|| < 1. \tag{2.20}$$

(ii) Note that

$$\underline{\sigma}(I+G^{-1}(j\omega)) = \left\{ \overline{\sigma}[(I+G^{-1}(j\omega))^{-1}] \right\}^{-1}$$

$$= \left\{ \overline{\sigma}[(I+G(j\omega))^{-1} G(j\omega)] \right\}^{-1} = \overline{\sigma}[I-(I+G(j\omega))^{-1}] \tag{2.21}$$

Therefore Lemma 1 and (2.18) imply that

$$|| I - (I+G)^{-1}|| \; ||\Delta G|| < 1. \tag{2.22}$$

Q.E.D.

Remarks

1.   Notice the analogy between Theorem 3 concerning robust stability of linear systems and the classical result quoted previously concerning robust inversion of matrices (or bounded operators).

2.   Theorem 3 for the case of rational transfer function matrices is the result of Doyle alluded to previously. Doyle's proof is completely different, however, depending on the multivariable Nyquist Theorem, so that the connections with the inversion issue are only implicit.

3.   The quantity $\underline{\sigma}(I+G(j\omega))$ or $\underline{\sigma}(I+G^{-1}(j\omega))$ is easily computed and plotted as a function of ω. Doyle has made great use of this technique in the analysis of multivariable feedback systems. Such a plot plays much the same role for determining MIMO robustness properties as the more classical Bode, etc. plots in SISO design.

4.   The quantity $\underline{\sigma}^{-1}(I+G(j\omega))$ is the generalization of the classical Bode SISO sensitivity function of changes in the closed-loop transfer function with respect to changes in the open-loop transfer function in the following sense. Let y_1 denote the output of the system of Figure 1 for a given input and y_2 the corresponding output of the system of Figure 2 for the same input. Then one can show

$$y_1(j\omega) - y_2(j\omega) = (I+G(j\omega))^{-1}\Delta G(j\omega)(G(j\omega) + \Delta G(j\omega))^{-1}y_2(j\omega)$$

$$(2.23)$$

so that

$$||y_1(j\omega) - y_2(j\omega)|| \leq \frac{1}{\underline{\sigma}(I+G(j\omega))} \, || \Delta G(j\omega)(I+G(j\omega) + \Delta G(j\omega))^{-1}|| \times$$

$$||y_2(j\omega)|| . \qquad\qquad (2.24)$$

Consequently, the percentage change in the closed-loop transfer function matrix is attenuated from the percentage change in the open loop transfer

function matrix by the factor $\underline{\sigma}^{-1}(I + G(j\omega))$

5. On the other hand, the perturbations defining the SISO gain and phase margins are multiplicative rather than additive, so that $\underline{\sigma}(I + G^{-1}(j\omega))$ is more appropriate as a measure of the tolerance of the feedback system to model uncertainty.

6. It is in general impossible to express $\underline{\sigma}(I + G^{-1}(j\omega))$ in terms of $\underline{\sigma}(I + G(j\omega))$.

7. Consider the feedback system in Figure 1 with

$$G(s) = -G(sI - A)^{-1}B \qquad (2.25)$$

where

$$G = B'K \qquad (2.26)$$

$$O = A'K + KA + C'C - KBB'K \qquad (2.27)$$

(We assume [A,B] controllable and [A,C] observable so that a unique positive definite solution of (2.27) exists.) The well known equality [15]

$$[I + G(-sI - A)^{-1}B]' \ [I + G(sI - A)^{-1}B] = I + [C(-sI-A)^{-1}B]' \ [C(sI - A)^{-1}B],$$

$$(2.28)$$

which follows from (2.27) after a little manipulation, shows that the system is robust to additive perturbations. Safonov and Athans [6] have shown that the system of Figure 3 with $G(s)$ defined by (2.25) is stable for $\Delta G(j\omega)$ satisfying

$$\overline{\sigma}(\Delta G(j\omega)) \leq 1/2 \text{ for all } \omega \geq 0;$$

this can also be inferred directly using the inequality [19]

$$\underline{\sigma}(I + G^{-1}(j\omega)) \geq \frac{\underline{\sigma}(I + G(j\omega))}{1 + \underline{\sigma}(I + G(j\omega))} \qquad (2.29)$$

together with (2.28) and Theorem 3.

