A New Approach to Multistage Serial Inventory Systems

by

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Abstract

We consider a single product multistage serial inventory system with several installations, say $N-1, \ldots, 1$. Installation $N-1$ intakes exogenous supply of a single commodity. For $i \in \{1, \ldots N-2\}$, installation i is supplied by shipments from installation $i+1$. Demands for the finished good occur at installation 1. Demands that cannot be filled immediately are backlogged. We assume holding costs at each installation which are linear functions of inventory, as well as a constant cost for each unit of backlogged demand, per period.

Clark and Scarf (1960) showed that over a finite horizon an echelon basestock policy is optimal. Federgruen and Zipkin (1984) extend their result to the infinite-horizon case for both discounted and average costs.

We present a new approach to this multistage serial inventory management problem, and give new proofs of these results by introducing and solving a simple Travel Time problem, using Dynamic Programming.

This approach is motivated by the fact that the exact cost-to-go function of the related Travel Time problem can be easily computed using a straightforward recursive procedure (instead of using the typical value iteration or policy iteration methods).

Moreover, this cost-to-go function gives various insights useful for a group of more complex multistage inventory problems. In this regard, we discuss how this cost-to-go function can be used to develop good Approximate Dynamic Programming algorithms for a number of complex multistage serial inventory problems.

The results obtained suggest that the idea of introducing a related "Travel Time" problem and our algorithm to solve this problem can be used as a building block of a new approach to solve large scale multistage inventory management problems.

This thesis was part of a research effort to find a fast algorithm to get very good robust suboptimal solutions to large scale multistage inventory management problems.

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Chapter 1

Introduction

1.1 Preliminary Discussion

A supply chain is a network of facilities that performs the functions of procurement of material, transformation of material to intermediate and finished products, and distribution of finished products to customers.

Different sources of uncertainty exist along a supply chain. They include demand (volume and mix), process (yield, machine downtimes, transportation and reliability) and supply.

Inventories are often used to protect the chain from these uncertainties. Inventories stored at different points of the supply chain have differing impacts on the cost and service performance of the chain.

Inventories at various points have different values (a higher value for finished goods and a lower value for raw materials). Also, inventories at various points have different degrees of flexibility (more flexibility in the form of raw materials because they can be turned into different alternative finished products without incurring a significant penalty). Finally, inventories at various points have different levels of responsiveness (finished goods can be shipped to customers without delay, whereas some lead time is needed to transform materials into finished goods before shipments can be made).

A major challenge to supply chain managers is how to control inventories and costs along the chain while maximizing customer service performance. They have a difficult time determining how much safety stock to hold and when to initiate orders for material from upstream sites. Indeed, incoming part availability and part delivery performance are the most important problems managers face today and they have to be addressed with decision support models. Since incoming parts at one site are often supplied from another site within the company, one can characterize the problem as one of managing leadtime uncertainties throughout the supply chain.

1.2 Problem Statement

Consider a serial multistage inventory system, with the following characteristics.

- There are several stages, or stocking points, arranged in series. The first stage receives supplies from an external source. Demand occurs only at the last stage. Demands that cannot be filled immediately are backlogged.
- There is only one product.
- To move units to a stage from its predecessor, the goods must pass through the supply system, representing production or transportation activities. The cost for a shipment to each stage is linear in the shipment quantity.
- There is an inventory holding cost at each stage and a backorder penalty cost at the last stage. The horizon is infinite, all data are stationary, and the objective is to minimize total (discounted or average) cost. Information and control are centralized.

  We focus on a basic system, where time is discrete, demand is a Poisson process, and each stage's supply system generates a constant leadtime. However, virtually all the results remain valid for a system with i.i.d demands. Also, since an assembly system can be reduced to an equivalent series system (Rosling 1989), the results apply there too.

1.3 Literature Review

We refer the reader to the review by Graves (1988) on production planning models in a multisite network, and to the paper by Gallego and Zipkin (1999) on stock positioning and performance estimation in serial production-transportation systems.

The analysis of multisite inventory networks can be categorized in two ways: those that are managed with complete centralized control, and those that are managed through decentralized control. Clark and Scarf's (1960) paper on centralized control forms the basis of most subsequent work. They considered a series system, assuming discrete time with a finite horizon and nonstationary data. Under periodic review inventory control with no setup costs, they were able to show that an order-up-to policy at each node is optimal. Federgruen and Zipkin (1984) adapted the results to the stationary, infinite-horizon setting and pointed out that the algorithm becomes simpler there. Rosling (1989) and Langenhoff and Zijm (1990) provided streamlined statements of the results.

The continuous review version of this problem has been addressed by Debodt and Graves (1985), where a reorder-point, fixed-lot size inventory control mechanism is used. Reorders are triggered on the echelon inventory position. Chen and Zheng (1994) further streamlined the results.

1.4 Summary of Contributions

In this thesis, we present a new approach to the multistage serial inventory management problem (see chapter 3), and give a new proof of the optimality of an echelon basestock policy (see chapter 5).

We introduce and solve a related "Travel Time" problem using Dynamic Programming (see chapter 4). We show how the exact cost-to-go function of this related "Travel Time" problem can be easily computed using a straightforward recursive procedure (instead of using the typical value iteration or policy iteration methods).

We have implemented our new approach on a simple example and show how convenient and simple to use this approach is (see chapter 6).

Finally, we discuss how this new approach can be used to develop good Approximate Dynamic Programming algorithms for a number of complex multistage serial inventory problems (see chapter 7). We discuss systems in which:

- The installations have different leadtimes
- The echelons have capacity constraints

- We are restricted to Open Loop policies
- The installations have stochastic leadtimes
- We have access to the actual demand a few periods before the due date
- There are two classes of customers with different Priority levels

The results obtained suggest that the idea of introducing a related "Travel Time" problem and our algorithm to solve this problem can be used as a building block of a new approach to solve large scale multistage inventory management problems.

Chapter 2

Background

2.1 Overview

In the following sections one can find a short analysis of the theories, methods and tools assumed as known in the rest of the thesis.

The basic tool used is the theory of Dynamic programming. We refer the reader to the book *Dynamic Programming and Optimal Control* by Dimitri P. Bertsekas [1].

2.2 Exact Dynamic Programming

The Dynamic Programming methods presented here concern a discrete-time finite-state system. They assume a cost function which is additive over time and depends on the states visited and possibly on the controls chosen.

Consider a decision making problem. Assume that time is broken down into a series of stages, and a control decision is made at the beginning of each stage. Let us assume that at time k all the information about the current state of the system is summarized in a variable $x(k)$. Also, let $u(k)$ be the control chosen. Let N be the total number of state transitions in the system (*horizon*). Suppose that we are given the dynamics of the system:

$$x(k+1) = f(x(k), u(k), w(k)), \text{ where } k = 0, 1, ..., N-1 \text{ is the time index,}$$
$$x(k) \text{ is the state vector at time } k,\ u(k) \text{ is the control input at time } k, \tag{2.1}$$
$$\text{and } w(k) \text{ is a random disturbance.}$$

The control $u(k)$ is constrained to be in a set of admissible controls $U_k(x(k))$ and is usually chosen by a rule of the form:

$$u(k) = \mu_k(x(k)) \tag{2.2}$$

A policy $\pi = \{\mu_0, \mu_1, ...\}$ is a collection of functions μ_k. A policy is admissible if the functions μ_k are such that $\mu_k(x(k)) \in U_k(x(k))$ for all states $x(k)$.

The probability distribution of $w(k)$ is allowed to depend on $x(k)$ and $u(k)$, but not on $x(k-1), ..., x(0), u(k-1), ..., u(0), w(k-1), ..., w(0)$.

At each stage k, a cost $\alpha^k g(x(k), u(k), w(k))$ is incurred, where g is a given function (*cost per stage*) and α a positive scalar with $0 < \alpha \leq 1$ (*discount factor*). Having a discount factor $\alpha < 1$ means that future costs matter less than if the same costs were incurred at the present time.

Given an initial state $x(0)$ and an admissible policy π, Equations (2.1) and (2.2) make $x(k), u(k), w(k)$ random variables with well-defined distributions. Moreover, the sequence $x(0), x(1), ...$ is a Markov process.

Given an initial state $x(0)$ and an admissible policy π, the cumulative *cost-to-go* $J^\pi(x(0))$ is the expected cost:

$$J^\pi(x(0)) = E\{\sum_{k=0}^{N} \alpha^k g(x(k), \mu_k(x(k)), w(k))\} \tag{2.3}$$

When there is explicit knowledge of cost structure and the transition probabilities, we say that we have a *model* of the system. In problems with a moderate number of states, a *Lookup Table* representation of the cost-to-go function is used, in the sense that a separate variable $J(x)$ is kept in memory for each state x.

When the cost accumulates indefinitely, we say that we have an *infinite horizon* problem. There are four principal classes of infinite horizon problems. In this thesis, we consider mainly *stochastic shortest path problems* or *discounted problems with bounded cost per stage.* In the case of stochastic shortest path problems, $\alpha = 1$ but there is a special cost-free termination state; once the system reaches that state, it remains there at no further cost. In the case of discounted problems with bounded cost per stage, $\alpha < 1$ and the absolute cost per stage $|g(., u, w)|$ is bounded.

In infinite horizon problems, the total expected cost (*cost-to-go*) associated with an initial state $x(0)$ and a *policy* $\pi = \{\mu_0, \mu_1, ...\}$ is defined by

$$J^{\pi}(x(0)) = \lim_{N \to \infty} E\{\sum_{k=0}^{N} \alpha^k g(x(k), \mu_k(x(k)), w(k))\} \tag{2.4}$$

The optimal cost starting from state $x(0)$, that is, the minimum of $J^{\pi}(x(0))$ over all π, is denoted by $J^*(x(0))$. Of particular interest in infinite horizon problems are stationary policies, which are policies of the form $\pi = \{\mu, \mu, ...\}$. For brevity, we refer to $\{\mu, \mu, ...\}$ as the stationary policy μ. The corresponding cost from state $x(0)$ is denoted by $J^{\mu}(x(0))$. We say that μ^* is optimal if $J^{\mu^*}(x(0)) = J^*(x(0))$ for all states $x(0)$. In infinite horizon problems, one can typically find an optimal policy which is stationary.

For most finite state space infinite horizon problems of interest and in particular for all models discussed in the subsequent sections, the following holds for all states x,

$$J^*(x) = \min_{u \in U(x)} E\{g(x, u, w) + \alpha J^*(f(x, u, w))\} \tag{2.5}$$

The above equation is known as *Bellman's equation.* It can be viewed as a functional equation for the cost-to-go function J^*.

In particular, Bellman's equation holds for all discounted problems with bounded cost per stage.

In case of stochastic shortest path problems, if we can guarantee that at least under an optimal policy, termination occurs with probability 1, then the Bellman's equation holds.

Chapter 3

The Multistage Serial Inventory System

3.1 Overview of the Problem

We consider a single product multi-echelon inventory system. The higher echelon, referred to as installation $N-1$, intakes exogenous supply of a single commodity. Installation i is supplied by shipments from installation $i+1$, $i \in \{1, ..N-2\}$. The demand for the finished good occurs at the lower echelon, called installation 1.

Besides the $N-1$ physical installations, we have two artificial installations in our model. Stage N refers to the exogenous supply installation feeding installation $N-1$ and we assume that this stage contains an infinite number of parts waiting to enter the system. Stage 0 refers to the terminal installation containing all the parts that have already been delivered to customers.

Stocks in each installation are reviewed and production decisions are made periodically. Instantaneous, perfect information about inventory at all levels is assumed. Unfilled demand at installation 1 is backordered incurring a linear penalty cost, at a rate of p per backordered part at each period. A linear holding cost of h_i per part is assessed on inventory at installation i, $i = 0, ...N$. We assume that the system does not incur holding cost for parts located at the terminal installation (Stage 0) and at the exogenous supply installation (Stage N). Moreover, it is less expensive to hold a part at installation i than to hold it at installation $i-1$, $i \in \{2, ..N-1\}$.

The underlying assumptions on the sequence of events are:

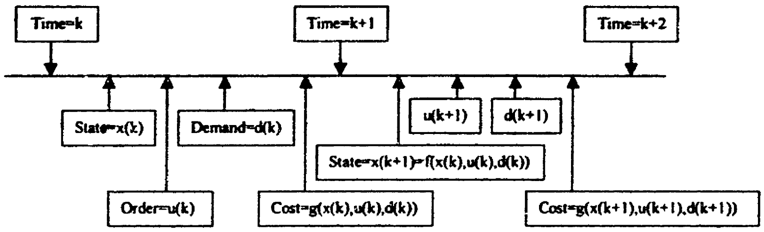


Figure 3-1: Holding as well as penalty costs are incurred after demand is observed in each period. Moreover, in each period, the current decisions are taken before demand is observed. Shipments arrive at the beginning of a period after costs are assessed in the prior period, and before the current decisions.

Condition 1 *Holding as well as penalty costs are incurred after demand is observed in each period. Moreover, in each period, the current decisions are taken before demand is observed.*

Condition 2 *Orders and shipments arrive, following their respective leadtimes (assumed to be one time unit in our model), at the beginning of a period, that is, after costs are assessed in the prior period, and before the current decisions.*

Condition 3 *Demand (actual and backlogged) is satisfied immediately as long as you have enough stock available in installation* 1.

