The Picard Scheme

THE PICARD SCHEME

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Abstract. This article introduces, informally, the substance and the spirit of Grothendieck's theory of the Picard scheme, highlighting its elegant simplicity, natural generality, and ingenious originality against the larger historical record.

1. Introduction

A scientific biography should be written
in which we indicate the "flow" of mathematics . . .
discussing a certain aspect of Grothendieck's work, indicating possible roots,
then describing the leap Grothendieck made from those roots to general ideas,
and finally setting forth the impact of those ideas.
Frans Oort [60, p. 2]

Alexander Grothendieck sketched his proof of the existence of the Picard scheme in his February 1962 Bourbaki talk. Then, in his May 1962 Bourbaki talk, he sketched his proofs of various general properties of the scheme. Shortly afterwards, these two talks were reprinted in [31], commonly known as **FGA**, along with his commentaries, which included statements of nine finiteness theorems that refine the single finiteness theorem in his May talk and answer several related questions.

However, Grothendieck had already defined the Picard scheme, via the functor it represents, on pp. 195-15, 16 of his February 1960 Bourbaki talk. Furthermore, on p. 212-01 of his February 1961 Bourbaki talk, he had announced that the scheme can be constructed by combining results on quotients sketched in that talk along with results on the Hilbert scheme to be sketched in his forthcoming May 1961 Bourbaki talk. Those three talks plus three earlier talks, which prepare the way, were also reprinted in [31].

Moreover, Grothendieck noted in [31, p. C-01] that, during the fall of 1961, he had discussed his theory of the Picard scheme in some detail at Harvard in his term-long seminar, which David Mumford and John Tate continued in the spring. In November 2003, Mumford kindly lent me his own folder of notes from talks given by each of the three, and notes written by each of them. Virtually all the content was published long ago.

Those notes contain a rudimentary form of the tool now known as *Castelnuovo–Mumford regularity.* Grothendieck mentions this tool in his commentaries [31, p. C-10], praising it as the basis for an "extremely simple" proof of a bit weaker version of

It is a great pleasure to thank my good friend Luc Illusie for carefully reading an earlier version of this paper, and making numerous comments, which led to many significant improvements. It is a pleasure as well to thank Michael Artin, Arthur Mattuck, Barry Mazur, David Mumford, Brian Osserman, Michel Raynaud, and the referee for their comments, which led to many other important improvements.

his third finiteness theorem. Mumford sharpened the tool in his book [46, Lect. 14], so that it yields the finiteness of the open subscheme of the Hilbert scheme that parameterizes all closed subschemes with given Hilbert polynomial.

Grothendieck [31, p. 221-1] correctly foresaw that the Hilbert scheme is "destined to replace" Chow coordinates. As he [31, p. 195-14] put it, they are "irremediably insufficient," because they "destroy the nilpotent elements in parameter varieties." Nevertheless, he [31, p. 221-7] had to appeal to the theory of Chow coordinates to prove the finiteness of the Hilbert scheme. So after he received a prepublication edition of [46], he wrote a letter on 31 August 1964 to Mumford in which he [48, p. 692] praised the theory in Lecture 14 as "a significant amelioration" of his own.

Mumford made use of the finiteness of the Hilbert scheme in his construction of the Picard scheme over an algebraically closed field in [46, Lect. 19], whereas Grothendieck took care to separate existence from finiteness, giving an example in [31, Rem. 3.3, p. 232-07] over a base curve of a Picard scheme with connected components that are not of finite type.

Mumford's book [46] was based closely on the lovely course he gave at Harvard in the spring of 1964. It was by far the most important course I ever took, due to the knowledge it gave me and the doors it opened for me. During the academic year of 1966–67, I was a Postdoc under Grothendieck at the IHES (Institut des Hautes Études Scientifiques). When he learned from me that I had taken that course and had advanced some of the finiteness theory in my thesis [35, Ch. II], he asked me to write up proofs of his nine finiteness theorems for SGA6 [8, Exps. XII, XIII].

Grothendieck, perhaps, figured that I had learned how to prove his nine theorems at Harvard, but in fact I had not even heard of them. At any rate, he told me very little about his original proofs, and left me to devise my own, which I was happy to do. There is one exception: the first theorem, which concerns generic relative representability of the Picard scheme. Its proof has a very different flavor, as it involves nonflat descent, Oort dévissage, and representability of unramified functors. Grothendieck asked Michel Raynaud to lecture on this theorem and to send me his lecture notes, which I wrote up in [8, Exp. XII].

My experience led me to study Grothendieck's construction of the Picard scheme, and to teach the whole theory a number of times. Further, in collaboration with Allen Altman, Mathieu Gagné, Eduardo Esteves, Tony Iarrabino, and Hans Kleppe, I extended some of Grothendieck's theory to the compactified Picard scheme. The underlying variety had been introduced via Geometric Invariant Theory in 1964 by Alan Mayer and Mumford in [44, § 4]. The scheme has been studied and used by many others ever since then.

Thus my experience is like the experiences of Nicholas Katz and Barry Mazur, which were described by Allyn Jackson in [33, p. 1054]. Katz said that Grothendieck assigned him the topic of Lefschetz pencils, which was new to him, but "he learned a tremendous amount from it, and it had a big effect on my future." Mazur said that Grothendieck asked him this question posed earlier by Gerard Washnitzer: "Can the topology of algebraic variety vary with the complex embedding of its field of definition?" Mazur, then a differential topologist, added, "But for me, it was precisely the right kind of motivation to get me to begin to think about algebra."

Both Katz and Mazur then confirmed my impression that our experiences were typical. Jackson quotes Katz as saying that Grothendieck got visitors interested in something, but with "a kind of amazing insight into what was a good problem to give to that particular person to think about. And he was somehow mathematically incredibly charismatic, so that it seemed like people felt it was almost a privilege to be asked to do something that was part of Grothendieck's long range vision of the future." Similarly, Mazur said that Grothendieck had an instinct for "matching people with open problems. He would size you up and pose a problem that would be just the thing to illuminate the world for you. It's a mode of perceptiveness that's quite wonderful and rare."

I spent the summer of 1968 at the IHES. Grothendieck invited me to his home in Massy-Verrières to discuss my drafts for my contributions to SGA 6. His comments ranged from providing insight into the theory of bounded families of sheaves to criticizing my starting sentences with symbols.[1] Again, my experience was typical:[2] Jackson [33, p. 1054] quotes Luc Illusie as saying, that Grothendieck often worked at home with colleagues and students, making a wide range of apposite comments on their manuscripts.

One time, Grothendieck found that I didn't know some result treated in EGA ([29] and sequels). So he gently advised me, for my own good, to read a little EGA every day, in order to familiarize myself with its content. After all, he pointed out, he had been writing EGA as a service to people like me; now it was up to us to take advantage of this resource. That experience supports a statement Leila Schneps made in [64, p. 16]: "The foundational work that Grothendieck and [Jean] Dieudonné were undertaking [was] in the service of all mathematicians, of mathematics itself. The strong sense of duty and public service was felt by everyone around Grothendieck."

As Grothendieck stated on p. 6 of [29], he planned to develop in EGA the ideas he sketched in [31]. He did not succeed. Nevertheless, those ideas have become a basic part of Algebraic Geometry. So they were chosen as the subject of a summer school held 7–18 July 2003 at the ITCP (International Center for Theoretical Physics) in Trieste, Italy. The first Bourbaki talk reprinted in [25] was not covered; it treats Grothendieck's generalization of Serre duality for coherent sheaves, so is somewhat apart and was already amply developed in the literature.

The lectures were written up, and published in [25]. As stated on p. viii, "this book fills in Grothendieck's outline. Furthermore, it introduces newer ideas whenever they promote understanding, and it draws connections to subsequent developments." In particular, I wrote about the Picard scheme, beginning with a 14-page historical introduction, which served as a first draft for the present article.

[1] Many years later, Jean-Pierre Serre told me that he had taught Grothendieck not to start sentences with symbols.

[2] Grothendieck gave me another project during my Postdoc. On April 18 and 25 that year, he talked in his seminar at the IHES on his Standard Conjectures and Theory of Motives. He asked me to write up his talks, gave me copies of his notes on related matters, and invited me to his home a year later, in the summer of 1968, to discuss my draft. That work too led me to learn some good mathematics and to write several articles, although they are more expository. Also, it led to my co-chairing an organizing committee for an AMS summer research conference in 1991.

However, my experience was the exception that proved the rule: Grothendieck had already asked others to write up his talks; they tried, and gave up! Also, curiously he never told me about his talk on the Standard Conjectures at a conference in Bombay, India, in January 1968, let alone offer me his notes. Moreover, in the conference proceedings, his writeup cites a talk of mine at the IHES, which I never gave, crediting me for an observation; but it is due to Saul Lubkin, and credited to him in my writeup [36, p. 361], which Grothendieck critiqued in his home that summer.

Mumford stated the goal of his book [46] on pp. vii–viii: "a complete clarification of . . . the so-called[3] Completeness of the Characteristic Linear System of a good complete algebraic system of curves on a surface. . . . Until about 1960, no algebraic proof of this purely algebraic theorem was known. . . . [Then] a truly amazing development occurred:" by combining his results on the Hilbert scheme and the Picard scheme with Cartier's result, "that group schemes in characteristic 0 are reduced," Grothendieck [31, pp. 221-23,24] obtained in February 1961 an enlightening, purely algebraic proof. "The key. . . is the systematic use of nilpotent elements."

Grothendieck had, moreover, reversed history: he proved Completeness via the Picard scheme. By contrast, in December 1904 Federigo Enriques and sometime in 1905 Francesco Severi gave algebraic proofs of Completeness from scratch. In the first half of 1905, on the basis of Enriques's work, Guido Castelnuovo **introduced** the Picard variety in order to prove the Fundamental Theorem of Irregular Surfaces. It asserts the surprising equality of the four basic invariants: the dimension of the Picard variety, the irregularity, the number of independent Picard integrals of the first kind, and half the first Betti number. Grothendieck's theory, without reference to Completeness, also yields the first part of the Fundamental Theorem, that the dimension of the Picard variety is equal to the irregularity in characteristic 0.

Both Enriques's and Severi's proofs have serious gaps, as Severi himself noted in 1921.[4] Severi then proved a more restricted version of Completeness, but one sufficient for Castelnuovo's work. Severi's proof was based on Henri Poincaré's construction of a key system of curves. That construction appeared in 1910, 1911; it is rigorous, but analytic. After 1921, finding a fully rigorous, purely algebraic proof of a suitable version of Completeness became a major endeavor — undertaken by Enriques, Severi, and others — until Grothendieck finally settled the matter. Section 2 explains more fully the history and meaning of Completeness and of the Fundamental Theorem; Section 5 elaborates on Grothendieck's proof.

When Grothendieck worked on his theory of the Picard scheme, the general algebro-geometric theory of the Picard variety had been under active development for nearly fifteen years. More than twenty mathematicians had worked on various aspects. Grothendieck clarified and settled a number of issues. Section 3 explains those issues in chronological order. Sections 4 and 5 give more detail about Grothendieck's advances, which involve many great innovations.

One issue was a topic of conversation between Grothendieck and Jacob Murre sometime in the academic year 1960/61. Murre told Schneps about it, and she [64, pp. 1–2] quoted him as saying, "A very important unsolved question . . . [was] the behavior of the Picard variety if the original variety . . . moved in a system and moreover — and worse — in[to] characteristic $p > 0$. . . . I asked Grothendieck whether he could explain this behavior. . . . He said he would certainly [do so]. . . . Then, in 1962, Grothendieck completely solved the question. . . . I attended his Bourbaki lectures, and needless to say, I was very impressed!"

As it happens, much earlier, in his 1958 talk [28, p. 118] at the ICM (International Congress of Mathematicians, Grothendieck said, "We shall not give here the precise

[3]On p. 8, the theorem is formulated as "problem (B)," and two analytic solutions are outlined. On p. 157, a more precise version is formulated as the "Fundamental Theorem," and given its first algebraic proof. On p. 169, an important special case is proved, following Grothendieck's somewhat different algebraic treatment. However, none of those is called the "Theorem of Completeness."

[4]In 1949, Severi [72, p. 40] lamented the fact that "this annoying episode was taken as an article of indictment for the [crime of] lack of rigor in Italian algebraic geometry!"

definition of a 'relative Picard schema', but . . . if this schema exists then it behaves in the simplest conceivable way with respect to change of base-space." In his February 1960 Bourbaki talk [31, pp. 195-15, 16], he added that "in particular, the Picard schemes of the fibers" of a system are the fibers of the relative Picard scheme, once existence is proved. In his February 1961 talk [31, pp. 212-01], as noted above, he announced his proof of existence.

Thus when Grothendieck had succeeded in settling a major issue, such as the Behavior of the Picard Variety in a Family or the Completeness of the Characteristic System, he noted the advance, but did not tout it. Cartier [12, p. 17] describes Grothendieck's philosophy as follows: "Grothendieck was convinced that if one has a sufficiently unifying vision of mathematics, if one can sufficiently penetrate the essence of mathematics and the strategies of its concepts, then particular problems are nothing but a test; they do not need to be solved for their own sake."

One beautiful illustration of Grothendieck's "unifying vision" is provided by his theory of the Picard functor. It is the *functor of points* of the Picard scheme— that is, the functor whose values are the sets of maps from a variable source into the scheme. Often, a functor of points is said to provide nothing more than another way of expressing the universal property of a fine moduli scheme. That statement is true for the Hilbert scheme, but a half-truth for the Picard scheme.

What is the universal property of the Picard scheme? The naive answer falls short! However, Grothendieck saw the hidden common thread in descent of the base field, Galois cohomology, and sheaf theory; he concluded that any functor of points must be a sheaf for the fpqc Grothendieck topology. Thus the right Picard functor has to be the sheaf associated to the naive Picard functor, regarded as a presheaf. More work with the functor leads to the construction of the Picard scheme. It is automatically compatible with base change, because the Picard functor is so. Sections 4 and 5 explain all that theory.