3. Application to Singular Perturbation Theory

In the previous section we have discussed the robustness of the stability property of a linear dynamic system to model variations. In this section we will consider a particular form of model variation due to a singular perturbation.[1]

We consider systems of the form

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t)$$

$$\varepsilon\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t)$$

$$(3.1)$$

where $\varepsilon > 0$ is a small parameter, and it is assume that the matrix A_{22} is stable (has eigenvalues with negative real parts). We define the so called degenerate system

$$\dot{x}_{1d}(t) = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_{1d}(t) \qquad (3.2)$$

associated with (3.1). This system is a reduced order system that neglects certain high-frequency or parasitic effects incorporated in the model (3.1). It has been shown that the stability of (3.2) (in the sense that the eigenvalues of the system matrix have negative real parts) is insensitive to these effects in the sense that there exists $\varepsilon_o > 0$ such that (3.1) is stable for all $0 < \varepsilon < \varepsilon_o$ if (3.2) is [17]. We propose to examine the robustness of the stability (in the input-output sense of Section 2) of (3.2) to the parasitic effects present in (3.1).

We begin by Laplace - transforming equations (3.1) (assuming zero initial condition).

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[1] See [16] for an excellent survey of results in singular perturbation theory.

$$x_1(s) = (sI - A_{11})^{-1} A_{12} x_2(s) \qquad (3.3)$$

$$x_2(s) = (\varepsilon sI - A_{22})^{-1} A_{21} x_1(s) \qquad$$

$$= [I - \varepsilon s(\varepsilon sI - A_{22})^{-1}](-A_{22}^{-1}A_{21}) x_1(s) \qquad (3.4)$$

To apply the input-output stability results of the previous section it is necessary to apply a test input to the system. This is most conveniently done as illustrated in Figure 4, although other locations are possible[1].

Figure 4 closely resembles Figure 3 with

$$G(s) = A_{22}^{-1} A_{21} (sI - A_{11})^{-1} A_{12} \qquad (3.5)$$

$$\Delta G(s,\varepsilon) = -\varepsilon s(\varepsilon sI - A_{22})^{-1} \qquad (3.6)$$

except that the perturbation is post-multiplicative rather than pre-multiplicative. However, assuming G(s) has full rank as a rational matrix, it is easily verified that the analysis of the preceding section is essentially unaffected. Thus we have the following result.

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[1] To insure the equivalence of the input-output stability analysis with the condition that the system matrix of (3.1) has eigenvalues with negative real parts, it is necessary to have the conditions

$$\left\{ \begin{bmatrix} A_{11} & A_{12} \\ \dfrac{A_{21}}{\varepsilon} & \dfrac{A_{22}}{\varepsilon} \end{bmatrix} , \begin{bmatrix} A_{12} \\ 0 \end{bmatrix} \right\} \qquad \text{controllable,}$$

$$\left\{ \begin{bmatrix} 0 & I \end{bmatrix} , \begin{bmatrix} A_{11} & A_{12} \\ \dfrac{A_{21}}{\varepsilon} & \dfrac{A_{22}}{\varepsilon} \end{bmatrix} \right\} \qquad \text{observable.}$$
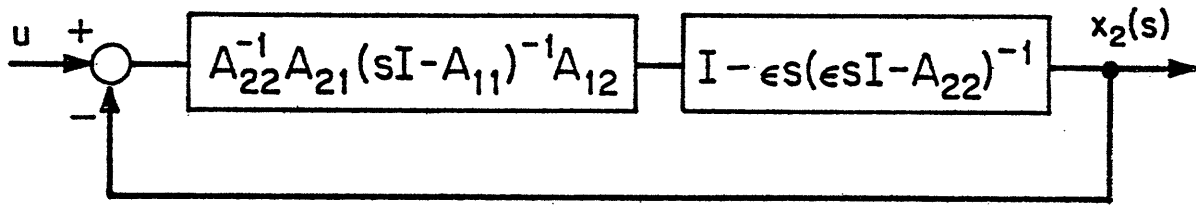
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Figure 4. Singular Perturbation in the Frequency Domain.

THEOREM 4

Assume that the system of Figure 4 is stable for $\varepsilon = 0$. Then it remains stable for all $\varepsilon > 0$ satisfying the inequality

$$\overline{\sigma}(\Delta G(j\omega,\varepsilon)) \quad \leq \quad \underline{\sigma}(I + G^{-1}(j\omega)) \tag{3.7}$$

for all $\omega > 0$.

The use of Theorem 4 is illustrated by the following examples.

