# Alfvén Wing Electric Fields Generated by a Conducting Object Moving Through a Magnetized Plasma

by

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Abstract

A conducting body travelling through a plasma in the presence of a magnetic field will produce electrical currents which will in turn radiate electromagnetic waves in the plasma. Because the object's velocity is greater than the velocity of some components of the waves, a shocked wing, analogous to Cerenkov radiation, is formed. We find the electromagnetic fields from the low frequency contribution, for which magnetohydrodynamic (MHD) approximations are valid, to the spectrum of the waves.

A cold plasma model is used in order to solve Fourier transformed electrodynamic equations for the electric field in terms of induced current in the solid conductor. The field is then inverse transformed to get the field as a function of position and time for specific case of a large cylinder, with its principal axis normal to the plane containing the ambient magnetic field B and the velocity v of the moving object. To simplify mathematical calculations we consider only the case where v is perpendicular to B.

# Introduction

In this paper, we calculate the electric fields in a magnetized plasma, generated by an electrically conducting object moving perpendicular to the magnetic field lines. The interaction of the object with the magnetized plasma causes the object to emit a whole spectrum of plasma waves. Our calculations are restricted to the low frequency range of waves, for which magnetohydrodynamic (MHD) approximations are acceptable.

An observer in the reference frame moving with the conductor with the velocity $\vec{v}$, detects an electric field $\vec{E}=\vec{v}\times\vec{B}/c$ perpendicular to both the object's velocity and to the direction of the magnetic field. In a vacuum this would result in a static charge separation within the conductor. For a conductor travelling through a plasma, however, the charges are able to leak off, and for certain physical situations, a current loop is completed outide the conductor, allowing a current to flow inside the conducting body. The current, moving through the plasma with the object, carries with it the electromagnetic disturbance that propagates in the form of plasma waves. Because a magnetized plasma is an anisotropic medium, the velocity of the waves are strongly dependent on the direction of their propogation. This leads to the possibility that the conductor can move at a higher velocity than the phase velocity of the Alfvén mode propagating at large angles with respect to the direction of the magnetic field lines, and "wings" analogous to Cerenkov radiation are produced as the conductor drags its plasma waves behind it (see page 16 for definition of the Alfvén mode).

This effect was examined by Drell, Foley, and Ruderman, in their 1965 study (ref. 4) of the Echo weather satellite, to account for Echo's anomalously high orbital decay rate in terms of energy loss from MHD waves. It is now expected that any large conducting structures in low earth orbit will radiate these plasma waves. Proposed space shuttle missions using tethered satellite systems will involve conducting cable tens or hundreds of kilometers long, and will generate potential difference on the order of kilovolts. This will correspond to significant radiation of energy through plasma waves.

The Drell-Foley-Ruderman explanation for Echo did not gain wide acceptance until in situ observations of the magnetic field perturbation (Acuna *et al*., ref. 1) and later of the plasma flow perturbation (Belcher *et al*., ref. 3, Barnett, ref. 2) generated by Io (an electrically conducting moon of Jupiter) in the Jovian magnetosphere. These observations confirmed the existence of the "Alfvén wing" predicted by Drell et al. Thus not only are these waves generated by man made objects, but a dramatic natural speciman of them also exists. The interaction of metallic asteroids with pulsar magnetospheres may be another illustration of the effect.

The way in which the current circuit is closed outside of the conducting body is interesting. Charge can flow freely only along the direction of the magnetic field, which in the cases described above leads down into the atmosphere. There is a region in the upper atmosphere where there is a high enough particle density for some current to leak from one field line to another, though not yet enough neutral gas for charge carriers to recombine. Thus current can flow off one side of the orbiting object, down a magnetic field line to a lower altitude, horizontally to another magnetic field line, and up the field line to the other side of the conductor, where it closes the circuit.

Drell, et al. did not evaluate the electric fields for our problem. The purpose of their paper was to show the existence of Alfvén wings and to derive an order of magnitude expression for the radiated power. In this paper, we find an expression for the electric field in the case of a cylindrical conductor.

Like Drell, et al., we use a two component, singly ionized cold plasma approximation in our calculations. The cold plasma approximation, by neglecting the effects of the pressure gradients, makes the problem computationally tractable. We restrict ourselves to the case of a homogeneous, infinite plasma with a constant magnetic field that is perpendicular to the (also constant) velocity of the conductor. In practice, this is equivalent to assuming that the scale length for changes in these parameters is much larger than scale lengths for the conductor. We also restrict our study to the case of $v \ll c_A \ll c$, where v is the conductor speed, c is the speed of light, and $c_A$ is the Alfvén velocity, a fundamental parameter of the plasma to be defined below. Finally, for the environments that interest us, such as the terrestrial ionosphere or Jovian magnetosphere, we assume that the plasma particle density is high enough so that the plasma frequency $\omega_p$ is much greater than the ion cyclotron frequency $\Omega_i$ (all of which are defined below).

Though many of the calculations will be applicable to the whole spectral range, our final results will only be valid for the very low frequency, MHD domain (i.e., $\omega \ll \Omega_i$). The plasma frequency is defined as

$$\omega_p = \sqrt{4\pi n_e e/m_e} \tag{1}$$

where $n_e$, e, and $m_e$ are the number density, charge, and mass of the electron, respectively. The cyclotron frequency is

$$\Omega = qB/(mc) \tag{2}$$

where B is the magnetic field and q and m are the charge and mass of the species (either electron or ion). Finally, the Alfven speed is defined as

$$c_A = B/\sqrt{4\pi m_i n} \tag{3}$$

where m is mass of an ion.

## Fourier Transformed Electric Field

The derivation of the electric and magnetic fields for the Alfvén wings starts with the statement of the Fourier transformed Maxwell equations, which lead to an equation for the electric field in terms of a source current and dielectric tensor. These quantities are calculated, utilizing the appropriate MHD approximations, and the electric field, a function of frequency and wave number, is Fourier inverse transformed to get the field as a function of position and time.

The four dimensional Fourier transforms we use are

$$\vec{F}(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int \vec{F}(\vec{x},t)e^{-i(\vec{k}\cdot\vec{x}-\omega t)}d^3x\,dt \tag{4}$$

$$\vec{F}(\vec{x},t) = \frac{1}{(2\pi)^2}\int \vec{F}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{x}-\omega t)}d^3k\,d\omega \tag{5}$$

When equations are transformed in this way, $\vec{\nabla}$ operators are replaced with ik and $\partial/\partial t$ is replaced with $-i\omega$. We combine the Maxwell equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \tag{6}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j} \tag{7}$$

in order to get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}(\partial^2/\partial t^2)\vec{E} - (4\pi/c^2)\frac{\partial}{\partial t}\vec{j} \tag{8}$$

Fourier transforming, this becomes

$$\vec{k}\times(\vec{k}\times\vec{E}) + (\omega/c)^2\vec{E} + (4\pi i\omega/c^2)\vec{j} = 0 \tag{9}$$

with $\vec{E}$, $\vec{j}$ functions of $(\vec{k},\omega)$ rather than of $(\vec{x},t)$. The current term may be divided up into a source term $\vec{j}_s$ and a plasma term $\vec{j}_p$: $\vec{j} = \vec{j}_s + \vec{j}_p$. Equation (9) becomes

$$\vec{k}\times(\vec{k}\times\vec{E}) + (\omega/c)^2\vec{E} + (4\pi i\omega/c^2)\vec{j}_p = -(4\pi i\omega/c^2)\vec{j}_s \qquad (10)$$

We can define a dielectric tensor K such that

$$K\vec{E} = \vec{E} + \frac{4\pi i}{\omega}\vec{j}_p \qquad (11)$$

To find an expression for K we will use the Fourier analyzed Lorentz force equation to find the velocities (as a function of $\vec{k},\omega$) of the particles in the plasma, from which we will derive $\vec{j}_s$ as a function of $\vec{E}$. This information will allow us to find K from its definition (for the following, see Stix, ref. 6).

The Lorentz force law for a particle is

$$m\dot{\vec{v}} = q(\vec{E} + \vec{v}\times\vec{B}/c) \qquad (12)$$

where $\vec{v}$, m and q are the velocity, mass, and charge of the particle. We orient the coordinate system so that $\vec{B}=B_0\hat{x}_3$, and the one vector equation (12) is equivalent to three scalar equations

$$\dot{v}_1 = (q/m)(E_1+v_2B_0/c) \qquad (13a)$$

$$\dot{v}_2 = (q/m)(E_2-v_1B_0/c) \qquad (13b)$$

$$\dot{v}_3 = (q/m)E_3 \qquad (13c)$$

Fourier transforming,

$$v_1(\vec{k},\omega) = (iq/\omega m)(E_1(\vec{k},\omega)+v_2(\vec{k},\omega)B_0/c) \qquad (14a)$$

$$v_2(\vec{k},\omega) = (iq/\omega m)(E_2(\vec{k},\omega)-v_1(\vec{k},\omega)B_0/c) \qquad (14b)$$

$$v_3(\vec{k},\omega) = (iq/\omega m)E_3(\vec{k},\omega) \qquad (14c)$$

The first two equations of this set can be rearranged to give

$$v_1 = \frac{q}{m}\,\frac{-i\omega E_1 + \Omega E_2}{\Omega^2 - \omega^2} \qquad (15a)$$

$$v_2 = -\frac{q}{m}\,\frac{\Omega E_1 + i\omega E_2}{\Omega^2 - \omega^2} \qquad (15b)$$

Where, as defined above, $\Omega \equiv qB/mc$. The plasma current density $\vec{j}_p$ for the $k^{th}$ species is given by $\vec{j}_p = n_k q_k \vec{v}_k$, where $n_k \equiv$ number density, and $q_k \equiv$ particle charge. Note that for our model the velocity of the species is simply the velocity already given in equations (13a) to (15b); we are studying a cold plasma in its proper frame. For a two component plasma,

$$\vec{j}_p = n_e q_e \vec{v}_e + n_i q_i \vec{v}_i, \tag{16}$$

where the subscripts e and i refer to electrons and ions, respectively.