In short, Section 2 gives a historical introduction to two venerable theorems: the Theorem of Completeness of the Characteristic System, and the Fundamental Theorem of Irregular Surfaces. Section 3 gives a historical introduction to the inadequate algebro-geometric theory of the Picard variety. Please note: these two introductions are **not** meant to be either serious historical studies or rigorous mathematical surveys, but simply fascinating informal accounts, providing background material for comprehending the nature and extent of Grothendieck's advances.

Section 4 explains Grothendieck's innovative theory of the Picard functor, culminating in his main construction of the Picard scheme. Finally, Section 5 explains how the theory of the Picard scheme enabled Grothendieck and others to provide enlightening treatments of the issues discussed in Sections 2 and 3. The discussions in Section 4 and 5 are mathematically rigorous, but just introductory. Sources for more information are given at the beginning of each of Sections 2–5.

There are three minor mathematical novelties below: (1) the proof on p. 10 of the equivalence of the 19th century definition of the arithmetic genus of a surface and the modern definition, (2) the algebro-geometric treatment on p. 24 of Severi's 1921 version of Completeness, and (3) the "nearly formal" treatment on p. 26 of the Albanese variety, including duality, intriguingly announced by Grothendieck on p. 232-14 of his February 1962 Bourbaki talk [31].

2. Irregular Surfaces

But to demonstrate the power of modern abstract ideas
to solve older very concrete problems,
I think that this example is unmatched.
David Mumford [50, p. 7]

In the quotation above, *this example* refers to Grothendieck's treatment of the Theorem of Completeness of the Characteristic System. In fact, the example is the centerpiece of Mumford's article [50] in this volume. Moreover, Mumford notes that Completeness yields the Fundamental Theorem of Irregular Surfaces. Thus if we are to appreciate the full significance of Grothendieck's contribution, then we must review the history of those two main theorems. We do so in this section. First we pursue, intuitively, the spirit of the original work. Then we treat that work rigorously, beginning at the end of this section and continuing in Section 5.

A number of historical reviews are already available, and served as a basis for the account here. Notably, in 1906, Castelnuovo and Enriques wrote[5] one [14] at the request of Emile Picard to be an appendix to Tome II of his book [62] with Georges Simart. In 1934, Oscar Zariski reviewed various aspects of the development in different places in his celebrated book [78]. Those reviews are fairly technical. In 1994, Fabio Bardelli [7] wrote a more informal review of the developments through 1934. In 1974, Dieudonné published a masterful history of algebraic geometry, which touches on these theorems in particular. The book was translated as [21] by Judith Sally, and supplemented with an extensive annotated bibliography.

In 2011, Mumford [49] carefully analyzed the mathematics in a 1936 paper by Enriques on Completeness. Mumford, in his introduction, stated his conclusion: "Enriques must be credited with a nearly complete [algebro-]geometric proof using, as did Grothendieck, higher order infinitesimal deformations. . . . Let's be careful: he certainly had the correct ideas about infinitesimal geometry, though he had no idea at all about how to make precise definitions."[6] Mumford's article is preceded by a lovely article by Donald Babbitt and Judith Goodstein [6], which focuses on the times, lives, and personalities of Enriques and his colleagues; please also see their related articles [5] and [26]. All the articles mentioned above give many precise references, which are not repeated here.

Around 1865, Alfred Clebsch caused a sea change in algebraic geometry, turning it away from the concrete study of particular curves and surfaces, and toward the abstract study of their *birational invariants* — the numbers that depend only on their field of rational (or global meromorphic) functions.

In 1868, Clebsch considered a connected *smooth*[7] complex projective algebraic surface $\widetilde{X}$ of large degree n. He studied it via its general projection in 3-space, which is a surface

$$X : f(x, y, z) = 0 \quad \text{and} \quad n := \deg f$$

with "ordinary" singularities, none at infinity, and no point at infinity on the z-axis.

[5] Please also see their encyclopedia article [15] and Castelnuovo's historical note [23, pp. 339–353].

[6] Mumford elaborated in his resumeé at the end: "Although [Enriques] gave [infinitesimal deformations] names, they remained in limbo, without substance, because he did not think of what it meant to have a function on them. Grothendieck realized that functions on such objects should be rings with nilpotent elements, and this gave life to these infinitesimal deformations."

[7] Also called *nonsingular,* $\widetilde{X}$ is defined by polynomials with Jacobian matrix of maximal rank.

Clebsch found the algebraic double integrals on $\widetilde{X}$ of the *first kind* — that is, those finite on any bounded analytic domain of integration — to be of the form

$$\iint \frac{h(x,y,z)}{\partial f/\partial z}\,dx\,dy$$

where h is a polynomial of degree at most $n-4$ vanishing on the singular locus,

$$\Gamma : f,\ \partial f/\partial x,\ \partial f/\partial z,\ \partial f/\partial z = 0,$$

a curve of double points. The number of linearly independent such integrals became known as the *geometric genus* and denoted by p_g.

Clebsch asserted without proof that p_g is a birational invariant. In 1870, his student, Max Noether, gave an algebraic proof. In 1869, Arthur Cayley worked out a formula for the number of independent h; essentially, he found an explicit expression for $F(n-4)$ where F is the Hilbert polynomial of the homogeneous ideal of Γ. The value $F(n-4)$ was later called the *arithmetic genus* and denoted by p_a.[8]

In 1871, Hieronymous Zeuthen used Cayley's formula to prove algebraically that p_a too is a birational invariant. Also in 1871, Cayley observed that, if X is a ruled surface with plane section of genus g, then $p_a = -g \le 0$, although $p_g = 0$.

The disagreement between p_g and p_a came as a surprise. In 1875, Noether explained it: $F(n-4)$ is the number of independent h only if n is suitably large. In any case, $p_g \ge p_a$. Moreover, if X is smooth or rational, then $p_g = p_a$. It was thought that, as a rule, $p_g = p_a$, and when so, X was dubbed *regular*. The failure of X to be regular is quantified by the difference $p_g - p_a$; so it became known as the *irregularity*. Zariski [78, p. 75] denoted it by q; following suit, set

$$q := p_g - p_a.$$

In 1884, Picard initiated the study of algebraic simple integrals

$$\int P(x,y,z)\,dx + Q(x,y,z)\,dy$$

that are *closed*, or $\partial P/\partial y = \partial Q/\partial x$; they became known as *Picard integrals*. He proved that there are only finitely many independent such integrals of the *first kind*, those finite on any bounded analytic path of integration; use[9] s to denote their number. Picard noted that, if X is smooth, then $s = 0$.

In 1894, Georges Humbert considered an *algebraic system,* or *algebraic family,* of curves. Its members are the zeros on X of a polynomial

$$\varphi(x,y,z;\lambda_0,\dots,\lambda_t)$$

where the λ_i satisfy polynomial equations, which define the *parameter variety* Λ. The curves can all contain common subcurves; some of them are included as *fixed components* of the system, and the others, omitted. The system is said to be *linear* if there are homogeneous polynomials $\varphi_i(x,y,z)$ of the same degree with

$$\varphi = \lambda_0\varphi_0 + \cdots + \lambda_t\varphi_t.$$

Humbert proved a remarkable result: if $s = 0$, then every algebraic system is a subsystem of a linear system. That result inspired Castelnuovo to prove in 1896 that, if $q = 0$, then again every algebraic system is a subsystem of a linear system

[8] Cayley [16] denoted it by D, and called it the *deficiency*. Picard and Simart [62, p. 88] denoted it by p_n, and called it the *numerical genus*. Those definitions soon fell into disuse.

[9] Castelnuovo and Enriques [14, p. 495] used q, whereas Zariski [78, p. 162] used r_0.

under a certain restriction, which Enriques removed in 1899.

In 1897, Castelnuovo fixed a linear system of curves on X. Let r be its *dimension,* the number of linearly independent restrictions $\varphi_i|X$ diminished by 1. Castelnuovo studied its *characteristic* linear system, the system cut out by the φ_i on a general member curve D_η, assuming D_η is *irreducible*, that is, not the union of two smaller curves. The characteristic system has dimension has $r-1$.

Castelnuovo formed the *complete,* or largest, linear system on D_η containing the characteristic system. Let δ be the amount, termed the *deficiency,* by which the dimension of the characteristic system falls short of the dimension of its complete linear system. Castelnuovo proved that

$$\delta \leq q, \tag{1}$$

with equality if the linear system consists of all *hypersurface sections* of high degree, namely if the $\varphi_i(x, y, z)$ generate all homogeneous polynomials of that degree.

In February 1904, Severi extended Castelnuovo's work. Severi fixed an algebraic system of curves on X, and a general member D_η. He assumed that D_η is irreducible and that $D_\lambda \neq D_\mu$ for all distinct λ, $\mu \in \Lambda$. As λ approaches η along a path in Λ, the intersections $D_\lambda \cap D_\eta$ approach a limit, which depends only the tangent vector at η to the path. The various limits form a linear system on D_η, parameterized by the projectivized tangent space to Λ at η. Thus Severi constructed the *characteristic linear system* of the algebraic system. Set $R := \dim \Lambda$. Then this characteristic system is of dimension $R-1$.

In the algebraic system, form the largest linear subsystem containing D_η. Denote its dimension by r. Form its characteristic linear system. Let δ be its deficiency. Then its complete linear system has dimension $r-1+\delta$, and it also contains the characteristic system of the algebraic system. Thus Severi proved that

$$R \leq r + \delta, \tag{2}$$

with equality if and only if the latter characteristic system is complete.

In December 1904 Enriques and sometime in 1905 Severi each constructed an algebraic system with $R = r + q$. Both constructions are short and delicate. Both rely on the completeness of the characteristic system of certain[10] algebraic systems. Both are flawed, as Severi himself pointed out in 1921. In 1934, Zariski [78, pp. 99–102] reviewed those constructions, "in order to analyze the assumption on which they are based and for which as yet an algebro-geometric proof is not available."

In 1910 and 1911 using a new method of "normal functions," Poincaré gave a rigorous analytic construction of an algebraic system with $r = 0$ and $R = q$. His construction was simplified and developed by Severi in 1921 and Solomon Lefschetz in 1921 and 1924. In 1934, Zariski [78, pp. 169–173] reviewed that work too.[11] He [78, p. 102] noted that "the value of the construction of such a system is greater than that of mere example; indeed it is an essential step in the theory."

Zariski then derived Severi's May 1905 theorem[12] that, if there is one system with $R = r + q$, then $R = r + q$ holds for every complete system whose general member

[10] Severi [72, p. 41] noted that both he and Enriques believed at the time that they had proved every complete algebraic system with irreducible general member has a complete characteristic system!

[11] Please also see the reviews of Mumford [46, pp. 9–10] and Dieudonné [21, p. 53].

[12] In December 1904, Enriques proved the theorem under the more stringent, but still sufficient, hypothesis that D_η is "regular," later renamed "regular and nonspecial." Please see Fn. 43 on p. 23.

D_η is *arithmetically effective*; namely, a certain lower semi-continuous combination of its numerical characters is nonnegative, a common condition (please see p. 24). Hence, by (1) and (2), if D_η is irreducible too, then its characteristic system is complete and $\delta = q$. By Bertini's Theorem, usually D_η is irreducible.

The **Theorem of Completeness** came[13] to mean the following assertion:

(3) *Every complete algebraic system whose general member is arithmetically effective and irreducible has a complete characteristic system.*

Moreover, (3) is equivalent to the existence of at least one system with $R = r + q$, and in turn to the existence of suitably many such systems.

On 16 January 1905 in the C. R. Paris, Enriques [22, pp. 134–135] announced that Severi had just proved $q \geq s$ and $q = b - s$, where b is the number of independent Picard integrals of the *second kind*, those with polar singularities. It was known before 1897 that b is equal to the first Betti number; please see [78, p. 157]. In the same issue of the C. R., Picard [61] proved $q = b - s$ independently.[14]

In the next issue on 23 January, Castelnuovo [13] outlined the last step in this direction. He gave the details in three notes in the Rend. Accad. Lincei of 21 May and 4 and 8 June 1905. Specifically, he took a complete algebraic system with arithmetically effective (in fact, regular) general member, fibered it into linear systems, and formed the quotient, P say. Then P is projective, and P is of dimension q as $R = r + q$, an equation he considered proved. Morover, P is, up to isomorphism, independent of the choice of algebraic system, and sum (union) of curves induces an addition of points of P, turning P into a commutative group variety.

Hence, by a general 1895 theorem of Picard, completed in 1901 by Painlevé, P is an *Abelian variety*: P is parameterized by q *Abelian functions*, or $2q$-ply periodic functions of q variables, with a common lattice of periods. Castelnuovo proved that these functions induce independent Picard integrals on X. Therefore, $q \leq s$. Thus Castelnuovo obtained the **Fundamental Theorem of Irregular Surfaces**:

$$\dim P = q = s = b/2.$$

In 1905, the term "Abelian variety" was not yet in use. So naturally enough, Castelnuovo termed P the *Picard variety* of X.[15]

In 1903, Severi [69, § 26] discovered a remarkable expression for p_a in terms of a different Hilbert polynomial. Say the smooth surface $\widetilde{X}$ is a subvariety of some higher dimensional projective space $\mathbb{P}^N$. Form the Hilbert polynomial $\widetilde{F}(\nu)$ of the homogeneous ideal of $\widetilde{X}$. Then $\widetilde{F}(0) - 1 = p_a$.

Serre, in his 1954 ICM talk [66, pp. 286–291], announced a theory of coherent algebraic sheaves, inspired by the analytic work of Friedrich Hirzebruch, Kunihiko Kodaira, and Donald Spencer. In particular, Serre proved the Euler characteristic

[13] According to Severi [72, p. 42], in 1921 he derived the theorem essentially in this form from Poincaré's construction.

[14] Picard presented Enriques's note to the Academy, but explained in Fn. ([1]) on p. 122 of [61] that he had completed his own note before receiving Enriques's.

