Plural Predication

by

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Abstract

My thesis consists of three self-contained but interconnected papers. In the first one, 'Word and Objects', I assume that it is possible to quantify over absolutely everything, and show that certain English sentences containing collective predicates resist paraphrase in first-order languages and even in first-order languages enriched with plural quantifiers. To capture such sentences I develop a language containing *plural predicates*.

The introduction of plural predicates leads to an extension of Quine's criterion of ontological commitment. I argue that theories containing plural predicates can have plural ontological commitments in addition to singular ones. In this sense, I argue that the subject-matter of ontology is richer than one might have thought.

Plural predicates turn out to be tremendously fruitful. For example, they provide us with natural formalizations for English plural definite descriptions and generalized quantifiers. They also allow us to state important set theoretic propositions, and give a formal semantics for second-order languages. Such a formal semantics is developed in the second paper, 'Toward a Theory of Second-Order Consequence', which is a collaboration with Gabriel Uzquiano.

In the third paper, 'Frege's Unofficial Arithmetic', I consider an application of plural predicates to the philosophy of mathematics. By developing a suggestion of the later Frege, I show that any arithmetical predicate can be transformed into a plural predicate in such a way that the arithmetical predicate is true of the number of the Fs just in case the plural predicate is true of the Fs themselves.

The transformation is important both because it can be put to use by nominalists about arithmetic and neo-Fregeans, and because it provides the foundations for an account of applied arithmetic.

Thesis Supervisor: Vann McGee
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A mis abuelos

Contents

Chapter 1

Word and Objects

The aim of this essay is to show that the subject-matter of ontology is richer than one might have thought. Our route will be indirect. We will argue that there are circumstances under which standard first-order regimentation is unacceptable, and that more appropriate varieties of regimentation lead to unexpected kinds of ontological commitment.

1.1 Introduction

Quine has taught us that ontological inquiry—inquiry as to what there is—can be separated into two distinct tasks.[1] On the one hand, there is the problem of determining the ontological commitments of a given theory; on the other, the problem of deciding what theories to accept. The objects whose existence we have reason to believe in are then the ontological commitments of the theories we have reason to accept.

Regarding the latter of these two tasks, Quine holds that our overall scientific theory is to be accepted on the basis of its ability to fit and arrange raw experience, together with considerations of simplicity. I will have nothing to say about such issues here.

[1] See Quine (1948).

As for the former task, Quine maintains that a first-order theory is committed to the existence of an object satisfying a certain predicate if and only if some object satisfying that predicate must be admitted among the values of the theory's variables in order for the theory to be true. Quine's criterion is extremely attractive, but it applies only to theories that are couched in first-order languages. Offhand this is not a serious constraint, because most of our theories have straightforward first-order regimentations. But here we shall see that there is a special kind of tension between regimenting our discourse in a first-order language and allowing our quantifiers to range over absolutely everything. We will proceed on the assumption that absolutely unrestricted quantification is possible, and show that an important class of English sentences resists first-order regimentation. This will lead us to develop alternate languages of regimentation, languages containing *plural* quantifiers and predicates. It will also lead us to set forth a more inclusive criterion of ontological commitment.

The possibility of quantifying over everything is readily challenged by *anti-realists*, who believe that what there is in some way depends upon our conception of the world. It is open for them to argue that, whenever our conception of the world specifies a totality we might quantifier over, it necessarily yields objects that lie outside this totality. But if one is a *realist*, and believes that what there is does not depend on our conception of the world, then the idea that unrestricted quantification is possible has considerable force. For what is there, on the realist picture, to stop our quantifiers from encompassing everything there is? In order to deny the possibility of unrestricted quantification, a realist would have to defend a thesis about our *linguistic abilities* to the effect that we are not capable of talking about everything at once.[2]

Insofar as the possibility of quantifying over everything is a tenet of realism, our discussion will reveal realist constraints on the adequacy of first-order regimentation and, therefore, realist constraints on ontological inquiry.

[2] See Glanzberg (unpublished). For an excellent discussion on quantifying over everything see Cartwright (1994).

1.2 Regimentation

I use 'regimentation' in Quine's sense.[3] Ordinary discourse is plagued with unclarities and ambiguities. Usually they are harmless. But under special circumstances—such as the practice of scientists and philosophers—they may interfere with our goals. When we regiment, we paraphrase the sentences of our original discourse into sentences with fewer unclarities or ambiguities. There is no presupposition of synonymy, or of sameness of 'logical form'. It is only required that, to our satisfaction, whatever we hoped to achieve by way of our original sentences can be achieved closely enough by way of their paraphrase. In many cases, this means that truth-conditions must be preserved, but we needn't assume this in general. No questions about our original sentences are settled with regimentation: the old discourse is surrendered in favor of the new.

Regimentation is important for the purposes of ontological inquiry because our theories are not always expressed in ways that allow us to assess their ontological commitments. But we may be able to regiment them using languages for which some criterion of ontological commitment is available. In doing so no light is shed on the commitments of the original theories, but as long as we are willing to surrender them in favor of their regimentations, we will be in a position to determine what our ontological commitments are.

A language of regimentation needn't be a fragment of natural language. It is sufficient that it be well understood. For instance, we may attempt to eliminate ambiguity by adding subscripts to the pronouns of some suitable fragment of English. The resulting language is not itself a fragment of English, but it will presumably be well understood by any English speaker. Formalisms such as first-order logic can also be used for regimenting. We may regard '$\exists x_i$', '$x_i = x_j$', '$\neg$' and '$\wedge$' as abbreviating the expressions 'there is an object$_i$ such that', 'it$_i$ is identical to it$_j$', 'it is not the case that ... ' and 'it is both the case that ... and ... '(respectively). Not all of the latter are part of English, but they will be well understood by any English speaker

[3]See Quine (1960), section 33.

familiar with the relevant subscripting conventions.

First-order languages naturally suggest themselves as languages of regimentation. Besides enabling the application of Quine's criterion of ontological commitment, they provide us with grammatical simplicity, notational perspicuity and enormous expressive power. So much so that there is some pull towards thinking that we should *always* choose a first-order language as our language of regimentation. Quine and others seem to have adopted just this view.[4] Nonetheless, the choice of a language of regimentation should be made on the basis of its ability to further our goals. And, depending on the circumstances, first-order languages may not turn out to be the best candidates for the job.

The adequacy of regimentation is constrained only by our needs. Not so for the adequacy of a criterion of ontological commitment. Once we have settled upon a language of regimentation and accepted a theory couched in that language, ontological commitments are forced upon us. They cannot be chosen on the basis of their ability to further our goals.

1.3 Critics

Suppose we agree that our language of regimentation is to be first-order. How might we regiment the Geach-Kaplan sentence?

(GK) Some critics admire only one another.

One option is to introduce the first-order predicate '$P(\ldots)$' as an abbreviation for the English '... is such that some critics admire only one another' and go on to paraphrase (GK) as '$\forall x P(x)$'. But normally we expect a paraphrase to preserve some of the logical connections of the original sentence. For instance, we might want the existence of critics to be derivable from our paraphrase of (GK). And, of course, '$\exists x \text{CRITIC}(x)$' isn't derivable from '$\forall x P(x)$'.

[4]See, for instance, Quine (1948) and Davidson (1967).

Is it possible to find a first-order paraphrase of (GK) which preserves all the logical connections we might be interested in? Kaplan answered part of this question in the negative.[5] He proved that there is no first-order sentence that is true in precisely the same models as the following second-order sentence, which is reminiscent of (GK) when 'Axy' is read 'x admires y' and all quantifiers are restricted to critics:

(1) $\exists X(\exists x Xx \wedge \forall x \forall y[Xx \wedge Axy \rightarrow x \neq y \wedge Xy])$).

But now suppose we were to agree that a sentence φ can only be a regimentation of (GK) if it meets the following condition:

(*) φ is true in a model just in case (1) is.

Then, by Kaplan's proof, there is no way of regimenting (GK) in a first-order language.

However, there is usually no reason to impose conditions as stringent as (*) on our regimentations. We might, for example, paraphrase (GK) as a first-order version of the following:

(2) There is a (non-empty) *set* of critics z such that, for any x and y, if $x \in z$ and x admires y, then $x \neq y$ and $y \in z$.

A first-order version of (2) does not meet condition (*) because there are models that verify (1) with domains containing no sets. But it is possible for a sentence to serve as a paraphrase even if it doesn't preserve all the original's logical properties. All that is required is that, to our satisfaction, whatever we hoped to achieve by way of the original can be achieved closely enough by way of the paraphrase. Thus, if not all of (GK)'s logical properties are important for our present purposes, there needn't be an obstacle for paraphrasing (GK) as (2). Moreover, solid intuitions about the logical properties of (GK) run out well before forcing anything like (*) upon us, and some of the intuitions we do have easily fade away in the presence of an otherwise attractive paraphrase.

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[5]See Boolos (1984), p. 57.

1.4 Collective Predicates

There is a well-known distinction between collective and distributive readings of English predicates. For instance, 'The children carried the piano' can be taken to mean either that the children carried the piano *together*, or that *each* of the children is such that she carried the piano. In the former case, the predicate '... carried the piano' is understood collectively; in the latter, it is understood distributively. In general, we shall say that an occurrence of the predicate '... (are) P' is understood distributively in 'they (are) P' just in case 'they (are) P' can be paraphrased as 'each of them (is) P'. Otherwise, we shall say that '... (are) P' is understood collectively.[6]

Throughout this paper it will be convenient to eliminate the sort of ambiguity that afflicts sentences containing predicates which are open to both collective and distributive readings. We shall do so by stipulating that predicates are to be understood according to their *collective* readings whenever there is any risk of ambiguity. Also, we shall sometimes speak of collective and distributive *predicates* instead of collective and distributive *readings* of predicates.

Attention to collective predicates sheds light on the reason why finding a first-order paraphrase for the Geach-Kaplan sentence is not entirely straightforward. Consider the following sentence, which is presumably an uncontroversial paraphrase of (GK),

(3) There are some critics such that, for any x and y, if x is one of them and x admires y, then $x \neq y$ and y is one of them.

Note that each of the three occurrences of the predicate '... is one of ...' in (3) must be understood collectively with respect to its second argument-place. But there is no *direct* way of paraphrasing collective English predicates into first-order languages because first-order predicates do not admit plural arguments. In order to find a first-order paraphrase for (3) some deviousness is required. One possibility is to replace

[6] This explication can be made to encompass sentences such as 'the Fs (are) P' or 'some Fs (are) P' by pretending that they are abbreviations for 'the Fs are such that they (are) P' and 'some Fs are such that they (are) P' (respectively). Also, by extending our characterization in the obvious way, we can speak of an n-adic predicate being understood distributively or collectively with respect to each of its argument-places.

plural quantification over critics with singular quantification over *sets* of critics, and replace '...is one of ...' with the set-theoretic '... ϵ ...'; we then get:

(4) There is a *set* of critics z such that z has at least one member and, for any x and y, if $x \epsilon z$ and x admires y, then $x \neq y$ and $y \epsilon z$;

which can easily be paraphrased into a first-order language (in fact, (4) is (2) from section 1.3).

More generally, we could replace plural quantification over critics with singular quantification over objects that serve as *surrogates* for critics: sets of critics, or classes of critics, or 'plural objects' composed of critics, or events involving critics. To ensure firstorderizability, we shall require of our surrogates that they admit of a 'membership' relation with the feature that s is a surrogate for the Fs just in case the Fs are all and only the 'members' of s.[7]

Thus, (3) might be paraphrased as:

(5) There is a *surrogate* z with only critics as members such that z has at least one member and, for any x and y, if x is a member of z and x admires y, then $x \neq y$ and y is a member of z.

As one would expect, (5) is equivalent to (4) when our surrogates of choice are sets.

In many cases, the surrogate-method (as we shall call it) is an extremely effective way of producing first-order paraphrases for sentences containing '...is one of ...'. But George Boolos has shown that it cannot always be made to work.[8] He noted that although the following is obviously true,

(6) There are some sets such that, for any y, y is one of them just in case $y \notin y$;

[7]This is an extremely weak requirement. It is equivalent to a *uniqueness* condition according to which, if s is a surrogate for the Fs and the Gs are not the Fs, then s is not a surrogate for the Gs. Yet it is not without consequences. For instance, we cannot have it that s is a surrogate for the Fs just in case s is the *mereological sum* of the Fs, since the Fs and Gs might not be the same objects but share a mereological sum. Requirements on surrogates are liberalized in section 1.7.

[8]See Boolos (1985b), Boolos (1984) and Boolos (1985a).

it gets assigned a necessarily false paraphrase by the surrogate method when surrogates are taken to be sets:

(7) There is a set x such that, for every y, $y \in x$ just in case y is a set such that $y \notin y$.

The problem generalizes. For, given any non-trivial[9] choice of surrogates σ, the following is true:

(8) There are some σ-surrogates such that, for any y, y is one of them just in case y is a σ-surrogate which is not a member of itself.

But it gets assigned a necessarily false paraphrase by the surrogate method when our choice of surrogates is σ:

(9) There is a σ-surrogate x such that, for every y, y is a member of x just in case y is a σ-surrogate which is not a member of itself.

Thus, for any non-trivial choice of surrogates, we can find a sentence that cannot be paraphrased by appeal to those surrogates.

Friends of the surrogate method have a way of avoiding this conclusion. They can claim that the quantifier 'for every y' in (9) doesn't really range over *all* surrogates, and that x is outside this range. They might add that the quantifiers in (9) are systematically ambiguous, or that their range is indeterminate. But this move is blocked if we are allowed to assume that the domain of (8) consists unequivocally of everything there is.

1.5 Bernays's Principle

The conclusions of the preceding section are not as strong as one might have hoped. We saw that, for any non-trivial choice of surrogates, there is a sentence that cannot be

[9]On the assumption that there are at least four individuals, say that a choice of surrogates σ is *non-trivial* only if every two individuals have a surrogate

paraphrased by appeal to those surrogates. But this is compatible with the view that every sentence can be paraphrased by appeal to some choice of surrogates. Moreover, a sentence might have a first-order paraphrase even if the surrogate-method fails. Thus, (8) can be paraphrased as 'there exists a nonselfmembered σ-surrogate', which is certainly firstorderizable.

In this section we shall make a stronger case for the view that there are sentences resisting first-order paraphrase.

Cantor's Theorem is well-known. It states that there is no function from a set onto its power-set. Less well-known is a certain kind of extension of this result. Intuitively, the thought is that there is no function from the objects there are onto the 'pluralities' of objects there are. This goes beyond Cantor's Theorem because there is no set containing all objects.

So far, however, our proposition has not been properly expressed. To begin with, we have said nothing about what a 'plurality' is supposed to be. Moreover, functions are normally taken to be sets, so it is unclear just what one might mean by 'function' in the present context. In order to express our proposition properly, we need a piece of notation. If the Gs are some ordered pairs, we shall say that the Gs *map* x onto the Fs if, for every y, $\langle x, y \rangle$ is one of the Gs just in case y is one of the Fs. Our proposition is then this:

(BP) Given any ordered pairs, there are some things onto which the ordered pairs map nothing.

A proof of (BP) is provided in the appendix. As far as I know, Paul Bernays was the first to set-forth this kind of result,[10] so I shall refer to it as *Bernays's Principle.*

I submit that, when our domain consists of everything there is, Bernays's Principle has no first-order paraphrase. Unfortunately, I have no proof that this is so. In fact, I haven't the slightest idea what such a proof would look like. Note, for example, that a Kaplan-style nonfirstorderizability result is not what we are looking for. The lesson of section 1.3 is that a sentence may have an acceptable first-order paraphrase even if

[10]See Bernays (1942), pp. 137-8.

it is nonfirstorderizable in Kaplan's sense.[11] The best I can do to forward my claim is show that, when our domain consists of absolutely everything, the surrogate-method does not succeed in firstorderizing Bernays's Principle, no matter what surrogates we chose.

If we attempt to paraphrase Bernays's Principle in accordance with the surrogate-method what we get is the following:

(10) $\forall\alpha\exists\beta\forall x\neg\forall y(\langle x,y\rangle$ is a member of $\alpha \leftrightarrow y$ is a member of $\beta)$;

where Greek letters range over the surrogates of our choice, σ-surrogates say. Let an ordered pair be one of the Ss just in case its first component is a σ-surrogate and its second component is a member of that σ-surrogate. On the assumption that our domain consists of everything there is, it is an instance of Bernays's Principle that there are some things onto which the Ss map nothing. If there is to be a corresponding instance of (10), we must assume that there is a σ-surrogate for the Ss—call it ρ.[12] What we get is then:

(11) $\forall x\neg\forall y(\langle x,y\rangle$ is a member of $\rho \leftrightarrow y$ is a member of $\gamma)$,

for some surrogate γ. But, if ρ exists, the following is a consequence of our definition of the Ss:

(12) $\forall y(\langle\gamma,y\rangle$ is a member of $\rho \leftrightarrow y$ is a member of $\gamma)$.

[11] In fact, a Kaplan-style result isn't available for Bernays's Principle. In order to prove such a result, we would have to select some second-order sentence χ and insist that a formula φ can only be a first-order paraphrase of Bernays's Principle if it meets the following condition:

(*) φ is true in a model just in case χ is.

To conclude the proof we would then have to show that there is no first-order formula true in precisely the same models as χ. But presumably (*) is only plausible if we chose χ to be something along the following lines,

(1) $\neg\exists X\forall Y\exists u\forall v(X(u,v) \leftrightarrow Yv)$;

Unfortunately, (1) is a *theorem* of second-order logic, so there are plenty of first-order formulas which are true in precisely the same models as (1).

[12] In fact, this assumption may fail, as it does when we choose sets as our surrogates.