The cost parameters are:

$$
\begin{aligned}
h_i &= \text{holding cost rate per part and per period, for installation } i \qquad (3.1)\\
p &= \text{penalty cost rate per backordered part and per period}\\
\alpha &= \text{discount rate, } 0 < \alpha \leq 1\\
h_N &= h_0 = 0\\
0 &\leq h_{N-1} \leq \dots \leq h_1\\
0 &< p
\end{aligned}
$$

The demand parameters are:

$\phi =$ distribution function of the demand d in each period. (3.2)

Demands in different periods are assumed independent.

$\lambda =$ average demand per period.

The state variables are:

$$X = (b, x_1, x_2, ..., x_{N-1}) \in \{0, 1, ...\}^{N-1} \text{ where} \quad (3.3)$$

$b =$ cumulative backlogged demand.

$x_i =$ inventory at installation i.

The control variables are:

$$S = (s_2, ..., s_N) \in \{0, 1, ...\}^{N-1} \text{ where} \quad (3.4)$$

$s_i =$ shipment from installation i to $i-1$.

$s_i = \nu_i(b, x_1, x_2, ..., x_{N-1})$

$\nu_i =$ shipment rule for installation i, $i \in \{2, ..., N\}$

$\mu = (\nu_2, ..., \nu_N)$

$\mu =$ shipment rule for the system

$\pi = \{\mu, \mu, ...\}$

$\pi =$ stationary shipment policy

The constraints are:

$$\begin{aligned} bx_1 &\doteq 0 \\ x_i &\geq 0 \\ s_i &\geq 0 \\ x_i &\geq s_i \end{aligned} \quad (3.5)$$

The first constraint states that, demand (actual and backlogged) is satisfied immediately if and only if you have enough stock available in installation 1 (see page 2). The last constraint states that, it is impossible to ship from installation i more than the current stock at installation i (see Equation (3.5)).

The following equation specifies the dynamics of the system at time t, under a particular policy:

$$b(t) = -\min\{0, x_1(t-1) + s_2(t-1) - d(t-1) - b(t-1)\}$$
$$x_1(t) = \max\{0, x_1(t-1) + s_2(t-1) - d(t-1) - b(t-1)\}$$
$$x_i(t) = x_i(t-1) + s_{i+1}(t-1) - s_i(t-1),\ i = 2, ..., N-1$$

The problem is to find an optimal shipment policy π, which minimizes the expected total inventory and penalty costs incurred by the system. Actually, the optimal policy defines a rule to transfer in the most cost-efficient way, the parts from installation N to installation 0.

Clark and Scarf (1960), assuming a finite planning horizon, show that an optimal policy for a two echelon inventory system can be computed by decomposing the problem into separate single-echelon problems. The problem for the lower echelon includes only its "own" costs, ignoring all others. The optimal policy and expected cost function for each period are then used to define a convex "induced penalty cost" function for each period. This function is added to the higher echelon's holding and ordering costs to form the second problem. The optimal policy for the system can be interpreted as prescribing a largely decentralized ("pull") system, where each outlet "orders" up to its own critical number s, whenever its inventory falls below that critical number. Federgruen and Zipkin (1984) show that the qualitative result of Clark and Scarf extends to the infinite horizon case under the criterion of discounted cost and for the long-term average cost.

Chapter 4

A new Approach

4.1 Overview of the New Approach

We treat the demands in a different way. We consider an infinite *list* of orders waiting to be submitted to the system. The idea is to view each single demand as the submission of one *order* to the system. Indeed, when an order is submitted to the system, a demand occurs and the order is deleted from the list of orders waiting to be submitted. Then, when there is a product available at installation 1, the demand is filled. Otherwise, the order is backordered and we pay a penalty. The number of orders submitted to the system in each period is a random variable drawn according to the distribution of the demand. Moreover, we view the list of orders as a typical ordered waiting list in the sense that we submit first, orders located at the beginning of the list.

This approach appears to be very powerful because we can use the dynamics of the evolution of the rank of each specific order in our list of orders to *monitor* our inventory system. Indeed, we consider a specific order in the list of orders waiting to be submitted. At the beginning of the first period ($t = 0$), we *assign* one part to this order. Assume that, at $t = 0$, this specific order is the K^{th} on the list and that the assigned part is located at installation N. In the subsequent periods, a number of orders will be submitted to the system and erased from the list; therefore the *rank* of that specific order will never increase. Thus, after a finite number of periods, this specific order will be submitted (when the rank reaches 0), eventually backordered (if the assigned part has not yet reached installation 1) and filled (when the assigned part becomes available at installation 1).

We then use dynamic programming to find an optimal movement scheme in the inventory system for the corresponding part, so as to minimize the total cost incurred to fill this specific order, given the dynamics of the evolution of the rank of the specific order, period after period.

We prove that the optimal movement scheme for the assigned part can be easily computed using a straightforward recursive procedure (instead of using the typical value iteration or policy iteration methods). We then show how to deduce from this optimal movement scheme, an optimal policy for the original multistage inventory management problem.

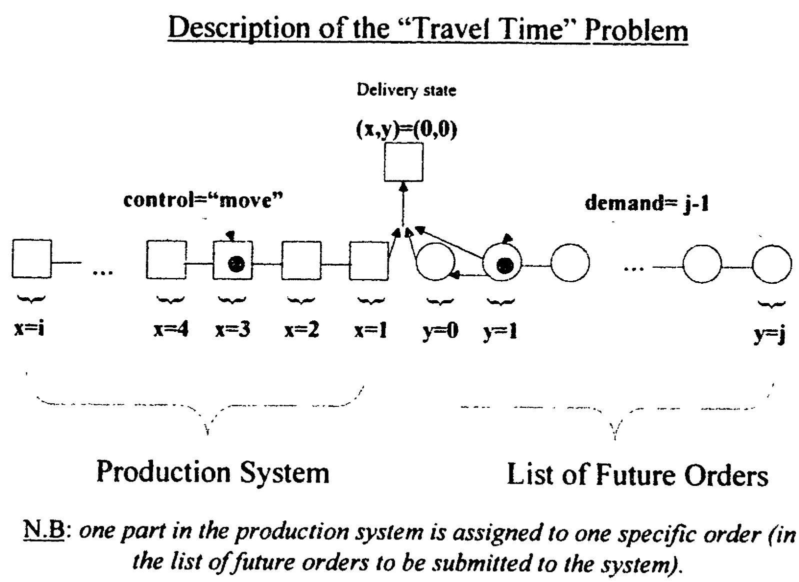


Figure 4-1: The delivery state $(0,0)$ can be reached in a period k if and only if in the prior period, the state was $(1,j)$, $j \geq 0$, the demand d_{k-1} satisfied $d_{k-1} \geq j$ and the control u_{k-1} was "*Move if possible*". These conditions correspond to a situation in which the part was ready ($i = 1$), the decision has been to deliver the part ($u_{k-1} = 1$) and the customer shows up ($d_{k-1} \geq j$).

4.2 Description of the Related "Travel Time" Problem

Let us formally introduce the related "Travel Time" problem that will serve as the building block of our approach to multistage serial inventory systems.

We consider the problem of finding an optimal movement scheme for the part assigned to a specific order in the list of orders. Assume that this specific order is the K^{th} on the list of orders at the beginning of the period $t = 0$. Assume also that demands in each period (number of customer arrivals) are *i.i.d.*

The essence of the problem is to trade off moving the part too early and incurring high inventory costs with starting moving too late and incurring high penalty costs. We start operating our system at the beginning of the period $t = 0$ with only one part in the inventory system (the part assigned to the specific order). Recall that in the subsequent periods, a number of orders will be submitted to the system and erased from the list; Thus, after a finite number of periods, the rank of the order will reach 0 (submission).

The cost data of the problem are:

$$\begin{aligned} h_i &= \text{holding cost rate for installation } i \text{ inventory.} && (4.1)\\ p &= \text{penalty cost rate for backorders at installation 1.} \end{aligned}$$

We assume that these cost factors are positive. Later, we will also assume that they are related in certain ways, depending on other factors, that preclude it being optimal never to order (see Equation (4.15)).

The other parameters are:

$$\begin{aligned} \alpha &= \text{discount rate, } 0 < \alpha \leq 1. && (4.2)\\ \phi &= \text{distribution function of the demand in each period.} \end{aligned}$$

The state variables are:

$$\begin{aligned} &(x, y) \in \{1, .., N\} \times \{0, .., K\} \cup \{(0, 0)\} \text{ with} && (4.3)\\ &x = \text{location of the part in the inventory system.}\\ &y = \text{current rank of the specific order in the list.} \end{aligned}$$

Indeed, at the beginning of the period t, $x(t)$ is the location of the part and $y(t)$ is the current rank of the specific order.

The control policy $\mu(x, y)$ is constrained to be in the set of admissible controls:

$$\begin{aligned}
\mu(x,y) &\in \{0,1\} \text{ where} \\
\mu(x,y) &= \begin{cases} 0 \text{ means "keep part in current installation"} \\ 1 \text{ means} \begin{cases} \text{"move part to the next installation, if possible", for } x = 1 \\ \text{"move part to the next installation", otherwise} \end{cases} \end{cases}
\end{aligned} \tag{4.4}$$

When $\mu(1, y) = 1$ in the above equation, the part actually moves only if the customer shows up (see Equation (4.6)). Later, we show that the option $\mu(1, y) = 0$ is never chosen (see Equation (4.13)).

From the state $(x(t), y(t))$ we move to the state

$$\begin{aligned}
&(x(t+1), y(t+1)) = f(x(t), y(t), \mu(x(t), y(t)), d(t)) \\
&= (f_x(x(t), y(t), \mu(x(t), y(t)), d(t)), f_y(x(t), y(t), \mu(x(t), y(t)), d(t)))
\end{aligned} \tag{4.5}$$

with:

$$\begin{aligned}
y(t+1) &= \begin{cases} 0 \text{ if } y(t) = 0 \\ \max\{y(t) - d(t), 0\} \text{ if } y(t) \geq 1 \end{cases} \\
x(t+1) &= \begin{cases} 0 \text{ if } (x(t), y(t)) = (0,0) \\ 1 \text{ if } x(t) = 1 \text{ and } y(t+1) \geq 1 \\ x(t) - \mu(x(t), y(t)) \text{ otherwise} \end{cases}
\end{aligned} \tag{4.6}$$

where the demand $d(t)$ is a random variable with distribution ϕ.

The initial state of the system is $(x(0), y(0))$. The transition probabilities can be easily derived from Equation (4.6).

The cost incurred at state $(x(t), y(t))$ is given by:

$$g(x(t), y(t), \mu(x(t), y(t)), d(t)) = \begin{cases} 0 \text{ if } (x(t+1), y(t+1)) = (0,0) \\ p + h_{x(t+1)} \text{ if } y(t+1) = 0 \text{ and } x(t+1) \geq 1 \\ h_{x(t+1)} \text{ otherwise} \end{cases} \quad (4.7)$$

i.e.:

$$g(i,j,u,d) = \begin{cases} 0 \text{ if } i = 0 \text{ (part delivered)} \\ 0 \text{ if } i = 1,\ u = 1,\ j - d \leq 0 \text{ (delivering)} \\ h_1 + p \text{ if } i = 1,\ u = 0,\ j - d \leq 0 \text{ (part \& customer ready, delivery not ordered)} \\ h_1 \text{ if } i = 1,\ j - d \geq 1 \text{ (part waiting for the customer)} \\ h_{i-u} + p \text{ if } i \geq 2,\ i - u \geq 1,\ j - d \leq 0 \text{ (customer waiting)} \\ h_{i-u} \text{ if } i \geq 2,\ i - u \geq 1,\ j - d \geq 1 \text{ (part not ready, customer not waiting)} \end{cases} \quad (4.8)$$

In the following table, we spell out the sequence of states and the costs for the (3) scenarios in which the part is ready at the same time (case 1), after (case 2) and before (case 3) the customer shows up:

	Period	*State*	*Control*	*Demand*	*Cost*	*Delivery ?*
case 1	$k=0$	$(2,3)$	$\mu(2,3)=1$	$d_0=1$	$g(2,3,1,1)=h_1$	*no*
	$k=1$	$(1,2)$	$\mu(1,2)=1$	$d_1=2$	$g(1,2,1,2)=0$	*yes*; *state* $=(0,0)$
case 2	$k=0$	$(2,3)$	$\mu(2,3)=1$	$d_0=1$	$g(2,3,1,1)=h_1$	*no*
	$k=1$	$(1,2)$	$\mu(1,2)=0$ *	$d_1=2$	$g(1,2,0,2)=p+h_1$	*no*; part late
	$k=2$	$(1,0)$	$\mu(1,0)=1$	d_2	$g(1,0,1,d_2)=0$	*yes*; *state* $=(0,0)$
case 3	$k=0$	$(2,3)$	$\mu(2,3)=1$	$d_0=1$	$g(2,3,1,1)=h_1$	*no*
	$k=1$	$(1,2)$	$\mu(1,2)=1$	$d_1=1$ *	$g(1,2,1,1)=h_1$	*no*; part ready
	$k=2$	$(1,1)$	$\mu(1,1)=1$	$d_2=1$	$g(1,1,1,1)=0$	*yes*; *state* $=(0,0)$

(4.9)

We now formulate the problem as an optimization of the expected cost over all admissible stationary policies:

$$\min_{\mu} \underset{d(0),d(1),\ldots}{E} \{\sum_{t=0}^{\infty} \alpha^t g(x(t), y(t), \mu(x(t), y(t)), d(t))\} \tag{4.10}$$

Let $J^*(x,y)$ denote the optimal cost to transfer the part from state (x,y) to the delivery state $(0,0)$. For $0 < \alpha < 1$, we have a discounted problem with bounded cost per stage and Bellman's equation holds (see page 12). For $\alpha = 1$, we have a stochastic shortest path problem. We can notice from Equation (4.6) that the state $(x,y) = (0,0)$ is an absorbing state. Moreover, $0 \leq x(t+1) \leq x(t) \leq N$ and $0 \leq y(t+1) \leq y(t) \leq K$. Finally, when $\alpha = 1$, the only stationary policies that do not yield an infinite cost are the policies which lead, in a finite number of steps, the system from the initial state $(x(0), y(0))$ to the absorbing state $(x,y) = (0,0)$ (see Equations (4.8) and (3.1)). Therefore, Bellman's equation holds for the case $\alpha = 1$ as well.