[15] Castelnuovo [13, p. 221] explained that "out of respect for Picard's profound research on surfaces [sic] admitting a group of birational automorphisms, [he] proposes calling the variety P (and [a certain] group G_d) *the Picard variety (and Picard group) associated to the surface* X."

Andre Weil [76, I, p. 572] discussed his own use of the term "Picard variety" in his commentary on his 1950 paper on Abelian varieties. Weil said, "Historically speaking, it would have been justified to give it Castelnuovo's name, but it was a matter of tampering as little as possible with common usage rather than rendering due homage unto this master."

of the twisted structure sheaf $\chi(\mathcal{O}_{\widetilde{X}}(\nu))$ is equal to $\widetilde{F}(\nu)$. Thus $p_a = \chi(\mathcal{O}_{\widetilde{X}}) - 1$;[16] so p_a is independent of the embedding of $\widetilde{X} \subset \mathbb{P}^N$ and of the projection $\widetilde{X} \to \mathbb{P}^3$.

Further, Serre Duality yields this equality of dimensions of cohomology groups: $h^i(\mathcal{O}_{\widetilde{X}}) = h^{2-i}(\Omega^2_{\widetilde{X}})$ for all i where $\Omega^2_{\widetilde{X}}$ is the sheaf of algebraic 2-forms. However, $p_g = h^0(\Omega^2_{\widetilde{X}})$, essentially by definition, and $h^0(\mathcal{O}_{\widetilde{X}}) = 1$ as $\widetilde{X}$ is connected. Thus

$$p_a = \chi(\Omega^2_{\widetilde{X}}) - 1, \quad p_g = h^2(\mathcal{O}_{\widetilde{X}}), \quad q = h^1(\mathcal{O}_{\widetilde{X}}).$$

Above, the first equation is a form of Severi's discovery. Here is a proof of it using Grothendieck's generalization of Serre Duality. Let ω_X be the dualizing sheaf. Since $X \subset \mathbb{P}^3$ and $\Omega^3_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$, duality theory and simple computation yield

$$\omega_X = \mathit{Ext}^1(\mathcal{O}_X\,,\ \Omega^3_{\mathbb{P}^3}) = \mathcal{O}_X(n-4).$$

Hence, duality for the finite map $\pi\colon \widetilde{X} \to X$ and elementary manipulation yield

$$\pi_*\Omega^2_{\widetilde{X}} = \mathit{Hom}(\pi_*\mathcal{O}_{\widetilde{X}}\,,\ \omega_X) = \mathfrak{C}(n-4) \quad \text{where} \quad \mathfrak{C} := \mathit{Hom}(\pi_*\mathcal{O}_{\widetilde{X}}\,,\ \mathcal{O}_X).$$

Thus $\chi(\Omega^2_{\widetilde{X}}) = \chi(\mathfrak{C}(n-4))$.

Here, $\mathfrak{C}$ is the conductor; it is the ideal sheaf on X of the curve Γ of double points. Let $\mathfrak{C}_0$ be the ideal sheaf on $\mathbb{P}^3$ of Γ. Then $p_a = \chi(\mathfrak{C}_0(n-4))$, essentially by definition. Form the standard exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{\times f} \mathfrak{C}_0(n-4) \to \mathfrak{C}(n-4) \to 0.$$

By Serre's Computation, $\chi(\mathcal{O}_{\mathbb{P}^3}(-4)) = -1$. Thus $p_a = \chi(\mathfrak{C}(n-4)) - 1$, as desired.

Recall $q = h^1(\mathcal{O}_{\widetilde{X}})$. Also, $s = h^0(\Omega^1_{\widetilde{X}})$ essentially by definition. So Hodge Theory yields $q = s$ and $q = b/2$, but Hodge Theory is not algebraic. However, a p-adic algebraic proof that $q = s$ was given by Kirti Joshi [34]. Further, if by b is meant the dimension of the first Grothendieck étale cohomology group, then a standard algebraic argument yields $\dim P = b/2$; see [36, Lem. 2A7, p. 375] for example. The latter argument also works in positive characteristic, but the equations $\dim P = q$ and $q = s$ may fail. In 1955, Jun-ichi Igusa gave an example with $\dim P = 1$ but $q = s = 2$; in 1958, Serre [66, p. 529] gave one with $\dim P = s = 0$ but $q = 1$.

Finally, $H^1(\mathcal{O}_{\widetilde{X}})$ is always the Zariski tangent space at 0 to the Picard scheme by Grothendieck's theory, and over $\mathbb{C}$ the Picard scheme is smooth by Cartier's theorem; so $\dim P = q$. Thus the Fundamental Theorem of Irregular Surfaces can be proved algebraically over $\mathbb{C}$, and the proof does not involve the Theorem of Completeness of the Characteristic System. Yet, the latter theorem has taken on a life of its own, and Grothendieck's work is heavily involved in proving both theorems algebraically. All that work is discussed further in Section 5.

3. The Picard Variety

Ever since 1949, I considered the construction
of an algebraic theory of the Picard variety
as the task of greatest urgency in abstract algebraic geometry.
André Weil [76, II, p. 537]

[16] Afterwards, it became common to define p_a by this formula.

Up to 1949, Weil worked primarily in Number Theory and Algebraic Geometry.[17] That work culminated in proofs of the Riemann hypothesis for curves in 1948 and in the formulation of his celebrated conjectures for arbitrary dimension in 1949. Next, he led "the construction of an algebraic theory of the Picard variety." In turn, that theory led Grothendieck to develop his theory of the Picard scheme. However, the Weil Conjectures themselves motivated much of Grothendieck's work. In particular, they led to the notion of a Grothendieck topology, which, as noted in the introduction, is fundamental for the very definition of the Picard functor; that functor is the subject of Section 4.

In the present section, so that we may better appreciate Grothendieck's advances, let us consider in chronological order up to 1962, what was sought and what was proved about the Weil conjectures and the Picard variety. Good secondary sources include Dieudonné's history [21] for all of it, Mazur's 1974 expository article [43] for the Weil conjectures, and the explanatory comments and historical notes in Serge Lang's 1959 book [40] for the Picard variety. Again, as those sources contain many primary references, those references are not always repeated here.

In his 1921 thesis, which was published in 1924, Emil Artin developed an analogue of the classical Riemann hypothesis, in effect, for a hyperelliptic curve over a prime field of odd characteristic. In 1929, Friedrich (F-K) Schmidt generalized Artin's work to all curves over all finite fields, recasting it in the algebro-geometric style of Richard Dedekind and Heinrich Weber. In 1882, they had viewed a curve as the set of discrete valuation rings in a finitely generated field of transcendence degree 1 over $\mathbb{C}$, but their approach works in any characteristic. In particular, Schmidt ported their proof of the Riemann–Roch theorem, and used it to prove that Artin's Zeta Function satisfies a natural functional equation.

In 1936, Helmut Hasse proved Artin's Riemann hypothesis in genus 1 via an analogue over finite fields for the theory of elliptic functions. Then he and Max Deuring noted that to extend the proof to higher genus would require developing a similar analogue for the nineteenth century theory of correspondences between complex curves.

Their work inspired Weil.[18] In each of two notes, [76, I, pp. 257–259] of 1940 and [76, I, pp. 277–279] of 1941, he sketched a different proof of the Riemann hypothesis in any genus. In both, the key is a certain positivity theorem for correspondences. It was found over $\mathbb{C}$ by Castelnuovo[19] in 1906, and proved over a field of any characteristic by Weil in two ways: in 1940 by algebraizing Adolf Hurwitz's transcendental theory of 1886, and in 1941 by porting to positive characteristic the algebro-geometric theory in Severi's textbook [70] of 1926.

To provide the details, Weil had to redo the foundations of Algebraic Geometry over a field of arbitrary characteristic. The first instalment [73] appeared in 1946. Building on ideas of Emmy Noether, Bartel van der Waerden, and Schmidt from the

[17] "Weil was far from confining himself to" those subjects, as Serre [67, pp. 523–526] noted, citing Weil's work in real and complex analysis, representation theory, and differential geometry.

[18] The interaction among the three and others has attracted a lot of study. One delightful and well-documented report was published by Michèle Audin in 2012 as [4]. It describes the political, social, and personal circumstances at the time, while focusing on three reviews of Weil's first note.

[19] In both notes, Weil cites only Severi. In his commentary on the second note [76, I, p. 553], Weil wrote, "it's one of Castelnuovo's most beautiful discoveries (see his *Memorie Scelete,* no. XXVIII, pp. 509–517). But I didn't read Castelnuovo until 1945 in Brazil; then I realized that Severi in the *Trattato* ([70, pp. 286–287]) had not given his elder due credit."

1920s, Weil fixed a *universal domain* Ω, a field of infinite transcendence degree over the prime field. Then a *projective variety* X is the locus of zeros with coordinates in Ω of homogeneous polynomials with coefficients in a variable *coefficient field* or *field of definition* k, a subfield of Ω over which Ω has infinite transcendence degree. Also X is *absolutely irreducible*, not the union of two smaller such loci. Then Weil formed *abstract* varieties by patching pieces of projective varieties.

Finally, Weil treated *cycles*. They are the formal $\mathbb{Z}$-linear combinations of subvarieties. Those of codimension 1 are called (*Weil*) *divisors*, and play a major role in the theory of the Picard variety. Weil developed a calculus of cycles, including intersection products, inverse images, and direct images.

In 1948, Weil published two books. In the first [74], he completed his note of 1941. He reproved the Riemann–Roch theorem, and developed an elementary theory of correspondences for curves. To prove Castelnuovo's theorem, he used his full calculus of cycles on products of numerous copies of the curve. The proof is "the most complicated part of the" book, as Otto Schilling observed in his Math Review [MR0027151]. Then Weil proved the Riemann hypothesis.[20]

Weil's proof inspired three others. First, in his 1953 thesis under Hasse, Peter Roquette translated it into the more arithmetic language of Schmidt, and simplified it to involve the product of just two different curves. Second, in 1958, Arthur Mattuck and John Tate applied the Riemann–Roch theorem for surfaces, which had been proved in any characteristic by Zariski in 1952 and by Serre in 1956. Mattuck and Tate proved the version of Castelnuovo's theorem for the product of two curves that Severi [70] gave on p. 265. They dubbed it the *inequality of Castelnuovo–Severi*. Then they rederived the Riemann hypothesis, thus showing that it is a fairly simple consequence of the general theory of algebraic surfaces.

Third, right as Mattuck and Tate finished their paper, Grothendieck [27, p. 208], "attempting to understand the full import of their method," found that it produces an index theorem on any surface, which yields the Castelnuovo–Severi inequality. According to Grothendieck however, Serre pointed out to him that he had proved an algebraic version of William Hodge's 1937 analytic index theorem, and moreover that the same version had already been proved the same way by Beniamino Segre in 1937 and independently by Jacob Bronowski in 1938.

In Weil's second book [75] of 1948, he completed his note of 1940. He developed the abstract theory of *Abelian varieties*, which he defined as the group varieties that are *complete*, the abstract equivalent of "compact." He proved that they are commutative, and that a map between two is a homomorphism plus a translation.

Weil constructed the Jacobian J of a smooth curve C of genus g by patching together copies of an open subset of the symmetric product $C^{(g)}$. Given a prime l different from the characteristic, he constructed, out of the points on J of order l^n for all $n \geq 1$, an l-adic representation of the ring of correspondences, which, over $\mathbb{C}$, is equivalent to the representation on the first cohomology group. He proved that the trace of this representation is positive definite, and recovered Castelnuovo's theorem. Finally, he reproved the Riemann hypothesis for curves.

Weil left open, as Lang [40, p. 17] noted, two important questions: (i) Is J defined over the given coefficient field of C? (ii) Is every Abelian variety projective? Both

[20] It is extraordinarily important. Dieudonné [21, p. 83] gave one reason why: it "allows proofs, in analytic [sic] number theory, of 'the best possible' upper bounds, inaccessible" otherwise, such as this bound on a Kloosterman sum: $\left|\sum_{x=1}^{p-1} \exp\left(\frac{2\pi i}{p}(x+x^{-1})\right)\right| \leq p^{1/2}$ for any prime p.

questions were soon answered affirmatively: (i) by Wei-Liang Chow, who announced his answer in 1949 but published it in 1954, and (ii) by Teruhisa Matsusaka in 1953. In 1954, Weil gave a much simpler and more direct answer to (ii).

In 1956, in order to handle (i), Weil addressed the general question of finding a smaller coefficient field, but only in the case where the resulting field extension is finitely generated and separable. In turn, Weil's work inspired Grothendieck to develop his general Descent Theory, which he then sketched in his December 1959 Bourbaki talk [31, 190]. Grothendieck said on p. 190-1 that he was also inspired by Cartier's subsequent treatment [10, §4] of purely inseparable extensions, but that "due to the lack of the language of schemes, and especially the lack of nilpotents, Cartier could not express the basic commonality of the two cases."

In 1949, Weil published his celebrated conjectures about the zeta function of a variety of arbitrary dimension. He did not involve a hypothetical cohomology theory outright, but one is implicit. Moreover, one was credited to him explicitly in Serre's 1956 "Mexico paper" [66, p. 502] and in Grothendieck's 1958 ICM talk [28, p. 103], where the term "Weil cohomology" appears, likely for the first time.

Grothendieck then announced "an approach [to Weil cohomology]... suggested to [him] by the connections between sheaf-theoretic cohomology and cohomology of Galois groups on the one hand, and the classification of unramified coverings of a variety on the other..., and by Serre's idea that a 'reasonable' algebraic principal fiber space... should become locally trivial on some covering unramified over a given point." Thus, on p. 104, he announced the birth of *Grothendieck topology*.

In 1950, Weil published a remarkably prescient note [76, I, pp. 437–440] on Abelian varieties. For each normal[21] projective variety X of any dimension in any characteristic, he said that there ought to be two associated Abelian varieties, the *Picard variety* P and the *Albanese*[22] *variety* A, with the following properties:

Universality: The Picard variety P parameterizes the linear equivalence classes of all divisors on X algebraically equivalent[23] to 0. There is a rational map[24] from X into A, defined wherever X is smooth, such that every rational map from X into an Abelian variety factors uniquely, up to translation, through it.

