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On the Behavior of the Threshold Operator for Bandlimited Functions

Holger Boche, Ullrich J. Mönich

ABSTRACT. One interesting question is how the good local approximation behavior of the Shannon sampling series for the Paley–Wiener space \mathcal{PW}_π^1 is affected if the samples are disturbed by the non-linear threshold operator. This operator, which is important in many applications, sets all samples whose absolute value is smaller than some threshold to zero. In this paper we analyze a generalization of this problem, in which not the Shannon sampling series is disturbed by the threshold operator but a more general system approximation process, where a stable linear time-invariant system is involved. We completely characterize the stable linear time-invariant systems that, for some functions in \mathcal{PW}_π^1 , lead to a diverging approximation process as the threshold is decreased to zero. Further, we show that if there exists one such function then the set of functions for which divergence occurs is in fact a residual set. We study the pointwise behavior as well as the behavior of the L^∞ -norm of the approximation process. It is known that oversampling does not lead to stable approximation processes in the presence of thresholding. An interesting open problem is the characterization of the systems that can be stably approximated with oversampling.

Math Subject Classifications. 94A20, 94A12.

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1. Introduction

A well-known fact about the convergence behavior of the Shannon sampling series for functions in the Paley–Wiener space \mathcal{PW}_π^1 is Brown’s theorem, which states the uniform convergence on compact subsets of \mathbb{R} [7, 8, 9].

Theorem 1 (Brown). *For all $f \in \mathcal{PW}_\pi^1$ and $\tau > 0$ fixed we have*

$$\lim_{N \rightarrow \infty} \max_{t \in [-\tau, \tau]} \left| f(t) - \sum_{k=-N}^N f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0. \quad (1.1)$$

The truncation of the series in (1.1) is done in the domain of the function f because only the samples $f(k)$, $k = -N, \dots, N$ are taken into account. In contrast, it is also possible to control the truncation of the series in the codomain of f by considering only the samples $f(k)$, $k \in \mathbb{Z}$, whose absolute value is larger than or equal to some threshold $\delta > 0$. This leads to the approximation process

$$(A_\delta f)(t) := \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}. \quad (1.2)$$

In general, $A_\delta f$ is only an approximation of f , and we want the function $A_\delta f$ to be close to f if δ is sufficiently small. In this paper we analyze a more general approximation process

$$(A_\delta^T f)(t) := (T A_\delta f)(t) = \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} f(k) h_T(t-k), \quad (1.3)$$

where additionally a linear time-invariant (LTI) system T is applied. Clearly, (1.2) is a special case of (1.3) with T being the identity operator. Surprisingly, the approximation errors of the approximation processes (1.2) and (1.3) do not always decrease as the threshold δ tends to zero, i.e., as more and more samples are used for the approximation. Depending on the function $f \in \mathcal{PW}_\pi^1$ and the LTI system T , the approximation process $(A_\delta^T f)(t)$ can diverge unboundedly, even for fixed $t \in \mathbb{R}$, as δ tends to zero.

Thresholding and quantization, which is closely related to thresholding, are two fundamental operations in digital signal processing because in digital circuits all signals can only be represented with a limited resolution and hence must be quantized [12].

2. Notation

In order to continue the discussion, we need some preliminaries and notation. Let \hat{f} denote the Fourier transform of a function f , where \hat{f} is to be

understood in the distributional sense. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all to the p th power Lebesgue integrable functions on \mathbb{R} , with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite.

For $\sigma > 0$ let \mathcal{B}_σ be the set of all entire functions f with the property that for all $\epsilon > 0$ there exists a constant $C(\epsilon)$ with $|f(z)| \leq C(\epsilon) \exp((\sigma + \epsilon)|z|)$ for all $z \in \mathbb{C}$. The Bernstein space \mathcal{B}_σ^p consists of all functions in \mathcal{B}_σ , whose restriction to the real line is in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. A function in \mathcal{B}_σ^p is called bandlimited to σ . By the Paley-Wiener-Schwartz theorem, the Fourier transform of a function bandlimited to σ is supported in $[-\sigma, \sigma]$. For $1 \leq p \leq 2$ the Fourier transformation is defined in the classical and for $p > 2$ in the distributional sense. It is well known, that $\mathcal{B}_\sigma^p \subset \mathcal{B}_\sigma^s$ for $1 \leq p \leq s \leq \infty$. Hence, every function $f \in \mathcal{B}_\sigma^p$, $p \leq \infty$, is bounded.

For $\sigma > 0$ and $1 \leq p \leq \infty$ we denote by \mathcal{PW}_σ^p the Paley-Wiener space of functions f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega) e^{iz\omega} d\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for \mathcal{PW}_σ^p , $1 \leq p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi) \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p d\omega)^{1/p}$.

We also need the following concepts from metric spaces [22]. A subset G of a metric space X is said to be nowhere dense in X if the closure $[G]$ does not contain a non-empty open set of X . G is said to be of the first category (or meager) if G is the countable union of sets each of which is nowhere dense in X . G is said to be of the second category (or nonmeager) if it is not of the first category. The complement of a set of the first category is called a residual set. Sets of first category may be considered as ‘‘small’’. According to Baire’s theorem [22] we have that in a complete metric space, the residual set is dense and a set of the second category. One property that shows the richness of residual sets is the following: The countable intersection of residual sets is always a residual set. In particular we will use the following fact in our proof. In a complete metric space an open and dense set is a residual set because its complement is nowhere dense.

3. Stable LTI Systems

Since our analyses involve stable linear time-invariant (LTI) systems, we briefly review some definitions and facts. A linear system $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ is called stable if the operator T is bounded, i.e., if

$$\|T\| = \sup_{\|f\|_{\mathcal{PW}_\pi^1} \leq 1} \|Tf\|_{\mathcal{PW}_\pi^1} < \infty.$$

Furthermore, it is called time-invariant if $(Tf(\cdot - a))(t) = (Tf)(t - a)$ for all $f \in \mathcal{PW}_\pi^1$ and $t, a \in \mathbb{R}$.