Inserting (16) into (11), and combining this with (14c) and (15), we get

$$K = \begin{bmatrix} S & iD & 0 \\ -iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \tag{17}$$

with

$$S = 1 + \frac{\omega_p^2}{\Omega_e^2 - \omega^2} \tag{18}$$

$$P = 1 - \frac{\omega_p^2}{\omega^2} \tag{19}$$

$$D = -\frac{\omega \omega_p^2}{\Omega_e \Omega_i^2} \tag{20}$$

If we eliminate $\vec{j}_p$ in (7) using (8), we see that (7) has the form

$$A\vec{E} = -(4\pi i \omega / c^2)\vec{j}_s \tag{21}$$

We want to solve equation (21) for $\vec{E}$ given $\vec{j}_s$. For the sake of simplicity, let $\vec{k}$ be entirely in the $x_1$-$x_3$ plane, with $k_1$

$= k\sin\theta$, $k2 = 0$, $k_3 = k\cos\theta$, and $k_r^2 = k_1^2 + k_2^2$. Then

$$A = \begin{bmatrix} (\frac{\omega}{c})^2 S - k^2\cos^2\theta & -i(\frac{\omega}{c})^2 D & k^2\sin\theta\cos\theta \\ i(\frac{\omega}{c})^2 D & (\frac{\omega}{c})^2 S - k^2 & 0 \\ (\frac{\omega}{c})^2 k^2\sin\theta\cos\theta & 0 & (\frac{\omega}{c})^2 P - k^2\sin^2\theta \end{bmatrix} \qquad (22)$$

Thus we have

$$\vec{E} = -(4\pi i\omega/c^2)A^{-1}\vec{j}_s \qquad (23)$$

where

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} bf & i(\omega/c)^2 Df & -bd \\ -i(\omega/c)^2 Df & af-d^2 & i(\omega/c)^2 Dd \\ -bd & -i(\omega/c)^2 Dd & ab-(\omega D/c)^2 \end{bmatrix} \qquad (24)$$

and

$$|A| = \det A = abf - (\omega/c)^4 D^2 f - bd^2 \qquad (25)$$

with $a = (\omega/c)^2 S - k^2\cos^2\theta$, $b = (\omega/c)^2 S - k^2$, $f = (\omega/c)^2 P - k^2\sin^2\theta$, and $d = k^2\sin\theta\cos\theta$. For convenience later, we define the matrix $M = |A| A^{-1}$. Originally we had assumed that $k_2 = 0$. To derive the electric field when this is not the case, we use the rotation matrix

$$R = (1/k_r)\begin{bmatrix} k_1 & -k_2 & 0 \\ k_2 & k_1 & 0 \\ 0 & 0 & k_r \end{bmatrix} \qquad (26)$$

From now on denote the original A by $A_0$. Since multiplying either $\vec{E}$ (or $\vec{j}_s$) in one frame by R gives us $\vec{E}$ (or $\vec{j}_s$) in the rotated frame, we find that $A^{-1}$ in the rotated frame is $A^{-1} = RA_0^{-1}R^{-1}$.

## The Source Current

The specific expression for $\vec{J}_s$ is dependent on the shape of the conducting object. There are some general attributes of the source current that should be clear. In the frame of reference in which the conducting object is at rest (i.e., the plasma is flowing past it), the source current is a stationary, time-independent current localized to within the object. Transforming the electric fields from the plasma frame to the (non-relativistic) object reference frame, and denoting the object frame with a prime, we have $\vec{E}' = \vec{E} - \vec{v} \times \vec{B}/c$. For a good conductor, $E'=0$. We use the Galilean transformation $x_1' = x_1 - vt$, $x_2' = x_2$, $x'_3 = x_3$, and $t' = t$. Figure 1 illustrates the orientation of the cylinder, magnetic field, and coordinate system.

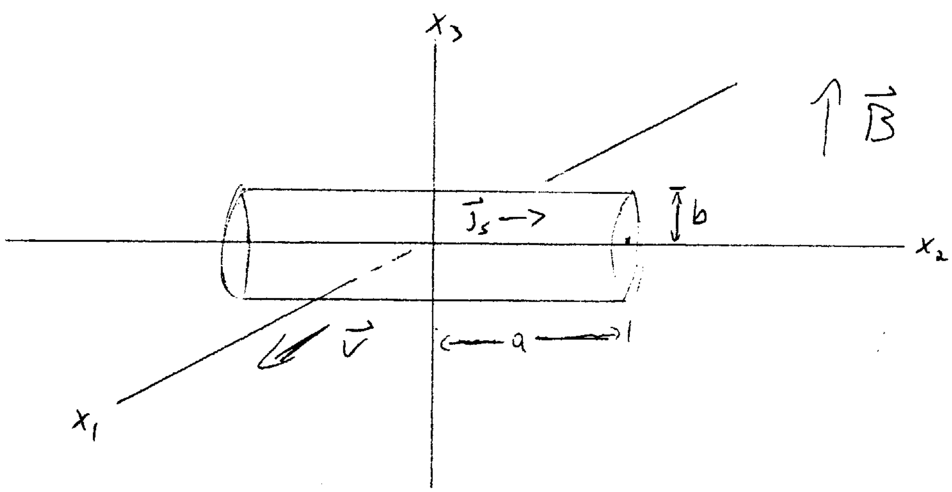


Fig. 1 -- Cylinder in a magnetic field

The specific system we will examine is a cylinder of length 2a and radius b, the axis of which is along the $x_2$ direction. We assume that $\vec{j}_s$ is parallel to $\vec{v}\times\vec{B}$ (i.e., along the $x_2$ axis) and is uniform. Thus

$$j_s(x') = \begin{cases} J & |y| < b \\ 0 & |y| > b \end{cases} \tag{27}$$

Fourier transforming $j_s$, we get

$$j_s(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} j_s(\vec{x},t)e^{-i(\vec{k}\cdot\vec{x}-\omega t)}d^3x\,dt \tag{28}$$

changing variables from x to x',

$$j_s(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} j_s(\vec{x}')e^{-i(\vec{k}\cdot\vec{x}'-[\omega-vk_1]t)}d^3x'\,dt \tag{29}$$

Because $\int_{-\infty}^{\infty}\exp\{i(\omega-vk_1)t\}dt = 2\pi\delta(\omega-vk_1)$,

$$j_s(k,\omega) = \frac{1}{(2\pi)^{3/2}}\quad \delta(\omega-vk_1)\int_{-\infty}^{\infty} j_s(\vec{x}')e^{-i\vec{k}\cdot\vec{x}'}d^3x \tag{30}$$

or

$$j_s(k,\omega) = 2\pi\delta(\omega-vk_1)j_s'(k) \tag{31}$$

The integral is best done in a cylindrical coordinate system in which $r^2 = x_1'^2 + x_3{}^2$, $k_r^2 = k_1^2 + k_2^2$, $\phi_x = \tan^{-1}(x_3/x_1)$, and $\phi_k = \tan^{-1}(k_3/k_1)$.

$$\vec{x} = x_2\hat{x}_2 + r(\hat{x}_1\sin\phi_x + \hat{x}_3\cos\phi_x) \tag{32}$$

$$\vec{k} = k_2\hat{x}_2 + k_r(\hat{x}_1\sin\phi_k + \hat{x}_3\cos\phi_k) \tag{33}$$

$$\vec{k}\cdot\vec{x}' = x_2k_2 + rk_r\cos\phi \quad (\phi\equiv\phi_x-\phi_k) \tag{34}$$

In these coordinates,

$$j_s(\vec{k},\omega) = \frac{J}{(2\pi)^{3/2}}\,\delta(\omega-vk_1)\int_{-a}^{a}\int_{0}^{b}\int_{0}^{2\pi} e^{i(k_2x_2+k_r r\cos\phi)}d\phi\, r\,dr\,dx_2 \tag{35}$$

Integrating first over $x_2$,

$$j_s(\vec{k},\omega) = \frac{J}{\pi\sqrt{2\pi}k_2} \delta(\omega - vk_1)\sin(ak_2)\int_0^b\int_0^{2\pi} e^{ik_r r\cos\phi} d\phi\, r\, dr \quad (36)$$

this is similar to an integral expression for the Bessel function

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{iz\cos\theta} d\theta \quad (37)$$