We may now state Bellman's equation for this problem:

$$\begin{aligned} J^*(0,0) &= 0 \\ J^*(x,y) &= \min_{u \in \{0,1\}} \underset{d}{E}\{g(x,y,u,d) + \alpha J^*(f(x,y,u,d))\} \end{aligned} \tag{4.11}$$

4.3 From the Layered Structure of Bellman's Equation to an Efficient Algorithm

Recall that the state $(x,y) = (0,0)$ is the only absorbing state. Moreover, $0 \leq f_x(i,j,u,d) \leq i$ and $0 \leq f_y(i,j,u,d) \leq j$. Indeed, Bellman's equation defined for the "Travel Time" problem has a special *layered structure* in the sense that, from state $(x,y) = (i,j)$ the system can not reach any state $(x,y) = (i',j')$ with $i' > i$ or $j' > j$. Therefore, the exact optimal value function and the optimal policy can be (algebraically) computed by using the following recursion over the state space:

$$
\begin{aligned}
&J^*(0,0) = 0 \\
&\text{For } i = 1 \text{ to } N, \text{ for } j = 0 \text{ to } K \\
&J^*(i,j) = \min \begin{cases} \underset{d}{E}[g(i,j,0,d) + \alpha J^*(f(i,j,0,d))] \\ \underset{d}{E}[g(i,j,1,d) + \alpha J^*(f(i,j,1,d))] \\ \text{with } d \text{ the number of arrivals} \end{cases}
\end{aligned}
\tag{4.12}
$$

Definition 4 *Define* $p_{d_0} = P_{d=}(d(t) = d_0)$.

The previous equations can be expanded as shown below:

$$
\begin{aligned}
&J^*(0,0) = 0 \\
&J^*(1,j) = \min \begin{cases} \sum_{d=0}^{j-1} p_d(h_1 + \alpha J^*(1, j-d)) + \sum_{d=j}^{\infty} p_d(p + h_1 + \alpha J^*(1,0)) \\ \sum_{d=0}^{j-1} p_d(h_1 + \alpha J^*(1, j-d)) + \sum_{d=j}^{\infty} p_d(0 + \alpha J^*(0,0)) \end{cases} \\
&J^*(i,j) = \min \begin{cases} \sum_{d=0}^{j-1} p_d(h_i + \alpha J^*(i, j-d)) + \sum_{d=j}^{\infty} p_d(p + h_i + \alpha J^*(i,0)) \\ \sum_{d=0}^{j-1} p_d(h_{i-1} + \alpha J^*(i-1, j-d)) + \sum_{d=j}^{\infty} p_d(p + h_{i-1} + \alpha J^*(i-1,0)) \end{cases} \quad \text{if } i \geq 2
\end{aligned}
\tag{4.13}
$$

Recall that it was specified in Equation (4.6) that for the special case of states $(1,j)$, the part is ready to be delivered but we have to hold it (in the system) at installation 1 until the customer shows up. Moreover, in the next section, we prove that $J^*(1,0) = J^*(0,0) = 0$ (see Equation(4.16)). Therefore, as stated earlier, it is always optimal to have $\mu^*(1,j) = 1$ because in Equation (4.13)

$$
J^*(1,j) = \sum_{d=0}^{j-1} p_d(h_1 + \alpha J^*(1, j-d)) + \sum_{d=j}^{\infty} p_d(0 + \alpha J^*(0,0)) \tag{4.14}
$$

4.4 An Efficient algorithm to solve Bellman's Equation

- For $(i,0)$:

The part is late and the customer is waiting to be served (order backordered). For $\alpha < 1$, we have to impose a condition on the penalty cost p to make it always optimal to transfer the part, i.e. $\mu^*(i,0) = 1$.

Condition 5 *When $\alpha < 1$, we want $\mu^*(i,0) = 1$ for all i. So we have to impose:*

$$\sum_{k=1}^{i-1} \alpha^{i-1-k}(h_k + p) < \sum_{k'=1}^{i} \alpha^{i-k'}(h_{k'} + p) \text{ for all } i \in \{2, ...N\} \tag{4.15}$$

Using the above condition, Equation (4.13) yields $J^*(0,0) = 0$, $J^*(1,0) = \min\{\frac{h_1+p}{1-\alpha}, 0\} = 0$ and:

$$\begin{cases} J^*(0,0) = 0 \\ J^*(1,0) = \min\{\frac{h_1+p}{1-\alpha}, 0\} = 0 \\ J^*(2,0) = \min\{\frac{h_2+p}{1-\alpha}, h_1 + p + \alpha 0\} = h_1 + p \\ \vdots \\ J^*(i,0) = \sum_{k=1}^{i-1} \alpha^{i-1-k}(h_k + p) \end{cases} \tag{4.16}$$

It turns out that the above result is also true for $\alpha = 1$.

- For $(1,j)$ with $j \geq 1$:

  The part is ready to be delivered but the customer has not yet shown up. As stated in Equation (4.13), $\mu^*(1,j) = 1$. Indeed, from Equation (4.14), we see that $J^*(1,1) = p_0 h_1/(1-\alpha p_0)$. Moreover $J^*(1,j) = \sum_{d=0}^{j-1}(p_d)(h_1 + \alpha J^*(1,j-d)) + (1 - \sum_{d=0}^{j-1} p_d)(h_0 + \alpha J^*(0,0))$. Thus:

$$J^*(1,j) = \begin{cases} \frac{p_0 h_1}{1-\alpha p_0} \text{ for } j = 1 \\ \frac{\sum_{d=1}^{j-1}(p_d)(h_1+\alpha J^*(1,j-d)) + p_0 h_1}{1-\alpha p_0} \text{ otherwise} \end{cases} \tag{4.17}$$

- For (i,j) with $i \geq 2$ and $j \geq 1$:

At each iteration of the recursion defined by (4.13), the $J^*(i,j)$ equation involves on the right hand side only $J^*(i',j')$ with $i' \leq i$ and $j' \leq j$. Indeed, at each iteration, there is only one unknown in the equation for $J^*(i,j)$ and we have to solve a generic equation of the form:

$$J^*(i,j) = \min \begin{cases} c_1 + p_0(g(i,j,0,0) + \alpha J^*(i,j)) \text{ with } c_1 \geq 0, 0 \leq p_0 < 1 \\ c_2 \text{ with } c_2 \geq 0 \end{cases} \tag{4.18}$$

In the above equation, c_2 represents the cost-to-go from (i,j) to $(0,0)$ for the policy $\pi = \{\mu_{\max}, \mu^*, \mu^*, ...\}$ with $\mu_{\max}(i',j') = 1$ for all (i',j'), and with μ^* an optimal policy. p_0 represents the probability of having zero arrivals. c_1 represents the cost to go from (i,j) to $(0,0)$ given at least one arrival, for the policy $\pi = \{\mu_{\min}, \mu^*, \mu^*, ...\}$ with $\mu_{\min}(i',j') = 0$ for all (i',j').

This class of generic equations can be solved in a closed form and we obtain:

$$J^*(i,j) = \min \begin{cases} \frac{c_1 + p_0 g(i,j,0,0)}{1 - \alpha p_0} \\ c_2 \end{cases} \tag{4.19}$$

Chapter 5

Characteristics of the Optimal Policy

5.1 Main Result: Optimal Policy for the "Travel Time" Problem

Definition 6 *Recall that the state space S is $\{1,..,N\} \times \{0,..,K\} \cup \{(0,0)\}$ and that a policy μ is a mapping of S into $\{0,1\}$. The "Move" Zone Z_M^{μ} of a policy μ is $\{(i,j) \in S | \mu(i,j) = 1\}$ and the "Wait" Zone Z_W^{μ} is $\{(i,j) \in S | \mu(i,j) = 0\}$.*

The following result will allow us to describe, using a dividing line (see the next section, page (42)), the qualitative structure of an optimal policy for the "Travel Time" problem. This key result is actually the main result of the thesis because we will use it to establish the optimality of basestock policies in the original multistage inventory problem (see the next section, page (42)).

Claim 7 *Let μ^* be an optimal policy in which we break ties by setting $\mu^*(i,j) = 1$. Then:*

- *For all $i \in \{0,..,N\}$, $(i,0) \in Z_M^{\mu^*}$.*
- *If $(i,j_1) \in Z_M^{\mu^*}$ and $j \leq j_1$ then $(i,j) \in Z_M^{\mu^*}$.*
- *If $(i,j_1) \in Z_W^{\mu^*}$ and $j \geq j_1$ then $(i,j) \in Z_W^{\mu^*}$.*

Definition 8 *The dividing line is the function* $l^{\mu^*} : i \in \{0,..,N\} \rightarrow l(i) = \max_{j \in Z_M^{\mu^*}} j$.

The idea behind the proof of this result is based on the (intuitive) fact that it is never optimal to have the part assigned to the j^{th} prospective customer in a lower location than the current location of the part assigned to the $(j-1)^{th}$. Mathematically, we will establish that $g(i,j-1,u_1,d) + g(i-1,j,u_2,d) = g(i,j,u_1,d) + g(i-1,j-1,u_2,d)$, for every u_1, u_2, and $J^*(i,j-1) + J^*(i-1,j) \geq J^*(i,j) + J^*(i-1,j-1)$.

Proof.

Remark 9 *Recall that (see Equation (4.8)):*

$$g(i,j,u,d) = \begin{cases} h_0 = 0 \text{ if } i-u \leq 0 \text{ and } j-d \leq 0 \text{ (part delivered)} \\ h_1 \text{ if } i-u \leq 0 \text{ and } j-d > 0 \text{ (part waiting for the customer)} \\ h_{i-u} + p \text{ if } i-u > 0 \text{ and } j-d \leq 0 \text{ (customer waiting)} \\ h_{i-u} \text{ if } i-u > 0 \text{ and } j-d > 0 \text{ (part not ready and customer not waiting)} \end{cases} \tag{5.1}$$

Therefore for $i-1 > 1$, *and for every choice of* u_1, u_2*:*

$$\begin{aligned} g(i,j-1,u_1,d) + g(i-1,j,u_2,d) &= h_{i-u_1} + h_{i-1-u_2} + p(1_{\{j-1-d\leq 0\}} + 1_{\{j-d\leq 0\}}) \\ g(i,j,u_1,d) + g(i-1,j-1,u_2,d) &= h_{i-u_1} + h_{i-1-u_2} + p(1_{\{j-d\leq 0\}} + 1_{\{j-1-d\leq 0\}}) \end{aligned} \tag{5.2}$$

So for $i-1 > 1$, *and for every choice of* u_1, u_2*:*

$$g(i,j-1,u_1,d) + g(i-1,j,u_2,d) = g(i,j,u_1,d) + g(i-1,j-1,u_2,d) \tag{5.3}$$

Remark 10 *Recall that (see Equation (4.6)):*

$$y(t+1) = f_y(x(t),y(t),\mu(x(t),y(t)),d(t)) = \begin{cases} 0 \text{ if } y(t) = 0 \\ \max\{y(t)-d(t),0\} \text{ if } y(t) \geq 1 \end{cases}$$
$$x(t+1) = f_x(x(t),y(t),\mu(x(t),y(t)),d(t)) = \begin{cases} 0 \text{ if } (x(t),y(t)) = (0,0) \\ 1 \text{ if } x(t) = 1 \text{ and } y(t+1) \geq 1 \\ x(t) - \mu(x(t),y(t)) \text{ otherwise} \end{cases} \tag{5.4}$$

Therefore for $i-1>1$:

$$\begin{cases} f_x(i,j-1,u_1,d) = \begin{cases} i \text{ for } u_1 = 0 \\ i-1 \text{ for } u_1 = 1 \end{cases} = f_x(i,j,u_1,d) \\ f_x(i-1,j,u_2,d) = \begin{cases} i-1 \text{ for } u_2 = 0 \\ i-2 \text{ for } u_2 = 1 \end{cases} = f_x(i-1,j-1,u_2,d) \end{cases} \tag{5.5}$$

$$\begin{cases} f_y(i,j-1,u_1,d) = \max(j-1-d,0) = f_y(i-1,j-1,u_2,d) \\ f_y(i-1,j,u_2,d) = \max(j-d,0) = f_y(i,j,u_1,d) \end{cases} \tag{5.6}$$

So we can combine the above result to get, for $i-1>1$:

$$\begin{cases} f(i,j-1,u_1,j-1) = f(i,j,u_1,j) \\ f(i-1,j-1,u_2,j-1) = f(i-1,j,u_2,j) \end{cases} \tag{5.7}$$

Definition 11 *Define*

$$Q(i,j,u) = \sum_{d=0}^{j-1} p_d(g(i,j,u,d) + \alpha J^*(f(i,j,u,d))) + \sum_{d=j}^{\infty} p_d(g(i,j,u,d) + \alpha J^*(f(i,j,u,d))) \tag{5.8}$$