And, of course, (11) and (12) are in contradiction.

When our domain consists of absolutely everything, the surrogate-method does not succeed in firstorderizing Bernays's Principle, no matter what surrogates we chose. Perhaps it has some other first-order paraphrase. If so, I have been unable to find it. As far as I can tell, Bernays's Principle cannot be regimented in a first-order language.

1.6 PFO Languages

An alternative language of regimentation suggests itself. Let a *plural first-order* language (PFO for short) be the result of enriching a first-order language with plural quantifiers and variables, and a dyadic predicate '$\prec$', which takes a singular variable in first argument-place and a plural variable in its second. Our plural variables are 'xx_1', 'xx_2', etc. '$\exists xx_i$' is to be interpreted as 'there are some objects$_i$ such that', and '$x_i \prec xx_j$' is to be interpreted as 'it$_i$ is one of them$_j$'. Thus, for instance, '$\exists x_i \exists xx_j (x_i \prec xx_j)$' is to be read

(13) there is an object$_i$ and some objects$_j$ such that it$_i$ is one of them$_j$.

(A formal characterization of PFO languages is provided in the appendix.)

PFO languages are an excellent means for regimenting English sentences containing the collective predicate '...is one of ...'. For instance, Bernays's Principle can be paraphrased as:

(14) $\forall xx \exists yy \forall u \exists v \neg(\langle u, v\rangle \prec xx \leftrightarrow v \prec yy)$.

The Geach-Kaplan sentence also has a natural PFO paraphrase:

(15) $\exists xx \forall y \forall z[(y \prec xx \wedge \textsc{Admires}(y, z)) \rightarrow (y \neq z \wedge z \prec xx)]$;

(where our domain of discourse consists of critics).

Our interpretation of PFO languages makes use of a convention that was introduced in Boolos (1984) and is now standard in the literature. There is some pull towards thinking that the English 'there are some Fs such that so-and-so' is only true if there are at least *two* Fs such that so-and-so. But it is by no means evident that this should be so. One could argue, for example, that an utterance of 'there are some Fs such that so-and-so' is *pragmatically inappropriate*, but true, when it is known that there is only one F such that so-and-so, in much the same way that the utterance of a disjunction can be pragmatically inappropriate, but true, when it is known of one of the disjuncts that it is true. Boolos's convention is to sidestep this controversy altogether and stipulate that 'there are some Fs such that so-and-so' is to be true just in case there are *one or more* Fs such that so-and-so. I will assume that Boolos's convention is in place throughout the remainder of this essay.

1.7 Beyond PFO

The question now arises whether PFO languages can be used to regiment English sentences involving collective predicates *other* than '... is one of ...'. In this section I will make a case for the view that, under certain circumstances, they cannot. Consider the following sentences,

(16) The seashells *are scattered*;

(17) The Peano Axioms *imply Fermat's Last Theorem*;

(18) The mechanics *repaired the car*;

(19) The musicians *will perform the symphony*;

(20) The philosophers *mingled with* the mathematicians;

(21) The soldiers *are between* the students *and* the administrators;

(22) The seashells are *mixed together with* the rocks.

How might one paraphrase (16)–(22) into a PFO language? A natural thing to do is paraphrase (16) as:

(16′) The set of seashells is scattered;

which, in turn, has an straightforward first-order paraphrase and, hence, a straightforward PFO paraphrase:

(23) $\exists x(\textsc{Set}(x) \wedge \forall y(y \in x \leftrightarrow \textsc{Seashell}(y)) \wedge \textsc{Scattered}(x))$.

More generally, one might claim that there is an object which serves as a *surrogate* for the seashells: a set of seashells, or a class of seashells, or a 'plural object' composed of seashells, or an event involving seashells. When discussing first-order paraphrases in section 1.4, we required that surrogates admit of a 'membership' relation. But now we can be more generous. All we shall require is that 'x is a surrogate for the Fs' have a PFO paraphrase. This allows us to treat the mereological sum of the Fs as a surrogate for the Fs.[13]

With the machinery of surrogates at hand, one might hold that talk of the seashells being scattered is not to be paraphrased by predicating something of the seashells themselves. Rather, it is to be paraphrased by predicating something of their surrogate. In the case of (16) we get:

(16″) The seashells' surrogate is scattered.

The surrogate-method faces an important difficulty. Suppose that our domain consists of absolutely everything. Then it follows from Bernays's Principle that, no matter what surrogates we chose, at least one of the following must be the case:

(α) There are some objects with no surrogate.

(β) There are some objects—the Fs—and some objects—the Gs—such that the Fs are not the Gs but the Fs have the same surrogate as the Gs.

A couple of examples should make clear why this causes trouble for friends of the surrogate-method.

[13] On the more restrictive conception of surrogates it cannot be done. See footnote 7.

First, suppose we chose sets as our surrogates. That is, whenever we have some things, we will let their surrogate be the *set* containing precisely those things. Then set-extensionality guarantees that (β) will never be the case, so (α) follows from Bernays's Principle: there must be some things with no surrogate. In particular, it turns out that the cardinals have no surrogate (since there is no set of all cardinals). This is a problem because, although the following sentence is intuitively true,

(24) The cardinals are scattered among the ordinals;

the surrogate-method would paraphrase it as something necessarily false:

(25) There is a *set* with precisely the cardinals as members and it is scattered among the ordinals.

Second, suppose we decide to let mereological sums be our surrogates. That is, whenever we have some things, we let their surrogate be their mereological sum. Since any objects whatsoever have a mereological sum, (α) cannot be the case, so (β) follows from Bernays's principle: there are some objects and some other objects such that the former have the same mereological sum as the latter. That this is a problem can be illustrated as follows. Suppose that there are a few scattered plies of sand on a table. Then it is true of the piles of sand, but false of the grains of sand which make up the piles, that they are scattered. But, if we take mereological sums to be our surrogates, this fact cannot be captured by the surrogate-method, since the mereological sum of the piles is precisely the same object as the mereological sum of the grains of sand.

Given a choice of surrogates σ, let us say that the Fs are a *problem case* for σ if either the Fs have no σ-surrogate, or there are some things which are distinct from the Fs but have the same σ-surrogate. As it turns out, problem cases are far from scarce: it is easy to verify that there are 'more' problem cases than non-problem cases. So, no matter what surrogates we choose, we are at risk of coming across sentences—such as our examples—that ge˙ the wrong truth-value when paraphrased in accordance with the surrogate-method. Of course, we may sometimes be able chose our surrogates in such a way that problem cases turn out not to be important in the relevant context. But it is not possible in general. Unless a non-trivial Reflection Principle is assumed to hold, the formal semantics described in section 1.14.2 is an example of a case in which it cannot be done.[14]

A particularly sophisticated version of the surrogate-method, developed by James Higginbotham and Barry Schein,[15] deserves special attention. Their method of paraphrase uses *events* as surrogates. For example, they paraphrase 'Those boys built a boat' as:

(26) There is an event E such that (*a*) E is a boat-building and (*b*) for every x, x is an *agent* of E just in case x is one of those boys.

This proposal works nicely for many special cases. But it cannot be made to work generally, on pain of generating a version of Russell's Paradox. The problem emerges if we predicate something collectively of the events that are not 'agents' of themselves. For instance,

(27) The events that are not agents of themselves have little in common.

Higginbotham and Schein would have us paraphrase (27) as:

(28) There is an event E such that (*a*) E is a having-little-in-common,[16] and (*b*) for every x, x is an agent of E just in case x is not an agent of itself.

But clause (*b*) of (28) implies a contradiction.[17]

It might be replied that it is somehow illegitimate to speak of events that are 'agents' of other events. Unfortunately, this restriction also undermines the generality of Higginbotham and Schein's proposal. For it would be unable to account for sentences like 'Events of this magnitude are very rare'.

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[14]For further discussion, see chapter 2 of this thesis.

[15]See Higginbotham (1998), Higginbotham and Schein (1989) and Schein (1993).

[16]A having-little-in-common is an awkward event indeed. This speaks against Higginbotham and Schein's account, not against my argument.

[17]Byeong-Uk Yi makes a similar point in footnote 34 of Yi (1999).

We have seen that the surrogate-view faces important difficulties. Other ways of paraphrasing English sentences with collective predicates into PFO languages might do better. I can only report that I have been able to find any.

1.8 Plural Predicates

There is a natural way of extending PFO languages to accommodate collective English predicates such as those considered in the preceding section. Let PFO$^+$ languages be the result of extending PFO languages with *plural predicates*, that is, predicates taking plural variables in some of their argument places. Plural predicates are interpreted in terms of collective English predicates. Thus, '**Scattered**(xx_i)' might be interpreted as 'they$_i$ are scattered', and '**Surrounded**(x_i, xx_j)' might be interpreted as 'it$_i$ is surrounded by them$_j$'. (A formal characterization of PFO$^+$ languages is provided in the appendix. Byeong-Uk Yi examines languages of this kind at some length in Yi (unpublished).)

PFO$^+$ languages have enormous expressive power. In this section we shall see that they provide us with a natural way of regimenting English sentences containing plural definite descriptions, such as (16) – (22) from section 1.7.

There is a familiar procedure for regimenting sentences with *singular* definite descriptions. As an example, consider 'The sailor carried John home'. If we follow Russell's advice, this sentence can be formalized in a first-order language as:

(29) $\exists x[\forall y(\text{S}\textsc{ailor}(y) \leftrightarrow x = y) \wedge \text{C}\textsc{arried}\text{J}(x)]$.[18]

The following definitional equivalence is frequently introduced:

$$\psi(\iota_x[\varphi(x)]) \;\equiv_{def}\; \exists x[\forall v(\varphi(v) \leftrightarrow x = v) \;\wedge\; \psi(x)].^{19}$$

Thus, (29) is equivalent to:

[18]Here and elsewhere, I assume that the context makes clear what the variables should be taken to range over and how the non-logical predicates should be interpreted.

[19]In this and other definitions I ignore differences of scope for the sake of simplicity.

(30) $\textsc{CarriedJ}(\iota_x[\textsc{Salior}(x)])$.

Something similar can be done for *plural* definite descriptions. Here we shall focus on the simplest case: sentences of the form 'the Fs are G', where 'F' is a count noun. Richard Sharvy has set forth a general account of plural definite descriptions which can be naturally framed in a PFO$^+$ language.[20]

Consider 'The sailors carried John home'. When '... carried John home' is understood collectively, it might be paraphrased as:

(31) $\exists yy[\forall x(x \prec yy \leftrightarrow \textsc{Sailor}(x)) \wedge \mathbf{CarriedJ}(yy)]$;

where '**CarriedJ**(...)' abbreviates the collective reading of '... carried John home'. This suggests the following notation:

$$\psi(\pi_x[\varphi(x)]) \equiv_{df} \exists yy[\forall x(x \prec yy \leftrightarrow \varphi(x)) \wedge \psi(yy)].$$

Thus, (31) is equivalent to

(32) $\mathbf{CarriedJ}(\pi_x[\textsc{Sailor}(x)])$.

When '... carried John home' is understood distributively in 'The sailors carried John home', a slightly different paraphrase suggests itself:

(33) $\forall y(y \prec \pi_x[\textsc{Sailor}(x)]) \to \textsc{CarriedJ}(y))$;

where '$\textsc{CarriedJ}(\ldots)$' is the singular counterpart of '**CarriedJ**(...)'.[21] It is easy to verify that (33) amounts to nothing more than:

(34) $\forall x(\textsc{Sailor}(x) \to \textsc{CarriedJ}(x))$.

Nonetheless, we may introduce the following piece of notation:

(35) $\mathbf{CarriedJ}^D(xx) \equiv_{df} \forall y(y \prec xx \to \textsc{CarriedJ}(y))$;

[20] See Sharvy (1980).

[21] See section 1.8. When singular and plural PFO$^+$ predicates are spelled the same, I shall assume that they are counterparts.

(and similarly for other predicates). Thus, (33) may be rewritten as:

(36) $\mathbf{CarriedJ}^D(\pi_x[\textsc{Sailor}(x)])$.

The machinery we have set forth allows us to give PFO$^+$ paraphrases for a substantial class of English sentences containing plural definite descriptions. For instance, 'The seashells are scattered' can be paraphrased as:

(37) $\mathbf{Scattered}(\pi_x[\textsc{Seashell}(x)])$;

and 'The fugitives crossed the border' can be paraphrased as:

(38) $\mathbf{CrossedBorder}^D(\pi_x[\textsc{Fugitive}(x)])$.

1.9 Generalized Quantifiers

PFO$^+$ languages also provide us with the resources for regimenting English sentences with generalized quantifiers. Consider the following examples:

(a) *Almost half of* the monkeys became infected;

(b) *Many of* the bills are counterfeit;

(c) *Few of* the students have any patience left.

They can be paraphrased as:

(a') The monkeys who became infected are *almost half of* the monkeys;

(b') The counterfeit bills are *many of* the bills;

(c') The students who have any patience left are *few of* the students.

Accordingly, (a)–(c) can be formalized in a suitable PFO$^+$ language as:

(a'') $\mathbf{AlmostHalfOf}(\pi_x[\textsc{Monkey}(x) \wedge \textsc{Inf}(x)], \pi_x[\textsc{Monkey}(x)])$;

(b'') $\mathbf{ManyOf}(\pi_x[\text{BILL}(x) \wedge \text{COUNTERFEIT}(x)], \pi_x[\text{BILL}(x)])$;

(c'') $\mathbf{FewOf}(\pi_x[\text{STUDENT}(x) \wedge \text{PATIENCE}(x)], \pi_x[\text{STUDENT}(x)])$;

where all the non-logical predicates in (a'')–(c'') are to be understood in the obvious way; in particular, '$\mathbf{AlmostHalfOf}(xx_i, xx_j)$' is interpreted as 'they$_i$ are almost half of them$_j$', '$\mathbf{ManyOf}(xx_i, xx_j)$' is interpreted as 'they$_i$ are many of them$_j$', and '$\mathbf{FewOf}(xx_i, xx_j)$' is interpreted as 'they$_i$ are few of them$_j$'.

The possibility of formalizing generalized quantifiers in terms of plural predicates is to be expected. In Barwise and Cooper's influential article on the subject, a determiner such as 'Many of' is interpreted as a binary relation taking a set S as its first argument and one of S's subsets as its second.[22] Thus, 'Many of the Fs are G' is true just in case 'Many of' holds between the set of Fs and its subset consisting of the F-and-Gs. But Barwise and Cooper's assumption that the Fs form a *set* is uncalled for. It would be better to think of 'Many of' as a two-place plural predicate, and say that 'Many of the Fs are G' is true just in case 'Many of' holds between the Fs and the F-and-Gs. That is what the present proposal amounts to.[23]

So far we have only considered quantifiers of the form 'Q of the Fs', whose definite description ensures the existence of Fs. What about quantifiers of the form 'Q Fs'? These come in two different flavors, depending on whether the absence of Fs makes 'Q Fs are G' true or false. If the latter, 'Q Fs are G' may be paraphrased as 'Q of the Fs are G'. (For instance, 'Many Fs are G' can be paraphrased as 'Many of the Fs are G'.) If the former, 'Q Fs are G' may be paraphrased as 'Either there are no Fs, or Q of the Fs are G'. (For instance, 'All Fs are G' can be paraphrased as 'either

[22]See Barwise and Cooper (1981). In fact, Barwise and Cooper say that determiners are to be interpreted as functions that take a set S as an argument and deliver a subset of the power-set of S, but the two formulations are equivalent.

[23]I am ignoring a complication. Note that 'Not all of the Fs are G' is true when none of the Fs are G. On Barwise and Cooper's proposal, this is accounted for by letting the determiner 'not all', understood as a two-place singular predicate, hold between the set of Fs and the empty set. But a similar move is not available when 'not all' is formalized as a plural predicate, because plural predicates do not admit of 'empty' arguments. Instead, one might formalize 'Not all of the Fs are Gs' as a PFO$^+$ version of 'Either there are Fs but no F-and-Gs, or the F-and-Gs are not all of the Fs'. A limit case is 'None of the Fs are G', which is true *only* when no Fs are G. It can still be formalized as 'either there are Fs but no F-and-Gs, or the F-and-Gs are none of the Fs', but the second clause is redundant.

there are no Fs, or all of the Fs are G'.) In any case, quantifiers of the form 'Q Fs' reduce to quantifiers of the form 'Q of the Fs'. They may therefore be ignored with no loss of generality.

If our account is along the right lines, then every formula of a first-order language with generalized quantifiers can be transformed into an equivalent PFO$^+$ formula. The relevant transformation is stated formally in the appendix. For illustration, consider 'Many of the bills are counterfeit'. In a first-order language with generalized quantifiers it may be formalized as:

$$[\textsc{ManyOf}\ x : \textsc{Bill}(x)]\ (\textsc{Counterfeit}(x)),$$

for which our transformation yields

$$\mathbf{ManyOf}(\pi_x[\textsc{Bill}(x) \wedge \textsc{Counterfeit}(x)], \pi_x[\textsc{Bill}(x)]).$$

Our transformation also works for sentences with iterated generalized quantifiers. Consider, for instance, 'Most of the candidates alienate many of the voters':

$$[\textsc{MostOf}\ x : \textsc{Candidate}(x)]\ [\textsc{ManyOf}\ y : \textsc{Voter}(y)]\ (\textsc{Alienate}(x, y));$$

it is transformed into:

$$\mathbf{MostOf}(\pi_x[\textsc{Candidate}(x) \wedge \mathcal{A}(x)], \pi_x[\textsc{Candidate}(x)]),$$

where $\mathcal{A}(x)$ is:

$$\mathbf{ManyOf}(\pi_y[\textsc{Voter}(y) \wedge \textsc{Alienate}(x, y)], \pi_y[\textsc{Voter}(y)]).$$

1.10 Truth and Satisfaction

In this section we will set forth definitions of truth and satisfaction for PFO$^+$ languages.

