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# Universal crossovers between entanglement entropy and thermal entropy

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We postulate the existence of universal crossover functions connecting the universal parts of the entanglement entropy to the low-temperature thermal entropy in gapless quantum many-body systems. These scaling functions encode the intuition that the same low-energy degrees of freedom which control low-temperature thermal physics are also responsible for the long-range entanglement in the quantum ground state. We demonstrate the correctness of the proposed scaling form and determine the scaling function for certain classes of gapless systems whose low-energy physics is described by a conformal field theory. We also use our crossover formalism to argue that local systems which are “natural” can violate the boundary law at most logarithmically. In particular, we show that several non-Fermi-liquid phases of matter have entanglement entropy that is at most of order  $L^{d-1} \log(L)$  for a region of linear size  $L$  thereby confirming various earlier suggestions in the literature. We also briefly apply our crossover formalism to the study of fluctuations in conserved quantities and discuss some subtleties that occur in systems that spontaneously break a continuous symmetry.

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## I. INTRODUCTION

Recently, an exchange of ideas between quantum-information science and many-body physics has led to an improved understanding of the “corner” of Hilbert space in which ground states of local Hamiltonians reside. One of the most important tools for investigating the properties of many-body ground states is entanglement entropy, defined as the von Neumann entropy of the reduced density matrix of a spatial subsystem. The ubiquitous presence of a boundary law for the entanglement entropy, as reviewed in Refs. 1 and 2, has provided a rough guide to the entanglement properties of quantum ground states. This rough intuition led to a new class of quantum states generically called tensor network states (Refs. 3–5) as well as new insights into the classification and identification of many-body phases and phase transitions in Refs. 6–10.

In this paper we show that by considering the relationship between thermal and entanglement entropy we can place significant constraints on ground-state entanglement structure for “natural” systems. One of our main motivations is to characterize the possible violations of the boundary law for entanglement entropy at zero temperature. There have been many constructions of anomalously entangled ground states in the quantum information community, e.g., Ref. 11, but what do these have to do with ordinary quantum systems relevant for laboratory studies? There are also motivations from the study of mutual information in quantum systems at finite temperature in Refs. 12 and 13. Interesting critical phenomena are visible in the mutual information, and a first step toward understanding these numerical results is a more complete understanding of the temperature dependence of the von Neumann entropy of a single region.

Our basic assumption that connects thermodynamics with entanglement is that the same low-energy degrees of freedom are responsible for both long-range entanglement and low-temperature thermal physics. To give a concrete example,

in one-dimensional relativistic critical systems, while the high-energy physics contributes an area law term to the entanglement entropy, only the low-energy modes contribute to the  $\log L$  entanglement and low-energy thermal properties as discussed in Refs. 14–16. Indeed, there is a universal crossover function that interpolates between the zero-temperature entanglement entropy and the finite-temperature thermal entropy of a given subregion (consisting of a single interval). A related example is provided by quantum-impurity models where a localized impurity interacts with an effectively one-dimensional metallic system (the  $s$ -wave channel) as studied in Ref. 17. We study and generalize this crossover phenomenon in a variety of critical systems in different dimensions. A related approach that makes some connection between entanglement and thermal quantities appears in Ref. 18 where an area law up to logarithmic corrections was proven for a variety of systems.

In more detail, we will make the following assumptions throughout this paper. We always study gapless systems since it is obvious (though not rigorous) that generic gapped phases obey a boundary law. Our primary assumption is that there exists a universal crossover function that relates thermal and entanglement entropy. This crossover function is only defined up to analytic boundary law terms coming from high-energy physics. We also assume that the system does not possess extensive ground-state degeneracy and that the Hamiltonian is not fine tuned (beyond the tuning necessary to reach criticality). We will mostly consider translation-invariant states, but we do discuss disordered states in Sec. VI. In short, we want to consider sensible gapless ground states of local Hamiltonians, but in an effort to be precise, we give the above assumptions as sufficient criteria for “sensible.” Finally, let us note that the renormalization group perspective on entanglement structure permeates our entire discussion.

As we said, our crossover functions will only be well defined up to terms analytic in region size  $L$  which respect the boundary law, e.g., nonnegative powers of  $L$ . These

nonuniversal terms may *a priori* contain nontrivial functions of the cutoff and temperature. Later we will revisit what sorts of cutoff-dependent terms are allowed. The motivation for this assumption is that nonuniversal contributions come from high-energy physics which is effectively short ranged. The nonuniversal contributions, as a function of  $L$ , should only show terms that could appear in a gapped phase. The “gap” from the RG perspective is simply the energy scale above which we have integrated out high-energy degrees of freedom. Of course, this is just an assumption; there is no rigorous proof, and should our assumptions be violated in some system, then our considerations will not apply.

We should also emphasize that our considerations apply in their simplest form only at the critical point. We must be considering sufficiently low energy physics so that all irrelevant operators have flowed to zero under the RG. In terms of practical contact with lattice models, our results would apply to systems at a critical point and for sufficiently large regions and low temperatures. These regions should also taken to be smooth with no corners or other sharp features.

We study the von Neumann entropy  $S(L, T) = -\text{Tr}(\rho_L \log \rho_L)$  of a real-space region of linear size  $L$  in  $d$  spatial dimensions as a function of temperature,  $T$ , and region geometry. Recall that at zero temperature most gapless quantum systems in  $d > 1$  dimensions satisfy a boundary law for the entanglement entropy  $S_L \sim L^{d-1}$  with the coefficient of this term nonuniversal (see Ref. 1). However, there are gapless systems that violate the boundary law for entanglement entropy including free fermions with a Fermi surface,<sup>19-21</sup> Landau Fermi liquids,<sup>22</sup> and Weyl fermions in a magnetic field at weak and strong coupling.<sup>22</sup> These examples have an entanglement entropy  $\sim L^{d-1} \log(L)$ .

It is of enormous interest to generalize this result to understand the entanglement structure of *non-Fermi-liquid* ground states of matter. Many such states share with the Fermi liquid the crucial feature that there are gapless excitations that reside at a surface in momentum space. However, unlike in a Fermi liquid there is no description of these excitations in terms of a Landau quasiparticle picture. Such states were suggested to also violate the area law for the entanglement entropy based on a heuristic argument that views that gapless momentum space surface as a collection of effective one-dimensional systems.<sup>21</sup> If the area law is indeed violated, can the violation be stronger than in a Fermi liquid?

An example of a non-Fermi-liquid state with a gapless surface in momentum space was studied numerically in Ref. 23. The second Renyi entropy of a wave function (obtained by Gutzwiller projecting a free Fermi sea) for a gapless quantum spin liquid phase of an insulating spin system in two dimensions was calculated using Monte Carlo methods. The second Renyi entropy was shown to obey a behavior consistent with  $L \log(L)$ . Given the current limitations on system size it is hard to distinguish this from a power-law violation of the area law. It is therefore important to have a general understanding of how seriously the area law can be violated in such a spin liquid state.

The quantum spin liquid phase discussed above is expected to be described by a low-energy effective theory with a Fermi surface of emergent fermionic spin-1/2 particles (spinons) coupled to an emergent  $U(1)$  gauge field. Similar effective field

theories describe Bose metals, some quantum critical points in metals, and other exotic gapless systems. In all these cases the violation of the boundary law is suggested by heuristic arguments. Based on the analogy with Fermi liquids we might guess that the violation is logarithmic.<sup>21</sup> It is clearly important to have a firm argument for the correctness of this guess. Providing such an argument is one of the purposes of this paper. What about other non-Fermi-liquid states where the effective theory is not yet understood? We will address a class of such states that have a critical Fermi surface with appropriate scaling properties<sup>24</sup> to discuss the scaling constraints on their entanglement structure. In all these cases we argue that the  $L^{d-1} \log(L)$  is the fastest possible parametric scaling with  $L$  in  $d$  dimensions.

Besides the von Neumann entropy, we also investigate the scaling behavior of fluctuations in conserved quantities as in Refs. 19, 25, and 26. Here the structure is slightly richer, but the basic conclusions are very similar. In phases with unbroken symmetry, the fluctuations in the conserved quantity generating the symmetry scale no faster than  $L^{d-1} \log(L)$  at zero temperature, again under the assumption that the same low-energy modes responsible for thermal fluctuations also give rise to these zero-temperature fluctuations.

This paper is organized as follows. We begin with a discussion of the crossover behavior in the simplest conformally invariant case in  $d$  dimensions. Next we discuss the case of codimension 1 critical manifolds relevant for Fermi liquids, and then we discuss the general structure including higher codimension critical manifolds. Finally, we turn to a discussion of fluctuations in conserved quantities. We conclude with a discussion of possible violations of our scaling formalism.

## II. SCALING FORMALISM: INTRODUCTION

### A. Conformal symmetry

Consider a local quantum system with Hamiltonian  $H$  at finite temperature  $T = \beta^{-1}$  so that the entire system is in the mixed state  $\rho(T) \propto \exp(-\beta H)$ . As  $\beta \rightarrow 0$  we recover the ground state up to corrections exponential in the gap to the first excited state. We will be exclusively interested in systems where  $H$  is either in a gapless phase or at a critical point. Thus we will always have some notion of scaling symmetry although we will often not have the full power of the conformal group.

