# Advanced Stochastic Processes.

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## LECTURE 13
## Ito integral. Properties

### Lecture outline

- Definition of Ito integral
- Properties of Ito integral

### 13.1. Ito integral. Existence

We continue with the construction of Ito integral. We combine the results of Propositions 1-3 from Lecture 12 and prove the following result.

**Proposition 1.** Given any process $X \in H^2$ there exists a sequence of simple processes $X_n \in S^2$ such that

$$\lim_n \mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] = 0. \tag{13.1}$$

**Proof.** Given any large $n > 0$, by Proposition 3 from Lecture 12, there exists a bounded process $X_n^1 \in H^2$ satisfying

$$\mathbb{E}[\int_0^T (X(t) - X_n^1(t))^2 dt] \le \frac{1}{n}.$$

Applying Proposition 2 from Lecture 12, for each $n$ there exists a bounded continuous process $X_n^2 \in H^2$ satisfying

$$\mathbb{E}[\int_0^T (X_n^1(t) - X_n^2(t))^2 dt] \le \frac{1}{n}.$$

Finally, applying Proposition 1 from Lecture 12, for each $n$ there exists a simple process $X_n^3 \in H^2$ satisfying

$$\mathbb{E}[\int_0^T (X_n^2(t) - X_n^3(t))^2 dt] \le \frac{1}{n}.$$

Let $a_j = X_n^j(t) - X_n^{j+1}(t), j = 0, 1, 2$, where $X_n^0(t)$ stands for $X(t)$. Then

$$
\begin{aligned}
(X(t) - X_n^3(t))^2 &= (X(t) - X_n^1(t) + X_n^1(t) - X_n^2(t) + X_n^2(t) - X_n^3(t))^2 \\
&= \sum_j a_j^2 + 2\sum_{j<k} a_j a_k \\
&\le \sum_j a_j^2 + \sum_{j<k}(a_j^2 + a_k^2) \\
&= 3\sum_j a_j^2.
\end{aligned}
$$

Combining, we conclude

$$\mathbb{E}[\int_0^T (X(t) - X_n^3(t))^2 dt] \le \frac{9}{n}.$$

We have constructed a sequence $X_n^3(t) \in S^2$ of processes such that

$$\lim_n \mathbb{E}[\int_0^T (X(t) - X_n^3(t))^2 dt] = 0.$$

□

Now, given a process $X \in H^2$, we fix any sequence of simple processes $X_n \in S^2$ which satisfies (13.1). Recall, that we already have defined Ito integral for simple processes $I_T(X_n)$.

**Proposition 2.** Suppose a sequence of simple processes $X_n \in S^2$ satisfies (13.1). Then the sequence of random variables $I_T(X_n)$ is Cauchy.

**Proof.** We begin by establishing the following simple result, which incidently holds for every two simple processes.

**Lemma 13.2.** *Ito integral defined for simple processes is a linear functional: for every m,n*

$$I_T(X_m) - I_T(X_n) = I_T(X_m - X_n). \tag{13.3}$$

**Proof.** Given two partitions $\Pi_1 : 0 = t_0^1 < \cdots < t_{r(m)}^1 = T$ and $\Pi_2 : 0 = t_0^2 < \cdots < t_{r(n)}^2 = T$, consider a superimposed partition $\Pi : 0 = t_0 < \cdots < t_r = T, r \le r(m) + r(n)$. Then we have for $X_m - X_n$ that

$$
\begin{aligned}
I_T(X_m - X_n) &= \sum_{j \le r-1} (X_m(t_j) - X_n(t_j))(B(t_{j+1}) - B(t_j)) \\
&= \sum_{j \le r-1} X_m(t_j)(B(t_{j+1}) - B(t_j)) - \sum_{j \le r-1} X_n(t_j)(B(t_{j+1}) - B(t_j))
\end{aligned}
$$

Observe, however, that

$$\sum_{j \le r-1} X_m(t_j)(B(t_{j+1}) - B(t_j)) = \sum_{j \le r(m)-1} X_m(t_j^1)(B(t_{j+1}) - B(t_j)),$$

because the term corresponding to $t_j$ between the points of partition $\Pi_1$ contributes zero. Similar assertion holds for $X_n$ with respect to partition $\Pi_2$. We conclude that (13.3) holds. □

Now we return to proving the proposition. We have using the lemma above and Ito isometry

$$\begin{aligned}\mathbb{E}[(I_T(X_m) - I_T(X_n))^2] &= \mathbb{E}[I_T^2(X_m - X_n)] \\ &= \mathbb{E}[\int_0^T (X_m(t) - X_n(t))^2 dt] \\ &\le 2\mathbb{E}[\int_0^T (X(t) - X_m(t))^2 dt] + 2\mathbb{E}[\int_0^T (X(t) - X_n(t))^2 dt].\end{aligned}$$

But since the sequence $X_n$ satisfies (13.1), the assertion of the proposition holds. □

Now we can formally state the definition of Ito integral.