Remark 12 *We have (see equation (4.8), (4.6)):*

$$\begin{aligned} g(i,j,u,d) &= g(i,j,u,j) \text{ for } d \geq j \\ f(i,j,u,d) &= f(i,j,u,j) \text{ for } d \geq j \end{aligned} \tag{5.9}$$

Therefore:

$$\begin{aligned} Q(i,j,u) &= \sum_{d=0}^{j-1} p_d(g(i,j,u,d) + \alpha J^*(f(i,j,u,d))) + \sum_{d=j}^{\infty} p_d(g(i,j,u,j) + \alpha J^*(f(i,j,u,j))) \\ &= \sum_{d=0}^{j-1} p_d(g(i,j,u,d) + \alpha J^*(f(i,j,u,d))) + (1 - \sum_{d=0}^{j-1} p_d)(g(i,j,u,j) + \alpha J^*(f(i,j,u,j))) \end{aligned} \tag{5.10}$$

Definition 13 *Define for* $i \geq 2$

$$W^i : j \rightarrow W^i(j) = J^*(i,j) - J^*(i-1,j) \tag{5.11}$$

Remark 14 *We have*

$$J^*(i,j) = \min_{u_1=0,1} Q(i,j,u_1)$$
$$W^i(j) = \min_{u_1=0,1} \max_{u_2=0,1} Q(i,j,u_1) - Q(i-1,j,u_2) \tag{5.12}$$

Lemma 15

$$W^i : j \to W^i(j) \text{ is nonincreasing over } \{0,..,K\} \text{ for all } i \in \{2,..,N\} \tag{5.13}$$

We will prove the above lemma by induction over $k \in \{3,4,...\}$ that for all i such that $2 \leq i \leq k-1$, we have:

$$W^i(.) \text{ is a non increasing function of } j, \text{ for } j \in \{0,..,k-i\} \tag{5.14}$$

i.e.:

$$W^i(j-1) \geq W^i(j), \text{ for } j \in \{0,..,k-i\} \tag{5.15}$$

which is equivalent to (see Equation (5.11)):

$$J^*(i,j-1) - J^*(i-1,j-1) \geq J^*(i,j) - J^*(i-1,j), \text{ for } 2 \leq i \leq k-1 \text{ and } j \in \{0,..,k-i\} \tag{5.16}$$

which can be rewritten as:

$$J^*(i,j-1) + J^*(i-1,j) \geq J^*(i,j) + J^*(i-1,j-1) \tag{5.17}$$

Let us assume the induction hypothesis to be true for $k = k_0 - 1$. For all i such as $2 \leq i \leq k_0 - 2$, $W^i(.)$ is a non increasing function of j, for $j \in \{0,..,k_0-1-i\}$, i.e.:

$$J^*(i,j-1) + J^*(i-1,j) \geq J^*(i,j) + J^*(i-1,j-1) \tag{5.18}$$

Consider i, j such as $2 \leq i \leq k_0 - 1$ and $0 \leq j \leq k_0 - i$. We want to prove that:

$$J^*(i, j-1) + J^*(i-1, j) \geq J^*(i, j) + J^*(i-1, j-1) \tag{5.19}$$

We notice from Equation (5.18) that the above result is true for all (i, j) such that $i+j \leq k_0 - 1$. Therefore, we have to establish the result only for i, j such that $2 \leq i \leq k_0 - 1$, $0 \leq j \leq k_0 - i$ and $i + j = k_0$. First, we start with the case $3 \leq i \leq k_0 - 1$, $0 \leq j \leq k_0 - i$ and $i + j = k_0$. Then, we deal with the case $i = 2, j = k_0 - 2$ (see page (34)). It is convenient to treat this case separately due to the specificity of the dynamics of the system for $x = 1$. Third, we prove the validity of the induction hypothesis for $k_0 = 3$ (see page (37)). Finally (see page (40)), we use the lemma to prove the claim (7).

- **First, assume that $3 \leq i \leq k_0 - 1$, $0 \leq j \leq k_0 - i$ and $i + j = k_0$.**

$$\begin{aligned}
&(Q(i, j-1, u_1) + Q(i-1, j, u_2)) - (Q(i, j, u_1) + Q(i-1, j-1, u_2)) = \\
&\sum_{d=0}^{j-1} p_d(g(i, j-1, u_1, d) + g(i-1, j, u_2, d)) - \sum_{d=0}^{j-1} p_d(g(i, j, u_1, d) + g(i-1, j-1, u_2, d)) \\
&+(1 - \sum_{d=0}^{j-1} p_d)(g(i, j-1, u_1, j-1) + g(i-1, j, u_2, j)) \\
&-(1 - \sum_{d=0}^{j-1} p_d)(g(i, j, u_1, j) + g(i-1, j-1, u_2, j-1)) \\
&+\sum_{d=0}^{j-1} p_d \alpha(J^*(f(i, j-1, u_1, d)) + J^*(f(i-1, j, u_2, d))) \\
&-\sum_{d=0}^{j-1} p_d \alpha(J^*(f(i, j, u_1, d)) + J^*(f(i-1, j-1, u_2, d))) \\
&+(1 - \sum_{d=0}^{j-1} p_d)\alpha(J^*(f(i, j-1, u_1, j-1)) + J^*(f(i-1, j, u_2, j))) \\
&-(1 - \sum_{d=0}^{j-1} p_d)\alpha(J^*(f(i, j, u_1, j)) + J^*(f(i-1, j-1, u_2, j-1)))
\end{aligned} \tag{5.20}$$

Then, by using Equation (5.3) to cancel all the terms involving $g(.)$, we obtain:

$$
\begin{aligned}
&(Q(i,j-1,u_1)+Q(i-1,j,u_2))-(Q(i,j,u_1)+Q(i-1,j-1,u_2)) = \\
&\sum_{d=0}^{j-1} p_d\alpha(J^*(f(i,j-1,u_1,d))+J^*(f(i-1,j,u_2,d))) \\
&-\sum_{d=0}^{j-1} p_d\alpha(J^*(f(i,j,u_1,d))+J^*(f(i-1,j-1,u_2,d))) \\
&+(1-\sum_{d=0}^{j-1} p_d)\alpha(J^*(f(i,j-1,u_1,j-1))+J^*(f(i-1,j,u_2,j))) \\
&-(1-\sum_{d=0}^{j-1} p_d)\alpha(J^*(f(i,j,u_1,j))+J^*(f(i-1,j-1,u_2,j-1)))
\end{aligned}
\tag{5.21}
$$

Then, by using Equation (5.7), we obtain:

$$
\begin{aligned}
&(Q(i,j-1,u_1)+Q(i-1,j,u_2))-(Q(i,j,u_1)+Q(i-1,j-1,u_2)) = \\
&\sum_{d=0}^{j-1} p_d\alpha(J^*(f(i,j-1,u_1,d))+J^*(f(i-1,j,u_2,d))) \\
&-\sum_{d=0}^{j-1} p_d\alpha(J^*(f(i,j,u_1,d))+J^*(f(i-1,j-1,u_2,d)))
\end{aligned}
\tag{5.22}
$$

From now on (and until Equation (5.41)) we let (u_1, u_2) be such that (see Equation (5.12)):

$$
\begin{cases}
Q(i,j-1,u_1) = J^*(i,j-1) \\
Q(i-1,j,u_2) = J^*(i-1,j)
\end{cases}
\tag{5.23}
$$

Then given u_1 (respectively u_2) needs not be optimal for (i,j) (respectively $(i-1,j-1)$). We have (see Equation (5.12)):

$$
\begin{cases}
Q(i,j,u_1) \geq J^*(i,j) \\
Q(i-1,j-1,u_2) \geq J^*(i-1,j-1)
\end{cases}
\tag{5.24}
$$

From Equations (5.5),(5.6), we can see that for $i \geq 3$ and $0 \leq d \leq j-1$:

$$
\begin{cases}
f(i,j-1,u_1,d) = (i-u_1, j-1-d) \\
f(i-1,j,u_2,d) = (i-1-u_2, j-d) \\
f(i,j,u_1,d) = (i-u_1, j-d) \\
f(i-1,j-1,u_2,d) = (i-1-u_2, j-1-d)
\end{cases}
\tag{5.25}
$$

We will consider four (4) different cases, and prove Equation (5.19) for each case.

Case 1: $(u_1 = 1, u_2 = 1)$

We see from Equation (5.25) that we can use the induction hypothesis for $d \in \{0, .., j-1\}$:

$$J^*(f(i, j-1, u_1, d)) + J^*(f(i-1, j, u_2, d)) \geq J^*(f(i, j, u_1, d)) + J^*(f(i-1, j-1, u_2, d)) \tag{5.26}$$

In that case, we can then conclude from Equation (5.22) that:

$$Q(i, j-1, u_1) + Q(i-1, j, u_2) \geq Q(i, j, u_1) + Q(i-1, j-1, u_2) \tag{5.27}$$

By using Equations (5.23),(5.24):

$$J^*(i, j-1) + J^*(i-1, j) \geq J^*(i, j) + J^*(i-1, j-1) \tag{5.28}$$

Case 2: $(u_1 = 1, u_2 = 0)$

We see from Equation (5.25) that for $d \in \{0, .., j-1\}$:

$$J^*(f(i, j-1, u_1, d)) + J^*(f(i-1, j, u_2, d)) = J^*(f(i, j, u_1, d)) + J^*(f(i-1, j-1, u_2, d) \tag{5.29}$$

In that case, we can then conclude from Equation (5.22) that:

$$Q(i, j-1, u_1) + Q(i-1, j, u_2) = Q(i, j, u_1) + Q(i-1, j-1, u_2) \tag{5.30}$$

By using Equations (5.23),(5.24):

$$J^*(i, j-1) + J^*(i-1, j) \geq J^*(i, j) + J^*(i-1, j-1) \tag{5.31}$$

Case 3: $(u_1 = 0, u_2 = 0)$

We see from Equation (5.25) that we can use the induction hypothesis for $d \in \{1, .., j-1\}$:

$$J^*(f(i,j-1,u_1,d)) + J^*(f(i-1,j,u_2,d)) \geq J^*(f(i,j,u_1,d)) + J^*(f(i-1,j-1,u_2,d)) \tag{5.32}$$

In that case, we can then conclude from Equation (5.22) that:

$$\begin{aligned} &Q(i,j-1,u_1) + Q(i-1,j,u_2) - p_0\alpha(J^*(i,j-1) + J^*(i-1,j)) \\ &\geq Q(i,j,u_1) + Q(i-1,j-1,u_2) - p_0\alpha(J^*(i,j) + J^*(i-1,j-1)) \end{aligned} \tag{5.33}$$

By using Equations (5.23),(5.24):

$$(1-p_0\alpha)(J^*(i,j-1) + J^*(i-1,j)) \geq (1-p_0\alpha)(J^*(i,j) + J^*(i-1,j-1)) \tag{5.34}$$

Since $(1-p_0\alpha) > 0$, we can cancel the $(1-p_0\alpha)$ factor in the above expression, which yields the desired inequality.

Case 4: $(u_1 = 0, u_2 = 1)$

In that case, $f_x(i,j,u_1,d) - f_x(i-1,j-1,u_2,d) = 2$. We see from Equation (5.25) that by using the induction hypothesis twice for $d \in \{1, .., j-1\}$:

$$J^*(f(i,j-1,u_1,d)) + J^*(f(i-1,j,u_2,d)) \geq J^*(f(i,j,u_1,d)) + J^*(f(i-1,j-1,u_2,d)) \tag{5.35}$$

In that case, we can then conclude from Equation (5.22) that:

$$\begin{aligned} &Q(i,j-1,u_1) + Q(i-1,j,u_2) - p_0\alpha(J^*(i,j-1) + J^*(i-2,j)) \\ &\geq Q(i,j,u_1) + Q(i-1,j-1,u_2) - p_0\alpha(J^*(i,j) + J^*(i-2,j-1)) \end{aligned} \tag{5.36}$$

By using Equations (5.23),(5.24) and by subtracting:

$$p_0\alpha(J^*(i-1,j) + J^*(i-1,j-1)) \tag{5.37}$$

from both sides of the previous inequality, we obtain:

$$
\begin{aligned}
&(1-p_0\alpha)(J^*(i,j-1)+J^*(i-1,j)) - p_0\alpha(J^*(i-1,j-1)+J^*(i-2,j)) \\
&\geq (1-p_0\alpha)(J^*(i,j)+J^*(i-1,j-1)) - p_0\alpha(J^*(i-1,j)+J^*(i-2,j-1))
\end{aligned}
\tag{5.38}
$$

Using the induction hypothesis at $(i-1,j)$, we have:

$$
J^*(i-1,j-1)+J^*(i-2,j) \geq J^*(i-1,j)+J^*(i-2,j-1) \tag{5.39}
$$

We can then conclude that:

$$
(1-p_0\alpha)(J^*(i,j-1)+J^*(i-1,j)) \geq (1-p_0\alpha)(J^*(i,j)+J^*(i-1,j-1)) \tag{5.40}
$$

Again, we can cancel the positive term $(1-p_0\alpha)$ to obtain the desired inequality.