The most natural way to proceed is to expand upon the standard definition of satisfaction for first-order languages. As before, we let variable assignments associate an object in our domain with each singular variable, but we also let variable assignments associate *multiple* objects in our domain with each plural variable. Variable assignments are therefore treated as *relations* rather than functions. If σ is an assignment and 'vv' a plural variable, 'σ('vv', x)' and 'σ('vv', y)' may be true even if $x \neq y$ (though, of course, if 'v' is a singular variable, σ behaves like a total function, so that 'σ('v', x)' holds for precisely one x).

Relations are standardly taken to be *sets* of ordered pairs. But this will not do for our purposes. Problems arise when our domain of discourse consists of too many objects to form a set. Since we want the sentence '$\exists xx \forall y(y \prec xx)$' to turn out to be true, we need a variable assignment that associates every object in our domain with the plural variable 'xx'. But such an assignment would contain an ordered pair $\langle$'xx', $y\rangle$ for every object y in our domain, and would therefore have too many members to be a set.

Fortunately, Boolos has found a way out of this difficulty.[24] Instead of taking a variable assignment to be a certain *set* of ordered pairs, we shall consider the ordered-pairs themselves, and have *them* play the role of assigning values to our variables. Thus, we shall say of some ordered pairs that they form a plural variable assignment just in case a certain plural predicate '**Assignment**(xx)' is true of them. And, instead of treating the satisfaction relation as a first-order predicate 'Sat(φ, σ)', which holds between a formula and a set of ordered-pairs, we shall take satisfaction to be a two-place plural predicate, '**Sat**(φ, xx)', which holds between a formula and the ordered pairs forming a plural variable assignment. Once Boolos's modification is in place, the definitions of truth and satisfaction proceed along familiar lines (see appendix for details).

Our formal semantics yields an important *stability* result: the satisfaction predicate for a PFO$^+$ language can always be defined within another PFO$^+$ language. First-order languages are also stable in this sense, but PFO languages are not. In general, the satisfaction predicate for a PFO language can only be defined within a PFO$^+$ language.[25] This suggests that if the realm of first-order regimentation is to

[24]See Boolos (1985a).

[25]Matters would be otherwise if there existed a set w with the feature that an arbitrary set of

be left behind, PFO^+ languages are a more natural stopping point than their PFO counterparts.

1.11 Ontological Commitment

The goal of ontology may be regarded as more or less ambitious. The less ambitious goal is to discover, of each predicate, whether there are objects it is true of. The more ambitious goal requires us to start by dividing our predicates into those that pick out 'basic' ontological properties and those that do not. It might be argued, for instance, that '...is an abstract object' and '...is an electron' pick out basic ontological properties, but that '...is owned by my uncle Hector' and '...is such that all whales are mammals' do not. We must then discover which of the elite predicates are instantiated and, if possible, go on to give an account of how they are related to their non-elite counterparts.

In what follows we will conceive of ontology in the more modest sense. When we speak of a theory's being ontologically committed to objects satisfying a certain predicate, there will be no presupposition that the predicate expresses a basic ontological property.

The formal semantics we set forth in the preceding section allows us to introduce a useful piece of notation. Let us say that x is the possible value of a singular PFO^+ variable v just in case x is the object which the ordered-pairs forming some plural variable assignment associate with v. Similarly, we shall say that the Fs are the possible values of a plural PFO^+ variable vv just in case the Fs are the objects which the ordered-pairs forming some plural variable assignment associate with vv.

PFO formulas is satisfied by the ordered-pairs of a given plural variable assignment just in case it is satisfied by the restriction of that assignment to w. This would allow us to regard variable assignments as *sets* of ordered pairs, and to frame our definition of satisfaction in a first-order language. Unfortunately, the existence of w is provably inconsistent with the axioms of second-order ZFC. A partial result holds if, *given the ordered pairs of a plural variable assignment*, there is always a set w' with the feature that an arbitrary set of PFO formulas is satisfied by the ordered-pairs of a plural variable assignment differing from the former in at most the values of singular variables just in case it is satisfied by the restriction of the latter assignment to w'. This would require the existence of a Π^2_0 indescribable cardinal, which is independent from the axioms of set-theory (if consistent with them). For an excellent discussion of these issues see Shapiro (1987).

To forestall any ambiguities, we shall always use 'the Fs are the possible values of a variable' to mean that the Fs are *together* the possible values of a variable.

With this machinery on board, we may set forth a criterion of ontological commitment for PFO^+ languages. We begin by emulating Quine's original proposal:

> A theory couched in a PFO^+ language is committed to the existence of an object satisfying a certain singular predicate if and only if, some object satisfying that predicate must be admitted as a possible value of one of the theory's singular variables in order for the theory to be true;

but to this we add:

> the theory is committed to the existence of objects satisfying a *plural* predicate if and only if some objects satisfying that predicate must be admitted as the possible values of one of the theory's plural variables in order for the theory to be true.

A PFO^+ theory might be committed to the existence of elephants. This will be the case whenever some object satisfying the singular predicate '$\text{ELEPHANT}(x)$' must be admitted as the possible value of a singular variable in order for the theory to be true. But it could also be committed to the existence of children who together carried the piano. This will be the case whenever some objects satisfying the plural predicate '$\mathbf{CarriedPiano}(xx)$' must be admitted as the possible values of a plural variable in order for the theory to be true.

An especially interesting case of plural ontological commitment is cardinality. For instance, a theory is committed to the existence of infinitely many things if some objects satisfying the plural predicate '$\mathbf{InfiniteInNumber}(xx)$' must be admitted as the possible values of a plural variable in order for the theory to be true.[26]

[26]For more on the use of plural predicates to describe cardinality see chapter 3 of this thesis.

1.12 A One-Sorted Language

The predicates of PFO$^+$ languages are sharply divided into those which are plural and those which are not. In this respect, PFO$^+$ languages are a poor mirror of English. For consider the following sentences,

(39) The children carried the piano;

(40) John carried the piano

Intuitively, we are employing the *same* English predicate in (39) and (40), first to say something about the children (collectively) and then to say something about John. For instance, we expect it to follow logically from both (39) and (40) that the piano was carried. This intuition is not preserved when we paraphrase (39) and (40) into a PFO$^+$ language as

(39′) $\exists xx(\forall y(y \prec xx \leftrightarrow \textsc{Child}(y)) \wedge \mathbf{CarriedPiano}(xx))$; and

(40′) $\textsc{CarriedPiano}(\text{John})$.

For '$\mathbf{CarriedPiano}(xx)$' and '$\textsc{CarriedPiano}(x)$' are two different PFO$^+$ predicates: the former is plural and the latter is not.

In order to do better justice to the intuition that (39) and (40) have a predicate in common we may appeal to Boolos's convention,[27] and paraphrase (40) as:

(40″) $\exists xx\forall y((y \prec xx \leftrightarrow y = \text{John}) \wedge \mathbf{CarriedPiano}(xx))$;

Now we get what we wanted because (40″) shares the plural predicate '$\mathbf{CarriedPiano}(xx)$' with (39′).

This sort of move can be carried out quite generally. Whenever we have singular and a plural PFO$^+$ predicates which correspond to the same English predicate, we can eliminate the former and have the latter do the work of both. In fact, singular predicates with no corresponding plural can also be eliminated. If '$\mathrm{P}(x)$' is a singular predicate, we may introduce a plural predicate '$\mathbf{P}^*(xx)$' to play its role. All we need to do is pick '$\mathbf{P}^*(xx)$' so that the following is true:

[27] See section 1.6.

(41) $\forall x \forall yy[\forall z(z \prec yy \leftrightarrow z = x) \longrightarrow (\mathbf{P}^*(yy) \leftrightarrow \mathrm{P}(x))]$.[28]

Whenever (41) holds, we shall say that '$\mathbf{P}^*(xx)$' is a *plural counterpart* of '$\mathrm{P}(x)$', and that '$\mathrm{P}(x)$' is the *singular counterpart* of '$\mathbf{P}^*(xx)$'.

By bringing in plural counterparts, we have the option of eliminating all singular variables and predicates from the language. We introduce the plural predicate '$xx_i \preceq xx_j$' as an abbreviation for 'they$_i$ are some of them$_j$', and set forth the following definition:

$$\mathbf{1}(xx) \equiv_{df} \forall yy(yy \preceq xx \rightarrow xx \preceq yy).$$

This guarantees that '$\mathbf{1}(\ldots)$' is true of some objects just in case there is only one of them—recall Boolos's convention! We then apply the following transformation:

- $Tr(\neg\varphi) = \neg Tr(\varphi)$;
- $Tr(\varphi \wedge \psi) = Tr(\varphi) \wedge Tr(\psi)$;
- $Tr(\exists xx_i(\varphi)) = \exists xx_{2i}\ Tr(\varphi)$
- $Tr(\exists x_i(\varphi)) = \exists xx_{2i+1}\ (\mathbf{1}(xx_{2i+1}) \wedge Tr(\varphi))$;
- $Tr(x_i \prec xx_j)) = (xx_{2i+1} \preceq xx_{2j})$;
- $Tr(x_i = x_j) = (xx_{2i+1} \preceq xx_{2j+1})\ \wedge\ (xx_{2j+1} \preceq xx_{2i+1})$;
- if $\mathbf{P}^*$ is the plural counterpart of P, $Tr(\mathrm{P}(x_{i_1}, \ldots, x_{i_m}, xx_{j_1}, \ldots, xx_{j_n})) = \mathbf{P}^*(xx_{2i_1+1}, \ldots, xx_{2i_m+1}, xx_{2j_1}, \ldots, xx_{2j_n})$.

It is therefore possible to formulate PFO$^+$ languages as one-sorted languages consisting solely of logical connectives, parenthesis, and *plural* variables and predicates. But one can continue to use *singular* variables and predicates by employing the definitional equivalences induced by our transformation:

[28]Besides requiring that it satisfy (41), we may let '$\mathbf{P}^*(xx)$' behave as we please, but the following seems like a natural further constraint:

(42) $\forall xx[(\mathbf{P}^*(xx) \leftrightarrow \forall y(y \prec xx \rightarrow \mathrm{P}(y))]$.

- $\exists x_i(\varphi) \equiv_{df} \exists xx_i\ (\mathbf{1}(xx_i) \wedge \varphi)$;
- $x_i \prec xx_j \equiv_{df} (xx_i \preceq xx_j)$;
- $x_i = x_j \equiv_{df} (xx_i \preceq xx_j) \wedge (xx_j \preceq xx_i)$;
- if $\mathbf{P}^*$ is the plural counterpart of P,
  $\mathrm{P}(x_{i_1}, \ldots, x_{i_m}, xx_{j_1}, \ldots, xx_{j_n}) \equiv_{df} \mathbf{P}^*(xx_{i_1}, \ldots, xx_{i_m}, xx_{j_1}, \ldots, xx_{j_n})$.

In English, it is natural to think of 'something' as a generic quantifier, and of 'someone' as specialized: 'someone' is a variant of 'something' which we use to indicate that the objects we are quantifying over are persons. In a one-sorted PFO$^+$ language, we treat plural quantifiers as generic and singular quantifiers as specialized in much the same way. Singular quantifiers are variants of plural quantifiers which we use to indicate that the possible values of a variable are always a single object. Similarly, in a one-sorted PFO$^+$ language we treat plural predicates as generic and singular predicates as specialized. Singular predicates are variants of plural predicates which are used to indicate that the possible values of admissible arguments are always a single object.

Thinking of PFO$^+$ languages as one-sorted therefore eliminates the need for an account of the relation between *singular* quantification and predication and *plural* quantification and predication. The former are simply a special case of the latter. It also eliminates the need for separating the ontological commitments of PFO$^+$ theories in two. Our criterion reduces to the following:

> A theory couched in a PFO$^+$ language is committed to the existence of objects satisfying a plural predicate if and only if some objects satisfying that predicate must be admitted as the possible values of one of the theory's plural variables in order for the theory to be true.

Singular ontological commitments are now a special case of plural ontological commitments: a one-sorted PFO$^+$ theory is committed to the existence of an elephant if objects satisfying the plural predicate '$\mathbf{1}(xx) \wedge \mathbf{Elephants}(xx)$' must be admitted as the possible values of a plural variable in order for the theory to be true.

1.13 Back to First-Order Languages

In this section we will see that, by making a certain conceptual leap, it is possible to treat one-sorted PFO$^+$ languages as first-order languages.[29]

Syntactically, one-sorted PFO$^+$ languages are no different than their first-order counterparts—the fact that bold fonts are used for predicates and double letters for variables is of no importance whatsoever. But there are semantic differences. Whereas a plural variable assignment may associate several values with a PFO$^+$ variable, a first-order variable assignment always associates a single value with a first-order variable.

In order to treat one-sorted PFO$^+$ languages as first-order languages, we shall therefore modify our formal semantics so that, with respect to an assignment, each variable is associated with precisely one 'plurality'. This is the easy part. The hard part will be to elucidate the status of 'pluralities'.

We begin by imposing a structural constraint. If pluralities are to serve their purpose, they must form an *atomic mereology* over some dyadic relation $\triangleleft$, in which the atoms consist of everything there is.[30] Accordingly, the role played by the Fs on our original semantics will be played on the new semantics by the plurality with precisely the Fs as atoms. (This is true, in particular, when there is only one F: the role played by an object x on the original semantics will be played on the new semantics by the plurality with x as its unique atom—that is, x itself.) And, on the new semantics, '$\preceq$' will be interpreted as $\triangleleft$.

Our structural constraint on pluralities gives rise to a serious difficulty. Any atomic mereology with more than one atom includes non-atoms, so there must be non-atomic pluralities. But since everything there is is an atomic plurality, there can be no non-atomic pluralities after all. Contradiction!

In order to make sense of non-atomic pluralities we have to take a conceptual

[29]The basic insight is due to Vann McGee.

[30]In other words, we shall demand (*a*) that $\triangleleft$ be transitive over pluralities, (*b*) that everything there is be an *atomic* plurality (that is, a plurality satisfying '$\forall y(y \triangleleft x \rightarrow x \triangleleft y)$'), (*c*) that every plurality x be such that some atomic plurality bears $\triangleleft$ to x, and (*d*) that, if each of the As is an atomic plurality, then the As have a unique $\triangleleft$-*fusion* (that is, there is a unique plurality x such that each of the As bears $\triangleleft$ to x and any plurality y which bears $\triangleleft$ to x is such that some A bears $\triangleleft$ to y).

leap. We need to allow for the idea that the subject-matter of ontology is richer than expected. Our starting-point is the platitude that everything there is is an individual. After the conceptual leap, we take the position that, in discovering what individuals there are, we have not fully answered the question of what there is. We have only addressed the small parcel of our ontology that consists of *atomic* pluralities. To fully answer the question of what there is we must give an account of what *pluralities* there are, both atomic and non-atomic.

On the new conception, not everything is an individual. But this is not because the extension of 'individual' has changed: at the beginning of the inquiry we took everything—*absolutely* everything—to be an individual, and whatever we counted as an individual then we count as an individual now. Nor, of course, is it because the world has changed. Instead, we have changed our conception of 'everything', by forcing the singular quantifier 'there is a plurality such that it is so-and-so' to behave like the plural quantifier 'there are some individuals such that they are so-and-so'.

Admittedly, this is not an easy leap to take. But to a certain extent it is encouraged by the conclusions we have reached in preceding sections. We have seen that there is more to the task of assessing the ontological commitments of a PFO^+ theory than the question of what *singular* predicates must be satisfied in order for a the theory to be true: there is also the question of what *plural* predicates must be satisfied in order for the theory to be true. In this sense, we have found that the subject-matter of ontology is richer than expected.

When we take one-sorted PFO^+ theories to be first-order theories, and admit non-atomic pluralities as a part of our ontology, what we do is transfer the unexpected richness in the subject-matter of ontology from one place to another. Instead of countenancing a new (plural) kind of commitment to the inhabitants of a familiar part of our ontology—the realm of individuals—we countenance a familiar (singular) kind of commitment to the inhabitants of a new part of our ontology—the realm of non-atomic pluralities.

Those who endorse the view that if something looks like a first-order language, then it *is* a first-order language will certainly welcome the idea of treating one-sorted

PFO$^+$ languages as first-order languages. And I do not think there is anything to prevent them from doing so. But I also think there is nothing to *force* such a move. It seems to me that our conception of ontology does not constrain things one way or the other. The question whether PFO$^+$ languages should be treated as first-order languages does not give rise to a substantial dispute.

Before bringing this section to a close, there are a few points that deserve mention. First, the fact that pluralities have a mereological *structure* should not be taken to mean that a non-atomic plurality is a *mereological sum*. For, in the traditional philosophical sense, a mereological sum is an *individual*.[31] Since only atomic pluralities are individuals, it follows that every mereological sum must be an atomic plurality.

Second, our treatment of one-sorted PFO$^+$ languages as first order languages does not, by itself, license the use of plural quantification over non-atomic pluralities. *Singular* quantification over non-atomic pluralities is justified by a reinterpretation of English plural quantification. What a parallel defense of *plural* quantification over non-atomic pluralities would call for is a reinterpretation of *super-plural* English quantification, and no reason has been given here to think that super-plural English quantifiers exist. It would be a mistake to protest that, once non-atomic pluralities are on board, our grasp of ordinary English plural quantification automatically yields plural quantification over non-atomic pluralities. For English plural quantification over *individuals*—that is, English plural quantification over atomic pluralities—is the only kind of English plural quantification there is. Perhaps there is a way of enriching our current use of plural quantification so as to have it encompass non-atomic pluralities, but such a move has not been justified by anything in the present discussion.

