

DUALITY FOR PROJECTIVE VARIETIES

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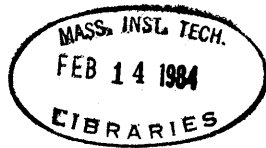
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ABSTRACT

We develop the theory of duality for projective varieties defined over fields of arbitrary characteristic. A central concept in this work is that of reflexivity and our main tool is the rank of certain local Hessians which provides a numerical criterion for reflexivity. Many of our results are necessary and sufficient conditions for reflexivity. We also analyze the reflexivity of a general hypersurface section of a given variety. Toward the classification of non-reflexive varieties, we determine all smooth in codimension one hypersurfaces with rank zero local Hessians and we solve the classification problem for a special class of these varieties.

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1. INTRODUCTION

The theory of the duality of projective varieties is a subject whose time has come again. It began in classical antiquity as the theory of polar reciprocation, in which a point and a line in the same plane correspond through the mediation of a conic, and it progressed for centuries as part of the theory of conics. With the intense development of geometry that began around the turn of the 19th century, the theory changed in two important ways: the mediation of the conic was eliminated, and the theory became a central part of algebraic geometry. It remained central throughout the 19th century, but then, for about fifty years, as interest in the intrinsic properties of algebraic varieties grew, interest in duality waned. In the mid 1950's the current renaissance began, and with every passing year there have been concurrently greater awareness, greater development, and greater application of the theory.

The current revival of activity in the theory of the duality, and in the larger theory of the conormal variety, is due mainly to their importance in four areas: (1) Lefschetz theory -- whose needs inspired the contributions of Wallace [1956] and [1958], Moishezon [1967] and Katz [1973]; (2) the theory of the ranks and the classification of varieties embedded in a projective space -- some recent contributions here are found in Mumford [1978], Piene [1978], Kleiman [1981], Lascoux [1981], Urabe [1981], and Holme [1978], Griffiths-Harris [1979], Fujita-Roberts [1981], Zak [1983], Pardini [1983], Lanteri [1984], Lanteri-Struppa [1984]; (3) the theory of singularities, stratifications, and constructible functions -- this area is nicely surveyed in Merle [1982], Teissier [1983]; and finally, (4) the needs of enumerative geometry, which inspired the development of the subject in Hefez-Sacchiero [1983], in Kleiman [1984], and in the present work.

Consider a reduced embedded projective variety defined over an algebraically closed field of arbitrary characteristic. Its conormal variety may be defined as the closure in the graph of the point-hyperplane incidence correspondence of the set of all point-hyperplane pairs such that the point is a simple point of the variety and the hyperplane contains the tangent space at that point. The dual variety is defined as the image in the dual projective space of the conormal variety. The original variety is called reflexive if its conormal variety is equal to that of the dual variety. In other words, reflexivity means that a hyperplane H is tangent at a point P if and only if in the dual projective space the hyperplane corresponding to P is tangent to the dual variety at the point corresponding to H . Lastly, the original variety will be called ordinary if it is reflexive and if its dual is a hypersurface. All these notions are defined more formally and in greater generality in Section 2.

Reflexivity is a stronger and more useful condition on the variety than the condition that it be equal to its dual's dual. Reflexivity is a non-trivial condition. For instance, in characteristic $p > 0$, the smooth curve with equation $y = x^{p+1} + x^p$ is not reflexive, but it is equal to its dual's dual; see Wallace [1956], last section. Reflexivity is the rule. Indeed, there are many theorems giving frequently satisfied conditions that imply reflexivity. For example, the Segre-Wallace criterion, recalled in Discussion (2.4,iv), asserts that a variety is reflexive when the characteristic is zero and the variety is irreducible and whenever else the map from the conormal variety to the dual variety is separable (that is, smooth on a dense open subset of the conormal variety). Several theorems giving conditions that imply reflexivity are main results in the present work, and they will be introduced below.

Two of the main results of the present work were announced previously. The first is Theorem (4.13), and it concerns the ranks of an irreducible embedded projective variety. The ranks are fundamental

extrinsic invariants. They appear, notably, in the contact formula, which expresses the number of n -folds in an r -parameter family that touch r given varieties in general position just in terms of the ranks of the given varieties and the characteristic numbers of the family; see Fulton-Kleiman-MacPherson [1983]. Now, Theorem (4.13) asserts that, if the variety is irreducible, then the i th rank is nonzero precisely when i lies in a certain interval. This result is relatively easy to prove, and it is surprising perhaps that it was not found before. The proof proceeds by assuming the result for a hyperplane section by induction on the dimension of the variety.

The second result that was previously announced is now part of Theorem (5.9). This result was used in an essential way in Fulton-Kleiman-MacPherson [1983]; namely, it was used to show that, in the enumeration of the contact formula, bitangencies never occur if the characteristic is 0 nor, more generally, if the varieties in question are reflexive and the characteristic is different from 2. The result itself asserts that a general hyperplane section of a reflexive variety (of dimension at least 1) is again reflexive, unless the characteristic is 2 and some component of the dual variety is a hypersurface. In addition, Theorem (5.9) asserts a converse; namely, in any characteristic, the original variety is reflexive if the hyperplane section is reflexive and if no component of the dual variety is a hypersurface.

Theorem (5.9), as stated so far, is sharp. Indeed, in any characteristic $p > 0$, there is a variety such that the dual variety is a hypersurface and the section is ordinary yet the variety is not ordinary; see Remark (5.11,i). Moreover, if the characteristic is 2 and if the variety is ordinary, then necessarily the hyperplane section is not reflexive, although the section's dual is necessarily a hypersurface; see Theorem (5.9,ii). Nevertheless, something more can be said, and this is done in Section 5 through the introduction and study of a new notion, semi-ordinariness. In particular, Theorem (5.9,ii) asserts in addition that every component of the section is

semi-ordinary, and Theorem (5.9,iv) asserts that, if every component of the variety is semi-ordinary, then the section is ordinary. Hence, as Corollary (5.10) states, in any characteristic, a general hyperplane section of an ordinary variety (of dimension at least 2) is again ordinary.

It is furthermore reasonable to conjecture that, given an ordinary variety in characteristic 2, a general hyperplane section's general tangent hyperplane is tangent at a unique point of contact, as is the case in all other characteristics. This conjecture implies that, in characteristic 2 as well, there are no bitangencies in the enumeration of the contact formula; see Fulton-Kleiman-MacPherson [1983].

Theorem (4.10), a third main result, gives two properties that together characterize ordinariness. The first is a well-known property of an ordinary variety X (cf. Wallace [1958], Lemma d, p. 5); namely, the dual of the section of any component of X by a general hyperplane M is equal to the cone of tangent lines to the dual of that component drawn from the point representing M . The second property of X is that the tangent hyperplane at a general point of the dual of X is not a component of the dual of the section of X by its tangent hyperplane represented by that point. This property is established in characteristic different from 2 on the basis of Theorem (5.9,i); the proof will work without change in characteristic 2 if the above conjecture is established. Conversely, it is proved that, in any characteristic, these two properties imply ordinariness. The first property alone implies ordinariness if the characteristic is 0 but it does not do so more generally; see Remark (4.11).

Theorem (5.9) is supplemented by Theorem (5.6), a fourth main result. It asserts that, if the characteristic is different from 2 or if each component of the variety is of odd dimension, then the section by a general hypersurface of degree at least 2 is ordinary. This theorem too is sharp; namely, if the characteristic is 2 and the dimension of some component is even, then necessarily the section is not ordinary; see Corollary (3.4). In fact, the section's dual is

still a hypersurface; this fact was proved by Ein ([1984], II, Theorem (1.3)) in answer to a question posed by the present authors before they had proved Theorem (5.6). In Ein's proof the characteristic and the parity of the dimensions never enter the discussion. On the other hand, the proof of Theorem (5.6) is logically independent of Ein's work.

Theorem (3.5) is a fifth main result. Already in the special case of a plane curve, it appears to be new. In this case, the theorem asserts that, if the curve is not reflexive, then at a general point the order of contact of the tangent line is equal to the inseparable degree of the function field of the curve over that of its dual curve, and conversely. In the general case, it is necessary to assume that the dimension of the original irreducible variety X and the dimension of its dual variety sum to the dimension of the ambient projective space; in other words, each irreducible component V of the locus of points of contact of a general tangent hyperplane H is a component of the intersection of H and X . The reasonableness of this hypothesis is discussed in Remark (3.6). Now, Theorem (3.5) asserts that, if X is not reflexive, then the multiplicity of appearance of V in the intersection is equal to the inseparable degree of the function field of the conormal variety over that of the dual variety, and conversely. On the other hand, the multiplicity of appearance of V is equal to 2 if X is reflexive; this conclusion follows from Theorem (3.2) and Proposition (2.2,ii).

Section 4 contains some minor results relating the size and shape of the dual of a variety to that of a hypersurface section. These results and their proofs are for the most part not really new, but they are difficult to reference. At any rate, they are presented systematically, and they are needed in the proofs of some of the main theorems.

To prove the main results, two principal methods are used. One of these is a method of specialization, which is new in the theory of the duality. For example, Lemma (4.3) implies that there always exists a hyperplane that is tangent to the section of a given variety by a

general hypersurface M but that is nowhere tangent to the variety. This assertion is obvious in the case that M is the union of a general plane and a general hypersurface of degree one less than that of M ; in the proof, M is degenerated into such a union. Similarly, in the proofs of the parts of Theorems (5.6) and (5.9) that deal with the reflexivity of the section, M is degenerated into a convenient special hypersurface. In the same way, it is possible to generalize these theorems to the case so that M is replaced by a general complete intersection. To justify this method of specialization, the basic theory of Sections 2 and 3 is set over an arbitrary noetherian base scheme. As it happens, it takes no additional effort to develop the theory in this generality.

The second principal method of proof is a refinement of the mode of employment of the Hessian used by Wallace [1956] and by Katz [1973], who pioneered the theory of the duality in arbitrary characteristic. The basic properties of the Hessian are given in Propositions (2.2) and (2.6) and Theorem (3.2). These properties include bounds on the size of the Hessian, the openness of the condition that the bounds are achieved, and the connection with reflexivity and ordinariness. The special case of Theorem (3.2) in which the base is a field and in which the variety is a hypersurface was discovered by Wallace, that in which the dual variety is a hypersurface was discovered by Katz. Theorem (3.2) moreover relates the reflexivity (resp. ordinariness) of the general member of a family to that of a single member. The additional key to the task of relating the reflexivity (resp. ordinariness) of a variety to that of a general hypersurface section is Lemma (5.2), which compares the two corresponding Hessian ranks.

The first five sections are fruits of joint work with Steven Kleiman.

Section 6 contains a study of reflexiveness, ordinariness and quasi-ordinariness for Segre and Grassmann varieties. Our analysis for Segre varieties is complete and the results are summarized in Theorem (6.3). On the other hand, our analysis for Grassmannians is rather

incomplete because of the difficulty we found in computing the rank of certain matrices. We have answers in the case of Grassmannians of d planes in \mathbb{P}^n when d and n are odd, namely $G_{d,n}$ is ordinary, and when $d=1$ and n is even, in this case we find that $G_{1,n}$ is always reflexive and its dual is of codimension 3.

Section 7 deals with the hypersurfaces which are completely opposite to reflexive varieties, that is, hypersurfaces with rank zero local Hessians. The main result here is Theorem (7.4) which allows us to describe through Corollary (7.7) the hypersurfaces in characteristic $p > 2$, which are smooth in codimension 1 and with rank zero local Hessians. Namely they are of the form

$$Y_0^q P_0(Y_0^p, \dots, Y_{n+1}^p) + \dots + Y_{n+1}^q P_{n+1}(Y_0^p, \dots, Y_{n+1}^p) = 0, \quad (1.1)$$

where q is a positive power of p and the L_i 's are linear polynomials. The central result is Theorem (8.8) which describes the m -Jacobian schemes of this family. There is also in this section a new proof for an old theorem of H. Hasse, Theorem (8.10), which is fundamental for the next section.

Section 9 has as main result Corollary (9.11) which asserts that for fixed q , all non-singular hypersurfaces of type (1.1) are projectively equivalent.

The results stated in Theorem (7.4), in its corollaries, and in Corollary (9.11) in the case of plane curves and for $q = p$, were first proved by R. Pardini [1983].

Finally, in section 10, we illustrate our previous results with projective curves. The new result we have is Proposition (10.7) which describes the tangent cone at a point of the dual of a projective curve.

2. BASIC THEORY

(2.1) Setup. Fix a noetherian ground scheme S .

Let A be an S -scheme of finite type, k a field, and x a k -point of A . For each subscheme C of A containing x , set

$$h(C/A/S;x) = \dim_x A/S - \dim_k \Omega_{C/S}^1(x). \quad (2.1.1)$$

Let B be a subscheme of A containing x . Assume that B is defined in a neighborhood of x by a single equation,

$$B : f = 0 \quad \text{about } x.$$

Assume that A/S is smooth at x of relative dimension n . Then

$\text{Sing}(B)$

will denote the subscheme of B defined in a neighborhood of x by the $(n-1)$ th Fitting ideal of $\Omega_{B/S}^1$.

(2.2) Proposition. Let D_1, \dots, D_n be a basis of the first partial derivative operators on $\mathcal{O}_{A,x}$. Then

(i) $\text{Sing}(B) : f, D_1 f, \dots, D_n f = 0$ in A at x .

(ii) $h(\text{Sing}(B)/A/S;x) = \text{rank}[(D_j D_i f)(x)]$.

Proof. In view of the conormal-cotangent sheaf sequence of B in A , it is clear that $\text{Sing}(B)$ is defined in A at x by the vanishing of f and of its partial derivatives $D_i f$. The corresponding Jacobian matrix is then

$$[(D_i f)(x); (D_j D_i f)(x)].$$

Its rank is clearly equal to $h(\text{Sing}(B)/A/S;x)$ in view of the conormal sheaf-cotangent sheaf sequence of $\text{Sing}(B)$ in A . The first column of this matrix vanishes. Hence the assertion holds.

(2.3) Setup continued. Let E be a locally free sheaf on S of constant rank $N+1$. Denote by E^* the dual sheaf, and set

$$P = \mathbb{P}(E) \quad \text{and} \quad P^* = \mathbb{P}(E^*).$$

Let X be a closed subscheme of P . Denote by

$$X^{\text{sm}} = (X/S)^{\text{sm}}$$

the S -smooth locus. In Kleiman [1984], it was assumed that X^{sm} is dense in X ; in the present work, it has turned out to be more convenient to proceed without this hypothesis.

Let $C(X/P)$ or, for short, CX denote the conormal scheme. It is defined in terms of the normal sheaf $\underline{N}(X^{\text{sm}}/P)$ as the closure in $P \times P^*$ of

$$CX^{\text{sm}} = \mathbb{P}(\underline{N}(X^{\text{sm}}/P)(-1)) .$$

Let $(X/P)'$ or X' denote the scheme-theoretic image of CX in P^* ; it is the smallest closed subscheme of P^* factoring the projection. It is called the dual (or reciprocal) of X . Let

$$q : CX \rightarrow P \quad \text{and} \quad q' : CX \rightarrow P^*$$

denote the projections. The same symbols q and q' will also be used to denote the corresponding maps

$$q : CX \rightarrow X \quad \text{and} \quad q' : CX \rightarrow X'$$

when it is convenient and there is no possibility of confusion.

The conormal scheme CX' of X' may be viewed as another closed subscheme of $P \times P^*$. If

$$CX = CX' \quad ,$$

then X will be called reflexive in P . If X is reflexive and if X' is a hypersurface, then X will be called ordinary in P .

Let T be an S -scheme. Let H be a hyperplane of P_T . Then H corresponds to a T -point H' of P^* , and this (standard) correspondence is bijective. Moreover, if H' is viewed as a subscheme of P_T^* , rather than as a map of T into P^* , then H' is just the dual of H . Now, the scheme-theoretic fiber of CX over H' is embedded by the projection q_T into X_T , because CX lies in $X \times P^*$. The corresponding subscheme of X_T

$$X_H = q_T q_T'^{-1} H'$$

will be called the H-contact locus.

(2.4) Discussion. (i) Clearly CX^{sm} is the full inverse image $q^{-1}X^{\text{sm}}$ in CX ; in particular, it is a dense open subscheme. Clearly CX^{sm} is S -smooth of constant relative dimension $N-1$. Clearly, if X is reduced (resp. irreducible), then CX and X' are too.

(ii) Let $T \rightarrow S$ be an arbitrary base change. Let W be an irreducible component of X_T such that

$$W \cap (X^{\text{sm}})_T \neq \emptyset .$$

It is easy to prove, see Kleiman [1984], (3.9,ii), that then CW is an irreducible component of $(CX)_T$; whence, $W' \subset (X_T)'$.

Suppose that S is reduced. Then, by Kleiman [1984], (3.10) (adjusted to do away with the hypothesis that X^{sm} is dense and dominates S), there exists a topologically dense open subscheme S^0 of S such that, if the image of T lies in S^0 , then

$$C(X_T) = (CX)_T \text{ and } (X_T)' = (X')_T .$$

(Of course, if X^{sm} does not dominate S , then these schemes may be empty.) In particular, if $S = \text{Spec}(k)$ and k is a field, then the formation of CX and that of X' commute with any field extension K/k ;

whence, then X is reflexive (resp. ordinary) if and only if X_K is so.

(iii) Let X_1, X_2, \dots be the irreducible generically smooth components of X . It is clear that X is reflexive if and only if (a) each X_i is reflexive and (b) $X_i^! \subset X_j^!$ implies that $i = j$. Note that (b) is equivalent to (b') every component of CX dominates a component of X' .

(iv) The Segre-Wallace criterion asserts that X is reflexive if and only if the projection $q' : CX \rightarrow X'$ is smooth on a dense open subscheme of CX . In (4.4) of Kleiman [1984], the criterion is established under the hypothesis that X'^{sm} is dense in X' ; however, this hypothesis is automatically satisfied if q' is smooth on a dense open subscheme of CX because CX^{sm} is a dense open subscheme that is S -smooth.

(v) The following statements are equivalent:

- (a) X is ordinary.
- (b) Each generically smooth component of X is ordinary.
- (c) The projection $q' : CX \rightarrow X'$ is birational.
- (d) q' is etale on a dense open subscheme of CX' .

Indeed, (a) and (b) are equivalent by (iii), because (iii,b) obviously holds when the X_i are ordinary. Now, (a) implies (c) because the normal sheaf of X'^{sm} in P^* is invertible. Trivially, (c) implies (d). Finally, (d) implies (a) by the Segre-Wallace criterion (iv).

(2.5) Lemma. Let T be an S -scheme, and H a hyperplane of P_T . Then

$$X_H \cap (X^{\text{sm}})_T = \text{Sing}((X^{\text{sm}})_T \cap H) \quad (2.5.1)$$

as schemes, where the scheme structure on the right is defined at a point x by the $(\dim_x X/S - 1)$ -th Fitting ideal of the sheaf of differentials.

Proof. It suffices to prove that

$$CX^{\text{sm}} = \text{Sing}(((X^{\text{sm}} \times P^*) \cap I)/P^*) \quad (2.5.2)$$

where I is the universal hyperplane on $P \times P^*$ (that is, the graph of the point-hyperplane incidence correspondence), because (2.5.1) may be obtained from (2.5.2) by taking the fiber of each side over the T -point H' of P^* representing H .