We have proven that the induction step is valid for $3 \leq k_0-1$, $3 \leq i \leq k_0-1$, $0 \leq j \leq k_0-i$ and $i+j=k_0$:

$$
(J^*(i,j-1)+J^*(i-1,j)) \geq (J^*(i,j)+J^*(i-1,j-1)) \tag{5.41}
$$

- **Second, let's treat the special case $i=2, j=k_0-2$. Recall that we treat this case separately due to the specificity of the dynamics of the system for $x=1$.** As in the general case, we can derive an equality similar to Equation (5.22):

$$
\begin{aligned}
&(Q(i,j-1,u_1)+Q(i-1,j,u_2)) - (Q(i,j,u_1)+Q(i-1,j-1,u_2)) = \\
&p_{j-1}(g(i,j-1,u_1,j-1)+g(i-1,j,u_2,j-1)) \\
&-p_{j-1}(g(i,j,u_1,j-1)+g(i-1,j-1,u_2,j-1)) \\
&+\sum_{d=0}^{j-1} p_d\alpha(J^*(f(i,j-1,u_1,d))+J^*(f(i-1,j,u_2,d))) \\
&-\sum_{d=0}^{j-1} p_d\alpha(J^*(f(i,j,u_1,d))+J^*(f(i-1,j-1,u_2,d)))
\end{aligned}
\tag{5.42}
$$

From now on (and until Equation (5.55)) we let (u_1, u_2) be such that (see Equation (5.12)):

$$\begin{cases} Q(i, j-1, u_1) = J^*(i, j-1) \\ Q(i-1, j, u_2) = J^*(i-1, j) \end{cases} \tag{5.43}$$

Then given that u_1 (respectively u_2) need not be optimal for (i, j) (respectively $(i-1, j-1)$), we have (see Equation (5.12)):

$$\begin{cases} Q(i, j, u_1) \geq J^*(i, j) \\ Q(i-1, j-1, u_2) \geq J^*(i-1, j-1) \end{cases} \tag{5.44}$$

Recall that in the last installation, the optimal control is to move the part if possible, so:

$$u_2 = 1 \tag{5.45}$$

Moreover, recall that in the last installation the part actually moves only if there is delivery, i.e. if the prospective customer actually shows up (see Equation (4.6)). Therefore, from Equations (5.5),(5.6), we can see that for $i = 2$ and $0 \leq d \leq j-1$:

$$\begin{cases} f(i, j-1, u_1, d) = (i - u_1, j-1-d) \\ f(i-1, j, u_2, d) = (i-1, j-d) \\ f(i, j, u_1, d) = (i - u_1, j-d) \\ f(i-1, j-1, u_2, d) = \begin{cases} (i-1-u_2, j-1-d) \text{ if } d = j-1 \\ (i-1, j-1-d) \text{ otherwise} \end{cases} \end{cases} \tag{5.46}$$

We will consider two (2) different cases, and prove Equation (5.19) for each case.

Case 1: $(u_1 = 0)$

We see from Equation (5.25) that we can use the induction hypothesis for $d \in \{1, .., j-2\}$:

$$J^*(f(i, j-1, u_1, d)) + J^*(f(i-1, j, u_2, d)) \geq J^*(f(i, j, u_1, d)) + J^*(f(i-1, j-1, u_2, d)) \tag{5.47}$$

In that case, we can then conclude from Equations (5.42) and (5.46) that:

$$\begin{aligned}
&Q(i,j-1,u_1)+Q(i-1,j,u_2)-(Q(i,j,u_1)+Q(i-1,j-1,u_2))\\
&\geq p_{j-1}(g(i,j-1,u_1,j-1)+g(i-1,j,u_2,j-1))\\
&-p_{j-1}(g(i,j,u_1,j-1)+g(i-1,j-1,u_2,j-1))\\
&+p_{j-1}\alpha(J^*(i-u_1,0)+J^*(i-1,1))\\
&-p_{j-1}\alpha(J^*(i-u_1,1)+J^*(i-1-u_2,0))\\
&+p_0\alpha(J^*(i-u_1,j-1)+J^*(i-1,j))\\
&-p_0\alpha(J^*(i-u_1,j)+J^*(i-1,j-1))\\
&=p_{j-1}((p+h_2)+h_1)-p_{j-1}(h_2+0)\\
&+p_{j-1}\alpha(J^*(i,0)+J^*(i-1,1)-(J^*(i,1)+J^*(i-2,0)))\\
&+p_0\alpha(J^*(i,j-1)+J^*(i-1,j))-p_0\alpha(J^*(i,j)+J^*(i-1,j-1)) \text{ given that } (u_1,u_2)=(0,1)
\end{aligned} \tag{5.48}$$

We can use again the induction hypothesis:

$$J^*(i,0)+J^*(i-1,1)-(J^*(i,1)+J^*(i-1,0))\geq 0 \tag{5.49}$$

We can use the fact that (see Equation (4.16)):

$$J^*(i-2,0)=J^*(0,0)=0=J^*(1,0)=J^*(i-1,0) \tag{5.50}$$

By using Equations (5.23), (5.24) and (5.48):

$$\begin{aligned}
&(1-p_0\alpha)(J^*(i,j-1)+J^*(i-1,j))-(1-p_0\alpha)(J^*(i,j)+J^*(i-1,j-1))\\
&\geq p_{j-1}(p+h_1)+p_{j-1}\alpha(J^*(i,0)+J^*(i-1,1)-(J^*(i,1)+J^*(i-1,0)))\\
&\geq 0
\end{aligned} \tag{5.51}$$

Case 2: $(u_1=1)$

We see from Equation (5.25) that for $d \in \{0, .., j-2\}$:

$$\begin{aligned}
&J^*(f(i,j-1,u_1,d)) + J^*(f(i-1,j,u_2,d)) - (J^*(f(i,j,u_1,d)) + J^*(f(i-1,j-1,u_2,d))) = \\
&p_d\alpha(J^*(i-u_1,j-1-d) + J^*(i-1,j-d)) - p_d\alpha(J^*(i-u_1,j-d) + J^*(i-1,j-1-d)) \\
&= p_{j-1}\alpha(J^*(i-1,j-1-d) + J^*(i-1,j-d)) - p_d\alpha(J^*(i-1,j-d) + J^*(i-1,j-1-d)) \\
&= 0
\end{aligned} \tag{5.52}$$

In that case, we can then conclude from Equations (5.42) and (5.46) that:

$$\begin{aligned}
&Q(i,j-1,u_1) + Q(i-1,j,u_2) - (Q(i,j,u_1) + Q(i-1,j-1,u_2)) = \\
&p_{j-1}(g(i,j-1,u_1,j-1) + g(i-1,j,u_2,j-1)) \\
&-p_{j-1}(g(i,j,u_1,j-1) + g(i-1,j-1,u_2,j-1)) \\
&+p_{j-1}\alpha(J^*(i-u_1,0) + J^*(i-1,1)) - p_{j-1}\alpha(J^*(i-u_1,1) + J^*(i-1-u_2,0)) \\
&= p_{j-1}((p+h_1)+h_1) - p_{j-1}(h_1+0) \\
&+p_{j-1}\alpha(J^*(i-1,0) - J^*(i-2,0)) \text{ given that } (u_1,u_2) = (1,1) \\
&= p_{j-1}(p+h_1) \text{ given that } J^*(1,0) = J^*(0,0) = 0 \text{ (see Equation (4.16))} \\
&\geq 0
\end{aligned} \tag{5.53}$$

By using Equations (5.23),(5.24):

$$J^*(i,j-1) + J^*(i-1,j) \geq J^*(i,j) + J^*(i-1,j-1) \tag{5.54}$$

We have proven that the induction hypothesis is valid for $i = 2, j = k_0 - 2$:

$$(J^*(i,j-1) + J^*(i-1,j)) \geq (J^*(i,j) + J^*(i-1,j-1)) \tag{5.55}$$

- **Third, let's prove the validity of the induction hypothesis for $k_0 = 3$. We have to check that W^2 is non increasing over $\{0,1\}$, i.e.** $W^2(0) - W^2(1) \geq 0$. We know that:

$$\begin{aligned}
&J^*(2,1) = \min_{u_1=0,1} Q(2,1,u_1) \text{ with} \\
&Q(2,1,u_1) = p_0(g(2,1,u_1,0) + \alpha J^*(f(2,1,u_1,0))) \\
&+(1-p_0)(g(2,1,u,1) + \alpha J^*(f(2,1,u,1)))
\end{aligned} \tag{5.56}$$

We will consider two (2) different cases, and prove Equation (5.19) for each case.

Case 1:

$$u_1^* = \arg\min_{u_1=0,1} Q(2,1,u_1) = 1 \tag{5.57}$$

Then:

$$\begin{aligned}
&J^*(2,1) = Q(2,1,1) = \\
&p_0(g(2,1,1,0) + \alpha J^*(f(2,1,1,0))) + (1-p_0)(g(2,1,1,1) + \alpha J^*(f(2,1,1,1))) \\
&= p_0(h_1 + \alpha J^*(1,1)) + (1-p_0)(h_1 + p + \alpha J^*(1,0)) \text{ (see Equations (4.8),(5.25))} \\
&= p_0(h_1 + \alpha J^*(1,1)) + (1-p_0)(h_1 + p + \alpha 0) \text{ because } J^*(1,0) = 0 \text{ (Equation (4.16))}
\end{aligned} \tag{5.58}$$

From Equations (4.17), (4.16):

$$\begin{aligned}
&W^2(0) - W^2(1) = J^*(2,0) - J^*(1,0) - (J^*(2,1) - J^*(1,1)) = \\
&(h_1 + p) - 0 - (Q(2,1,u_1^* = 1) - J^*(1,1)) \\
&= (h_1 + p) - ((p_0 h_1 + p_0 \alpha J^*(1,1) + (1-p_0)(p + h_1) - J^*(1,1)) \\
&= (h_1 + p) - ((\alpha p_0 - 1)J^*(1,1) + h_1 + (1-p_0)p) \\
&= (1 - \alpha p_0)J^*(1,1) + p_0 p \\
&= (1-\alpha p_0)\tfrac{p_0 h_1}{1-\alpha p_0} + p_0 p \text{ because } J^*(1,1) = \tfrac{p_0 h_1}{1-\alpha p_0} \text{ (Equation (4.17))} \\
&= p_0(h_1 + p) \\
&\geq 0
\end{aligned}$$

We obtain the desired inequality:

$$W^2(0) - W^2(1) \geq 0 \tag{5.59}$$

Case 2:

$$u_1^* = \arg\min_{u_1=0,1} Q(2,1,u_1) = 0 \tag{5.60}$$

Then:

$$Q(2,1,1) - Q(2,1,0) \geq 0 \tag{5.61}$$

But:

$$\begin{aligned} Q(2,1,u_1) &= p_0(g(2,1,u_1,0) + \alpha J^*(f(2,1,u_1,0))) \\ &+(1-p_0)(g(2,1,u,1) + \alpha J^*(f(2,1,u,1))) \end{aligned} \tag{5.62}$$

By using the above with Equations (4.8),(5.25):

$$\begin{aligned} &Q(2,1,1) - Q(2,1,0) = \\ &p_0(h_1 - h_2) + \alpha p_0(J^*(1,1) - J^*(2,1)) \\ &+(1-p_0)(h_1 + p - (h_2 + p)) + \alpha(1-p_0)(J^*(1,0) - J^*(2,0)) \\ &= \alpha p_0[(J^*(2,0) - J^*(1,0)) - (J^*(2,1) - J^*(1,1))] \\ &-(h_2 - h_1) - \alpha(J^*(2,0) - J^*(1,0)) \end{aligned} \tag{5.63}$$

Equations (5.61),(5.63) yield:

$$\begin{aligned} &\alpha p_0[(J^*(2,0) - J^*(1,0)) - (J^*(2,1) - J^*(1,1))] \\ &\geq (h_2 - h_1) + \alpha(J^*(2,0) - J^*(1,0)) \\ &= (h_2 - h_1) + \alpha(h_1 + p) \text{ (see Equation (4.16))} \\ &= (h_2 + p) - (h_1 + p) + \alpha(h_1 + p) \end{aligned} \tag{5.64}$$

Recall that by hypothesis, we have (see Equation (4.15)):

$$(h_2 + p) - (h_1 + p) + \alpha(h_1 + p) \geq 0 \tag{5.65}$$

Recall that $\alpha p_0 \geq 0$ and $W^2(0) - W^2(1) = (J^*(2,0) - J^*(1,0)) - (J^*(2,1) - J^*(1,1))$. Therefore from Equation (5.64):

$$W^2(0) - W^2(1) \geq 0 \tag{5.66}$$

We have proven that the induction hypothesis is valid for $k_0 = 3$, i.e. that W^2 is non increasing over $\{0, 1\}$. **This result completes the proof of the lemma (5.13).** ∎

Proof. Finally, let's now finish the proof of the claim (7). We have:

$$\begin{aligned} \mu^*(1,j) &= 1 \text{ for all } j \in \{0,..,K\} \\ \mu^*(i,0) &= 1 \text{ for all } i \in \{0,...,N\} \end{aligned} \tag{5.67}$$

Indeed, the optimal control is 1 when the customer is waiting for the part or when the part is waiting for the customer (see (4.15)).