Finally, it is worth noting that we must modify the standard model theory for first-order languages if our novel interpretation is to preserve the logical properties of PFO$^+$ sentences. Specifically, we must require that every model form an atomic mereology over '$\preceq$'.

Throughout the remainder of this essay we will consider a family of applications

[31] See Leonard and Goodman (1940).

for PFO^+ languages. Our discussion will not depend on whether they receive a plural two-sorted interpretation (as in section 1.8), a plural one-sorted interpretation (as in section 1.12), or a first-order interpretation (as in the present section). We assume the former for the sake of concreteness.

1.14 Applications

1.14.1 Second-Order Logic

A second-order formulation of standard (Zermelo-Fraenkel) set theory is highly desirable. It enables us to express the general principles underlying the first-order schemes of separation and replacement.[32] In addition, Vann McGee has shown that, when our domain of discourse consists of absolutely everything, there is an extension of second-order set theory that characterizes the set-theoretic universe up to isomorphism.[33] Yet there is some debate as to whether it is legitimate to use second-order languages for the study of set theory. The reason is that, on one standard interpretation—Quine's interpretation—second-order languages are nothing but 'set-theory in sheep's clothing'.[34] More precisely, they are two-sorted first-order languages in which variables of the first sort range over the elements of a certain *set* S, and variables of the second sort range over the subsets of S.[35] Second-order languages are therefore useless when our variables range over objects which are too many to form a set. And this is certainly the case in the intended interpretation of set theory.

It is tempting to overcome this difficulty by taking second-order languages to be *class*-theory in sheep's clothing, that is, by understanding them as two-sorted first-order languages in which variables of the first sort range over the elements of a certain *class* C, and variables of the second sort range over the sub-classes of C. Doing

[32]For more on the expressive limitations of first-order languages and the role of second-order languages in overcoming them, see chapter 5 of Shapiro (1991).

[33]See McGee (1997)

[34]See Quine (1986).

[35]However, the *model-theory* for second-order languages must differ from that of two-sorted first-order languages. I owe this observation to Gabriel Uzquiano.

so, however, only postpones our difficulties. For, in making second-order languages available for the study of set-theory, we have made them unavailable for the study of *class*-theory, which now takes center-stage.

Fortunately, Boolos has given us a new way of interpreting second-order logic, one which does not run into trouble when our variables range over objects which are too many to form a set. Boolos's original proposal involves a translation method from second-order formulas into English,[36] but we may obtain identical results by introducing the following definitional equivalences into PFO languages:

- $X_i(x_j) \equiv_{df} x_j \prec xx_i$;
- $\exists X_i(\varphi) \equiv_{df} \exists xx_i(\varphi) \vee \varphi^*$,
  where φ^* is the result of substituting $\ulcorner x_j \neq x_j \urcorner$ everywhere for $\ulcorner X_i(x_j) \urcorner$ (or its notational variants).

The complication in our second definition is needed to accommodate the fact that, although '$\exists X \forall y \neg X(y)$' is a theorem of second-order logic, '$\exists xx \forall y \neg y \prec xx$' is necessarily false (since it is impossible for there to be some objects such that no object is one of them).

On this interpretation, second-order formulas are definitional variants of PFO formulas. Hence, in contexts where the expressive power of second-order ZFC is important, PFO languages turn out to be excellent languages of regimentation.

A point is worth mentioning. So far we have accounted only for *monadic* second-order variables. But, as Boolos points out, relation variables can be incorporated into his scheme by appealing to ordered pairs.[37]

1.14.2 Model Theory for Second-Order Languages

A standard model for the first- or second-order language of set theory is an ordered pair $\langle D, I \rangle$. Its domain, D, is a non-empty set, and its interpretation function, I,

[36] See Boolos (1984).

[37] Treat $\ulcorner \exists R^n_i \varphi \urcorner$ as a notational variant for $\ulcorner \exists X_i \varphi \urcorner$, and $\ulcorner R^n_i(x_1, \ldots, x_n) \urcorner$ as a notational variant for $\ulcorner X_i(\langle x_1, \ldots, x_n \rangle) \urcorner$ (where '$\langle \ldots \rangle$' is the ordered n-tuple function). For more on polyadic second-order logic, see Rayo and Yablo (forthcoming).

assigns a binary relation on D to the two-place predicate letter 'ϵ'. A sentence is then said to be *valid* if it is true in all standard models.

It is a familiar point that this does not correspond to our intuitive notion of validity. What we would like to say is, roughly, that a sentence is valid if it is true no matter what its domain of discourse is, and no matter how its non-logical vocabulary is interpreted. But there are no standard models corresponding to certain domains of discourse and interpretations of 'ϵ'. For instance, there is no model $\langle D, I\rangle$ such that D contains all sets and I assigns to 'ϵ' the set of all pairs $\langle x, y\rangle$ for x a member of y, because it is a theorem of ZFC that there is no set of all sets and that there is no set of all pairs $\langle x, y\rangle$ for x a member of y. Among other things, this opens the alarming possibility of a false sentence which is true in all standard models.

There is therefore no immediate guarantee of the adequacy of standard model theory. If it does turn out to be adequate it will be in virtue of non-trivial set-theoretic principles, not merely in virtue of our definitions.

In fact, it is possible to improve upon the standard model theory. By building upon Boolos's work,[38] Gabriel Uzquiano and I have set forth a formal semantics for second-order set theory that is intuitively adequate.[39] We proceed by rejecting the idea that the domain of a model must be a *set* of objectss. Instead we focus attention on the objects themselves, and let *them* function as our domain. Accordingly, we reject the idea that the interpretation function of a model must be a *set* of ordered-pairs. We let the ordered-pairs themselves provide an interpretation for 'ϵ'. To accommodate these changes, we take the satisfaction predicate to be a *plural* predicate '$\mathbf{Sat}(x, yy, zz)$'.[40] Thus, although our formal semantics cannot be formulated within a PFO language, it can easily be captured within a PFO$^+$ language.

With an intuitively adequate model theory at hand, it is natural to ask whether every intuitively satisfiable set of second-order formulas is satisfied by some standard model. A version of this proposition was first set forth by Georg Kreisel, so we shall

[38]See Boolos (1985a).

[39]See chapter 2 of this thesis. Similar ideas have been set forth in unpublished manuscripts by Josep Macià Fabrega and Byeong-Uk Yi.

[40]Variable assignments are dealt with as in section 1.10.

call it *Kreisel's Principle.*[41]

Alternative formulations of Kreisel's Principle—often under the guise of Reflection Principles—have played a significant role in the development of set theory. Nonetheless, the literature suggests that (without the aid of proper classes) there is no way of expressing Kreisel's Principle within a PFO (or second-order) language.[42] On the other hand, it is easily captured within PFO$^+$ languages. All we need is the plural predicate '$\mathbf{Sat}(x, yy, zz)$', from our novel formal semantics.

If true, Kreisel's Principle guarantees the adequacy of standard model theory. But only its restriction to first-order formulas is provable within standard set theory. In its unrestricted second-order form, it is demonstrably independent from the axioms of set theory (if consistent with them).[43]

We may therefore rest assured that standard first-order model theory is adequate. In particular, the first-order version of Kreisel's Principle guarantees that every first-order sentence which is true in all standard models is true.[44] However, without the unrestricted version of Kreisel's Principle, we have no assurance that standard second-order model theory is adequate. For all we know, there is a false second-order sentence which is true in all standard models. Because of this, our novel formal semantics is a significant improvement over standard second-order model theory.

Without further logical resources, it is not possible to extend our formal semantics to encompass PFO$^+$ languages.[45] Intuitively, the problem is that there are 'too many' possible semantic values for plural predicates.

PFO$^+$ regimentation might therefore turn out to be unstable in the strong sense that it may not be generally possible to formulate the notion of *truth-in-a-model* for a given PFO$^+$ language in another PFO$^+$ language.[46] In contrast, we know that

[41] See Kreisel (1967) pp. 152-7.

[42] Shapiro provides an excellent discussion in chapter 6.3 of Shapiro (1991).

[43] See Shapiro (1991), chapter 6.3.

[44] This also follows from the first-order Completeness Theorem, which is not available in the second-order case.

[45] Yi (unpublished) provides a formal semantics for PFO$^+$ languages. Unfortunately, Yi appeals to an 'interpretation function' that exceeds the resources of PFO$^+$ languages.

[46] This is not to be confused with the fact that a (sufficiently strong) PFO$^+$ language cannot be used to formulate its own formal semantics.

first-order languages are stable in this respect. We have just seen that the first-order version of Kreisel's Principle legitimizes the use of standard first-order model theory, which can be formalized in a first-order language. To attain strong stability, a friend of PFO^+ regimentation might be tempted to postulate a strengthened version of Kreisel's Principle. But such a move would presumably require some sort of independent motivation.

Fortunately, we have seen in section 1.10 that PFO^+ languages turn out to be stable in the weaker sense that a Tarskian definition of *truth* for a given PFO^+ language can be defined in another PFO^+ language.

1.15 Conclusions

We have assumed that it is possible to quantify over absolutely everything, and found that certain English sentences containing collective predicates resist both first-order and PFO paraphrase. To capture such sentences we introduced PFO^+ languages, which may contain arbitrary plural predicates.

PFO^+ languages turn out to be tremendously fruitful. They allow us to give a formal semantics for second-order languages and state important set theoretic propositions; they also provide us with natural formalizations for English plural definite descriptions and generalized quantifiers. I believe this makes a solid case for the use of PFO^+ languages as languages of regimentation.

In leaving first-order regimentation behind, we were led to enrich Quine's criterion of ontological commitment. It emerged that PFO^+ theories can have plural ontological commitments in addition to singular ones. In this sense, we discovered that the subject-matter of ontology is richer than one might have thought.

We noted that this unexpected ontological richness can be accounted for in different ways. On one construal, the singular is regarded as a special case of a plural and, accordingly, plural ontological commitments are taken to be the only kind of ontological commitments a PFO^+ theory can have. On a more radical view, the unexpected ontological richness is accommodated by interpreting PFO^+ variables as first-order variables ranging over ‘pluralities’, not all of which should be counted as individuals.

Chapter 2

Toward a Theory of Second-Order Consequence

(with Gabriel Uzquiano)

There is little doubt that a second-order axiomatization of Zermelo-Fraenkel set theory plus the axiom of choice (ZFC) is desirable. One advantage of such an axiomatization is that it permits us to express the principles underlying the first-order schemata of separation and replacement. Another is its *almost*-categoricity: $\mathcal{M}$ is a model of second-order ZFC if and only if it is isomorphic to a model of the form $\langle V_\kappa, \in \cap (V_\kappa \times V_\kappa)\rangle$, for κ a strongly inaccessible ordinal.

We obtain similar benefits when we allow for the existence of Urelemente. The axioms of second-order ZFC with Urelemente (ZFCU) are not able to specify the structure of the universe up to isomorphism, but Vann McGee has recently shown that, provided one takes the range of its quantifiers to be unrestricted, the addition of an axiom that states that the Urelemente form a set to the axioms of ZFCU will characterize the structure of the universe of *pure* sets up to isomorphism.[1] In sum, there is much to be gained from the ability to employ second-order quantification in

[1]This categoricity result is stated and proved in McGee (1997). A little inspection of the proof reveals that what is required for the result to be provable is that one can prove that there is a 1-1 correspondence between the universe of *pure* sets and the universe of discourse.

the context of set theory.

What is much more controversial is that we can, with a clear conscience, develop set theory within a second-order language. The standard interpretation of second-order quantification takes second-order variables to range over the sets of individuals which first-order variables range over. This interpretation may be convenient for the development of second-order arithmetic, but it will not do for the purpose of developing set theory in a second-order language. The reason is not difficult to state. When we do set theory, we take our first-order variables to range over all sets. But if we take our second-order variables to range over sets of sets in the range of the first-order variables, then second-order comprehension will fail. A simple instance of second-order comprehension such as $\exists X \forall y\ (Xy \leftrightarrow y \notin y)$ will be false on account of Russell's paradox, according to which no set contains all and only those sets that are not members of themselves.

A different approach would be to take the second-order variables of the language to range not over sets, but rather over classes. An instance of comprehension such as $\exists X \forall y\ (Xy \leftrightarrow y \notin y)$ would then be taken to amount to the existence of a class of all and only those sets that are not members of themselves. One difficulty with this approach is that it would be in tension with the attitude of most set theorists, who seem to regard their subject as the most comprehensive theory of collections. There are no collections other than sets, and even if it is, on occasion, convenient to speak of proper classes, i.e., collections that are 'too big' to form sets, such talk is not to be taken literally.

An interpretation of second-order quantification that avoids commitment to proper classes, and still makes second-order logic available for the development of set theory, is therefore preferable to one that takes second-order variables to range over classes. In Boolos (1984), George Boolos offered just such an interpretation. He proposed to understand second-order quantification in terms of English plural quantification.[2]

[2] Boolos (1984) and Boolos (1985a) make use of English plural quantification to interpret monadic second-order quantification, but rely on the availability of ordered pairs to interpret *polyadic* second-order quantification. A more direct interpretation of polyadic quantification is given in Rayo and Yablo (forthcoming).

Accordingly, he read an instance of comprehension such as $\exists X \forall y\ (Xy \leftrightarrow y \notin y)$ as the truism that there are some sets such that a set is one of them just in case it is not a member of itself.[3] The advantage of Boolos' plural interpretation is that, as he argued, it verifies verifies all instances of second-order comprehension, and legitimizes the development of second-order set theory.

In a later article, Boolos (1985a), he made use of the apparatus of plurals to give an account of the truth- and validity-conditions of second-order formulas of set theory. He provided definitions of truth and of a notion of validity he called 'supervalidity,' which were aimed to show that commitment to classes is not necessary to develop a rigorous semantics for the language of second-order set theory. But there was an important drawback: Boolos' definitions of truth and validity didn't generalize to a definition of logical consequence.

The purpose of this note is to present an account of the truth- and validity-conditions of second-order formulas which can be generalized to an account of the conditions under which a second-order formula is a logical consequence of a set of second-order formulas.

There are two desiderata our semantics should satisfy. First, in the spirit of the plural interpretation of second-order set theory, it should commit us to no entities other than sets, which are the objects in the range of the first-order variables of the language. The second desideratum concerns the connection between truth, satisfaction and validity, and will require some explanation.[4]

A standard model for the language of first-order set theory is an ordered pair $\langle D, I \rangle$. Its domain, D, is a non-empty set, and its interpretation function, I, assigns a set of ordered pairs to the two-place predicate '$\in$.' A sentence is true in $\langle D, I \rangle$ just in case it is satisfied by all assignments of first-order variables to members of D and second-order variables to subsets of D; a sentence is satisfiable just in case it is true in

[3]More precisely, Boolos' reading is 'Either there are no sets that are not self-identical, or there are some sets such that a set is one of them just in case it is not a member of itself'.

[4]The classical discussion of the connection between truth and second-order validity can be found in Kreisel (1967). Shapiro (1991) (Sections 6.1 and 6.3), and Etchemendy (1990) (Chapter 11) discuss some of the issues raised by Kreisel.

some standard model; finally, a sentence is valid just in case it is true in all standard models.

The stipulation that D and I be sets is not without consequence. An immediate effect of this stipulation is that no standard model provides the language of set theory with its intended interpretation. In other words, there is no standard model $\langle D, I\rangle$ in which D consists of all sets and I assigns the standard element-set relation to '$\in$.' For it is a theorem of ZFC, that there is no set of all sets, and that there is no set of ordered-pairs $\langle x, y\rangle$, for x an element of y.

Therefore, on the standard definition of model, it is not at all obvious that the validity of a sentence is a guarantee of its truth; similarly, it is far from evident that the truth of a sentence is a guarantee of its satisfiability in some standard model. If there is a connection between satisfiability, truth, and validity, it is not one that can be 'read off' standard model theory.

This is not a problem in the first-order case, since set theory provides us with two reassuring results for the language of first-order set theory. One result is the first-order completeness theorem, according to which first-order sentences are provable, if true in all models. Granted the truth of the axioms of the first-order predicate calculus and the truth preserving character of its rules of inference, we know that a sentence of the first-order language of set theory is true, if it is provable. Thus, since valid sentences are provable and provable sentences are true, we know that valid sentences are true. The connection between truth and satisfiability immediately follows: if ϕ is unsatisfiable, then $\neg\phi$, its negation, is true in all models, and hence valid. Therefore, $\neg\phi$ is true, and ϕ is false.[5]

The other comforting result is a principle of reflection, provable within first-order ZFC. According to this principle, for each sentence ϕ of first-order set theory, there is a standard model of the form $\langle V_\kappa, \in \cap (V_\kappa \times V_\kappa)\rangle$, for some ordinal κ, such that ϕ is true if and only ϕ is true in that model. Thus, suppose a sentence ϕ of first-order set theory is false. Then $\neg\phi$ will be true, and, by the reflection principle, true in some

[5]See Kreisel (1967) (pp. 89-93), Boolos (1985b) (p. 84), and Cartwright (1994). The argument is discussed in Shapiro (1991) and Shapiro (1987), (Section 6.3).

standard model of the form $\langle V_\kappa, \in \cap (V_\kappa \times V_\kappa)\rangle$, for some ordinal κ. ϕ will be false in that model, and hence not valid.