Consider now a region  $R$  of linear size  $L$  inside a larger many-body system. The complement of region  $R$  is denoted  $\bar{R}$ . The reduced density matrix of  $R$  is

$$\rho_R(L, T) = \text{Tr}_{\bar{R}}(\rho(T)) \quad (1)$$

and the von Neumann entropy of this reduced density matrix is

$$S_R(L, T) = -\text{Tr}_R(\rho_R \log \rho_R). \quad (2)$$

We will also be interested in generalizations of the von Neumann entropy called Renyi entropies labeled by a parameter  $n$ :

$$S_n = \frac{1}{(1-n)} \log(\text{Tr}(\rho^n)). \quad (3)$$

The limit  $n \rightarrow 1$  of  $S_n$  is simply  $S_R$ , the von Neumann entropy of  $\rho_R$ .

Let us initially consider the special case of a conformal field theory in  $d$  spatial dimensions. Two simple limits exist. As  $LT \rightarrow 0$  (a nonuniversal velocity is suppressed) the von Neumann entropy recovers the usual entanglement entropy of the ground state. We know from earlier studies that the entanglement entropy may contain a mixture of universal and cutoff-dependent terms; see Refs. 27–29 for representative calculations. For example, the boundary law term, going as  $L^{d-1}$ , is nonuniversal, but there are universal logarithmic terms in  $d = 1, 3, 5, \dots$  dimensions. In  $d = 2, 4, \dots$  dimensions the universal logarithmic term is replaced by a constant term. On the other hand, as  $LT \rightarrow \infty$  we recover the usual thermal entropy going as  $(LT)^d$ .

Using our basic assumption we write the entropy of region  $R$  as

$$S_R(L, T) = T^\phi f_R(LT) + \dots, \quad (4)$$

where  $\dots$  stands for the aforementioned addition or subtraction of nonuniversal terms involving the momentum cutoff  $\Lambda$ . Let us now determine the properties of  $f_R$  and the exponent  $\phi$ . For the moment we suppress the region dependence writing  $f_R$  as  $f$ . First, as  $LT \rightarrow \infty$  we must recover the extensive thermal entropy and hence  $f(x \rightarrow \infty) \sim x^d$ . This further implies that  $\phi = 0$  to obtain the correct temperature dependence of the entropy. In the opposite limit as  $LT \rightarrow 0$  the only possibility for a nonzero and finite contribution is  $f(x) \rightarrow \text{constant}$  or  $f(x) \rightarrow \log x$ . The possibility of the logarithm is allowed since the  $T$  appearing in the logarithm can be replaced by  $\Lambda$  at the expense of a nonuniversal term.

We conclude that our scaling assumption is consistent with either a universal constant term or a universal logarithm in the entanglement entropy of a conformal field theory at zero temperature. Indeed, both these possibilities are realized; the logarithmic term obtains for  $d$  odd and the constant for  $d$  even. This is also an appropriate time to mention the possibility of shape dependence; for example, the fact that sharp corners in  $d = 2$  produce logarithmic corrections is completely consistent with our scaling framework (see Refs. 28 and 30). It is important to note that the constant term in odd dimensions is only meaningful in the absence of corners. Other types of universal terms such as  $(\log L)^p$  ( $p \neq 1$ ) or  $L^{d-1+\delta}$  ( $\delta > 0$ ) are not allowed unless they violate our assumptions and are unrelated to the thermal physics. Of course, this conclusion is very natural from the renormalization group point of view.

We briefly elaborate on this point and discuss the structure of high-energy contributions in more detail. Locality demands that all high-energy contributions be proportional to integrals of local geometric data over the boundary. Consider an entangling surface  $\partial R$  in  $d$  dimensions. We may use coordinates  $u^a$  ( $a = 1, \dots, d-1$ ) in terms of which the surface is  $x^i(u^a)$  and the induced metric is  $h_{ab} = \partial_a x^i \partial_b x^j \delta_{ij}$  (we only consider flat space here; the generalization is straightforward). In addition to the intrinsic geometry of the surface we also have extrinsic geometry related to the embedding of the surface into flat space. For example, a cylinder has extrinsic curvature but no intrinsic curvature whereas a sphere has both. The extrinsic geometry is controlled by the extrinsic curvature

which is given in terms of the normal vector  $n^i$  and the projector onto the surface  $P_j^i = \delta_j^i - n^i n_j$  as  $K_{ij} = P_i^k \partial_k n_j$ . Now an important constraint for global pure states is the requirement that  $S(R) = S(\bar{R})$ , and since we consider here only high-energy contributions that are independent of the low-energy physics, we may still demand this symmetry at finite temperature for the terms of interest. Since the only difference between the boundary of  $R$  and  $\bar{R}$  is the direction of the normal  $n$  we conclude that only even powers of  $n$  can appear. Thus only even powers of the extrinsic curvature and hence only even powers of derivatives can appear (the same is true for intrinsic terms). Roughly speaking, we must form fully contracted invariants involving the normal vector and the gradient, but requiring the normal vector to appear with only even powers forces the same for gradients due to rotation invariance. This explains the general even/odd structure of universal terms via a simple scaling argument—e.g., Ref. 31.

Let  $r$  denote the length scale of interest along the RG flow. The infinitesimal contribution to the entanglement entropy from degrees of freedom at scale  $r$  is of the form described above:

$$r \frac{dS}{dr} = \frac{L^{d-1}}{r^{d-1}} \left( c_0 + c_2 \frac{r^2}{L^2} + \dots \right), \quad (5)$$

where we have used the appropriate logarithmic measure for  $r$ . The presence of only even corrections comes from our argument above. Performing this integral from the UV  $r = \epsilon$  to the IR  $r = L$  gives the desired structure. More generally, one should cut this integral off at  $\min(L, 1/T)$  where, roughly speaking, the von Neumann entropy becomes thermal in nature. This folklore argument has been spelled out in some detail in Ref. 32. It provides an alternate perspective on the assumptions about nonuniversal terms that we discussed before; however, we still prefer our original assumptions since these generalize more easily to non-Fermi liquids where the cutoff dependence may not be as simple and other length scales may exist.

Returning to our main development, the simplest example of such a crossover function occurs in  $d = 1$  conformal field theories where the single-interval case is dictated by conformal invariance. The result is

$$S(L, T) = \frac{c}{3} \log \left( \frac{\beta \Lambda}{\pi} \sinh \left( \frac{\pi L}{\beta} \right) \right), \quad (6)$$

and we see immediately that this form is consistent with our general scaling hypothesis. The high-energy cutoff  $\Lambda$  can be shifted by a boundary-law-respecting term, but the thermal physics and long-range entanglement are independent of the precise choice of  $\Lambda$  (again up to a nonuniversal dimensional conversion factor).

## B. Conformal field theories in $d > 1$ : Holographic calculation

We have already mentioned one concrete example of such a crossover function in  $1 + 1$  dimensional conformal field theory. In general the computation of such a crossover function is a highly nontrivial task for interacting conformal field theories in spatial dimensions  $d > 1$ . However, one set of examples where a computation is possible is provided by holography. For an introduction to this set of ideas, see

Ref. 33. For our purposes it suffices to mention three facts. First, certain strongly interacting conformal theories living in flat space are dual to theories of gravity in a curved higher dimensional space known as anti-de Sitter (AdS) space. The UV of the conformal field theory lives at the boundary of the gravitational spacetime. Second, there is a simple prescription to compute the entropy in such theories, at least in a special limit. Briefly, we must compute the “area” (in Planck units) of a minimal “surface” in the extended gravitational geometry that terminates in the UV on the boundary of the region of interest. Third, finite-temperature effects are dual to placing a black hole in the gravitational spacetime.

Putting all these facts together permits a geometric calculation of the entropy of the field theory that is precisely of the form we assumed. We will now give the details of this calculation. Consider first the case of CFT<sub>1+1</sub>.  $x$  and  $t$  are the CFT directions while  $r$  is the emergent scale coordinate ( $r \rightarrow 0$  is the UV boundary). The gravitational geometry in this case is the AdS<sub>2+1</sub> black hole with metric

$$ds^2 = \frac{L_\Lambda^2}{r^2} \left( -f dt^2 + \frac{1}{f} dr^2 + dx^2 \right), \quad (7)$$

where  $f(r) = 1 - r^2/r_0^2$ . We study minimal curves  $r(x)$  terminating in an interval of length  $\ell$  at finite temperature. The minimal length is given by

$$J = \int_{-\ell/2}^{\ell/2} \frac{dx}{r} \sqrt{1 + (dr/dx)^2 f^{-1}}. \quad (8)$$

Minimizing this length with respect to  $r$  gives an equation of motion which may immediately be integrated to yield a conserved quantity

$$\frac{1}{r \sqrt{1 + (dr/dx)^2 f^{-1}}} = \frac{1}{r_m}, \quad (9)$$

where  $r_m$  is the maximum depth achieved by the curve. Solving this for  $dr/dx$  gives

$$dr/dx = \sqrt{\left(\frac{r_m^2}{r^2} - 1\right) f} = \frac{\sqrt{(r_m^2 - r^2) f}}{r}. \quad (10)$$

We may now rewrite the length as

$$J = 2 \int_\epsilon^{r_m} \frac{dr r}{\sqrt{(r_m^2 - r^2) f}} \frac{r_m}{r^2}, \quad (11)$$

while the parameter  $r_m$  is determined from

$$\ell = 2 \int_\epsilon^{r_m} \frac{dr r}{\sqrt{(r_m^2 - r^2) f}}. \quad (12)$$

$\epsilon$  is a UV cutoff.