For a given $i \in \{2,..,N\}$, let us consider the biggest j_0 such as $\mu^*(i,j) = 1$ for all $j \in \{0,...,j_0\}$. We then have:

$$\mu^*(i, j_0 + 1) = 0 \tag{5.68}$$

For all $k \in \{1,...,N-1-j_0\}$,

$$\begin{aligned} & Q(i, j_0+1+k, 0) - Q(i, j_0+1+k, 1) = \\ & \sum_{d=0}^{j_0+k} p_d(h_i - h_{i-1} + \alpha(J^*(i, j_0+1+k-d) - J^*(i-1, j_0+1+k-d))) \\ & + \sum_{d=j_0+1+k}^{\infty} p_d(h_i - h_{i-1} + \alpha(J^*(i,0) - J^*(i-1,0))) \\ & = h_i - h_{i-1} + \sum_{d=0}^{j_0+k} p_d \alpha W^i(j_0+1+k-d) + \sum_{d=j_0+1+k}^{\infty} p_d \alpha W^i(0) \\ & \leq h_i - h_{i-1} + \sum_{d=0}^{j_0} p_d \alpha W^i(j_0+1-d) + \sum_{d=j_0+1}^{j_0+k} p_d \alpha W^i(0) + \sum_{d=j_0+1+k}^{\infty} p_d \alpha W^i(0) \\ & = Q(i, j_0+1, 0) - Q(i, j_0+1, 1) \end{aligned} \tag{5.69}$$

But $\mu^*(i, j_0+1) = 0$, so:

$$Q(i, j_0+1, 0) - Q(i, j_0+1, 1) < 0 \tag{5.70}$$

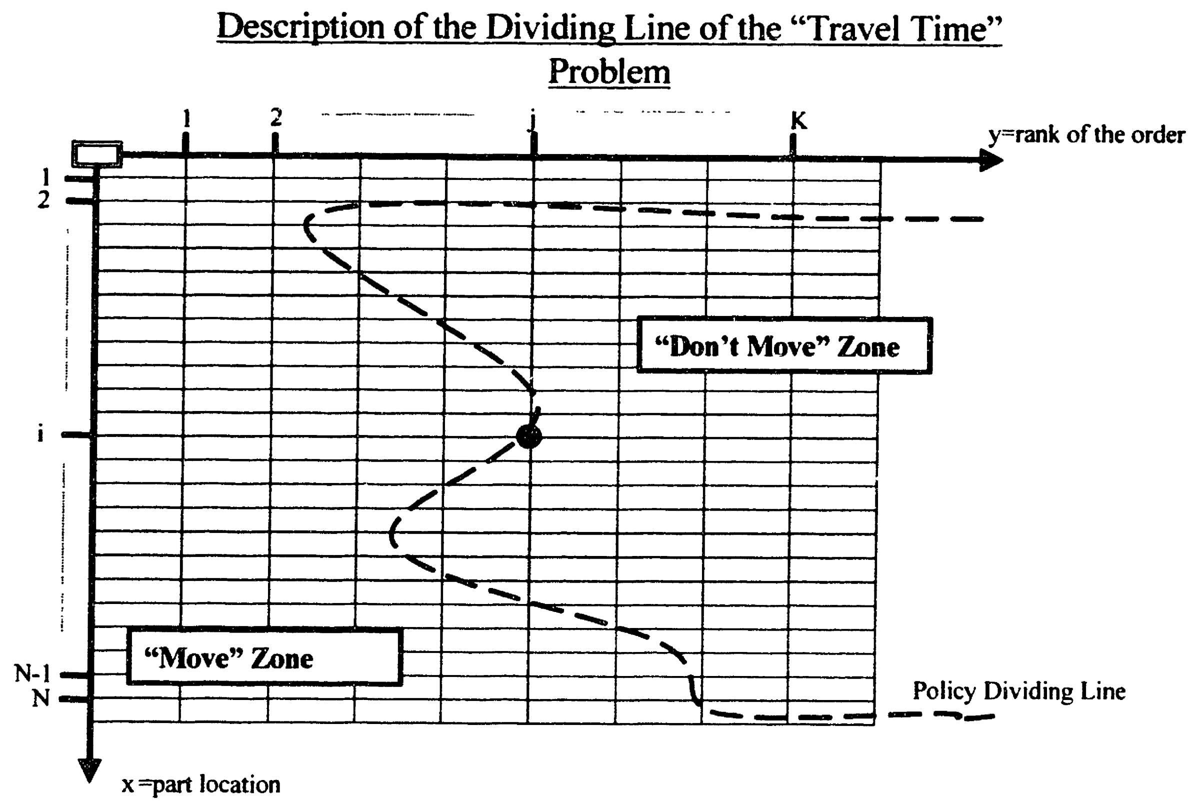


Figure 5-1: If the state (i, j) is in the "Don't Move" zone (respectively "Move"), the optimal control is $u = 0$ (respectively $u = 1$).

Therefore:

$$J^*(i, j_0 + 1 + k) = Q(i, j_0 + 1 + k, 0) \tag{5.71}$$

And:

$$\mu^*(i, j_0 + 1 + k) = 0 \tag{5.72}$$

This result completes the proof of the claim (7). ■

5.2 Optimal Policy for the Multistage Serial Inventory Management Problem

We explain now how an optimal policy for the original multistage serial inventory management problem (see Equation(3.4)) can be read from the dividing line of the related "Travel Time" problem.

Claim 16 *For each physical installation i, where $i \in \{1,..,N-1\}$, a critical-number policy is optimal. The critical number is given by $l^{\mu^*}(i+1)$ where $l^{\mu^*}(.)$ is the dividing line obtained by solving the related "Travel Time" problem.*

Proof. Let X be the state of the system. For a given installation i, where $i \in \{1,..,N-1\}$, define $n = \sum_{k=1}^{i} x_k$, where x_k is the inventory at installation K. n is the number of parts in echelon i.

If $n+1 > l^{\mu^*}(i+1)$ then $\mu^*(i+1, n+1) = 0$. Therefore, from the definition of $l^{\mu^*}(.)$ we have $\mu^*(i+1, n_1) = 0$, for $n_1 \geq n+1$. In that case, it is optimal to request no shipment from installation $i+1$ to i.

If $n+1 \leq l^{\mu^*}(i+1)$ then $\mu^*(i+1, n+1) = 1$. Moreover, from the definition of $l^{\mu^*}(.)$ we have $\mu^*(i+1, n_1) = 1$, for $n_1 \leq l^{\mu^*}(i+1)$. In that case, it is optimal to request shipment of $l^{\mu^*}(i+1) - n$ parts from installation $i+1$ to i. ∎

Claim 17 *Let J_0 be the optimal cost per part delivered. Then:*

$$J_0 = \lim_{K \to +\infty} J^*(N, K) \tag{5.73}$$

Proof. Recall that the holding cost at installation N, $h_N = 0$. By definition of J^*, the cost to go function of the related "Travel Time" problem, $\lim_{K \to +\infty} J^*(N, K)$ is the lowest expected cost to transfer a part from installation N (exogenous supply) to installation 0 (delivery) if we were given an infinite amount of time to execute the transfer. ∎

Remark 18 *Let J_0 be the optimal cost per part delivered and A a positive integer. Recall that*

λ is the average demand in each period.

if A is very large relative to 1, then $J^*(N, AN\lambda)$ is a very tight, practical approximation of J_0
$$(5.74)$$

Let us conclude this section with a few results on the convergence of $\lim_{K \to +\infty} J^*(N, K)$. From Equation (5.73), we have $J_0 = \lim_{K \to +\infty} J^*(N, K)$. Recall that (see Equation (5.10))

$$\begin{aligned} J^*(N,K) = & \sum_{d=0}^{K-1} p_d(g(N,K,u^*,d) + \alpha J^*(f(N,K,u^*,d))) \\ & +(1 - \sum_{d=0}^{K-1} p_d)(g(N,K,u^*,K) + \alpha J^*(f(N,K,u^*,K))) \\ & \text{where } u^* = \mu^*(N,K) \end{aligned} \tag{5.75}$$

- If $h_{N-1} > 0$ then it is not optimal to order an infinite amount of parts at installation $N-1$. Therefore, for K large enough, we have:

$$\mu^*(N,K) = 0 \tag{5.76}$$

Moreover, we have:

$$\begin{cases} g(N,K,0,d) + \alpha J^*(f(N,K,0,d)) = 0 + \alpha J^*(N, K-d) \text{ for } 0 \le d \le K-1 \\ g(N,K,0,K) + \alpha J^*(f(N,K,0,K)) = p + \alpha J^*(N,0) \end{cases} \tag{5.77}$$

Then:

$$J^*(N,K) - \sum_{d=0}^{K-1} p_d \alpha J^*(N, K-d) = (1 - \sum_{d=0}^{K-1} p_d)(p + \alpha J^*(N,0))$$

We also have:

$$\lim_{K \to +\infty} (1 - \sum_{d=0}^{K-1} p_d)(g(N,K,0,K) + \alpha J^*(f(N,K,0,K))) = 0 \tag{5.78}$$

Therefore:

$$J_0 = \lim_{K \to +\infty} \sum_{d=0}^{K-1} p_d \alpha J^*(N, K-d) \tag{5.79}$$

- Moreover, if $h_{N-1} = 0$ it can be shown using similar argument (with $\mu^*(N, K) = 1$) that:

$$J_0 = \lim_{K \to +\infty} \sum_{d=0}^{K-1} p_d \alpha J^*(N-1, K-d) \tag{5.80}$$

Chapter 6

Numerical Results

In the previous chapter, we have explained how an optimal policy for the original multistage serial inventory problem can be read from the dividing line. Thus, having a clear idea of the shape of the dividing line is an important issue.

6.1 Shape of the Dividing Line

The dividing line can take various shapes, depending on the holding cost function, the penalty cost rate, the discount factor and the customer arrival rate.

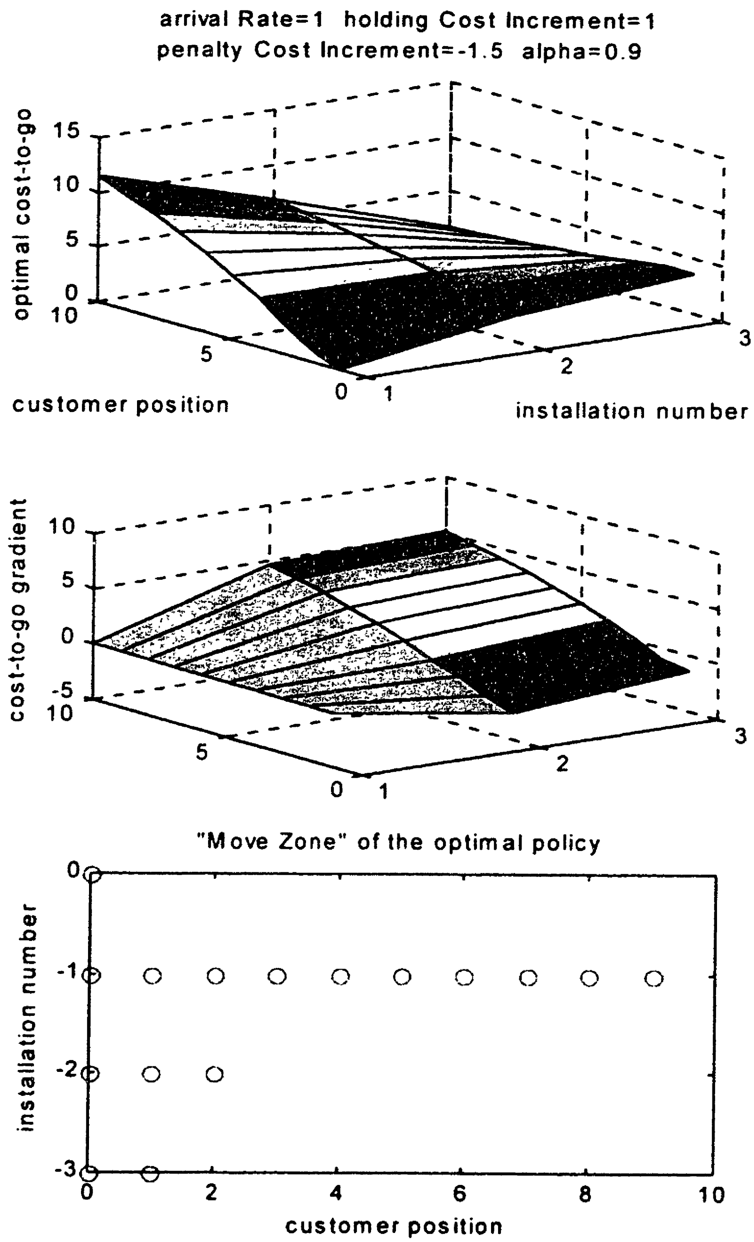
To illustrate this result, we have applied our algorithm to the following classical inventory optimization problem. We consider a two-echelon inventory system. Installation 2 is the depot and installation 1 is the retail outlet. The depot places orders for exogenous supply of a single commodity. We introduce two artificial installations and assume that this exogenous supply is delivered from installation 3. Moreover, we represent the action of delivering a part to a customer by moving that specific part to installation 0. We assume that demands in each period are Poisson with parameter λ. Our model is consistent with the actual dynamics of the real system described, if we assume infinite capacity for each installation and a leadtime of one time unit for all shipments.