For every stable LTI system $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$ there exists exactly one

function $\hat{h}_T \in L^\infty[-\pi, \pi]$ such that

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) e^{i\omega t} d\omega \quad (3.1)$$

for all $f \in \mathcal{PW}_\pi^1$. Conversely, every function $\hat{h}_T \in L^\infty[-\pi, \pi]$ defines a stable LTI system $T : \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^1$. We have $h_T = T \text{sinc}$, where

$$\text{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

The operator norm of a stable LTI system T is given by $\|T\| = \|\hat{h}_T\|_\infty$. Furthermore, it can be shown that the representation (3.1) with $\hat{h}_T \in L^\infty[-\pi, \pi]$ is also valid for all stable LTI systems $T : \mathcal{PW}_\pi^2 \rightarrow \mathcal{PW}_\pi^2$. Therefore, every stable LTI system that maps \mathcal{PW}_π^1 in \mathcal{PW}_π^1 maps \mathcal{PW}_π^2 in \mathcal{PW}_π^2 , and vice versa. Note that $\hat{h}_T \in L^\infty[-\pi, \pi] \subset L^2[-\pi, \pi]$, and consequently $h_T \in \mathcal{PW}_\pi^2$.

If the samples $\{f(k)\}_{k \in \mathbb{Z}}$ of a function f are known perfectly, we can use

$$\sum_{k=-N}^N f(k) h_T(t-k) \quad (3.2)$$

to obtain an approximation of Tf . The conditions under which (3.2) converges to Tf for $f \in \mathcal{PW}_\pi^1$ as N goes to infinity were analyzed in [2]. In this paper we analyze the approximation behavior of (3.2) for functions in \mathcal{PW}_π^1 when the samples are disturbed by the threshold operator.

4. The Threshold Operator and Basic Properties

Before we state our main results, we precisely introduce the threshold operator and discuss some of its basic properties. For complex numbers $z \in \mathbb{C}$, the threshold operator κ_δ , $\delta > 0$, is defined by

$$\kappa_\delta z = \begin{cases} z & |z| \geq \delta \\ 0 & |z| < \delta. \end{cases}$$

Furthermore, for continuous functions $f : \mathbb{R} \rightarrow \mathbb{C}$, we define the threshold operator Θ_δ , $\delta > 0$, pointwise, i.e., $(\Theta_\delta f)(t) = \kappa_\delta f(t)$, $t \in \mathbb{R}$.

In this paper, the threshold operator κ_δ is applied on the samples $\{f(k)\}_{k \in \mathbb{Z}}$ of functions $f \in \mathcal{PW}_\pi^1$, which gives the disturbed samples $\{\kappa_\delta f(k)\}_{k \in \mathbb{Z}}$. This is, of course, equivalent to applying the threshold operator Θ_δ on the function f itself and then taking the samples, i.e., $\{(\Theta_\delta f)(k)\}_{k \in \mathbb{Z}}$. Then, the resulting samples $\{(\Theta_\delta f)(k)\}_{k \in \mathbb{Z}}$ are used to build

an approximation

$$(A_\delta^T f)(t) := \sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k) h_T(t-k) = \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} f(k) h_T(t-k), \quad (4.1)$$

of the system output Tf . By A_δ^T we denote the operator that maps $f \in \mathcal{PW}_\pi^1$ to $A_\delta^T f$ according to (4.1). If $f \in \mathcal{PW}_\pi^1$ we have $\lim_{t \rightarrow \infty} f(t) = 0$ by the Riemann-Lebesgue lemma, and it follows that the series in (4.1) has only finitely many summands, which implies $A_\delta^T f \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_\pi^1$. In general, $A_\delta^T f$ is only an approximation of Tf , and we want the function $A_\delta^T f$ to be close to f if δ is sufficiently small.

Since the series in (4.1) uses all “important” samples of the function, i.e., all samples that are larger or equal than δ , one could expect A_δ^T to have an approximation behavior similar to the approximation process (3.2). But, as we will see, A_δ^T exhibits a significantly different behavior.

The analysis of the approximation processes (4.1) is difficult, because the operator A_δ^T has several properties, which complicate its treatment.

1. For every $\delta > 0$, $A_\delta^T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^2$ is a non-linear operator.
2. For every $\delta > 0$, the operator $A_\delta^T: \mathcal{PW}_\pi^1 \rightarrow \mathcal{PW}_\pi^2$ is discontinuous, i.e., there exist a function $f \in \mathcal{PW}_\pi^1$ and a constant C_1 such that for every $\epsilon > 0$ there exists a function $g_\epsilon \in \mathcal{PW}_\pi^1$ satisfying $\|f - g_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$ and $\|A_\delta^T f - A_\delta^T g_\epsilon\|_{\mathcal{PW}_\pi^2} \geq C_1$.
3. For certain $f \in \mathcal{PW}_\pi^1$, the operator A_δ^T is also discontinuous with respect to δ , i.e., there exist a function $f \in \mathcal{PW}_\pi^1$ and a $t \in \mathbb{R}$ such that $\lim_{h \rightarrow 0} (A_{\delta+h}^T f)(t) \neq (A_\delta^T f)(t)$.

For fixed $t \in \mathbb{R}$, $\delta > 0$, and stable LTI system T , the mapping $f \mapsto (A_\delta^T f)(t)$ defines a functional on \mathcal{PW}_π^1 . This functional is not sublinear. A sublinear functional U on a general Banach space X satisfies

$$|U(f+g)| \leq |Uf| + |Ug| \quad (4.2)$$

and

$$|U(\lambda f)| = |\lambda| |Uf| \quad (4.3)$$

for all $f, g \in X$ and $\lambda \in \mathbb{C}$. It is easy to show that for $(A_\delta^T f)(t)$ it is not possible to obtain equations like (4.2) and (4.3).

Sequences of non-linear operators have been extensively studied since the fundamental paper [1] by Banach and Steinhaus. The central assumption in [1] was that the operators are sublinear, i.e., fulfill (4.2) and (4.3). Further, in [1] the sequences of operators were analyzed for fixed t . In [10, 11, 18] conditions were discussed that allow results for more general sets $T \subset \mathbb{R}$. For convergence almost everywhere a new approach was developed in [19]

that extends the theorem of Banach and Steinhaus. All papers [10, 11, 18, 19] have in common that they need the sublinearity of the involved operators. It is clear that these results cannot be applied here, because approximation process with thresholding $(A_\delta^T f)(t)$ is not sublinear. We will analyze $(A_\delta^T f)(t)$ for fixed $t \in \mathbb{R}$ in Section 5 and the peak value $\|A_\delta^T f\|_\infty$ in Section 6.

5. Behavior for Fixed t

In this section we analyze the behavior of $(A_\delta^T f)(t)$ for fixed $t \in \mathbb{R}$ as the threshold δ is decreased to zero.

Definition 1. Let Φ be the set of all continuous, positive, and monotonically decreasing functions ϕ defined on $(0, 1]$ that satisfy $\lim_{\delta \rightarrow 0} \phi(\delta) = \infty$ and $\phi(\delta) \geq 1$ for all $0 < \delta \leq 1$.