Changing variables to  $w = r^2/r_m^2$  allows us to rewrite the integral for  $\ell$  as

$$\ell = r_0 \int_0^1 dw \frac{1}{\sqrt{(1-w)(\alpha^2 - w)}}, \quad (13)$$

where we have safely put  $\epsilon = 0$  and  $\alpha = r_0/r_m \geq 1$ . Performing the integral we obtain

$$\ell/r_0 = \log \left( \frac{\alpha^2 - 1}{(\alpha - 1)^2} \right) = \log \left( \frac{\alpha + 1}{\alpha - 1} \right). \quad (14)$$

We may now solve for  $\alpha$  in terms of  $\ell$ :

$$\alpha = \frac{1 + e^{-\ell/r_0}}{1 - e^{-\ell/r_0}}. \quad (15)$$

Returning now to the length  $J$  we find

$$J = \frac{r_0}{r_m} \int_\epsilon^1 dw \frac{1}{\sqrt{(1-w)(\alpha^2 - w)}} \frac{1}{w}. \quad (16)$$

Doing this integral gives

$$J = 2 \log \left( \frac{2r_0}{\epsilon} \sinh \left( \frac{\ell}{2r_0} \right) \right). \quad (17)$$

To compute the entropy we append the factor  $\frac{L_\Lambda}{4G_N}$  and use the relation  $c = \frac{3L_\Lambda}{2G_N}$  to obtain

$$S = \frac{c}{3} \log \left( \frac{2r_0}{\epsilon} \sinh \left( \frac{\ell}{2r_0} \right) \right). \quad (18)$$

We may now determine  $r_0$  in terms of the temperature. Zooming in near the horizon and writing  $r = r_0 - \rho$  we have

$$ds^2 \sim \frac{L_\Lambda^2}{r_0^2} \left( -\frac{2\rho}{r_0} dt^2 + \frac{r_0}{2\rho} d\rho^2 \right). \quad (19)$$

Changing variables to  $u = 2\sqrt{\rho}$  the near-horizon metric is brought into the form

$$ds^2 \sim -\frac{u^2}{r_0^2} dt^2 + du^2. \quad (20)$$

Demanding periodicity in imaginary time gives  $\beta/r_0 = 2\pi$  or  $2r_0 = 1/(\pi T)$ . Plugging this into our entropy formula reproduces the usual crossover function

$$S(\ell, T) = \frac{c}{3} \log \left( \frac{1}{\pi T \epsilon} \sinh(\pi T \ell) \right). \quad (21)$$

Let us now consider regions in higher dimensional conformal field theories. For a  $d + 1$  dimensional CFT the relevant metric is the AdS <sub>$d+2$</sub>  black hole

$$ds^2 = \frac{L_\Lambda^2}{r^2} \left( -f dt^2 + \frac{1}{f} dr^2 + dx_d^2 \right), \quad (22)$$

where  $f(r) = 1 - r^{d+1}/r_0^{d+1}$ .

We consider striplike regions in the field theory of cross-section  $A$  and width  $\ell$  ( $A$  has units of length <sup>$d-1$</sup> ). The minimal

surface area is

$$\sigma = A \int_{-\ell/2}^{\ell/2} \frac{dx}{r^d} \sqrt{1 + (dr/dx)^2 f^{-1}}. \quad (23)$$

As before, we may immediately integrate the equation of motion to yield the conserved quantity

$$\frac{1}{r^d \sqrt{1 + (dr/dx)^2 f^{-1}}} = \frac{1}{r_m^d}. \quad (24)$$

$r_m$  has the same meaning as before. Solving for  $dr/dx$  we find

$$dr/dx = \frac{\sqrt{(r_m^{2d} - r^{2d}) f}}{r^d}. \quad (25)$$

Putting these facts together gives two integrals

$$\sigma = 2A \int_{\epsilon}^{r_m} \frac{dr r^d}{\sqrt{(r_m^{2d} - r^{2d}) f}} \frac{1}{r^d} \frac{r_m^d}{r^d} \quad (26)$$

and

$$\ell = 2 \int_{\epsilon}^{r_m} \frac{dr r^d}{\sqrt{(r_m^{2d} - r^{2d}) f}}. \quad (27)$$

We may set  $\epsilon = 0$  in the integral for  $\ell$  which gives a cutoff-independent relation  $r_m = r_m(r_0, \ell)$ . Of course, when  $r_0 \rightarrow \infty$  ( $f \rightarrow 1$ ) we have  $r_m \propto \ell$ .

For the area integral we write

$$\frac{\sigma}{2A} = \int_{\epsilon}^{r_m} dr \left[ \left( \frac{r_m^d}{r^d \sqrt{(r_m^{2d} - r^{2d}) f}} - \frac{1}{r^d} \right) + \frac{1}{r^d} \right]. \quad (28)$$

We have essentially subtracted off the UV-sensitive boundary law term so that the integral in parentheses converges as  $\epsilon \rightarrow 0$ . The  $\epsilon$  dependence is now trivial to extract and we find

$$\frac{\sigma}{A} = \frac{2}{d-1} \frac{1}{\epsilon^{d-1}} + (\sigma/A)_{\text{fin}}(\ell, r_0). \quad (29)$$

But this is of the required crossover form: a universal crossover function plus a boundary-law-respecting piece sensitive to UV physics.

We may put this function into the precise form considered above by scaling the variables appropriately. In Eq. (27) set  $r_m = \ell g(\ell T)$  and  $w = r/r_m$  to obtain

$$1 = g(\ell T) \int_0^1 dw \frac{w^d}{\sqrt{1 - w^{2d}} \sqrt{1 - k(\ell T g(\ell T))^{d+1} w^{d+1}}} \quad (30)$$

with  $k$  some constant. This is an implicit equation for the scaling function  $g(x)$  that can easily be shown to have the properties claimed above. In particular, it shows relativistic length-energy scaling. Plugging this scaling form into (28) shows that  $(\sigma/A)_{\text{fin}}$  has the form

$$T^{d-1} \frac{1}{(\ell T g(\ell T))^{d-1}} \left[ 2 \int_0^1 dw \left( \frac{1}{w^d \sqrt{1 - w^{2d}} \sqrt{1 - k(\ell T g(\ell T))^{d+1} w^{d+1}}} - \frac{1}{w^d} \right) - \frac{2}{d-1} \right] = T^{d-1} f(\ell T). \quad (31)$$

One comment is necessary: The overall factor  $T^{d-1}$  differs from the  $\phi$  obtained above simply because we are here working in a limit where  $A$  is bigger than all other scales. Repeating our general analysis above predicts  $\phi = d - 1$  since we must have  $f(x \rightarrow \infty) \rightarrow x$ . Similarly, we have  $f(x \rightarrow 0) \rightarrow x^{-(d-1)}$  to compensate the vanishing powers of  $T$ .

### C. Nonrelativistic scale invariance

In this section we discuss critical theories with dynamical exponent  $z \neq 1$ . The entanglement structure of these theories has recently been analyzed in Ref. 34. The dynamical exponent controls the relative scaling of space and time leading to the invariant form  $LT^{1/z}$  where again we suppress a nonuniversal dimensionful parameter. The thermal entropy of such a theory scales as  $L^d T^{d/z}$  as follows simply from the requirement of dimensionlessness and extensivity. Let us again introduce a universal scaling function following the assumptions above. We write the entropy as

$$S(L, T) \sim T^\phi f(LT^{1/z}) + \dots,$$

and use the limit  $LT^{1/z} \rightarrow \infty$  to establish that  $f(x) \rightarrow x^d$  and  $\phi = 0$ . The rest of the analysis for the nonrelativistic case

is unchanged and again we are permitted at most universal constant or logarithmic terms in the entanglement entropy.

One can also perform a holographic computation in this setting using so called Lifshitz geometries. In fact, the spatial part of the metric is unchanged at zero temperature; hence the structure of the entanglement entropy is identical. For example, even space dimensions have subleading constants while odd space dimensions have subleading logs. These zero-temperature solutions may also be generalized into finite-temperature black hole solutions, at least for certain values of  $z$ . Reference 35 contains a nice example of such a Lifshitz black hole with  $d = z = 2$  and metric

$$ds^2 = -f \frac{dt^2}{r^{2z}} + \frac{1}{f} \frac{dr^2}{r^2} + \frac{dx_2^2}{r^2}, \quad (32)$$

where  $f = 1 - r^2/r_0^2$ . As claimed, the only difference between this metric and the relativistic examples above is in the  $r$  dependence of the  $dt^2$  term and the different power of  $r$  appearing in  $f$ . The same manipulations establish a nearly identical crossover structure to the relativistic case except that the argument of all scaling functions is  $\ell T^{1/2}$  instead of  $\ell T$ . As always, a dimensionful constant has been suppressed.