The situation changes drastically when we venture into a second-order language. There is no completeness theorem for second-order logic. Nor do the axioms of second-order ZFC imply a reflection principle which ensures that if a sentence of second-order set theory is true, then it is true in some standard model. Thus there may be sentences of the language of second-order set theory that are true but unsatisfiable, or sentences that are valid, but false. To make this possibility vivid, let Z be the conjunction of all the axioms of second-order ZFC. Z is surely true. But the existence of a model for Z requires the existence of strongly inaccessible cardinals. The axioms of second-order ZFC doesn't entail the existence of strongly inaccessible cardinals, and hence the satisfiability of Z is independent of second-order ZFC. Thus, Z is true, but its unsatisfiability is consistent with second-order ZFC.[6]

One could be tempted to opt for the advantages of theft over honest toil and postulate a second-order reflection principle. But it would be somewhat disappointing if we had to rely on a non-trivial hypothesis which—no matter how plausible—is not susceptible of a proof from currently accepted axioms in order to establish what ought to be obvious: that a sentence is true if it is valid and that it is satisfiable, if it is true.

The second desideratum of our theory is therefore this: it should make plain the connection between validity, satisfiability and truth.

Boolos' semantics satisfies this desideratum. On his definition of supervalidity, a sentence of second-order set theory is supervalid if it is true no matter what sets we take its quantifiers to range over and no matter what ordered pairs of sets we take '$\in$' to denote. The definition, however, is schematic: to each sentence of set theory ϕ he associated a second-order sentence ϕ^* such that ϕ is supervalid just in

[6]For those who view the existence of strongly inaccessible cardinals as a very plausible hypothesis and are thus not persuaded by the example, Vann McGee described in McGee (1992) another candidate to be a second-order sentence which is true, yet unsatisfiable. Very roughly, McGee's sentence is the result of conjoining Z with an axiom to the effect that the set-theoretic universe can't be embedded into a strictly larger universe.

case ϕ^* is true. It takes very little—universal instantiation and substitution—to show that a sentence ϕ is true if ϕ^* is true. This yields an immediate connection between validity and truth. Unfortunately, as Boolos put it, "it would seem that there is no obvious way to generalize the notion of supervalidity to a notion of superconsequence or supersatisfiability.'[7]

We shall now present our alternative account of second-order validity, which *can* be extended to an account of logical consequence while satisfying the two desiderata we just laid down. Like Boolos, we shall understand second-order quantification in terms of plural quantification. Moreover, we will make use of a primitive satisfaction predicate which takes predicates in some of its argument places. In this respect, our definitions will not be unlike Boolos' definition of truth for the language of second-order ZFC, as he himself made use of a satisfaction predicate which took predicates in some of its argument places in his definition.

To a large extent, the success of our proposal depends on whether it is possible to give an adequate account of the new sort of predicate it requires. Boolos made a convincing case for the view that plural quantification can be used to understand second-order quantification, but it is not obvious that English provides us with the resources to make sense of predicates which take first-order predicates in their argument places. We propose to understand them in terms of collective English predicates. In 'The rocks rained down,' for example, 'rained down' is not predicated of a particular object such as this rock or that rock. Nor is it predicated of some peculiar complex object made up by these rocks or those. Rather, it is predicated of these rocks or those.[8] Similarly, with 'The ordinals do not outnumber the cardinals' or 'The sets possessing a rank exhaust the universe.'[9]

An adequate justification of such predicates would take us far beyond the scope of this paper, but has been taken up elsewhere by one of us.[10] A similar position has

[7]Boolos (1985a) (pp. 86-87).

[8]This is Boolos' example. In Boolos (1985b) he hinted at the possibility of plural predication.

[9]Another Boolosian example.

[10]See chapter 1.

been developed in print by Byeong-Uk Yi.[11]

Stewart Shapiro has developed a semantics for the language of ZFC in a language augmented with the primitive satisfaction predicate '$\mathbf{sat}(P, q, R, m)$,' which takes class-variables in some of its argument places.[12] Our proposal will be equivalent to Shapiro's when his quantification over classes is interpreted as plural quantification over sets, and his predicate '$\mathbf{sat}(P, q, R, m)$' is interpreted as a collective plural predicate.

It is now time to explain the thought underlying our proposal. Even from the standpoint of the standard model-theoretic semantics, this much is uncontroversial: A standard model for the language of set-theory is determined by the individuals that constitute its domain, and by the ordered pairs of individuals that its interpretation function assigns to '$\in$.' To require, in addition, that the individuals over which our variables range (or that the ordered pairs assigned to '$\in$') form a *set* strikes us as a somewhat artificial feature of the standard definition of a model. The core of our proposal is that we conceive of a model, not as a single set-theoretic object, but rather as given by the value*s* of a second-order variable 'M.' Accordingly, we take satisfaction to be a relation that a formula ϕ bears, not to a certain structured set, but to the values of 'M'. These objects will encode a specification of the individuals over which our first-order quantifiers are to range and a specification of the ordered pairs that are to be assigned to '$\in$.'[13]

There are several ways in which the proposal can be implemented. The option we favor takes a model to be given by ordered pairs of two different types: (1) ordered pairs of the form $\langle `\forall', x\rangle$, which are taken to encode the fact that $x$ is to be within the range of our quantifiers, and (2) ordered pairs of the form $\langle `\in', \langle x, y\rangle\rangle$, which are taken encode the fact that $\langle x, y\rangle$ is part of the interpretation of '$\in$.' We impose the requirement that if a model is given by some ordered pairs which include $\langle `\in', \langle x, y\rangle\rangle$,

[11]Yi (1999).

[12]He develops this semantics in section 6.1 of his Shapiro (1991).

[13]This is an extremely natural move to make. In fact, similar ideas have been set forth independently by two other philosophers concerned with English plurals and their relation to standard logic: Josep Macià Fabrega and Byeong-Uk Yi. Their unpublished manuscripts are 'Plural Quantification and Second-Order Quantification,' and 'The Language and Logic of Plurals,' respectively.

then $\langle `\forall', x\rangle$ and $\langle `\forall', y\rangle$ must also be among these ordered pairs. Formally, we take 'M is a model,' where 'M' is a monadic second-order variable, to abbreviate the following formula of second-order set theory:

$$\exists x\ M\,\langle `\forall', x\rangle \wedge$$
$$\forall x\ (Mx \rightarrow (\exists y\ x = \langle `\forall', y\rangle \vee \exists w \exists z\ x = \langle `\in', \langle w, z\rangle\rangle\,))\ \wedge$$
$$\forall w \forall z\ (M\,\langle `\in', \langle w, z\rangle\rangle \rightarrow M\,\langle `\forall', w\rangle \wedge \langle `\forall', z\rangle\,)$$

Recall that, for us, a second-order variable such as 'M' is a plural variable. Thus, when we speak of *a model M*, we are not to be taken to speak of an object of some sort or another. Rather, we should be taken to speak of some sets (the values of the variable 'M'), which happen to satisfy the above formula. In a similar vein, we will sometimes we say that *the domain of a model M consists of the Fs*; this should be read: 'for every x, $\langle `\forall', x\rangle$ is one of the values of '$M$' if and only if $x$ is one of the Fs'. Finally, when we say *a model M assigns interpretation R to* '$\in$', this locution should be read: 'For every $x$ and $y$, $\langle `\in', \langle x, y\rangle\rangle$ is one of the values of 'M' if and only if x bears R to y'.

According to our definition, there is a model whose domain consists of all sets and which assigns the standard element-set relation to '$\in$'. It is given by a second-order variable whose values are the ordered pairs with '$\forall$' as their first-component and a set as their second component, and the ordered pairs with '$\in$' as their first component and an ordered pair $\langle x, y\rangle$ for x an element of y as their second component.

Although it is in the second-order case that the proposal deserves the most interest, it is best for expository purposes to begin by giving definitions of first-order truth, validity, and logical consequence, and later extend the proposal to the second-order case.

A first-order variable assignment is a map from the first-order variables of the language into the domain of a model. Since there are denumerably many first-order variables in the language, the axioms of infinity and replacement guarantee that such maps are sets. Accordingly, we may use a first-order variable to range over variable assignments. Let us take 's is a variable assignment with respect to model M' to abbreviate the formula:

$\forall v_j\ (v_j \text{ is a variable} \rightarrow \exists! x\ \ \langle v_j, x\rangle \in s)\ \wedge$

$\forall x\ (x \in s \rightarrow \exists v_i \exists y (v_i \text{ is a variable} \wedge x = \langle v_i, y\rangle \wedge M\ \langle `\forall\text{'}, y\rangle\ ))$

Since variable assignments are functions, we shall say that '$s(v_i) = x$' holds whenever it is the case that $\langle v_i, x\rangle \in s$. A v_i-variant of a variable assignment s is a variable assignment t that agrees with s except perhaps in the value it assigns to v_i. Thus, we will take 't is a v_i-variant of s' to abbreviate the first-order formula:

s is a variable assignment $\wedge$ t is a variable assignment $\wedge$

$\forall v_j\ ((v_j \text{ is a first-order variable} \wedge v_j \neq v_i) \rightarrow t(v_j) = s(v_j))$

We are now in a position to introduce the predicate: 's satisfies ϕ with respect to M.' Note that this predicate takes first-order variables in two of its argument places, and a second-order variable in its third. Our satisfaction predicate is implicitly defined by the following axioms:

(0) s is a variable assignment with respect to M,

(1) if ϕ is $v_i = v_j$, then s satisfies ϕ with respect to M iff: $s(v_i) = s(v_j)$,

(2) if ϕ is $v_i \in v_j$, then s satisfies ϕ with respect to M iff: $M\ \langle `\in\text{'}, \langle s(v_i), s(v_j)\rangle\rangle$,

(3) if ϕ is $\neg\psi$, then s satisfies ϕ with respect to M iff: s does not satisfy ψ with respect to M,

(4) if ϕ is $(\psi \wedge \chi)$, then s satisfies ϕ with respect to M iff: s satisfies ψ with respect to M and s satisfies χ with respect to M,

(5) if ϕ is $\exists v_i \psi$, then s satisfies ϕ with respect to M iff: $\exists t$ (t is a v_i-variant of $s \wedge t$ satisfies ψ with respect to M).

With our implicit definition of satisfaction in place, we can provide an explicit definition for the predicate 'ϕ is true in M':

ϕ is true in M iff:

$\forall s$ (s is a variable assignment with respect to $M \rightarrow$

s satisfies ϕ with respect to M)

Truth is a special case of *truth in a model*: a sentence is true just in case it is true in the model whose domain consists of all sets and whose interpretation function assigns the standard element-set relation to '$\in$'. Finally, we provide explicit definitions of validity and logical consequence:

ϕ is valid iff:

$\forall M$ (M is a model $\rightarrow \phi$ is true in M)

ϕ is a logical consequence of Γ iff:

$\forall M[M$ is a model $\rightarrow \forall\psi \in \Gamma$ (ψ is true in $M \rightarrow \phi$ is true in M)]

We now extend the proposal to encompass second-order languages. Since the values assigned to second-order variables may encompass too many sets to form a set, second-order variable assignments cannot be sets. Instead, we will use a second-order variable S. The values of S will be ordered pairs with a variable in their first component and a member of the domain in their second component. If v_k is a first-order variable, we stipulate that S is to be true of exactly one pair of the form $\langle v_k, x\rangle$; if V_k is a second-order variable S may be true of several pairs $\langle V_k, x\rangle$ (or none). We shall say that x is the assignment of v_k with respect to S if $\langle v_k, x\rangle$ is among the values of S, and that x is an assignment of V_k with respect to S if $\langle V_k, x\rangle$ is among the values of S. Formally, we let 'S is a variable assignment with respect to M' abbreviate a second-order formula:

$\forall v_i$ (v_i is a first-order variable $\rightarrow \exists! y\ S\langle v_i y\rangle$) $\wedge$

$\forall x$ ($Sx \rightarrow$

[$\exists v_i$(v_i is a first-order variable $\wedge \exists y$ (M $\langle$'$\forall$'$, y\rangle \wedge x = \langle v_i, y\rangle$)) $\vee$

$\exists V_i$ (V_i is a second-order variable $\wedge \exists y$ (M $\langle$'$\forall$'$, y\rangle \wedge x = \langle V_i, y\rangle$))]

Thus, when we say 'S is a variable assignment with respect to M' we are not speaking of an object of some sort, as grammatical form would suggest. What we mean is that the values of the second-order variable 'S' satisfy the above formula. We let '$S(v_i) = x$' abbreviate '$S\langle v_i, x\rangle$.' Moreover, we let 'x is the value of v_i with respect to S' abbreviate '$S\langle v_i, x\rangle$,' and 'x is a value of V_i with respect to S' abbreviate '$S\langle V_i, x\rangle$.' A v_i-variant of a variable assignment S is a variable assignment that agrees with S

except perhaps in the value it assigns to v_i. Thus, we take 'T is a v_i-variant of S' to abbreviate the second-order formula:

S is a variable assignment $\wedge$ T is a variable assignment $\wedge$
$\forall v_j$ $((v_j$ is a first-order variable $\wedge\, v_j \neq v_i) \rightarrow T(v_j) = S(v_j)) \wedge$
$\forall V_i$ $(V_i$ is a second-order variable $\rightarrow \forall x\ (T\langle V_i, x\rangle \leftrightarrow S\langle V_i, x\rangle\,))$

In a similar fashion, a V_i-variant of a variable assignment S is a variable assignment that agrees with S except perhaps in the values it assigns to V_i. Thus, we take 'T is a V_i-variant of S' to abbreviate the second-order formula:

S is a variable assignment $\wedge$ T is a variable assignment $\wedge$
$\forall v_i$ $(v_i$ is a first-order variable $\rightarrow T(v_i) = S(v_i)) \wedge$
$\forall V_j$ $((V_j$ is a second-order variable $\wedge\, V_j \neq V_i) \rightarrow$
$\forall x\ (T\langle V_j, x\rangle \leftrightarrow S\langle V_j, x\rangle\,))$

We are now in a position to define satisfaction for the language of second-order set theory. The new satisfaction predicate, 'S satisfies ϕ with respect to M,' differs from its first-order counterpart in that it takes two second-order variables as arguments instead of one. It is implicitly defined by axioms analogous to (0)–(5):

(0′) S is a variable assignment with respect to M,

(1′) if ϕ is $v_i = v_j$, then S satisfies ϕ with respect to M iff: $S(v_i) = S(v_j)$,

(2′) if ϕ is $v_i \in v_j$, then S satisfies ϕ with respect to M iff: $M\langle \text{`}\in\text{'}, \langle S(v_i), S(v_j)\rangle\rangle$,

(3′) if ϕ is $\neg\psi$, then S satisfies ϕ with respect to M iff: S does not satisfy ψ with respect to M,

(4′) if ϕ is $(\psi \wedge \chi)$, then S satisfies ϕ with respect to M iff: S satisfies ψ with respect to M and S satisfies χ with respect to M,

(5′) if ϕ is $\exists v_i \psi$, then S satisfies ϕ with respect to M iff: $\exists T$ (T is a v_i-variant of $S \wedge T$ satisfies ψ with respect to M).

Two further axioms have no first-order analogues:

(6′) If ϕ is $V_i v_j$, then S satisfies ϕ with respect to M iff: $S\langle V_i, S(v_j)\rangle$,

(7′) if ϕ is $\exists V_i\ \phi$, then S satisfies ϕ with respect to M iff: $\exists T$ (T is a V_i-variant of $S \wedge T$ satisfies ψ with respect to M).

With our implicit definition of satisfaction in place, we may explicitly define *truth in a model* and *truth* as before. And our definitions of consequence and validity carry over to the second-order case without incident. Since the result of extending second-order ZFC with axioms (0′)–(7′) allows us to define a truth predicate for second-order ZFC, it follows from Tarski's Theorem on the undefinability of truth that axioms (0′)–(7′) yield a genuine extension of second-order ZFC.

We have managed to give a formal semantics for the second-order language of set theory without expanding our ontology to include classes that are not sets. The obvious alternative is to invoke the existence of proper classes. One can then tinker with the definition of a standard model so as to allow for a model with the (proper) class of all sets as its domain and the class of all ordered-pairs $\langle x, y\rangle$ (for x an element of y) as its interpretation function.[14] The existence of such a model is in fact all it takes to render the truth of a sentence of the language of set theory an immediate consequence of its validity.

One difficulty with this move is that it requires us to countenance the existence of proper classes.[15] Another concerns the *instability* of the semantics that results. For once one takes the existence of proper classes at face value, class theory takes center stage, and one must acknowledge that there is as much reason to provide a semantics for the language of class theory as there is for the language of second-order set theory. One may be tempted to postulate the existence of collections more encompassing than classes. One could then use 'superclasses' to give a model theory for the first-order

[14]This sort of account is developed in Shapiro (1991), Section 6.1. See also Shapiro (1987). Shapiro leaves open the question of whether talk of classes is to be taken literally.

[15]This is provided that one takes talk of proper classes literally; that is, one takes it to involve singular reference to set-like entities other than sets. An alternative to this would be, for example, to understand talk of classes in terms of plural reference to sets in which case the move just described would collapse into a version of our own proposal. The view that talk of classes is best understood in terms of plural reference to sets is defended in Uzquiano, 'A No-Class Theory of Classes.'

theory of classes. But this is only to postpone the problem. It will arise again as soon as one tries to give a model theory for the language of superclass theory.

What is worse, this sort of move is of no help at all if one tries to give a semantics for a language whose variables range over *all* the class-like entities there are, not just those lying below some level or other of a hierarchy of more and more encompassing collections.

The semantics we have developed faces an analogous instability. The cost of avoiding ontological expansion is 'ideological' expansion. In order to obtain a semantics for the language of second-order ZFC we had to move into the realm of third-order logic, by introducing a satisfaction predicate that takes first-order predicates as arguments. In a similar way, we would be forced to resort to an even higher-order satisfaction predicate in order to give a semantics for a language augmented with a predicate that takes first-order predicates as arguments. The situation is quite general. When ontological expansion is avoided and reflection principles are absent, the *logical resources* that are needed to produce a model theory for a given language are strictly greater than the logical resources of that language. This is problematic because there is no guarantee that the use of such logical resources can be made legitimate. In particular, it is doubtful that they can be interpreted in terms of English locutions we antecedently understand.