To prove (2.5.2), it suffices to work locally on S ; so we may assume that S is affine and that $P = \mathbb{P}_S^N$. Let t_0, \dots, t_N be a system of homogeneous coordinates for P , and let t_0^*, \dots, t_N^* be the dual system for P^* . Then I is given by

$$I : t_0 t_0^* + \dots + t_N t_N^* = 0 . \quad (2.5.3)$$

To prove (2.5.2), it suffices to show that each point x of X^{sm} has a neighborhood in X over which (2.5.2) holds. Set

$$m = N - \dim_x(X/S)$$

and let F_1, \dots, F_m be homogeneous polynomials in t_0, \dots, t_N that define X in a neighborhood U of x in P .

Let D_0, \dots, D_N denote the partial-derivative operators with respect to t_0, \dots, t_N . Consider the $(m+1) \times (m+1)$ -minors of the matrix

$$\begin{bmatrix} t_0^* & \dots & t_N^* \\ D_0 F_1 & \dots & D_N F_1 \\ \vdots & \ddots & \vdots \\ D_0 F_m & \dots & D_N F_m \end{bmatrix} .$$

On the one hand, these minors define

$$CX^{\text{sm}} \cap (U \times P^*)$$

in $U \times P^*$, because their vanishing is just the condition that the first row of the matrix belong to the space spanned by the remaining rows, which is just the conormal space in question. On the other hand, these minors define the singular locus of the P^* -scheme

$$((X^{\text{sm}} \cap U) \cap P^*) \cap I ,$$

because the matrix is just the Jacobian matrix of a system of $m+1$ equations defining the scheme in $U \times P^*$.

(2.6) Proposition. Let (x, H') be a k -point of CX^{sm} where k is a field. Denote by H the hyperplane of P_k corresponding to H' , and set

$$h(x, H) = h(\text{Sing}(X_k \cap H)/X_k/k; x) .$$

- (i) Then $h(x, H) \leq [\dim_x X/S + \dim_{H'} X'/S - (N-1)] \leq \dim_x X/S$.
- (ii) There is an open (possibly empty) subset RX (resp. OX) of CX^{sm} such that (x, H') belongs to it if and only if in (i) the first and second (resp. first and third) terms are equal.
- (iii) The image in S of any open subset of RX (resp. of OX) is open.
- (iv) If X is reflexive (resp. ordinary), then RX (resp. OX) is dense in CX .
- (v) If RX (resp. OX) is dense in CX and if the open subset of CX on which $q' : CX \rightarrow X'$ is flat is dense, then X is reflexive (resp. ordinary).
- (vi) Each component of RX (resp. of OX) dominates a component of X' .

Proof. It follows immediately from (2.5) and then from (2.3.1) and (2.1.1) that

$$h(x, H) = h(X_H/X_k/k; x) = \dim_x X/S - \dim_k \Omega_{CX/X'}^1(x, H') . \quad (2.6.1)$$

On the other hand, the standard theory of smoothness yields that

$$\dim_{(x, H')} CX/X' \leq \dim_k \Omega_{CX/X'}^1(x, H') ,$$

that equality holds if CX/X' is smooth at (x, H') , and that if CX/X' is flat at (x, H') and if equality holds, then CX/X' is smooth at (x, H') . Now, by standard dimension theory,

$$0 \leq [\dim_{(x,H')} CX/S - \dim_{H'} X'/S] \leq \dim_{(x,H')} CX/X' , \quad (2.6.2)$$

and the second and third terms are equal if CX/X' is flat at (x,H') .

Since

$$\dim_{(x,H')} CX/S = N-1 ,$$

(see (2.4,i)), therefore (i) holds. Note for future use that if the first and second terms in (i) are equal, then the second and third terms in (2.6.2) are equal.

The points z of CX at which $\dim_{k(z)} \Omega_{CX/X'}^1(z)$ (resp. $\dim_{q'(z)} X'/S$) achieves a local minimum form an open subset. Therefore (ii) holds because of (2.6.1) and (i).

Obviously, (iii) holds because RX (resp. OX) is open in CX^{sm} by (ii) and because the structure map $CX^{sm} \rightarrow S$ is smooth, so open.

It follows from the Segre-Wallace criterion (2.4,iv) and from the discussion in the first paragraph above that (iv) and (v) hold.

Finally, to prove (vi), let Z be the closure of a component of RX (resp. of OX), and assume that (x,H') lies over the generic point of Z . Then

$$\dim_{(x,H')} Z/S - \dim_{H'} q'Z/S = \dim_{(x,H')} Z/q'Z .$$

Moreover, since (x,H') is a point of RX , the second and third terms in (2.6.2) are equal by the last statement of the first paragraph. Hence

$$\dim_{H'} q'Z/S = \dim_{H'} X'/S .$$

Now, (iii) implies that H' lies over the generic point of a component of S . It follows that $q'Z$ is a component of X' . Thus (vi) holds.

3. THE HESSIAN CRITERION. GENERIC ORDER OF CONTACT.

(3.1) Setup. Use the notation and hypotheses of (2.1) and (2.3).

(3.2) Theorem (The Hessian Criterion). Assume that X is reduced.

Then the following statements are equivalent:

(a) X is reflexive (resp. ordinary).

(b) There is a dense open subscheme RX (resp. OX) of CX^{sm} such that a k -point (x, H') of CX^{sm} , where k is a field, belongs to RX (resp. OX) if and only if

$$h(\text{Sing}(X_k \cap H)/X_k/k; x) = \dim_x X/S + \dim_{H'} X'/S - (N-1) \quad (3.2.1)$$

(resp. $h(\text{Sing}(X_k \cap H)/X_k/k; x) = \dim_x X/S$)

where H is the hyperplane of P_k corresponding to H' .

(b') Each component of CX^{sm} supports some k -point (x, H') of CX^{sm} for some field k , which may vary with the component, such that (3.2.1) holds.

(c) There exists an open subscheme S^0 of S that is dense in the image of X^{sm} , and for each k -point of S^0 where k is a field, the corresponding fiber X_k is reflexive (resp. ordinary) in P_k .

Proof: Consider the open subscheme RX (resp. OX) of CX^{sm} introduced in (2.6,ii). Then (a) implies (b) by (2.6,iv). Trivially (b) implies (b'). Now, assume (b'). Then RX (resp. OX) meets each component of CX^{sm} . Being open, it is therefore dense in CX . Now, since X is reduced, so are CX and X' . Hence, by the lemma of generic flatness (EGAIV-(6.9.3), p. 154), q' is flat over a dense open subset of X' . Therefore, the open subscheme of CX on which q' is flat is dense by (2.6,vi), because RX (resp. OX) is dense. Consequently, (2.6,v) implies (a).

To prove the equivalence of (a) and (c), we obviously may replace X by the closure of X^{sm} and S by the (scheme-theoretic) image of X .

Thus we may assume that each component of X dominates a component of S and that S is reduced (as X is). Then, there is a dense open subscheme S_1 of S such that, for each k -point of S_1 where k is a field,

$$C(X_k) = (CX)_k, C((X_k)') = C((X')_k) \text{ and } C((X')_k) = (CX')_k \quad (3.2.2)$$

by (2.4,ii) applied to X and to X' .

Assume (a). Then (3.2.2) implies (c) with $S^0 = S_1$. Conversely, assume (c). Then (3.2.2) implies that, over each point of the dense open subscheme $S^0 \cap S_1$ of S , the fiber of CX and that of CX' are equal. Since CX (resp. CX') is reduced and since each of its components dominates a component of S , therefore $CX = CX'$. Thus (a) holds.

(3.3) Corollary (Wallace [1956], 6.2). Assume that X is a reduced hypersurface. Set $n = N-1$. For a k -point x of X^{sm} , where k is a field, and for a system of inhomogeneous coordinate functions t_1, \dots, t_N for X at x ordered so that dt_1, \dots, dt_n form a basis of $\Omega_{X_k/k}^1$ at x , let D_1, \dots, D_n denote the corresponding partial differential operators of the dual basis, and consider the condition on x and t_1, \dots, t_N that

$$\text{rank}[(D_j D_i t_N)(x)] = \dim_x X'/S \quad .$$

Then the following statements are equivalent:

- (a) X is reflexive.
- (b) There is a dense open subscheme WX of X^{sm} such that a k -point x of X^{sm} , where k is a field, belongs to WX if and only if x plus any given properly ordered system t_1, \dots, t_N satisfy the condition.
- (b') Each component of X^{sm} supports some k -point x for some field k such that x plus some properly ordered system t_1, \dots, t_N satisfy the condition.

Proof. Let x be a k -point of X^{sm} , where k is a field, and t_1, \dots, t_N an inhomogeneous coordinate system for X_k at x . Then the dt_i generate $\Omega_{X_k/k}^1$ at x ; so, reordering them, we may assume that dt_1, \dots, dt_n form a basis.

Denote by H the hyperplane tangent to X_k at x . Then for some a_i in k

$$X_k \cap H : a_1 t_1 + \dots + a_n t_n + a_N t_N = 0 .$$

If the dual basis of dt_1, \dots, dt_n is D_1, \dots, D_n , then by (2.2)

$$h(\text{Sing}(X_k \cap H)/X_k/k; x) = \text{rank}[a_N (D_j D_i t_N)(x)] .$$

Now, $a_N \neq 0$; otherwise, $a_1 dt_1 + \dots + a_n dt_n$ would be 0, whence all the a_i would be 0. Thus in particular, for a fixed x , the stated condition does not depend on the choice of t_1, \dots, t_N . Finally, it is now evident that (3.2) implies the assertion with $WX = q(RX)$.

(3.4) Corollary (Katz [1973], note on p. 3 and Prop. 3.3). Assume that $S = \text{Spec}(k)$ where k is a field of characteristic 2 and that some generically smooth component of X is of odd dimension. Then X is not ordinary.

Proof: Suppose that X were ordinary. Then by (3.2) there would be a k -point x in the appropriate component of CX^{sm} , for some field k , such that

$$h(\text{Sing}(X_k \cap H)/X_k/k; x) = \dim_x X/S$$

is odd. However, this number is by (2.2) equal to the rank of a skew-symmetric matrix whose diagonal terms are all 0 because $\text{char}(k)$ is 2. And the rank of such a matrix is even (Bourbaki [1959], cor. 3, p. 81).

(3.5) Theorem (Generic Order of Contact). Assume that S is the spectrum of a field k , that X is irreducible and geometrically reduced, and that

$$\dim(X) + \dim(X') = N . \tag{3.5.1}$$

Let K be an algebraically closed field containing k , let H' be a K -point of X' , and let H denote the corresponding hyperplane of P_K . Consider the condition on H' that each irreducible component V of the contact locus X_H meets X_K^{sm} and be such that

$$i(V, H.X_K, P_K) = [k(CX) : k(X')]_i \quad (3.5.2)$$

where the term on the left is the intersection multiplicity and the term on the right is the inseparable degree of the extension of function fields. Now, the following statements are equivalent:

- (a) X is not reflexive.
- (b) There exists a dense open subset of X' such that, if H' belongs to it, then the above condition is satisfied.
- (b') If H' lies over the generic point of X' , then the condition is satisfied.
- (b'') There exists a scheme-theoretic point of X' such that, if H' lies over it, then the condition is satisfied.

Proof. Trivially, (b) implies (b'). Trivially, (b') implies (b''). By EGA IV₃-(9.8.6), p. 86, there exists an open, but possibly empty, subset of X' such that, if H' belongs to it, then the condition is satisfied; hence, (b') implies (b). Note, moreover, that whether or not the condition is satisfied depends only on the image of H' in X' .

If V meets X^{sm} , then by (a trivial case of) the criterion of multiplicity 1

$$i(V, H.X_K, P_K) \geq 2$$

because $H \cap X_K$ is singular along V by (2.5). On the other hand, the right hand side of (3.5.2) is equal to 1 if and only if X is reflexive by the Segre-Wallace criterion (2.4,iv). Hence (b'') implies (a).

Finally, assume (a). To prove (b'), let E be a generic linear space of dimension $N-2-n$, where $n = \dim(X)$; that is, E is a subspace of

P_L , where L is a field containing k , and E corresponds to an L -point lying over the generic point of the appropriate Grassmannian. Let x be a K -point of X_L , where K is an algebraically closed field containing L , such that x lies over the generic point of X_L . Let H denote the hyperplane of P_K spanned by E_K and the tangent n -space to X_K at x . Let V denote the irreducible component of $H \cap X_K$ containing x . Since the corresponding point H' lies over the generic point of X' and since V is an irreducible component of X_H meeting X_K^{sm} , it is clear that it will suffice to show that now (3.5.2) holds.

Choose a system of affine coordinates for P_L such that E lies in the hyperplane at infinity and is cut out of it by the vanishing of the first $n+1$ coordinate functions and such that $x = (0, \dots, 0, 1)$. Denote the restrictions of these functions to X_L by t_0, t_1, \dots, t_n . Then for some a_i in K we have at x ,

$$X_K \cap H : f \equiv a_0 t_0 + a_1 t_1 + \dots + a_n t_n = 0 . \quad (3.5.3)$$

Now, X is not a linear space because of (3.5.1). Hence, X_K does not lie in H . Hence, some a_i is nonzero; assume $a_0 \neq 0$. Then dt_1, \dots, dt_n form a basis of $\Omega_{X_L/L}^1$ at x . Let D_1, \dots, D_n denote the dual basis. Then, by (3.2), (2.2) and (3.5.1), because X is not reflexive,

$$(D_j D_i t_0)(x) = 0 \quad \text{for } i, j = 1, \dots, n$$

as x lies over a generic point of X_L , therefore in the function field of X_L

$$D_j D_i t_0 = 0 . \quad (3.5.4)$$

Consider the Hasse differential operators $D_{(i)}$ where

$$(i) = (i_1, \dots, i_n) .$$

They compose according to the following rule (EGA IV₄, 16.11.2.2, p.54):

$$D_{(j)} D_{(i)} = \binom{(i+j)}{(i)} D_{(i+j)}$$

where the coefficient is the usual product of binomial coefficients. The coefficient is not divisible by $p = \text{char}(k)$ so long as $i_\ell < p$ for all ℓ . Therefore, it follows from (3.5.4) that there exists a power p^e and an integer m where $1 \leq m \leq n$ such that

$$D_{(i)} t_0 = 0 \quad \text{if} \quad 2 \leq i_\ell < p^e \quad \text{for} \quad \ell = 1, \dots, n \quad (3.5.5)$$

$$D_{(i)} t_0 \neq 0 \quad \text{where} \quad i_\ell = 0 \quad \text{if} \quad \ell \neq m \quad \text{and} \quad i_m = p^e. \quad (3.5.6)$$

Note that it is not possible that $D_{(i)} t_0 = 0$ for all (i) such that $2 \leq i_\ell$ for some ℓ ; indeed, such vanishing occurs only if X is linear, but by assumption X is not reflexive.

Consider the Taylor expansion at x of the function f of (3.5.3). The zeroth and first order coefficients vanish because $X_K \cap H$ is singular at x . Now, for any (i) , the corresponding coefficient is

$$(D_{(i)} f)(x) .$$

Hence (3.5.5), (3.5.6) and (3.5.3) imply that f lies in the p^e th power of the maximal ideal of X_K at x but not in its (p^e+1) th power.

Consider a reduced equation $g = 0$ for V in X_K at x . Then

$$f = ug^r \quad (3.5.7)$$

for some unit u in the local ring of X_K at x . Since V is a component of X_H , we may replace x by a simple point of V if necessary. Since x is a simple point of X_K , then g is in the maximal ideal of x but not in its square. Hence, by the preceding paragraph

$$r = p^e . \quad (3.5.8)$$

In view of (3.5.7) and (3.5.8), obviously for all i

$$D_i f = (D_i u) g^r = (D_i u) (u^{-1}) f .$$

However, the $D_i f$ together with f generate the ideal of X_H in the local ring of X_K at x by (2.5) and (2.2,i). Therefore f alone generates. Thus $H \cap X_K$ and X_H are scheme-theoretically equal at x . Since X_H is isomorphic to the fiber of CX over the K -point H' of X' representing H and since H' lies over the generic point of X' , therefore (3.5.2) holds, and the proof is complete.

(3.6) Remark. As in (3.5), assume that S is the spectrum of a field k and that X is irreducible and geometrically reduced. Let K be an algebraically closed field containing k , let H' be K -point of X' , and let H denote the corresponding hyperplane of P_K . Then, the contact locus X_H is a subvariety of X_K that is isomorphic to the fiber of CX over H' ; see (2.3.1). Hence

$$\dim(X) \geq \dim(X_H) \geq \dim(CX) - \dim(X') .$$

Therefore

$$\dim(X) + \dim(X') \geq N-1 . \quad (3.6.1)$$

Equality holds in (3.6.1) if and only if X is a linear space (as is well known). Indeed, consider the tangent space T to X_K at a simple K -point. Clearly, T' is contained in X'_K , and the two are equal,

on the one hand, if and only if T is equal to X_K , and on the other, if and only if T' and X' have the same dimension.

Hypothesis (3.5.1) is that the sum in (3.6.1) be equal to N .

Suppose that the sum is strictly greater than N and that H' is a general point of X' . Let V be any irreducible component of $H \cap X_K$. Then, since H is tangent to X_K precisely along X_H , and since

$$\dim(X_H) = (N-1) - \dim(X') < \dim(X)-1 = \dim(V) ,$$

by (a trivial case of) the criterion of multiplicity 1,

$$i(V, H.X_K, P_K) = 1 .$$

In particular, the conclusion of (3.5), that this intersection number be equal to the inseparable degree

$$[k(CX) : k(X')]_i$$

if and only if X is not reflexive, is false. Indeed, the inseparable degree is equal to 1 if and only if X is reflexive by the Segre-Wallace criterion (2.4,iv).

4. COMPARATIVE SIZE AND LOCATION - NONVANISHING OF THE RANKS

(4.1) Setup. Keeping the notation and hypotheses of (2.1) and (2.3), assume that the ground scheme S is the spectrum of a field k , and assume that X is a geometrically reduced closed subscheme of the projective N -space P over k , such that each component of X is of dimension at least 1 and at most $(N-1)$.

Unless there is an explicit indication to the contrary in a particular discussion, let M be a hypersurface of arbitrary degree, and set

$$Y = X \cap M .$$

Assume moreover that M is general; that is, as is conventional, assume that the k -point representing M in the appropriate projective space lies outside of a certain proper closed (k -) subscheme which will appear, though usually, implicitly, in the discussion at hand. If k is infinite, then obviously such M will exist.

(4.2) Lemma. The subscheme $q^{-1}Y$ of CX is geometrically reduced, it is irreducible (resp. geometrically irreducible) if X is so and if the dimension of X is at least 2, and its image $q'q^{-1}Y$ in the dual scheme X' is such that

$$q'q^{-1}Y \subsetneq Y' .$$

Proof: Since M is general, it is not hard to see (1) that $q^{-1}(Y - X^{sm})$ is nowhere dense in $q^{-1}Y$, (2) that $Y \cap X^{sm}$ is irreducible (resp. geometrically irreducible) if X is so and is of dimension at least 2, (3) that

$$Y \cap X^{sm} = Y^{sm}$$

and (4) that $q^{-1}Y$ passes through no point of CX of depth 1 and codimension ≥ 2 (there are finitely many such points; see for instance Kleiman [1966], Example 3, p. 311). Since CX^{sm} is locally a product of X^{sm} and a projective space (2) implies that $q^{-1}(Y \cap X^{\text{sm}})$ is irreducible (resp. geometrically irreducible) if X is so and is of dimension at least 2. Similarly, (3) implies that $q^{-1}(Y \cap X^{\text{sm}})$ is smooth; hence (4) implies $q^{-1}Y$ is geometrically reduced.