For fixed $t \in \mathbb{R}$, we want to characterize the stable LTI systems T for which the set

$$\mathcal{D}_1(T, t, \phi) := \left\{ f \in \mathcal{PW}_\pi^1 : \limsup_{\delta \rightarrow 0} \frac{|(A_\delta^T f)(t)|}{\phi(\delta)} = \infty \right\}$$

is non-empty, and, in the case where $\mathcal{D}_1(T, t, \phi)$ is non-empty, we are interested in structure of this set. The function $\phi \in \Phi$ is introduced in the above expression in order to describe the divergence speed of $(A_\delta^T f)(t)$.

The next theorem is our first main result.

Theorem 2. Let T be an stable LTI system, $t \in \mathbb{R}$, and $\phi \in \Phi$. Then we have $\mathcal{D}_1(T, t, \phi) \neq \emptyset$ if and only if $\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$. Further, if $\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$ then $\mathcal{D}_1(T, t, \phi)$ is a residual set.

The proofs of Theorem 2 and the required lemmas are done for stable LTI systems T with real-valued impulse response h_T . However, the transition to complex-valued h_T is straight forward.

For the proof of Theorem 2 we need the following three lemmas. Lemma 1 is a simple technical lemma, the proof of which is omitted.

Lemma 1. Let T be an stable LTI system, $t \in \mathbb{R}$, $\phi \in \Phi$, and $f \in \mathcal{PW}_\pi^1$. If

$$\sup_{0 < \delta < 1} \frac{|(A_\delta^T f)(t)|}{\phi(\delta)} = \infty,$$

then we have

$$\limsup_{\delta \rightarrow 0} \frac{|(A_\delta^T f)(t)|}{\phi(\delta)} = \infty.$$

In Lemma 3, which is the key lemma, we use a modified version of the threshold operator. In contrast to the threshold operator that sets all sam-

ples whose absolute value is smaller than δ to zero, we consider a threshold operator that sets all samples whose absolute value is smaller than or equal to δ to zero. This operator leads to the sampling series

$$(\bar{A}_\delta^T f)(t) := \sum_{\substack{k=-\infty \\ |f(k)| > \delta}}^{\infty} f(k)h_T(t-k) \quad (5.1)$$

and the set

$$\mathcal{D}_2(T, t, \phi) := \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta \rightarrow 0} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} = \infty\}.$$

Lemma 2 connects the sets $\mathcal{D}_1(T, t, \phi)$ and $\mathcal{D}_2(T, t, \phi)$.

Lemma 2. *Let T be an stable LTI system, $t \in \mathbb{R}$, and $\phi \in \Phi$. We have $\mathcal{D}_1(T, t, \phi) = \mathcal{D}_2(T, t, \phi)$.*

Now we are in the position to state the key lemma.

Lemma 3. *Let T be a stable LTI system, $t \in \mathbb{R}$, and $\phi \in \Phi$. If*

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$$

then, for all $M \in \mathbb{N}$,

$$\mathcal{D}_2(T, t, \phi, M) = \{f \in \mathcal{PW}_\pi^1 : \sup_{0 < \delta < 1} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} > M\}$$

is a residual set.

In order to improve the readability, we postpone the proofs of the Lemmas 2 and 3 and start with the proof of Theorem 2.

Proof of Theorem 2. Let $\phi \in \Phi$ be arbitrary but fixed.

We prove the “ \Rightarrow ” direction of the if and only if assertion by showing that $\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty$ implies $\mathcal{D}_1(T, t, \phi) = \emptyset$. Thus, let T be a stable LTI system and $t \in \mathbb{R}$ such that $\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty$. For all $\delta > 0$ and $f \in \mathcal{PW}_\pi^1$ we have

$$\begin{aligned} |(\bar{A}_\delta^T f)(t)| &\leq \sum_{\substack{k=-\infty \\ |f(k)| > \delta}}^{\infty} |f(k)h_T(t-k)| \\ &\leq \|f\|_{\mathcal{PW}_\pi^1} \sum_{k=-\infty}^{\infty} |h_T(t-k)| \\ &< \infty. \end{aligned}$$

This shows that

$$\limsup_{\delta \rightarrow 0} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} < \infty$$

for all $f \in \mathcal{PW}_\pi^1$. Thus, we have $\mathcal{D}_2(T, t, \phi) = \emptyset$, which in turn implies that $\mathcal{D}_1(T, t, \phi) = \emptyset$, because of Lemma 2.

Next, we prove the second assertion of the theorem, i.e., that $\mathcal{D}_1(T, t, \phi)$ is a residual set if $\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$. This also proves the “ \Leftarrow ” direction of the if and only if assertion. Let T be a stable LTI system and $t \in \mathbb{R}$ such that $\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$. From Lemma 3 we know that all sets $\mathcal{D}_2(T, t, \phi, M)$, $M \in \mathbb{N}$ are residual sets. It follows that

$$\{f \in \mathcal{PW}_\pi^1 : \sup_{0 < \delta < 1} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} = \infty\} = \bigcap_{M \in \mathbb{N}} \mathcal{D}_2(T, t, \phi, M)$$

is a residual set, because the countable intersection of residual sets is a residual set. Similar to Lemma 1, it can be shown that

$$\mathcal{D}_2(T, t, \phi) = \{f \in \mathcal{PW}_\pi^1 : \sup_{0 < \delta < 1} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} = \infty\}.$$

Finally, application of Lemma 2 completes the proof. \square

Proof of Lemma 2. Let $f \in \mathcal{D}_2(T, t, \phi)$ be arbitrary but fixed. By the definition of $\mathcal{D}_2(T, t, \phi)$, we have

$$\limsup_{\delta \rightarrow 0} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} = \infty.$$