Example 1

$$\left.\begin{array}{l} \dot{x}_1(t) = x_1(t) + 2\,x_2(t) \\[2mm] \varepsilon\dot{x}_2(t) = -x_2(t) - x_1(t) \end{array}\right\} \tag{3.8}$$

$$G(s) = \frac{2}{s-1} \tag{3.9}$$

$$\Delta G(s,\varepsilon) = \frac{\varepsilon s}{\varepsilon s+1} \tag{3.10}$$

In this case for which G and ΔG are scalars, the condition (3.7) is equivalent to

$$\left|1 + G^{-1}(j\omega)\right| > \left|G(j\omega,\varepsilon)\right| \tag{3.11}$$

or

$$\left|1 + G(j\omega)\right| > \left|G(j\omega)\Delta G(j\omega,\varepsilon)\right| \tag{3.12}$$

for all $\omega > 0$.

The condition (3.12) has an interesting graphical interpretation. Specifically the Nyquist locus of $G(j\omega)$ must avoid the critical point -1 by at least the distance $\left|G(j\omega)\Delta G(j\omega)\right|$ (Figure 5). It is easily verified that for $\varepsilon = +1$ and $\omega = \sqrt{2}$, we have $\left|1 + G(j\omega)\right| = \left|G(j\omega)\Delta G(j\omega)\right|$ so that $\varepsilon_o = 1$. It can be directly verified that the system (3.8) becomes unstable for $\varepsilon = 1$.
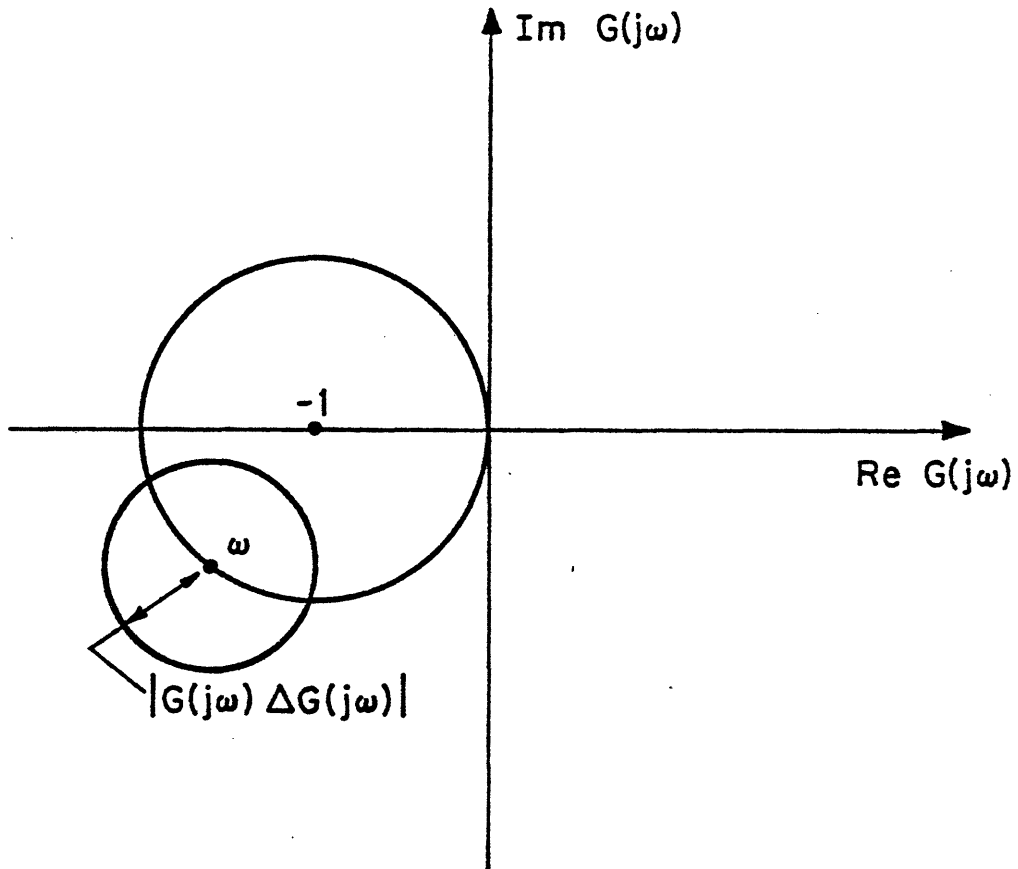
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Figure 5.   Illustrating the Condition (3.12).