In view of (3) clearly any hyperplane tangent to X at a point of Y^{sm} is also tangent to Y ; in other words,

$$q^{-1}(Y \cap X^{\text{sm}}) \subset CY^{\text{sm}} .$$

Now, the set on the left is dense in $q^{-1}Y$ by (1) and the set on the right is dense in CY by definition. Hence

$$q'q^{-1}Y \subset Y' .$$

Finally, $Y' \not\subset q'q^{-1}Y$ because $q'q^{-1}Y \subset X'$ and because of (4.3) next.

(4.3) Lemma. Y' is not contained in any given subset Z of P^* .

Proof. First suppose that $\text{deg}(M) = 1$. Since M is general, then $M' \notin Z$. However, obviously $M' \in Y'$. Thus $Y' \not\subset Z$.

Suppose that $\text{deg}(M) \geq 2$. Degenerate M into $M_1 \cup M_2$, where M_1 is a general hyperplane and M_2 is a general hypersurface of degree one less than that of M . For $i = 1, 2$ set

$$Y_i = Y \cap M_i .$$

Since the corresponding family that degenerates Y into $Y_1 \cup Y_2$ is flat along $Y_1 \cup Y_2$, the smooth locus of the family meets $Y_1 \cup Y_2$ in its smooth locus. It follows, see (2.4,ii), that CY specializes to

$$CY_1 \cup CY_2 \cup D_1 \cup \dots \cup D_m$$

where the D_i are suitable subvarieties of $P \times P^*$.

Now, $Y'_1 \not\subset Z$ because Y_1 is a general hyperplane. It follows that $Y' \not\subset Z$.

(4.4) Lemma. (i) Let M be an arbitrary hypersurface. If no component of X' is a hypersurface, then the subset $q'q^{-1}Y$ of X' is equal to X' .

(ii) Let M be a hypersurface that contains a general point of each component of X . If X' is a hypersurface and if each component of CX dominates a component of X' , then $q'q^{-1}Y$ is of pure codimension 1 in X' .

Proof. (i) Since no component of X' is a hypersurface, the fibers of q' are all of dimension at least 1. Now, q embeds these fibers in X (their images are the contact loci.) Since M is a hypersurface, it therefore meets each of these images. So $q^{-1}Y$ meets each fiber of q' . So $q'q^{-1}Y$ is all of X' .

(ii) Since X' is a hypersurface, each component of X' has the same dimension as CX . Since each component of CX dominates a component of X' , therefore q' is finite on a dense open subset of CX , say U . Since M contains a general point of each component of X , clearly $U \cap q^{-1}Y$ is dense in $q^{-1}Y$. Hence, the restriction of q' is finite on a dense open subset of $q^{-1}Y$. Therefore, $q'q^{-1}Y$ is of pure codimension 1 in X' , as $q^{-1}Y$ is so in CX .

(4.5) Proposition. If X' is a hypersurface and each component of CX dominates a component of X' , then Y' is a hypersurface distinct from X' ; if no component of X' is a hypersurface, then $X' \subsetneq Y'$.

Proof. The assertions are immediate consequences of (4.2), (4.3) and (4.4).

(4.6) Lemma. Let M be an arbitrary hyperplane.

(i) Then Y' is a cone; its vertex is the point M' and its base is the dual $(Y/M)'$ of Y in M . Moreover, Y' is the cone of lines from M' to $q'q^{-1}Y$.

(ii) Projection from M' induces a dominating map,

$$\pi : (q'q^{-1}Y - M') \rightarrow (Y/M)' , \quad \pi(H') = (H \cap M/M)' .$$

Moreover, a k -point L' of $(Y/M)'$ belongs to the image of π if $Y_L \not\subset X_M$.

(iii) If $M' \not\subset X'$, then projection from M' induces a finite and surjective map,

$$\pi : q'q^{-1}Y \rightarrow (Y/M)' , \quad \pi(H') = (H \cap M/M)' .$$

Proof. Obviously, we may replace k by any extension field, and so we may assume that k is algebraically closed.

Fix a k -point (y, H') of CY^{sm} such that $H \neq M$. Then any hyperplane in the pencil generated by H and M is obviously tangent to Y at y . Hence Y' is a cone with vertex M' , because a dense open subset is. Its base is contained in $(Y/M)'$ because $H \cap M$ is a hyperplane in M that is tangent to Y at y . Note moreover that projection from M' carries H' to $(H \cap M/M)'$.

Given a hyperplane L of M that is tangent to Y at a simple k -point y , consider the span H of L and the tangent space $T_y Y$ (note that y is necessarily a simple point of X). Note that $T_y Y$ does not lie in M and that it meets M in $T_y Y$. It is now evident that H is tangent to X , that H is a hyperplane, and that $H \cap M$ is equal to L . Hence the base of the cone Y' contains a dense subset of $(Y/M)'$. Moreover, this dense subset lies in the image of $(q'q^{-1}Y - M')$ under projection from M' . Therefore (i) and the first part of (ii) hold.

To prove the second part of (ii), let y be a k -point of Y_L that does not belong to X_M . Consider the rational correspondence from $q^{-1}Y$ onto $C(Y/M)$ induced by $l_Y \times \pi$. Let (z, H') be a k -point of $q^{-1}Y$ that corresponds to (y, L') . Obviously, $z = y$. So $H' \neq M'$ because $y \notin X_M$. Obviously, $\pi(H') = (L/M)'$.

Finally, (i) and (ii) obviously imply (iii).

(4.7) Proposition. Let M be a hyperplane. (i) Assume that M contains a general point of each component of X and that $M' \not\subset X'$ or

or $M' \in X'$ but $q'q^{-1}Y$ is not a cone with M' as vertex. If X' is a hypersurface and if each component of CX dominates a component of X' , then the dual $(Y/M)'$ of Y in M is a hypersurface in the dual projective space M^* .

(ii) Assume that $M' \notin X'$ or that $M' \in X'$ but X' is not a cone with M' as vertex. If no component of X' is a hypersurface, then

$$\dim(Y/M)' = \dim(X') \quad ;$$

in fact, $(Y/M)'$ is equal to the projection of X' from M' .

Proof. The assertions follow immediately from (4.4) and (4.6).

(4.8) Proposition (Landman [1976]). Let M be a hyperplane. If no component of X' is a hypersurface, then Y' is the cone of lines from M' to X' ; moreover, if $M' \notin X'$ or if $M' \in X'$ but X' is not a cone with M' as its vertex, then $X' \subsetneq Y'$ and

$$\dim(Y') = \dim(X') + 1$$

Proof. The first assertion follows immediately from (4.4) and (4.6); the second assertion follows immediately from the first.

(4.9) Lemma. Let M and H be arbitrary hyperplanes, and let x be a simple k -point of X . Suppose that both M and H pass through x and that they are transverse to X at x . Then H is tangent to $X \cap M$ at x if and only if M is tangent to $X \cap H$ at x .

Proof. The hypotheses imply that both $X \cap M$ and $X \cap H$ are smooth at x and that their tangent spaces at x are given by

$$T_x(X \cap M) = (T_x X) \cap M \quad \text{and} \quad T_x(X \cap H) = (T_x X) \cap H.$$

Hence, H is tangent to $X \cap M$ at x if and only if H contains $(T_x X) \cap M$; whence, if and only if $(T_x X) \cap H$ is equal to $(T_x X) \cap M$ because H does not contain $T_x X$; whence by symmetry, if and only if M is tangent to $X \cap H$ at x .

(4.10) Theorem. (i) (cf. Wallace [1958], Lemma d, p. 5) If X is ordinary, then

(a) for any general hyperplane M and for any irreducible component X_i of X , the dual Y'_i of the section $X_i \cap M$ is equal to the cone of tangent lines from M' to X_i .

(ii) If X is ordinary and if the characteristic is different from 2, then

(b) for any general hyperplane H tangent to X , the hyperplane tangent to X' at H' is not a component of $(X \cap H)'$.

(iii) Conversely, if the ground field k is algebraically closed of any characteristic, then (a) and (b) together imply that X is ordinary.

Proof. (i) Obviously, we may assume that X is irreducible. Since M is general, $q^{-1}Y$ meets any given non-empty open subset of CX . Hence the set

$$U = X'^{\text{sm}} \cap q'q^{-1}Y$$

is nonempty. Obviously U is open in $q'q^{-1}Y$, and by (4.2) $q'q^{-1}Y$ is irreducible. Moreover, Y' is the cone of lines from M' to $q'q^{-1}Y$ by (4.6). Therefore, to prove that Y' is the cone of tangent lines from M' to X' , it suffices to replace k by its algebraic closure and to prove that, for each k -point (y, H') of CX'^{sm} , conditions (A) and (E) below are equivalent.

Note that since X' is a hypersurface, the tangent space to X' at H' is all of the hyperplane y' . Hence each of the following conditions is obviously equivalent to the next one:

(A) The line from M' to H' is tangent to X' at H' .

(B) The hyperplane y' contains the point M' .

(C) The point y is contained in the hyperplane M .

(D) The point (y, H') of $CX' = CX$ lies in the subset $q^{-1}Y$.

(E) The point H' of X'^{sm} lies in the subset $q'q^{-1}Y$.

(ii) Let I denote the graph of the point-hyperplane incidence correspondence and consider the closed subscheme A of $X \times X' \times P^*$ defined by

$$A = (CX)_{x_p} I .$$

Obviously, each K -point of A , where K is an extension field of k , is a triple (y, H', G') such that the point y lies in both the H -contact locus X_H and the hyperplane G . Obviously, A is a bundle of projective $(N-1)$ -spaces over CX . Obviously, each component of A projects onto P^* because each component of X is of dimension at least 1 and so intersects every hyperplane G .

It will be shown below that there is a dense open subset U of A such that each K -point (y, H', G') of U has the following three properties:

- (1) G is transverse to X_K^{sm} ,
- (2) y is the entire support of the H -contact locus $(X_K \cap G)_H$,
- (3) the G -contact locus $(X_K \cap H)_G$, if nonempty, is contained in the smooth locus $(X_K \cap H)^{sm}$.

Assume for the moment that U exists.

Fix a dense open subset U' of X^{sm} contained in the image of U , and consider a k -point H' of U' . Let K be an algebraically closed field containing k , and lift H' to a K -point (y, H', G') of U . Then, because y lies in X_H and X is ordinary, y' is the hyperplane tangent to X' at H' . Moreover, G' lies in y' . However, G' does not lie in $(X \cap H)'$. Indeed, if it did, then G would be tangent to $X \cap H$ at a simple K -point x by (3). Hence (1) and (4.9) would imply that H is tangent to $X \cap G$ at x . Therefore, (2) would imply that $x = y$. However, x is a simple point of $X \cap H$, and y is not. Thus (b) holds.

The subset U is obtained by intersecting three other dense open subsets of A , which correspond to the three properties. The first of these, U_1 , is obtained by starting with a nonempty open subset of P^* whose K -points G' have Property (1) and then taking the subset's inverse image in A . This inverse image is dense because each component of A projects onto P^* .

The second subset U_2 exists because X is ordinary and the characteristic is different from 2. Indeed, these hypotheses imply,

according to (5.9,i) and (3.2,a \Leftrightarrow c), that the total space of the family of hyperplane sections of X , namely $(X \times P^*) \cap I$, is ordinary in I/P^* . Hence there is a dense open subset V_2 of the conormal scheme on which the map to the dual scheme is birational. Moreover, it may be assumed that V_2 lies over a nonempty open subscheme W of P^* such that the formation of the conormal scheme, the dual scheme and the map between them commutes with base-change to the fiber over each point of W , see (2.4,ii). Then, for any extension field K of k , the K -points of V_2 are triples (y, L', G') such that G is a hyperplane such that $X \cap G$ is ordinary in G , and L is a hyperplane of G such that the L -contact locus $(X \cap G)_L$ is just the reduced subscheme supported at the point y . In view of (4.6), it is evident that there is a natural map to the present conormal scheme from $A-\Delta$, where Δ is the diagonal of $X \times_P X$, and that the inverse image of V_2 may be taken as U_2 ; that is, U_2 is dense and the K -points of U_2 have Property (2).

The third subset U_3 is constructed as follows. Consider the family T/X' of sections of X by its tangent hyperplanes. Form the singular locus of T/X' , form its inverse image D in the conormal scheme $C(T/P \times X')$, and form the image C of D in the dual scheme T' . View T' as a subscheme of $X' \times P^*$, and consider the inverse image B of C in A . Now, there is a dense open subset V' of X' such that the formation of $C(T/P \times X')$ and of T' commutes with base-change over each point of V' , see (2.4,ii). Take U_3 to be the intersection of the inverse image in A of V' with $(A-B)$. It is evident that the K -points of U_3 have Property (3). Finally, U_3 is dense by reason of dimension; indeed, D is nowhere dense in $C(T/P \times X')$, both $C(T/P \times X')$ and A are pure and they have the same dimension, and the map from A into $X' \times P^*$ is birational onto its image. The proof of (ii) is now complete.

(iii) Obviously, (a) and (b) will continue to hold if X is replaced by any one of its irreducible components. Hence, by (2.4,v) we may assume that X is irreducible.

Note that X' is a hypersurface. Indeed, if not, then (a) and (4.4) and (4.6,iii) would imply that for any general hyperplane M , the cone

of all lines from M' to X' is equal to the cone of tangent lines; in other words, projection of X' from M' is everywhere ramified. However, the projection is generically unramified because the center M' is in general position.

Note next that the dual X'' of X' is not a point. Indeed, otherwise, X' would be a hyperplane. Whence, for a general hyperplane M , the cone of tangent Y' lines from M' to X' would be empty. So X would be a finite set of points, contrary to a hypothesis in (4.1).

In view of the above two notes and of (a) and (b), it is clear that there exists a dense open subset U of $C(X'^{\text{sm}})$ whose k -points (y, H') are such that

- (1) X_H is finite,
- (2) y' , which is just the hyperplane tangent to X' at H' , is not a component of $(X \cap H)'$, and
- (3) for any general hyperplane M through y , the cone of tangent lines from M' to X' is equal to Y' .

To establish (iii), it obviously suffices to prove that y lies in X_H .

Suppose that y does not lie in X_H . Let M be a general hyperplane through y . Then M does not contain any point of X_H by (1). Moreover, M' does not lie in $(X \cap H)'$ by (2), because M' is a general point of y' . Hence, there exists an open neighborhood V of H' in P^* whose k -points G' are such that

- (4) M' does not lie in $(X \cap G)'$ and
- (5) M does not contain any point of X_G .

The line from M' to H' is tangent to X' at H' , because M' lies in y' and y' is the hyperplane tangent to X' at H' . Hence this line lies in Y' by (3). So H' lies in Y' . So the intersection of V and Y' is nonempty. Let G be a general k -point of this intersection. Then G is tangent to Y at a simple point x of Y . Hence M' lies in $(X \cap G)'$ by (5) and (4.9). This conclusion contradicts (4). Thus (iii) holds.

(4.11) Remark. (i) If in (4.10,iii) the characteristic is 0, then (b) is superfluous; (a) alone implies that X is ordinary. The proof is simple. Indeed, by the first paragraph of the proof of (4.10,iii), we

may assume that X is irreducible; by the second, X' is a hypersurface. Since the characteristic is 0, the Segre-Wallace criterion (2.4,iv) now implies that X is ordinary.

Moreover, in characteristic 0, it is also easier to prove that (4.10,ii) holds. Indeed, consider a hyperplane H tangent to X at a general k -point x . Since X is ordinary, x' is the hyperplane tangent to X' at H' . Suppose that x' is a component of $(X \cap H)'$. Then x' is the dual of some (reduced and irreducible) component Z of $X \cap H$. Now, Z is reflexive by the Segre-Wallace criterion (2.4,iv) because the characteristic is 0. Hence $Z = x$. So $\dim_x X = 1$. However, then x supports an isolated, nonreduced component of $X \cap H$, in contradiction to the fact that Z is reduced.

(ii) If in (4.10,iii) the characteristic is $p > 0$, then (a) alone does not imply that X is ordinary. If it did, then (a) would imply (b) at least if $p \neq 2$ by (4.10,ii). However, (a) does not imply (b). In fact, the following statement will be proved next:

(4.11.1) If (b) fails, if X' is a hypersurface but not a hyperplane, and if X is irreducible, then (a) holds.

After (4.11.1) is proved a specific example illustrating it will be discussed. Finally, there is a short discussion of the irreducible varieties X , other than the points, such that X' is a hyperplane. For such an X , (a) fails. However, (b) may either fail or hold; specific examples will be discussed. In particular, (4.11.1) is sharp.

Throughout this remark, k will be algebraically closed.

To prove (4.11.1), note that Y' is irreducible because X is irreducible, and note that Y' is equal to the cone of all lines from M' to $q'q^{-1}Y$ by (4.6,i). Hence, to prove (a), it suffices to show that, if a line L' contains M' and is tangent to X' at a smooth k -point H' , then L' meets X' at a k -point G' of $q'q^{-1}Y$. Indeed, then (1) Y' will contain the cone of tangent lines from M' to X' , because, by definition, the cone of tangent lines is the closure of the cone of these lines L' .

Furthermore, then (2) the cone of tangent lines is a hypersurface, because X' is a hypersurface but not a hyperplane and because the points of contact H' are just the smooth k -points of X' in the polar locus C , where projection from M' ramifies. Note that the set of smooth points of X in C form a dense (open) subset of C ; in fact, each component of C meets any given open subset of X' , because M' is a general point of P^* . (For the theory of polar loci, see for example Kleiman [1977], IV, B.) Since Y' is irreducible, (1) and (2) imply (a). Moreover, because (b) fails, and again because each component of the polar locus C meets any given open subset of X' , we may assume that the hyperplane tangent to X' at H' , denote it by y' , is a component of Z' , where $Z = X \cap H$.

Projection from H' induces a map from $(Z' - H')$ onto $(Z/H)'$ by (4.6,i) with H as M . Now, $H' \in L' \subset y'$ because L' is tangent to X' at H' . Hence, $(L/H)' \in (Z/H)'$ where $L = M \cap H$, because $y' \subset Z'$, so $M' \in Z'$. We may assume that $Z_L \not\subset X_H$; indeed, otherwise, some point of X_H would lie in L , and so H' may be taken as G' . Therefore, (4.6,ii) implies that there exists a k -point G' of $q'q^{-1}Z$ such that $G' \cap H = L$. Obviously, G' lies in $q'q^{-1}L$ and in L' . Thus the proof is complete.

A specific example of an irreducible variety X such that X' is a hypersurface but not a hyperplane and such that (b) fails is the smooth variety defined by

$$X : x_0^{p+1} + \dots + x_N^{p+1} = 0 . \quad (4.11.2)$$

Indeed, as a k -point $x = (x_0, \dots, x_N)$, the tangent hyperplane is

$$H : x_0^p x_0 + \dots + x_N^p x_N = 0 .$$

Hence, X' has the same equation as X . In particular, X' is a smooth hypersurface but not a hyperplane.

Consider $Z = X \cap H$. It is smooth except at x because, obviously, H is tangent to X only at x . Let $z = (z_0, \dots, z_N)$ be a k -point of Z .

Since z lies in H , it satisfies the following equation:

$$x_0^p z_0 + \dots + x_N^p z_N = 0 .$$

Raising this equation to the p th power yields an equation that shows that the hyperplane tangent to X at z contains the point,

$$x^p = (x_0^p, \dots, x_N^p) .$$

Hence, $q'q^{-1}z$ lies in the hyperplane tangent to X' at H' . Now, Z' is obviously equal to the cone of lines from H' to $q'q^{-1}z$; see (4.6,i). Therefore, Z' is contained in the hyperplane tangent to X' at H' .