Duality: If X is an Abelian variety, so that $X = A$, then A is the Picard variety of P; such a pair, A and P, are called *dual* Abelian varieties.

Also, A and P are *isogenous*, or finite coverings of each other, and of dimension equal to the irregularity [sic]. If X is arbitrary, then A and P are dual; in fact, the universal map $X \to A$ induces the canonical isomorphism from the Picard variety of A to P. If X is a curve, then both A and P coincide with the Jacobian.

In the note, Weil said that he had complete treatments of P and A for a smooth

[21] *Normal* means the singular locus has codimension at least 2, and (Zariski's Main Theorem) a rational map is defined everywhere if its graph projects finite-to-one onto X (so isomorphically).

[22] In 1913, Severi introduced and studied A over the complex numbers. Nevertheless, much to Severi's dismay, Weil [76, I, p. 571] named A after Severi's student, Giacomo Albanese, ostensibly because, in 1934, Albanese viewed A as a quotient of a symmetric power of X. However, Weil [76, I, p. 562] left the impression that rather it is because he owed a debt of gratitude to Albanese for enriching the library of the University of Saõ Paulo, Brazil, with works of Castelnuovo, Torelli and others, which were new to Weil and from which he "profited amply."

[23] *Algebraic equivalence* and *linear equivalence* are just the equivalence relations generated by the algebraic systems and the linear systems.

[24] A *rational map* is given by the ratio of two polynomials, and is *defined* at a point if, for some choice of the two, the denominator does not vanishes there.

complex X, and sketches in general. The sketches rest on two criteria for linear equivalence of divisors in terms of linear-space sections. The criteria were found in 1906 by Severi, and reformulated in the note by Weil, who referred to pp. 104–105, 164–165 in Zariski's book [78]; please see Mumford's comments [78, p. 120] as well. Weil announced proofs of the criteria in 1952, and gave the details in 1954.

In 1951, Matsusaka gave the first algebraic construction of P. He extend the coefficient field k in order to apply two of Weil's results: one of the equivalence criteria and the construction of the Jacobian. Both applications involve the *generic curve*, the section of X by a linear space of appropriate dimension defined over a transcendental extension of k. In 1952, Matsusaka gave a second construction; it does not require extending k, but does require X to be smooth.

Both of Matsusaka's constructions are like Castelnuovo's: first Matsusaka constructed a complete algebraic system of sufficiently positive divisors, and then he formed the quotient modulo linear equivalence. To parameterize the divisors, he used the theory of "Chow coordinates," which was developed in 1938 by Chow and van der Waerden and was under refinement by Chow. In 1952, Matsusaka also used Chow coordinates to form the quotient. Further, he made the first construction of A, again using the Jacobian of the generic curve, but he did not relate A and P.

Also in 1952, in § II of his paper *On Picard Varieties* [76, II, pp. 73–102], Weil refined the sense in which P parameterizes classes of divisors. Working complex analytically, he constructed "an algebraic family of divisors on X, parameterized by P, containing one and only one representative of each class."

Weil did not name that family of divisors. However, the same year, André Néron and Pierre Samuel [56] constructed,[25] in any characteristic, a similar family, which they named a *Poincaré family* citing [76, II, pp. 73–102] in a way suggesting the name[26] is due to Weil. The family is defined by a divisor D on $X \times P$, which is called a *Poincaré divisor* by Lang [40, p. 114]. Moreover, Lang showed that the pair (P, D) is unique, P up to isomorphism and D up to addition of a "trivial" divisor.

In 1955, Chow constructed A and P in a new way, as what he called respectively the "image" and the "trace" of the Jacobian of a generic curve on X. Also, he proved that, indeed, the universal map $X \to A$ induces an isomorphism of the Picard variety of A onto P.

In a course at the University of Chicago, 1954–55, Weil gave a more complete and elegant treatment, based on the "see-saw principle," which he adapted from Severi, and on his own Theorem of the Square and Theorem of the cube. This treatment became the core of Lang's 1959 book [40]. The idea is to construct A first using the generic curve, and then to construct P as a quotient of A modulo a finite subgroup. Thus there is no need for Chow coordinates.

In 1958, Serre [66, p. 555] worked over a fixed algebraically closed coefficient field k of any characteristic. He reproved Igusa's 1955 bound $\dim A \le h^0(\Omega^1_X)$, and obtained a simple direct construction of A over k, not using the generic curve.

In 1958 Cartier [10] and in 1959 Nishi [57] independently proved Weil's conjectured duality of A and P: in any characteristic, each is the others Picard variety.

Between 1952 and 1957, Maxwell Rosenlicht published a remarkable series of

[25] Unfortunately, Lang [40, p. 175] felt that he had to write: "It can not be said that the Picard variety is constructed in [56] because this paper begins by a false statement concerning the birational invariance. This is a delicate point, when the varieties involved have singularities."

[26] Of course, here and below, Poincaré's name is used to honor his work mentioned above.

papers, inspired by Severi's 1947 monograph [71], which treated curves with double points. Treating a curve with arbitrary singularities, Rosenlicht generalized the notions of linear equivalence and differentials of the first kind. Then he constructed a *generalized Jacobian* J over $\mathbb{C}$ by integrating and in arbitrary characteristic by patching. It is not an Abelian variety, but an extension of the Jacobian J_0 of the desingularized curve by an affine algebraic group. In 1962, Frans Oort [59] gave another construction, which gives J as a successive extension of J_0 by additive and multiplicative groups.

For arithmetic applications, Tate suggested, according to Lang [40, p. 176], doing this. Given finitely many simple points on X, consider the divisors avoiding them. Form linear equivalence classes via functions congruent to 1 to given multiplicities at the points. Finally, seek a generalized Picard variety to parameterize these classes.

In 1959, Serre published a textbook [65] on the case $\dim X = 1$ and its application to Lang's class field theory over function fields. In particular, Serre recovered Rosenlicht's generalized Jacobian of an X with one singular point[27], constructed by identifying given points with given multiplicities on a given smooth curve.

In 1962, Murre [52] constructed Tate's generalized Picard variety P by adapting Matsusaka's two constructions. Thus Murre obtained P for any (normal) X via patching and for any smooth X directly over the same ground field.

In 1956, Igusa established the compatibility of specializing a curve with specializing its generalized Jacobian, possibly under reduction mod p, provided the general curve is smooth and the special curve has at most one node. Igusa explained that, in 1952, Néron had studied the total space of such a family of Jacobians, but had not explicitly analyzed the special fiber.[28] Igusa's approach is, in spirit, like Castelnuovo's, Chow's, and Matsusaka's before him.

In 1960, Claude Chevalley [19] constructed a Picard variety using Weil divisors locally defined by one equation; they are called *Cartier divisors* in honor of Cartier's 1958 Paris thesis [11]. First, Chevalley constructed a *strict* Albanese variety; it is universal for *regular* maps, ones defined everywhere, into Abelian varieties. Then he took its Picard variety to be that of X. He noted his Picard and Albanese varieties need not be equal to those of a desingularization of X. By contrast, Weil's P and A are birational invariants, and his universal map $X \to A$ is a rational map. In 1962, Conjeerveram Seshadri [68] generalized Chevalley's construction to an X with arbitrary singularities, recovering Rosenlicht's generalized Jacobian.

In 1961, Mattuck [42] took, on a smooth X over an algebraically closed field, a complete algebraic system Σ of suitably positive divisors C. He parameterized Σ by the *Chow variety* H, the locus of points given by the Chow coordinates of the $C \in \Sigma$. He fixed a $D \in \Sigma$, and took the class of the difference $C - D$ for $C \in \Sigma$, to get a rational map $\alpha\colon H \to P$. In order that α be defined everywhere, he reembedded X in another projective space, because Murre [51] had just proved that then H is smooth, so normal.

Mattuck proved that α is a projective bundle[29] and has a section. The section corresponds to a refined Poincaré divisor: not only does it define a Poincaré family, but it contains no fiber of α, so cuts each fiber in a divisor. Finally, Mattuck

[27] The singularity cannot be arbitrary; for example, it cannot be a planar triple point.

[28] For a comprehensive discussion of the "Néron model" and its connection to the Picard scheme (and Picard algebraic space) along with historical notes and references to the original sources, please see the textbook [9] of Siegfried Bosch, Werner Lütkebohmert, and Raynaud.

[29] Earlier, in 1956, Kodaira [38] obtained a similar result analytically.

studied the case that X is a curve of genus $g > 1$, so that H is the n-th symmetric power of X where n is the degree of the divisors. He proved that α is a bundle[30] if $n > 2g-2$, and that α has a section[31] if $n > 4g$, but no section if $n = 2g-1$ and the divisor classes modulo algebraic equivalence on P form a group of $\mathbb{Q}$-rank 1. So it seems unlikely that α has a section if $n = 2g-1$ and X has general moduli.

Thus, in 1962, the algebraic theory of the Picard variety was indirect, involved, and incomplete. There were competing definitions and constructions, each with advantages and disadvantages. There was a lot of fussing with fields of definition. There were loose ends. Notably, there was no fully satisfactory way to parameterize divisors or to construct quotients. So there was not enough machinery to prove the Completeness of the Characteristic System or to treat, in general, the behavior of the (generalized) Picard variety in a family. Grothendieck brilliantly handled those issues in the way explained in the next two sections.

4. The Picard Functor

Grothendieck certainly did not feel
that he was attempting to use powerful techniques
in order to obtain stronger results by generalizing.
What he perceived himself as doing was simplifying situations and objects,
by extracting the fundamental essence of their structure.
Leila Schneps [64, p. 3]

Many times, Grothendieck proceeded "by extracting the fundamental essence" of existing theories, and then developing his own versions, for example his theories of schemes, representable functors, the Hilbert scheme, and the Picard scheme. Parts of those theories belong to the theory of the Picard functor. Those parts are treated in depth in [32] and [25], and they are introduced in this section.

To begin, here is a bit more informal history. Starting in 1937, Zariski made deep use of the local ring of all rational functions that are finite at a given point of a variety with a fixed field of definition. Inspired by Zariski's work, Chevalley developed an intrinsic theory of abstract varieties V in his paper [17], submitted on 2 July 1954. On p. 2, he called the set of all the local rings of the points of V its *model,* and then developed a theory of models. Earlier, in January 1954, he lectured on his theory at Kyoto University, according to Masayoshi Nagata [55, p. 78], who then generalized it by replacing the field of definition by a Dedekind domain.

In 1944, Zariski topologized the set of all valuation rings of the field of rational functions of a variety in order to use the finite-covering property to pass from local uniformization to global desingularization. In 1949, Weil [76, I, pp. 411–413] observed that his abstract varieties support what he called the *Zariski Topology*, whose closed sets are the subvarieties and their finite unions.

Weil used the Zariski topology to define locally trivial fiber spaces. He discussed the natural bijective correspondence between the line bundles on a smooth variety and its linear equivalence classes of divisors. Then, in his 1950 paper on Abelian varieties [76, I, pp. 438–439], he suggested that those line bundles might be used to develop, for any abstract variety, a version of Severi's generalized Jacobian.

As already noted in Section 2, Serre, in his 1954 ICM talk, announced a theory

[30] Atiyah [3, p. 451] wrote in 1957 that this case is "well known."

[31] In fact, $n > 2g-1$ suffices; please see p. 26 below.

of coherent algebraic sheaves. In fact, he worked over an arbitrary algebraically closed field k, of any transcendence degree over the prime field, and used k both as the field of definition and as the field of coordinates.[32] Moreover, he worked only with projective space and its subvarieties, which he allowed to be reducible, and he viewed as the closed sets of a topology, which he too called the "Zariski topology."

In 1955, Serre presented the details in his celebrated paper *Faisceaux algébriques cohérents* [66, pp. 310–391]. He also generalized the notion of abstract variety via Henri Cartan's notion[33] of *ringed space*. It is a topological space X endowed with a *sheaf of rings* $\mathcal{O}_X$, called the *structure sheaf*: over each open set, its sections form a ring; for each smaller open set, the restriction map is a ring homomorphism. To be a variety, X must be covered by finitely many open subsets, each of which, when endowed with the restriction of $\mathcal{O}_X$, is isomorphic to an *affine variety*; the latter's space is a closed subspace of an affine space, and its structure sheaf has, as its sections over an open set U, the rational functions defined everywhere on U.

Both Serre and Chevalley downgraded rational maps, preferring maps[34] that are defined everywhere. For Serre, a map of varieties $\varphi\colon X \to Y$ is a map of ringed spaces: φ is a continuous map equipped with a map φ^* relating the two structure sheaves, so that a section f of $\mathcal{O}_Y$ over an open set V yields a section $\varphi^* f$ of $\mathcal{O}_X$ over $f^{-1}V$ in a way respecting addition, multiplication, and restriction.

Grothendieck "extracted the fundamental essence of" those ideas, and developed a theory of *schemes*.[35] By February 1956 (see [20, p. 32]), he was working with ringed spaces that have an open covering by *affine schemes*, or *ring spectra*.[36] The spectrum of a ring R is the set of all its prime ideals $\mathfrak{p}$. Its topology is generated by its *principal open subsets* $D(f)$ for all $f \in R$, where $D(f) := \{\mathfrak{p} \not\ni f\}$. Over $D(f)$, the sections of the structure sheaf are the fractions a/f^n for all $a \in R$ and $n \geq 0$. In 1956, Grothendieck took R to be Noetherian, but in EGA I [29], which appeared in 1961, R is an arbitrary commutative ring. In any case, R may have nilpotents.

Moreover, Grothendieck worked with Cartier's generalization in [11, p. 206] of coherent sheaves, the *quasi-coherent* sheaves. He [20, p. 32] told Serre why: they are "technically very convenient because they have the relevant properties of coherent sheaves without requiring the finiteness (on an affine, they correspond to all the modules over the coordinate ring, and not just the finitely generated modules)."

