Advanced Stochastic Processes.

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LECTURE 15

Martingale property of Ito integral and Girsanov theorem

Lecture outline

- Continuity of Ito integral
- Martingale property of Ito integral. Martingale representation theorem.
- Girsanov's theorem.

15.1. Continuity and martingale properties of Ito process

Ito process is a process of the form $X(t) = X(0) + \int_0^t U(s)ds + \int_0^t V(s)dB(s)$. The first part of this expression, $\int_0^t U(s)ds$ is a continuous function of t (recall from real analysis that Reimann integral is continuous in t). What about the Ito integral part? It turns out that there always exists a *continuous version* of the Ito integral.

Theorem 15.1. *Given $V \in H^2$ and $T > 0$ there exists an a.s. continuous stochastic process $Y(t)$ such that $Y(t) = \int_0^t V(s)dB(s)$ a.s. for all $0 \le t \le T$. Namely, there exists a continuous version of $\int_0^t V(s)dB(s)$ on every finite interval.*

Note that the theorem does not imply that every version of $\int_0^t V(s)dB(s)$ is continuous. Can you think about two process which are a.s. equal to each other but one is a.s. continuous while the other is discontinuous everywhere?

Proof. Recall that $\int_0^t V(s)dB(s)$ is the $\mathcal{L}_2$ limit along any sequence of simple processes V_n satisfying

$$\mathbb{E}[\int_0^t (V_n(t) - V(t))^2 dt] \to 0. \tag{15.2}$$

Fix any such sequence V_n and consider their corresponding Ito integral $I_t(X_n)$.

Lemma 15.3. *$I_t(V_n) = \int_0^t V_n(t)dB(t)$ is a martingale.*

Proof. Let $\Pi_n : 0 = t_0 < t_1 < \cdots < t_n = t$ be the partition corresponding to V_n. Fix $s < t$. Increase the partition by an extra point $t_k = s$. Observe that $I_t(X_n)$ with respect to the new partition is the same as the original one. Also $\mathcal{F}_s = \mathcal{F}_{t_k}$. We have

$$\begin{aligned}
\mathbb{E}[I_t(V_n)|\mathcal{F}_s] &= \mathbb{E}[\sum_j V_n(t_j)(B(t_{j+1}) - B(t_j))|\mathcal{F}_s] \\
&= \mathbb{E}[\sum_{j\geq k} V_n(t_j)(B(t_{j+1}) - B(t_j))|\mathcal{F}_s] + \mathbb{E}[\sum_{j\leq k-1} V_n(t_j)(B(t_{j+1}) - B(t_j))|\mathcal{F}_s] \\
&= \mathbb{E}[\sum_{j\geq k} V_n(t_j)\mathbb{E}[B(t_{j+1}) - B(t_j)|\mathcal{F}_{t_j}]|\mathcal{F}_s] + \sum_{j\leq k-1} V_n(t_j)(B(t_{j+1}) - B(t_j)) \\
&= 0 + I_s(V_n),
\end{aligned}$$

where the last equality holds since $\sum_{j\leq k-1} V_n(t_j)(B(t_{j+1}) - B(t_j)) \in \mathcal{F}_s$. Therefore $I_t(V_n)$ is indeed a martingale. □

We fix $\epsilon > 0$. From (15.2) it follows that there exists n_0 such that for all $m, n \geq n_0$

$$\mathbb{E}[\int_0^T (V_n(t) - V_m(t))^2 dt < \epsilon^3.$$

Fix any such pair m, n. From Lemma 15.3 we have $I_t(V_n) - I_t(V_m) = I_t(V_n - V_m)$ is also martingale. Applying the Doob-Kolmogorov inequality, we obtain

$$\begin{aligned}
\mathbb{P}(\sup_{0\leq t\leq T} |I_t(V_n) - I_t(V_m)| \geq \epsilon) &\leq \frac{\mathbb{E}[(I_T(V_n) - I_T(V_m))^2]}{\epsilon^2} \\
&= \frac{\mathbb{E}[(I_T(V_n - V_m))^2]}{\epsilon^2} \\
&= \frac{\mathbb{E}[\int_0^T (V_n(t) - V_m(t))^2 dt]}{\epsilon^2} \\
&< \epsilon.
\end{aligned}$$

This means that we can construct a subsequence n_k along which

$$\mathbb{P}(\sup_{0\leq t\leq T} |I_t(V_{n_{k+1}}) - I_t(V_{n_k})| \geq \frac{1}{2^k}) \leq \frac{1}{2^k}.$$

In particular, the sum of these probabilities is finite. Applying the Borel-Cantelli Lemma, for almost all samples ω, there exists $k(\omega)$ such that for all $k > k(\omega)$

$$\sup_{0\leq t\leq T} |I_t(V_{n_{k+1}(\omega)}) - I_t(V_{n_k}(\omega))| < \frac{1}{2^k}.$$

Now we need a short digression into real analysis. We define a sequence of functions $a_k(t), t \in [0, T]$ to be *uniformly* Cauchy if for every $\epsilon > 0$ there exists k_0 such that for all $k, m > k_0$ we have $\sup_{0\leq t\leq T} |a_k(t) - a_m(t)| < \epsilon$. The following is the result from real analysis.