A simple condensed matter example of such a system is provided by spinless fermions  $c_r$  hopping in one dimension.

Suppose there is a nearest neighbor hopping  $t_1$  and third neighbor hopping  $t_3$ . The Hamiltonian is thus

$$H = -t_1 \sum_r (c_r^+ c_{r+1} + \text{H.c.}) - t_3 \sum_r (c_r^+ c_{r+3} + \text{H.c.}). \quad (33)$$

By setting  $t_3 = t_1/3$  we can arrange to have the Fermi velocity  $dE/dk$  and  $d^2E/dk^2$  vanish at  $k = \pm\pi/2$ . If we also set the chemical potential to zero then we have a one-dimensional free fermion system with Fermi points having dynamical exponent  $z = 3$  (the first nonvanishing derivative). However, the ground-state wave function is identical to the usual  $z = 1$  filled Fermi sea and hence all entanglement properties are unchanged. The crossover function now interpolates between the usual entanglement entropy and a thermal entropy going like  $T^{1/3}$ .

### III. SCALING FORMALISM: CODIMENSION 1

Now we turn to the case where the low-energy degrees of freedom reside on a codimension 1 subspace in momentum space. By contrast, the scale-invariant theories in the previous section had low-energy degrees of freedom only a single point in momentum space (or finite set of isolated points). Examples of systems with a codimension 1 gapless surface includes Fermi liquids with a  $d - 1$  dimensional Fermi surface in  $d$  dimensions, Bose metals, spinon Fermi surfaces, and much more. Later we will consider the case of a general codimension gapless manifold.

#### A. Review of Fermi liquids

The low-energy physics of a Fermi liquid is, for many purposes, effectively one dimensional (an exception to this rule is provided by zero sound which requires the full Fermi surface to participate). Thus Fermi liquids violate the boundary law for entanglement entropy because one-dimensional gapless systems violate the boundary law. The anomalous term has been found to be universal in the sense that it depends only on the geometry of the interacting Fermi surface and not on any Landau parameters. Remarkably, this term also controls the finite-temperature entropy to leading order in  $T/E_F$ . The universal part of the entanglement entropy and the low-temperature thermal entropy are connected by a universal scaling function which can be calculated using one-dimensional conformal field theory.

We work in  $d = 2$  for concreteness. Consider a real-space region  $A$  of linear size  $L$  in a Fermi liquid with spherical Fermi sea  $\Gamma$ . The entanglement entropy for this region scales as  $S_L \sim k_F L \log L + L/\epsilon + \dots$  with the boundary-law-violating term universal and the subleading term nonuniversal. The precise value of the boundary-law-violating term is expressed in terms of the geometry of the real-space boundary  $\partial A$  and the Fermi surface  $\partial\Gamma$  as

$$S_L = \frac{1}{2\pi} \frac{1}{12} \int_{\partial A} \int_{\partial\Gamma} dA_x dA_k |n_x \cdot n_k| \log(L), \quad (34)$$

where  $n_x$  and  $n_k$  are unit normals to  $\partial A$  and  $\partial\Gamma$ . The intuition behind this formula is simply that the Fermi surface in a box of size  $L$  is equivalent to roughly  $k_F L$  gapless modes that each contribute  $\log L$  to the entanglement entropy. This formula is known as the Widom formula because of

its relation to a conjecture of Widom in signal processing. The Widom formula has not yet been rigorously proven, but it has been checked numerically and can be obtained simply from the one-dimensional point of view. To generalize to finite temperature we must replace the zero-temperature one-dimensional entanglement entropy by the general result at finite temperature given by

$$S_{1+1}(L, T) = \frac{c + \tilde{c}}{6} \log \left( \frac{\beta v \Lambda}{\pi} \sinh \left( \frac{\pi L}{\beta v} \right) \right). \quad (35)$$

Fermi liquids are described by many nearly free chiral fermions with  $c = 1$  and  $\tilde{c} = 0$ . The marginal forward scattering interactions do not change the number of low-energy modes, and hence the mode-counting picture still works quantitatively. However, we will only require a much more crude scaling assumption for our results.

Following this one-dimensional result a higher dimensional Fermi liquid also possesses a universal crossover between the low-temperature thermal entropy and the universal part of the entanglement entropy. This scaling function depends only on the geometry of the real-space region  $A$  and on the shape of the Fermi surface  $\partial\Gamma$ . For a spherical Fermi surface and spherical real-space region of radius  $L$  this universal crossover function is given by

$$S(L, T) = \frac{1}{2\pi} \frac{1}{12} \int_{\partial A} \int_{\partial\Gamma} dA_x dA_k |n_x \cdot n_k| \times \log \left( \sinh \left( \frac{\pi 2L |n_x \cdot n_k|}{\beta v_F(n_k)} \right) \right). \quad (36)$$

#### B. Non-Fermi liquids

We now consider the entanglement structure of non-Fermi-liquid states. We will restrict attention to the class of such states that have a codimension 1 gapless surface in momentum space, a critical Fermi surface, but where there is no Landau quasiparticle description of the excitations. A general scaling formalism has been developed for these states in Ref. 24. Of primary importance to us is the scaling of the thermal entropy. Our considerations will apply to any non-Fermi liquid falling into the general scaling formalism of Ref. 24 irrespective of the detailed low-energy theory. However, to be concrete let us consider a Fermi surface coupled to a gauge field in  $d = 2$ .

Recently there has been a controlled calculation of the properties of this system in terms of the gauge field dynamical critical exponent  $z_b$  and the number  $N$  of fermion flavors in Ref. 36 following important earlier work in Refs. 37 and 38. The expansion parameters are  $\epsilon = z_b - 2$  and  $1/N$  with a controlled limit possible as  $N \rightarrow \infty$  with  $\epsilon N$  fixed. This system was found to possess a critical Fermi surface. Following the intuition for Fermi liquids, this system will violate the boundary law for entanglement entropy because of the presence of many gapless one-dimensional degrees of freedom. However, this situation is not a trivial generalization of the Fermi liquid case because the system lacks a quasiparticle description.

Thermodynamic quantities can be understood roughly in terms of many one-dimensional gapless degrees of freedom on the Fermi surface with a dynamical critical exponent  $z_f \neq 1$ . The thermodynamic entropy is predicted to be  $S \sim k_F T^{1/z_f}$

( $k_F$  just measures the size of the Fermi surface). Additionally, the low-energy theory is such that only antipodal patches of the critical Fermi surface couple strongly to each other. With our current knowledge, we cannot formulate the patch theory as a truly one-dimensional theory; nevertheless thermodynamic quantities are correctly captured. Furthermore, although the Fermi surface curvature must be kept in all existing formulations, this curvature enjoys a non-renormalization property which makes it into a kind of gauge variable: Dispersing perpendicular to the patch normal is roughly like changing patches. However, we reiterate that our results depend only on the thermodynamic entropy being given by  $S \sim k_F T^{1/z_f}$ .

Before going on, let us digress for a moment to give a more detailed defense of this nontrivial assumption. In the patch framework of Refs. 36–38, each patch of the Fermi surface is coupled strongly only to the gauge field whose momentum is orthogonal to the local Fermi velocity. We can think of the gauge field (which originally sat at the origin of momentum space) as being broken up into pieces labeled by momentum and with each piece being moved out to its respective patch. Furthermore, this theory has a decoupling limit as the patch size goes to zero where different patches do not communicate. This decoupling is stronger than that which occurs in the Fermi liquid where forward scattering still mixes the patches. Each patch is a strongly coupled system of fermions plus gauge field with a thermal entropy density proportional to  $(k_F \Delta \theta) T^{1/z_f}$ . This entropy is obtained by estimating the free energy per unit volume as

$$F \sim \frac{T}{\ell_{\perp}(T)\ell_{\parallel}(T)}, \quad (37)$$

where  $\ell_{\perp}$  and  $\ell_{\parallel}$  are thermal length scales perpendicular and parallel to the local Fermi normal.  $1/\ell_{\perp} \sim k_F \Delta \theta$  sets the extent of the patch along the Fermi surface,  $1/\ell_{\parallel} \sim T^{1/z_f}$  sets the scaling along the Fermi normal, and as long as  $\Delta \theta \rightarrow 0$ , this accounting is exact. Note that the “bosonic” dynamical exponent  $z_b$  does not appear. We might have thought that  $\ell_{\perp}(T) \sim T^{-1/z_b}$  because this is the local direction of the gauge field momentum, but we must explicitly consider patches much smaller than this scale to reach the decoupling limit where we can reliably analyze the theory.