Thus, for every $M > 0$ there exists a $\delta_M \in (0, 1)$ such that

$$\frac{|(\bar{A}_{\delta_M}^T f)(t)|}{\phi(\delta_M)} > M.$$

Let $\mathcal{T}(M) = \{k \in \mathbb{Z} : |f(k)| > \delta_M\}$ and $\underline{f}_M = \min_{k \in \mathcal{T}(M)} |f(k)|$. Then it follows that $\underline{f}_M > \delta_M$. For all $\tilde{\delta}_M$ with $\min\{\underline{f}_M, 1\} > \tilde{\delta}_M > \delta_M$ we have

$$\begin{aligned} (A_{\tilde{\delta}_M}^T f)(t) &= \sum_{\substack{k=-\infty \\ |f(k)| \geq \tilde{\delta}_M}}^{\infty} f(k) h_T(t-k) \\ &= \sum_{\substack{k=-\infty \\ |f(k)| > \delta_M}}^{\infty} f(k) h_T(t-k) \\ &= (\bar{A}_{\delta_M}^T f)(t). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \sup_{0 < \delta < 1} \frac{|(A_\delta^T f)(t)|}{\phi(\delta)} &\geq \frac{|(A_{\tilde{\delta}_M}^T f)(t)|}{\phi(\tilde{\delta}_M)} \\ &\geq \frac{|(\bar{A}_{\delta_M}^T f)(t)|}{\phi(\delta_M)} \\ &> M, \end{aligned} \tag{5.2}$$

where we used the fact that ϕ is monotonically decreasing in the second inequality. Since (5.2) is valid for all $M > 0$, it follows that

$$\sup_{0 < \delta < 1} \frac{|(A_\delta^T f)(t)|}{\phi(\delta)} = \infty,$$

and, due to Lemma 1, that

$$\limsup_{\delta \rightarrow 0} \frac{|(A_\delta^T f)(t)|}{\phi(\delta)} = \infty.$$

This shows that $f \in \mathcal{D}_1(T, t, \phi)$, which implies that $\mathcal{D}_2(T, t, \phi) \subset \mathcal{D}_1(T, t, \phi)$.

Next, we prove the converse inclusion, i.e., $\mathcal{D}_2(T, t, \phi) \supset \mathcal{D}_1(T, t, \phi)$. Let $f \in \mathcal{D}_1(T, t, \phi)$ be arbitrary but fixed. According to the definition of $\mathcal{D}_1(T, t, \phi)$ there exists a sequence $\{\delta_n\}_{n \in \mathbb{N}}$ of positive numbers, satisfying $1 > \delta_n > \delta_{n+1}$, $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} \delta_n = 0$ such that

$$\lim_{n \rightarrow \infty} \frac{|(A_{\delta_n}^T f)(t)|}{\phi(\delta_n)} = \infty.$$

Let $\mathcal{F} = \{f(k) : k \in \mathbb{Z}\}$. Since $f \in \mathcal{PW}_\pi^1$, we have $\lim_{|t| \rightarrow \infty} f(t) = 0$ on the real axis, which implies that zero is the only possible limit point of \mathcal{F} . Hence, for every $n \in \mathbb{N}$ there exists a $\rho_n > 0$ such that $\delta_n - \rho_n > \delta_{n+1}$,

$$\frac{2}{\phi(\delta_n - \rho_n)} \geq \frac{1}{\phi(\delta_n)}, \tag{5.3}$$

and $\mathcal{F} \cap (\delta_n - \rho_n, \delta_n) = \emptyset$. Thus, we have

$$\{k \in \mathbb{Z} : |f(k)| \geq \delta_n\} = \{k \in \mathbb{Z} : |f(k)| > \delta_n - \rho_n\},$$

and it follows that

$$\frac{2|(\bar{A}_{\delta_n - \rho_n}^T f)(t)|}{\phi(\delta_n - \rho_n)} = \frac{2|(A_{\delta_n}^T f)(t)|}{\phi(\delta_n - \rho_n)} \geq \frac{|(A_{\delta_n}^T f)(t)|}{\phi(\delta_n)},$$

where we used (5.3) in the last inequality. Consequently, we have

$$\lim_{n \rightarrow \infty} \frac{2|(\bar{A}_{\delta_n - \rho_n}^T f)(t)|}{\phi(\delta_n - \rho_n)} = \infty$$

which in turn implies that

$$\limsup_{\delta \rightarrow 0} \frac{|(\bar{A}_\delta^T f)(t)|}{\phi(\delta)} = \infty,$$

i.e., $f \in \mathcal{D}_2(T, t, \phi)$. \square

Proof of Lemma 3. Let $\phi \in \Phi$ and $M \in \mathbb{N}$ be arbitrary but fixed. Further, let T be a stable LTI system and $t \in \mathbb{R}$ such that $\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$.

We first show that $\mathcal{D}_2(T, t, \phi, M)$ is an open set. Let $f_1 \in \mathcal{D}_2(T, t, \phi, M)$ be arbitrary. We have to show that there exists an $\tilde{\epsilon} > 0$ such that, for any $f \in \mathcal{PW}_\pi^1$ with $\|f - f_1\|_{\mathcal{PW}_\pi^1} < \tilde{\epsilon}$ we have $f \in \mathcal{D}_2(T, t, \phi, M)$. By assumption, there exists a $0 < \delta_M < 1$ such that

$$\frac{|(\bar{A}_{\delta_M}^T f_1)(t)|}{\phi(\delta_M)} > M. \quad (5.4)$$

Furthermore, let $\mathcal{T}(M) = \{k \in \mathbb{Z} : |f_1(k)| > \delta_M\}$ and $\underline{f}_{1,M} = \min_{k \in \mathcal{T}(M)} |f_1(k)|$. Next, we choose $\tilde{\delta}_M = \delta_M + (\underline{f}_{1,M} - \delta_M)/2$. Then we have $\tilde{\delta}_M > \delta_M$ and

$$\{k \in \mathbb{Z} : |f_1(k)| > \tilde{\delta}_M\} = \mathcal{T}(M). \quad (5.5)$$

It follows that $(\bar{A}_{\tilde{\delta}_M}^T f_1)(t) = (\bar{A}_{\delta_M}^T f_1)(t)$. Further, since ϕ is monotonically decreasing, we have

$$|(\bar{A}_{\tilde{\delta}_M}^T f_1)(t)| - M\phi(\tilde{\delta}_M) > 0,$$

because of (5.4). Next, we choose some $\tilde{\epsilon}$ that satisfies

$$0 < \tilde{\epsilon} < \min \left(\frac{|(\bar{A}_{\tilde{\delta}_M}^T f_1)(t)| - M\phi(\tilde{\delta}_M)}{\|h_T\|_\infty |\mathcal{T}(M)|}, \tilde{\delta}_M - \delta_M \right). \quad (5.6)$$

Let $f \in \mathcal{PW}_\pi^1$ with $\|f_1 - f\|_{\mathcal{PW}_\pi^1} < \tilde{\epsilon}$ be arbitrary but fixed. We have $|f_1(k) - f(k)| < \tilde{\epsilon}$, $k \in \mathbb{Z}$. It follows, for all $k \in \mathbb{Z}$ with $|f(k)| > \tilde{\delta}_M$, that

$$|f_1(k)| \geq |f(k)| - |f(k) - f_1(k)| > \tilde{\delta}_M - \tilde{\epsilon} > \delta_M,$$

i.e., $k \in \mathcal{T}(M)$. Conversely, $k \in \mathcal{T}(M)$ implies $f_1(k) \geq \underline{f}_{1,M}$, and it follows that

$$\begin{aligned} |f(k)| &\geq |f_1(k)| - |f(k) - f_1(k)| > \underline{f}_{1,M} - \tilde{\epsilon} \\ &> \underline{f}_{1,M} - \tilde{\delta}_M + \delta_M = \tilde{\delta}_M. \end{aligned}$$