Assume $x_0 = 1$ and view X_1, \dots, X_N as a system of homogeneous coordinates for H . Then Z is given in H by the equation,

$$(x_1^p X_1 + \dots + x_N^p X_N)^{p+1} + x_1^{p+1} + \dots + x_N^{p+1} = 0 ,$$

which may be rewritten as

$$X_1 L_1^p + \dots + X_N L_N^p = 0$$

where the L_i are linear homogeneous polynomials. Obviously the L_i^p are the partial derivatives of the left side of this equation. Hence the L_i vanish simultaneously only at x , because x is the unique singular point of Z . Moreover, the Gauss map, which carries a simple k -point z of Z to the tangent space of Z at z , factors as the composition of projection from x followed by the Frobenius p -th power map. Hence, (Z/H') , which is equal to the closure of the image of the Gauss map, is a hypersurface if and only if Z is not a cone with x as vertex.

A short, straightforward computation shows that every line L in H , through x , intersects Z at x with multiplicity at least p . Moreover, such an L intersects Z at x with multiplicity at least $p+1$ if and only if L lies on the hyperplane

$$N : x_0^{1/p} X_0 + \dots + x_N^{1/p} X_N = 0 \quad . \quad (4.11.3)$$

Now, a line L that intersects Z at x with multiplicity at least $p+1$ and that contains a second point of Z must lie on Z by Bezout's theorem. Hence, Z is a cone with x as vertex if and only if $x = x^{p^2}$.

It is evident in view of (4.6,i) that $(Z/H)'$ is a hypersurface if and only if Z' is one. Hence, together the conclusions of the above three paragraphs yield that Z' is equal to the hyperplane tangent to X' at H' if and only if $x \neq x^{p^2}$. Thus (b) fails.

In this example, it is easy to check (a) directly. Indeed, consider any hyperplane M which is transverse to X ; that is, $M' \not\subseteq X'$. Say

$$M : a_0 X_0 + \dots + a_N X_N = 0 \quad .$$

Suppose that the line L' from M' to H' is tangent to X' at H' . Then the point x^{p^2} lies in M ; in other words,

$$a_0 x_0^{p^2} + \dots + a_N x_N^{p^2} = 0 \quad .$$

Using this equation and the equation of X , it is easy to see that L' also meets X' at the point $G' = (y_0^p, \dots, y_N^p)$ where

$$y_i^p = (a_0^{p+1} + \dots + a_N^{p+1}) x_0^p = (a_0 x_0 + \dots + a_N x_N)^p a_i \quad .$$

Multiplying both sides of the preceding equation by a_i^p and summing up gives

$$a_0^p y_0^p + \dots + a_N^p y_N^p = 0 \quad .$$

So the point $y = (y_0, \dots, y_N)$ lies in M . Hence y belongs to $Y = X \cap M$. Therefore, G' belongs to $q'q^{-1}Y$. Thus the cone of tangent lines from M' to X' is included in the cone of lines from M' to $q'q^{-1}Y$. The opposite

inclusion may be checked similarly. Thus the two cones are equal. However, the second cone is equal to Y' by (4.6,i). Thus (a) holds.

Finally, consider an irreducible variety X , not a point, such that X' is a hyperplane. For it, (a) fails. Indeed, any line tangent to X' must lie in X' , so the cone of tangent lines from a point M' off X' is empty. However, since X is not a point, $Y = X \cap M$ is nonempty. Moreover, Y is geometrically reduced, because M' lies off X' and because X is geometrically reduced as X' is nonempty. Hence, Y' is nonempty.

A specific example of an irreducible variety Z , not a point, such that Z' is a hyperplane, and for which (b) fails is the section $Z = X \cap H$ of the variety X defined in (4.11.2) by the hyperplane H tangent to X at a k -point x such that $x \neq x^{p^2}$ and such that

$$x_0^{p^3+1} + \dots + x_N^{p^3+1} \neq 0 \quad (4.11.4)$$

provided $N \geq 4$. Indeed, by the discussion of X above, Z' is the hyperplane $(x^{p^2})'$, and Z is smooth except at x . Since Z is smooth except at x and since it is a hypersurface in H , which is a projective space of dimension ≥ 3 , obviously Z is irreducible.

Let y be a general k -point of Z . Set $G = T_y X$ and $W = Z \cap G$. Then $x \notin G$ because $Z' \neq x'$ as $x \neq x^{p^2}$. Moreover, in view of the above analysis of the Gauss map of Z , it is clear that G is not tangent to Z at any simple point aside from y . Hence W is smooth except at y . Now, it is evident that the equation of W in G is of the same form as the equation of Z in H . Furthermore, (4.6,i) implies that $W' \subset Z'$. To prove that $W' = Z'$, it therefore suffices to prove that W is not a cone with y as vertex, by the argument used above to prove that $Z' = (x^{p^2})'$.

The point x^{p^2} lies on G , on H and on X , so on W . Hence, if W were a cone with y as vertex, then the line L joining y and x^{p^2} would lie on W , so on X . Now, (4.11.4) implies that y does not satisfy the equation

$$y_0^{1/p} x_0^{p^2} + \dots + y_N^{1/p} x_N^{p^2} = 0$$

because the points y that satisfy this equation form a closed set that does not contain x : So $x^{p^2} \notin N$, where

$$N : y_0^{1/p} x_0 + \dots + y_N^{1/p} x_N = 0 .$$

Hence, L intersects X with multiplicity p at y by the discussion about (4.11.3). So L does not lie on X . Thus (b) fails.

A specific example of an irreducible variety X , not a point, such that X' is a hyperplane, and for which (b) holds is the even dimensional, smooth quadric hypersurface X in characteristic 2. By the general Plucker formula (Kleiman [1977], IV, 49, p. 357),

$$\deg(q')\deg(X') = 2 .$$

Now, X is not ordinary by (3.4). Hence X' is a hyperplane.

Consider any k -point x of X . Let H denote the hyperplane tangent to X at x , and set $Z = X \cap H$. Then Z is a quadric hypersurface with a singular point at x . Hence Z is a cone with x as vertex by Bezout's theorem. It is now evident that Z' lies in the hyperplane x' , but $Z' \neq x'$. Thus (b) holds.

(4.12) Setup continued. Recall (Fulton-Kleiman-MacPherson [1985], (1.3)) that the i th rank of X is defined as the nonnegative integer

$$r_i = \int q^* C_1(\mathcal{O}(1))^{i-1} q'^* C_1(\mathcal{O}(1))^{N-1-i} [CX],$$

where N is the dimension of the ambient projective space. It is evident that $r_i = r_i(X)$ behaves additively in X .

(4.13) Theorem. Assume that X is irreducible. Then $r_i \neq 0$ if and only if $i \in [(N-1-\dim X'), \dim X]$.

Proof. It is easy to see using the projection formula with respect to q' and q that

$$r_i = 0 \quad \text{if } i \notin [(N-1-\dim X'), \dim X]$$

$$r_i \neq 0 \quad \text{if } i = (N-1-\dim X'), \dim X .$$

In particular, the assertion holds if X is of dimension 1 (or 0).

Proceeding by induction on $\dim X$, assume the assertion holds for a general hyperplane section Y of X . Now, it is known that

$$r_i = r_{i-1}(Y/M) \quad \text{if } i \geq 1$$

(see Piene [1978], (4.2); note however, that r_i is equal to Piene's $(n-i)$ -th class, where $n = \dim X$.) The assertion is now easy to prove; use (4.7) and consider separately the two cases in which X' is and is not a hypersurface.

5. COMPARATIVE REFLEXIVITY

(5.1) Setup. Keep the notation and hypotheses of (4.1); however, also allow $X = P$. Set $p = \text{char}(k)$.

(5.2) Lemma. Fix a k -point x of X^{sm} . Let H be a hypersurface not containing X . Assume that H is tangent to X at x ; that is, x belongs to the singular locus $\text{Sing}(X \cap H)$. Set $n = \dim_x X$. Set

$$h_X = h(\text{Sing}(X \cap H)/X/k; x) .$$

(i) Assume that H is general of degree at least 2.

(a) If either $p \neq 2$ or n is even, then $h_X = n$.

(b) If $p = 2$ and n is odd, then $h_X = n-1$.

(ii) Let M be a general hypersurface containing x . Set $Y = X \cap M$, and set

$$h_Y = h(\text{Sing}(Y \cap H)/Y/k; x) .$$

(a) If $h_X < n$, then $h_Y = h_X$.

(b) If $h_X = n$ and $p \neq 2$, then $h_Y = n-1$.

(c) If $h_X = n$ and $p = 2$, then $h_Y = n-2$.

Proof. Choose inhomogeneous coordinates t_1, \dots, t_N centered at x for the ambient projective space P such that the restrictions $u_i = t_i|_X$ for $i = 1, \dots, n$ are local parameters for X at x . Say $H : F = 0$, and set $f = F/X$. Since H is tangent to X at x , the Taylor series for f begins, at least, with a quadratic form Q in u_1, \dots, u_n . Let D_1, \dots, D_n denote the partial derivative operators with respect to u_1, \dots, u_n . Then $(D_j D_i f)(x)$ is equal to $D_j D_i Q$. So (2.2,ii) yields that

$$h_X = \text{rank}[D_j D_i Q]_{i,j \leq n} \tag{5.2.1}$$

Since the right side is lower semi-continuous in H , so is h_X .

To prove (i)(a) (resp. (b)), it suffices by lower-semicontinuity to give one H such that $h_X = n$ (resp. $h_X = n-1$, note that in case (b) $h_X < n$ because h_X is even by Bourbaki [1959], cor. 3, p. 81). Now, obviously any H such that

$$Q = u_1^2 + \dots + u_n^2 \quad \text{if } p \neq 2,$$

or

$$Q = u_1 u_{s+1} + u_2 u_{s+2} + \dots + u_s u_{2s} \quad \text{if } n = 2s \text{ (resp. if } n = 2s+1 \text{ and } p=2)$$

is such that $h_X = n$ (resp. $h_X = n-1$), because of (5.2.1)

To prove (ii), fix H and make a linear change of coordinates with coefficients in k so that

$$Q = a_1 u_1^2 + \dots + a_r u_r^2 \quad \text{if } p \neq 2, \quad (5.2.2)$$

or

$$Q = \sum_{i=1}^s (a_i u_i^2 + u_i u_{s+i} + b_i u_{s+i}^2) + \sum_{i=2s+1}^n c_i u_i^2 \quad \text{if } p = 2,$$

where r and s are suitable integers and the a 's, b 's and c 's are suitable elements of k ; such a change is possible by Bourbaki [1959], §6, no. 1, Thm. 1, p. 90, and Exer. 27, p. 112.

Since M is general, the restrictions $v_i = u_i|_Y$ for $i = 1, \dots, n-1$ are local parameters for Y at x . Let E_1, \dots, E_{n-1} denote the corresponding partial derivative operators. Then (2.2,ii) yields that

$$h_Y = \text{rank}[E_j E_i Q(v_1, \dots, v_n)]_{i,j \leq n-1} \quad (5.2.3)$$

If $h_X < n$, then in view of (5.2.1) clearly in (5.2.2) $r < n$ if $p \neq 2$, and $2s < n$ if $p = 2$. Hence in view of (5.2.2) and (5.2.3)

obviously $h_Y = h_X$.

Assume $h_X = n$. If $p \neq 2$, then $r = n$ and $a_i \neq 0$ for all i in (5.2.2). So, if the equation of Y in X were $u_n = 0$, then h_Y would be $n-1$ in view of (5.2.3). Hence, by lower semi-continuity, since Y is general, $h_Y = n-1$.

If $p = 2$, then $2s = n$ in (5.2.2). Moreover, h_Y is even, so at most $n-2$. Hence, to complete the proof, it suffices by lower semi-continuity to observe that, if the equation of Y in X were $u_n = 0$, Then h_Y would be $n-2$.

(5.3) Discussion. Return, for this discussion alone, to the general setup of (2.1) and (2.3).

(i) The subscheme X will be called semi-ordinary if X is not ordinary but there exists a dense open subset U of CX^{sm} such that for every K -point (x, H') of U , where K is any field, we have

$$h(\text{Sing}(X_K \cap H)/X_K/K; x) \geq \dim_x(X) - 1. \quad (5.3.1)$$

(ii) It is clear from the proof of (2.6,ii) that the subset U of CX^{sm} whose K -points (x, H') satisfy (5.3.1) is open (but possibly empty). It is now easy to check that the following conditions, which are similar to those in (3.2), are equivalent:

- (a) X is semi-ordinary
- (b) X is not ordinary and every component of CX^{sm} has a K -point (x, H') for which (5.3.1) holds.
- (c) There exists an open dense subscheme S^0 of S such that, for each K -point of S^0 , the corresponding fiber X_K is semi-ordinary in P_K .

(iii) For $S = \text{Speck}$ where k is a field, it follows from (2.6,i) that if X is semi-ordinary, then every component of X' has codimension less or equal than 2 in P .

(iv) Every curve over a field is either ordinary or semi-ordinary (but not both).

(5.4) Theorem. Reembed X via the d -fold Veronese map for some $d \geq 2$.

(i) (Cf. Katz [1973], Thm. 2.5) If $p \neq 2$ or if each component of X is of even dimension, then X is ordinary.

(ii) If $p = 2$ and if some component of X is of odd dimension, then X is semi-ordinary.

Proof. Clearly it suffices in view of (2.4,ii and v) (resp. (5.3,ii)) to treat the case in which k is algebraically closed and X is irreducible. Fix a general k -point (x, H') of CX^{sm} . Under the Veronese map $X \cap H$ is identified with the section of X by a general hypersurface of degree d and tangent to X at x . Hence, (i) (resp. (ii)) results from (5.2,i,a) and (3.2) (resp. (5.2,i,b), (5.3,ii) and (3.4)).

(5.5) Remark. If X is irreducible, then (5.4) and (5.3,iii) yield that X' is a hypersurface, except possibly if $p = 2$ and $\dim(X)$ is odd, in which case $\text{codim}(X') \leq 2$. Probably X' is always a hypersurface; possibly this conjecture may be proved by modifying Ein's argument, Ein [1984], II, (1.3).

(5.6) Theorem. Let M be a general hypersurface of degree at least 2, and consider $Y = X \cap M$.

(i) If $p \neq 2$ or if each component of X is of odd dimension, then Y is ordinary.

(ii) If $p = 2$ and if some component of X is of even dimension, then Y is semi-ordinary. In fact, if X is irreducible, if x is a given k -point of X^{sm} and if M is general containing x , then, in (i), Y is ordinary, and in (ii), Y is semi-ordinary.

Proof. (i) It is not hard to reduce the question to the case in which X is irreducible (see (2.4,v)) and of dimension at least 2. In this case, the family of section Y of X by the hypersurfaces M of given degree and not containing X (resp. and containing x) is flat, and its total space is irreducible. Hence by (2.4,ii) and (3.2) it suffices to find one such Y defined over a suitable extension field K of k and a K -point (y, H') of CY^{sm} such that

$$h(\text{Sing}(Y \cap H)/Y/K; y) = \dim_y Y \quad (5.6.1)$$

(resp. and such that $y = x$).

Let K be the algebraic closure of k . Let y be a K -point of X^{sm} (resp. set $y = x$). Let H be any hyperplane defined over K and transversal to X_K at y . Set

$$Z = X_K \cap H .$$

Let M be a general hypersurface defined over K and tangent to Z at y . Set

$$Y = X_K \cap M .$$

Then (y, H') is a K -point of CY^{sm} . Finally, (5.2,i,a) yields that

$$h(\text{Sing}(Z \cap M)/Z/K; y) = \dim_y Z \quad (5.6.2)$$

Since $Z \cap M = Y \cap H$ and $\dim_y Z = \dim_y Y$, therefore, (5.6.1) holds by (2.1.1), and the proof of (i) is complete.

(ii) Note that (i) takes care of the components of X of odd dimension, hence we may assume by (5.3,ii) that X is irreducible of even dimension. By (3.4) we have that Y is not ordinary. Now, the proof proceeds as in (i) but with two changes. First, (5.6.1) has to be replaced by

$$h(\text{Sing}(Y \cap H)/Y/K; y) = \dim_y Y - 1 .$$

This condition is by (5.3,ii) sufficient for Y to be semi-ordinary. The second change is that (5.6.2) has to be replaced by

$$h(\text{Sing}(Z \cap M)/Z/K; y) = \dim_y Z - 1 ;$$

this condition is satisfied by (5.2,i,b).

(5.7) Remark. In (5.6,ii) as well as in (5.6,i), Y' is a hypersurface; both cases are covered by (1.3) of Ein [1984], II. In particular, therefore in (5.6,ii), Y is in fact not reflexive.

(5.8) Proposition. Let M be a hyperplane and Y a geometrically reduced, closed subscheme of M . Then Y is reflexive (resp. ordinary, resp. semi-ordinary) in the ambient projective N -space if and only if Y is reflexive (resp. ordinary, resp. semi-ordinary) in M .

Proof. By (2.4,ii) and (5.3,ii) we may assume that k is algebraically closed. Now, let (y, H') be a k -point of CY^{sm} such that $H \not\supset Y$. Set $L = M \cap H$. Obviously,

$$h(\text{Sing}(Y \cap H)/Y/k; y) = h(\text{Sing}(Y \cap L)/Y/k; y) . \quad (5.8.1)$$

Now, Y' is the cone over $(Y/M)'$ with vertex M' by (4.6); so

$$\dim_y Y + \dim_{H'} Y' - (N-1) = \dim_y Y + \dim_{(L/M)'} (Y/M)' - (N-2),$$

and projection of Y' from M' , which carries H to L , maps onto $(Y/M)'$. Therefore the assertions follow from (3.2) and (5.3,ii).

(5.9) Theorem. Let M be a general hyperplane, and consider $Y = X \cap M$.

(i) If $p \neq 2$ and if X is ordinary, then Y is ordinary in P and in M .

(ii) If $p = 2$ and X is ordinary, then every component of Y is semi-ordinary in P and in M , and every component of Y' (resp. of $(Y/M)'$) is a hypersurface.

(iii) Assume that no component of X' is a hypersurface. Then

$$\dim(Y') - 1 = \dim(Y/M)' = \dim X' \quad (5.9.1)$$

and X is reflexive if and only if Y is reflexive in P (resp. in M).

(iv) If every component of X is semi-ordinary, then Y is ordinary.

Proof. If $X = P$, then (i), (ii) and (iv) do not apply, while (iii) and (iv) is trivial; so we may assume that $X \neq P$. By (2.4,ii) and (5.3,ii) we may assume that k is algebraically closed. Consider the

irreducible components X_i of X , and set $Y_i = X_i \cap M$. Then the Y_i are the irreducible components of Y . Hence, we may assume that X is irreducible -- in (i), (ii) and (iv), because of (2.4,v); and in (iii), because of (2.4,iii) and also because of (4.8); indeed, (4.8) implies that $X_i' \subset X_j'$ if and only if $Y_i' \subset Y_j'$.

Recall from (4.5) and (4.7) that if X' is a hypersurface, then Y' and $(Y/M)'$ are also. Recall from (4.7) and (4.8) that if no component of X' is a hypersurface, then (5.9.1) holds. Recall from (5.8) that Y is reflexive (resp. ordinary, resp. semi-ordinary) in P if and only if it is so in M .

Suppose that X is reflexive. To prove that the general hyperplane section Y of X is reflexive in (i) and (iii), and semi-ordinary in (ii), it suffices by an argument similar to that in (5.6) to find one hyperplane section Y and a k -point (x, H') of CY^{sm} such that

$$h(\text{Sing}(Y \cap H)/Y/k; x) = \dim_x Y + \dim_{H'} Y' - (N-1) \quad (5.9.2)$$

(resp. $h(\text{Sing}(Y \cap H)/Y/k; x) = \dim_x Y - 1$) .