The generality is vast, but not idle. Murre put it as follows, according to Schneps [64, p. 2]: "Undoubtedly, people did see in the mid 50's that one could generalize a lot of things to schemes, but Grothendieck saw that such a generalization was not only possible and natural, but necessary to explain what was going on, even if one started with varieties."

The spectrum S of a field k is a single point, but S has k as structure sheaf.

[32] Cartier [11] generalized Serre's theory to an arbitrary field of definition, and studied the effect of extending it. He took the field of coordinates to be a universal domain.

[33] Cartan had used the notion to define C^∞-manifolds. He and Serre had used it to define complex analytic manifolds.

[34] Serre called them *regular maps,* a traditional term; Chevalley [18, p. 219] used *morphisms.*

[35] Cartier [12, Fn. 8] noted, "This word results from a typical epistemological shift from one thing to another: for Chevalley, who invented the name in 1955, it indicated the 'scheme' or 'skeleton' of an algebraic variety, which itself remained the central object. For Grothendieck, the 'scheme' is the focal point, the source of all the projections and all the incarnations."

[36] Cartier [12, Fn. 29] explained, "It was [Israel] Gelfand's fundamental idea [of 1938] to associate a normed commutative algebra to a space. . . . The term 'spectrum' comes directly from Gelfand."

Call k the *field of definition* of a scheme X if there is a distinguished map $X \to S$. On the other hand, given a universal domain Ω extending k, let T be its spectrum. Then a *point* of X with "coordinates" in Ω, or an Ω-*point* of X, is just a map $T \to X$ that respects the distinguished maps $X \to S$ and $T \to S$. Cartier [12, p. 20] described this new situation as being "admirable simplicity — and a very fruitful point of view — but a complete change of paradigm!"

For example, say X is the spectrum of the residue ring R of the polynomial ring $k[u_1, \dots, u_n]$ modulo the ideal generated by polynomials f_λ. Then the inclusion $k \to R$ defines the map $X \to S$. Further, since Ω is a universal domain, a zero $(c_1, \dots, c_n)$ of the f_λ in Ω^n amounts exactly to a k-algebra map $\gamma\colon R \to \Omega$ that carries the residue of u_i to c_i. In turn, γ amounts exactly to an S-map $T \to X$.

However, there's no need to restrict S and T to be the spectra of fields, and good reason not to. Moreover, the right setting for this theory, as for any theory whose principals are objects and maps, is Category Theory. It is not simply a convenient language for expressing abstract ideas, but more importantly, an effective tool, which eases the work at hand and affords new possibilities. Grothendieck recognized as much, and promoted Category Theory.

Thus, given a *base* scheme S, an S-*scheme* is a scheme equipped with a map to S, its *structure map*. An S-*map* is a map between S-schemes that commutes with the two structure maps. The category of S-schemes has products:[37] the product of X and Y is an S-scheme $X \times_S Y$ with a distinguished pair of S-maps to X and Y, the *projections*, such that composition with them sets up a bijection from the S-maps $T \to X \times_S Y$ to the pairs of S-maps $T \to X$ and $T \to Y$.

The product $X \times_S Y$ is determined, formally, up to unique isomorphism. It is constructed by patching together the spectra of the tensor products of the rings of affines that cover S, X, and Y. It can also be viewed as the result of *base change* of X when Y is viewed as a new base. By contrast, Cartier [12, p. 20] noted that, "in both [Serre's and Chevalley's] cases, the two fundamental problems of products and base change could only be approached indirectly."

Another way to view a map $X \to S$ is as a family X/S with S as parameter space and X as total space. Its members are the *fibers*, the preimages X_s of the points $s \in S$. More precisely, $X_s := X \times_S Y$ where Y is the spectrum of the residue field k_s of the stalk $\mathcal{O}_{S,s}$, which is the local ring of all functions that are each defined on some neighborhood of s. Grothendieck discovered that, many properties of the X_s vary continuously when X is S-*flat*; that is, $\mathcal{O}_{X,x}$ is $\mathcal{O}_{S,s}$-flat for all $s \in S$ and $x \in X_s$. Often, he considered what he called a *geometric fiber*, which is the product $X \times_S Y'$ where Y' is the spectrum of an algebraically closed field containing k_s.

An S-map $T \to X$ is called a T-*point* of X, and the set of all of them is denoted by $X(T)$ or $h_X(T)$. An S-map $T' \to T$ induces a set map $h_X(T) \to h_X(T')$. Thus h_X is a contravariant functor from the category of S-schemes to the category of sets; it is called the *functor of points* of X.

The contravariant functors H from S-schemes to sets themselves form a category. The assignment $X \mapsto h_X$ is a functor from S-schemes into the latter category. This functor is an embedding by Yoneda's Lemma. Given an H, if an X is found with $h_X = H$, then H is said to be *representable* by X. If so, then $H(X)$ contains a *universal* element W, which corresponds to the identity map of X; namely, each

[37] Mac Lane [41, p. 76] said that he himself, in 1948 and 1950, formulated "the idea that . . . products could be described by universal properties of their projections."

element $Y \in H(T)$ defines a unique S-map $\varphi\colon T \to X$ with $H(\varphi)W = Y$. In other words, the T-points of X *represent* the elements of $H(T)$. Conversely, if an X is found with a universal $W \in H(X)$, then there's a canonical isomorphism $h_X = H$.

The first important example is the functor $P(\mathcal{E})$, where $\mathcal{E}$ is an arbitrary quasi-coherent sheaf on S. For each S-scheme T, the set $P(\mathcal{E})(T)$ is the set of invertible quotients $\mathcal{L}$ of the pullback $\mathcal{E}_T$; *invertible* means that $\mathcal{L}$ is the sheaf of sections of a line bundle. The functor $P(\mathcal{E})$ is representable by an S-scheme $\mathbf{P}(\mathcal{E})$. Automatically, $\mathbf{P}(\mathcal{E})$ carries a *universal* invertible quotient of $\mathcal{E}_{\mathbf{P}(\mathcal{E})}$, denoted $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$; namely, each invertible quotient $\mathcal{L}$ of $\mathcal{E}_T$ defines a unique S-map $\varphi\colon T \to \mathbf{P}(\mathcal{E})$ with $\varphi^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = \mathcal{L}$. Moreover, $\mathbf{P}(\mathcal{E}) \times_S Y = \mathbf{P}(\mathcal{E}_Y)$ for any S-scheme Y. In particular, the fiber $\mathbf{P}(\mathcal{E})_s$ over $s \in S$ is the projective space of 1-dimensional quotients of the vector space $\mathcal{E}_s \otimes k_s$ where $\mathcal{E}_s$ is the stalk of $\mathcal{E}$ at s.

For convenience, **assume from now on** that all schemes are *locally Noetherian*, or covered by affine schemes on Noetherian rings, and that S is *Noetherian*, or covered by finitely many such. Assume **also** that X is projective over S; that is, X can be embedded as a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some coherent sheaf $\mathcal{E}$ on S.

Consider the *Hilbert functor* $\mathrm{Hilb}_{X/S}$, treated by Grothendieck in his May 1961 Bourbaki talk [31, Exp. 221]. For each S-scheme T, the set $\mathrm{Hilb}_{X/S}(T)$ is the set of T-flat closed subschemes Y of $X \times_S T$. Moreover, for each polynomial $F(\nu)$ with rational coefficients, $\mathrm{Hilb}_{X/S}$ has a subfunctor $\mathrm{Hilb}^F_{X/S}$; namely, for all T, the set $\mathrm{Hilb}^F_{X/S}(T)$ is the set of Y such that Y_t has Hilbert polynomial F for all $t \in T$, that is, $F(\nu) = \chi(\mathcal{O}_{Y_t}(\nu))$ where $\mathcal{O}_{Y_t}(\nu)$ is the pullback to Y_t of the νth tensor power of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Grothendieck proved that $\mathrm{Hilb}_{X/S}$ is representable by a locally Noetherian S-scheme $\mathbf{Hilb}_{X/S}$, the *Hilbert scheme*. In fact, $\mathbf{Hilb}_{X/S}$ is the disjoint union of projective S-schemes $\mathbf{Hilb}^F_{X/S}$, which represent the functors $\mathrm{Hilb}^F_{X/S}$.

Automatically, $X \times_S \mathbf{Hilb}_{X/S}$ has a *universal* subscheme W; namely, each T-flat closed subscheme Y of $X \times_S T$ defines a unique S-map $T \to \mathbf{Hilb}_{X/S}$ with $W \times_{\mathbf{Hilb}_{X/S}} T = Y$. Note that the $\mathbf{Hilb}^F_{X/S}$ depend on the choice of embedding of X in some $P(\mathcal{E})$, but $\mathbf{Hilb}_{X/S}$ does not. Thus the Hilbert scheme is a noble replacement for Chow coordinates; the latter only parameterize the cycles on a variety V, and depend on the choice of embedding of V in projective space.

A subscheme R of $X \times_S X$ defines a *flat and projective equivalence relation* if each projection $R \to X$ is flat and projective and if, for each S-scheme T, the subset $h_R(T)$ of $h_X(T) \times h_X(T)$ defines a set-theoretic equivalence relation. Grothendieck found two constructions of a *quotient* X/R in the strongest sense of the term. Namely, first, an S-map $X \to Z$ factors through X/R if and only if the two compositions $R \rightrightarrows X \to Z$ are equal; if so, then $X/R \to Z$ is unique. So by "abstract nonsense," X/R is determined up to unique isomorphism. Second, the quotient map $X \to X/R$ is flat and projective, and the canonical map $R \to X \times_{X/R} X$ is an isomorphism.[38]

Grothendieck's first construction [31, p. 212-15] uses quasi-sections to reduce to the case where X is affine and each $R \to X$ has finite fibers. However, his second construction [31, p. 232-13] is easier and more elegant. It proceeds as follows: R lies in $\mathrm{Hilb}_{X/S}(X)$, so defines a map $\varphi\colon X \to \mathbf{Hilb}_{X/S}$; the graph Γ_φ is a closed subscheme of the universal subscheme W; finally, by Grothendieck's Descent Theory, Γ_φ descends to a closed subscheme of $\mathbf{Hilb}_{X/S}$, which is the desired X/R.

Assume from now on that X is **also** S-flat. Then $\mathrm{Hilb}_{X/S}$ has an important

[38]It follows that $h_{X/R}$ is the fpqc sheaf associated to h_X/h_R in the sense discussed below.

subfunctor $\mathrm{Div}_{X/S}$; namely, for each S-scheme T, let $\mathrm{Div}_{X/S}(T)$ consist of the *effective Cartier divisors* D in $\mathrm{Hilb}_{X/S}(T)$, that is, the flat subschemes D whose ideal $\mathcal{I}_D$ is locally generated by one nonzerodivisor; equivalently, $\mathcal{I}_D$ is invertible as an abstract sheaf. Grothendieck [31, p. 232-10] proved that $\mathrm{Div}_{X/S}$ is representable by an open subscheme $\mathbf{Div}_{X/S}$ of $\mathbf{Hilb}_{X/S}$.

Given an invertible sheaf $\mathcal{L}$ on X, define[39] a subfunctor $\mathrm{LinSys}_{\mathcal{L}/X/S}$ of $\mathrm{Div}_{X/S}$: for each S-scheme T, let $\mathrm{LinSys}_{\mathcal{L}/X/S}(T)$ consist of the D in $\mathrm{Div}_{X/S}(T)$ for which there is an invertible sheaf $\mathcal{M}$ on T such that the inverse $\mathcal{I}_D^{-1}$ is isomorphic to the tensor product on $X \times_S T$ of the pullbacks of $\mathcal{L}$ and $\mathcal{M}$.

Assume in addition from now on the geometric fibers of X/S are *integral*; that is, each affine ring of each geometric fiber is an integral domain. Grothendieck [31, p. 232-11] proved that then there is a coherent sheaf $\mathcal{Q}$ on S, determined up to unique isomorphism, such that $\mathrm{LinSys}_{\mathcal{L}/X/S}$ is representable by $\mathbf{P}(\mathcal{Q})$. Hence, $\mathbf{P}(\mathcal{Q})$ is equal to a closed subscheme of $\mathbf{Div}_{X/S}$. Also, if $H^1(\mathcal{L}|X_s) = 0$ at $s \in S$, then s has a neighborhood on which $\mathcal{Q}$ is *free*, or isomorphic to $\mathcal{O}_S^r$ for some r.

In general, what makes a functor H representable? Say $H = h_X$. Then given any S-scheme T and any open covering $\{T_\lambda\}$ of T, two maps $T \to X$ are equal if their restrictions to each T_λ are equal. Furthermore, maps $\varphi_\lambda \colon T_\lambda \to X$ are the restrictions of a single map $T \to X$ if, for all λ and μ, the restrictions of φ_λ and φ_μ to $T_\lambda \cap T_\mu$ are equal. In other words, as U ranges over the open sets of T, the $H(U)$ form a sheaf. The latter condition does not explicitly involve X. So it makes sense for any H, representable or not. If it is satisfied, H is called a *Zariski sheaf.*

Here's another formulation. Let T' be the disjoint union of the T_λ, and consider the induced map $T' \to T$. Then $T' \times_T T'$ is the disjoint union of the $T_\lambda \cap T_\mu$. So the condition to be a Zariski sheaf just means that the induced sequence of sets

$$H(T) \to H(T') \rightrightarrows H(T' \times_T T') \tag{4}$$

is *exact*; that is, the first map is injective, and its image consists of the elements of $H(T')$ whose two images are equal in $H(T' \times_T T')$.

Grothendieck's Descent Theory yields more. Let $T' \to T$ be an *fpqc* map; namely, it is flat and surjective, and the preimage of any affine open subscheme is a finite union of affine open subschemes. If H is representable, remarkably (4) is still exact; in other words, H is an *fpqc sheaf.* Indeed, the *fpqc Grothendieck topology* may be defined as the refinement of the Zariski topology with the fpqc maps as additional generalized open coverings. In particular, H is an *étale sheaf*, the notion obtained by requiring the maps to be *étale*, that is, flat, unramified and locally of finite type.