The original Hertz-Millis theory was obtained by integrating out the gapless fermions to obtain a nonlocal action for the gauge field. From this point of view, one might have expected an entanglement structure similar to the  $z \neq 1$  scale invariance discussed above. However, this is clearly incorrect since we are interested in the entanglement entropy of all the low-energy degrees of freedom, not just in the bosonic sector. Thus we should not “integrate out” or trace over the fermion degrees of freedom (nor should we integrate out the boson). However, as we already argued above, the full strongly coupled system should be understood in the patch framework where the integral over the Fermi surface is explicit. This is all to say that our assumption of a universal crossover function involving  $z_f$  and not  $z_b$  (although these numbers are related) should correctly capture boundary law violations in the entanglement entropy.

As usual, we write the von Neumann entropy as

$$S(L, T) = k_F T^{\phi} f(LT^{1/z}), \quad (38)$$

where  $z = z_f$  is the fermion dynamical critical exponent and  $\phi$  is an exponent to be determined.  $k_F$  just measures the size of the Fermi surface. It does not enter into the scaling argument in a nontrivial way. Now from thermodynamics we know that for  $x = LT^{1/z} \rightarrow \infty$  we must have  $S \sim T^{1/z} L^2$ . The  $L$  dependence requires that  $f(x) \sim x^2$  as  $x \rightarrow \infty$ , and the  $T$  dependence forces us to choose  $\phi = -1/z$ . As  $x = LT^{1/z} \rightarrow 0$  we must have  $f(x) \sim x$  in order to cancel the diverging  $T$  dependence. More generally, we must have  $f(x) = x \tilde{f}(x)$  with  $\tilde{f}(x) = a + b \log x$  as before. In particular, powers of  $\log$  are not allowed because these would produce  $T \rightarrow 0$  divergent terms that are supposed to be finite and universal. This demonstrates that these non-Fermi-liquid states may violate the boundary law at most logarithmically. An example of such a crossover function has recently been provided in the holographic context in Ref. 39.

Let us also make some more detailed speculations. The  $z_b = 2$  critical Fermi surface actually corresponds to a marginal Fermi liquid, so for this theory with  $N$  flavors we suspect that the boundary-law-violating term has the usual Fermi liquid form

$$S_L = N \times \frac{1}{2\pi} \frac{1}{12} \int_{\partial A} \int_{\partial \Gamma} dA_x dA_k |n_x \cdot n_k| \log(L). \quad (39)$$

At finite  $\epsilon = z_b - 2$  we expect modifications of the prefactor due to  $\epsilon$  dependent corrections. However, it is likely that the geometric dependence of the integral remains unchanged. Indeed, the different patches in the critical Fermi surface decouple much more strongly than they do in a Fermi liquid. We also note this geometrical form has recently been verified in a holographic setup with log violations of the boundary law. Depending on how precisely the critical Fermi surface is effectively one dimensional, a more structured crossover function of the form

$$S_L = \frac{1}{2\pi} \frac{T^{-1/z} C(N, \epsilon)}{12} \int_{\partial A} \int_{\partial \Gamma} dA_x dA_k |n_x \cdot n_k| f_{\epsilon, N}(LT^{1/z}) \quad (40)$$

may be expected. The function  $f_{\epsilon, N}$  would play the role of  $\log(\sinh(\pi x))$  in the Fermi liquid case. This detailed geometric form may be too strong a requirement in general, but the general scaling form in the previous paragraph is certainly reasonable.

Similar scaling arguments can be made for the Renyi entropy  $S_n = \frac{1}{1-n} \log(\text{Tr}(\rho^n))$ . We know that the Renyi entropy at finite temperature has a specific relationship to the thermal entropy because of the simple scaling with  $T$  of the finite-temperature free energy. The complete result is

$$S_n(T) = \frac{n - \frac{1}{n^{1/z_f}}}{n - 1} \frac{1}{1 + \frac{1}{z_f}} S(T), \quad (41)$$

where  $S(T)$  is the thermal entropy. This  $n$  dependence of the Renyi entropy actually holds for all  $T$  and  $L$  in the  $z = 1$  case of a Fermi liquid. It would be interesting to determine whether this is also true for the  $z \neq 1$  theory. Because the Renyi entropy is potentially much easier to calculate numerically and analytically we believe it is a useful target for future work.



#### IV. SCALING FORMALISM: GENERAL CODIMENSION

As a simple example of what we have in mind, consider a free fermion system tuned so that the ‘‘Fermi manifold’’ is of codimension  $q$  in the  $d$ -dimensional momentum space.  $q = 1$  is the generic case, a Fermi line in  $d = 2$  and a Fermi surface in  $d = 3$ .  $q = 2$  in  $d = 2$  and  $q = 3$  in  $d = 3$  correspond to Dirac points. The interesting case of  $q = 2$  in  $d = 3$  is the problem of Fermi lines where the zero-energy locus is one dimensional in the three-dimensional momentum space. The codimension is a useful parameter because it tells us the effective space dimension of the local excitations in momentum space. For example, the Fermi surface case always  $q = 1$  indicates that the excitations are effectively moving in one dimension (radially). Similarly, the case  $q = 2$  in  $d = 3$  corresponds to modes that move in two effective dimensions since there is no dispersion along the Fermi line. Just as we could calculate the entropy of Fermi surfaces by integrating over the contributions of one-dimensional degrees of freedom, we can obtain the entropy of these higher codimension systems by integrating over the contributions of dimension  $q$  degrees of freedom. We now make this explicit with a scaling argument.

Suppose that there exists a universal scaling function connecting the entanglement entropy to the thermal entropy for these codimension  $q$  free Fermi systems. If we assume a generic band structure then the dispersion may be linearized near the Fermi manifold yielding a dynamical exponent  $z = 1$ . Thus we write the von Neumann entropy of region  $R$  as

$$S(L, T) = k_F^{d-q} T^\phi f(LT), \quad (42)$$

where the factor of  $k_F^{d-q}$  accounts for the size of the Fermi manifold. The thermal entropy of such a system scales as  $k_F^{d-q} L^d T^q$  and hence as usual we require  $f(x) \rightarrow x^d$  as  $x \rightarrow \infty$ . This also fixes  $\phi = q - d < 0$ . Requiring regularity in the limit as  $T \rightarrow 0$ , we see that  $f(x) \rightarrow x^{d-q} \tilde{f}(x)$  as  $x \rightarrow 0$  where  $\tilde{f}(x) = a + b \log x$ . Thus we discover that such systems may have a universal term proportional to  $(k_F L)^{d-q}$  with either a constant or logarithmic prefactor. We emphasize that this is precisely what one expects from integrating a  $q$ -dimensional contribution over the Fermi manifold. In particular, we expect a constant prefactor for  $q$  even and a logarithmic prefactor for  $q$  odd because the  $q$ -dimensional system has  $z = 1$  and hence resembles a relativistic scale-invariant theory of the type we considered in Sec. II. These statements may be checked in the free fermion case because the entanglement entropy can be computed exactly; however, we defer a full discussion of this case to a future publication.

We conclude this section by noting that our conclusion is unmodified even if we have an interacting theory with a codimension  $q$  gapless manifold and with general  $z \neq 1$ . The scaling function has the form

$$S = k_F^{d-q} T^\phi f(LT^{1/z}) \quad (43)$$

with  $\phi = (q - d)/z$  which is obtained by matching to the thermal entropy  $k_F^{d-q} L^d T^{q/z}$ . Although we do not rigorously prove that the thermal entropy is always of this form, such a form does follow from a very general scaling analysis in momentum space and we know of no exceptions. We still predict a universal term, constant or logarithmic, proportional

to  $(k_F L)^{d-q}$  at zero temperature. For an example of such a transition, see Ref. 40, which considered a quantum critical point between a line nodal metal and a paired superconductor.

#### V. FLUCTUATIONS OF CONSERVED QUANTITIES

In this section we give a brief description of our scaling formalism as applied to an interesting observable: the fluctuations of a conserved charge. These considerations are motivated by the direct experimental accessibility of such fluctuations as well as by a desire to illustrate the general nature of our arguments. As the primary example, consider a conserved number operator  $N$  that may be written as a sum of local densities,  $N = \sum_r n_r$  in a lattice model or  $N = \int d^d x \rho(x)$  in the continuum. Unless otherwise specified we will restrict the discussion to phases where the corresponding symmetry is unbroken; i.e., we assume that the ground state is not a superfluid. The  $N$  operator commutes with the Hamiltonian and we may label energy eigenstates with different values of  $N$ . Hence  $N$  itself need have no fluctuations in the ground state. However, we can consider the restricted operator  $N_R = \int_R \rho$  which need not commute with the Hamiltonian and may have fluctuations. An interesting measure of the correlations between  $R$  and its environment is thus the quadratic fluctuations in  $N_R$

$$\Delta N_R^2 = \langle (N_R - \langle N_R \rangle)^2 \rangle. \quad (44)$$

We expect in a gapped phase of matter that these fluctuations satisfy a boundary law  $\Delta N_R^2 \sim L^{d-1}$ . What happens in a gapless phase? To understand this let us first relate these fluctuations to the structure factor. From the definition we have

$$\Delta N_R^2 = \int_{x, x' \in R} \langle (\rho(x) - \langle \rho \rangle)(\rho(x') - \langle \rho \rangle) \rangle, \quad (45)$$

where  $\rho(x)$  is the density of particles and  $\langle \rho \rangle$  is the average density. In a translation-invariant system the equal-time density-density correlator  $\langle \rho(x)\rho(x') \rangle = C(x - x')$  is a function only of the separation  $x - x'$ . In terms of a function  $\Theta_R(x)$  which is 1 for  $x \in R$  and 0 otherwise, we may then write

$$\Delta N_R^2 = \int_{x, x'} \Theta_R(x)\Theta_R(x')C(x - x') \quad (46)$$

$$= \int \frac{d^d k}{(2\pi)^d} |f(k)|^2 S(k). \quad (47)$$

Here  $f(k) = \int_x e^{-ik \cdot x} \Theta_R(x)$  is the Fourier transform of  $\Theta$  and  $S(k)$ , the equal-time structure factor, is the Fourier transform of  $C(x)$ . For  $L$  large,  $f(k)$  will be sharply peaked near  $k \approx 0$ . Thus (as expected)  $\Delta N_R^2$  will be determined by the small- $k$  behavior of the equal-time structure factor.