It is a fact of life that higher-order languages are unstable in the above sense. The present proposal does not tell us how to address this situation. But it does show how much can be done with the logical resources that the apparatus of plural quantification and plural predication makes available.[16]

We have stressed the implicit character of our definition of satisfaction. But it should be mentioned that our implicit definition of satisfaction can be transformed into an explicit one if we help ourselves to quantification over predicates that take first-order predicates as arguments, that is, if we help ourselves to third-order quantifiers. Let $\Sigma(\mathbf{R})$ be the result of conjoining axioms (0′)–(7′) and replacing the satisfaction

[16]For an interesting discussion of issues relating to instability, see Weir (1998), section 5.

predicate by a suitable third-order variable '**R**'.[17] We may then say that S satisfies ϕ with respect to M iff $\forall \mathbf{R}[\Sigma(\mathbf{R}) \rightarrow \mathbf{R}(S, \phi, M)]$ holds. Unfortunately, the apparatus of plurals does not seem to provide us with the resources necessary to understand third-order quantification. An interpretation of third-order quantification in terms of English non-nominal quantification is set forth in Rayo and Yablo (forthcoming), but it is sure to be somewhat controversial.

We should like to conclude by reporting three comforting results concerning our implicit definition of satisfaction.

The first result shows that the axioms that implicitly define satisfaction uniquely pin down its extension. Suppose that suitable versions of axioms (0′)–(7′) hold of the predicates 'S satisfies$_1$ ϕ with respect to M' and 'S satisfies$_2$ ϕ with respect to M'. Then, for every formula ϕ, every model M, and every variable assignment S, S satisfies$_1$ ϕ with respect to M just in case S satisfies$_2$ ϕ with respect to M. The proof of this result is a straightforward induction on the complexity of formulas.

The second result is the derivability of all instances of Tarski's schema T. A little symbol manipulation should convince the reader that if ϕ is a sentence of the language of second-order set theory and 'p' is a translation of ϕ into the metalanguage, then

$$\phi \text{ is true} \leftrightarrow p$$

is a derivable consequence of our definitions.

The third and last result is just that our semantics sanctions common deductive systems for second-order languages. More precisely, given a standard axiomatic system for second-order logic (e.g., the system indicated in Frege's *Begriffsschrift*), it can be shown that if ϕ is a sentence of the language of second-order set theory and Γ is a set of such sentences, then ϕ is a superconsequence of Γ if it is a deductive consequence of Γ. The proof proceeds by verifying the supervalidity of the deductive axioms and the fact that the rules of inference preserve supervalidity.

[17]Since '**R**' is to take the place of the satisfaction predicate, it must be a three-place third-order variable taking second-order variables in its first and third argument places and a first-order variable in its second argument place.

It is a consequence of Gödel's Incompleteness Theorem that we cannot hope for a converse of this proposition. Given any recursively axiomatizable axiom system for second-order logic, we know how to construct second-order sentences that are supervalid but not provable. The proof of this result is analogous to the incompleteness proof for full second-order logic.[18]

[18]For a proof of the incompleteness of full second-order logic see section 4.2 of Shapiro (1991)

Chapter 3

Frege's Unofficial Arithmetic

In *The Foundations of Arithmetic*, Frege held the view that number-terms refer to objects.[1] Later in his life, however, he seems to have been open to other possibilities:

> Since a statement of number based on counting contains an assertion about a concept, in a logically perfect language a sentence used to make such a statement must contain two parts, first a sign for the concept about which the statement is made, and secondly a sign for a second-order concept. These second-order concepts form a series and there is a rule in accordance with which, if one of these concepts is given, we can specify the next. But still we do not have in them the numbers of arithmetic; we do not have objects, but concepts. How can we get from these concepts to the numbers of arithmetic in a way that cannot be faulted? Or are there simply no numbers in arithmetic? Could the numbers help to form signs for these second-order concepts, and yet not be signs in their own right?[2]

To illustrate Frege's point, let us consider the statement of number 'there are three cats'. It might be paraphrased in a first-order language as:[3]

(1) $(\exists_3 x)[\textsc{Cat}(x)]$.

[1]This is reflected in his definition of number. See, for instance §67.

[2]*Notes for Ludwig Darmstaedter*, pp. 366-7. I have substituted 'second-order' for 'second-level'.

[3]As usual, '$(\exists_1 x)[\varphi(x)]$' is defined as '$\exists x(\varphi(x) \wedge \forall y(\varphi(y) \rightarrow x = y))$', and (for $n > 1$) '$(\exists_n x)[\varphi(x)]$' is defined as '$\exists x(\varphi(x) \wedge (\exists_{n-1} y)[\varphi(y) \wedge y \neq x])$'.

If its logical form is to be taken at face value, (1) can be divided into two main logical components: first, the predicate 'CAT(...)', which for Frege refers to the (first-order) concept *cat*; and, second, the quantifier-expression '$(\exists_3 x)[\ldots(x)]$', which for Frege refers to a second-order concept (specifically, the second-order concept which is true of the first-order concepts under which precisely 3 objects fall).[4] Significantly, Frege would regard neither of these components as referring to an *object*.

Let us now consider a close cousin of 'there are three cats', namely, 'the number of the cats is three'. This sentence might be paraphrased as:

(2) the number of the cats = 3.

If its logical form is to be taken at face value, (2) cannot be divided into a predicate and a quantifier-expression, like (1). Instead, Frege would take 'the number of the cats' and '3' to be *names*, referring to numbers (which he regarded as objects).

Frege saw a deep connection between sentences like (1)—in which something is predicated of a *concept*—and sentences like (2)—in which something is predicated of the *number* associated with that concept. An effort to account for this connection was a main theme in his philosophy of arithmetic. But, after the discovery that Basic Law V leads to inconsistency, he found much reason for dissatisfaction with his original proposal. As evidenced by the quoted passage, he no longer felt confident about the possibility of getting from concepts to their numbers 'in a way that cannot be faulted'.

Towards the end of the passage, Frege considers an alternative: the view that there really are no numbers in arithmetic, and that—appearances to the contrary—numerals are not names of objects. They do not even instantiate a legitimate logical category, they are merely *orthographic* components of expressions standing for second-order concepts. The grammatical form of a sentence like (2) is therefore not indicative of its logical form. Presumably, 'the number of the cats = 3' is to be divided into two main logical components. First, the expression '... cats', which refers to the (first-

[4]For Frege, a first-order concept is a concept that takes objects as arguments, and an $(n+1)$th-order concept is a concept that takes nth-order concepts as arguments. See Frege (1964), §21. Unless otherwise noted, we shall use 'concept' to mean '*first-order* concept'.

order) concept *cat*; and, second, the expression 'the number of the ... = 3', which refers to a second-order concept (specifically, the second-order concept which is true of the first-order concepts under which precisely 3 objects fall). The numeral '3' is merely an orthographic component of 'the number of the ... = 3', in much the same way that 'cat' is an orthographic component of 'caterpillar'. The outermost logical form of (2) is therefore identical to that of (1). If, in addition, it turns out that the logical form of 'the number of the ... = 3' corresponds to that of '$(\exists_3 x)[\ldots(x)]$', then the logical form of (1) is identical to that of (2).

It is unfortunate that Frege never spelled out his *unofficial* proposal (as we shall call it) in any detail. In particular, he said nothing about how first-order arithmetic might be understood. Luckily, Harold Hodes has developed and defended a version of the Unofficial Proposal.[5] On Hodes's reconstruction, a sentence '$\mathrm{F}(n)$' of the language of first-order arithmetic is to be regarded as abbreviating a higher-order sentence '$(\mathsf{F}X)((\exists_n x)[Xx])$', where '$(\exists_n x)[\ldots x]$' is a quantifier-expression, and '$(\mathsf{F}X)(\ldots X \ldots)$' refers to a *third*-order concept. For instance, the first-order sentence 'PRIME(19)' abbreviates a certain higher-order sentence '$(\mathsf{Prime}\, X)((\exists_{19} x)[Xx])$'.

Hodes does not explicitly tell us how to deal with quantified sentences. But he makes it sound as though they would involve quantification over second-order concepts.[6] More specifically, they would involve quantification over *finite cardinality object-quantifiers*: the referents of quantifier-expressions of the form '$(\exists_n x)[\ldots x]$'. Thus, the first-order '$\exists z$PRIME(z)' would abbreviate the result of replacing the position occupied by '$(\exists_{19} x)[\ldots x]$' in '$(\mathsf{Prime}\, X)((\exists_{19} x)[Xx])$' by a variable ranging over finite cardinality object-quantifiers, and binding the new variable with an initial existential quantifier.

If this is right, Hodes's account of first-order arithmetic requires *third*-order quantification. And the obvious extension to second-order arithmetic would call for *fourth-*

[5]See his Hodes (1984)

[6]He says, for instance, "In making what appears to be a statement about numbers one is really making a statement primarily about cardinality object-quantifiers; what appears to be a first-order theory about objects of a distinctive sort really is an encoding of a fragment of third-order logic." See Hodes (1984) p. 143.

order quantification. These are pretty extravagant logical resources.

Here we shall see that more modest resources will do. We will develop a version of the Unofficial Proposal within *second*-order logic, and show that it can be used to account for first- and second-order arithmetic. This, in itself, is a surprising result. But it is especially important in light of the fact that, although the use of higher-order languages is often considered problematic, recent work has done much to assuage concerns about certain second-order resources.[7]

3.1 A Transformation

To simplify our presentation, we will proceed on the assumption that the positive integers exist. (In section 3.4 we drop this assumption.) It will also be convenient to begin by taking second-order quantifiers to range over (first-order) concepts, as Frege did.

We will work within a second-order language L, with a suitably unrestricted domain (such as our current domain of discourse). In particular, we assume that the domain of L includes the positive integers. Besides the standard variables and predicates,[8] we take L to contain a specialized arithmetical vocabulary. It is to include the first-order arithmetical variables 'm_1', 'm_2', etc., whose range is restricted to the

[7]See especially Boolos (1984), Boolos (1985a) and Boolos (1985b). See also McGee (2000) and Rayo and Yablo (forthcoming).

[8]We take L to contain first-order variables 'x_1', 'x_2', ... and, for n a positive integer, n-place second-order variables 'V_1^n', 'V_2^n', ... As a precaution against variable clashes, we divide our monadic second-order variables in two: the 'V_{2i-1}^1'—which we abbreviate 'Z_i'—will be paired with first-order arithmetical variables; the 'V_{2i}^1'—which we abbreviate 'X_i'—will be used for more general purposes. Similarly, we divide our dyadic second-order variables in two: the 'V_{2i-1}^2'—which we abbreviate 'R_i'—will be paired with second-order arithmetical variables; the 'V_{2i}^2'—which we abbreviate 'R_i^2'—will be used for more general purposes. Finally, when $n > 2$ we use to 'R_i^n' abbreviate 'V_i^n'. Also to avoid variable clashes, we will sometimes appeal to the introduction of unused variables. We employ 'w', 'v' and 'u' as unused first-order variables, 'W', 'V' and 'U' as unused monadic second-order variables, and, for each n (to be determined by context), we employ 'R' as an unused n-place relation variable. (It is worth noting that appeal to unused variables could be avoided by renumbering subscripts.) It will often be convenient regard 'x', 'y', and 'z' as arbitrary first-order variables and 'X', 'Y' and 'Z' as arbitrary (monadic) second-order variables.

L also contains the following: (a) the identity symbol '='; (b) predicate-letters 'P_i^n'; (c) logical connective-symbols '$\neg$' and '$\wedge$' (with '$\vee$', '$\rightarrow$' and '$\leftrightarrow$' defined in the usual way); (d) the quantifier-symbol '$\exists$' (with '$\forall$' defined in the usual way); and (e) parenthesis.

positive integers, together with the arithmetical predicates '$1^N(m_i)$', '$S^N(m_i, m_j)$', '$+^N(m_i, m_j, m_k)$' and '$\times^N(m_i, m_j, m_k)$', interpreted in the obvious way.[9] We shall say that a formula is part of the *first-order arithmetical fragment* of L, A^1, just in case it contains no non-arithmetical predicates and all of its variables are first-order arithmetical variables. (It is easy to verify that A^1 is a notational variant of the language of first-order arithmetic.)

Finally, we introduce some definitions, all of which are couched in purely logical vocabulary:

(D1) $X \approx Y \ \equiv_{df} \ \exists R\ [\forall w(Xw \rightarrow \exists! v(Yv \wedge Rwv)) \wedge \forall w(Yw \rightarrow \exists! v(Xv \wedge Rvw))]$;

(The objects falling under X are as many as the objects falling under Y)

(D2) $\Sigma(X, Y) \ \equiv_{df} \ \exists w[\neg Xw \wedge \forall v(Yv \leftrightarrow (Xv \vee v = w))]$;

(The objects falling under Y are the objects falling under X and one more)

(D3) $\mathbf{F}(X) \ \equiv_{df} \ \exists w\ (Xw) \ \wedge \ \neg\exists W(\Sigma(W, X) \wedge W \approx X)$;

(There are finitely many objects falling under X)

(D4) $\exists^F X\ (\ldots) \ \equiv_{df} \ \exists X\ (\mathbf{F}(X) \ \wedge \ \ldots)$;

(There is a concept X such that there are finitely many objects falling under X and ...)

(D5) $\mathbf{1}(X) \ \equiv_{df} \ \exists! w(Xw)$;

(There is precisely one object falling under X)

(D6) $\mathbf{S}(X, Y) \ \equiv_{df} \ \forall W(\Sigma(X, W) \rightarrow Y \approx W)$;

(The objects falling under Y succeed the objects falling under X in number)

(D7) $+(X, Y, Z) \ \equiv_{df} \ \forall W[(W \approx Y \ \wedge \ \forall u(Wu \rightarrow \neg Xu)) \rightarrow$
$\forall V \forall u(Vu \leftrightarrow (Xu \vee Wu)) \rightarrow V \approx Z)]$;

[9] On the intended interpretation, '$1^N(m_i)$' is true iff m_i is the number 1, '$S^N(m_i, m_j)$' is true iff the successor of m_i is m_j, '$+^N(m_i, m_j, m_k)$' is true iff the sum of m_i and m_j is m_k, and '$\times^N(m_i, m_j, m_k)$' is true iff the product of m_i and m_j is m_k.

(D8) $\times(X, Y, Z) \equiv_{df} \exists R\, [\forall w \forall v((Xw \wedge Yv) \rightarrow \exists! u(Zu \wedge Rwvu)) \wedge$
$\forall u(Zu \rightarrow \exists! w \exists! v(Xw \wedge Yv \wedge Rwvu))]$,

Our simplified version of the Unofficial Proposal is based on the observation that every formula of A^1 can be transformed into a purely logical formula of L, as follows:

- $Tr(\neg\psi) = \neg Tr(\psi)$;
- $Tr(\psi \wedge \theta) = (Tr(\psi) \wedge Tr(\theta))$;
- $Tr(\exists m_i\ (\psi)) = \exists^F Z_i\ (Tr(\psi))$;
- $Tr(m_i = m_j) = Z_i \approx Z_j$;
- $Tr(1^N(m_i)) = \mathbf{1}(Z_i)$;
- $Tr(S^N(m_i, m_j)) = \mathbf{S}(Z_i, Z_j)$;
- $Tr(+^N(m_i, m_j, m_k)) = +(Z_i, Z_j, Z_k)$;
- $Tr(\times^N(m_i, m_j, m_k)) = \times(Z_i, Z_j, Z_k)$.

As an example, consider '$\forall n \forall m(n + m = m + n)$'. It can be expressed in A^1 as:

$$\forall m_1\ \forall m_2\ \forall m_3\ \forall m_4\ (+^N(m_1, m_2, m_3)\ \wedge\ +^N(m_2, m_1, m_4) \rightarrow m_3 = m_4);$$

and the result of applying our transformation is:

$$\forall^F Z_1 \forall^F Z_2 \forall^F Z_3 \forall^F Z_4\ (+(Z_1, Z_2, Z_3)\ \wedge\ +(Z_2, Z_1, Z_4) \rightarrow Z_3 \approx Z_4).$$ [10]

In general, the connection between φ and $Tr(\varphi)$ can be characterized as follows:

> **[Con]** Suppose $\varphi(m_i)$ is a formula of A^1 and let $\psi(Z_i)$ be $Tr(\varphi(m_i))$. If there are finitely many Fs, then $\varphi(m_i)$ is true of the number of the Fs just in case $\psi(Z_i)$ is true of the Fs.[11]

[10] I assume that '$\forall^F Z_i\ (\varphi)$' abbreviates '$\neg\exists^F Z_i\ (\neg\varphi)$'.

[11] More generally, suppose $\varphi(m_{i_1}, \ldots, m_{i_n})$ is a formula of A^1 and let $\psi(Z_{i_1}, \ldots, Z_{i_n})$ be $Tr(\varphi(m_{i_1}, \ldots, m_{i_n}))$; suppose, moreover, that there are finitely many F_1s, finitely many F_2s, ..., and finitely many F_ns. Then $\varphi(m_{i_1}, \ldots, m_{i_n})$ is true when m_{i_1} is the number of the F_1s, m_{i_2} is the number of the F_2s, ..., and m_{i_n} is the number of the F_ns just in case $\psi(Z_{i_1}, \ldots, Z_{i_n})$ is true when the Z_{i_1}s are the F_1s, the Z_{i_2}s are the F_2s, ..., and the Z_{i_n}s are the F_ns.