Proposition 1. Suppose a_k is a uniform Cauchy sequence of continuous functions on $[0, T]$. Then $a_k(t)$ converges everywhere on $[0, T]$ to some continuous function $a(t)$.

Applying this proposition to our case we conclude that for almost all ω the point-wise limit of $I_t(V_{n_k}(\omega))$ exists and is continuous on $[0, T]$. We call this limit $Y(t, \omega)$ and now we show that $Y(t) = I_t(V)$ a.s. From the definition of Ito integral, we have that the subsequence $I_t(V_{n_k})$

converges to $I_t(V)$ in $\mathcal{L}_2$. This implies convergence in probability. On the other hand a.s. convergence also implies convergence in probability. Therefore Y_t and $I_t(V)$ are two limits in probability of $I_t(V_{n_k})$. Therefore (check that this is indeed the case) they are a.s. equal to each other.

We have created a version Y_t of $I_t(V)$ which is a.s. continuous. □

From now on assume that we are dealing only with continuous versions of $I_t(V)$.

Another important property of $I_t(V)$ is that it is a martingale.

Theorem 15.4. *For every $V \in H^2$, $I_t(V)$ is a martingale.*

Proof. We first need to check $\mathbb{E}[|I_t(V)|] < \infty$. This will be true, provided that $\mathbb{E}[I_t^2(V)] < \infty$. But by Ito isometry $\mathbb{E}[I_t^2(V)] = \mathbb{E}[\int_0^t V^2(s)ds] < \infty$ since $V \in H^2$.

Fix $s < t$. Take any sequence of simple processes V_n satisfying (15.2). Let $Y_s = \mathbb{E}[I_t(V)|\mathcal{F}_s]$. Consider

$$\begin{aligned}\mathbb{E}[(Y_s - I_s(V))^2] &= \mathbb{E}[(Y_s - I_s(V_n) + I_s(V_n) - I_s(V))^2] \\ &\le 2\mathbb{E}[(Y_s - I_s(V_n))^2] + 2\mathbb{E}[(I_s(V_n) - I_s(V))^2]\end{aligned}$$

By definition, we have $\lim_n \mathbb{E}[(I_s(V_n) - I_s(V))^2] = 0$. On the other hand, since by Lemma 15.3 we have $I_s(V_n) = \mathbb{E}[I_t(V_n)|\mathcal{F}_s]$, then

$$\begin{aligned}\mathbb{E}[(Y_s - I_s(V_n))^2] &= \mathbb{E}[(\mathbb{E}[I_t(V)|\mathcal{F}_s] - \mathbb{E}[I_t(V_n)|\mathcal{F}_s])^2] \\ &= \mathbb{E}[(\mathbb{E}[I_t(V) - I_t(V_n)|\mathcal{F}_s])^2] \\ &\le \mathbb{E}[\mathbb{E}[(I_t(V) - I_t(V_n))^2|\mathcal{F}_s]] \\ &= \mathbb{E}[(I_t(V) - I_t(V_n))^2],\end{aligned}$$

where we use conditional Jensen's inequality in the second step. But again $\lim_n \mathbb{E}[(I_t(V) - I_t(V_n))^2] = 0$ by definition. Combining, we conclude that $\mathbb{E}[(Y_s - I_s(V))^2] = 0$ or $Y_s = \mathbb{E}[I_t(V)|\mathcal{F}_s] = I_s(V)$, or $I_t(V)$ is indeed a martingale. □

15.2. Martingale Representation Theorem and Girsanov' theorem

We established in the previous section that Ito integral is a martingale. It turns out a converse statement is true. The proof of this fact is complicated, it is based on some complex analytic techniques and we skip it.

Theorem 15.5 (**Martingale Representation Theorem**). *Suppose $M(t) \in \mathcal{F}_t$ is a martingale with a.s. continuous sample paths. Then there exists a unique process $X(t) \in H^2$ such that*

$$M(t) = \mathbb{E}[M(0)] + \int_0^t X(s)dB(s).$$

The martingale representation theorem is based on a related Dudley's Theorem and is an important tool in finance. Another related tool is Girsanov's theorem, which we now discuss.