(Watson, p. 48, ref. 7). Inserting this into (36) we get

$$j_s(\vec{k},\omega) = \frac{2J}{\sqrt{\pi}k_2} \delta(\omega - vk_1)\sin(ak_2)\int_0^a J_0(rk_r)\, r\, dr \quad (38)$$

Luckily,

$$\int_0^1 x^{n+1} J_n(cx)\, dx = \frac{1}{c} J_{n+1}(c) \quad (39)$$

(Gradshteyn, p 683, ref. 5), so

$$j_s(\vec{k},\omega) = \frac{\sqrt{2}Jb}{\sqrt{\pi}k_2 k_r} \delta(\omega - vk_1)\sin(ak_2) J_1(bk_r) \quad (40)$$

Let the total current be $I_s = \pi b^2 J$. Then our expression for $j_s$ is

$$j_s(\vec{k},\omega) = \frac{2I_s}{\pi\sqrt{\pi}b} \frac{J_1(bk_1)}{k_2 k_r} \sin(ak_2)\delta(\omega - vk_1) \quad (41)$$

Determinant of A

Rotations do not affect the determinant of a matrix, so det A = det $A_0$. As we shall later see, the form of det A0 is particularly simple in the MHD regime, allowing for simple factorization. Without any approximation,

$$\det A = (\tfrac{\omega}{c})^2\{k^4(P\cos^2\theta+S\sin^2\theta)-(\tfrac{\omega}{c})^2k^2(PS[1+\cos^2\theta] +[S^2-D^2]\sin^2\theta)+(\tfrac{\omega}{c})^4P(S^2-D^2)\} \quad (42)$$

where $k^2 = k_3^2 + k_\perp^2$, where $k_\perp^2 = k_1^2 + k_2^2$. The determinant is quadratic in $k_3^2$, so

$$\det A = (\tfrac{\omega}{c})^2P\{k_3^4+([1-S/P]k^2-2(\tfrac{\omega}{c})^2S)k_3^2+\tfrac{S}{P}k^2 +(\tfrac{\omega}{c})^2([D^2-S^2]/P-S)k^2+(\tfrac{\omega}{c})^2(S^2-D^2)\} \quad (43)$$

Written in factored form,

$$\det A = (\tfrac{\omega}{c})^2P(k^2-\Lambda_1)(k_3^2+\Lambda_2) \quad (44)$$

with

$$\Lambda_1 = (\tfrac{\omega}{c})^2-k^2\tfrac{S}{P}+\tfrac{1}{2}k^2(1-S/P)(\sqrt{1+\varepsilon}-1) \quad (45)$$

$$\Lambda_2 = k^2-(\tfrac{\omega}{c})^2S+\tfrac{1}{2}k^2((1-S/P)(\sqrt{1+\varepsilon}-1) \quad (46)$$

and

$$\varepsilon = \frac{4\omega^2PD^2(\omega^2P-c^2k^2)}{c^4k^4(S-P)^2} \quad (47)$$

In low earth orbit or for the Jovian environment around Io, $\omega_p^2 \gg \Omega_e^2$, and $v^2 \ll (\Omega_i\Omega_e c/\omega_p^2)^2 \ll c^2$, in which case $\varepsilon \ll 1$, so that

$$\Lambda_1 = -\tfrac{S}{P}(k^2 - (\tfrac{\omega}{c})^2P) \quad (48)$$

and

$$\Lambda_2 = k^2 - (\omega/c)^2S \quad (49)$$

In the MHD regime, we have further simplification. Let us begin by finding a simpler expression for S. From the definition of $\omega_p$ and $\Omega$, we have

$$\Omega_i/\Omega_e = m_e/m_i << 1 \tag{50}$$

The MHD approximation is valid for very low frequencies: $\omega << \Omega_i, \omega_p$. Thus,

$$S = 1 + \omega_p^2/(\Omega_i\Omega_e) \tag{51}$$

Since $\omega_p{}^2 >> \Omega_i\Omega_e$, we have

$$S = \omega_p^2/(\Omega_i\Omega_e). \tag{52}$$

Similarly, $P = 1 - (\omega_p/\omega)^2$ and $\omega^2 << \omega_p^2$, so our expression for $\Lambda_1$ becomes

$$\Lambda_1 = (\omega/c)^2 S \tag{53}$$

If we define a new variable with dimensions of velocity,

$$c_A = c/\sqrt{S} \tag{54}$$

then

$$\det A = -\frac{c^2\omega_p^2}{c_A^2\omega^6}(\omega^2 - c_A^2k_3^2))(\omega^2 - c_A{}^2k^2) \tag{55}$$

What is the physical significance of $c_A$? The equation $\det A_0 = 0$ is the condition for a solution to the homogeneous equation $A_0\vec{E}=0$. Physically, this corresponds to the propagation of plasma waves through a sourceless medium ($j_s = 0$), and $\det A_0 = 0$ gives the dispersion relation between $\vec{k}$ and $\omega$. In this dispersion relation, we get two solutions, $\omega/k = \pm c_A$ and $\omega/k = \pm c_A \cos\theta$. But $\omega/k = v_p$, the phase velocity of the wave; these solutions of the dispersion equation imply two modes of propagation of the wave: one for which $v_p$ is independent of the angle of propagation, the other for which $v_p$ varies with $\cos\theta$. The former mode is known as the "fast mode," the latter is the "Alfvén mode," and $c_A$ is the "Alfvén velocity." It is instructive to calculate the group velocity of the Alfvén mode waves, for this measures the velocity of signal or energy propagation due to Alfven waves. Group velocity is given by $\vec{v}_g = \vec{\nabla}_k \omega$, and $\omega = k_3$, so $\vec{v}_g = c_A \hat{x}_3$. Though the phase velocity of Alfvén waves can have components in any direction, signal and energy only propogate along the magnetic field lines, with velocity $c_A$.

This leads to a compelling model of Alfvén waves as disturbances of the magnetic field lines themselves, analogous to waves propogating along strings. Indeed, in magnetohydrodynamics, the magnetic field lines can be said to be "frozen into" the plasma, and the Alfvén mode is purely transverse (no compression) like the transverse displacements on a string.

This "string" model for Alfvén waves is further illustrated by a look at the definition of the Alfvén velocity. Using our approximation for S, we have

$$c_A = \sqrt{c^2 \Omega_e \Omega_i / \omega_p^2} \qquad (56)$$

For a singly ionized plasma, $n_e = n_i$, and since $n_i m_i = \rho_i$, the ion density of the plasma (and approximately the total density, since the electrons contribute negligible mass),

$$c_A^2 = \frac{B_0^2/4\pi}{\rho} \qquad (57)$$

$B_0^2/4\pi$ is twice the magnetic energy density, or magnetic tension of the field lines, while $\rho$ is the mass density. This is completely analogous to the formula for the wave speed in a wire, $v^2 = T/\lambda$, where T is tension and $\lambda$ is mass/length.

## Integrals for E(x,t)

Knowing $\vec{E}(\vec{k},\omega)$ (at least in principle), we can calculate $\vec{E}(\vec{x},t)$ from the inverse Fourier transform (5).

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left(-\frac{4\pi i\omega}{c^2}\right)A^{-1}\vec{j}_s e^{i(\vec{k}\cdot\vec{x}-\omega t)}d^3k\,d\omega \qquad (58)$$

In the reference frame of a conducting object moving with velocity $\vec{v}$, the current density vector is not a function of frequency. In the proper frame of the plasma, the current density is $\vec{j}_s(\vec{k},\omega) = 2\pi\delta(\omega-\vec{v}\cdot\vec{k})\vec{j}_s'(\vec{k})$. The conductor is moving in the $\hat{x}_1$ direction, so $\vec{j}_s(\vec{k},\omega) = 2\pi\delta(\omega-vk_1)\vec{j}_s'(\vec{k})$, hence

$$\vec{E}(\vec{x},t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left(-\frac{4\pi ivk_1}{c^2}\right)A^{-1}\vec{j}_s'(\vec{k})e^{i(\vec{k}\cdot\vec{x}-k_1vt)}d^3k \qquad (59)$$

$$E(x,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left(-\frac{4\pi ivk_1}{c^2\det A}\right)RM^{-1}R^{-1}\vec{j}_s'(\vec{k})e^{i(\vec{k}\cdot\vec{x}-k_1vt)}d^3k \qquad (60)$$

In the MHD range, as we saw, the poles of the denominator of our integrand have a particularly simple form. Inserting the MHD expression for det $A_0$ into the Fourier integral (60), and remembering that due to the delta function $\omega = vk_1$, we get

$$\vec{E}(\vec{x},t) = -\frac{4\pi i}{\omega_e^2}\left(\frac{c_A}{c}\right)^2\int_{-\infty}^{\infty}\frac{v^5k_1^5RM^{-1}R^{-1}\vec{j}_s(\vec{k})e^{i(\vec{k}\cdot\vec{x}-k_1vt)}}{(v^2k_1^2-c_A^2k_3^2)(v^2k_1^2-c_A^2k^2)}d^3k \qquad (61)$$

The integral (61) for $\vec{E}(\vec{x},t)$ has poles at the points in k space for which $k_1 = \pm c_Ak_3/v$ or $k_1 = \pm c_Ak/v$. The latter pole is purely imaginary for real k's because $k_1 = k\sin\theta$, so for the latter pole to exist $\sin\theta = c_A/v$, which is larger than one.