Since k is algebraically closed, by (3.2, $a \Rightarrow b$) there exists a k -point (x, H') of CX^{sm} such that

$$h(\text{Sing}(X \cap H)/X/k; x) = \dim_x X + \dim_{H'} X' - (N-1) \quad (5.9.3)$$

Let M be a general hyperplane containing x , and set $Y = X \cap M$. Then (x, H') is a k -point of CY^{sm} . Therefore, (i) holds by (5.2,ii,b), and (ii) holds by (5.2,ii,c), while half of the second assertion of (iii) holds by (5.2,ii,a).

Suppose now that X' is not a hypersurface and that Y is reflexive in M . By (3.2, $a \Rightarrow b$), there exists a k -point (x, L') of $C(Y^{\text{sm}}/M)$ such that

$$h(\text{Sing}(Y \cap L)/Y/k; x) = \dim_x Y + \dim_{L'} (Y/M)' - (N-2) .$$

Then x lies in X^{sm} , and by (4.7) there is a hyperplane H tangent to X at x such that $L = M \cap H$ and $\dim_L (Y/M)' = \dim_H X'$. Hence X is reflexive by (5.2,ii,a) and (3.2, $b' \Rightarrow a$).

To prove (iv), let (x, H') be a k -point of CX^{sm} such that

$$h(\text{Sing}(X \cap H)/X/k; x) = \dim(X) - 1.$$

Then (5.2,ii,a) yields that

$$h(\text{Sing}(Y \cap H)/Y/k; x) = \dim(X) - 1 \quad (= \dim Y) .$$

Therefore, Y is ordinary by (3.2, $b' \Rightarrow a$).

(5.10) Corollary. In any characteristic, if X is ordinary of dimension ≥ 2 , then a general codimension -2 linear section of X is also ordinary in P and in M .

(5.11) Remarks. (i) In any positive characteristic p , it may happen that X is irreducible, X' is a hypersurface and Y is ordinary, but that X is not ordinary. Indeed, in view of (5.9,iv), it suffices to exhibit an irreducible semi-ordinary variety X such that X' is a hypersurface.

If $p = 2$, the smooth quadric hypersurface X in \mathbb{P}^4 , given by

$$X : vw = xy + xz + yz \quad ,$$

has the required properties, as is easy to verify.

If $p \geq 3$, consider the surface

$$X : z = x^2 + y^{p+1} .$$

The hyperplane H tangent to X at (x_0, y_0, z_0) is given by

$$H : z = 2x_0 x + y_0^p y - x_0^2 .$$

Hence X' is the locus of the point $(2x_0, y_0^p, -x_0^2)$. Therefore X' is a surface. Moreover X' is not birationally equivalent to X under the duality correspondence, so X is not ordinary. Now, take $P = (0,0,0)$ and $H : z = 0$. Then $h(P, H) = 1$, so X is semi-ordinary.

(ii) The proofs of (5.9,i) and (5.9,ii) work without change when M is a general hypersurface of any degree. Thus they yield a weaker version of (5.6).

(iii) Probably more than stated in (5.9,ii) is true. It is reasonable to conjecture that, if $p = 2$ and X is ordinary, then, as is the case in all other characteristics, a general tangent hyperplane to Y is tangent at a unique point of contact. Moreover, the map $q' : CY \rightarrow Y'$ should be purely inseparable of degree 2. If the first part of this conjecture is true, the proof of (4.10,ii) will work in any characteristic.

(iv) Assertion (5.9,iii) is dual to a result (Lem. 3, p. 334) of Wallace [1956], which asserts that, if X is not a hypersurface, then X is reflexive if and only if a general central projection of X is reflexive. Neither result implies the other, but Wallace's is easy to prove using (2.4,iv) and (2.5); for, a general hyperplane section maps birationally onto a general hyperplane section.

(5.12) Proposition. Let M be a general hyperplane. If Y is ordinary, then X is either ordinary or semi-ordinary.

Proof. Since Y is ordinary, then (Y/M) is ordinary by (5.8). By (2.4,ii and v) and (5.3,ii), we may assume that k is algebraically closed and that X is irreducible.

Let (x, L') be a k -point of $C(Y^{\text{sm}}/M)$ such that

$$h(\text{Sing}(Y \cap L)/Y/k; x) = \dim Y .$$

Obviously, there exists a k -point (x, H') of CX^{sm} such that $H \cap M = L$; hence

$$h(\text{Sing}(Y \cap H)/Y/k; x) = \dim Y . \quad (5.12.1)$$

If X is not ordinary, then by (5.2,ii,a)

$$h(\text{Sing}(X \cap H)/X/k; x) = h(\text{Sing}(Y \cap H)/Y/k; x) ;$$

hence, by (5.12.1)

$$h(\text{Sing}(X \cap H)/X/k; x) = \dim Y = \dim(X) - 1 .$$

Therefore X is semi-ordinary.

6. REFLEXIVITY FOR GRASSMANNIANS AND SEGRE VARIETIES

(6.1) Segre Varieties. Let n, m be positive integers with $n \leq m$. Let K be any field. The image of $\mathbb{P}_K^n \times \mathbb{P}_K^m$ in $\mathbb{P}_K^{(n+1)(m+1)-1}$ via the Segre embedding

$$\mathbb{P}_K^n \times \mathbb{P}_K^m \longrightarrow \mathbb{P}_K^{(n+1)(m+1)-1}$$

$$(X_0; \dots; X_n) \times (Y_0; \dots; Y_m) \longrightarrow (X_0 Y_0; \dots; X_i Y_j; \dots; X_n Y_m)$$

is the Segre variety $S_K(n, m)$. Calling w_{ij} , $(i, j) \in [0, n] \times [0, m]$, the coordinates of $\mathbb{P}_K^{(n+1)(m+1)-1}$, $S_K(n, m)$ is given by the following equations

$$w_{ij} w_{k\ell} = w_{kj} w_{i\ell}, \quad (i, j), (k, \ell) \in [0, n] \times [0, m].$$

In the open set $U = \{w_{00} \neq 0\}$, the equations of X are

$$w_{ij} - w_{0j} w_{i0} = 0 \quad (i, j) \in [0, n] \times [0, m]$$

where $w_{ij} = \frac{w_{ij}}{w_{00}}$. At the point $P = (1; 0; \dots; 0) \in S_K(n, m) \cap U$, the tangent space^{oo} to $S_K(n, m)$ is given by

$$w_{ij} = 0, \quad (i, j) \in [1, n] \times [1, m].$$

Hence a tangent hyperplane H to $S_K(n, m)$ at P has equation

$$H : \sum_{\substack{i>1 \\ j>1}} a_{ij} w_{ij} = 0$$

with $a_{ij} \in K$. Since w_{i0} and w_{0j} , $i=1, \dots, n$, $j=1, \dots, m$, are a regular system of parameters for $S_K(n, m)$ at P , we have that $H \cap X$ is

given locally in X at P by the equation

$$f = \sum_{\substack{i>1 \\ j>1}} a_{ij} w_{io} w_{oj} = 0 ,$$

and the hessian $(n+m) \times (n+m)$ matrix of f at P with respect to w_{io} and w_{oj} is made of the following blocks.

$$\left[\frac{\partial^2 f}{\partial w_{io} \partial w_{ko}} (P) \right] = [0] \quad \left[\frac{\partial^2 f}{\partial w_{io} \partial w_{oj}} (P) \right] = {}^t[(a_{ij})]$$

$$\left[\frac{\partial^2 f}{\partial w_{oj} \partial w_{io}} (P) \right] = [(a_{ij})], \quad \left[\frac{\partial^2 f}{\partial w_{oj} \partial w_{ol}} (P) \right] = [0]$$

Since for a general tangent hyperplane H at P , the a_{ij} are general elements in K , we have that

$$\text{rk}[(a_{ij})] = \min(n, m) = n .$$

Therefore

$$h(P, H) = 2n \tag{6.1.1}$$

To conclude our discussion we will need the following:

(6.2) Lemma* (Landman [1976]). Let X be an $(n+m)$ -dimensional variety ruled by \mathbb{P}^m over an n -dimensional variety Z in an N -dimensional projective space. Then

$$\dim X' \leq N - m - 1 + n .$$

Proof. Denoting by F_z the fiber over $z \in Z$, then clearly

* See also Mumford [1978]

$$X' \subset \bigcup_{z \in Z} F'_z$$

Hence

$$\dim X' \leq \dim F'_z + \dim Z = N - m - 1 + n$$

(6.3) Theorem. For an arbitrary field K , and for arbitrary positive integers m and n with $n \leq m$, we have

- (i) $S_K(n, m)$ is reflexive
- (ii) $\dim(S_K(n, m)') = (n+1)(m+1) - 2 - (m-n)$
- (iii) $S_K(n, m)$ is ordinary if and only if $n = m$
- (iv) $S_K(n, m)$ is semi-ordinary if and only if $n = m-1$.

Proof. From (2.6,i), (6.1.1) and (6.2) we have

$$2n = h(P, H) \leq \dim S_K(n, m) + \dim(S_K(n, m)') - (N-1) \leq 2n,$$

where $N = (n+1)(m+1) - 1$.

Now, (i) and (ii) follow from (3.2), while (iii) follows from (i), (ii) and (3.2), and (iv) follows from (5.3,ii).

(6.4) Grassmannians. Let $G_{d,n} \subset \mathbb{P}^N$, where $N = \binom{n+1}{d+1} - 1$, be the Grassmann variety of d -planes in \mathbb{P}_K^n embedded via the Plücker embedding.

Give \mathbb{P}^N coordinates $p(j_0, \dots, j_d)$ with $0 \leq j_0 < \dots < j_d \leq n$.

Then $G_{d,n}$ is cut out of \mathbb{P}^N by the following quadratic equations

$$(QR) \sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0, \dots, j_{d-1}, k_\lambda) p(k_0, \dots, k_\lambda^v, \dots, k_{d+1}) = 0$$

where j_0, \dots, j_{d-1} and k_0, \dots, k_{d+1} are any sequences of integers with $0 \leq j_\beta, k_\gamma \leq n$, k_λ^v means that the integer k_λ has been removed from the sequence, and $p(j_0, \dots, j_d)$ have to be interpreted as being skew-symmetric functions of j_0, \dots, j_d such that $p(j_0, \dots, j_d) = 0$ if any two j_β are equal (see for example Kleiman-Laksov [1972]).

Consider the following partition of the coordinates of \mathbb{P}^N in sets S_m , for $0 \leq m \leq d+1$, where

$$S_m = \{p(j_0, \dots, j_d) / \text{exactly } m \text{ of the integers } j_0, \dots, j_d \text{ are not among the integers } 0, 1, \dots, d\}.$$

If $j_\beta \in \{0, 1, \dots, d\}$, then from (QR) we get the relation

$$\begin{aligned} (-1)^{d-\beta} p(j_0, \dots, j_d) p(0, \dots, d) = \\ \sum_{\lambda=0}^d (-1)^\lambda p(j_0, \dots, j_\beta, \dots, j_d^\lambda) p(0, \dots, \lambda, \dots, dj_\beta) \end{aligned} \quad (6.4.1)$$

Let U be the open subset of \mathbb{P}^N defined by $p(0, \dots, d) \neq 0$. After making $p(0, \dots, d) = 1$, from (6.4.1) it is easy to see that every element of S_m can be expressed as a homogeneous polynomial of degree m in the elements of S_1 . In particular, S_1 is a regular system of parameters of $G_{d,n} \cap U$.

Now, it is clear that the quadratic term of the restriction to $G_{d,n}$ of the equation of a general tangent hyperplane to $G_{d,n}$ at $P = (1, 0, \dots, 0)$ is a general linear combination of the elements of S_2 . From (6.4.1) it follows, for $\alpha, \beta \in \{0, \dots, d\}$, that

$$\begin{aligned} p(0 \dots \overset{\vee}{\alpha} \dots \overset{\vee}{\beta} \dots d\mu\nu) = p(0 \dots \overset{\vee}{\beta} \dots d\mu) p(0 \dots \overset{\vee}{\alpha} \dots d\nu) - \\ - p(0 \dots \overset{\vee}{\alpha} \dots d\mu) p(0 \dots \overset{\vee}{\beta} \dots d\nu) \end{aligned} \quad (6.4.2)$$

If we put

$$X_{\alpha, \mu} = p(0 \dots \overset{\vee}{\alpha} \dots d\mu), \quad \alpha = 0, \dots, d; \quad \mu = d+1, \dots, n,$$

then S_1 is the set of these elements and from (6.4.2) we have

$$S_2 = \{x_{\beta,\mu} x_{\alpha,\nu} - x_{\alpha,\mu} x_{\beta,\nu}; \alpha, \beta = 0, \dots, d; \mu, \nu = d+1, \dots, n; \alpha < \beta \text{ and } \mu < \nu\} .$$

It follows then that the Hessian matrix associated to a general tangent hyperplane H to $G_{d,n}$ at P is given by

$$A = \begin{bmatrix} A_{00} & A_{01} & \dots & A_{0d} \\ A_{d0} & A_{d1} & \dots & A_{dd} \end{bmatrix} ,$$

where each block A_{ij} is an $(n-d) \times (n-d)$ matrix with entries in K , A is symmetric, A_{ii} , for $i = 0, \dots, d$, is the zero matrix, A_{ij} for $i < j$, is a general skew-symmetric matrix with zero diagonal entries.

Since $G_{d,n}$ is a homogeneous embedded variety, the rank of the Hessian matrix associated to a general tangent hyperplane at a general point is equal to the rank of the above A . Recall also that we have the following isomorphisms as embedded subvarieties of \mathbb{P}^N :

$$G_{d,n} \cong G_{n-d-1,n} \tag{6.4.3}$$

Now, to analyze the rank of A , it suffices, because of the above isomorphisms, to consider the following three cases:

- (a) $(d+1)$ and $(n-d)$ are even,
- (b) $(d+1)$ is odd and $(n-d)$ is even, and
- (c) $(d+1)$ and $(n-d)$ are odd.

Since the computation of the rank of a general A as above showed to be fairly complicated, we will do it only in some few cases and then state our conjectures for the remaining cases.

(6.5) Case (a). In this case we are able to compute the rank of A . Indeed, it is easy to produce a particular matrix A which is invertible, take for example

$$A = \begin{bmatrix} 0 & -B & & & 0 \\ B & 0 & & & \\ & & 0 & -B & \\ & & B & 0 & \\ & & & & \ddots & \\ & & & & & 0 & -B \\ & 0 & \dots & & & B & 0 \end{bmatrix},$$

where B is an $(n-d) \times (n-d)$ non-singular, skew-symmetric matrix with zero diagonal entries. Now, A corresponds to some tangent hyperplane H to $G_{d,n}$ at P , hence

$$h(P,H) = \text{rk}A = \dim G_{d,n}.$$

Therefore, from (3.2) it follows that $G_{d,n}$ is ordinary.

(6.6) Case (b). We will distinguish two subcases:

(b') $n - d = 2$. This is the same in view of (6.4.3) to study $G_{1,n}$ with n even. In this case A is equal to

$$\begin{bmatrix} 0 & -B \\ B & 0 \end{bmatrix},$$

where B is a skew-symmetric $(n-1) \times (n-1)$ matrix with zero diagonal entries. Hence a general such A has rank $2(n-2)$ (cf. Bourbaki [1959], Cor. 3, p. 81). Therefore for n even, $G_{1,n}$ is not ordinary.

If $\text{char}K = 0$, then $G_{1,n}$ is reflexive by the Segre-Wallace criterion, so for a general H we have, from (3.2), that

$$h(P,H) = \dim G_{1,n} + \dim G'_{1,n} - (N-1).$$

Since $h(P,H) = \text{rk}A = 2(n-2)$, it follows that

$$\dim G'_{1,n} = N-3.$$

Suppose now that $\text{char } K \neq 0$. D. Mumford [1978] computed the dimension of $G'_{1,n}$, his computation works in arbitrary characteristic and gives

$$\dim G'_{1,n} = N-3 .$$

It is easy to check that

$$h(P,H) = \dim G_{1,n} + \dim G'_{1,n} - (N-1) ,$$

hence by (3.2) it follows that $G_{1,n}$ is reflexive. So we proved the following result:

If n is even, then in arbitrary characteristic, $G_{1,n}$ is reflexive and $\dim G'_{1,d} = N-3$.

(b'') $n-d > 2$. It is not easy to determine the rank of a general A . We suspect that A is always invertible as some examples indicate. If this is true, then $G_{d,n}$ is ordinary.

(6.7) Case (c). Since also here it is hard to determine the rank of a general A , we will just state our conjecture.

$G'_{d,n}$ is a hypersurface and $G_{d,n}$ is ordinary if $\text{char } K \neq 2$ and semi-ordinary if $\text{char } K = 2$.

7. HYPERSURFACES WITH RANK ZERO LOCAL HESSIANS

(7.1) Keep the notations and hypotheses of (2.1) and (2.3).

(7.2) Definition. We will say that X has rank zero local Hessians, if for every K -point (x, H') of CX^{sm} , where K is a field, we have

$$h(x, H) = 0 .$$

From now up to (7.8), inclusive, we will assume that $S = \text{Spec}(k)$, where k is a field of characteristic $p > 0$. Note that if $\text{char}(k) = 0$, and X is geometrically reduced and irreducible, then X has rank zero local Hessians if and only if X is linear.

As always, K will denote an extension field of k . If

$$G \in K[Y_0, \dots, Y_N] ,$$

then G_i (resp. G_{ij}) will denote the partial derivative with respect to Y_i (resp. Y_j and Y_i) .

(7.3) Proposition. Let $p > 2$ and let $X : G = 0$, be a hypersurface in an $(n+1)$ -projective space P over k . If X has rank zero local Hessians, then for every algebraically closed field K we have the following set of identities on the K -points of X (hence on X_K)

$$2G_{ij}G_iG_j = G_i^2G_{jj} + G_j^2G_{ii} , \quad i, j = 0, \dots, n+1 . \quad (7.3.1)$$

Proof. The equalities are trivially satisfied on the rational points of $\text{Sing}(X_K)$. Let Q be a rational point of X_K^{sm} . We may assume that $Q = (1; a_1; \dots; a_{n+1})$ with $a_i \in K$, $i = 1, \dots, n+1$. Consider the Taylor expansion of G at Q :

$$G(1, y_1, \dots, y_{n+1}) = \sum_{i=1}^{n+1} G_i(Q) (y_i - a_i) + \frac{1}{2} \sum_{i, j=1}^{n+1} G_{ij}(Q) (y_i - a_i)(y_j - a_j) + \dots$$

Since by hypothesis X has vanishing Hessians, then

$$h(Q, (T_Q X)) = h(\text{Sing}(X_K \cap T_Q X_K)/X_K/K; Q) = 0 .$$

Now, since $\dim_Q(X_K) = \dim_Q(T_Q X_K)$, it follows from (2.1.1) that

$$h(\text{Sing}(X_K \cap T_Q X_K)/T_Q X_K/K; Q) = 0 .$$

This implies that the polynomial

$$\sum_{i,j=1}^{n+1} G_{ij}(Q) (y_i - a_i)(y_j - a_j) \quad (7.3.2)$$

vanishes at the K -points of

$$\sum_{i=1}^{n+1} G_i(Q) (y_i - a_i) = 0 . \quad (7.3.3)$$

Therefore the polynomial in the left side of (7.3.3) divides the polynomial in (7.3.2). Now, a straightforward computation, using Euler's identity for homogeneous polynomials, shows that

$$\sum_{i=1}^{n+1} G_i(Q) (y_i - a_i) = \sum_{i=0}^{n+1} G_i(Q) y_i ,$$

and that

$$\sum_{i,j=1}^{n+1} G_{ij}(Q) (y_i - a_i)(y_j - a_j) = \sum_{i,j=0}^{n+1} G_{ij}(Q) y_i y_j - 2(\deg(G)-1) \sum_{i=0}^{n+1} G_i(Q) y_i ,$$

where $y_0 = 1$. Hence there exist elements b_0, \dots, b_{n+1} in K such that

$$\sum_{i,j=0}^{n+1} G_{ij}(Q) Y_i Y_j = \left(\sum_{j=0}^{n+1} b_j Y_j \right) \left(\sum_{i=0}^{n+1} G_i(Q) Y_i \right) .$$

Identifying the corresponding terms we get the equalities

$$b_i G_i(Q) = G_{ii}(Q) ,$$

and

$$b_i G_j(Q) + b_j G_i(Q) = 2G_{ij}(a) .$$

These equalities imply (7.3.1) for all $i, j = 0, \dots, n+1$, at any smooth rational point Q of X_K . This completes the proof.