The *Picard group* $\mathrm{Pic}(X)$ is the group, under tensor product, of isomorphism classes of invertible sheaves on X. The *absolute Picard functor* Pic_X is defined by $\mathrm{Pic}_X(T) := \mathrm{Pic}(X \times_S T)$. It is never a Zariski sheaf, so never representable.

There is a sequence of ever more promising "Picard functors." First comes the *relative Picard functor* $\mathrm{Pic}_{X/S}$ defined by

$$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X \times_S T)/\,\mathrm{Pic}(T)$$

where $\mathrm{Pic}(T)$ acts via pullback. Following it are its *associated sheaves* in the Zariski, étale and fpqc topologies: $\mathrm{Pic}_{(X/S)(\mathrm{Zar})}$, $\mathrm{Pic}_{(X/S)(\text{ét})}$, and $\mathrm{Pic}_{(X/S)(\mathrm{fpqc})}$. Grothendieck [31, p. 232-03, (1.6)] formed them as direct limits. For example,

$$\mathrm{Pic}_{(X/S)(\mathrm{Zar})}(T) := \varinjlim_{T'} \mathrm{Pic}_{X/S}(T')$$

[39]This subfunctor appears in [31, p. 232-10], but the notation for it comes from [46, p. 93].

where T' ranges over the small category of all open coverings of T.

Recall that S is Noetherian and that X is a flat and projective S-scheme with integral geometric fibers. Grothendieck [31, pp. 232-4–6] proved[40] that *then the three canonical comparison maps are, respectively, injective, injective, and bijective:*

$$\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)(\mathrm{Zar})} \hookrightarrow \mathrm{Pic}_{(X/S)(\text{ét})} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)(\mathrm{fpqc})}\,.$$

Moreover, the first two maps are bijective if $X \to S$ has a section; the middle map is bijective if it just has a section on a Zariski neighborhood of each point of S.

A simple example shows that, in general, we must pass to the étale sheaf. Namely, in the real projective plane, consider the conic $X : u^2+v^2+w^2 = 0$. Let S and T be the spectra of $\mathbb{R}$ and $\mathbb{C}$. Then $T \to S$ is an étale covering. Moreover, $X \times_S T$ is the complex conic with the same equation; so X is isomorphic to the complex projective line. The latter's universal sheaf $\mathcal{O}(1)$ defines an element $\tau \in \mathrm{Pic}_{(X/S)(\text{ét})}(S)$, as the two pullbacks of $\mathcal{O}(1)$ to $X \times_S T \times_S T$ are isomorphic. And τ is not in the image of $\mathrm{Pic}_{(X/S)(\mathrm{Zar})}(S)$, as X has no S-points.

Grothendieck's main existence theorem [31, p. 232-06] says that $\mathrm{Pic}_{(X/S)(\text{ét})}$ *is representable by a scheme* $\mathbf{Pic}_{X/S}$. It is called the *Picard scheme*. Of course, if $\mathrm{Pic}_{(X/S)(\mathrm{Zar})}$ is representable, then it is an étale sheaf, so equal to $\mathrm{Pic}_{(X/S)(\text{ét})}$, and representable by $\mathbf{Pic}_{X/S}$. Similarly, if $\mathrm{Pic}_{X/S}$ is representable, then all four functors are equal, and representable by $\mathbf{Pic}_{X/S}$.

Grothendieck's proof is fairly simple at this point. Here is the idea. Fix an embedding of X in a $\mathbf{P}(\mathcal{E})$. Given any S-scheme T and any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}(\mathcal{E}) \times_S T$, let $\mathcal{F}(n)$ denote the tensor product of $\mathcal{F}$ and the pullback of the nth tensor power of the universal sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Let $\mathcal{I}$ be the ideal of the universal divisor on $X \times_S \mathbf{Div}_{X/S}$, and $\mathcal{I}^{-1}$ its inverse. Form the open subscheme $\mathbf{D}^+$ of $\mathbf{Div}_{X/S}$ on which all the higher direct images of $\mathcal{I}^{-1}(n)$ vanish for all $n \geq 0$.

Set $\mathcal{L} := \mathcal{I}^{-1}|(X \times_S \mathbf{D}^+)$. Then $\mathrm{LinSys}_{\mathcal{L}/X \times_S \mathbf{D}^+/\mathbf{D}^+}$ is representable by $\mathbf{P}(\mathcal{Q})$ where $\mathcal{Q}$ is a coherent sheaf on $\mathbf{D}^+$. Moreover, $\mathcal{Q}$ is *locally free*; that is, each point of $\mathbf{D}^+$ has neighborhood on which $\mathcal{Q}$ is free. Thus $\mathbf{P}(\mathcal{Q})$ is flat over $\mathbf{D}^+$.

Set $R := \mathbf{P}(\mathcal{Q})$. Then R is a closed subscheme of $\mathbf{D}^+ \times_S \mathbf{D}^+$. Moreover, for each S-scheme T, the subset $h_R(T)$ of $h_{\mathbf{D}^+}(T) \times h_{\mathbf{D}^+}(T)$ consists of the pairs of $D,\, D' \in \mathbf{D}^+(T)$ for which there is an invertible sheaf $\mathcal{M}$ on T such that the ideal $\mathcal{I}_D$ is isomorphic to the tensor product of $\mathcal{I}_{D'}$ and the pullback of $\mathcal{M}$. Thus $h_R(T)$ is a set-theoretic equivalence relation.

Although $\mathbf{D}^+$ isn't Noetherian, nevertheless it is the disjoint union of Noetherian subschemes, as $\mathbf{Hilb}_{X/S}$ is the disjoint union of the projective S-schemes $\mathbf{Hilb}^F_{X/S}$, and R decomposes compatibly. Consequently, the quotient $\mathbf{D}^+/R$ exists.

For each $m \geq 0$, let P_m be the fpqc subsheaf of $\mathrm{Pic}_{(X/S)(\mathrm{fpqc})}$ associated to the subfunctor of $\mathrm{Pic}_{X/S}$ whose value at T consists of the classes of invertible sheaves $\mathcal{L}$ on $X \times_S T$ for which all the higher direct images of $\mathcal{L}(n)$ vanish on T for all $n \geq m$, but the direct image doesn't vanish. Then P_0 is representable by $\mathbf{D}^+/R$.

Tensor product with the pullback of $\mathcal{O}_X(1)$ defines an isomorphism $P_{m+1} \xrightarrow{\sim} P_m$ for all $m \geq 0$. So the P_m are representable by (isomorphic) schemes U_m. Each inclusion $P_m \hookrightarrow P_{m+1}$ is representable by an open embedding $U_m \hookrightarrow U_{m+1}$. Finally, $\mathrm{Pic}_{(X/S)(\mathrm{fpqc})}$ is the "union" of the P_m; so is representable by the union, or rather

[40] In [31], Grothendieck did not consider $\mathrm{Pic}_{(X/S)(\text{ét})}$. However, his methods apply to it, and show that it is equal to $\mathrm{Pic}_{(X/S)(\mathrm{fpqc})}$, because in the case at hand, there exists an *étale quasi-section*, an S-map $S' \to X$ for which the structure map $S' \to S$ is étale.

direct limit, of the U_m. Thus Grothendieck proved his main existence theorem.

Commenting on his proof, Grothendieck [31, p. 232-13] noted that "the approach is basically the one followed by Matsusaka" (so by Igusa, Chow and Castelnuovo). Further, he [31, p. 232-14] noted that "the proof appeals neither to the preliminary construction of the Jacobians of curves ... nor to the theory of Abelian varieties, and thus differs in an essential way from the 'traditional' treatments in Lang's book [40] and Chevalley's paper [19].... That the construction of the Picard scheme ought to precede and not follow the theory of Abelian varieties is clear a priori from the fact that ... Rosenlicht's 'generalized Jacobians' are not Abelian varieties." More of Grothendieck's advances are highlighted in the next section.

5. The Picard Scheme

His [Grothendieck's] feeling was that "those people" made too strict assumptions and tried to prove too little.
Jacob Murre, quoted in [64, p. 2]

Above, Murre describes Grothendieck's feeling about the theory of the Picard variety: it was hampered by its developers' narrow vision. This section explains how Grothendieck's broader vision led to clarifying and settling a number of issues. Primarily, we focus on the two major issues: Behavior in a Family and Completeness of the Characteristic System. In addition, we consider some other issues mentioned earlier, especially Poincaré divisors and the Albanese variety. And we consider some other ways that other mathematicians enhanced Grothendieck's theory between 1962 and 1973, especially ways of generalizing his main existence theorem. For more discussion of those issues and some discussion of a lot of other issues of the same sort, please see [31], [25], [9], and [8].

As noted in the Introduction, Grothendieck explained the behavior of the Picard schemes of the members of a family as compatibility with base change. More precisely, if the functor $\mathrm{Pic}_{(X/S)(\mathrm{fpqc})}$ is representable by an S-scheme $\mathbf{Pic}_{X/S}$, then for any S-scheme S', the functor $\mathrm{Pic}_{(X\times S'/S')(\mathrm{fpqc})}$ is representable by the S'-scheme $\mathbf{Pic}_{X/S}\times_S S'$. In particular, if S' is the spectrum of the residue field k_s of $s\in S$, then the Picard scheme of the fiber X_s of X/S is just the fiber of $\mathbf{Pic}_{X/S}$.

Compatibility holds for this reason. For any S'-scheme T, the equation

$$\mathrm{Pic}_{X\times S'/S'}(T)=\mathrm{Pic}_{X/S}(T)$$

results from the definitions, because $(X\times_S S')\times_{S'} T = X\times_S T$. So the equation

$$\mathrm{Pic}_{(X\times S'/S')(\mathrm{fpqc})}(T)=\mathrm{Pic}_{(X/S)(\mathrm{fpqc})}(T)$$

follows, because a map of S'-schemes $T'\to T$ is a covering if and only if it is a covering when viewed as a map of S-schemes. However, the equation

$$\big(\mathbf{Pic}_{X/S}\times_S S'\big)(T)=\mathbf{Pic}_{X/S}(T)$$

holds, because the structure map $T\to S'$ is fixed. Since the right sides of the last two equations are equal, so are their left sides, as desired.

Until otherwise said near the end of the section, assume that S is the spectrum of an algebraically closed field k and that X is an integral and projective S-scheme. As is common, write "k-scheme," $\mathbf{Div}_{X/k}$, etc. for "S-scheme," $\mathbf{Div}_{X/S}$, etc.

In order to complete the discussion in Section 2 of the algebraic proofs of the Theorem of Completeness of the Characteristic System and of the Fundamental

Theorem of Irregular Surfaces, we must discuss what's called[41] the *Zariski tangent space* $\mathbb{T}_z(Z)$ to a k-scheme Z at a *rational point* z, a point whose residue field k_z is k. Let $\mathfrak{m}$ be the maximal ideal, and set $\mathbb{T}_z(Z) := \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,\, k)$.

Then $\mathbb{T}_z(Z)$ can be viewed as the vector space of k-derivations $\delta\colon \mathcal{O}_{Z,z} \to k$; indeed, $\delta(\mathfrak{m}^2) = 0$, and so δ corresponds to a linear map $\mathfrak{m}/\mathfrak{m}^2 \to k$. Let k_ε be the ring of *dual numbers,* the ring obtained by adjoining an element ε with $\varepsilon^2 = 0$. Then δ corresponds to the map of k-algebras $u\colon \mathcal{O}_{Z,z} \to k_\varepsilon$ given by $u(a) := \overline{a} + \delta(a)\varepsilon$ where $\overline{a} \in k$ is the residue of a. Finally, let S_ε be the spectrum of k_ε; it is the *free tangent vector*. Then u corresponds to a k-map $S_\varepsilon \to Z$, whose image is supported at z. Denote the set of such k-maps by $h_Z(S_\varepsilon)_z$. Then in summary $\mathbb{T}_z(Z) = h_Z(S_\varepsilon)_z$.

Often, if Z represents a given functor H, that is $h_Z = H$, then we can work out a useful description of $h_Z(S_\varepsilon)_z$ by viewing it as the subset of $H(S_\varepsilon)$ of elements whose image in $H(S)$ is z. For example, say $Z = \mathbf{Hilb}_{X/k}$ and $z \in Z$ represents $Y \subset X$. Then $h_Z(S_\varepsilon)_z$ is the set of S_ε-flat closed subschemes of $X \times_k S_\varepsilon$ whose fiber over S is Y. Say the ideal of Y is $\mathcal{I}_Y$. Working it out, Grothendieck [31, pp. 221-21–23] found $h_Z(S_\varepsilon)_z = H^0(\mathcal{N}_Y)$ where $\mathcal{N}_Y := \mathcal{H}om(\mathcal{I}_Y/\mathcal{I}_Y^2,\, \mathcal{O}_Y)$. If Y is a Cartier divisor D, then $\mathcal{N}_D$ is invertible on D, and

$$\mathbb{T}_z(\mathbf{Div}_{X/k}) = H^0(\mathcal{N}_D). \tag{5}$$

Let Λ parameterize a system of divisors on X including D, and say $\lambda \in \Lambda$ represents D; in other words, there is a map $\Lambda \to \mathbf{Div}_{X/k}$ carrying λ to z. It induces a map of vector spaces $\theta\colon \mathbb{T}_\lambda(\Lambda) \to \mathbb{T}_z(\mathbf{Div}_{X/k})$. If D is integral, then, owing to (5), the image of θ defines a linear system on D, the storied *characteristic linear system*. When is it complete? More generally, for any D, when is θ surjective?

Each version of the Theorem of Completeness provides conditions guaranteeing the existence of a Λ that is smooth at λ and for which θ is surjective. But then $\mathbf{Div}_{X/k}$ is smooth at z, owing to a simple general observation [37, p. 305]. Thus the conditions in question just guarantee that $\Lambda := \mathbf{Div}_{X/k}$ is smooth at z.