In the examples, $h_3 = 0$, $h_2 = holdingCostIncrement$, $h_1 = 2 \times holdingCostIncrement$,

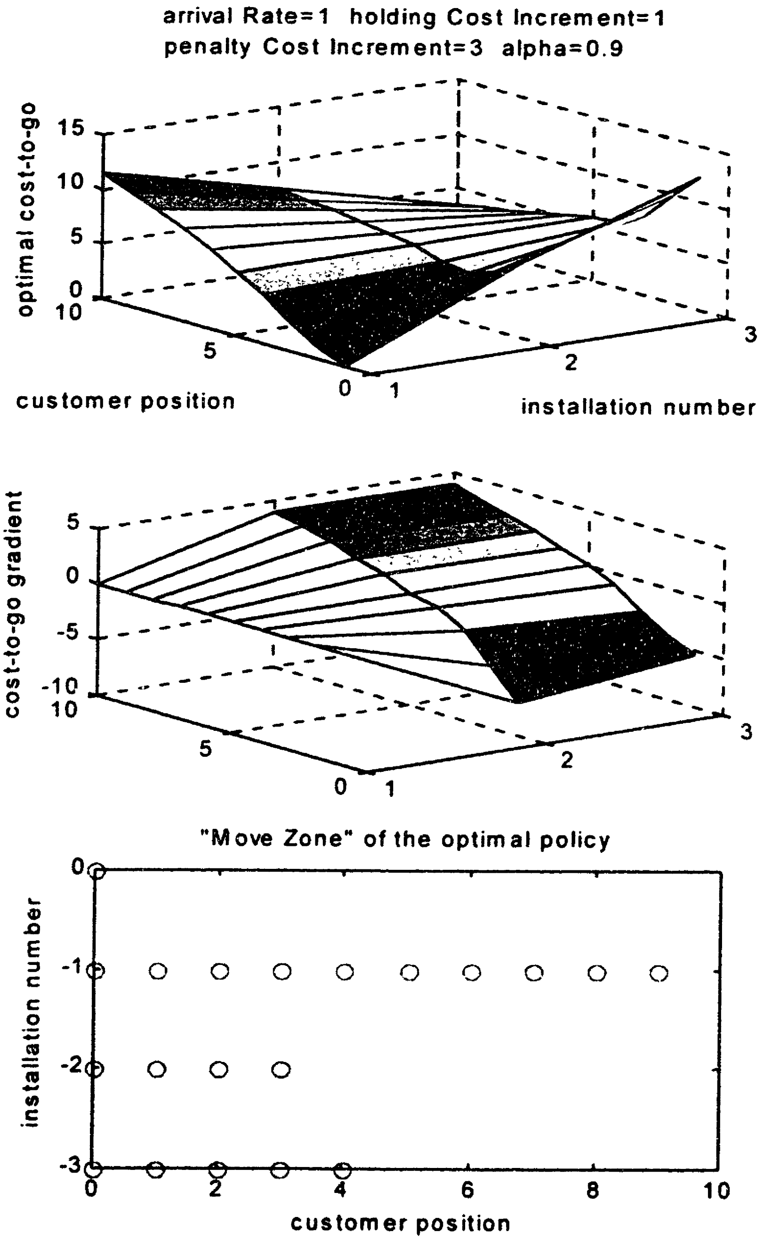
$h_0 = 0$, $p = h_1 + penaltyCostIncrement$, $\alpha = alpha$ and $\lambda = arrivalRate$.

$$
\begin{aligned}
h_3 &= 0 \\
h_2 &= holdingCostIncrement \\
h_1 &= 2 \times holdingCostIncrement \\
h_0 &= 0 \\
p &= h_1 + penaltyCostIncrement \\
\alpha &= alpha \\
\lambda &= arrivalRate
\end{aligned}
\tag{6.1}
$$

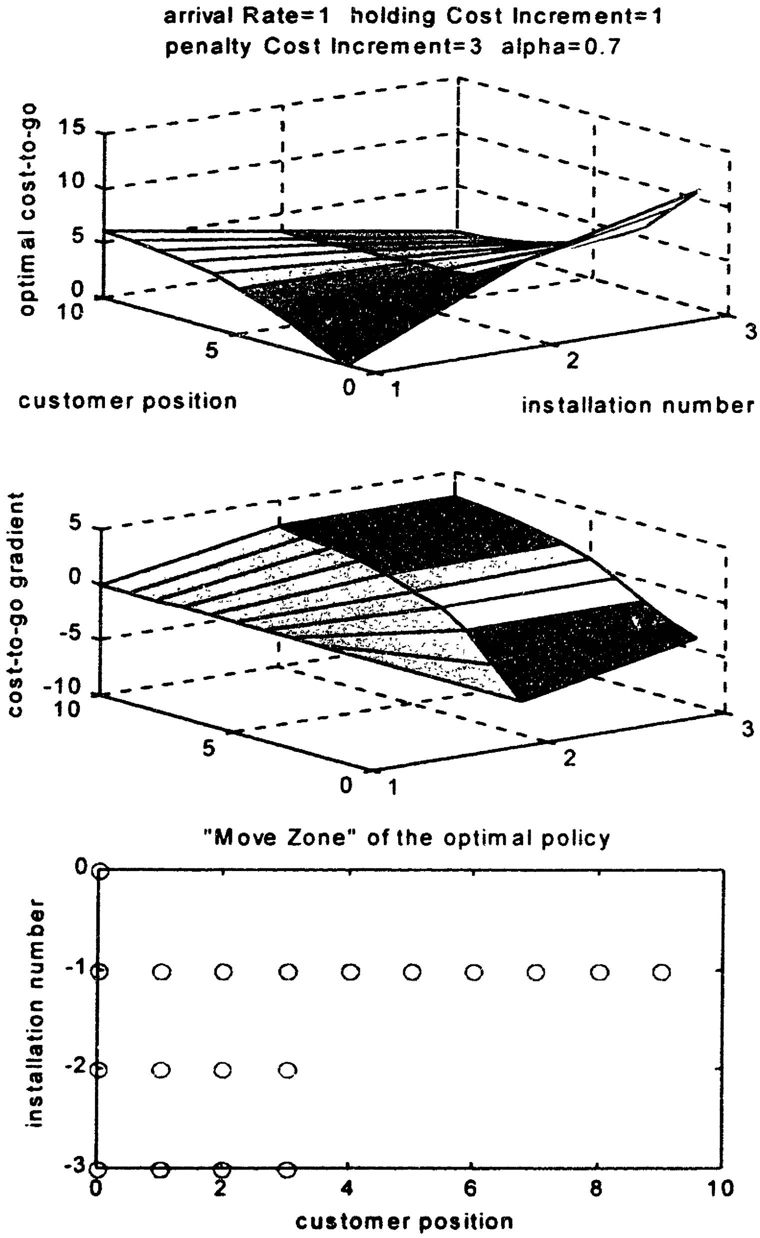
$J^*(0,0) = 0$ and for $(i,j) \in \{1,2,3\} \times \{0,..,9\}$, $J^*(i,j)$ is given by the Optimal cost-to-go function. $J^*(i,j) - J^*(i-1,j)$ is given by the cost-to-go gradient function. This important function gives us the cost of any inappropriate control decision. The optimal policy map gives all the states belonging to $Z_M^{\mu^*}$, the Move Zone of the optimal policy. It gives also the dividing line function $l^{\mu^*}(.)$. For each physical installation, i.e. 1 and 2, the critical-number can be read from the optimal policy map (see next chapter).



Impact of having $p < h_1$ on the shape of the "Dividing Line" and the cost-to-go. In this example, $p = 0.5$, $h_1 = 2$, $h_2 = 1$ and $\alpha = 0.9$. First, the basestock levels (claim (7)) are 2 units for installation 1 (depot) and 1 unit for installation 2 (warehouse). Second, the cost-to-go along the plane $i = 1$ (part waiting) is steeper than along the plane $j = 0$ (part late). Third, the cost-to-go gradient (lemma (5.13)) is non increasing along each plane $i = i_0$.



Impact of having $p > h_1$ on the shape of the "Dividing Line" and the cost-to-go. In this example, $p = 5$, $h_1 = 2$, $h_2 = 1$ and $\alpha = 0.9$. First, the basestock levels (claim (7)) are 3 units for installation 1 (depot) and 4 units for installation 2 (waıehouse). Second, the cost-to-go along the plane $i = 1$ (part waiting) is less steep than along the plane $j = 0$ (part late). Third, the cost-to-go gradient (lemma (5.13)) is non increasing along each plane $i = i_0$.

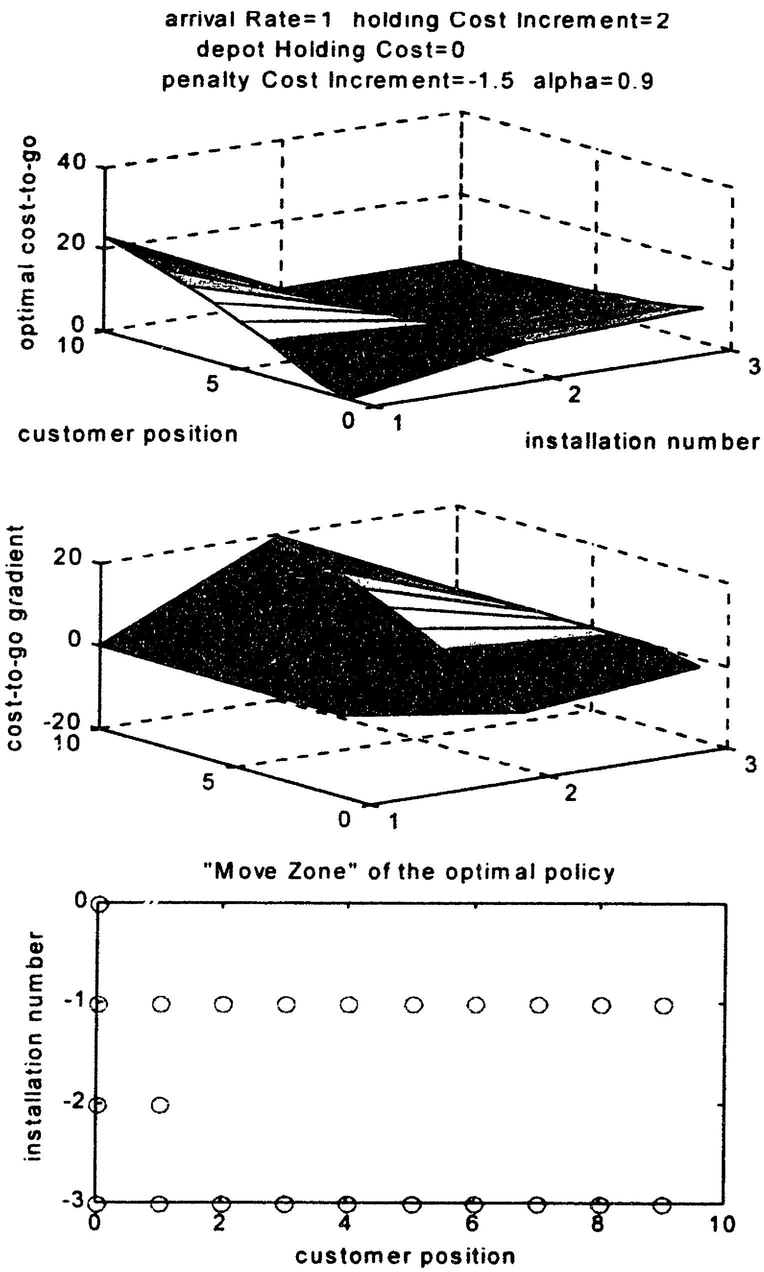


Impact of having a smaller discount factor α, on the shape of the "Dividing Line" and the cost-to-go. In this example, $p = 5$, $h_1 = 2$, $h_2 = 1$ and $\alpha = 0.7$. First, the basestock levels (claim (7)) are 3 units for installations 1 (depot) and 2 (warehouse). Second, the cost-to-go along the planes $i = 1$ (part waiting) and $j = 0$ (part late) are less steep than in the previous example. Third, the cost-to-go gradient is non increasing along each plane $i = i_0$.

In the following example, we consider the case where the depot holding cost is null, i.e. $h_3 = 0$, $h_2 = 0$, $h_1 = 2 \times holdingCostIncrement$, $h_0 = 0$ and $p = h_1 + penaltyCostIncrement$.

$$\begin{aligned}
&h_3 = 0 \\
&h_2 = 0 \\
&h_1 = 2 \times holdingCostIncrement \\
&h_0 = 0 \\
&p = h_1 + penaltyCostIncrement \\
&\alpha = alpha \\
&\lambda = arrivalRate
\end{aligned} \tag{6.2}$$

In that case, the critical-number for the depot is infinite and (in terms of polices) installation 2 becomes the "external supplier".



Impact of having $h_2 = h_3 = 0$, on the shape of the "Dividing Line" and the cost-to-go. In this example, $p = 2.5$, $h_1 = 4$, $h_2 = 0$ and $\alpha = 0.9$. First, the basestock levels (see claim (7)) are 1 unit for installation 1 (depot) and an infinite amout for installation 2 (warehouse). Second, the cost-to-go gradient (see lemma (5.13)) is non increasing along each plane $i = i_0$.

Chapter 7

Useful Insights on a Class of More Complex Multistage Serial Inventory Problems

The previous approach appears to be powerful as it provides us with useful insights on various more realistic multistage serial inventory models. The following sections highlight the flexibility of our approach.

7.1 What if the Installations have Different Leadtimes?

Recall that in the previous chapters, each installation was assumed to have a leadtime of one time unit, for all shipments (see page14). Let assume now that the leadtime for all shipments from installation i_0 is two time units.

The idea to deal with this constraint is to introduce one additional installation, installation $i_{transit}$, between installation i_0 and installation $i_0 - 1$. Let

$$h_{i_{transit}} = h_{i_0 - 1} \tag{7.1}$$

We then have to restrict the set of admissible controls at installation $i_{transit}$, in the definition of the related "Travel Time" problem:

$$\mu(i_{transit}, y) \in \{1\} \text{ where} \tag{7.2}$$

$$\mu(x,y) = \begin{cases} 0 \text{ means keep part in current installation} \\ 1 \text{ means move part to the next installation} \end{cases}$$

With these modifications, the leadtime between installation i_0 and installation $i_0 - 1$ is two time units and it is not possible to have a part stored in installation $i_{transit}$ in the related "travel time" problem. Our new approach can therefore be applied to the modified multistage serial inventory system.