Consider first one-dimensional boson or fermion systems described as Luttinger liquids. Then for small  $k$ , the (Euclidean) density correlation function behaves as

$$\chi(k, i\omega) = \frac{\kappa v_c^2 k^2}{\omega^2 + v_c^2 k^2}. \quad (48)$$

Here  $\kappa$  is the macroscopic charge compressibility, and  $v_c$  the charge velocity. The equal-time structure factor is obtained by

integrating over  $\omega$ . We thus get

$$S(k) = \frac{\kappa v_c |k|}{2} \quad (49)$$

for small  $k$ . Consider now a region  $R$  where  $-\frac{L}{2} < x < \frac{L}{2}$ . Then

$$f(k) = \frac{2 \sin\left(\frac{kL}{2}\right)}{k}. \quad (50)$$

We thus get

$$\Delta N_R^2 = \frac{2\kappa v_c}{\pi} \int_0^\Lambda dk \frac{1 - \cos(kL)}{k} \quad (51)$$

$$= \frac{2\kappa v_c}{\pi} \ln(L\Lambda). \quad (52)$$

Here  $\Lambda$  is a nonuniversal high-momentum cutoff. Thus the gapless  $d = 1$  Luttinger liquid violates the boundary law by a  $\ln(L)$  factor just like the entanglement entropy. However, unlike the latter the prefactor for the number fluctuations is nonuniversal and will vary continuously within the Luttinger liquid phase.

Next we consider gapless systems described by a conformal field theory in dimensions  $d > 1$ . Conserved densities in such a theory have scaling dimension  $d$ . It follows that their equal-time structure factor  $S(k) \sim |k|^d$  for small  $k$ . To study the number fluctuations it is simplest to consider a cylindrical geometry and a region  $R$  defined by  $-\frac{L}{2} < x^i < \frac{L}{2}$ ,  $i > 1$ . Then we may write

$$f(k) = \delta^{(d-1)}(k_\perp) \left( \frac{2 \sin\left(\frac{k_1 L}{2}\right)}{k_1} \right). \quad (53)$$

For the number fluctuations we then get

$$\Delta N_R^2 \sim L_\perp^{d-1} \int_{k_1} \frac{\sin^2\left(\frac{k_1 L}{2}\right)}{k_1^2} |k_1|^d. \quad (54)$$

For  $d > 1$  the  $k_1$  integral is convergent in the infrared as  $L \rightarrow \infty$ . Thus we simply get the boundary law  $\Delta N_R^2 \sim L_\perp^{d-1}$  without any multiplicative log correction. Note once again the similarity with the behavior of the entanglement entropy.

Next we consider Fermi liquids with a gapless Fermi surface. Here it is well known that the equal-time structure factor  $\propto |k|$ . Taking again a cylindrical geometry as above we see that this implies a multiplicative  $\ln(L)$  correction to the boundary law:

$$\Delta N_R^2 \sim L^{d-1} \ln L. \quad (55)$$

For a free Fermi gas, the coefficient in front of  $|k|$  in the structure factor (and hence the prefactor of the  $L^{d-1} \ln L$  term in  $\Delta N_R^2$ ) is  $\sim N(0)v_F$  where  $N(0)$  is the density of states at the Fermi surface and  $v_F$  is the Fermi velocity. This will be modified in an interacting Fermi liquid by Landau parameters. As in the  $d = 1$  example this is a difference with the entanglement entropy of the Fermi liquid where the leading term is expected to be the same as that of the free Fermi gas.<sup>25</sup>

It is physically obvious that the number fluctuations in an interacting Fermi liquid must have a coefficient that is modified by interactions. In a conventional metal we can also consider spin fluctuations. These will scale the same way as the number

fluctuations in a Fermi liquid

$$(\Delta S_R^z)^2 \sim L^{d-1} \ln L, \quad (56)$$

but with a different prefactor. The dependence on the Landau parameters is readily seen from the considerations below.

At asymptotic low energies and long wavelengths the density-density correlation function in the Fermi liquid is

$$\chi(k, \omega) = \frac{\chi_0(k, \omega)}{1 - f_{0s} \chi_0(k, \omega)}, \quad (57)$$

where  $\chi_0$  is the bare pair bubble and  $f_{0s}$  is proportional to a Landau interaction (see Ref. 41 for further details). For simplicity, let us consider a spherical  $d = 2$  Fermi liquid where we have

$$\chi_0(k, \omega) = \frac{N(0)}{2\pi} \int_0^{2\pi} d\theta \frac{vk \cos \theta}{i\omega - vk \cos \theta}, \quad (58)$$

where  $v$  is the renormalized Fermi velocity and  $N(0)$  is the density of states at the Fermi energy. As usual, we have

$$N(0) = \frac{1}{(2\pi)^2} \int d^2k \delta(E_k - \mu) = \frac{k_F}{2\pi v_F} = \frac{m}{2\pi}. \quad (59)$$

We note that  $m$  is the quasiparticle effective mass of the Fermi liquid. If we take the static limit  $\omega = 0$  and then the long-wavelength  $k = 0$  limit we recover the usual Fermi liquid compressibility

$$\kappa = \kappa_0 \frac{v_0}{v} \frac{1}{1 + F_{0s}} \quad (60)$$

with  $v_0$  the bare Fermi velocity,  $\kappa_0$  the bare compressibility, and  $F_{0s} = f_{0s} N(0)$  the usual Landau parameter. To compute the number fluctuations at zero temperature we must obtain the equal-time correlation function by integrating over all frequency. The result is proportional to  $|k|$  and a computable function of the Landau parameter  $F_{0s}$ . Below we will need the large- $F_0$  behavior of this function. In the Appendix we show that for large  $F_{0s}$ ,  $S(k) \propto |k|/\sqrt{F_{0s}}$ . A similar result holds for the spin fluctuations but with the Landau parameter  $F_{0a}$  replacing  $F_{0s}$ .

In general just as we did for the entropy we may use the scaling crossover between zero and nonzero temperatures to constrain the form of the zero-temperature fluctuations of conserved densities. Because the discussion is similar to the case of the entropy, we will not give any of the details here. However, a few points are worth mentioning. First, what we study is now the crossover between the zero-temperature number fluctuations and the finite-temperature number fluctuations as controlled by the thermodynamic compressibility. Indeed, precisely such a crossover argument was used in Ref. 25 to argue that the number fluctuations in a Fermi liquid are modified by Fermi liquid parameters (unlike the entanglement entropy). As before, we are allowed to subtract off any analytic in  $L$  boundary law contributions since they can be generated by high-energy degrees of freedom. For relativistic scale-invariant systems the finite-temperature fluctuations go like  $(LT)^d$  and we again find that only constant or logarithmic universal terms are allowed at  $T = 0$ . More generally, we conclude that  $L^{d-1} \log L$  is the most severe violation of the boundary law that is possible without breaking the symmetry, e.g., in critical Fermi surface systems that remain compressible (see Ref. 24).

It is particularly interesting to consider the fate of the number and spin fluctuations as interactions are tuned to drive a continuous quantum phase transition from the Fermi liquid to a Mott insulator. Such a transition can occur in frustrated (i.e., non-bipartite) lattices for instance by tuning the ratio of the on-site repulsion  $U$  to the hopping matrix element  $t$  of a Hubbard model at fixed half filling. Reference 42 developed a theory of the universal critical properties near such a continuous Mott transition to a spin liquid Mott insulator with a spinon Fermi surface. We now use this theory to discuss the behavior of the number and spin fluctuations as the Mott transition is approached from the Fermi liquid side.

Clearly in the Mott insulator the number fluctuations will obey a boundary law, but the nature of the spin fluctuations depend on the state. If the electronic Fermi surface survives into the Mott-insulating phase as a spinon Fermi surface then we would expect spin fluctuations to continue to violate the boundary law. More formally if

$$|k_F, \uparrow\rangle |k_F, \downarrow\rangle \tag{61}$$

is the Fermi gas wave function then the projected wave function

$$\prod_r (1 - n_{r\uparrow} n_{r\downarrow}) |k_F, \uparrow\rangle |k_F, \downarrow\rangle \tag{62}$$

captures the physics of the spinon Fermi surface. In this state the number fluctuations are completely quenched by the projection; however, the spin fluctuations continue to scale as  $L \log L$ .