Some conditions are necessary. Indeed, in 1943 Severi's student, Guido Zappa, found a smooth complex surface X such that $\mathbf{Div}_{X/k}$ has an isolated point z with $\dim \mathbb{T}_z(\mathbf{Div}_{X/k}) = 1$; so $\mathbf{Div}_{X/k}$ has nilpotents; for details, please see [46, pp. 155–156] or [25, p. 285]. Commenting, Grothendieck [31, pp. 221-24] wrote that this example "shows in a particularly striking way how varieties with nilpotents are needed to understand the phenomena of the most classical theory of surfaces."

Grothendieck then gave an enlightening proof of Kodaira's 1956 version [38] of Completeness. As to Kodaira's own proof, Kodaira and Spencer [39, p. 477] said that it's "based essentially on the theory of harmonic differential forms" [so not algebraic]; it's "indirect and does not reveal the real nature of the theorem."

Kodaira proved that, *if X and D are smooth and complex, and*[42] *if $h^1(\mathcal{I}_D^{-1}) = 0$, then $\mathbf{Div}_{X/k}$ is smooth at z, where $\mathcal{I}_D$ is the ideal of D*. For example, $h^1(\mathcal{I}_D^{-1}) = 0$ by Serre's computation if D is a hypersurface section of large degree. In 1904, Enriques studied the case that X is a surface and D is *regular* and *nonspecial*, meaning[43] $h^1(\mathcal{I}_D^{-1}) = h^2(\mathcal{I}_D^{-1})$ and $h^2(\mathcal{I}_D^{-1}) = 0$. Thus then Completeness holds.

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[41] In 1947, Zariski [77] introduced and studied $\mathfrak{m}/\mathfrak{m}^2$ for a variety in any characteristic, and called it the "local vector space." In Weil's math review of Zarisk's paper, Weil wrote: "the dual vector-space . . . seems to deserve to be called the 'tangent vector-space'."

[42] It is now common to let $\mathcal{O}_X(D)$ stand for $\mathcal{I}_D^{-1}$, but that practice is not followed here.

[43] At first, "regular" alone was used to mean $h^1(\mathcal{I}_D^{-1}) = 0$ and $h^2(\mathcal{I}_D^{-1}) = 0$.

Grothendieck proved Kodaira's theorem for any X and D as follows. Let $\mathcal{I}$ denote the ideal of the universal divisor on $X \times_k \mathbf{Div}_{X/k}$. Then $\mathcal{I}^{-1}$ is invertible. So it defines a map $\boldsymbol{\alpha}_{X/k}\colon \mathbf{Div}_{X/k} \to \mathbf{Pic}_{X/k}$, called the *Abel map*. Assume $H^1(\mathcal{I}_D^{-1}) = 0$. Then $\boldsymbol{\alpha}_{X/k}$ is smooth at z; see below. Hence $\mathbf{Div}_{X/k}$ is smooth at z if and only if $\mathbf{Pic}_{X/k}$ is smooth at $\boldsymbol{\alpha}_{X/k}(z)$, or equivalently by translation, everywhere. In characteristic zero, $\mathbf{Pic}_{X/k}$ is smooth by Cartier's Theorem [46, p. 167]. Thus $\mathbf{Div}_{X/k}$ is smooth at z in characteristic zero.

In positive characteristic, $\mathbf{Pic}_{X/k}$ can be nonreduced everywhere even if X is a smooth surface; see below. If so and $h^1(\mathcal{I}_D^{-1}) = 0$, then $\mathbf{Div}_{X/k}$ is nonreduced at z. Thus Completeness fails, even if D is a general hypersurface section of large degree.

Since k is algebraically closed, X has a rational point, so that $X \to S$ has a section. Set $P := \mathbf{Pic}_{X/k}$. Then $\mathrm{Pic}_{X/k}$ is representable by P. So $X \times_k P$ carries an invertible sheaf $\mathcal{P}$, called a *Poincaré sheaf*, whose class modulo $\mathrm{Pic}(P)$ is universal. In particular, there is an invertible sheaf $\mathcal{M}$ on P such that $\mathcal{I}^{-1}$ is isomorphic to the tensor product on $X \times_k \mathbf{Div}_{X/k}$ of the pullbacks of $\mathcal{P}$ and $\mathcal{M}$. Then $\mathrm{LinSys}_{\mathcal{P}/X\times P/P}$ is representable, on the one hand, by $\mathbf{Div}_{X/k}$ regarded as a P-scheme via $\boldsymbol{\alpha}_{X/k}$, and on the other, by $\mathbf{P}(\mathcal{Q})$ for some coherent sheaf $\mathcal{Q}$ on P. So $\mathbf{Div}_{X/k}$ and $\mathbf{P}(\mathcal{Q})$ are canonically isomorphic P-schemes. If $H^1(\mathcal{I}_D^{-1}) = 0$, then $\mathcal{Q}$ is free at $\boldsymbol{\alpha}_{X/k}(z)$, and so $\boldsymbol{\alpha}_{X/k}$ is smooth at z, as desired.

Grothendieck [31, pp. 236-16] asserted $\mathbb{T}_0(\mathbf{Pic}_{X/k}) = H^1(\mathcal{O}_X)$; for proofs, please see [46, pp. 163–164] and [25, pp. 281–282]. So $\dim \mathbf{Pic}_{X/k} \le h^1(\mathcal{O}_X)$, with equality if and only if $\mathbf{Pic}_{X/k}$ is smooth. That result is part of Grothendieck's contribution to the proof of the Fundamental Theorem of Irregular Surfaces. In the examples of Igusa and Serre recalled in Section 2, we have $\dim \mathbf{Pic}_{X/k} < h^1(\mathcal{O}_X)$; hence, $\mathbf{Pic}_{X/k}$ is not smooth at 0, so nonreduced everywhere.

Grothendieck noted smoothness holds if $H^2(\mathcal{O}_X) = 0$, owing to the Infinitesimal Criterion for Smoothness and a well-known computation; for details, please see [25, pp. 285–286]. For example, if X is a curve, then $\mathbf{Pic}_{X/k}$ is smooth, so of dimension g where $g := h^1(O_X)$. In positive characteristic, Mumford [46, pp. 193–198] proved $\mathbf{Pic}_{X/k}$ is smooth if and only if Serre's Bockstein operations [66, p. 505] all vanish.

Grothendieck did not consider the other versions of Completeness, but his work does provide a basis for proving them algebraically. First consider Severi's 1921 version (3) on p. 9. Given an invertible sheaf $\mathcal{L}$ on X, set $e(\mathcal{L}) := \chi(\mathcal{L}) - 1 - h^2(\mathcal{L})$. Call $\mathcal{L}$ *arithmetically effective* if $e(\mathcal{L}) \ge 0$. Vary $\mathcal{L}$. Then $\chi(\mathcal{L})$ is locally constant, and $h^2(\mathcal{L})$ is upper semi-continuous. So $e(\mathcal{L})$ is lower semi-continuous.

Hence there is an open subscheme of $\mathbf{Pic}_{X/k}$, say $\mathbf{P}_{\mathrm{ae}}$, that parameterizes the arithmetically effective $\mathcal{L}$ on X. Set $\mathbf{D}_{\mathrm{ae}} := \boldsymbol{\alpha}_{X/k}^{-1}\mathbf{P}_{\mathrm{ae}}$. Assume $\dim X = 2$. Then $\mathbf{D}_{\mathrm{ae}}$ surjects onto $\mathbf{P}_{\mathrm{ae}}$, since over a point representing an $\mathcal{L}$, the fiber has dimension $e(\mathcal{L}) + h^1(\mathcal{L})$, which is nonnegative. In these terms, a refined Version (3) says that, *if* $\mathbf{Pic}_{X/k}$ *is smooth too, then* $\mathbf{D}_{\mathrm{ae}}$ *is smooth on a dense open subset* U.

Since $h^0(\mathcal{L})$ is upper semi-continuous in $\mathcal{L}$, there is an open subscheme $V \subset \mathbf{P}_{\mathrm{ae}}$ that parameterizes the $\mathcal{L}$ where $h^0(\mathcal{L})$ has a local minimum. Set $U := \boldsymbol{\alpha}_{X/k}^{-1}V$. Recall that $\mathbf{Pic}_{X/k}$ carries a coherent sheaf $\mathcal{Q}$ such that $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/k}$. Then V is precisely the set of points of $\mathbf{P}_{\mathrm{ae}}$ at which the rank of $\mathcal{Q}$ has a local minimum. Suppose $\mathbf{Pic}_{X/k}$ is smooth. Then the restriction $\mathcal{Q}|V$ is locally free. Hence $U \to V$ is smooth. So U is smooth. Thus we have refined and proved Severi's version (3).

In 1944, Severi discovered another condition on D for Completeness to hold if $\mathbf{Pic}_{X/k}$ is smooth. The condition requires D to be *semi-regular*; namely, in the standard long exact sequence of cohomology

$$(6)\qquad \begin{aligned} &0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{I}_D^{-1}) \longrightarrow H^0(\mathcal{N}_D) \\ &\xrightarrow{\partial^0} H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{I}_D^{-1}) \xrightarrow{u} H^1(\mathcal{N}_D) \xrightarrow{\partial^1} H^2(\mathcal{O}_X), \end{aligned}$$

the map u is 0, or equivalently ∂^1 is injective. In particular, D is semi-regular if either $H^1(\mathcal{I}_D^{-1}) = 0$ or $H^1(\mathcal{N}_D) = 0$.

Severi worked with an integral D on a smooth surface X, and he formulated the condition in its dual form: the restriction $H^0(\Omega_X^2) \to H^0(\Omega_X^2|D)$ is surjective; in other words, the canonical system on X cuts out a complete system on D. In 1959, Kodaira and Spencer [39, p. 481] reformulated Severi's condition as $u = 0$ in any dimension. Then they proved that, *in the complex analytic case, if X and D are smooth and if $u = 0$, then $\mathbf{Div}_{X/k}$ is smooth at z.*

Grothendieck did not consider semi-regularity per se, but he [31, pp. 221-23] did observe that $H^1(\mathcal{N}_D)$ houses the obstruction to deforming D in X. Thus[44] *if $H^1(\mathcal{N}_D) = 0$, then $\mathbf{Div}_{X/k}$ is smooth at z in any characteristic* whether X and D are smooth or not. For example, if X is a curve, then $\mathbf{Div}_{X/k}$ is smooth everywhere; however, $\boldsymbol{\alpha}_{X/k}$ is not smooth at z if $\deg D < g$ where $g := h^1(\mathcal{O}_X)$, since $\dim \mathbf{Div}_{X/k} = \deg D$ by (5) and $\dim \mathbf{Pic}_{X/k} = g$ as noted above.

Mumford [46, pp. 157–159] explicitly computed the obstruction to deforming D, as well as its image under ∂^1. He proved that this image vanishes in characteristic 0 using an exponential. Therefore, if ∂^1 is injective, then $\mathbf{Div}_{X/k}$ is smooth at z. Cartier's Theorem is not involved, but recovered. Thus in 1966 Mumford gave the first algebraic proof that semi-regularity yields Completeness in characteristic 0.

In 1973, I [37] gave another algebraic proof, yielding a more refined statement: *assume $\mathbf{Pic}_{X/k}$ is smooth; then $\mathbf{Div}_{X/k}$ is smooth at z of dimension ρ where*

$$\rho := h^1(\mathcal{O}_X) - 1 + h^0(\mathcal{I}_D^{-1}) - h^1(\mathcal{I}_D^{-1})$$

if and only if D is semi-regular. My proof[45] does not use obstruction theory, but a short formal analysis, essentially due to George Kempf, of the scheme $\mathbf{P}(\mathcal{Q})$ above.

In passing, set $R := \dim_z \mathbf{Div}_{X/k}$ and note that (5) yields $R \le h^0(\mathcal{N}_D)$, with equality if and only if $\mathbf{Div}_{X/k}$ is smooth at z. Also, (6) yields $\rho \le h^0(\mathcal{N}_D)$, with equality if and only if D is semi-regular. Thus if $\mathbf{Div}_{X/k}$ is smooth at z, then D is semi-regular if and only if $R = \rho$.

Generalizing more of Section 2, set $\delta := \dim \operatorname{Coker}(\partial^0)$ and $q := h^1(\mathcal{O}_X)$. Then (6) yields $\delta \le q$, with equality if $H^1(\mathcal{I}_D^{-1}) = 0$. As $H^1(\mathcal{I}_D^{-1}) = 0$ if D is a hypersurface section of large degree, we have generalized Castelnuovo's result (1). Next, set $r := h^0(\mathcal{I}_D^{-1}) - 1$. Then (6) yields $h^0(\mathcal{N}_D) = r + \delta$. Hence $R \le r + \delta$, with equality if and only if $\mathbf{Div}_{X/k}$ is smooth at z. Thus we have generalized Severi's result (2). Finally, *if X is a surface, $\mathbf{Pic}_{X/k}$ is smooth and z lies in the open subset $U \subset \mathbf{D}_{\mathrm{ae}}$, then $\mathbf{Div}_{X/k}$ is smooth at z, and so* $R = r + \delta$, just as Severi discovered.

Grothendieck [31, pp. 2-12] proved the following basic properties of the connected component of 0 in $\mathbf{Pic}_{X/k}$, denoted $\mathbf{Pic}^0_{X/k}$. It is open and closed. It is irreducible.

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[44] Perhaps not surprisingly, the condition $H^1(\mathcal{N}_D) = 0$ is related to the flaw in the constructions of a good algebraic system made in 1904 by Enriques and in 1905 by Severi. In 1934, Zariski [78, p. 100] noted that both constructions rely on a certain assumption and that Severi's 1921 "criticism is to the effect that the available algebro-geometric proof of this assumption fails if" $H^1(\mathcal{N}_D) \ne 0$.