First let us review the change of measure technique. Given a probability measure $\mathbb{P}$ and a non-negative random variable ψ such that $\mathbb{E}[\psi] < \infty$, we may define a new probability measure

$$\mathbb{P}_2(A) = \frac{\mathbb{E}[\psi 1\{A\}]}{\mathbb{E}[\psi]}.$$

The following identity was established during the recitation. Given two fields $\mathcal{F} \supset \mathcal{G}$ and a random variable $X \in \mathcal{F}$ such that $\mathbb{E}_{\mathbb{P}}[|X|], \mathbb{E}_{\mathbb{P}_2}[|X|] < \infty$

$$\mathbb{E}_{\mathbb{P}_2}[X|\mathcal{G}] = \frac{\mathbb{E}_{\mathbb{P}}[\psi X|\mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[\psi|\mathcal{G}]} \tag{15.6}$$

Girsanov's theorem is an important statement which provides a translation between different probability measures on the same space $(\Omega, \mathcal{F})$ and filtration $\{\mathcal{F}_t\}$.

Thus suppose we have an Ito process $X(t) = X(0) + \int_0^t U(s)ds + \int_0^t V(s)dB(s)$. Recall that both $X(t), B(t) \in \mathcal{F}_t$ and correspond to the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will derive Girsanov's theorem first for the case $V(s) = 1$ and then state without proof the general case.

For now, consider the process $X(t) = X(0) + \int_0^t U(s)ds + dB(s)$ or $dX = Udt + dB$. Consider the following process

$$M(t) = \exp(-\int_0^t U(s)dB(s) - \frac{1}{2}\int_0^t U^2(s)ds).$$

This process is adapted to $\mathcal{F}_t$, but does it belong to H^2? Not necessarily. So we only consider the case when it does:

Definition 15.7. A process U is defined to satisfy Novikov's condition if for every t

$$\mathbb{E}[e^{\frac{1}{2}\int_0^t U^2(s)ds}] < \infty.$$

It turns out (we do not prove this) that this suffices to ensure $M \in H^2$. There is a reason we used the mnemonic M for the process – it gives rise to a martingale.

Proposition 2. Consider an Ito process $dX = Udt + dB$. The processes $M(t)$ and $Y(t) = M(t)X(t)$ are martingales.

Proof. It turns out (we skip the proof) that the Novikov condition implies $\mathbb{E}[M(t)] < \infty$. This implies $\mathbb{E}[Y(t)] < \infty$. By Ito's lemma $Y(t)$ is an Ito process since M, X are Ito processes. Using the Ito formula we have

$$dM = M(-UdB - \frac{1}{2}U^2dt) + \frac{1}{2}MU^2dt = -MUdB.$$

Since Ito integral is a martingale, $M(t)$ is a martingale. We use the multidimensional Ito formula applied to function $g(x, y) = xy$:

$$dY = XdM + MdX + (dMdX).$$

Therefore

$$dY = X(-MUdB) + M(Udt + dB) + (-MUdB)(Udt + dB) = (-XMU + M)dB$$

where we use $(-MUdB)(Udt+dB) = -MUUdBdt - MU(dB)^2 = -MUdt$ and this cancels with $MUdt$. We conclude that Y is an Ito integral (it does not have a non-Brownian component). We have established earlier that Ito integral is a martingale. □

Now let us consider a change of measure implied by the transformation $X \to MX$ meaning the following. First fix $T > 0$. Consider the random variable $M(T)$ and a new measure $\mathbb{P}_T = M(T)\mathbb{P}$ introduced by $M(T)$: for every $A \subset \Omega, A \in \mathcal{F}$ we let

$$\mathbb{P}_T(A) = \frac{\mathbb{E}[M(T)1\{A\}]}{\mathbb{E}[M(T)]} = \mathbb{E}[M(T)1\{A\}],$$

where the second equality follows since M is a martingale and therefore $\mathbb{E}[M(T)] = \mathbb{E}[M(0)] = 1$. This change of measure is called **Girsanov's** transformation. Intuitively every sample $\omega \in \Omega$ is reweighted from $\mathbb{P}(\omega)$ to $\mathbb{P}(\omega)M_T(\omega)$. We know that this transformation does introduce a new probability measure. The distribution of $X(T)$ under the new measure $\mathbb{P}_T$ is then the distribution of $X(T)M(T)$ under the old measure $\mathbb{P}$. This begs the question: how dependent is this transformation on the choice of T? What if we chose a different T? The wonderful thing about this transformation is that it "works across" times as the following lemma establishes.