The proof of this result is a straightforward induction on the complexity of formulas, and relies on the following three assumptions:

- *Hume's Principle*: if there are finitely many Fs and finitely many Gs, then '$m_i = m_j$' is true of the number of the Fs and the number of the Gs just in case '$Z_i \approx Z_j$' is true of the Fs and the Gs;
- if there are finitely many Fs, then '$1^N(m_i)$' is true of the number of the Fs just in case '$\mathbf{1}(Z_i)$' is true of the Fs;
- if there are finitely many Fs and finitely many Gs, then '$S^n(m_i, m_j)$' is true of the number of the Fs and the number of the Gs just in case '$S^n(Z_i, Z_j)$' is true of the Fs and the Gs.

It is essential that the domain of L include infinitely many objects.[12] Because of this, Tr does not reduce first-order arithmetic to pure second-order logic. But it comes close: to every truth of first-order arithmetic it maps a second-order consequence of the axiom of infinity.

It is possible to extend Tr so that it encompasses *second-order* arithmetic, by making use of a certain kind of coding.[13] Let us enrich L with second-order arithmetical variables 'M_1', 'M_2', etc. We shall say that a formula is part of the *second-order arithmetical fragment* of L, A^2, just in case it contains no non-arithmetical predicates and all of its variables are first- or second-order arithmetical variables. (It is easy to verify that A^2 is a notational variant of the language of second-order arithmetic.) Our transformation can be made to encompass A^2 by way of the following two clauses:[14]

- $Tr(\exists M_i\ (\varphi)) = \exists R_i\ (Tr(\varphi))$;

[12] Suppose, for instance, that the domain of L contained only one object. Then it will be true that

$$\forall^F Z_1\ \forall^F Z_2\ (\mathbf{1}(Z_1) \wedge \mathbf{S}(Z_1, Z_2) \longrightarrow Z_1 \approx Z_2),$$

even though this is the result of applying our transformation method to a version of '$1 = 2$' in A^1. Clearly, analogous problems arise when the domain of L contains any finite number of objects.

[13] Thanks to Vann McGee for pointing this out to me.

[14] Polyadic second-order quantification can be defined as monadic second-order quantification over sequences, which can be simulated within first-order arithmetic.

- $Tr(M_i m_j) = \exists W[Z_j \approx W \wedge \exists v \forall u(R_i(v,u) \leftrightarrow Wu)]$.

Our coding works by representing each arithmetical concept M_i by a dyadic relation R_i. Specifically, we represent the fact that a number m_j falls under M_i by having it be the case that some concept W under which precisely m_j objects fall be such that some individual v bears R_i to all and only the individuals falling under W. As one might expect, the extended transformation yields a suitable strengthening of [Con] for A^2.

We are finally in a position to state our simplified version of the Unofficial Proposal: *every formula of A^2 is to be eliminated in favor of its transformation.*

3.2 Impure Arithmetic

In section 3.4 we will have more to say about what the status of this 'elimination' might be. For now, let us return to Frege's starting point and focus our attention on *impure* arithmetic. Consider, for example, 'the cats are as many as the dogs'. The Frege of the *Foundations* would have us paraphrase it as

> The number belonging to the concept *cat* is the number belonging to the concept *dog*;

where 'the number belonging to the concept *cat*' and 'the number belonging to the concept *dog*' refer to *numbers*. On the Unofficial Proposal, this sentence should be eliminated in favor of 'The objects falling under the concept *cat* are as many as objects falling under the concept *dog*', or:

$$\hat{x}\ [\textsc{Cat}(x)] \approx \hat{x}\ [\textsc{Dog}(x)].$$

Syntactically, an expression of the form '$\hat{x}\ [\varphi(x)]$' takes the place of a monadic second-order variable. But the result of substituting '$\hat{x}\ [\varphi(x)]$' for 'Y' in a formula '$\Psi(Y)$' is to be understood as shorthand for:

$$\exists W\ (\forall x(W\,x \leftrightarrow \varphi(x)) \rightarrow \Psi(W)).$$

Consider now the *mixed* sentence 'the number of the cats is 3'. From the perspective of the *Foundations*, it might be paraphrased as:

> The number belonging to the concept *cat* is three.

But, on the Unofficial Proposal, this sentence should be eliminated in favor of:

$$\mathbf{3}(\hat{x}\ [\text{CAT}(x)]);$$

where numeral-predicates are defined in the obvious way:

- $\mathbf{2}(X) \equiv_{df} \forall^F Y(\mathbf{S}(Y, X) \rightarrow \mathbf{1}(Y))$;
- $\mathbf{3}(X) \equiv_{df} \forall^F Y(\mathbf{S}(Y, X) \rightarrow \mathbf{2}(Y))$;
- etc.

This sort of approach towards impure arithmetic can easily be generalized. Let us enrich L with a predicate '$\mathbf{N}(X_i,\ m_j)$' which takes a standard second-order variable in its first argument-place and an arithmetical first-order variable in its second argument-place. On its intended interpretation, '$\mathbf{N}(X_i,\ m_j)$' is true just in case m_j is the number of the objects falling under X_i. Consider 'The number belonging to the concept *cat* is three' as an example. It can now be formalized in L as:[15]

(3) $\exists m_1(\mathbf{N}(\hat{x}_1\ [\text{CAT}(x_1)],\ m_1)\ \wedge\ 3^N(m_1))$.

We can make our transformation encompass impure arithmetic by extending it with the following clauses:

- $Tr(\mathbf{N}(X_i,\ m_j)) = X_i \approx Z_j$;
- $Tr(\exists x_i\ (\varphi)) = \exists x_i\ (Tr(\varphi))$;

[15]In analogy with the above, we let '$3^N(m_1)$' be shorthand for:

$$\forall m_i \forall m_j\ ((1^N(m_i) \wedge S^N(m_i, m_j)) \rightarrow S^N(m_j, m_1)),$$

for 'm_i' and 'm_j' unused arithmetical variables. Other numeral-predicates in the arithmetical fragment of L are to be given a similar treatment.

- $Tr(\exists X_i\ (\varphi)) = \exists X_i\ (Tr(\varphi))$;
- $Tr(X_i x_j) = X_i x_j$;
- $Tr(\exists R^n_i\ (\varphi)) = \exists R^n_i\ (Tr(\varphi))$;
- $Tr(R^n_i(x_{j_1}, \ldots, x_{j_{n+1}})) = R^n_i(x_{j_1}, \ldots, x_{j_{n+1}})$;
- $Tr(x_i = x_j) \ = \ x_i = x_j$;
- $Tr(\mathrm{P}^n_j(x_{i_1}, \ldots, x_{i_n})) = \mathrm{P}^n_j(x_{i_1}, \ldots, x_{i_n})$.

As an example of the extended transformation, note that Tr converts (3) to:

$$\exists^F Z_1(\hat{x}_1\ [\textsc{Cat}(x_1)] \approx Z_1\ \wedge\ \mathbf{3}(Z_1));$$

or, equivalently:

$$\mathbf{3}(\hat{x}_1\ [\textsc{Cat}(x_1)]).$$

For further illustration, note that 'the number belonging to the concept *cat* is the number belonging to the concept *dog*' can be formalized in L as:

$$\exists m_1\ [\mathbf{N}(\hat{x}_1\ [\textsc{Cat}(x_1)],\ m_1) \wedge (\mathbf{N}(\hat{x}_1\ [\textsc{Dog}(x_1)],\ m_1)].$$

which Tr converts to:

$$\exists^F Z_1\ [\hat{x}_1\ [\textsc{Cat}(x_1)] \approx Z_1 \wedge \hat{x}_1\ [\textsc{Dog}(x_1)] \approx Z_1],$$

or, equivalently:

$$\hat{x}_1\ [\textsc{Cat}(x_1)] \approx \hat{x}_1\ [\textsc{Dog}(x_1)].$$

As one would expect, the extension of Tr yields a suitable strengthening of [Con].

It is worth emphasizing that our transformation is not defined for every formula of L. Notably, it is undefined for *mixed* identity statements '$x_i = m_j$'. This is as it should be. The view that numbers are objects lead Frege to the uncomfortable question of whether the number belonging to the concept *cat* is, for instance, Julius

Caesar. But on the Unofficial Proposal, such questions never arise, because number-terms do not refer to objects. 'The number belonging to the concept *cat* is the number belonging to the concept *dog*' is to be eliminated in favor of 'the concept *cat* is equinumerous with the concept *dog*', and 'the number belonging to the concept *cat* is 3' is to be eliminated in favor of 'the concept *cat* is equinumerous with any concept under which three objects fall'.

The question whether Julius Caesar is the number belonging to the concept *cat* isn't only uncomfortable because it appears to be nonsensical. It also underscores a problem Paul Benacerraf made famous, that if mathematical terms refer to objects, then nothing in our mathematical practice determines *which* objects they refer to.[16] A remarkable feature of the Unofficial Proposal is that it avoids Benacerraf's Problem altogether. It would, however, be a mistake to conclude from this that the Unofficial Proposal is the last word on Benacerraf's Problem, since the inscrutability of reference pervades far beyond arithmetic.

One would like to be able to count cats. For that purpose we introduced the formula '$\mathbf{N}(\hat{x}_i\ [\textsc{Cat}(x_i)], m_j)$' of L. We have seen that our transformation is defined for this formula, and that it can be eliminated on the Unofficial Proposal. But one would also like to be able to count *numbers*. One would like to say, for example, that the number of numbers falling under the concept *prime smaller than ten* is four. And, unfortunately, our transformation is not defined for a formula such as '$\mathbf{N}(\hat{m}_i\ [\textsc{Prime-less-than-}10(m_i)], m_j)$'.[17] To make up for the loss, we introduce a new predicate '$\mathbf{NN}(M_i, m_j)$', by appealing to the same sort of coding that allowed our transformation to encompass second-order arithmetical quantifiers.[18] Informally, '$\mathbf{NN}(M_i, m_j)$' is to abbreviate a formula of L to the effect that there is a binary relation R with the following properties:

[16]See Benacerraf (1965).

[17]In analogy with the above, we let the result of substituting '$\hat{m}_i\ [\varphi(m_i)]$' for '$M_j$' in a formula '$\Psi(M_j)$' be shorthand for

$$\exists M_k\ (\forall m_i(M_k\, m_i \leftrightarrow \varphi(m_i)) \rightarrow \Psi(M_i)).$$

for 'M_k' an unused variable.

[18]Thanks to Vann McGee for pointing this out to me.

- For any number n, $M_i n$ holds just in case some member of the domain of R is paired with exactly n objects;
- every member of the domain of R is paired with finitely many objects;
- for any x and y in the domain of R, if the objects paired with x are as many as the objects paired with y, then $x = y$;
- the domain of R contains exactly m_j objects.[19]

The new predicate allows us to say that the number of numbers falling under the concept *prime smaller than ten* is four. It also allows us to say that the number of numbers falling under the concept *prime smaller than six* is the number of objects falling under the concept *cat*:

$$\exists m_2\ (\mathbf{NN}(\hat{m}_1\ [\textsc{Prime-less-than-6}(m_1)], m_2)\ \wedge\ \mathbf{N}(\hat{x}_1\ [\textsc{Cat}(x_1)], m_2))$$

And, as desired, our transformation is defined for any formula '$\mathbf{NN}(M_i, m_j)$'.

We are now in a position to state a more general version of the Unofficial Proposal: *every formula of L for which Tr is defined should be eliminated in favor of its transformation.*

3.3 Interpreting Second-Order Languages

There is a steep price to be paid for the Unofficial Proposal in its present form: one is driven into the mysterious realm of Fregean Concepts. Fortunately, we have taken care to ensure that the outputs of our transformation are always formulas of L. So

[19]More precisely, '$\mathbf{NN}(M_i, m_j)$' is to abbreviate:

$$\begin{aligned}&\exists R\ [\forall m_k\ (M_i m_k \leftrightarrow \exists w \exists W\ (\forall v\ (Wv \leftrightarrow Rwv) \wedge \mathbf{N}(W, m_j))) \wedge \\ &\forall w \forall v\ (Rwv \rightarrow \exists^F W \forall u\ (Wu \leftrightarrow Rwu)) \wedge \\ &\forall w \forall v \forall W \forall V\ ((\exists u\ (Rwu) \wedge \forall u\ (Wu \leftrightarrow Rwu) \wedge \forall u\ (Vu \leftrightarrow Rvu) \wedge W \approx V) \rightarrow w = v) \wedge \\ &\exists W\ (\forall v\ (Wv \leftrightarrow \exists u\ (Rvu)) \wedge \mathbf{N}(W, m_j))\,];\end{aligned}$$

for m_k an unused variable.

talk of Fregean Concepts can be avoided simply by parting company with Frege in our interpretation of second-order logic.

Not anything will do. On Quine's interpretation, second-order logic is 'set-theory in sheep's clothing'. So one can only succeed in the task of eliminating number-terms from arithmetic if one has previously succeeded in the far more difficult task of eliminating *set*-terms from set-theory. Nor is any progress made by interpreting second-order logic as Boolos has suggested.[20] Our definitions (D1) and (D8) make essential use of *polyadic* second-order quantifiers, which Boolos treats as ranging (plurally) over *ordered n-tuples*. But ordered-pair-terms are presumably no less problematic than numbers-terms.

Some deviousness is needed to avoid Fregean Concepts without betraying the spirit of the Unofficial Proposal. One way of doing so is by defining second-order quantifiers implicitly, in terms of an *open-ended* schema, as in McGee's 'Everything'. Another is by interpreting second-order logic as in Rayo and Yablo's 'Nominalism through De-Nominalization'. Alternatively, one might argue that second-order logic is to be accepted as primitive, and taken at face value.

If one is skeptical towards these approaches, but endorses Boolos's work on plural quantifiers, a slightly different strategy suggests itself. We retain *monadic* second-order quantifiers, interpreted in a Boolosian fashion: '$\exists X$' is read 'there are some objects$_X$ such that', and 'Xy' is read 'it$_y$ is one of them$_X$'.[21] But we give up *polyadic* second-order quantifiers. To make up for the loss, we introduce atomic *plural* predicates: atomic predicates taking second-order variables as arguments.[22]

Instead of using definitions (D1) and (D8) above we now treat '$X \approx Y$' and '$\times(X, Y, Z)$' as atomic plural predicates. The former is to be interpreted by appeal to our pre-theoretic understanding of the English predicate '... are as many as ...'. Thus, as a first approximation, '$\exists X \exists Y (X \approx Y)$' might be read:

There are some objects$_X$ and some objects$_Y$ such that they$_X$ are as many

[20]See Boolos (1984) and Boolos (1985a).

[21]Boolos's original proposal incorporates a complication in order to allow for second-order variables to take 'empty' values (see Boolos (1984)). Here we shall not require this complication.

[22]I offer a detailed defense of plural predicates in chapter 1.

as them_Y.

But this is not exactly what we want. For when there are infinitely many Fs or infinitely many Gs, our pre-theoretic grasp of 'the Fs are as many as the Gs' is somewhat vague. It will do, for present purposes, to interpret '$X \approx Y$', not as 'they_X are as many as them_Y', but as the slightly more cumbersome

either they_Y are not finite in number, or they_X are as many as them_Y;

where '... are as many as ...' and '... are finite in number' are understood in accordance with their pre-theoretic readings.

An interpretation for '$\times(X, Y, Z)$' is provided implicitly, by way of the following axioms:

(A1) $\forall^F X \forall^F Y \forall^F Z\ [\mathbf{1}(Y) \rightarrow (\times(X, Y, Z) \leftrightarrow Z \approx X)]$;

(A2) $\forall^F X \forall^F Y \forall^F Z \forall^F W\ [\mathbf{S}(W, Y) \rightarrow (\times(X, Y, Z) \leftrightarrow$
$\forall^F V(\times(X, W, V) \rightarrow \forall^F U(+(V, X, U) \rightarrow Z \approx U)))]$.

We may then apply our transformation as before, and prove a suitable version of [Con]. It should be noted, however, that the present approach is unable to accommodate the coding techniques that allowed us to account for second-order arithmetic and to define the predicate '$\mathbf{NN}(M_\imath, m_\jmath)$'.

We have seen that Frege's Unofficial Proposal can be developed with surprisingly modest logical resources: full second-order quantification, or monadic second-order quantification together with atomic plural predicates. Let us now turn to the question of what it might be used to accomplish.

3.4 Applications

The Unofficial Proposal—the view that number-statements are to be eliminated in favor of their transformations—can take several different forms, depending on the sort of elimination one has in mind. On an approach like Hodes's, number-statements are taken to *abbreviate* their transformations. As a result, number-terms do not refer to objects, and there is room for rejecting the existence of numbers altogether. The Unofficial Proposal might therefore provide a basis for a nominalist philosophy of arithmetic.

It should be noted, however, that if there are only finitely many objects, then $Tr(\varphi)$ will not always have the truth-value that φ receives on its standard interpretation.[23] In order to avoid infinity assumptions, a nominalist might claim that a number-statement φ abbreviates 'necessarily, $(\xi \rightarrow Tr(\varphi))$' (where '$\xi$' is a sentence stating that there are infinitely many objects, such as '$\exists X\ (\neg\mathbf{F}(X)\ \wedge\ \exists x\ Xx)$'). On the plausible condition that it is *possible* for there to be infinitely many objects, 'necessarily, $(\xi \rightarrow Tr(\varphi))$' is true if and only if φ is true on its standard interpretation.[24]

A different approach towards the Unofficial Proposal might serve the purposes of the Neo-Fregean Program, championed by Crispin Wright and Bob Hale. Neo-Fregeans believe that Hume's Principle allows us to *reconceptualize* the state of affairs which is described by saying that the Fs are as many as the Gs, and that, on the reconceptualization, that same state of affairs is rightly described by saying that the *number* of the Fs is the *number* of the Gs.[25] A version of the Unofficial Proposal might allow Neo-Fregeans to make the more general claim that every number-statement φ describes—on the appropriate reconceptualization—the state of affairs which is otherwise described by $Tr(\varphi)$.