As the continuous Mott transition is approached from the Fermi liquid side Ref. 42 showed that the Landau parameter  $F_{0s}$  diverges rapidly as  $F_{0s} \sim |\delta|^{-\nu} \ln(\frac{1}{\delta})$  where  $\delta$  is the distance from the transition.  $\nu \approx 0.67$  is the correlation length exponent of the 3D XY universality class. As the coefficient of the  $L \ln L$  term in the number fluctuations is  $\propto 1/\sqrt{F_{0s}}$  for large  $F_{0s}$ , this determines how it vanishes on approaching the Mott critical point. Interestingly  $F_{0\alpha}$  is also predicted to diverge—albeit more weakly—as  $\ln(\frac{1}{\delta})$  as the transition is approached. Somewhat surprisingly this implies that the coefficient of the  $L \ln L$  term also vanishes for the spin fluctuations. However this result is only true for  $L \gg \xi_{LFL}$  where  $\xi_{LFL}$  is the length scale beyond which a Landau Fermi liquid description becomes legitimate. At scales  $L \ll \xi_{LFL}$  but bigger than all microscopic scales we still expect to have  $L \ln L$  behavior of the spin fluctuations though not of the density fluctuations. Thus for the spin fluctuations the coefficient of the  $L \ln L$  term depends on the order of limits associated with  $L, \xi$  going to  $\infty$ .

We wish to mention one final subtle point that arises in phases with broken symmetry and which is not properly captured in our thermodynamic treatment. Thus consider a superfluid phase where the particle number symmetry is broken. Besides the usual sound modes that possess an energy scaling as  $1/\ell$  ( $\ell$  is system size), there is also a zero mode with energy levels scaling like  $1/\ell^d$ . This zero mode plays an important role in finite-size systems, such as quantum Monte Carlo simulations, where it is responsible for ensuring that although the symmetry is broken, the many-body state has a definite particle number. In other words, it is related to the fact that the many-body ground state in a finite-size system is a properly a cat state unless the symmetry is broken explicitly.

This zero mode is not easily visible in thermodynamics but it does affect the entanglement entropy and number fluctuations in an important way.

Let us ignore the sound modes and ask for a state that captures the dynamics of the zero mode. Such a cat state has the form

$$|M\rangle \sim \int d\theta e^{-iM\theta} \bigotimes_r |\theta\rangle_r, \tag{63}$$

where  $|\theta\rangle_r$  is a state of definite phase on site  $r$ . We wish to trace out part of the system and compute the entropy of the remainder, but this problem has to be regulated because ambiguities are encountered in this procedure. Consider a simpler system consisting of  $p$  states per site with a  $Z_p$  symmetry relating them and where the many-body state is of the form  $\sum_{x=1}^p \bigotimes_r |x\rangle_r$ . If we now trace out part of the system we find a reduced density matrix for region  $R$  of the form

$$\rho_R = \text{tr}_{\bar{R}} \rho = \sum_x \bigotimes_{r \in R} |x\rangle \langle x|_r. \tag{64}$$

Perfect correlation with the environment has rendered the reduced density matrix completely diagonal. The entropy is now trivially  $S = \log p$ . To connect this model to the superfluid, we need only estimate the effective value of  $p$ . We do this by counting the effective number of orthogonal states in  $R$ . Now the many-body coherent state of the form

$$|\theta\rangle = \bigotimes_{r \in R} e^{-|\alpha|^2/2} \sum_n \frac{\alpha e^{in\theta}}{\sqrt{n!}} |n\rangle_r \tag{65}$$

has an overlap with a neighboring state of the form

$$\langle \theta | \theta' \rangle = \exp[|R| |\alpha|^2 (e^{-i(\theta-\theta')} - 1)]. \tag{66}$$

Expanding in small  $\theta - \theta'$ , the first real term is  $\exp -|R| |\alpha|^2 (\theta - \theta')^2/2$  and hence states greater than  $\Delta\theta \sim 1/\sqrt{|R| |\alpha|^2}$  are effectively orthogonal. Hence we may take  $p \sim 2\pi/\Delta\theta$  to give an entropy contribution of the form  $\log \sqrt{|R|} = \frac{d}{2} \log L$ . We also see that this mean-field cat state captures the extensive in subsystem size number fluctuation while maintaining the ground state with definite particle number.

We can also pin the order parameter to remove the anomalous contribution to the entanglement. For the superfluid, there is now no anomalous entropy although there are still extensive number fluctuations. On the other hand, in an antiferromagnet we may pin the Néel field to point in a particular direction. If we take as a mean-field state

$$|\text{Néel}\rangle = \bigotimes_{r \in A} |\uparrow\rangle_r \bigotimes_{r \in B} |\downarrow\rangle_r, \tag{67}$$

then we see immediately that there is no anomalous entanglement and the fluctuations of  $S_z$  are not extensive (they are zero). However, the fluctuations of  $S_x$  and  $S_y$  are still extensive. These simple considerations have been considerably developed in Ref. 43. A careful treatment including the interactions between the zero mode and the sound modes is also expected soon in Ref. 44. They find that the coefficient of the logarithm is somewhat modified from the simple-minded argument above.

## VI. DISCUSSION

Before concluding, let us point out some potential ways that one might violate the scaling forms we have developed. A nice example is provided by a certain degenerate spin chain. By fine-tuning the strength of the nearest neighbor couplings as a function of position along the chain, Ref. 11 showed that it was possible to find a ground state with entanglement entropy that scaled as the length of the interval. This was arranged by adjusting the couplings so that a real-space RG procedure always coupled the boundary spin inside the region with a spin outside the boundary; i.e., no spins formed singlets within the interval. This required exponentially decaying couplings and is clearly not generic.

Another route is provided by large ground-state degeneracy provided we use the completely mixed state. Recently such systems have received a lot of attention due to results showing non-Fermi-liquid behavior in certain holographic systems. However, we emphasize that nothing is special about the holographic setting; it is but one example. A simple condensed matter example with “ground-state degeneracy” is provided by the spin-incoherent Luttinger liquid. This system is a Luttinger liquid where the spin energy scale is much less than the charge energy scale. At temperatures above the spin energy scale, the spin-incoherent state emerges where the spin degrees of freedom are totally disordered. Such a state has an extensive temperature-independent entropy, but it also cannot be a true ground state. To force the state to zero temperature, we must fine-tune an infinite number of relevant operators (the entire spin Hamiltonian) to zero. We call this an IR-incomplete theory since it cannot be smoothly connected to zero temperature. More generally, we can imagine intermediate-scale RG fixed points that control the physics over a wide range of energies but which cannot be interpreted as ground states due to an infinite fine tuning. We know one possibility is that such a state may have extensive entropy, but perhaps there are other possibilities where the entanglement entropy scales like  $L^{d-1+\delta}$  for  $0 < \delta < 1$ .

Another setting where violations might occur is in random systems. In one dimension we know that even at infinite-randomness fixed points the boundary law for the average entanglement entropy is violated no worse than in the conformal case. However, we do not know whether infinite-randomness fixed points would violate the boundary law in higher dimensions. We expect, based on the scaling argument above, that any finite- $z$  random fixed point will not violate the boundary law, and we suspect that infinite-randomness fixed points would not either, but we do not give a definite argument at this time. We also note that there are considerable subtleties in these systems, e.g., typical versus average values. Since the entanglement entropy of a region has a probability distribution  $p(S, L)$ , it would be interesting to determine whether the distribution was a function of  $S/\log(L)$  only or something more complicated. In any event, there are many open issues at such random fixed points, the thermodynamics does not obey the simple forms we have considered here, and so we do not have much else to say about these issues at this time.

Let us also briefly mention long-range interactions. If these interactions are due to massless fields with a nonsingular action within the physical description, e.g., fluctuating gauge fields

or other critical bosonic modes, then a proper renormalization group description is possible and the entanglement entropy should have no additional anomalous structure. Similarly, as long as the long-range forces present in the system have such an interpretation, even if they must be introduced as auxiliary fields, we might expect no new anomalies to appear. On the other hand, consider the “1d chain” where, in addition to nearest neighbor hoppings, every site can hop to every other site with the same strength. Calling such a system one dimensional is a perversion, but it is an extreme form of long-range interactions. Clearly such a system can be expected to violate the one-dimensional boundary law more than logarithmically. The task that emerges is thus to understand where the crossover point is, as a function of the interaction range, to conventional one-dimensional behavior. One quite interesting situation where these considerations are directly applicable is momentum-space entanglement where the region  $R$  is some subset of momentum space instead of position space. This topic, which has already received some preliminary attention, deserves its own exposition which we will present elsewhere.

Finally, we note that from the general codimension section our formalism can in some sense encompass states that violate the boundary law more seriously than logarithmically provided that  $q$  is effectively less than 1. For example,  $q = 0$  roughly describes a state with gapless excitations everywhere in some region of finite measure in momentum space. This is a lot like the situation with ground-state degeneracy. Nevertheless, as we have already said, we know of no example of a sensible ground state with  $q < 1$ .