The other poles are relevent, and their existence suggests the possibility of evaluating the $k_3$ integration by means of a closed contour integral in the complex $k_3$ plane, using the residue theorem to find the integral on a path around the poles.

Considerations involving minute corrections due to dissipative processes (e.g., collisions) allow us to determine which of the poles is to be included in the integration path. We find that the pole at $k_3 = vk_1/c_A$ is actually at $k_3 = vk_1/c_A + i\varepsilon$, where $\varepsilon$ is some small, positive number. The other pole is at $k^3 = -vk_1/c_A - i\varepsilon$.

Our $k_3$ integral is

$$\int_{-\infty}^{\infty} \vec{E}(\vec{k}) e^{ik_3 x_3} dk_3 \tag{62}$$

From this we can complete a closed loop with a semi-circular path in either the upper or lower half-plane of the complex $k_3$ plane. For integration over this semi-circle, let $k_3 = R \exp(i\alpha)$, where we take the limit of $R \to \infty$, and where $\alpha$ goes from either 0 to $\pi$ ($\alpha > 0$) or from 0 to $-\pi$ ($\alpha < 0$). Since $E(k)$ is of order $1/k_3^2$ or lower, as long as the $\exp(ix_3 k_3)$ term is bounded, the contribution of the semicircle to the total closed contour integral is 0.

If $x_3 > 0$, $\exp(ix_3 k_3) = \exp(-Rx_3 \cos \alpha)$; this goes to 0 for $R \to \infty$ for the upper semicircle. Similarly, for $x_3 < 0$, we would choose the lower semicircle. In either case, the residue theorem and Cauchy's theorem tell us that if $g(z)$ is analytic,

$$\oint \frac{g(z)}{z-z_0}\, dz = 2\pi i g(z_0) \tag{63}$$

For $x_3 > 0$, only the pole at $vk_1/c_A + i\varepsilon$ is inside the loop; for $x_3 < 0$, only the pole at $-vk_1/c_A - i\varepsilon$ is inside. Taking the limit as $\varepsilon \to 0$,

$$\int_{-\infty}^{\infty} \vec{E}(k1,k2,k3)e^{ik_3x_3}dk_3$$

$$= 2\pi i\left(-\frac{4\pi i}{\omega_e{}^2}\right)\left(\frac{c_A}{c}\right)^2 \frac{v^5k_1^5RM^{-1}R^{-1}\vec{j}_s(k_1,k_2,\frac{v}{c_A}k_1 \operatorname{sgn}(x_3))}{2vk_1(v^2k_1^2-c_A^2k^2)\operatorname{sgn}(x_3)} \tag{64}$$

or

$$\vec{E}(\vec{x},t) = \sqrt{2\pi}\iint_{-\infty}^{\infty}\frac{C\vec{j}_s e^{i(\bar{x}_1k_1+x_2k_2)}}{k^2(1-S/P)P\ \omega k_3}\ dk_1dk_2 \tag{65}$$

where C is the a matrix derived from $RM^{-1}R^{-1}$ and $\bar{x}_1 = x_1-vt+(v/c_A)\ x_3$.

As shown above, only the $\hat{x}_2$ component of the vector $\vec{j}_s$ is nonzero for the case we are studying, so only $(C^{-1})_{12}$, $(C^{-1})_{22}$, and $(C^{-1})_{32}$ contribute to the integrand. For all frequencies, the expressions for these are

$$C_{21} = (1-S/P)(k^2-(\tfrac{\omega}{c})^2P)k_1k_2 \tag{66}$$

$$C_{22} = (1-S/P)(k^2-(\tfrac{\omega}{c})^2P)k^2 \tag{67}$$

$$C_{23} = (1-S/P)k^2k_2k_3 \tag{68}$$

## Evaluation for $\vec{E}(\bar{x},t)$

The expressions in the MHD regime for the electric fields due to our source are

$$E_1 = \frac{\sqrt{2\pi}}{c^2} c_A \iint_{-\infty}^{\infty} \frac{j_s k_1 k_2}{k_1^2+k_2^2} e^{i(\bar{x}_1 k_1 + x_2 k_2)} dk_1 dk_2 \qquad (69)$$

$$E_2 = -\frac{\sqrt{2\pi}}{c^2} c_A \iint_{-\infty}^{\infty} \frac{j_s k_2^2}{k_1^2+k_2^2} e^{i(\bar{x}_1 k_1 + x_2 k_2)} dk_1 dk_2 \qquad (70)$$

$$E_3 = -\frac{\sqrt{2\pi}}{\omega_p^2} v \iint_{-\infty}^{\infty} j_s k_1 k_2 e^{i(\bar{x}_1 k_1 + x_2 k_2)} dk_1 dk_2 \qquad (71)$$

As shown before, the source factor for a cylinder of length 2a and radius b is ($k_r = k_1^2+k_3^2 = k_1\sqrt{1+(v/c_A)^2} \cong k_1$

$$j_s = \frac{2I_s}{\pi\sqrt{\pi}b} \frac{J_1(bk_1)\sin(ak_2)}{k_1 k_2} \qquad (72)$$

All three definite integrals can be calculated analytically. Combining (69) and (72), we have

$$E_1 = -\frac{2I_s c_A}{\pi b c^2} \iint_{-\infty}^{\infty} \frac{(bk_1)\sin(ak_2)}{k_1^2+k_2^2} e^{i(\bar{x}_1 k_1 + x_2 k_2)} dk_1 dk_2 \qquad (73)$$

Separating the double integral and only taking the terms that symmetric with respect to $k_1$ and $k_2$,

$$E_1 = \frac{8I_s c_A}{\pi b c^2} \int_0^{\infty} dk_1 J_1(bk_1)\sin(\bar{x}_1 k_1) \int_0^{\infty} dk_2 \frac{\sin(ak_2)\sin(x_2 k_2)}{k_1^2+k_2^2} \qquad (74)$$

Using a table of integrals, we find

$$\int_0^{\infty} \frac{\sin(ak_2)\sin(x_2 k_2)}{k_1^2 + k_2^2} dk_1 = \frac{\pi}{4|k_1|}\left(e^{-|(a-x_2)k_1|} - e^{-|(a+x_2)k_1|}\right) \qquad (75)$$

(Gradshteyn & Ryzhik, p 715, ref. 5) which gives us

$$E_1 = \frac{2I_s c_A}{bc^2} \int_0^{\infty} \frac{J_1(bk_1)}{k_1} \left(e^{-|a-x_2|k_1} - e^{-|a+x_2|k_1}\right)\sin(\bar{x}_1 k_1) dk_1 \qquad (76)$$

Again turning to the integral table(Gradshteyn & Ryzhik, p. 763, ref. 5),

$$E_1 = \frac{2I_s c_A}{bc^2}\left[\frac{\bar{x}_1}{b}(1-r_1)-\frac{\bar{x}_1}{b}(1-r_2)\right] \quad (77)$$

or

$$E_1 = (2I_s c_A/b^2c^2)\bar{x}_1(r_2-r_1) \quad (78)$$

where

$$\bar{x}_1^2 = \frac{b^2}{1-r_1^2} - \frac{(a-x_2)^2}{r_1^2}, \quad \bar{x}_1^2 = \frac{b^2}{1-r_2^2} - \frac{(a+x_2)^2}{r_2^2} \quad (79)$$

Solving for $r_1$ and $r_2$, we find that

$$E_1 = C_0 \mathrm{sgn}(\bar{x}_1)\left[\sqrt{\sqrt{A_-^2+B_-^2}-A_-} - \sqrt{\sqrt{A_+^2+B_+^2}-A_+}\right] \quad (80)$$

where

$$C_0 = 2I_s c_A/(bc)^2 \quad (81)$$

$$A = (x_2 \pm a)^2 + b^2 - x_1^2 \quad (82)$$

$$B = 2x1(x_2 \pm a). \quad (83)$$

Similarly, for $E_2$, we have

$$E_2 = -\frac{2I_s c_A}{\pi bc^2}\iint_{-\infty}^{\infty}\frac{k_2}{k1}\,\frac{J_1(bk_1)\sin(ak_2)}{k_1^2+k_2^2}e^{i(\bar{x}_1k_1+x_2k_2)}dk_1dk_2 \quad (84)$$

Separating the $k_1$ and $k_2$ integrations,

$$E_2 = -\frac{4I_s c_A}{\pi bc^2}\int_{-\infty}^{\infty}dk_1\frac{1}{k_1}J_1(bk_1)e^{i\bar{x}_1k_1}\int_{-\infty}^{\infty}dk_2\frac{k_2\sin(ak_2)\cos(x_2k_2)}{k_1^2+k_2^2} \quad (85)$$