7.2 What if the Echelons have Capacity Constraints?

Let assume now that echelon i_0 has a maximum capacity of C parts. The idea to deal with this constraint is again to restrict the set of admissible controls at all installations of the echelon in the definition of the related "travel time" problem:

$$\mu(i,y) \in \{0\} \text{ for } i \in \{1,..,i_0+1\} \text{ and } y > C \text{ where} \tag{7.3}$$

$$\mu(x,y) = \begin{cases} 0 \text{ means keep part in current installation} \\ 1 \text{ means move part to the next installation} \end{cases}$$

With these modifications, the states (i,y), with $i \in \{1,..,i_0\}$ and $y > C$, would never be reached in the related "Travel Time" and the capacity constraint is enforced. Our new approach can therefore be applied to the modified multistage serial inventory system.

7.3 What if we are Restricted to the Class of Open Loop Policies?

Let assume now that we want to find the best policy within the class of open loop policies. This category of policies are special because they are easy to implement. The idea to deal with this constraint is again to restrict the set of admissible controls at all installations except installation N, in the definition of the related "travel time" problem:

$$\mu(i,y) \in \{1\} \text{ for } i \in \{1,..,N-1\} \text{ where} \tag{7.4}$$

$$\mu(x,y) = \begin{cases} 0 \text{ means keep part in current installation} \\ 1 \text{ means move part to the next installation} \end{cases}$$

With these modifications in the related "Travel Time" problem, the production on any part located in echelon $N-1$ can never be paused. Thus, each policy is determined only by the production starting time. Our new approach can therefore be applied to the modified multistage serial inventory system.

7.4 What if the Installations have Stochastic Leadtimes?

Let assume now that for any given installation, the mechanism of part transfer is not deterministic but follows a *geometric* probability law with parameter δ. Indeed, any "move" order will be a "success" with probability δ and a "fail" with probability $1-\delta$, with the understanding that a part is transferred to the next installation only after a "success".

We can modify and use our approach to get a very good approximately optimal policy. The idea is to change the transition probabilities in the definition of the "related travel time" problem. However, the resulting "dividing line" will yield a policy that need not be optimal because the decision applied to a given part and its associated customer does not depend explicitly on the state of the other parts and associated customer.

Let $x(t)$ be the location of the part and $y(t)$ the current rank of the specific order, at the beginning of the period t. The control policy $\mu(x,y)$ is constrained to be in the set of admissible controls:

$$\mu(x,y) \in \{0,1\} \text{ where} \tag{7.5}$$

$$\mu(x,y) = \begin{cases} 0 \text{ means "keep part in current installation"} \\ 1 \text{ means "try to move part to the next installation"} \end{cases} \tag{7.6}$$

From the state $(x(t), y(t))$ we may move to the state:

$$\begin{aligned}&(x(t+1), y(t+1)) = f(x(t), y(t), \mu(x(t), y(t)), d(t))\\&= (f_x(x(t), y(t), \mu(x(t), y(t)), d(t)), f_y(x(t), y(t), \mu(x(t), y(t)), d(t)))\end{aligned} \tag{7.7}$$

with:

$$y(t+1) = \begin{cases} 0 \text{ if } y(t) = 0 \\ \max\{y(t) - d(t), 0\} \text{ if } y(t) \geq 1 \end{cases} \tag{7.8}$$

$$x(t+1) = \begin{cases} 0 \text{ if } (x(t), y(t)) = (0,0) \\ 1 \text{ if } x(t) = 1 \text{ and } y(t+1) \geq 1 \\ x(t) - g(t) \times \mu(x(t), y(t)) \text{ otherwise} \end{cases}$$

where the demand $d(t)$ is a random variable with distribution ϕ and $g(t)$ is a bernouilli random variable with parameter δ.

We notice that from state $(x, y) = (i, j)$ the system can not reach any state $(x, y) = (i', j')$ with $i' > i$ or $j' > j$. Therefore the layered structure of the modified related "Travel Time" problem is maintained. Thus, the exact optimal value function and the optimal policy of the modified "travel time problem" can be (algebraically) computed by using the following recursion over the state space:

$$\begin{aligned}&J^*(0,0) = 0\\&\text{For } i = 1 \text{ to } N, \text{ for } j = 0 \text{ to } K\\&J^*(i,j) = \min \begin{cases} \underset{d}{E}[g(i,j,0,d) + \alpha J^*(f(i,j,0,d))] \\ \underset{d}{E}[g(i,j,1,d) + \alpha J^*(f(i,j,1,d))] \\ \text{with } d \text{ the number of arrivals} \end{cases}\end{aligned} \tag{7.9}$$

For (i, j) with $i \geq 2$ and $j \geq 1$, at each iteration of the recursion defined by (7.9), the $J^*(i, j)$ equation involves on the right hand side only $J^*(i', j')$ with $i' \leq i$ and $j' \leq j$. Indeed, at each iteration, there is only one unknown in the $J^*(i, j)$ equation and we have to solve a generic equation of the form:

$$J^*(i,j) = \min \begin{cases} c_1 + p_0(g(i,j,0,0) + \alpha J^*(i,j)) \text{ with } c_1 \geq 0, 0 \leq p_0 < 1 \\ \delta \times c_2 + (1-\delta) \times (c_1 + p_0(g(i,j,0,0) + \alpha J^*(i,j))) \text{ with } c_2 \geq 0 \end{cases} \tag{7.10}$$

In the above equation, c_2 represents the cost-to-go from (i,j) to $(0,0)$ given that the first attempt to move is a success, for the policy $\pi = \{\mu_{\max}, \mu, \mu, ...\}$ with $\mu_{\max}(i',j') = 1$ for all (i',j'). p_0 represents the probability of having zero arrival. c_1 represents the cost to go from (i,j) to $(0,0)$ given at least one arrival, for the policy $\pi = \{\mu_{\min}, \mu, \mu, ...\}$ with $\mu_{\min}(i',j') = 0$ for all (i',j').

This class of generic equations can be solved in a closed form and we obtain:

$$J^*(i,j) = \min \begin{cases} \frac{c_1 + p_0 g(i,j,0,0)}{1-\alpha p_0} \\ \frac{c_1 + p_0 g(i,j,0,0) + \delta \times (c_2 - c_1 - p_0 g(i,j,0,0))}{1 - \alpha p_0 - \delta \times \alpha p_0} \end{cases} \tag{7.11}$$

Our new approach can therefore be applied to obtain very good insights on the behavior of multistage serial inventory systems with geometric leadtime.

7.5 What if we have access to the Actual Demand a few periods before the Due Date?

Let assume now that for an order submitted to the system at time t_0, the due date is $t_0 + f_0$, with f_0 being the customer lead time (a given positive integer). Indeed, the customer expects to be delivered a part exactly f_0 periods after having submitted his order to the system. If the part is ready before f_0 periods, the system incurs storage costs. On the other hand, if the part is not ready to be delivered after exactly f_0 periods, the system incurs a penalty fee. We can notice that the case $f_0 = 0$ corresponds to case described in the previous chapters.

We can use our approach to get an optimal policy. The idea is to introduce a "related travel time" problem which takes into account the dynamics of the system described above.

The state variables for the "related travel time" problem are:

$$(x, y) \in \{1, .., N\} \times \{-f_0, .., K\} \cup \{(0, -f_0)\} \text{ with} \tag{7.12}$$

$x =$ location of the part in the inventory system.

$y =$ current rank of the specific order in the list.

Indeed, at the beginning of the period t, $x(t)$ is the location of the part and $y(t)$ is the current rank of the specific order.

The control policy $\mu(x, y)$ is constrained to be in the set of admissible controls:

$$\mu(x, y) \in \{0, 1\} \text{ where} \tag{7.13}$$

$$\mu(x, y) = \begin{cases} 0 \text{ means "keep part in current installation"} \\ 1 \text{ means "move part to the next installation"} \end{cases} \tag{7.14}$$

From the state $(x(t), y(t))$ we may move to the state:

$$\begin{aligned} &(x(t+1), y(t+1)) = f(x(t), y(t), \mu(x(t), y(t)), d(t)) \\ &= (f_x(x(t), y(t), \mu(x(t), y(t)), d(t)), f_y(x(t), y(t), \mu(x(t), y(t)), d(t))) \end{aligned} \tag{7.15}$$

with:

$$y(t+1) = \begin{cases} -f_0 \text{ if } y(t) = -f_0 \\ y(t) - 1 \text{ if } -f_0 < y(t) < 1 \\ \max\{y(t) - d(t), 0\} \text{ if } y(t) \geq 1 \end{cases} \tag{7.16}$$

$$x(t+1) = \begin{cases} 0 \text{ if } (x(t), y(t)) = (-f_0, 0) \\ 1 \text{ if } x(t) = 1 \text{ and } y(t+1) \geq -f_0 + 1 \\ x(t) - \mu(x(t), y(t)) \text{ otherwise} \end{cases}$$

where the demand $d(t)$ is a random variable with distribution ϕ.

The initial state of the system being $(x(0), y(0))$. These equations are consistent with the characteristics of the modified problem. Indeed, when $y(t) \geq 1$ the order has not yet been submitted to the system. The order is submitted to the system when $y(t) = 0$. For $y(t) \leq 0$ the dynamic of the system is deterministic. We start incurring the penalty cost exactly f_0 periods after the submission date.

The cost incurred when $(x, y) = (i, j)$ is given by:

$$g(i,j,u,d) = \begin{cases} 0 \text{ if } f(i,j,u,d) = (0, -f_0) \\ p + h_{f_x(i,j,u,d)} \text{ if } f_y(i,j,u,d) = -f_0 \text{ and } f_x(i,j,u,d) \geq 1 \\ h_{f_x(i,j,u,d)} \text{ otherwise} \end{cases} \tag{7.17}$$

Our new approach can then be apply because the "related travel time" problem still has a special *layered structure*. We have to use the procedure (4.13) three times.

First, we have $J^*(0, -f_0) = 0$, $J^*(1, -f_0) = \min\{\frac{h_1+p}{1-\alpha}, 0\} = 0$ and:

$$\begin{cases} J^*(2, -f_0) = \min\{\frac{h_2+p}{1-\alpha}, h_1 + p + \alpha 0\} = h_1 + p \\ \vdots \\ J^*(i, -f_0) = \sum_{k=1}^{i-1} \alpha^{i-1-k}(h_k + p) \end{cases} \tag{7.18}$$

It turns out that the above result is also true for $\alpha = 1$.

Then, we can also notice that $J^*(1, -f_0 + 1) = \min\{h_1 + p + \alpha 0, 0 + \alpha 0\} = 0$. We can therefore get $J^*(i, j)$ for $-f_0 \leq j \leq 0$, $i \geq 1$.

- Finally, we can get $J^*(i, j)$ for $0 < j$, $i \geq 1$ and our new approach can be used to get an optimal policy.

7.6 What if there are Two Classes of Customers with Different Priority Levels?

Let assume now that, in addition to the normal demand distribution ϕ, the system incurred from time to time exceptional high priority demands. Assume also that the exceptional demand have to be served first, i.e. before any regular customer. Our new approach can be used to gain very good insights on the impact of these exceptional demands on the operating delays, penalty and holding costs.

For example, assume that at time t, the state of the system is $X(t) = (x_1, x_2, .., x_{N-1})$ (see (3.3)) and we have l exceptional high priority customers that have to be served first. We assign the first l parts to the exceptional high priority customers, the $(l+1)^{th}$ to the N^{th} parts to the first regular customers and l parts from the exogenous supply installation, i.e. installation N, to the last l^{th} regular customers.

Claim 19 *Let $X(t) = (x_1, x_2, .., x_{N-1})$ the state of the system at time t. Let $n = \sum_{i=1}^{N-1} x_i$. Let i_k be the location of the k^{th} part, $k = 1, ..n$, $i_1 \leq .. \leq i_n$. The impact of serving l exceptional high priority customers on the operating cost is:*

$$\begin{aligned}
&\sum_{k=1}^{l} (g(i_k, k, \mu^*(i_k, k), l) + \alpha J^*(f(i_k, k, \mu^*(i_k, k), l))) \\
&+ \sum_{k=l+1}^{n} (J^*(i_k, k-l) - J^*(i_k, k)) \\
&+ \sum_{k=n+1}^{l+n} (J^*(N, k-l) - J^*(N, k))
\end{aligned} \tag{7.19}$$

Our new approach can therefore be applied to obtain very good insights on the behavior of multistage serial inventory systems with two classes of customer priority level.

Chapter 8

Conclusion

We have presented a new approach to solve inventory management problems in a multistage serial supply chain. In this regard, we have introduced and solved a related "Travel Time" problem. The algorithm used is very fast due to the special structure of the related "Travel Time" problem. We have also proved that the optimal solution of the of the multistage serial inventory management problem can be read from the optimal solution of the simple related "travel time" problem. Finally, we have shown how powerful our approach was by explaining how it can be used to get some very useful insight for a class of more complex multistage serial inventory management problems.

The results obtained suggest that the idea of introducing a related "Travel Time" problem and our algorithm to solve this problem can be used as a building block of a new approach to solve large scale multistage inventory management problems.

Indeed, this thesis was part of a research effort to find a fast algorithm to get very good robust suboptimal solutions to large scale multistage inventory management problems.

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