Thus we have

$$\{k \in \mathbb{Z} : |f(k)| > \tilde{\delta}_M\} = \mathcal{T}(M). \quad (5.7)$$

Moreover, using (5.5) and (5.7), we obtain that

$$\begin{aligned} & |(\bar{A}_{\tilde{\delta}_M}^T f)(t) - (\bar{A}_{\tilde{\delta}_M}^T f_1)(t)| \\ &= \left| \sum_{\substack{k=-\infty \\ |f(k)| > \tilde{\delta}_M}}^{\infty} f(k)h_T(t-k) - \sum_{\substack{k=-\infty \\ |f_1(k)| > \tilde{\delta}_M}}^{\infty} f_1(k)h_T(t-k) \right| \\ &\leq \sum_{k \in \mathcal{T}(M)} |f_1(k) - f(k)| |h_T(t-k)| \\ &\leq \tilde{\epsilon} \|h_T\|_{\infty} |\mathcal{T}(M)| \end{aligned}$$

and consequently

$$|(\bar{A}_{\tilde{\delta}_M}^T f)(t)| \geq |(\bar{A}_{\tilde{\delta}_M}^T f_1)(t)| - \tilde{\epsilon} \|h_T\|_{\infty} |\mathcal{T}(M)| > M\phi(\tilde{\delta}_M),$$

where the last inequality is due to (5.6). Therefore

$$\sup_{0 < \delta < 1} \frac{|(\bar{A}_{\delta}^T f)(t)|}{\phi(\delta)} > M,$$

i.e., $f \in \mathcal{D}_2(T, t, \phi, M)$, for all $f \in \mathcal{PW}_{\pi}^1$ with $\|f_1 - f\|_{\mathcal{PW}_{\pi}^1} < \tilde{\epsilon}$.

Second, we show that $\mathcal{D}_2(T, t, \phi, M)$ is dense in \mathcal{PW}_{π}^1 . Let $f \in \mathcal{PW}_{\pi}^1$ be arbitrary. We have to show that for every $\epsilon > 0$ there exists a $f_{\epsilon} \in \mathcal{D}_2(T, t, \phi, M)$ such that $\|f - f_{\epsilon}\|_{\mathcal{PW}_{\pi}^1} < \epsilon$. Let $\epsilon > 0$ be arbitrary but fixed. Since \mathcal{PW}_{π}^2 is dense in \mathcal{PW}_{π}^1 , there exists a $f_{\epsilon}^{(1)} \in \mathcal{PW}_{\pi}^2$ with

$$\|f - f_{\epsilon}^{(1)}\|_{\mathcal{PW}_{\pi}^1} < \frac{\epsilon}{3}. \quad (5.8)$$

Moreover, there exists a $f_{\epsilon}^{(2)} \in \mathcal{PW}_{\pi}^2$ such that $f_{\epsilon}^{(2)}(k) \neq 0$ only for finitely many $k \in \mathbb{Z}$ and

$$\|f_{\epsilon}^{(1)} - f_{\epsilon}^{(2)}\|_{\mathcal{PW}_{\pi}^1} < \frac{\epsilon}{3}. \quad (5.9)$$

Let $\mathcal{Z}^+ = \{k \in \mathbb{Z} : h_T(t-k) \geq 0\}$ and $\mathcal{Z}^- = \{k \in \mathbb{Z} : h_T(t-k) < 0\}$. Then we have

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \sum_{\substack{k=-\infty \\ k \in \mathcal{Z}^+}}^{\infty} h_T(t-k) + \sum_{\substack{k=-\infty \\ k \in \mathcal{Z}^-}}^{\infty} (-h_T(t-k)), \quad (5.10)$$

and, according to the assumption $\sum_{k=-\infty}^{\infty} |h_T(t-k)| = \infty$, at least one of the sums on the right-hand side of (5.10) must be infinity. Without loss of

generality, we assume that

$$\sum_{\substack{k=-\infty \\ k \in \mathcal{Z}^+}}^{\infty} h_T(t-k) = \infty. \quad (5.11)$$

Let N denote the smallest natural number such that $f_\epsilon^{(2)}(k) = 0$ for all $|k| > N$. For $0 < \eta < 1$ and $L \in \mathbb{N}$, $L > N$, consider the function

$$h^+(\tau, \eta, L) := \sum_{k=-2L+1}^{2L-1} h^+(k, \eta, L) \frac{\sin(\pi(\tau-k))}{\pi(\tau-k)},$$

where

$$h^+(k, \eta, L) = \begin{cases} 1 + \eta, & k \in \mathcal{Z}^+ \cap [-L, L], \\ 1 - \eta, & k \in \mathcal{Z}^- \cap [-L, L], \\ 2 - \frac{|k|}{L}, & L < |k| < 2L. \end{cases}$$

We have

$$h^+(\tau, \eta, L) = h^+(\tau, 0, L) + \eta \underbrace{\sum_{\substack{k=-L \\ k \in \mathcal{Z}^+}}^L \frac{\sin(\pi(\tau-k))}{\pi(\tau-k)}}_{=: u^+(\tau, L)} - \eta \underbrace{\sum_{\substack{k=-L \\ k \in \mathcal{Z}^-}}^L \frac{\sin(\pi(\tau-k))}{\pi(\tau-k)}}_{=: u^-(\tau, L)},$$

and it follows that

$$\begin{aligned} & \|h^+(\cdot, \eta, L)\|_{\mathcal{PW}_\pi^1} \\ & \leq \|h^+(\cdot, 0, L)\|_{\mathcal{PW}_\pi^1} + \eta \|u^+(\cdot, L)\|_{\mathcal{PW}_\pi^1} + \eta \|u^-(\cdot, L)\|_{\mathcal{PW}_\pi^1}. \end{aligned}$$