[45] The proof works over any Noetherian S, and yields this more general result: *let D be a divisor on an (integral) geometric fiber of X/S, and assume $\mathbf{Pic}_{X/S}$ is smooth; then $\mathbf{Div}_{X/S}$ is smooth of relative dimension ρ at the point representing D if and only if D is semi-regular.*

Forming it commutes with base change. It is *quasi-projective*; that is, it is an open subscheme of a projective k-scheme. Moreover, it is projective if X is normal. Of course, the *Picard variety* of X is the set of points of $\mathbf{Pic}^0_{X/k}$ with coordinates in a given universal domain. If X is a curve, then $\mathbf{Pic}^0_{X/k}$ is its *generalized Jacobian*.

Define a *Poincaré divisor* to be a Cartier divisor Δ on $X \times P$, where P is some connected component of $\mathbf{Pic}_{X/k}$, such that Δ yields a section $P \to \mathbf{Div}_{X/k}$. Such Δ abound, as is shown next, generalizing Mattuck's work mentioned on p. 15.

Given any connected component P' of $\mathbf{Pic}_{X/k}$, notice it's a translate of $\mathbf{Pic}^0_{X/k}$, so quasi-projective. Hence there's an n such that, for any invertible sheaf $\mathcal{L}$ on X represented by a point of P', we have $h^0(\mathcal{L}(n)) > \dim \mathbf{Pic}_{X/k}$ and $h^1(\mathcal{L}(n)) = 0$ where $\mathcal{L}(n)$ is the nth twist by $\mathcal{O}_X(1)$. Let P be the translate of P' defined by $\mathcal{O}_X(n)$. Since P is quasi-projective, it has a universal sheaf $\mathcal{O}_P(1)$.

Recall $\mathbf{Pic}_{X/k}$ carries a coherent sheaf $\mathcal{Q}$ such that $\mathbf{P}(\mathcal{Q}) = \mathbf{Div}_{X/k}$. Notice the restriction $\mathcal{Q}|P$ is locally free of rank $h^0(\mathcal{L}(n))$, so of rank more than $\dim P$. Take m so that $Hom(\mathcal{Q}|P, \mathcal{O}_P)(m)$ is generated by its global sections, so by finitely many. Then a general linear combination of the latter vanishes nowhere on P by a well-known lemma, [3, p. 426] or [46, p. 148], due to Serre. So there's a surjection $\mathcal{Q}|P \twoheadrightarrow \mathcal{O}_P(m)$. It defines a section $P \to \mathbf{P}(\mathcal{Q})$, and so a Poincaré divisor Δ.

Suppose also that X is a curve. Set $g := h^1(\mathcal{O}_X)$ and recall $g = \dim \mathbf{Pic}_{X/k}$. Take P to be any connected component of $\mathbf{Pic}_{X/k}$ that parameterizes invertible sheaves $\mathcal{L}$ on X of $\deg \mathcal{L} > 2g - 1$. Then $h^0(\mathcal{L}) > g$ and $h^1(\mathcal{L}) = 0$. So similarly there is a section $P \to \mathbf{P}(\mathcal{Q})$, and so a Poincaré divisor Δ.

Here's an introduction[46] to the scheme-theoretic theory of the Albanese variety. Assume X is normal. Then $\mathbf{Pic}^0_{X/k}$ is projective. Let P denote its *reduction*, namely, the subscheme defined by the nilradical of the structure sheaf of $\mathbf{Pic}^0_{X/k}$. It too is a *group scheme*; that is, its T-points form a group for all T. So P is smooth. Call any such connected smooth projective group scheme an *Abelian variety*.

If X is an Abelian variety, then $\mathbf{Pic}^0_{X/k}$ is already reduced. Mumford [45] gave a proof on pp. 117–118, which he attributed to Grothendieck on p. 115. Then $\mathbf{Pic}^0_{X/k}$ is denoted by $\widehat{X}$ or X^*, and called the *dual Abelian variety*.

In general, let Y be another integral and projective k-scheme, and fix rational points $x \in X$ and $y \in Y$. Then a k-map $f\colon Y \to P$ with $f(y) = 0$ is defined by an invertible sheaf $\mathcal{L}$ on $X \times_k Y$ whose restriction to $X \times_k y$ is $\mathcal{O}_X$. At first, $\mathcal{L}$ is only determined modulo $\mathrm{Pic}(Y)$, but normalize $\mathcal{L}$ as follows: restrict it to $x \times_k Y$, pull the restriction back to $X \times_k Y$ via the projection, and replace $\mathcal{L}$ by its tensor product with the inverse of the pullback.

Let Q be the reduction of $\mathbf{Pic}_{Y/k}$. By symmetry, $\mathcal{L}$ corresponds to a k-map $g\colon X \to Q$ with $g(x) = 0$. Plainly, this correspondence $f \longleftrightarrow g$ is *functorial*: given a similar triple (Z, z, R) and a k-map $h\colon Z \to Y$ with $g(z) = y$, the composition $fh\colon Z \to P$ corresponds to the composition $h^* \circ g\colon X \to R$ where $h^*\colon P \to R$ is the k-map induced by pullback of invertible sheaves, which is a group homomorphism.

For a moment, take $Y := P$ and $y := 0$ and $f := 1_p$. Since P is an Abelian

[46]Doubtless, Grothendieck had something similar in mind when he [31, p. 232-14] wrote, "the theory of Abelian varieties, and more generally of Abelian schemes, becomes much simpler once we have a general theory of the Picard scheme at our disposal. In particular, the theory of duality for Abelian schemes, and notably results like Cartier's, thus become nearly formal (cf. for example the forthcoming notes to the Mumford–Tate seminar at Harvard in the spring term of 1962)." However, it seems that nothing like the present introduction appears in Mumford's personal notes to the seminar, or has already appeared in print.

variety, Q is its dual P^*. Set $A := P^* = Q$ and $a := g$. Then A is called the *Albanese variety* of X, and there is a canonical k-map $a\colon X \to A$.

The map $a\colon X \to A$ is the *universal example* of a map $g\colon X \to Q$ where Q is the reduction of $\mathbf{Pic}_{Y/k}$ for some integral and projective k-scheme Y; that is, any such g factors uniquely through a. Here's why. Say g corresponds to $h\colon Y \to P$. Then by functoriality, $1_P \circ h$ corresponds to $h^* \circ a$.

By definition, A^* is the Albanese of P. Moreover, the canonical map $p\colon P \to A^*$ is an isomorphism, since by "abstract nonsense," a universal example is determined up to unique isomorphism, and $1_P\colon P \to P$ is trivially another universal example. In fact, p^{-1} is just the map a^* induced by the canonical map $a\colon X \to A$, because by functoriality, $1_A \circ a$ corresponds to $a^* \circ p$. Thus A and P are dual to each other.

Suppose X is an Abelian variety; take $x := 0$. Mumford [47, p. 125] constructed X^* as a quotient of X by a finite subgroup; so X and X^* are isogenous. Mumford [47, Cor., p. 43, 132] proved that $a : X \to X^{**}$ is an isomorphism of groups and of schemes. Hence, for any X, the map $a\colon X \to A$ is the *universal example* of a map $X \to B$ where B is an Abelian variety, because $B = Q$ if $Y = B^*$.

Suppose finally that X is a smooth curve of genus $g > 0$. Then X is a component of $\mathbf{Div}_{X/k}$. So the Abel map restricts to a k-map $X \to \mathbf{Pic}_{X/k}$. Its image lies in the connected component parameterizing the sheaves $\mathcal{L}$ of degree 1. Fix an $\mathcal{L}$. Translating by $\mathcal{L}^{-1}$ yields a map $\alpha\colon X \to P$. It is an embedding by general principles, since its fibers are finite and $X = \mathbf{P}(\mathcal{Q})$ for some coherent sheaf $\mathcal{Q}$ on P. It is proved (in a more general form) in [24, Thm. 2.1, p. 595] that $\alpha^*\colon P^* \to P$ is an isomorphism, which is independent of the choice of $\mathcal{L}$.

To end this article, let's consider some important ways in which Grothendieck and others generalized the existence theorem culminating Section 4. First, let k be an arbitrary field. On pp. 232-15–17 in [31], Grothendieck outlined a construction of $\mathbf{Pic}_{X/k}$ for any projective k-scheme X. He used that earlier theorem plus a method of *relative representability*, by which $\mathbf{Pic}_{X/k}$ is constructed from $\mathbf{Pic}_{X'/k}$ for a suitable surjective k-map $X' \twoheadrightarrow X$. The method employs two main tools: nonflat descent and Oort dévissage. The former refers to descent along maps not required to be flat; however, key objects are flat. The second tool was introduced by Oort in [58] to construct $\mathbf{Pic}_{X/k}$ from $\mathbf{Pic}_{X'/k}$ where X' is the reduction of X.

On p. 232-17 in [31], Grothendieck, in effect, made two conjectures: first, $\mathbf{Pic}_{X/k}$ exists for any proper k-scheme X; second, given any surjective k-map between proper k-schemes, the induced map on Picard schemes is affine.

The first conjecture was proved in 1964 by Murre [53], who thanked Grothendieck for help. However, instead of using relative representability, Murre identified seven conditions that are necessary and sufficient for the representability of a functor from schemes over a field to Abelian groups. Then he checked the seven for $\mathrm{Pic}_{(X/k)(\mathrm{fpqc})}$.

From now on, assume S is Noetherian and X is a flat and proper S-scheme.

Murre [53, p. 5] said that Grothendieck too proved the first conjecture. In 1965, Murre [54] sketched Grothendieck's proof of the following key intermediate result: *let $\mathcal{F}$ be a coherent sheaf on X, and $S_{\mathcal{F}}$ the functor of all S-schemes T such that the pullback $\mathcal{F}_T$ is T-flat; then $S_{\mathcal{F}}$ is representable by an unramified S-scheme of finite type.* The proof involves identifying and checking eight conditions that are necessary and sufficient for representability by a scheme of the desired sort.

In 1966, Raynaud [8, Exp. XII] gave Grothendieck's proof of another key intermediate result: *assume S is integral and let $X' \twoheadrightarrow X$ be a surjective map of proper*

S-schemes; then there's a nonempty open subscheme $V \subset S$ such that $\mathbf{Pic}_{X' \times V/V}$ and $\mathbf{Pic}_{X \times V/V}$ exist, and the induced map between them is quasi-affine. The proof does indeed involve suitably general versions of nonflat descent and Oort dévissage. As corollaries, that result yields Grothendieck's two conjectures.

If the geometric fibers of X/S are not all integral, then $\mathbf{Pic}_{X/S}$ need not exist. On p. 236-01 in [31], Grothendieck described an example of Mumford's; one geometric fiber is integral, but another is a pair of conjugate lines. On the other hand, on p. viii in [46], Mumford asserted this theorem: *Assume X/S is projective, and its geometric fibers are reduced and connected; assume the irreducible components of its ordinary fibers are geometrically irreducible; then $\mathbf{Pic}_{X/S}$ exists.* He said the proof is like the one on pp. 133–149, involving his theory of independent 0-cycles.

On p. 236-01, Grothendieck attributed a slightly different theorem to Mumford, and referred to the Mumford–Tate seminar. Mumford's seminar notes contain a precise statement of the theorem and a rough sketch of the proof. However, he crossed out the hypothesis that the geometric fibers are connected, and made the weaker assumption that the ordinary fibers are connected.

On p. 236-13, Grothendieck wrote that "it is not ruled out that $\mathbf{Pic}_{X/S}$ exists" whenever[47] the direct image of $\mathcal{O}_{X \times T}$ is $\mathcal{O}_T$ for any T. "At least, this statement is proved for analytic spaces when X/S is also projective." Mumford's example shows the statement is false for schemes. Michael Artin's work shows it holds for *algebraic spaces*, which he introduced in 1968 in [1]. They are formed by gluing together schemes along open subsets that are isomorphic locally in the étale topology. Over $\mathbb{C}$, these open sets are locally analytically isomorphic; so a separated algebraic space is a kind of complex analytic space.

In 1969, Artin [2], inspired by Grothendieck and Murre, found five conditions on a functor that are necessary and sufficient for it to be representable by a well-behaved algebraic space. A key new ingredient is Artin's Approximation Theorem; it facilitates the passage from formal power series to polynomials. By checking that the conditions hold if the direct image of $\mathcal{O}_{X \times T}$ is always $\mathcal{O}_T$, Artin [2, Thm. 7.3, p. 67] proved[48] $\mathbf{Pic}_{X/S}$ exists as an algebraic space, a magnificent achievement. Also, he [2, Lem. 4.2, p. 43] proved that, if S is the spectrum of a field, then $\mathbf{Pic}_{X/S}$ is a scheme. Thus he obtained a third proof of Grothendieck's first conjecture.

As S and X are schemes, so are the fibers of X/S. Hence their Picard schemes exist. Furthermore, if the direct image of $\mathcal{O}_{X \times T}$ is always $\mathcal{O}_T$, then these Picard schemes form a family; its total space $\mathbf{Pic}_{X/S}$ is an algebraic space, but need not be a scheme. Thus Artin proved the definitive statement explaining the behavior of the Picard schemes of the members of a family.[49]

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[47] This condition holds if the geometric fibers of X/S are integral by [30, Prp. (7.8.6), p. 74]. For more about its significance when S is the spectrum of a discrete valuation ring, please see [63].

[48] In fact, he proved a more general theorem, in which S and X are algebraic spaces. That theorem and Grothendieck's theorem in Section 4 are the two main representability theorems for the Picard functor. Grothendieck used projective methods. Artin's work has a very different flavor. Moreover, it yields a major improvement of Murre's representability theorem stated above, and it yields the representability of the Hilbert functor and related functors in algebraic spaces.

[49] As the Picard varieties in the family are the points of the component $\mathbf{Pic}^0_{X_s/k_s}$ for $s \in S$, their behavior is explained by an open subspace $\mathbf{Pic}^0_{X/S}$ of $\mathbf{Pic}_{X/S}$ whose fibers are the $\mathbf{Pic}^0_{X_s/k_s}$. Such a $\mathbf{Pic}^0_{X/S}$ is observed in [9, p. 233] to exist when $\mathbf{Pic}_{X/S}$ is S-smooth along the 0-section.

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