Even if the Unofficial Proposal is to be abandoned altogether, it would be a mistake to neglect the connection between number-statements and their transformations. Any account of arithmetic that takes number-terms to refer to numbers must yield an account of how *applied* arithmetic is possible. Specifically, it must provide an explanation of how our account of the natural world can bear upon our knowledge of number-statements, and an explanation of how our knowledge of number-statements can bear upon our account of the natural world. But [Con] provides us with just such

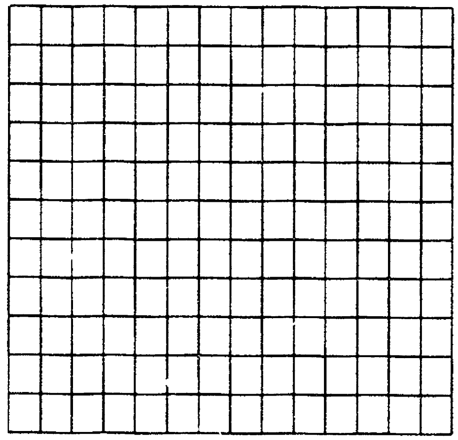
[23]See footnote 12.

[24]For more on modal strategies, see part II of Burgess and Rosen (1997). Hodes discusses a modal strategy in section III of Hodes (1984).

[25]See Wright (1997), section I.

an explanation. The following example should make this clear.

Let us suppose that the tiles on my kitchen floor are arranged as in the diagram below, and that I wish to know how many of them there are.



Let us say that the *rows* of tiles on my kitchen floor are the Rs, that the *columns* of tiles are the Cs, and that the tiles themselves are the Ts. By studying my kitchen floor, I can put myself in a position to conclude that '$\times(Z_i, Z_j, Z_k)$' holds of the Rs, the Cs and the Ts. And, on the assumption that the relevant second-order resources can be shown to be unproblematic, it is no mystery that this should be so. Just as the first-order sentence '$\forall x$ WHALE$(x) \rightarrow$ MAMMAL(x)' can be part of an account of the natural world pertaining to the taxonomical characteristics of whales, the second-order sentence '$\times(\hat{x}[\mathrm{R}(x)], \hat{x}[\mathrm{C}(x)], \hat{x}[\mathrm{T}(x)])$' can be part of an account of the natural world pertaining to the tiles on my kitchen floor.

Enter [Con] and we can make the leap to statements about numbers. For [Con] tells us that the second-order predicate '$\times(Z_i, Z_j, Z_k)$' holds of the Rs, the Cs and the Ts just in case the number predicate '$\times^N(m_i, m_j, m_k)$' holds of the number of the Rs, the number of the Cs and the number Ts. So it explains how our account of the natural world can bear upon our knowledge of number-statements: our knowledge of the truth of '$\times(\hat{x}[\mathrm{R}(x)], \hat{x}[\mathrm{C}(x)], \hat{x}[\mathrm{T}(x)])$' gives rise to knowledge that the number of the Ts is the product of the number of the Rs and the number of the Cs.

In a similar way, our knowledge of the truth of the second-order '$\mathbf{11}(\hat{x}[\mathrm{R}(x)])$' and

'$\mathbf{14}(\hat{x}[\mathrm{C}(x)])$' gives rise to knowledge that the number of the Rs is 11 and that the number of the Cs is 14. By applying a simple computation, we may therefore come to know that number of the Ts is 154. But [Con] tells us that '$154^N(m_i)$' is true of the number of the Ts just in case '$\mathbf{154}(Z_i)$' is true of the Ts themselves. So it explains how the knowledge of a number-statement can bear upon our account of the natural world: our knowledge that the number of the Ts is 154 gives rise to the knowledge that '$\mathbf{154}(\hat{x}[\mathrm{T}(x)])$' is true and, hence, to the knowledge that there are 154 tiles on my kitchen floor.

A few points about this example deserve mention. First, it is not intended as a description of our actual practice. It is intended as a rational reconstruction of the ways in which the knowledge of number-statements can interact with our account of the natural world. Second, it should not be taken to suggest that arithmetical computations can only be carried out by appeal to numbers. In fact, the opposite is true: [Con] makes clear that the derivation of a number-statement can always be reproduced within the framework of second-order logic. Finally, our example presupposes that [Con] can be known to be true. Accordingly, it presupposes that the three assumptions upon which [Con] is based can be known to be true—Hume's Principle in particular. Here we have remained silent on the question of how these assumptions should be justified. Different accounts of arithmetic are likely to answer it in different ways.

The example can easily be generalized. Suppose $\varphi(m_i)$ is a formula of L for which Tr is defined, and let $\psi(Z_i)$ be its transformation. [Con] tells us that $\varphi(m_i)$ is true of the number of the Fs just in case $\psi(Z_i)$ is true of the Fs themselves. Thus, an account of the natural world which implies a statement to the effect that $\psi(Z_i)$ is true of the Fs can give rise to the knowledge that $\varphi(m_i)$ is true of the number of the Fs and, converescly, the knowledge that $\varphi(m_i)$ is true of the number of the Fs can lead us to enrich our account of the natural world with a statement to the effect that $\psi(Z_i)$ is true of the Fs.

We discussed two variants of Frege's Unofficial Proposal, and saw that they might serve the purposes of a nominalist philosophy of arithmetic and of the Neo-Fregean

Program. We also saw that, even if the Unofficial Proposal is ultimately rejected, it can be used to shed light on the interaction between our knowledge of number-statements and our account of the natural world.

Appendix

1 *Proof of* (BP)

Suppose (BP) is false. Then there are some ordered-pairs, the Gs, such that, given any Fs, the Gs map something onto the Fs. That is, given any Fs, there is an x such that, for every y, $\langle x, y \rangle$ is one of the Gs if and only if y is one of the Fs.

Say that y is one of the Rs just in case $\langle y, y \rangle$ is not one of the Gs. We start by verifying that there is at least one R. Suppose for *reductio* that, for every y, $\langle y, y \rangle$ is one of the Gs. By hypothesis, there is an object α such that, for every x, $\langle \alpha, x \rangle$ is one of the Gs. By hypothesis again, this means that there is an object α^{α}, such that, for every x, $\langle \alpha^{\alpha}, x \rangle$ is one of the Gs just in case x is one of the things identical to α (i.e. just in case $x = \alpha$). Since there exist at least two objects in the world, α must be distinct from α^{α}. Hence, by the definition of α^{α}, $\langle \alpha^{\alpha}, \alpha^{\alpha} \rangle$ is not one of the Gs. Contradiction.

Now, by hypothesis, there is an object α^{R}, such that, for every x, $\left\langle \alpha^{R}, x \right\rangle$ is one of the Gs just in case x is one of the Rs. Suppose α^{R} is one of the Rs; then, by our characterization of the Rs α^{R} is not one of the Rs. So α^{R} is not one of the Rs. But then, again by our characterization of the Rs, α^{R} must be one of the Rs. Contradiction.

2 *Formal Characterization of PFO languages*

Let a PFO language (short for *plural first-order*) consist of these symbols: (*a*) *logical connectives:* '$\exists$', '$\neg$' and '$\wedge$'; (*b*) *singular variables:* 'x_1', 'x_2', etc.; (*c*) *plural variables:* 'xx_1', 'xx_2', etc.; (*d*) *logical predicates* '$=$' and '$\prec$'; (*e*) *singular non-logical predicates:* 'P^1_1', 'P^1_2', ..., 'P^2_1', 'P^2_2', ..., etc.; and (*f*) *auxiliaries:* '(' and ')'. The formulas of PFO languages are defined as follows:

- $\ulcorner x_i = x_j \urcorner$, $\ulcorner x_i \prec xx_j \urcorner$ and $\ulcorner P^n_i(x_{j_1}, \ldots, x_{j_n}) \urcorner$ are formulas;
- if $\ulcorner \varphi \urcorner$ and $\ulcorner \psi \urcorner$ are formulas then so are $\ulcorner \exists x_i(\varphi) \urcorner$, $\ulcorner \exists xx_i(\varphi) \urcorner$, $\ulcorner \neg\varphi \urcorner$ and $\ulcorner (\varphi \wedge \psi) \urcorner$;
- nothing else is a formula.

If φ is a PFO formula, we shall let it abbreviate the (subscripted) English sentence $Tr(\varphi)$, where $Tr(\ldots)$ is defined as follows:

- $Tr(\ulcorner \neg\varphi \urcorner)$ = 'it is not the case that' $\frown Tr(\ulcorner \varphi \urcorner)$;
- $Tr(\ulcorner \varphi \wedge \psi \urcorner)$ = 'it is both the case that' $\frown Tr(\ulcorner \varphi \urcorner) \frown$ 'and' $\frown Tr(\ulcorner \psi \urcorner)$;
- $Tr(\ulcorner \exists x_i(\varphi) \urcorner) = \ulcorner$there is an object$_i$ such that$\urcorner \frown Tr(\ulcorner \varphi \urcorner)$;
- $Tr(\ulcorner \exists xx_i(\varphi) \urcorner) = \ulcorner$there are some objects$_i$ such that$\urcorner \frown Tr(\ulcorner \varphi \urcorner)$;
- $Tr(\ulcorner x_i = x_j \urcorner) = \ulcorner$it$_i$ is identical to it$_j\urcorner$;
- $Tr(\ulcorner x_i \prec xx_j \urcorner) = \ulcorner$it$_i$ is one of them$_j\urcorner$;

in addition, non-logical predicates are to be translated into English in accordance with their intended interpretations. As it might be, $Tr(\ulcorner P^1_1(x_i) \urcorner) = \ulcorner$it$_i$ is red$\urcorner$, and $Tr(\ulcorner P^2_1(x_i, x_j) \urcorner) = \ulcorner$it$_i$ is bigger than it$_j\urcorner$.

As an example, note that (4) turns out to abbreviate something equivalent to (5),

(4) $\exists x_i \exists xx_j(x_i \prec xx_j)$;

(5) there is an object$_i$ and some objects$_j$ such that it$_i$ is one of them$_j$.

The expressions '$\rightarrow$', '$\leftrightarrow$', '$\forall$' and '$\vee$' are defined as usual. Also, we will sometimes use variables 'x', 'y', 'z', ... instead of 'x_1', 'x_2', ..., and variables 'xx', 'yy', 'zz', ... instead of 'xx_1', 'xx_2', ... Finally, we may use predicates such as 'RED' and 'BIGGER' in place of the non-logical predicate letters such as 'P_1^1' or 'P_1^2', and define constants and non-logical function letters out of relations in the ordinary way. A formal semantics for PFO languages is provided in section 1.14.2.

3 *Formal Characterization of PFO$^+$ languages*

Let PFO$^+$ languages be the result of extending PFO languages with *plural predicate letters:* $\ulcorner \mathbf{P}_1^{\langle m,n\rangle} \urcorner$, $\ulcorner \mathbf{P}_2^{\langle m,n\rangle} \urcorner$, ... (for $0 \leq m$ and $0 < n$). We then add the following clause to our characterization of formulas:

- $\ulcorner \mathbf{P}_i^{\langle m,n\rangle}(x_1, \ldots x_m, xx_1, \ldots, xx_n) \urcorner$ is a formula.

Plural predicates are to be understood as collective English predicates, in accordance with their intended interpretations. As it might be, $Tr(\ulcorner \mathbf{P}_1^{\langle 0,1\rangle}(xx_i)\urcorner) = \ulcorner$they$_i$ are scattered$\urcorner$, and $Tr(\ulcorner \mathbf{P}_1^{\langle 1,2\rangle}(x_i, xx_j, xx_k)\urcorner) = \ulcorner$it$_i$ is between them$_j$ and them$_k\urcorner$. In practice we shall use predicates such as '**Scattered**' and '**Between**' (in bold font) in place of plural predicates such as '$\mathbf{P}_1^{\langle 0,1\rangle}$' and '$\mathbf{P}_1^{\langle 1,2\rangle}$'.

4 *Generalized Quantifiers*

Let $Tr(\varphi)$ be the following transformation, from a first-order language with generalized quantifiers to an appropriate PFO$^+$ language:

- $Tr(\ulcorner x_i = x_j \urcorner) = \ulcorner xx_i \preceq xx_j \wedge xx_j \preceq xx_i \urcorner$;
- $Tr(\ulcorner \mathrm{P}(x_1, \ldots, x_n)\urcorner) = \ulcorner \mathrm{P}(x_1, \ldots, x_n)\urcorner$;
- $Tr(\ulcorner \psi \wedge \xi \urcorner) = Tr(\ulcorner \psi \urcorner) \wedge Tr(\ulcorner \xi \urcorner)$;
- $Tr(\ulcorner \neg\psi \urcorner) = \neg Tr(\ulcorner \psi \urcorner)$;
- $Tr(\ulcorner \exists x_i(\psi)\urcorner) = \ulcorner \pi_{x_i}[Tr(\psi)] \preceq \pi_{x_i}[x_i = x_i]\urcorner$;

- $Tr(\ulcorner[\mathrm{Q}x_i : \xi](\psi)\urcorner) = \ulcorner\mathbf{Q}^*(\pi_{x_i}[Tr(\xi) \wedge Tr(\psi)], \pi_{x_i}[Tr(\xi)])\urcorner$
  (where $\ulcorner[\mathrm{Q}x_i : \xi]\urcorner$ is the basic quantifier $\ulcorner$Q of the things$_i$ that are $\xi\urcorner$, and $\ulcorner\mathbf{Q}^*\urcorner$ is the plural predicate corresponding to the determiner $\ulcorner$Q$\urcorner$).

A rather cumbersome induction on the complexity of formulas shows that, if φ is a sentence from a first-order language with generalized quantifiers, then φ and $Tr(\varphi)$ are equivalent.

5 *Definitions of Truth and Satisfaction for PFO⁺ Languages*

We work within a PFO$^+$ language. For the sake of simplicity, we assume that the domain of discourse of the metalanguage is the same as the domain of discourse of the object language. Let '$\mathbf{Assignment}(xx)$' abbreviate the following:

$$\forall y(y \prec xx \rightarrow \exists z(y = \langle v, z\rangle \text{ for } v \text{ a variable})) \wedge$$
$$\forall v(v \text{ is a singular variable} \rightarrow \exists! z(\langle v, z\rangle \prec xx)) \wedge$$
$$\forall v(v \text{ is a plural variable} \rightarrow \exists z(\langle v, z\rangle \prec xx))$$

Next, define the satisfaction predicate '$\mathbf{Sat}(\varphi, yy)$' implicitly, by way of the following axioms:

- $\mathbf{Sat}(\ulcorner\neg\psi\urcorner, yy) \leftrightarrow \neg\mathbf{Sat}(\ulcorner\psi\urcorner, yy)$;
- $\mathbf{Sat}(\ulcorner\psi \wedge \theta\urcorner, yy) \leftrightarrow \mathbf{Sat}(\ulcorner\psi\urcorner, yy) \;\wedge\; \mathbf{Sat}(\ulcorner\theta\urcorner, yy)$;
- $\mathbf{Sat}(\ulcorner\exists x_i\psi\urcorner, yy) \leftrightarrow \exists tt[\mathbf{Assignment}(tt) \;\wedge\; \forall v((v \text{ is a variable } \wedge\; v \neq \ulcorner x_i\urcorner) \rightarrow \forall w(\langle v, w\rangle \prec yy \leftrightarrow \langle v, w\rangle \prec tt)) \;\wedge\; \mathbf{Sat}(\ulcorner\psi\urcorner, tt)]$;
- $\mathbf{Sat}(\ulcorner\exists xx_i\psi\urcorner, yy) \leftrightarrow \exists tt[\mathbf{Assignment}(tt) \wedge \forall v((v \text{ is a variable} \wedge v \neq \ulcorner xx_i\urcorner) \rightarrow \forall w(\langle v, w\rangle \prec yy \leftrightarrow \langle v, w\rangle \prec tt)) \;\wedge\; \mathbf{Sat}(\ulcorner\psi\urcorner, tt)]$;
- $\mathbf{Sat}(\ulcorner x_i \prec xx_j\urcorner, yy) \leftrightarrow \forall z(\langle\ulcorner x_i\urcorner, z\rangle \prec yy \rightarrow \langle\ulcorner xx_i\urcorner, z\rangle \prec yy)$;
- $\mathbf{Sat}(\ulcorner \mathrm{P}(x_i)\urcorner, yy) \leftrightarrow \forall z(\langle\ulcorner x_i\urcorner, z\rangle \prec yy \rightarrow \mathrm{P}^*(z))$, where '$\mathrm{P}^*(\ldots)$' is a translation of '$\mathrm{P}(\ldots)$' into the metalanguage;

- $\mathbf{Sat}(\ulcorner \mathbf{P}(xx_i) \urcorner, yy) \leftrightarrow \forall zz \forall w((\langle \ulcorner xx_i \urcorner, w \rangle \prec yy \leftrightarrow w \prec zz) \rightarrow \mathbf{P}^*(zz))$, where '$\mathbf{P}^*(\ldots)$' is a translation of '$\mathbf{P}(\ldots)$' into the metalanguage.

Finally, truth is defined in terms of satisfaction in the usual way:

$$\textsc{True}(\varphi) \equiv_{def} \forall yy(\mathbf{Assignment}(yy) \rightarrow \mathbf{Sat}(\varphi, yy)).$$

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