We first evaluate the integral in $k_2$, using

$$\int_0^{\infty}\frac{k_2\sin(ck_2)}{k_1^2+k_2^2}dk_2 = \frac{\pi}{2}e^{-|ck_1|}\,\mathrm{sgn}(c) \quad (86)$$

(Gradshteyn & Ryzhik p. 406, ref. 5) leaving

$$E_2 = -\frac{2I_s c_A}{\pi bc^2}\int_{-\infty}^{\infty}\frac{1}{k_1}J_1(bk_1)e^{i\bar{x}_1k_1}\,\frac{\pi}{2}\left[e^{-|(a-x_2)k_1|}\,\mathrm{sgn}(a-x_2)\right.$$

$$+e^{-|(a+x_2)k_1|}\operatorname{sgn}(a+x_2)\Big]\,dk_1 \tag{87}$$

Since the integrand is symmetric, only the cosine part of $\exp(i\bar{x}_1k_1)$ contributes to the integral. Using the exponential form of cosine, and knowing that

$$\int_0^\infty e^{-ck_1}J_1(bk_1)(1/k_1)dk_1 = (1/b)(\sqrt{c^2+b^2}-c) \tag{88}$$

is true for $c > 0$, we can integrate (87) to get

$$\begin{aligned}E_2 = -(I_s c_A/b^2c^2)[&\operatorname{sgn}(a-x_2)(\sqrt{(|a-x_2|-i\bar{x}_1)^2+b^2}\\ &+\sqrt{(|a-x_2|+i\bar{x}_1)^2+b^2}-2|a-x_2|)+\\ &\operatorname{sgn}(a+x_2)(\sqrt{(|a+x_2|-i\bar{x}_1)^2+b^2}+\sqrt{(|a+x_2|+i\bar{x}_1)^2+b^2}\\ &-2|a+x_2|)]\end{aligned} \tag{89}$$

We can rewrite (89) in terms of the A's and B's defined above. Since

$$\sqrt{A+iB}+\sqrt{A-iB} = \sqrt{2}\sqrt{\sqrt{A^2+B^2}+A} \tag{90}$$

the final expression for $E_2$ is

$$\begin{aligned}E_2 = -C_0[&\operatorname{sgn}(a-x_2)\sqrt{\sqrt{A_-^2+B_-^2}+A_-}\\ &+\operatorname{sgn}(a+x_2)\sqrt{\sqrt{A_+^2+B_+^2}+A_+}-4a/\sqrt{2}]\end{aligned} \tag{91}$$

The final expression for $E_3$ has particularly simple form. We must evaluate the integral

$$E_3=-\frac{2I_s v}{\pi b\omega_p^2}\iint_{-\infty}^{\infty} J_1(bk_1)\sin(ak_2)e^{i(\bar{x}_1k_1+x_2k_2)}dk_1dk_2 \tag{92}$$

which can be written

$$\begin{aligned}E_3=-\frac{2I_s v}{\pi b\omega_p^2}\int_{-\infty}^{\infty}dk_1J_1(bk_1)e^{i\bar{x}_1k_1}(2i)\int_{-\infty}^{\infty}dk_2(&e^{i(x_2+a)k_2}\\ &-e^{-i(x_2+a)k_2}).\end{aligned} \tag{93}$$

Using the definition of the delta function, we get

$$E_3= -\frac{2I_s v}{\pi b\omega_p^2}(2\pi)[\delta(x_2+a)-\delta(x_2+a)](2i)\int_{-\infty}^{\infty}J_1(bk_1)e^{i\bar{x}_1k_1}dk_1 \tag{94}$$

or

$$E_3=\frac{16I_s v}{b\omega_p}[\delta(x_2+a)-\delta(x_2-a)]\int_0^\infty J_1(bk_1)\sin(\bar{x}_1k_1)dk_1 \qquad (95)$$

(ref. 5, p. 730). Using the table of integrals, we see that

$$\int_0^\infty J_n(bk_1)\sin(\bar{x}_1k_1)dk_1=\begin{cases}\dfrac{\sin(n\sin^{-1}(\bar{x}_1/b))}{b^2-\bar{x}_1^2} & |\bar{x}_1|<b\\ \dfrac{b^n\cos(n\pi/2)}{\sqrt{\bar{x}_1^2-b^2}(\bar{x}_1+\sqrt{\bar{x}_1^2-b_2})^n} & |\bar{x}_1|>b\end{cases} \qquad (96)$$

We restate the other two components of E along with the $x_3$ component:

$$E_1 = C_0\,\mathrm{sgn}(\bar{x}_1)\left[\sqrt{\sqrt{A_-^2+B_-^2}-A_-}-\sqrt{\sqrt{A_+^2+B_+^2}-A_+}\right] \qquad (97)$$

$$E_2 = -C_0[\mathrm{sgn}(a-x_2)\sqrt{\sqrt{A_-^2+B_-^2}+A_-}$$

$$+\ \mathrm{sgn}(a+x_2)\sqrt{\sqrt{A_+^2+B_+^2}+A_+}-4a/\sqrt{2}] \qquad (98)$$

$$E_3=\begin{cases}\dfrac{16I_s v}{b\omega_p}[\delta(x_2+a)-\delta(x_2-a)]\dfrac{\bar{x}_1}{b}\dfrac{1}{b^2-\bar{x}_1^2} & (|\bar{x}_1|<b)\\ 0 & (|\bar{x}_1|>b)\end{cases} \qquad (99)$$

where $C_0= 2I_s c_A/(bc)^2$, $A=(x_2\pm a)^2+b^2-x_1^2$, $B=2x_1(x_2\pm a)$, and $\bar{x}_1 = x_1 - vt + (v/c_A)|x_3|$.

## Results

The first thing we can discover from our results is the confirmation that a "wing" is indeed an appropriate description for the electric fields due to plasma waves from the conducting object, as first described by Drell, et al. (ref. 4). The wing stems from the fact that E must be a function of $x_2$ and $\bar{x}_1$. This means that the fields are constant along the "V" shape of $x_1 - vt + (v/c_A)|x_3|$ =constant, or in the object's reference frame, along $x_1' + (v/c_A)|x_3|$ =constant.

This makes the $(\bar{x}_1, x_2)$ plane the most convenient one in which to plot the electric field lines. We do so for cylinders with a/b = 1, 10/3, and 10/1 (see figures 2 to ). Note that for large distances, the field lines are nearly circular; this is the field due to a line dipole. This limit is physically reasonable because the distant field is caused by the charge associated with the two currents induced in the plasma by the conducting object. These currents lead to a line dipole.

Figures (2 to ) also illustrate the degree to which our assumption that $\vec{j}_s' = j_s'\hat{x}_2$ is valid. Since $\vec{j}_s'$ is in the same direction as $\vec{E}'$ inside the conductor, we see that for a/b=1 or 10/3, $\vec{E}$ and $\vec{j}_s$ are reasonably well (but not exactly) approximated by vectors in the $\hat{x}_2$ direction, while for a/b=10 the approximation is better. Figures ( to ) are graphs of $E_1$, $E_2$, and E (= $E_1^2 + E_2^2$ , i.e., ignoring the δ-function from the $E_3$ component) as functions of $x_2$, plotted for our three values of a/b and for various $\bar{x}_1$. For these plots we set $C_0=1$. As can be seen from the analytic expressions for $\vec{E}$, as $x_2 \to \infty$, $E_1$ and $E_2 \to 0$. There is also a discontinuity at each endcap of the cylinder, reflecting the discontinuity in $\vec{j}_s'$.

## Suggestions for Further Study

The formulas we have derived for $\vec{E}$ detail the field due to low frequency waves in a cold plasma. In order to get a complete expression for the electric field, we would have to extend our calculations to include the complexity of higher frequency waves. The importance of the MHD contribution to the total electromagnetic field is related to the size and speed of the object. As noted earlier, for MHD, $\omega < \Omega_i$. Because the current is stationary in the reference frame of the conductor, $\omega = vk$, so $k_1 < \Omega_i/v$. The major contribution to the fields will be in this range if $L > v/\Omega_i$, where L is the scale length of the object. For low earth orbit, L is on the order of 20 m.

Another useful study would be the calculation of E due to the Alfven wings of a sphere. Also, one would like to correct the expressions for $\vec{E}$ in order to take into account the fact that the cold plasma model is only an approximation to actual warm plasmas.

An interesting experiment would be to measure the Alfven wings produced by the space telescope (to be launched in 1986) or due to a tethered satellite system. The MHD contributions calculated here could be compared to the measured electric fields.