Since $\|h^+(\cdot, 0, L)\|_{\mathcal{PW}_\pi^1} < 3$ [4], and $\|u^+(\cdot, L)\|_{\mathcal{PW}_\pi^1} < \infty$ as well as $\|u^-(\cdot, L)\|_{\mathcal{PW}_\pi^1} < \infty$ for all $L \in \mathbb{N}$, there exists a real number $\eta_0(L)$ with $0 < \eta_0(L) < 1$ such that

$$\|h^+(\cdot, \eta_0(L), L)\|_{\mathcal{PW}_\pi^1} < 3.$$

Next, consider the function

$$\tilde{h}^+(\tau, \eta, L) = h^+(\tau, \eta, L) - v^+(\tau, \eta)$$

with

$$\begin{aligned} v^+(\tau, \eta) &= (1 + \eta) \sum_{\substack{k=-N \\ k \in \mathcal{Z}^+}}^N \frac{\sin(\pi(\tau-k))}{\pi(\tau-k)} + (1 - \eta) \sum_{\substack{k=-N \\ k \in \mathcal{Z}^-}}^N \frac{\sin(\pi(\tau-k))}{\pi(\tau-k)} \\ &= (1 + \eta)u^+(\tau, N) + (1 - \eta)u^-(\tau, N), \end{aligned}$$

where N denotes the smallest natural number such that $f_\epsilon^{(2)}(k) = 0$ for all $|k| > N$. We have

$$\begin{aligned} \|v^+(\cdot, \eta)\|_{\mathcal{PW}_\pi^1} &\leq (1 + \eta)\|u^+(\cdot, N)\|_{\mathcal{PW}_\pi^1} + (1 - \eta)\|u^-(\cdot, N)\|_{\mathcal{PW}_\pi^1} \\ &\leq 2\|u^+(\cdot, N)\|_{\mathcal{PW}_\pi^1} + \|u^-(\cdot, N)\|_{\mathcal{PW}_\pi^1}, \end{aligned}$$

and consequently

$$\begin{aligned} \|\tilde{h}^+(\cdot, \eta_0(L), L)\|_{\mathcal{PW}_\pi^1} &\leq \|h^+(\cdot, \eta_0(L), L)\|_{\mathcal{PW}_\pi^1} + \|v^+(\cdot, \eta_0(L))\|_{\mathcal{PW}_\pi^1} \\ &< 3 + 2\|u^+(\cdot, N)\|_{\mathcal{PW}_\pi^1} + \|u^-(\cdot, N)\|_{\mathcal{PW}_\pi^1} =: C_2, \end{aligned}$$

where the constant C_2 is independent of L . Let $\mathcal{T}_2 = \{k \in \mathbb{Z} : |f_\epsilon^{(2)}(k)| \neq 0\}$ and $\underline{f}_\epsilon^{(2)} = \min_{k \in \mathcal{T}_2} |f_\epsilon^{(2)}(k)|$. Next, we analyze

$$G_\epsilon(t, L) = f_\epsilon^{(2)}(t) + \mu \tilde{h}^+(t, \eta_0(L), L),$$

where $\mu > 0$ is some real number that satisfies

$$\mu < \min \left\{ \frac{\epsilon}{3C_2}, \underline{f}_\epsilon^{(2)}, 1 \right\}.$$

By the choice of μ we have

$$\|f_\epsilon^{(2)} - G_\epsilon(\cdot, L)\|_{\mathcal{PW}_\pi^1} = \mu C_2 < \frac{\epsilon}{3} \quad (5.12)$$

for all $L > N$. Combining (5.8), (5.9), and (5.12), we see that

$$\|f - G_\epsilon(\cdot, L)\|_{\mathcal{PW}_\pi^1} < \epsilon \quad (5.13)$$

for all $L > N$, i.e., $G_\epsilon(\cdot, L)$ lies in the ϵ -ball around f . Further, for any $L > N$ we can find a $\delta_0(L)$ that fulfills

$$\max \left\{ (1 - \eta_0(L))\mu, \left(1 - \frac{1}{L}\right)\mu \right\} < \delta_0(L) < \mu. \quad (5.14)$$

Since $\delta_0(L) < \underline{f}_\epsilon^{(2)}$, by the definition of μ , it follows that

$$\begin{aligned} &(\bar{A}_{\delta_0(L)}^T G_\epsilon(\cdot, L))(t) \\ &= \sum_{\substack{k=-N \\ |G_\epsilon(k, L)| > \delta_0(L)}}^N G_\epsilon(k, L) h_T(t - k) + \sum_{\substack{N < |k| \leq L \\ |G_\epsilon(k, L)| > \delta_0(L)}} G_\epsilon(k, L) h_T(t - k) \\ &= \sum_{k=-N}^N f_\epsilon^{(2)}(k) h_T(t - k) + \mu(1 + \eta_0(L)) \sum_{\substack{N < |k| \leq L \\ k \in \mathcal{Z}^+}} h_T(t - k) \\ &= (T f_\epsilon^{(2)})(t) + \mu(1 + \eta_0(L)) \sum_{\substack{N < |k| \leq L \\ k \in \mathcal{Z}^+}} h_T(t - k). \end{aligned}$$

Hence, we have

$$\begin{aligned} |(\bar{A}_{\delta_0(L)}^T G_\epsilon(\cdot, L))(t)| &\geq \mu(1 + \eta_0(L)) \sum_{\substack{N < |k| \leq L \\ k \in \mathcal{Z}^+}} h_T(t - k) - |(Tf_\epsilon^{(2)})(t)| \\ &\geq \mu \sum_{\substack{N < |k| \leq L \\ k \in \mathcal{Z}^+}} h_T(t - k) - |(Tf_\epsilon^{(2)})(t)|, \end{aligned}$$

and consequently

$$\begin{aligned} \frac{|(\bar{A}_{\delta_0(L)}^T G_\epsilon(\cdot, L))(t)|}{\phi(\delta_0(L))} &\geq \frac{\mu}{\phi(\delta_0(L))} \sum_{\substack{N < |k| \leq L \\ k \in \mathcal{Z}^+}} h_T(t - k) - \frac{|(Tf_\epsilon^{(2)})(t)|}{\phi(\delta_0(L))} \\ &\geq \frac{\mu}{\phi(\mu/2)} \sum_{\substack{N < |k| \leq L \\ k \in \mathcal{Z}^+}} h_T(t - k) - |(Tf_\epsilon^{(2)})(t)|. \quad (5.15) \end{aligned}$$

where we used the fact that $\phi(\delta_0(L)) \geq 1$, which follows from $\phi(\delta) \geq 1$ for all $0 < \delta \leq 1$ and $0 < \delta_0(L) < 1$, and the fact that $\phi(\delta_0(L)) \leq \phi(\mu/2)$, which follows from (5.14), $L \geq 2$, and the monotonicity of ϕ . Due to the assumption (5.11), the right-hand side of (5.15) can be made arbitrarily large by choosing L large. Let $L_1 > N$ be the smallest L such that the right hand side of (5.15) is larger than M . It follows that $f_\epsilon = G_\epsilon(\cdot, L_1)$ is the desired function, because

$$\sup_{0 < \delta < 1} \frac{|(\bar{A}_\delta^T f_\epsilon)(t)|}{\phi(\delta)} \geq \frac{|(\bar{A}_{\delta_0(L_1)}^T f_\epsilon)(t)|}{\phi(\delta_0(L_1))} > M,$$

i.e., $f_\epsilon \in \mathcal{D}_2(T, t, \phi, M)$, and because $\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$, according to (5.13). \square