**Definition 13.4** (**Ito integral**). Given a stochastic process $X \in H^2$ and $T > 0$, its Ito integral $I_T(X)$ is defined to be the $\mathcal{L}_2$ limit of random variables $I_T(X_n)$, where $X_n \in S^2$ is any sequence of simple processes satisfying (13.1). Namely, $I_T(X)$ is the unique random variable satisfying

$$\lim_n \mathbb{E}[(I_T(X) - I_T(X_n))^2] = 0.$$

We write

$$I_T(X) = \int_0^T X(t) dB(t) = \int_0^T X(t, \omega) dB(t, \omega).$$

**Theorem 13.5.** *Ito integral is well defined. That is the $\mathcal{L}_2$ limit of $I_T(X_n)$ exists and does not depend on the choice of the sequence satisfying (13.1). Moreover, Ito integral satisfies Ito isometry*

$$\mathbb{E}[I_T^2(X)] = \mathbb{E}[\int_0^T X^2(t) dt].$$

**Proof.** We need to prove the consistency of the Ito integral definition. Namely, that such a limiting random variable exists and does not depend on the choice of the sequence $X_n$.

We have established in Proposition 1, the existence of a sequence $X_n$ satisfying (13.1). We have established in Proposition 2 that the sequence $I_T(X_n)$ is Cauchy. Applying Theorem 12.3 from Lecture 12 (Completeness of $\mathcal{L}_2$), there exists a unique (up to measure zero sets) random variable which is the $\mathcal{L}_2$ limit of $I_T(X_n)$.

Now we need to establish uniqueness. Namely, if $X'_n$ is another sequence satisfying (13.1), then the unique limit of the sequence $I_T(X'_n)$ is the same as that of the sequence $I_T(X_n)$.

**Problem 1.** Let $I_T(X)$ be a limit of $I_T(X_n)$ along some sequence of simple processes $X_n \in S^2$ satisfying (13.1). Establish that $I_T(X)$ satisfies Ito isometry:

$$\mathbb{E}[I_T^2(X)] = \mathbb{E}[\int_0^T X^2(t) dt].$$

Using this finish the proof of the theorem, by establishing the uniqueness of the limit.

□

## 13.2. Ito integral. Properties

### 13.2.1. Simple examples

Let us compute the Ito integral for a special case $X(t) = B(t)$. We will do this directly from the definition. Later on we will develop calculus rules for computing the Ito integral for many interesting cases.

We fix a sequence of partitions $\Pi_n : 0 = t_0 < \cdots < t_n = T$ and consider $B_n(t) = B(t_j), t \in [t_j, t_{j+1})$. Assume that $\lim_n \Delta(\Pi_n) = 0$. We first show that this is sufficient for having

$$\lim_n \mathbb{E}[\int_0^T (B(t) - B_n(t))^2 dt] = 0. \tag{13.6}$$

Indeed

$$\int_0^T (B(t) - B_n(t))^2 dt = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (B(t) - B(t_j))^2 dt.$$

We have

$$\begin{aligned} \mathbb{E}[\int_{t_j}^{t_{j+1}} (B(t) - B(t_j))^2 dt &= \int_{t_j}^{t_{j+1}} \mathbb{E}[(B(t) - B(t_j))^2] dt \\ &= \int_{t_j}^{t_{j+1}} (t - t_j) dt \\ &= \frac{(t_{j+1} - t_j)^2}{2}, \end{aligned}$$

implying

$$\mathbb{E}[\int_0^T (B(t) - B_n(t))^2 dt] = \frac{1}{2} \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \le \Delta(\Pi_n) \sum_{j=0}^{n-1} (t_{j+1} - t_j) = \Delta(\Pi_n) T.$$

By our assumption, though, the right-hand side of this expression converges to zero implying that (13.6).

Thus we need to compute the $\mathcal{L}_2$ limit of

$$I_T(B_n) = \sum_j B(t_j)(B(t_{j+1}) - B(t_j))$$

as $n \to \infty$. We use the identity

$$B^2(t_{j+1}) - B^2(t_j) = (B(t_{j+1}) - B(t_j))^2 + 2B(t_j)(B(t_{j+1}) - B(t_j)),$$

implying

$$B^2(T) - B^2(0) = \sum_{j=0}^{n-1} B^2(t_{j+1}) - B^2(t_j) = \sum_{j=0}^{n-1} (B(t_{j+1}) - B(t_j))^2 + 2 \sum_{j=0}^{n-1} B(t_j)(B(t_{j+1}) - B(t_j)),$$

But recall the quadratic variation property of the Brownian motion:

$$\lim_n \sum_{j=0}^{n-1} (B(t_{j+1}) - B(t_j))^2 = T$$

in $\mathcal{L}_2$ (recall that the only requirement for this convergence was that $\Delta(\Pi_n) \to 0$). Therefore, also in $\mathcal{L}_2$

$$\sum_{j=0}^{n-1} B(t_j)(B(t_{j+1}) - B(t_j)) \to \frac{1}{2}B^2(T) - \frac{T}{2}.$$

We conclude

**Proposition 3.** The following identity holds

$$I_T(B) = \int_0^T B(t)dB(t) = \frac{1}{2}B^2(T) - \frac{T}{2}.$$

### 13.2.2. Simple properties

The following properties follow in a straightforward way from the definition:

(a) $\int_0^T (a_1 X(t) + a_2 Y(t))dB(t) = a_1 \int_0^T X(t)dB(t) + a_2 \int_0^T Y(t)dB(t)$.
(b) $\mathbb{E}[\int_0^T X(t)dB(t)] = 0$.
(c) $\int_0^T X(t)dB(t)$ is $\mathcal{F}_T$ measurable.

## 13.3. Additional reading materials

- Course Packet. Chapter from Harrison's book "Brownian models and stochastic control".
- Øksendal [**1**], Chapter III.

# BIBLIOGRAPHY

1. B. Øksendal, *Stochastic differential equations*, Springer, 1991.