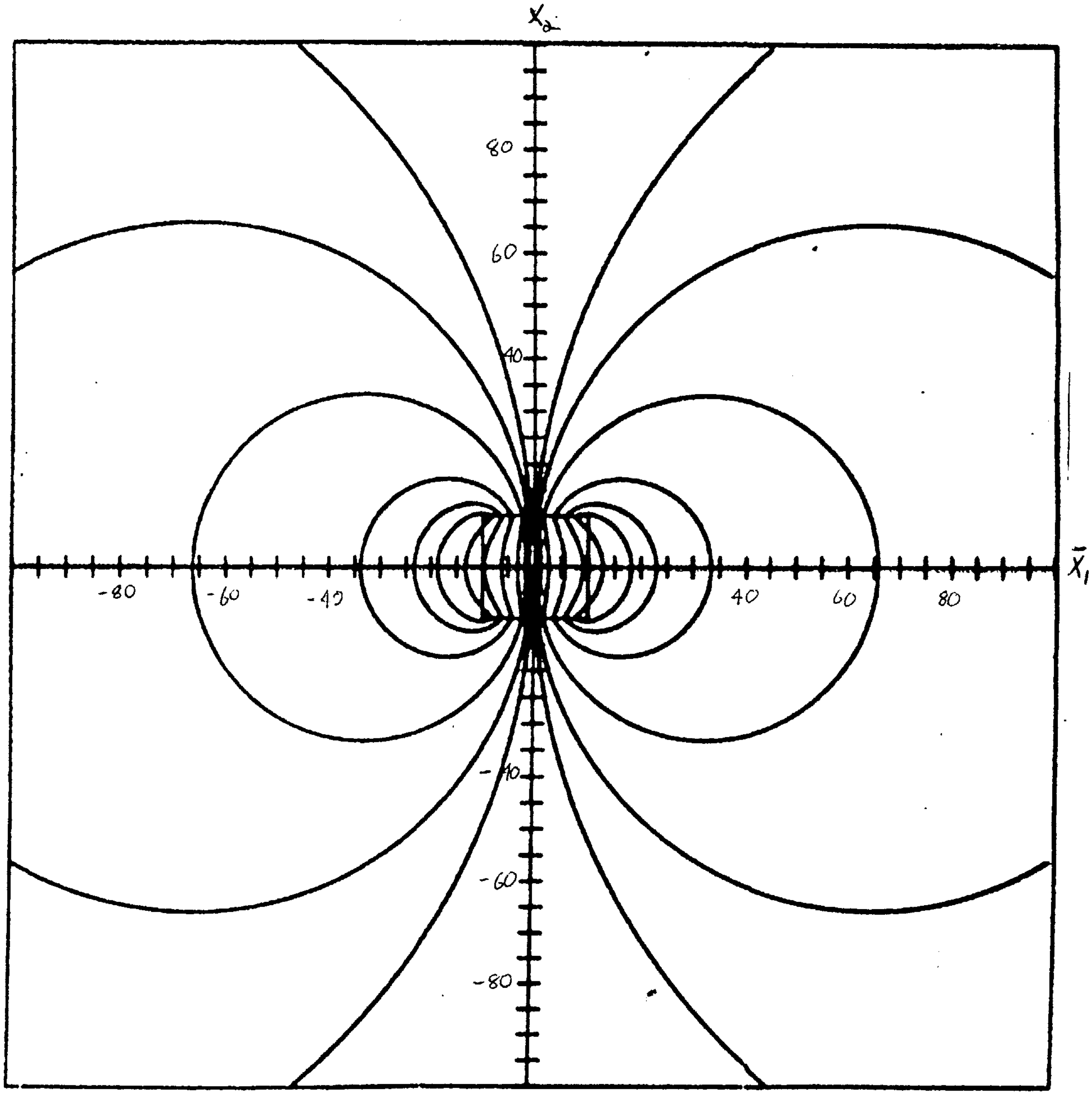


Fig 2 – Field lines for $\vec{E}$ ($a = 10$, $b = 10$)