Next, we want to apply Theorem 2. For the LTI system $T = Id$, where Id denotes the identity operator, we have $h_T = \text{sinc}$ and thus obtain, as a special case of (4.1), the sampling series

$$(A_\delta f)(t) := (A_\delta^{Id} f)(t) = \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} f(k) \frac{\text{sinc}(\pi(t - k))}{\pi(t - k)},$$

which is the Shannon sampling series that uses only the samples that are larger than or equal to the threshold δ . Since

$$\sum_{k=-\infty}^{\infty} |\text{sinc}(t - k)| = \infty$$

for all $t \in \mathbb{R} \setminus \mathbb{Z}$, the next corollary is an immediate consequence of Theorem 2.

Corollary 1. *Let $t \in \mathbb{R} \setminus \mathbb{Z}$ and $\phi \in \Phi$. Then*

$$\{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta \rightarrow 0} \frac{|(A_\delta f)(t)|}{\phi(\delta)} = \infty\}$$

is a residual set.

Corollary 1 shows, in particular, that for fixed $t \in \mathbb{R} \setminus \mathbb{Z}$ there exists a function $f \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{\delta \rightarrow 0} \frac{|(A_\delta f)(t)|}{\phi(\delta)} = \infty. \quad (5.16)$$

The next corollary strengthens this assertion. It states that there exists a universal function $f \in \mathcal{PW}_\pi^1$ such that we have divergence as in (5.16) for all $t \in \mathbb{R} \setminus \mathbb{Z}$.

Remark 1. Note that the technique from the proof of Theorem 2, where we took the countable intersection of residual sets, cannot be used here because

$$\bigcap_{t \in \mathbb{R} \setminus \mathbb{Z}} \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta \rightarrow 0} \frac{|(A_\delta f)(t)|}{\phi(\delta)} = \infty\}$$

is an uncountable intersection of residual sets, which is not necessarily a residual set again.

Corollary 2. *Let $\phi \in \Phi$. Then*

$$\{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta \rightarrow 0} \frac{|(A_\delta f)(t)|}{\phi(\delta)} = \infty \text{ for all } t \in \mathbb{R} \setminus \mathbb{Z}\}$$

is a residual set.

Proof. Let $t_1 \in \mathbb{R} \setminus \mathbb{Z}$ and $\phi \in \Phi$ be arbitrary but fixed. According to Corollary 1, there exists a residual set $G \subset \mathcal{PW}_\pi^1$ such that

$$\limsup_{\delta \rightarrow 0} \frac{|(A_\delta f)(t_1)|}{\phi(\delta)} = \infty$$

for all $f \in G$. We further have, for $t_2 \in \mathbb{R} \setminus \mathbb{Z}$, $0 < \delta < 1$, and $f \in G$ that

$$\begin{aligned} & \left| \frac{(A_\delta f)(t_1)}{\phi(\delta) \sin(\pi t_1)} - \frac{(A_\delta f)(t_2)}{\phi(\delta) \sin(\pi t_2)} \right| \\ &= \frac{1}{\phi(\delta) \pi} \left| \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} f(k) \frac{(-1)^k}{t_1 - k} - \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} f(k) \frac{(-1)^k}{t_2 - k} \right| \\ &\leq \frac{\|f\|_{\mathcal{PW}_\pi^1}}{\pi} \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} \frac{|t_2 - t_1|}{|t_1 - k| |t_2 - k|} =: C_3(t_1, t_2, f), \end{aligned}$$

where $C_3(t_1, t_2, f) < \infty$ is a constant that depends only on t_1 , t_2 , and f . It follows that

$$\frac{|(A_\delta f)(t_2)|}{\phi(\delta)} \geq \frac{|(A_\delta f)(t_1)|}{\phi(\delta)} \left| \frac{\sin(\pi t_2)}{\sin(\pi t_1)} \right| - C_4(t, t_2, f) \quad (5.17)$$

for all $t_2 \in \mathbb{R} \setminus \mathbb{Z}$, $0 < \delta < 1$, $f \in G$. Taking the limit superior on both sides of (5.17) gives

$$\limsup_{\delta \rightarrow 0} \frac{|(A_\delta f)(t_2)|}{\phi(\delta)} = \infty$$

for all $t_2 \in \mathbb{R} \setminus \mathbb{Z}$ and all $f \in G$. \square

6. Behavior of the L^∞ -Norm

In this section we study the behavior of $\|A_\delta^T f\|_\infty$, i.e., the L^∞ -norm of the approximation process, as the threshold δ is decreased to zero. The set of interest in this case is

$$\mathcal{D}_1^\infty(T, \phi) = \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta \rightarrow 0} \frac{\|A_\delta^T f\|_\infty}{\phi(\delta)} = \infty\}.$$

Theorem 3. *Let T be a stable LTI system and $\phi \in \Phi$. Then we have $\mathcal{D}_1^\infty(T, \phi) \neq \emptyset$ if and only if $h_T \notin \mathcal{B}_\pi^1$. Further, if $h_T \notin \mathcal{B}_\pi^1$ then $\mathcal{D}_1^\infty(T, \phi)$ is a residual set.*

For the proof of Theorem 3 we need the following lemma.

Lemma 4. *Let $h \in \mathcal{B}_\pi$. If*

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad (6.1)$$

and

$$\sum_{k=-\infty}^{\infty} \left| h\left(k + \frac{1}{2}\right) \right| < \infty \quad (6.2)$$

then we have $h \in \mathcal{B}_\pi^1$.

Lemma 4 follows directly from the fact that conditions (6.1) and (6.2) together correspond to oversampling with oversampling factor 2.

Proof of Theorem 3. Let $\phi \in \Phi$ be arbitrary but fixed.