(7.4) Theorem. Let $p > 2$. Let $X : G = 0$ be a hypersurface in \mathbb{P}_k^{n+1} which is smooth in codimension one. If X has rank zero local Hessians, then for every $i, j = 0, \dots, n+1$ we have that $G_{ij} = 0$ in $k[Y_0, \dots, Y_{n+1}]$.

Proof. If for some i , $G_i = 0$, then $G_{ij} = 0$ for all $j = 0, \dots, n+1$. Suppose now that $G_i \neq 0$. Let K be the algebraic closure of k . We denote by S_i the intersection of the hypersurfaces in \mathbb{P}_k^{n+1} defined by $G = 0$ and $G_i = 0$. Let V be an irreducible component of S_i . Since the codimension of $\text{Sing}(X_K)$ in X_K is greater or equal than 2, there exists an index j different from i such that V is not a component of S_j . So in $\mathcal{O}_{X_K, V}$ we have

$$\text{ord}_V(G_j) = 0 .$$

From this and from (7.3.1) we have that

$$\text{ord}_V(G_{ii}) \geq \text{ord}_V(G_i) . \quad (7.4.1)$$

Now, if $G_{ii} \neq 0$, then it follows from (7.4.1) that every irreducible component V of S_i is an irreducible component of the intersection S_{ii} in \mathbb{P}_K^{n+1} of $\{G = 0\}$ and $\{G_{ii} = 0\}$. Set $d = \deg G$. Then from Bézout's inequality (cf. Fulton [1984], Eg. 12.3.1, page 223) we have that

$$(d-2)d = (\deg G_{ii})(\deg G) \geq \sum_{\substack{V=\text{irred.} \\ \text{Comp. of } S_{ii}}} (\text{ord}_V(G_{ii}))(\deg V) \geq$$

$$\sum_{\substack{V=\text{irred.} \\ \text{Comp. of } S_{ii}}} (\text{ord}_V(G_i))(\deg V) \geq (\deg G_i)(\deg G) = (d-1)d ,$$

contradiction. Therefore, $G_{ii} = 0$ for all $i = 0, \dots, n+1$.

Now, for all i, j we have from (7.3.1) the following set of identities on X_K :

$$G_{ij}G_iG_j = 0 .$$

Note that the hypothesis on the singularities of X imply that X is geometrically irreducible, so from the above identity it follows that on X_K , either $G_i = 0$ or $G_j = 0$. But anyone of these conditions implies that $G_{ij} = 0$ in $k[Y_0, \dots, Y_{n+1}]$.

(7.5) Remarks. (i) For $n = 1$, (7.4) was first proved by R. Pardini [1983].

(ii) The hypothesis of regularity in codimension one is essential in theorem (7.4). Indeed, for $p > 2$ the plane curve $X : Y_0^{p-2}Y_1^2 + Y_2^p = 0$ is not reflexive because clearly the point $(0;0;1)$ is a strange point of X . Hence X has rank zero local Hessians but, nevertheless, $G_{11} = 2Y_0^{p-2} \neq 0$.

(iii) Theorem (7.4) has trivially the following converse:

Let $p > 2$, and let X be defined in \mathbb{P}_k^{n+1} by $G = 0$. If $G_{ij} = 0$ for all $i, j = 0, \dots, n+1$, then X has rank zero local Hessians.

(7.6) Corollary. The hypotheses being as in (7.4), we have that

$$p \mid (\deg(G) - 1) .$$

Proof. From (7.4) and from Euler's identity we have for all $j = 0, \dots, n+1$ that

$$0 = \sum_{i=0}^{n+1} G_{ij} Y_i = (\deg(G) - 1)G_j .$$

Since not all the G_j 's are zero, it follows that $p \mid (\deg(G) - 1)$.

(7.7) Corollary. The hypotheses being as in (7.4), there exist

homogeneous polynomials P_0, \dots, P_{n+1} in $k[T_0, \dots, T_{n+1}]$, all of those which are not zero of the same degree, such that

$$G(Y_0, \dots, Y_{n+1}) = Y_0 P_0(Y_0^P, \dots, Y_{n+1}^P) + \dots + Y_{n+1} P_{n+1}(Y_0^P, \dots, Y_{n+1}^P).$$

Proof. From (7.6) and Euler's identity we have that

$$G(Y_0, \dots, Y_{n+1}) = Y_0 G_0(Y_0, \dots, Y_{n+1}) + \dots + Y_{n+1} G_{n+1}(Y_0, \dots, Y_{n+1}).$$

Now, since by (7.4), for every i all the partial derivatives of G_i are zero, it follows that the G_i 's are homogeneous polynomials in Y_0^P, \dots, Y_{n+1}^P .

(7.8) Corollary. If $p \geq 3$, then the smooth hypersurfaces of degree $\ell p + 1$ in \mathbb{P}_k^{n+1} with rank zero local Hessians are the rational points of a non-empty open set S in a projective space over k of dimension $(n+2) \binom{\ell+n+1}{n+1} - 1$.

Proof. The dimension follows from the count of coefficients in (7.7), the openness comes from elimination theory and the non-emptiness follows by observing that the hypersurface

$$Y_0^{\ell p + 1} + \dots + Y_{n+1}^{\ell p + 1} = 0$$

has obviously the required properties.

(7.9) Let S be as in (7.8) and let V be an $(n+2)$ -dimensional k -vector space. Set $P = \mathbb{P}(V_S) = S \times_k \mathbb{P}(V)$. Define X to be the graph of the point-hypersurface incidence correspondence in P . So X is the total family of all smooth hypersurfaces in \mathbb{P}_k^{n+1} of degree $\ell p + 1$ and with vanishing Hessians.

Now, X/S is flat since X is a family of divisors parameterized by S (cf. Mumford [1966] p. 72). Since every fiber of X over S is smooth of dimension n , it follows that X/S is smooth of relative dimension n

(cf. EGAIV₄, 17.5.1, p. 67). It is clear that X has rank zero local Hessians and that we have an S -isomorphism $C(X/P) \simeq X$.

(7.10) m -Jacobians. Keep the notation and hypotheses of (2.1) and (2.3). Let X be a closed subscheme of $P = \mathbb{P}(E)$, smooth over S of relative dimension n . Let $Z = X \times_S P^*$. Let m be a positive integer. The m -Jacobian $J_m(X/P)$ is the subscheme of Z defined by

$$J_m(X/P) = \mathbb{P}(\text{Coker}(w^V)) ,$$

where w is the natural map from E_X to the sheaf of $(m-1)$ -principal parts $P_{X/S}^{m-1}(\mathcal{O}_X(1))$. It is standard theory (see for example Vainsencher [1981], page 404) that, locally, $J_m(X/P)$ is cut out of Z by the set of equations

$$D_{(i)} F = 0 , \quad |(i)| \leq m-1 , \quad (7.10.1)$$

where $i = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$, $|(i)| = i_1 + \dots + i_n$, the $D_{(i)}$'s are the Hasse differential operators of Z over P^* (cf. EGAIV₄, 16.11), and F is a local equation of the incidence correspondence I in Z .

It is known, and easy to verify from the definitions, that $J_m(X/P)$ commutes with base change. It follows easily from (2.5.2), (2.2.i) and the above description of $J_m(X/P)$ that

$$J_2(X/P) = C(X/P)$$

(7.11) Proposition. Keep the hypothesis of (7.10) and assume that all the residual fields of the points of S have the same characteristic $p > 0$. If X has rank zero local Hessians and if for some $s \in S$, X_s is not a linear subspace of P_s , then there exists a positive integer e such that

$$J_2(X/P) = \dots = J_{p^e}(X/P) \supset J_{p^{e+1}}(X/P) .$$

Proof. For every point $s \in S$, it follows by (2.1.1) that X_s has rank

zero local Hessians. Hence by the same argument we used in the proof of (a) \Rightarrow (b') in theorem (3.5), for every $s \in S$ for which X_s is non-linear, there exists a positive integer $e(s)$ such that

$$J_2(X_s/P_s) = \dots = J_{p^{e(s)}}(X_s/P_s) \supsetneq J_{p^{e(s)+1}}(X_s/P_s) .$$

put

$$e = \min\{e(s) \mid X_s \text{ is non-linear}\} .$$

By the commutativity of the formation of $J_m(X/P)$ with base change, we have for all $s \in S$ that

$$J_2(X/P)_s = \dots = J_{p^e}(X/P)_s ,$$

and for s_0 realizing the minimum e , that

$$J_{p^e}(X/P)_{s_0} \supsetneq J_{p^{e+1}}(X/P)_{s_0} .$$

This proves the proposition.

(7.12) Remark. In some cases we can give an interpretation for the integer p^e . For example, when $S = \text{Speck}$ and $\dim X + \dim X' = \dim P$, then the proof of (3.5) shows that

$$p^e = [k(CX) : k(X')]_i .$$

See also (8.8) for another result about p^e .

8. A CLASS OF HYPERSURFACES

(8.1) Let k be algebraically closed of characteristic $p > 0$, and let n be a positive integer. In this section we will study the smooth hypersurfaces M in an $(n+1)$ -projective space over k given by equations of the form

$$G(Y_0, \dots, Y_{n+1}) = Y_0 L_0(Y_0^q, \dots, Y_{n+1}^q) + \dots + Y_{n+1} L_{n+1}(Y_0^q, \dots, Y_{n+1}^q) = 0, \quad (8.1.1)$$

where $q = p^e$ for some positive integer e , and where for every $i = 0, \dots, n+1$,

$$L_i(T_0, \dots, T_{n+1}) = \sum_{j=0}^{n+1} a_i^j T_j$$

with $a_i^j \in k$ for $i, j = 0, \dots, n+1$.

If $p > 2$, these hypersurfaces have, by (7.5,iii), rank zero local Hessians. We do not make this restriction here.

The smoothness of M implies that

$$\det(a_i^j)_{i,j} \neq 0$$

Hence all the hypersurfaces like M , with fixed e , form a smooth family X over

$$S = \text{PGL}(n+1, k),$$

of relative dimension n , embedded in $P = S \times_k \mathbb{P}_k^{n+1}$.

(8.2) Notation. If $C = (C_i^j)_{i,j}$ is any matrix with entries in k and $r = p^s$, where s is any positive integer, then we define

$$C^{(r)} = ((C_i^j)^r)_{i,j} .$$

Note that $C^{(r)}$ is not the r -th power of C with respect to matrix product. It is clear that, whenever the matrices involved are compatible with the situation, we have the equalities:

$$(C+D)^{(r)} = C^{(r)} + D^{(r)} , \quad (C.D)^{(r)} = C^{(r)}.D^{(r)} , \quad ({}^t C)^{(r)} = {}^t(C^{(r)}) ,$$

$$(C^{-1})^{(r)} = (C^{(r)})^{-1} , \quad \det(C^{(r)}) = (\det C)^{(r)} .$$

If $C \in \mathbb{A}_k^{n+2} \setminus \{0\}$ and $D \in GL(n+2, k)$, then we will use the notations $[C]$ and $[D]$ to represent respectively the class of C and D in \mathbb{P}_k^{n+1} and in $PGL(n+1, k)$.

(8.3) With these notations, equation (8.1.1) for M can be written matrixially as follows

$$G(Y_0, \dots, Y_{n+1}) = YA({}^t Y)^{(q)} = 0 , \quad (8.3.1)$$

where

$$Y = (Y_0, \dots, Y_{n+1}) ,$$

and

$$A = (a_i^j)_{i,j} .$$

It is clear that $([A], [Y], [Y^*]) \in C(X/P)$ if and only if (8.3.1) and (8.3.2) below hold.

$$[Y^*] = [Y^{(q)} {}^t A] . \quad (8.3.2)$$

From (8.3.2) we get

$$[Y^{(q)}] = [Y^* \quad {}^t A^{-1}] . \quad (8.3.3)$$

Raising (8.3.1) to the q -th power and using (8.3.2) we get the following equation for M' , the dual of M ,

$$Y^* ({}^t A^{-1}) ({}^t Y^*)^{(q)} = 0 . \quad (8.3.4)$$

(8.4) Remarks. (i) From (8.3.4) it follows that M' is a smooth hypersurface of the same type of M . It is also clear that M satisfies biduality, i.e. $(M')' = M$; and that M is self dual if and only if $[A] = [{}^t A^{-1}]$.

(ii) From (8.3.3), it follows that for every $[Y^*] \in M'$, the contact locus of $[Y^*]'$ in M is just one point. In particular we have that the maps $q' : CM \rightarrow M'$ and $q' \circ q^{-1} : M \rightarrow M'$ are purely inseparable. From this and by the general Plücker formula,

$$\deg q' \cdot \deg X' = \deg X \cdot (\deg(X) - 1)^n ,$$

(see Kleiman [1977], IV, 49, p. 357), we have that

$$[k(X) : k(X')]_i = q^n .$$

(iii) Applying (8.3.2) twice, we see that the map

$$b : M \xrightarrow{\sim} CM \rightarrow M' \xrightarrow{\sim} CM' \rightarrow (M')' = M$$

is given by

$$[Y] \rightarrow [Y^{(q^2)} ({}^t A)^{(q)} A^{-1}]$$

(8.5) In the next theorem we will give explicit equations defining the schemes $J_2(X/P), \dots, J_q(X/P), \dots$, in $X \times_k \mathbb{P}_k^{n+1*}$.

Let U be the open set of X , over an affine open subset of S (which we still denote by S), defined by

$$Y_0 \neq 0 \quad \text{and} \quad (A({}^t Y)^{(q)})_{n+1} \neq 0 .$$

Set $V = U \times_k \mathbb{P}_k^{n+1*}$. From (8.3.2) it is clear that $C(X/P) \cap V$ is contained in the open subset v_0 of V defined by $Y_{n+1}^* \neq 0$. Set

$$y_i = \frac{Y_i}{Y_0} \Big|_V, \quad i = 1, \dots, n+1,$$

and set

$$y_i^* = \frac{Y_i^*}{Y_{n+1}^*} \Big|_{V_0}, \quad i = 0, \dots, n .$$

Let $D_{(\ell_1, \dots, \ell_n)}$, for $(\ell_1, \dots, \ell_n) \in \mathbb{Z}_+^n$, be the Hasse differential operators on V^n over $S \times_k \mathbb{P}_k^{n+1*}$ associated to y_1, \dots, y_n .

We denote by D_i^j , for $1 \leq i \leq n$ and $j \geq 0$, the differential operator $D_{(\ell_1, \dots, \ell_n)}$ such that $\ell_r = 0$ if $r \neq i$ and $\ell_i = j$. Clearly we have, for every permutation σ of $1, \dots, n$, that

$$D_{(\ell_1, \dots, \ell_n)} = D_{\sigma(n)}^{\ell_{\sigma(1)}} \circ \dots \circ D_{\sigma(n)}^{\ell_{\sigma(n)}} .$$

Let

$$F = y_0^* + y_1 y_1^* + \dots + y_n y_n^* + y_{n+1} = 0 \quad (8.5.1)$$

be the equation of the incidence correspondence I in V_0 .

(8.6) Lemma. For every $i = 1, \dots, n$, we have on $J_q(X/P) \cap V$,

$$D_i^{q,q} y_{n+1}^q = -y_i^*{}^q .$$

Proof. From (7.10.1) we have on $J_q(X/P)$ that

$$D_i^j F = 0 \quad \text{for} \quad 0 \leq j \leq q-1 \quad \text{and} \quad 1 \leq i \leq n .$$

From Leibniz formula for Hasse differential operators (see Teichmüller [1936], (4) p. 91) we have on $J_q(X/P) \cap V$, for all $i = 1, \dots, n$, that

$$D_i^q F^q = \sum_{j=0}^q (D_i^j F) (D_i^{q-j} F^{q-1}) = 0 .$$

Hence on $J_q(X/P) \cap V$ and for every $i = 1, \dots, n$, we have for F as defined in (8.5.1) that

$$0 = D_i^q (y_0^{*q} + y_1^q y_1^{*q} + \dots + y_n^q y_n^{*q} + y_{n+1}^q y_{n+1}^{*q}) = y_i^{*q} + D_i^q y_{n+1}^q .$$

From which the lemma follows.

(8.7) Proposition (i) If $\ell_1 + \dots + \ell_n > 1$ and for some i , $1 \leq \ell_i \leq q-1$, then

$$D_{(\ell_1, \dots, \ell_n)} F = 0 \quad \text{on} \quad V_0 .$$

(ii) For every $i = 1, \dots, n$, we have on $C(X/P) \cap V$ that

$$D_i^q F = [-(YA)_i + y_i^{*q} (YA)_{n+1}] [(A({}^t Y)^{(q)})_{n+1}]^{-1} ,$$

where $Y = (1, y_1, \dots, y_{n+1})$.

Proof. From (8.5.1) we have, for every (ℓ_1, \dots, ℓ_n) such that $\ell_1 + \dots + \ell_n > 1$,

$$D_{(\ell_1, \dots, \ell_n)} F = D_{(\ell_1, \dots, \ell_n)} y_{n+1} \quad (8.7.1)$$

Now, to compute the derivative in the right side of (8.7.1), we apply the operator $D_i^{\ell_i}$ to the equation (8.3.1) where $Y = (1, y_1, \dots, y_{n+1})$.

By Leibniz formula we have

$$0 = D_i^{\ell_i} (YA({}^tY)^{(q)}) = \sum_{j=0}^{\ell_i} (D_i^j Y) A(D_i^{\ell_i-j} ({}^tY)^{(q)}) \quad (8.7.2)$$

From Leibniz formula and by induction it is easy to check that

$$D_i^{\ell_i} y_j^q = \sum_{\nu=1}^{\ell_i} \sum_{\substack{\lambda_1 + \dots + \lambda_\nu = \ell_i \\ \lambda > 0}} \binom{q}{\nu} y_j^{q-\nu} D_i^{\lambda_1} y_j \dots D_i^{\lambda_\nu} y_j,$$

(cf. Teichmüller [1936], p. 92). Hence

$$D_i^{\ell_i} ({}^tY)^{(q)} = 0 \quad \text{for} \quad 1 \leq \ell_i \leq q-1.$$

So from (8.7.2) we have that

(i) if $1 \leq \ell_i \leq q-1$, then

$$0 = (D_i^{\ell_i} Y) A({}^tY)^{(q)} = (D_i^{\ell_i} y_i) (A({}^tY)^{(q)})_i + (D_i^{\ell_i} y_{n+1}) (A({}^tY)^{(q)})_{n+1} \quad (8.7.3)$$

Suppose that for some i , $\ell_i > 1$. Since $D_i^{\ell_i} y_i = 0$ and $(A({}^tY)^{(q)})_{n+1} \neq 0$ on V_0 , then

$$D_i^{\ell_i} y_{n+1} = 0 \quad \text{on} \quad V_0.$$

Suppose that for every i , $\ell_i \leq 1$. Then there exist i and j with $i \neq j$ such that $\ell_i = \ell_j = 1$, hence from (8.7.3) by differentiating the left and right sides with respect to D_j^1 , we get $D_j^1 D_i^1 y_{n+1} = 0$ on V_0 . So in any case we have on V_0 that

$$D_{(\ell_1, \dots, \ell_n)} y_{n+1} = D_1^{\ell_1} \circ \dots \circ \hat{D}_i^{\ell_i} \circ \dots \circ \hat{D}_j^{\ell_j} \circ \dots \circ D_n^{\ell_n} (D_j^{\ell_j} (D_i^{\ell_i} y_{n+1})) = 0,$$

and the proof of (i) is complete.