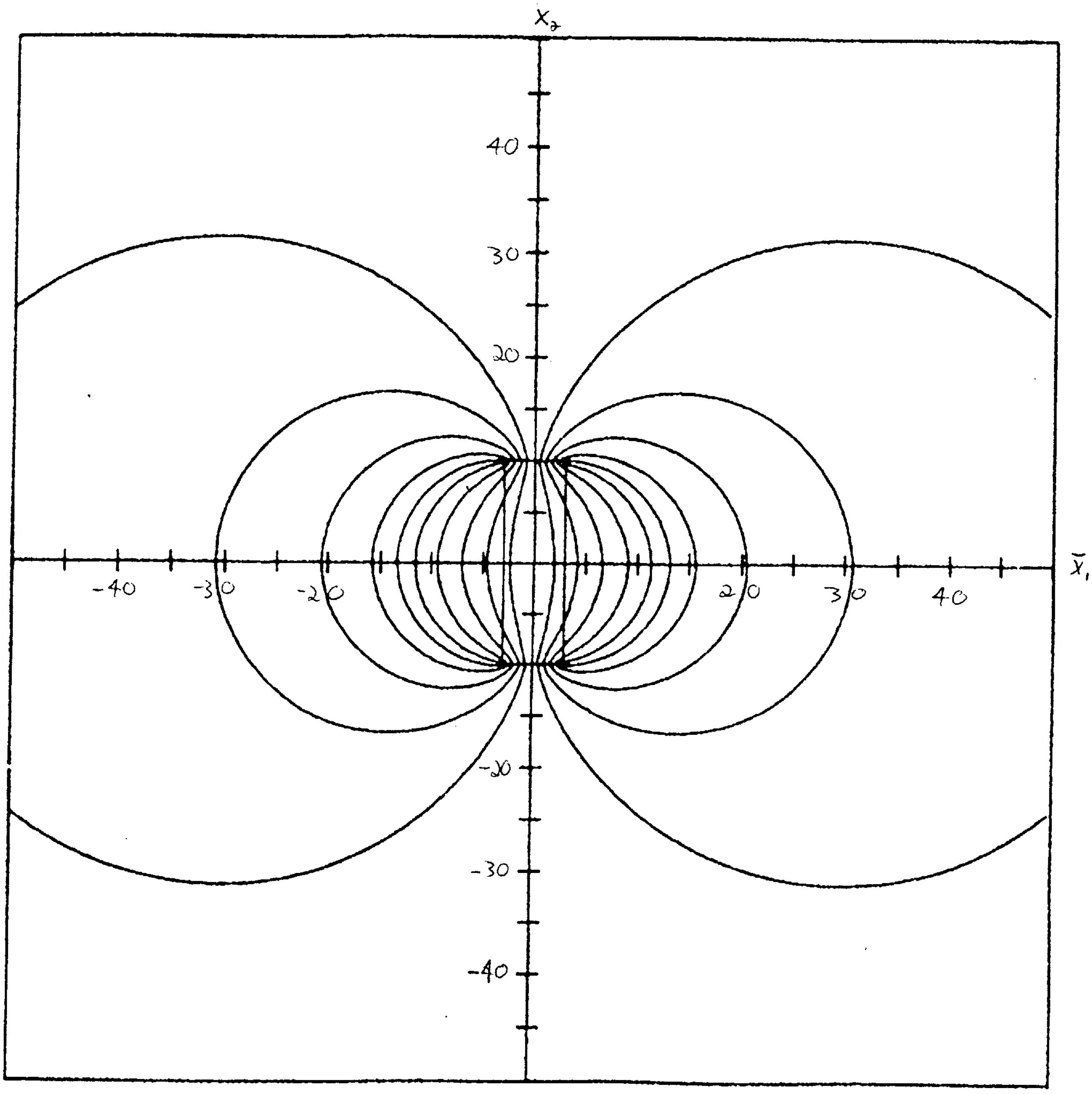


Fig 3 - Field lines for $\vec{E}$ $(a=10, b=3)$

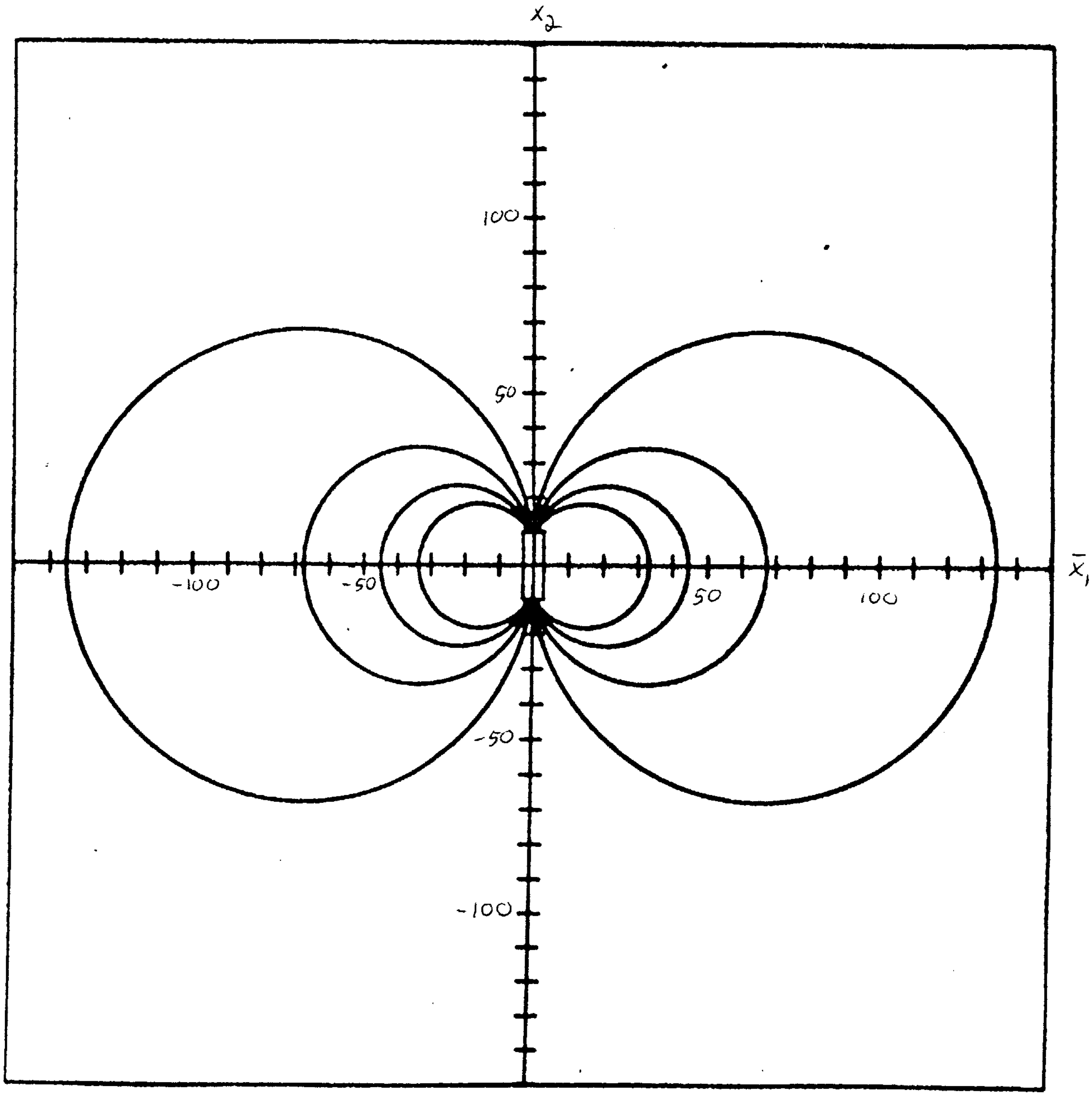


Fig 4 - Field lines for $\vec{E}$ $(a=10, b=3)$

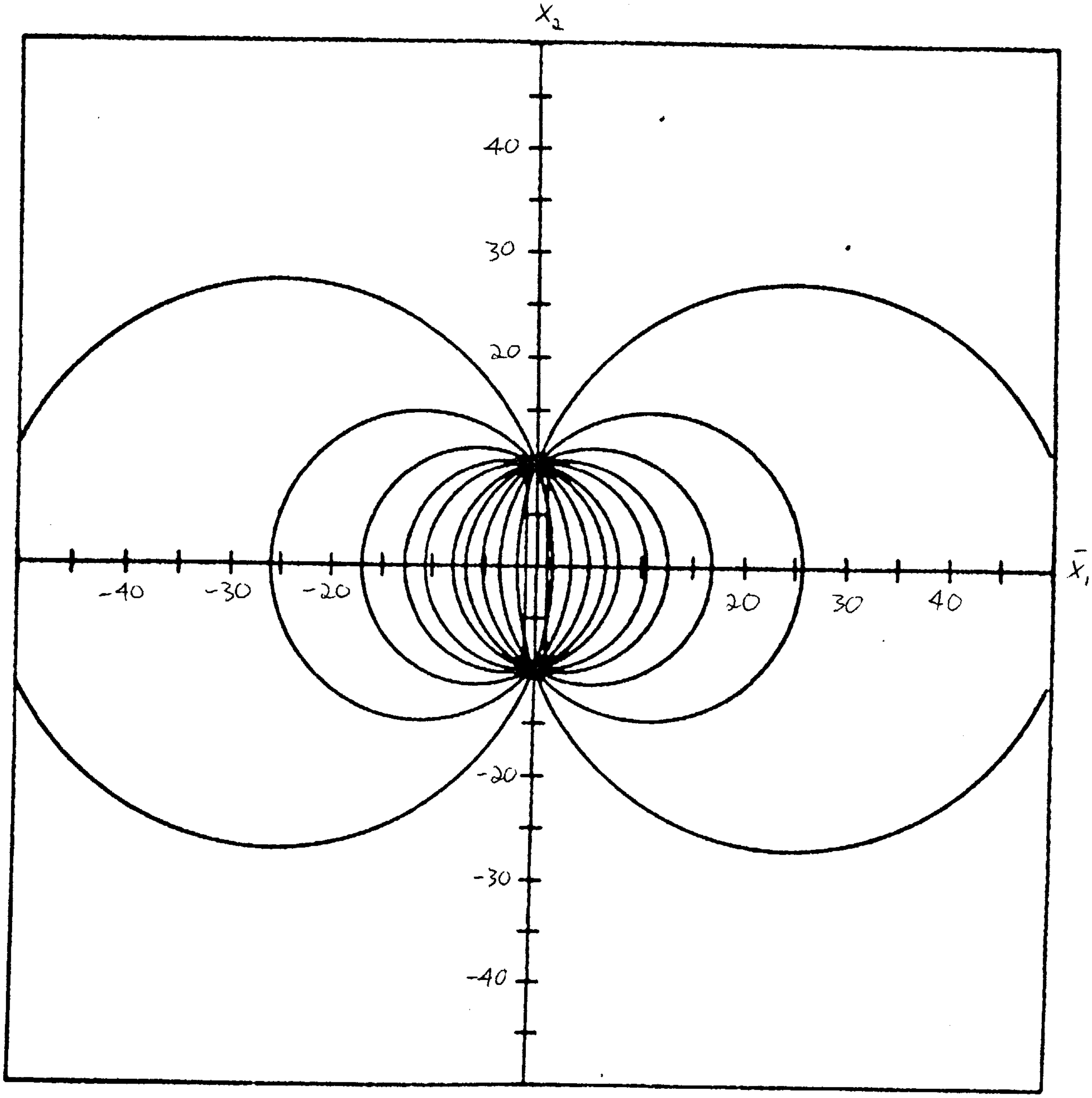


Fig. 5 — Field lines for $\vec{E}$ $(a=10,\ b=1)$

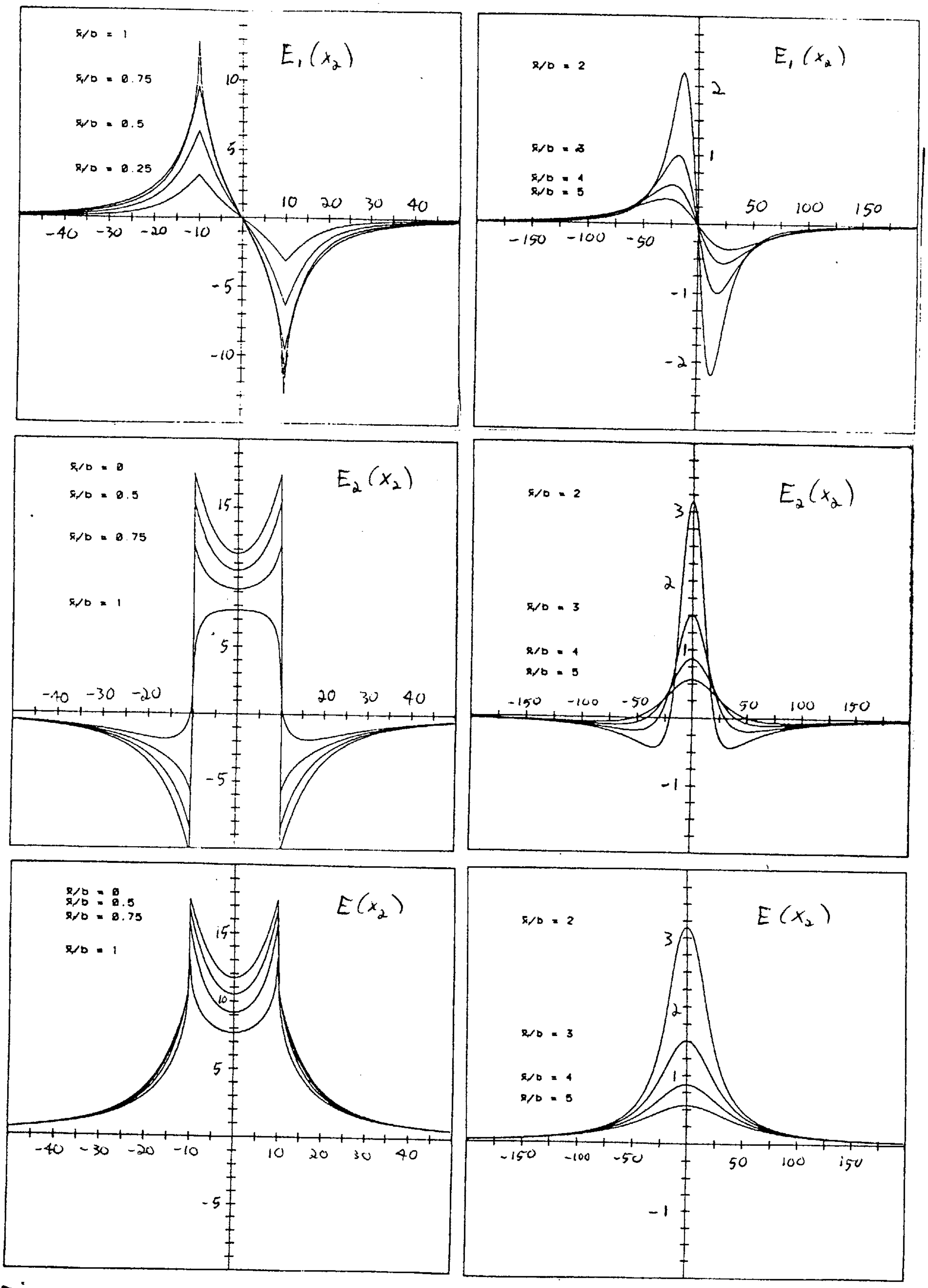