Example 2

$$\dot{x}_1 = -x_1 + x_2 \left.\begin{matrix}\\\\\\\end{matrix}\right\}$$

$$\varepsilon \dot{x}_2 = -x_1 - x_2$$

(3.13)

$$G(s) = \frac{1}{s+1}$$

(3.14)

$$\Delta G(s,\varepsilon) = \frac{\varepsilon s}{1+\varepsilon s}$$

(3.15)

As in the previous example, we will check (3.12). We have

$$\sqrt{2+\omega^2} \geq 2 > 1 \geq \frac{\varepsilon\omega}{\sqrt{1+\varepsilon^2\omega^2}}$$

(3.16)

so that the system is stable for all $\varepsilon \geq 0$!

4. Summary and Conclusions

In this paper we have considered the robustness of the stability property of a linear feedback system to variations in the system model. The approach was to generalize a fundamental result in numerical linear algebra concerning robust inversion of matrices to linear operators of the type arising in input-output stability theory. In the time-invariant case, this approach specializes to give sufficient conditions for stability under additive and multiplicative perturbations that are easily verified by computing the singular values of certain transfer function matrices. We then applied the robustness condition to the analysis of singularly perturbed systems. We were able to give an explicit, readily computable bound on the magnitude of the perturbation parameter ε that can be tolerated and still have a stability analysis of the reduced system valid for the full system.

The results of this paper are felt to be of interest for two reasons. First, as Doyle has previously pointed out in the time invariant case [7], the characterization and design of robustly stable MIMO feedback systems is a fundamental problem in control theory that has yet to be completely resolved. Second, as has been previously emphasized by Zames [18] and Safonov [5], a fundamental problem in large scale system theory is to give conditions for the success of designs based on multiple, aggregate models of a single large system - this is essentially a robustness problem.

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References

1.   W.M. Wonham: Linear Multivariable Control: A Geometric Approach. Springer-Verlag, New York (1974).

2.   N.R. Sandell, Jr. and M. Athans: On multivariable type-ℓ systems, Automatica 9, pp. 131-136 (1973).

3.   I.M. Horowitz: Synthesis of Feedback Systems, Academic Press, New York (1963).

4.   BACKGROUND INFORMATION AND USER GUIDE FOR MIL-F-9490D, Flight Control Systems - Design, Installation and Test of Piloted Aircraft, General Specification for, Air Force Flight Dynamics Laboratory, Technical Report AFFDL-TR-74-116, January 1975.

5.   M.G. Safonov: Robustness and Stability Aspects of Stochastic Multivariable Feedback System Design, Ph.D. Thesis, Massachusetts Institute of Technology (1977).

6.   M. Safonov and M. Athans: Gain and phase margins for multiloop LQG regulators. IEEE Trans. on Aut. Control AC-22, pp. 173-179 (1977).

7.   J.C. Doyle: Robustness of multiloop linear feedback systems. To appear.

8.   M.G. Safonov, personal communication.

9.   B.S. Garbow et.al.: Matrix Eigensystem Routines - EISPACK Guide Extension. Lecture Notes in Computer Science, Vol. 51, Springer-Verlag, New York (1977).

10.   N.R. Sandell, Jr., P. Variaya, M. Athans, and M.G. Safonov: Survey of decentralized control methods for large scale systems. IEEE Trans. on Aut. Control AC-23, pp. 108-128 (1978).

11.   H.K. Khalil and P.V. Kokotovic: Control strategies for decision makers using different models of the same system. IEEE Trans. on Aut. Control, AC-23, pp. 289-298, (1978).

12.   J.C. Willems, The Analysis of Feedback Systems. M.I.T. Press, Cambridge, Mass. (1971).

13.   C.A. Desoer and M. Vidyasagar: Feedback Systems: Input-Output Properties. Academic Press, N.Y., (1975).

14.   A.J. Laub: Linear multivariable control. Numerical considerations. Invited paper, American Mathematical Society Short Course on Control Theory, Providence, R.I., (1978). Also ESL-P- 833 Electronic Systems Laboratory, Massachusetts Institute of Technology.

15.   R.E. Kalman: When is a Linear System Optimal? Trans. ASME Ser.D: J. Basic Eng., Vol. 86, pp. 51-60 (1964).

16. P.V. Kokotovic, R.E. O'Malley, Jr., and P. Sannuti: Singular perturbations and order reduction in control theory - an overview. Automatic 12, 123-132 (1976).

17. C.A. Desoer and M.J. Shensa: Networks with very small and very large parasitics: natural frequencies and stability. Proc. IEEE, 12, 1933-1938, Dec. 1970.

18. G. Zames: On feedback hierarchies and complexity (or information) 1976 CDC, Clearwater, Florida, (1976).

19. D. W. Nuzman and N.R. Sandell, Jr., "An Inequality Arising in Robustness Analysis of Multivariable Systems", submitted for publication.