(ii) if $\ell_i = q$, then

$$\begin{aligned}
0 &= YA(D_i^q({}^tY)^{(q)}) + (D_i^q Y)A({}^tY)^{(q)} = & (8.7.4) \\
&= (YA)_i + (YA)_{n+1} (D_i^q y_{n+1}^q) + (D_i^q y_{n+1}^q) (A({}^tY)^{(q)})_{n+1}
\end{aligned}$$

Now, from (i) we have that $C(X/P) \cap V = J_q(X/P) \cap V$, so by (8.6) and (8.7.4), (ii) follows.

(8.8) Theorem. (i) $J_2(X/P) = \dots = J_q(X/P)$, scheme theoretically.
(ii) $([A], [Y], [Y^*])$ is a K -point of $J_{q+1}(X/P)$ if and only if the following conditions are satisfied.

$$(a) \quad YA({}^tY)^{(q)} = 0, \quad (b) \quad [Y^*] = [Y^{(q)}]{}^tA \text{ and}$$

$$(c) \quad [Y^{(q^2)}]({}^tA)^{(q)} = [YA].$$

(iii) $J_q(X/P) \supset J_{q+1}(X/P) \neq \emptyset$.

Proof. Note that $C(X/P)$ can be covered by sets like V_0 on which statements analogous to (8.7) hold. It is clear now that (i) follows from (8.7,i).

In view of (i) we have that on V_0 , $J_{q+1}(X/P)$ is cut out of $C(X/P)$ by $D_i^q F = 0$, $i = 1, \dots, n$. Since (a) and (b) are equivalent to $([A], [Y], [Y^*]) \in C(X/P)$, from (8.7,ii) we have that $([A], [Y], [Y^*]) \in J_{q+1}(X/P)$ if and only if (a), (b) and (c') below, are satisfied

$$(c') \quad (YA)_i = y_i^{*q}(YA)_{n+1}, \quad i = 1, \dots, n.$$

To complete the proof of (ii) we have only to show that (a), (b) and (c') are equivalent to (a), (b) and (c). Clearly (a), (b) and (c) imply (a), (b) and (c'). To show the converse, since $YA \neq 0$ (recall that A is invertible), it suffices to show that (c') also holds when $i = 0$. Multiplying both sides of (c') by y_i^q and summing the equalities for $i = 1, \dots, n+1$ (note that (c') is trivially satisfied when $i = n+1$), we get

$$\sum_{i=1}^{n+1} (YA)_i y_i^q = \left(\sum_{i=1}^{n+1} y_i^q y_i^{*q} \right) (YA)_{n+1} .$$

So, from (a) and (8.5.1) (which is implied by (a) and (b)), we get

$$(YA)_0 = y_0^{*q} (YA)_{n+1} ,$$

and the proof of (ii) is complete.

To prove (iii) it suffices to note that $C(X/P)_{Id} \neq J_{q+1}(X/P)_{Id} \neq \emptyset$, where Id is the identity matrix; indeed, the scheme on the left side is positive dimensional, while the scheme in the middle is zero-dimensional. Now observe that if $a \in k$ is such that $a^{q+1} = -1$, then for $Y = (0, \dots, 1, a)$ and $Y^* = (0, \dots, 1, a^q)$ we have $([Y], [Y^*]) \in J_{q+1}(X/P)_{Id}$.

(8.9) Let A be a K -point of S . The K -points of

$$J_{q+1}(X/P)_A = J_{q+1}(X_A/\mathbb{P}_K^{n+1})$$

have a nice geometric interpretation. From (8.4,iii) and (8.8,ii) it follows that $([Y], [Y^*])$ is a K -point of $J_{q+1}(X_A/\mathbb{P}_K^{n+1})$ if and only if $[Y]$ is a fixed point of the map

$$b : X_A \rightarrow X_A ,$$

defined in (8.4,iii).

The next theorem, essentially due to Hasse [1936] (Satz 10, 11), for which we offer a geometric proof, will imply that the map b (or any of its iterated) has finitely many fixed points.

(8.10) Theorem (Hasse [1936]) Let K be an algebraically closed field of characteristic $p > 0$. For every $B \in GL(N+1, K)$ and for every positive power q of p , there exists $T \in GL(N+1, K)$ such that

$$T^{(q)}_{BT^{-1}} = \text{Id} .$$

Proof. Set $S = \text{GL}(N+1, K)$, and let

$$W = \{(B, [Y]) \in S \times \mathbb{P}^N / \Lambda^2 \begin{pmatrix} Y^{(q)} B \\ Y \end{pmatrix} = 0\} \subset S \times \mathbb{P}^N .$$

It is clear that the projection $\pi : W \rightarrow S$ is proper. Let U_i be the open set in $S \times \mathbb{P}^N$ defined by $Y_i \neq 0$. It is clear that on $U_i \cap W$ we have that $(Y^{(q)} B)_i \neq 0$. If P is a point of W , we may assume that $P \in W \cap U_0$, so W is defined around P by the vanishing of

$$g_i = (Y^{(q)} B)_0 y_i - (Y^{(q)} B)_i , \quad i = 1, \dots, N ,$$

where $y_i = \frac{Y_i}{Y_0}$, $i = 1, \dots, N$, and $Y = (1, y_1, \dots, y_N)$.
Since

$$dg_i(P) = (P^{(q)} B)_0 dy_i, \quad i = 1, \dots, N ,$$

and $(P^{(q)} B)_0 \neq 0$, it follows that the $dg_i(P)$, $i = 1, \dots, N$, form a basis for $\Omega^1_{S \times \mathbb{P}^N / S}(P)$ over $k(P)$. Hence from the Jacobian criterion (SGA I, Exposé II, Thm. 4.10) we have that W/S is smooth at P of relative dimension zero. It follows that $\pi : W \rightarrow S$ is étale. In particular, π is quasi-finite, and since it is proper, by Chevalley's theorem (cf. EGAIII, 4.4.2), it follows that π is finite. So π is an étale covering.

Since π is an étale covering, all its geometric fibers have the same number of elements s (cf. EGAIV₄, Cor. 18.2.9). Therefore for any $B \in \text{GL}(N+1, K)$, the equation $[Y^{(q)} B] = [Y]$ has s solutions in \mathbb{P}^N_K . Since trivially the number r of solutions in \mathbb{A}^{N+1}_K of

$$Y^{(q)} B = Y \tag{8.10.1}$$

is related to s by the formula

$$r = s(q-1) + 1 ,$$

it follows that r is independent from B . Now, by taking $B = \text{Id}$ in (8.10.1), we find that

$$r = q^{N+1} . \quad (8.10.2)$$

Let u_1, \dots, u_ℓ be a maximal set of (linearly) independent solutions in A_K^{N+1} of (8.10.1). Hence all solutions of (8.10.1) are of the form

$$u = \lambda_1 u_1 + \dots + \lambda_\ell u_\ell ,$$

with $\lambda_i \in K$ such that $\lambda_i^q = \lambda_i$, $i = 1, \dots, \ell$. Therefore (8.10.1) admits q^ℓ solutions, and from (8.10.2) it follows that $\ell = N+1$.

Now, let $T \in \text{GL}(N+1, K)$ be such that

$$u_i = e_i T , \quad i = 1, \dots, N+1 ,$$

where $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$, \dots , $e_{N+1} = (0, 0, \dots, 1)$.
So

$$(e_i T)^{(q)}_B = e_i T , \quad i = 1, \dots, N+1 ,$$

therefore

$$T^{(q)}_B T^{-1} = \text{Id}.$$

(8.11) Corollary. Keep the hypotheses of (8.10). If $D \in A_K^{N+1}$, then the equation

$$Y^{(q)}_B = Y + D$$

has exactly q^{N+1} solutions in \mathbb{A}_K^{N+1} .

Proof. Let T be such that $T^{(q)}BT^{-1} = \text{Id}$ and put $Y = ZT$.

So far, $J_{q+1}(X/P)$ is only defined for positive integers n . We will define $J_{q+1}(X/P)$ for $n = 0$ by equations (a), (b) and (c) of (8.8,ii). The proof of (8.8,ii) shows that also when $n = 0$, we have that $J_{q+1}(X/P) \neq \emptyset$

(8.12) Corollary. For every K -point A of S , $J_{q+1}(X_A/\mathbb{P}_K^{n+1})$ is at most finite.

Proof. Obviously we may assume that K is algebraically closed. Put $B = {}^t_A^{(q)}A^{-1}$ and $D = 0$ in Corollary (8.11). Then the equation

$$Y^{(q^2)}({}^t_A^{(q)}A^{-1}) = Y \tag{8.12.1}$$

has exactly $q^{2(n+2)}$ solutions in \mathbb{A}_K^{n+2} . Therefore the equation (8.12.1) viewed in \mathbb{P}_K^{N+1} has finitely many solutions. This implies the result by (8.8,ii).

9. CLASSIFICATION PROBLEM FOR A CLASS OF HYPERSURFACES

(9.1) Keep the hypotheses and notations of section 8. In this section points mean closed points (hence rational). Let n be a non negative integer.

The action of $\text{PGL}(n+1, k)$ on \mathbb{P}_k^{n+1} by linear change of coordinates,

$$\text{PGL}(n+1, k) \times \mathbb{P}_k^{n+1} \longrightarrow \mathbb{P}_k^{n+1},$$

$$([T], [Y]) \longrightarrow [YT^*]$$

where T^* is the cofactor matrix of T (i.e., $T^* = (\det T)^{-1} {}^t T^{-1}$), induces a natural action on $\mathbb{P}_k^{(n+2)^2-1} \times_k I$, where I is the incidence correspondence in $\mathbb{P}_k^{n+1} \times \mathbb{P}_k^{n+1}$, given by

$$\text{PGL}(n+1, k) \times \mathbb{P}_k^{(n+2)^2-1} \times_k I \longrightarrow \mathbb{P}_k^{(n+2)^2-1} \times_k I \quad (9.1.1)$$

$$([T], ([A], [Y], [Y^*])) \longrightarrow ([{}^t \text{TAT}^{(q)}], [YT^*], [Y^*T]) .$$

Notice that

$$([T], [Y], [Y^*]) \longrightarrow ([YT^*], [Y^*T]) \quad (9.1.2)$$

is a transitive action of $\text{PGL}(n+1, k)$ on I .

(9.2) Remarks. (i) Let $W \subset \mathbb{P}_k^{(n+2)^2-1} \times_k I$ be defined by

$$W = \{([A], [Y], [Y^*]) / Y A ({}^t Y)^{(q)} = 0, \Lambda^2 \begin{pmatrix} Y^{(q)} & {}^t A \\ & Y^* \end{pmatrix} = 0, \Lambda^2 \begin{pmatrix} Y^{(q^2)} & {}^t A \\ & Y A \end{pmatrix}^{(q)} = 0\}$$

and let $\pi_1 : W \rightarrow \mathbb{P}_k^{(n+2)^2-1}$ and $\pi_2 : W \rightarrow I$ be the projections. From (8.8,ii) we have that $\pi_1^{-1}(S) = J_{q+1}(X/P)$, hence $J_{q+1}(X/P)$ is an open

subset of W .

(ii) The fibers of π_2 are not irreducible. For example, if $Y = (1, 0, \dots, 0)$ and $Y^* = (0, \dots, 0, 1)$, then we have

$$\pi_2^{-1}([Y]; [Y^*]) = \{[A]/A_i^0 = 0, i = 0, \dots, n+1\} \cup \{[A]/A_0^j = A_i^0 = 0, i, j = 0, \dots, n\}.$$

(iii) It can be shown that the action in (9.1.1) restricts to an action on W and that this action is compatible with π_2 and with the action in (9.1.2).

(9.3) The action in (9.1.1) restricts clearly to an action on $J_{q+1}(X/S)$ and this action is clearly compatible with $\pi_2 : J_{q+1}(X/P) \rightarrow I$ and with the action in (9.1.2). More precisely, if $([Z], [Z^*]) = ([YT^*], [Y^*T])$, then

$$\pi_2^{-1}([Z], [Z^*]) = {}^t_T(\pi_2^{-1}([Y], [Y^*]))T^{(q)} \quad (9.3.1)$$

(9.4) Proposition. $J_{q+1}(X/P)$ is irreducible and of the same dimension as S .

Proof. Let $P_0 = (1; 0; \dots; 0)$ and $P_0^* = (0; \dots, 0; 1)$. From (8.8,ii) we have that

$$F = \pi_2^{-1}(P_0; P_0^*) = \{[A] \in S/A_0^0 = A_1^0 = \dots = A_n^0 = A_0^1 = \dots = A_0^n = 0\}.$$

F is an open subset of a linear subspace of $\mathbb{P}^{(n+2)^2-1}$ of codimension $2n+1$.

We will show that every point of I has a neighborhood U such that $\pi_2^{-1}(U)$ is irreducible and is of the same dimension as S . This is sufficient to prove the proposition.

Let $([Y], [Y^*])$ be an arbitrary point of I . Fix a point (Q_{n+1}, Q_{n+1}^*) in I such that

$$[Y] \notin (Q_{n+1}^*)' \quad \text{and} \quad [Y^*] \notin Q_{n+1}' .$$

Fix n lines in \mathbb{P}_k^{n+1} , through Q_{n+1} , such that they generate $(Q_{n+1}^*)'$.
Let

$$U = I \setminus ((Q_{n+1}^*)' \times \mathbb{P}_k^{n+1*} \cup \mathbb{P}_k^{n+1} \times Q_{n+1}') .$$

Now, to every $([Z], [Z^*])$ in U , we may associate the points

$$Q_0 = [Z], \quad Q_1 = \ell_1 \cap [Z^*]', \quad \dots, \quad Q_n = \ell_n \cap [Z^*]', \quad Q_{n+1}$$

which form a reference frame of \mathbb{P}_k^{n+1} , hence there is a unique linear transformation $\tilde{T}_{(Z, Z^*)}$ of \mathbb{P}_k^{n+1} such that

$$\tilde{T}_{(Z, Z^*)}(Q_i) = P_i, \quad i = 0, \dots, n+1$$

where

$$P_0 = (1; \dots; 0), \quad P_1 = (0; 1; \dots; 0), \quad \dots, \quad P_{n+1} = (0; \dots; 0; 1) .$$

It is clear that the induced transformation maps $[Z^*]$ into P_0^* , and that we have an isomorphism

$$\pi_2^{-1}(U) \xrightarrow{\sim} U \times F$$

$$([A], [Z], [Z^*]) \longrightarrow ([Z], [Z^*], \tilde{T}_{(Z, Z^*)}(A))$$

Therefore $\pi_2^{-1}(U)$ is irreducible and clearly of the same dimension as S .

(9.5) Remark. The proof of proposition (9.4) also shows that $J_{q+1}(X/P)$ is smooth over k .

(9.6) Proposition. The projection $J_{q+1}(X/P) \rightarrow S$ is proper and surjective.

Proof. The map is proper because the projection $\pi_1 : W \rightarrow \mathbb{P}^{(n+2)^2-1}$ (see (9.2,i)) is proper and $\pi_1^{-1}(S) = J_{q+1}(X/P)$. Since the map is closed and it has a zero dimensional fiber (cf. (8.8,iii) and (8.12)), and since S and $J_{q+1}(X/P)$ are irreducible (cf. (9.4)), it follows that the map is surjective.

(9.7) Remark. It is possible to prove that the projection $J_{q+1}(X/P) \rightarrow S$ is an étale covering. We will not prove this result here since we will not need it in the sequel.

(9.8) Lemma. Let k be an algebraically closed field of characteristic p , and let n be a non negative integer. If $A \in GL(n+2, k)$, then there exist $T \in GL(n+2, k)$ and $B \in GL(n, k)$ such that

$${}^t_{TAT}(q) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & & 0 \\ \cdot & & & & \cdot \\ \cdot & & B & & \cdot \\ \cdot & & & & \cdot \\ 0 & & & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} .$$

Proof. For every closed point $[A]$ of S we have from (9.6) that there exists a closed point $([Z], [Z^*])$ in $J_{q+1}(X_A/\mathbb{P}_k^{n+1})$. By the transitivity of the action of $PGL(n+1, k)$ on I , there exists a closed point $[T_1]$ in $PGL(n+1, k)$ such that

$$([ZT_1^*], [Z^*T_1]) = (P, P^*) ,$$

where $P = (0; \dots; 0; 1)$ and $P^* = (1; 0; \dots; 0)$. Now, from the equations (a), (b) and (c) of (8.8,ii) which define $J_{q+1}(X/P)$, we have that

$${}^t_{T_1}AT_1(q) = C = \begin{pmatrix} C_0^0 & C_0^1 & & C_0^n & C_0^{n+1} \\ C_1^0 & C_1^1 & \dots & C_1^n & 0 \\ \vdots & & & \vdots & \vdots \\ \vdots & & & \vdots & \vdots \\ C_{n+1}^0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

with $C_0^{n+1} \cdot C_{n+1}^0 \neq 0$.

We write C in blocks as follows:

$$C = \begin{pmatrix} C_0^0 & C_0 & C_0^{n+1} \\ C^0 & B & 0 \\ C_{n+1}^0 & 0 & 0 \end{pmatrix},$$

where $C_0 = (C_0^1, \dots, C_0^n)$, $C^0 = \begin{pmatrix} C_1^0 \\ \vdots \\ C_n^0 \end{pmatrix}$, $B = \begin{pmatrix} C_1^1 & \dots & C_1^n \\ \vdots & \dots & \vdots \\ C_n^1 & \dots & C_n^n \end{pmatrix} \in GL(n, k)$.

Let

$$T = \begin{pmatrix} \lambda & 0 & 0 \\ {}^t_U & \text{Id} & 0 \\ w & V & \mu \end{pmatrix},$$

where $U = (u_1, \dots, u_n)$, $V = (v_1, \dots, v_n)$, with $\lambda, \mu, w, u_1, \dots, u_n, v_1, \dots, v_n$ indeterminates.

Now, by computing the product, we have

$${}^t_{TCT}(q) = \begin{pmatrix} d_0^0 & D_0 & \mu^q \lambda C_0^{n+1} \\ D^0 & B & 0 \\ \lambda^q \mu C_{n+1}^0 & 0 & 0 \end{pmatrix}.$$

where

$$d_0^0 = \lambda C_0^{n+1} w^q + \lambda^q C_{n+1}^0 w + \lambda^q (\lambda C_0^0 + UC^0) + (\lambda C_0 + UB) {}^t_U(q) \quad (9.8.1)$$

$$D^0 = \lambda^q (C^0 + C_{n+1}^0 \cdot t_V) + B \cdot t_U^{(q)} . \quad (9.8.2)$$

$$D_0 = \lambda C_0 + UB + \lambda C_0^{n+1} V^{(q)} . \quad (9.8.3)$$

First of all observe that the equations

$$\lambda^q \mu C_{n+1}^0 = \mu^q \lambda C_0^{n+1} = 1 \quad (9.8.4)$$

can be solved for λ and μ in k^* . Fix such a solution pair.