Fig 6 – $E_1(x_2)$, $E_2(x_2)$, $E(x_2)$ for $a = 10$, $b = 10$

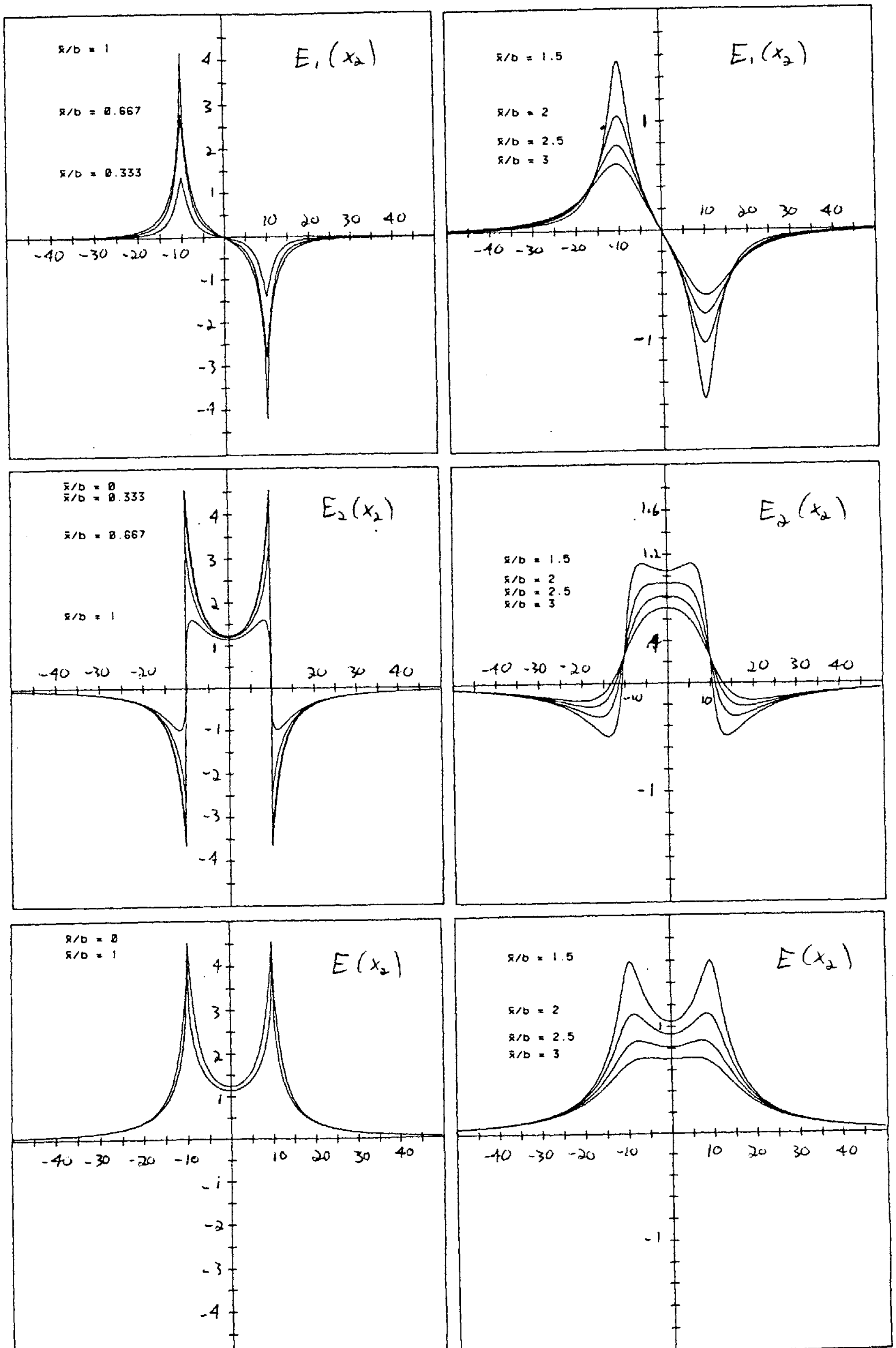


Fig 7 – $E_1(x_2)$, $E_2(x_2)$, $E(x_2)$, for $a = 10$, $b = 3$

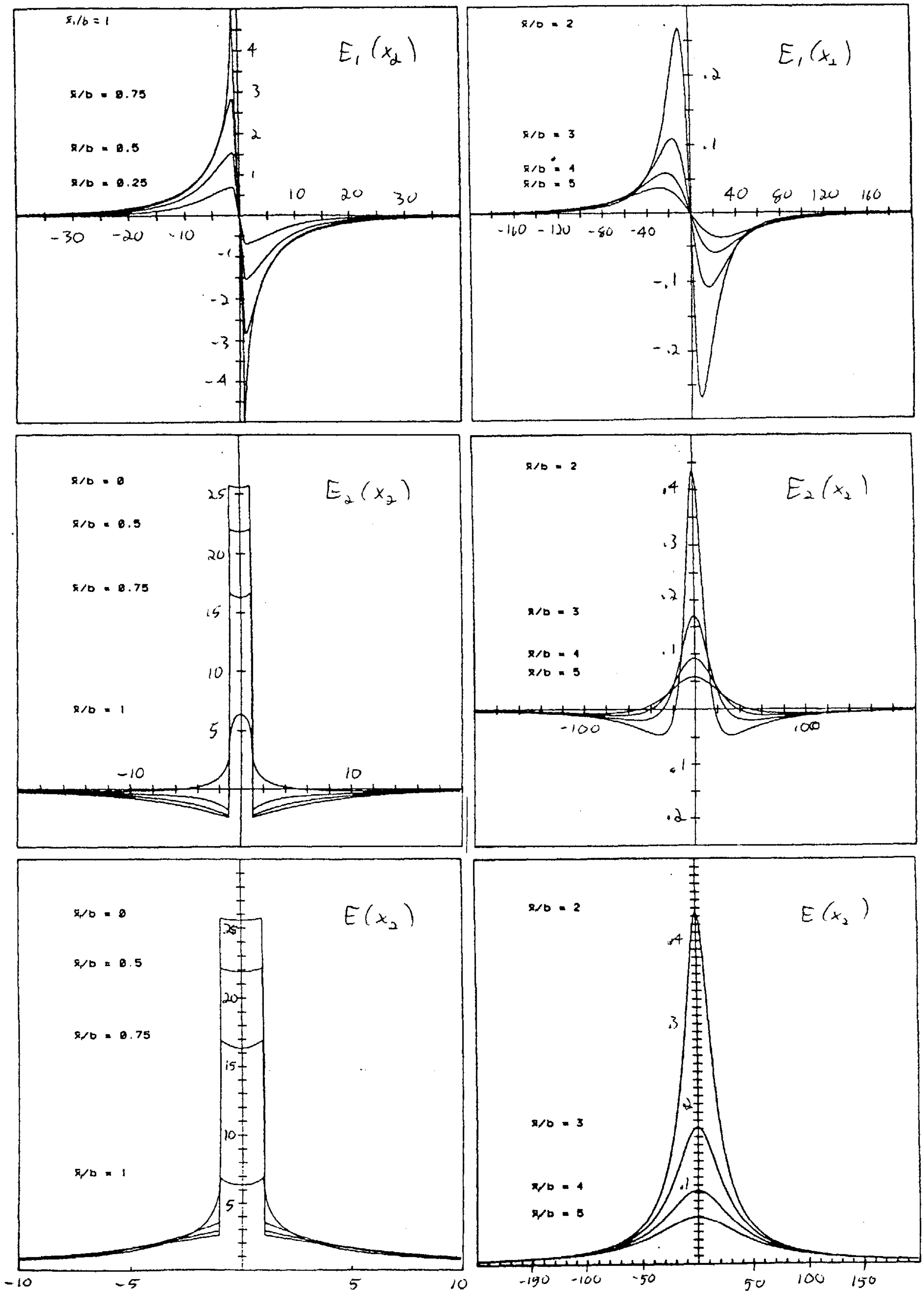


Fig 8 – $E_1(x_2)$, $E_2(x_2)$, $E(x_2)$ for $a=10$, $b=1$

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