We have argued that entanglement entropy and thermal entropy may be connected via a universal crossover function in gapless phases and at critical points. One major consequence of this assumption is that local quantum systems cannot violate the boundary law more than logarithmically. However, we hasten to add that should our assumptions be violated, we have no objection. In particular, possible loopholes escaping our conclusion include fine tuning in the Hamiltonian, systems with many degenerate ground states, and systems with long-range interactions. Models showing these characteristics may indeed be physically realistic in special cases; nevertheless, we argue that conventional gapless systems, even those with critical Fermi surfaces, will not violate the boundary law more than logarithmically. Actually calculating the entanglement entropy in a model of a critical Fermi surface, perhaps in an  $\epsilon$  expansion, and studying in more detail the entanglement properties for  $q > 1$  are projects we leave for the future.

Many conventional quantum critical points that describe symmetry-breaking transitions, such as the  $2 + 1$  XY critical point, fall into the category of conformal field theories discussed in Sec. II. However, it was recently shown in Ref. 45 that “deconfined” quantum critical points have a different entanglement structure due to proximate topologically ordered phases. For example, Ref. 45 discussed the  $XY^*$  critical point in  $2 + 1$  dimensions which has the same correlation length exponent as the XY transition but a different anomalous dimension for the order parameter. This arises because the order parameter  $\Phi$  is actually composite  $\Phi = b^2$  and it is the

“fractional” bosons  $b$  that undergo the  $XY$  transition. However, an important difference is that the  $b$  bosons are coupled to a  $Z_2$  gauge field and hence not all operators in the  $XY$  theory of the  $bs$  are gauge invariant. Furthermore, there must always exist somewhere in the high-energy spectrum gapped  $Z_2$  vortices. This is all to say that while the thermal entropy of  $XY$  and  $XY^*$  are identical, they have different crossover functions and different entanglement properties at zero temperature.

A potentially profitable generalization of our work here would be to include in the scaling formalism the effect of relevant operators that move away from the critical point. Similarly, it would be interesting to study the scaling structure of the full Renyi entropy in more detail. Scale invariance fixes the high temperature Renyi dependence, but the zero-temperature entanglement structure as a function of Renyi parameter could be rather rich. Perhaps our scaling approach could shed some light on this structure. The generalization to multiple regions is also open and is especially relevant to studies of mutual information. Early work for Fermi liquids can be found in Ref. 25 where a simple crossover function for the mutual information was found. One finds the usual zero-temperature result for short distances and exponential decay set by the thermal length at long distances.

It would also be quite interesting to develop a variational class of density matrices that encode the kind of crossover behavior we described here. Since the mutual information always obeys a boundary law at finite temperature,<sup>12</sup> such states could in principle look like a density matrix generalization of tensor network states, although presumably the bond dimension would have to grow as the temperature was lowered to account for systems that violate the boundary law at zero temperature. Perhaps the new branching MERA approach<sup>46</sup> could help? One could compute some of these crossover functions in field theory to see what sort of universal information is easily accessible. We have already done the analogous calculations in holographic theories, but there it was already known that the entanglement entropy contains relatively little information due to the large- $N$  limit.

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### APPENDIX: FLUCTUATIONS AT LARGE $F_0$

As discussed in the main text, the number fluctuations of a region can be obtained from a knowledge of the equal-time density-density correlation function. This correlation function, at general times, is

$$C(x,t) = \langle \rho(x,t)\rho(0,0) \rangle. \quad (\text{A1})$$

Working at finite temperature and inserting a complete set of states we have the usual formula

$$C(x,t) = \sum_{nm} \frac{e^{-\beta E_n}}{Z} \langle n|\rho(x)|m \rangle e^{i(E_n - E_m)t} \langle m|\rho(0)|n \rangle, \quad (\text{A2})$$

or upon Fourier transformation,

$$\begin{aligned} S(k,\omega) &= \int dx dt e^{i\omega t - ikx} C(x,t) \\ &= \sum_{nm} \frac{e^{-\beta E_n}}{Z} |\langle n|\rho(k)|m \rangle|^2 \delta(\omega - (E_m - E_n)). \end{aligned} \quad (\text{A3})$$

The structure factor may be obtained from the imaginary-time correlation function after analytic continuation. The Euclidean density correlator is

$$\chi(i\omega, k) = \frac{\chi_0(i\omega, k)}{1 - f_0 \chi_0(i\omega, k)} \quad (\text{A4})$$

in terms of the bare density correlator

$$\chi_0(i\omega, k) = \frac{N(0)}{A_d} \int d\Omega_d \frac{v_F k \cos \theta}{i\omega - v_F k \cos \theta}. \quad (\text{A5})$$

$d$  is the spatial dimension,  $A_d = \int d\Omega_d$  is the total solid angle, and  $f_0$  is the short-ranged interaction strength.

We expect the dominant contribution to come from the zero-sound pole at large  $F_0 = f_0 N(0)$ . To extract this pole, which occurs at  $i\omega = v_{ZS} k$ , we use the fact that  $v_{ZS} \gg v_F$  at large  $F_0$  so that typically  $\omega \gg v_F k$ . Thus we approximate  $\chi_0$  as

$$\begin{aligned} \chi_0 &\approx \frac{N(0)}{A_d} \int d\Omega_d v_F k \cos \theta \frac{v_F k \cos \theta}{(i\omega^2)} \\ &= -\frac{N(0)}{d} \left( \frac{v_F k}{\omega} \right)^2. \end{aligned} \quad (\text{A6})$$

Inserting this approximation into the full density correlator we obtain

$$\chi(i\omega, k) = -\frac{N(0)}{d} \frac{v_F^2 k^2}{\omega^2 + v_{ZS}^2 k^2} \quad (\text{A7})$$

with  $v_{ZS}/v_F = \sqrt{F_0/d}$ . Now setting  $i\omega = \omega + i\eta$  and taking the imaginary part we find

$$\begin{aligned} \text{Im} \chi(\omega + i\eta, k) &= -\frac{N(0)}{d} \frac{v_F^2 k^2}{v_{ZS} k} \frac{\pi}{2} \\ &\quad \times [\delta(\omega - v_{ZS} k) - \delta(\omega + v_{ZS} k)]. \end{aligned} \quad (\text{A8})$$

By the usual fluctuation-dissipation theorem, this spectral density is related to the structure factor above via  $S(q, \omega) = -\frac{2}{1 - e^{-\omega/T}} \text{Im} \chi$ , so after integrating over frequency at  $T = 0$ , the equal-time structure factor is

$$\begin{aligned} S(k) &= \int \frac{d\omega}{2\pi} S(k, \omega) \\ &= \frac{1}{2\pi} \frac{N(0)}{d} \frac{(v_F k)^2}{v_{ZS} k} \pi. \end{aligned} \quad (\text{A9})$$

We can obtain the same result directly in imaginary time by simply integrating over imaginary frequency, i.e., studying the equal imaginary time correlator. Using the expression for the zero-sound pole we find again

$$S(k) = \frac{1}{2\pi} \frac{N(0)}{d} \frac{(v_F k)^2}{v_{ZS} k} \pi. \quad (\text{A10})$$

As an example, consider the case of  $d = 2$ . We have  $N(0) = \frac{m}{2\pi}$  with  $m = k_F/v_F$ . Using  $v_{ZS}/v_F = \sqrt{F_0/2}$  in  $d = 2$  we have

$$S(k) = \frac{m}{8\pi} \frac{v_F}{\sqrt{F_0/2}} k = \frac{k_F k}{4\sqrt{2}\pi\sqrt{F_0}}. \quad (\text{A11})$$

Hence the amplitude of  $L \ln(L)$  in the number fluctuations decay like  $1/\sqrt{F_0}$  at large  $F_0$ . It is also simple to show that the addition of long-range interactions, like a Coulomb interaction, render the system incompressible in the usual way and completely suppress the  $L \ln(L)$  number fluctuations.

Finally, we can justify our focus on zero sound with a more complete calculation. Consider again the case  $d = 2$ .  $\chi_0$  takes the simple form

$$\chi_0 = \frac{N(0)}{2\pi} \int d\theta \frac{v_F k \cos \theta}{i\omega - v_F k \cos \theta}. \quad (\text{A12})$$

Symmetry forces the imaginary part to zero, so we have

$$\chi_0 = -N(0) \int \frac{d\theta}{2\pi} \frac{\cos^2 \theta}{u^2 + \cos^2 \theta} \quad (\text{A13})$$

with  $u = \omega/(v_F k)$ . Recalling that

$$\chi = \frac{\chi_0}{1 - f_0 \chi_0}, \quad (\text{A14})$$

we see that, at very large  $F_0$ ,  $\chi$  is going like  $1/F_0$  unless  $\chi_0$  is going to zero.  $\chi_0$  goes to zero at large  $u$  where it goes like

$$\chi_0 \approx -N(0) \frac{1}{2u^2}. \quad (\text{A15})$$

But this is nothing but the approximation which led us to the zero-sound pole, so our previous calculation gives the full answer at large  $F_0$ .

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