The system of equations

$$\begin{cases} D^0 = 0 \\ D_0 = 0 \end{cases}$$

is equivalent to the system

$$\begin{cases} \lambda^q C_{n+1}^0 V + U^{(q)} \cdot t_B + \lambda^q t_{C^0} = 0 \\ \lambda C_0^{n+1} V^{(q)} + U \cdot B + \lambda C_0 = 0 \end{cases}$$

which in view of (9.8.4) is equivalent to the following system:

$$V = -\mu U^{(q)} \cdot t_B + \mu \lambda^q t_{C^0} \quad (9.8.5)$$

$$U^{(q^2)} \cdot t_B^{(q)} \cdot B^{-1} = U + (\lambda C_0 - \lambda^{q^2} (t_{C^0})^{(q)}) B^{-1} . \quad (9.8.6)$$

Equation (9.8.6) has solutions in view of (8.11). Hence the system has a solution in U and V . Fix one such solution. Finally from (9.8.1) it is clear that $d_0^0 = 0$ has always a solution in w . Now the proof of the lemma is complete.

(9.9) Remark. The result stated in lemma (9.8) is equivalent to the following geometric statement: For every $A \in GL(n+2, k)$, $J_{q+1}(X_A / \mathbb{P}_k^{n+1})$ has two rational points (P, P^*) and (Q, Q^*) such that $Q \notin (P^*)'$ and

$P \notin (Q^*)'$. Unfortunately, we do not have a geometric proof for this.

(9.10) Theorem. Let k be an algebraically closed field of characteristic p , and let n be a non-negative integer. Given any $A \in GL(n+2, k)$ and any positive power q of p , there exists $T \in GL(n+2, k)$ such that

$${}^t_{TAT}(q) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ \cdot & & 1 & & \cdot \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & & 0 & 0 \end{pmatrix}$$

Proof. The proof is by induction separately on n even and n odd. The case $n=0$ follows immediately from (9.8). The case $n=1$ also follows from (9.8) by first putting A in the form

$$A' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and then by taking ${}^t_{TA'T}(q)$ where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with λ such that $\lambda^{q+1}b = 1$.

Assume now that the result is true for matrices in $GL(n, k)$. Let $A \in GL(n+2, k)$, then by (9.8) A can be put in the form

$$A' = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ \cdot & & B & & \cdot \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

with $B \in GL(n, k)$, so by the inductive hypothesis there exists $R \in GL(n, k)$ such that

$${}^t_{RBR}(q) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & & & & 0 \\ \cdot & & & & \cdot \\ \cdot & & 1 & & \cdot \\ \cdot & & \cdot & & \cdot \\ 0 & & & & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix},$$

so the matrix

$$T_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & 0 \\ \cdot & & R & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

does the job for A' , and the proof is complete.

(9.11) Corollary. Let k be an algebraically closed field of characteristic p , and let q be a fixed positive power of p . Then all non-singular hypersurfaces of type

$$Y_0 L_0(Y_0^q, \dots, Y_{n+1}^q) + \dots + Y_{n+1} L_{n+1}(Y_0^q, \dots, Y_{n+1}^q) = 0$$

with L_0, \dots, L_{n+1} linear polynomials with coefficients in k are projectively equivalent.

(9.12) Corollary. Let k be an algebraically closed field of characteristic $p > 2$. Then all non-singular hypersurfaces of degree $p+1$ with rank zero local Hessians are projectively equivalent to

$$X_0^{p+1} + \dots + X_{n+1}^{p+1} = 0$$

(9.13) Remark. This corollary for $n = 1$ was first proved by Pardini [1983].

10. DUALITY FOR PROJECTIVE CURVES

In this section we will illustrate our methods and results in the particular case of projective curves, and will obtain further results for these varieties.

(10.1) Hasse Differential Operators, Taylor Expansions. Let X be a reduced and irreducible curve over our algebraically closed field k . Let t be a section of \mathcal{O}_X over an open smooth subset U , such that dt generates $\Omega_{X/k}^1/U$. Consider the Hasse Differential Operators on \mathcal{O}_U (cf. EGAIV₄ 16.11), these are the differential operators $D_t^0, D_t^1, D_t^2, \dots$, on \mathcal{O}_U , uniquely determined by the formulas

$$D_t^n t^m = \binom{m}{n} t^{m-n} \quad (10.1.1)$$

These differential operators verify the relations:

$$D_t^m \circ D_t^n = D_t^n \circ D_t^m = \frac{(m+n)!}{n! m!} D_t^{n+m} . \quad (10.1.2)$$

Let f be a section of \mathcal{O}_U . Then to f there are associated the sections $D_t^1 f, D_t^2 f, \dots$, of \mathcal{O}_U , which allow us to expand f in power series at any closed point of U . Indeed we have in $\hat{\mathcal{O}}_{X,P}$

$$f = f(P) + D_t^1 f(P)(t - t(P)) + D_t^2 f(P)(t - t(P))^2 + \dots \quad (10.1.3)$$

If $\text{char } k = 0$, then the operators D_t^n , for $n \geq 1$, are determined, in view of formulas (10.1.2) by the single operator D_t^1 , in fact we have for all $n \geq 1$,

$$D_t^n = \frac{1}{n!} (D_t^1)^n .$$

If $\text{char } k = p > 0$, then D_t^1 no longer determine all the higher order differential operators, but some of them do, as we will see soon.

We had the occasion to use the Hasse Differential Operators and their properties in general, in (3.5), (7.10) and throughout section 8. However, we want to be more explicit in the case of curves because this could give an insight on the methods and results we described so far.

(10.2) Proposition (Dieudonné [1950]) Let $\text{chark} = p > 0$, and let

$$n = n_0 + n_1 p + n_2 p^2 + \dots + n_s p^s,$$

where the n_i are integers such that $0 \leq n_i < p$, $i = 0, \dots, s$, be the p -adic expansion of n . Then we have

$$D_t^n = \frac{1}{n_0! n_1! \dots n_s!} (D_t^{p^s})^{n_s} \circ \dots \circ (D_t^p)^{n_1} \circ (D_t^1)^{n_0}.$$

Proof. Use (10.1.2) and induction on s .

It follows by (10.2) that the family $D = (D_t^n)_n$ is determined by the operators $D_t^1, D_t^p, \dots, D_t^{p^s}, \dots$.

(10.3) Proposition. Let f be a regular function on U .

(i) If $D_t^{p^s} f = 0$ on U , then we have identically on U :

$$D_t^{p^s} f = D_t^{p^s+1} f = \dots = D_t^{p^{s+1}-1} f = 0$$

(ii) If $p \neq 2$ and $D_t^2 f = 0$ on U , then for all $\ell \geq 0$, we have identically on U :

$$D_t^{\ell p+2} f = \dots = D_t^{\ell p+p-1} f = 0.$$

Proof. (i) Let $0 \leq n < p^{s+1} - p^s$. Write $n = n_0 + n_1 p + \dots + n_s p^s$, with $0 \leq n_i < p$, $i = 0, \dots, s$. Since $n_s p^s \leq n < p^{s+1} - p^s$, it follows that $n_s < p-1$, therefore

$$p^{s+n} = n_0 + n_1 p_1 + \dots + (n_s + 1) p^s,$$

with $0 \leq n_0, n_1, \dots, n_{s-1}, (n_s + 1) < p$. Hence by (10.2) we have

$$D_t^{p^{s+n}} = \frac{1}{n_0! n_1! \dots (n_s + 1)!} (D_t^1)^{n_0} \circ (D_t^p)^{n_1} \circ \dots \circ (D_t^{p^s})^{n_s + 1}.$$

So if $D_t^{p^s} f = 0$ on U , then $D_t^{p^{s+n}} f = 0$ on U for $0 \leq n < p^{s+1} - p^s$.

(ii) Let $2 \leq n < p$ and suppose that $\ell = \ell_1 + \ell_2 p + \dots + \ell_{r+1} p^r$, with $0 \leq \ell_i < p$, $i = 1, \dots, r+1$, so

$$\ell p + n = n + \ell_1 p + \dots + \ell_{r+1} p^{r+1},$$

hence

$$\begin{aligned} D_t^{\ell p + n} &= \frac{1}{n! \ell_1! \dots \ell_{r+1}!} (D_t^{p^{r+1}})^{\ell_{r+1}} \circ \dots \circ (D_t^p)^{\ell_1} \circ (D_t^1)^n = \\ &= \frac{2}{n! \ell_1! \dots \ell_{r+1}!} (D_t^{p^{r+1}})^{\ell_{r+1}} \circ \dots \circ (D_t^p)^{\ell_1} \circ (D_t^1)^{n-2} \circ D_t^2. \end{aligned}$$

So if $D_t^2 f = 0$ on U , it follows that $D_t^{\ell p + n} f = 0$ on U for $2 \leq n < p$.

(10.4) Corollary (Taylor expansion). Let f be a regular function on U . If $\text{char } k = p = 2$ or $D_t^2 f \equiv 0$ on X , then there exists a positive power q of p such that for every closed point P in U we have

$$\begin{aligned} f &= f(P) + D_t^1 f(P) (t - t(P)) + D_t^q f(P) (t - t(P))^q + D_t^{q+1} f(P) (t - t(P))^{q+1} + \\ &D_t^{2q} f(P) (t - t(P))^{2q} + D_t^{2q+1} f(P) (t - t(P))^{2q+1} + \dots \end{aligned}$$

(10.5) Duality for plane curves. The tangent cone of the dual of a projective curve at a point. Let Y be another curve over k , and let $\phi : X \dashrightarrow Y$ be a rational dominating map. Let $\phi^* : k(Y) \rightarrow k(X)$ be the induced non-trivial homomorphism.

To every parameterization $h : k(X) \rightarrow k((T))$, we get a parameterization $h \circ \phi^*$ on Y . Let P be the place corresponding to h and

Q be the place corresponding to $h \circ \phi^*$. Let e_P be the ramification index of ϕ at P , so the order of imprimitivity of $h \circ \phi^*$ is equal to e_P times the order of imprimitivity of h . Hence if h is primitive, then the order of imprimitivity of $h \circ \phi^*$ is equal to e_P .

If X is complete, then it is well known that

$$\sum_{\phi(P)=Q} e_P = \deg \phi ,$$

and moreover, if Q is general, then there are precisely $[k(X) : k(Y)]_s$ places P in the counterimage of Q and these are such that $e_P = [k(X) : k(Y)]_i$. So if P is general, we have

$$e_P = [k(X) : k(Y)]_i \quad (10.5.1)$$

Suppose now that X is a non-linear plane projective curve, that $Y = X'$ is the dual of X and that ϕ is the composite map

$$\phi : X \xrightarrow{\sim} CX \rightarrow X' .$$

Suppose now that coordinates X_0, X_1, X_2 have been chosen for \mathbb{P}^2 such that the set U on which dx_1 , where $x_1 = \frac{X_1}{X_0} \Big|_X$, generates $\Omega^1_{X/k}/U$, is non empty.

If $P = [X_0(t); X_1(t); X_2(t)]$ is any place of X in any coordinate system, then the equation of the tangent line of P is given by

$$\begin{bmatrix} X_0 & X_1 & X_2 \\ X_0(t) & X_1(t) & X_2(t) \\ \dot{X}_0(t) & \dot{X}_1(t) & \dot{X}_2(t) \end{bmatrix} = 0 ,$$

hence, $Q = \phi(P)$ is given by

$$Q = [X_1(t)\dot{X}_2(t) - X_2(t)\dot{X}_1(t); \dot{X}_0(t)X_2(t) - \dot{X}_2(t)X_0(t); X_0(t)\dot{X}_1(t) - \dot{X}_0(t)X_1(t)]$$

If the center of the place P is $[1; a_0; b_0] \in U$, then we may write

$$P = [1; a_0 + t; b_0 + b_1 t + b_2 t^2 + \dots] ,$$

hence we have

$$\begin{aligned} Q &= [X_1(t)\dot{X}_2(t) - X_2(t); -\dot{X}_2(t); 1] & (10.5.2) \\ &= [a_0 b_1 - b_0 + 2a_0 b_2 t + (3a_0 b_3 + b_2)t^2 + (4a_0 b_4 + 2b_3)t^3 + \dots; -b_1 - 2b_2 t \\ &\quad - 3b_3 t^2 - \dots; 1]. \end{aligned}$$

If P is general, then $Q = (P)$ is centered at a simple point of X' , hence

$$e_P = \min\{\text{ord}(2a_0 b_2 t + 3a_0 b_3 + b_2)t^2 + \dots, \text{ord}(2b_2 t + 3b_3 t^2 + \dots)\} \quad (10.5.3)$$

(10.6) Remarks. (i) If $\text{chark} = 2$, then from (10.5.3) it follows that for a general P we have $e_P > 1$. Therefore it follows from (10.5.1) that ϕ is not separable, consequently no non-linear reduced and irreducible plane curve is reflexive. Since the dual of such a curve is also a plane curve, it follows that no such curve is ordinary. This is a particular case of (3.4).

(ii) Generic order of contact. Let

$$m = \min_n \{n/n \geq 2 \text{ and } D_t^n \left(\frac{X_2(t)}{X_0(t)} \right) \neq 0 \text{ on } U\} .$$

So for a general point P of X we have

$$m = m_P(X.T_P X) .$$

A point P will be called a flex (resp. a simple flex) of X , if $m_P(X.T_P X) \geq m+1$ (resp. $m_P(X.T_P X) = m+1$).

From (10.5.2), it follows that $e_P = 1$ (i.e., X is reflexive) if and only if $m=2$ and $\text{chark} \neq 2$.

Suppose now that $m \neq 2$ or $\text{chark} = 2$, then from corollary (10.4) it follows that a place P with general center P on X is such that

$$P = [1; a_0 + t; b_0 + b_1 t + b_q t^q + b_{q+1} t^{q+1} + \dots] \quad (10.6.1)$$

where $q = p^r$ for some $r \geq 1$, and since X is not linear, $b_q \neq 0$. Hence

$$\begin{aligned} \phi(P) = [a_0 b_1 - b_0 + (a_0 b_{q+1} - b_q) t^q + (a_0 b_{2q+1} - b_{2q}) t^{2q} + \dots; \\ -b_1 - b_{q+1} t^q - b_{2q+1} t^{2q} - \dots] , \end{aligned} \quad (10.6.2)$$

and therefore $e_P = q$. From (10.5.1), it follows that $q = [k(X) : k(X')]_i$. From (10.6.1), it is clear that $m = q$, so we may conclude that

$$m = [k(X) : k(X')]_i .$$

From the above discussion we have that X is not reflexive if and only if $m = [k(X) : k(X')]_i$. This is a particular case of theorem (3.5).

(iii) If X is a non-reflexive curve and P is a smooth point of X which is not a flex or it is a simple flex, then it is clear from (10.6.1) and (10.6.2) that the branch of X' corresponding to the place P of X centered at P , is non singular. Note that this is a peculiar property of non-reflexive curves, since for reflexive curves X , flexes on X always produce singular points on X'

The next proposition will describe the tangent cone at most points of the dual of a projective curve.

(10.7) Proposition. Let $X \subseteq \mathbb{P}_k^n$ be a reduced irreducible non-linear reflexive curve. Let H be a hyperplane not containing X and meeting it at simple points. Then the tangent cone to X' at H' is given as a cycle by

$$TC_{H'}(X') = \sum_{P \in X \cap H} r_P P',$$

where

$$r_P = \begin{cases} m_P(X.H) - 1, & \text{if } p \nmid m_P(X.H) \\ m_P(X.H), & \text{if } p \mid m_P(X.H) \end{cases}$$

Proof. Suppose $n = 2$.

Let Q be a branch of X' centered at H' . Q comes by duality from a branch P of X centered at some point $P \in X \cap H$ and with tangent line H . Since X is reflexive, the tangent line of Q is P' . The contribution of the point $P \in X \cap H$ to the tangent cone of X' at H' is then P' counted with the multiplicity $r_P = \text{ord } Q$.

Choose homogeneous coordinates of \mathbb{P}^2 such that P is given by

$$P = [1; t; t^\beta + \dots],$$

with $\beta > 1$. So $\beta = m_P(X.H)$. Now,

$$\phi(P) = [(\beta-1)t^\beta + \dots; \beta t^{\beta-1} + \dots; 1],$$

and since X is reflexive, hence $\phi : X \dashrightarrow X'$ is birational, it follows that $\phi(P)$ is primitive, hence

$$\text{ord } Q = \begin{cases} \beta - 1, & \text{if } p \nmid \beta \\ \beta, & \text{if } p \mid \beta. \end{cases}$$

Since $\beta = m_P(X.H)$, the result follows for $n = 2$.

Suppose now that $n \geq 3$.

Let P be any point of $X \cap H$, and let $\pi \subset \mathbb{P}^{n*}$ be a general plane through the point H' . The dual of π , $\pi' \subset \mathbb{P}^n$, is a linear space of dimension $n-3$ contained in the hyperplane H and not containing P .

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ denote the linear projection with center π' . By the principle of section and projection we have, in the duality of \mathbb{P}^2 , that

$$(f(X))' = X' \cap \pi \quad \text{and} \quad (f(P))' = P' \cap \pi \quad (10.7.1)$$

Now, since X is not contained in H and π' is general in H , it is easy to prove that f is a morphism on X which is birational onto its image (see for example Hefez-Sacchiero [1983] Lemma 1), and the proof proceeds exactly as in Hefez-Sacchiero [1983] Proposition 1. We reproduce it here for the convenience of the reader.

From the birationality of f and the fact that no tangent to X at P meets a general π' , it follows that, for the branch P of X centered at P , we have that $f(P)$ is non singular.

By a projection formula, which can be easily verified directly, we have

$$\text{ord}_P H = \text{ord}_{f(P)} f(H) . \quad (10.7.2)$$

From (10.7.1), the case $n=2$, and the fact that π' do not meet any secant of X lying on H , we have

$$\text{TC}_{H'}(X' \cap \pi) = \sum_{P \in X \cap H} r_P(P' \cap \pi) \quad (= (\sum_{P \in X \cap H} r_P P') \cap \pi) \quad (10.7.3)$$

where

$$r_P = \begin{cases} \text{ord}_{f(P)}(f(H)) - 1 & \text{if } p \neq \text{ord}_{f(P)} f(H) \\ \text{ord}_{f(P)} f(H) & \text{if } p = \text{ord}_{f(P)} f(H) \end{cases}$$

From (10.7.2) it follows that

$$r_p = \begin{cases} \text{ord}_p(H) - 1 & \text{if } p \nmid \text{ord}_p(H) \\ \text{ord}_p(H) & \text{if } p \mid \text{ord}_p(H) . \end{cases} \quad (10.7.4)$$

Since π is general we also have as a cycle

$$\text{TC}_{H'}(X' \cap \pi) = \text{TC}_{H'}(X') \cap \pi . \quad (10.7.5)$$

From (10.7.3) and (10.7.5) we get

$$\text{TC}_{H'}(X') \cap \pi = \left(\sum_{P \in X' \cap H} r_P P' \right) \cap \pi .$$

Now if one looks at the forms defining $\text{TC}_{H'}(X')$ and $\sum_{P \in X' \cap H} r_P P'$, it follows easily that

$$\text{TC}_{H'}(X') = \sum_{P \in X' \cap H} r_P P' .$$

This equation together with (10.7.4) completes the proof.

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