We prove the “ \Rightarrow ” direction of the if and only if assertion by showing that $h_T \in \mathcal{B}_\pi^1$ implies $\mathcal{D}_1^\infty(T, \phi) = \emptyset$. Thus, let T be a stable LTI system such that $h_T \in \mathcal{B}_\pi^1$. For all $\delta > 0$ and $f \in \mathcal{PW}_\pi^1$ we have

$$\begin{aligned} |(A_\delta^T f)(t)| &\leq \sum_{\substack{k=-\infty \\ |f(k)| \geq \delta}}^{\infty} |f(k)h_T(t-k)| \\ &\leq \|f\|_{\mathcal{PW}_\pi^1} \sum_{k=-\infty}^{\infty} |h_T(t-k)|. \end{aligned}$$

It follows, using Nikol’skii’s inequality [13, p. 49], that

$$\begin{aligned} \|A_\delta^T f\|_\infty &\leq \|f\|_{\mathcal{PW}_\pi^1} \sup_{t \in \mathbb{R}} \sum_{k=-\infty}^{\infty} |h_T(t-k)| \\ &\leq \|f\|_{\mathcal{PW}_\pi^1} \|h_T\|_1 \\ &< \infty, \end{aligned}$$

which implies $\mathcal{D}_1^\infty(T, \phi) = \emptyset$.

Next, we prove the second assertion of the theorem, i.e., that $\mathcal{D}_1^\infty(T, \phi)$ is a residual set if $h_T \notin \mathcal{B}_\pi^1$. This also proves the “ \Leftarrow ” direction of the if and only if assertion. Let the stable LTI system T be such that $h_T \notin \mathcal{B}_\pi^1$. Then we have

$$\sum_{k=-\infty}^{\infty} |h_T(k)| = \infty$$

or

$$\sum_{k=-\infty}^{\infty} \left| h_T \left(k + \frac{1}{2} \right) \right| = \infty,$$

according to Lemma 4. From Theorem 2 it follows that $\mathcal{D}_1(T, 0, \phi)$ or $\mathcal{D}_1(T, 1/2, \phi)$ is a residual set, which in turn implies that $\mathcal{D}_1^\infty(T, \phi)$ is a residual set. \square

The proof of Theorem 3 has also revealed the following corollary.

Corollary 3. *Let T be a stable LTI system and $\phi \in \Phi$. Then we have $\mathcal{D}_1^\infty(T, \phi) \neq \emptyset$ if and only if*

$$\sum_{k=-\infty}^{\infty} |h_T(k)| = \infty \quad \text{or} \quad \sum_{k=-\infty}^{\infty} \left| h_T \left(k + \frac{1}{2} \right) \right| = \infty.$$

Moreover, if $\mathcal{D}_1^\infty(T, \phi) \neq \emptyset$ then $\mathcal{D}_1^\infty(T, \phi)$ is a residual set.

7. Discussion

Corollary 3 and Theorem 2 together show the significant difference between the approximation behavior of A_δ^T and the approximation behavior of the Shannon sampling series

$$(S_N f)(t) := \sum_{k=-N}^N f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)},$$

which is described by Brown's theorem.

Although the L^∞ -norm of the Shannon sampling series $\|S_N f\|_\infty$ diverges for certain functions in \mathcal{PW}_π^1 , we still have, for fixed $t \in \mathbb{R}$, convergence for all functions in \mathcal{PW}_π^1 . In contrast, the divergence of the L^∞ -norm of $A_\delta^T f$ for one function $f \in \mathcal{PW}_\pi^1$ results in the divergence of $(A_\delta^T f)(t)$ for $t = 0$ or $t = 1/2$ and all functions from a residual set.

As for the divergence speed, we have the following difference. In [3] it was shown for the Shannon sampling series that there exists a constant $C_5 > 0$ such that for all $f \in \mathcal{PW}_\pi^1$ we have $\|S_N f\|_\infty \leq C_5 \log(N+1) \|f\|_{\mathcal{PW}_\pi^1}$ for all $N \in \mathbb{N}$, i.e., the growth speed of $\|S_N f\|_\infty$ is bounded above and cannot be arbitrarily fast. This is contrast to the approximation process $A_\delta^T f$ where the divergence can be arbitrarily fast as Theorems 2 and 3 have shown.

Further, since

$$\sum_{k=-\infty}^{\infty} |h_T(k)| = \infty \quad \text{or} \quad \sum_{k=-\infty}^{\infty} \left| h_T \left(k + \frac{1}{2} \right) \right| = \infty$$

implies, for all $t \in \mathbb{R}$, that

$$\sum_{k=-\infty}^{\infty} |h_T(t+k)| = \infty \quad \text{or} \quad \sum_{k=-\infty}^{\infty} \left| h_T \left(t + k + \frac{1}{2} \right) \right| = \infty,$$

we have the interesting situation that the divergence of the L^∞ -norm of $A_\delta^T f$ for one function $f \in \mathcal{PW}_\pi^1$ implies that, for all $\phi \in \Phi$ and all $t \in \mathbb{R}$, $D_1(T, \phi, t)$ or $D_1(T, \phi, t + 1/2)$ is a residual set.

Greedy approximation [20, 21, 16, 17, 14, 15] is a topic which seems to be related to the approximation with thresholding that is studied in this paper. However, in greedy approximations the truncation is usually performed in the frequency domain and divergence in the frequency domain does not always translate to divergence in the time domain. For example, it is easy to construct a sequence of \mathcal{PW}_π^2 -functions $\{f_n\}_{n \in \mathbb{N}}$ with uniformly bounded \mathcal{PW}_π^2 -norm, for which the corresponding sequence of Fourier transforms $\{\hat{f}_n\}_{n \in \mathbb{N}}$ diverges everywhere in $[-\pi, \pi]$. The uniform boundedness of the \mathcal{PW}_π^2 -norm implies the uniform boundedness of the $L^\infty(\mathbb{R})$ -norm. Hence, the results in greedy approximations cannot be simply transferred to our

problem. In [16, 17] the pointwise divergence of greedy approximations in the frequency domain is analyzed. This kind of approximation is interesting because it seems that there is a connection to the system approximation problem, where the impulse response is disturbed by the threshold operator, for signals in \mathcal{PW}_π^2 . Clearly, we cannot expect a pointwise divergence of the system approximation process in this setting, however, a decrease of the threshold will lead to an output signal of the system approximation that is worse concentrated in the time domain. Future research could focus on this aspect. A more detailed discussion about greedy approximation and its relation to approximation with thresholding can be found in [6].

In general, oversampling is suitable for improving the convergence behavior of approximation processes. However, it is known that, for certain stable LTI systems, oversampling cannot remove the divergence of the approximation process with thresholding [5]. It is an interesting open problem to characterize the systems that can be stably approximated with approximation processes that use oversampling.

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