Lemma 15.8. *For every $t \le T$, the measures $\mathbb{P}_t = M(t)\mathbb{P}$ and $\mathbb{P}_T = \mathbb{E}[M(T)]\mathbb{P}$ are identical on $\mathcal{F}_t$. In particular the distributions of $X(t)M(t)$ and $X(t)M(T)$ are identical.*

Proof. Fix $A \in \mathcal{F}_t$. We have using the martingale property of M from (2)

$$\mathbb{P}_T(A) = \mathbb{E}[M(T)1\{A\}] = \mathbb{E}[\mathbb{E}[M(T)1\{A\}|\mathcal{F}_t]] = \mathbb{E}[1\{A\}\mathbb{E}[M(T)|\mathcal{F}_t]] = \mathbb{E}[1\{A\}M(t)] = \mathbb{P}_t(A).$$

□

We turn to the first version of Girsanov's theorem. It basically states that the Girsanov change of measure "creates" a Brownian motion out of the process X:

Theorem 15.9 (Girsanov's Theorem I). *For every $T > 0$, the process $X(t), 0 \le t \le T$ is a Brownian motion with respect to the probability measure $\mathbb{P}_T$.*

Proof. The proof relies the the following so called **Levi** characterization of a Brownian motion

Proposition 3. A continuous process $B(t)$ is a standard Brownian motion if and only if it is a martingale and $B^2(t) - t$ is a martingale.

We use this characterization in order to prove Girsanov's theorem. The non-Brownian part of $X(t)$, which is $\int_0^t U(s)ds$ is a Riemann integral and is therefore a continuous function of t (a fact from real analysis). The Brownian part is an Ito integral. Earlier we showed that Ito integral is a continuous process (always has a continuous representation). Thus $X(t)$ is continuous. Now we show that $X(t)$ is a martingale w.r.t. $\mathbb{P}_T$. Fix $s < t$. Using (15.6)

$$\mathbb{E}_{\mathbb{P}_T}[X(t)|\mathcal{F}_s] = \frac{\mathbb{E}_{\mathbb{P}}[M(t)X(t)|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{P}}[M(t)|\mathcal{F}_s]} = \frac{M(s)X(s)}{M(s)} = X(s)$$

where we have used Proposition (2) and Lemma (15.8).

It remains to show that $X^2(t) - t$ is a martingale w.r.t. P_T. We use the Ito formula and the computations of Proposition 2 to compute $d(MX^2)$:

$$\begin{aligned} d(MX^2) &= X^2 dM + 2MXdX + \frac{1}{2}2X(dM)(dX) + \frac{1}{2}2M(dX)^2 \\ &= X^2(-MUdB) + 2MXUdt + 2MXdB - 2XMUdt + Mdt \\ &= X^2(-MUdB) + 2MXdB + Mdt \end{aligned}$$

On the other hand again using Ito formula

$$d(Mt) = tdM + Mdt = -tUMdB + Mdt$$

Therefore

$$d(M(X^2 - t)) = d(MX^2 - Mt) = X^2(-MUdB) + 2MXdB - tUMdB$$

Since this expression is an Ito integral, the process $M(X^2(t) - t)$ is a martingale. Again using (15.6) we show that under $\mathbb{P}_T$, $X^2(t) - t$ is martingale.

We showed that Levi condition for X is satisfied. Thus X is a Brownian motion w.r.t. $\mathbb{P}_T$. □

Let us turn to a more general version of Girsanov's Theorem. It corresponds to the case when the multiplicative component of the Brownian part in X is non-unit. We will state it without the proof. The proof details are similar.

Theorem 15.10 (**Girsanov's Theorem II**). *Consider an Ito process $dX = Udt + VdB$. Suppose there exists a process $\eta \in H^2$ satisfying $\eta(t)V(t) = U(t)$ a.s. and suppose the Novikov condition holds for η. Let*

$$M(t) = \exp(-\int_0^t \eta(s)dB(s) - \frac{1}{2}\int_0^t \eta^2(s)ds).$$

Fix $T > 0$ and consider the change of measures $\mathbb{P}_T = \eta(T)\mathbb{P}$. Then $\hat{B}(t) = \int_0^t \eta(s)ds + B(t)$ is a Brownian motion w.r.t. $\mathbb{P}_T$ and $dX(t) = V(t)d\hat{B}(t)$.

In the special case $V = 1$ we obtain $\eta = U$ and $X = \hat{B}$ and we recover the first version of Girsanov's theorem.

15.3. Additional reading materials

- Øksendal [**1**], Chapters III,IV, VIII

BIBLIOGRAPHY

1. B. Øksendal, *Stochastic differential equations*, Springer, 1991.