AN A PRIORI INEQUALITY FOR THE SIGNATURE OPERATOR

by

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ABSTRACT

Let X be a C^∞ manifold of dim $4k$, oriented, with boundary ∂X, where ∂X is not C^∞ but locally as the boundary of the product of two manifolds with ordinary C^∞ boundary.

For a special Riemannian metric on X (corresponding to a product metric near the boundary), we prove an a priori inequality for the signature operator (that is, $d+\delta$ acting between certain sub-bundles of the bundle of differential forms) using non-local boundary conditions on ∂X. These conditions are defined using eigenfunctions of essentially the tangential part of $d+\delta$ on the pieces of ∂X of dimension $4k-1$, subjected to boundary conditions on the piece of dimension $4k-2$.

Using this inequality, we define closed extensions of $d+\delta$ with finite dimensional kernels and closed images. We study such kernels and give other applications related to the Laplace operator.

Thesis Supervisor: I. M. Singer
Title: Professor of Mathematics

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Introduction

In [3], Atiyah, Patodi and Singer studied a non-local boundary value problem for certain kinds of first order elliptic differential operators on manifolds with C^∞ boundary, and gave a formula expressing the index of such problem. The formula is especially interesting as it contains two different contributions to the index: one of the same type as in the index theorem for closed manifolds (i.e. the integral over the manifold of certain characteristic form) and a term related to the spectrum of an elliptic self-adjoint differential operator acting on the boundary, essentially the tangential part of the operator on the interior, that they called the η invariant of the boundary (actually, there is a third term of very simple interpretation). One important operator which fits into the above group is the signature operator, that is $d+\delta$ acting between two subbundles of the bundle of differential forms, provided the manifold is taken with a Riemannian metric that is a product near the boundary.

If we now consider the product of two manifolds with boundary, say X and Y, due to the multiplicativity of the signature we can write the signature of $X \times Y$ using the above result for X and Y. We obtain an expression where one term is the usual integral over $X \times Y$, and three more terms that roughly can be interpreted as contributions from $X \times \partial Y$, $\partial X \times Y$ and $\partial X \times \partial Y$, involving spectrums for differential operators on these manifolds.

We consider in this work a manifold X whose boundary ∂X is not C^∞, but locally it is like the product case, that is $\partial X = \overline{Y}_1 \cup \overline{Y}_2$, where Y_j's are manifolds with C^∞ boundary. We attempted to obtain an index for $d+\delta$, acting between the usual bundles related to the signature, and subjected to certain non-local boundary conditions.

As suggested by the product case, we obtain spectrums and complete sets of eigenfunctions for the tangential parts of $d+\delta$ on Y_1 and Y_2, using again non-local boundary conditions in order to get self-adjoint operators. This is done in part I, where we also prove some estimates that we use afterwards.

When ∂X is C^∞, the existence of the index and the formula for it are obtained constructing an explicit parametrix. In our case, such construction cannot be carried out. Instead, we prove an a priori inequality for the first Sobolev space. The presence of "corners" on ∂X forces us to use stronger norm than $H^{1/2}$ on the boundary to take into account compatibility conditions at the corner. Part II consists of the proof of this inequality, which is the main result of this work.

Using this, we define closed, densely defined operators (A_+ and A_-), which are extensions of $d+\delta$ on certain differential forms. These operators have finite dimensional kernels and closed images, and we get $A_- \subseteq A_+^*$. We didn't succeed in proving that they have an index (implied by $A_- = A_+^*$). If proven, this last equality would give some indication about the contribution of ∂X to this index (using heat equation methods). The main problem when one tries to prove $A_+^* = A_-$ comes from the difficulty in describing completely the set of restrictions to ∂X of the elements in the domains of A_+ and A_-. For the C^∞ case this can be done, as we remark at the end of Part III, so one can prove at least the existence of an index without using the parametrix. We also describe some properties of the elements in ker A_+, as well as other consequences of the a priori inequality like a weak Hodge theorem and a set of generalized eigenfunctions for the Laplace operator on X (in the sense that they are not necessarily C^∞ on the closure of X).

I. Operators on faces of the boundary.

Let Y be a $4k-1$ dimensional manifold with C^∞ boundary ∂Y, oriented and with a Riemannian metric that is a product near the boundary.

We use $< , >$ for the inner product of differential forms induced by the Riemannian metric, dv for the volume element and

$(\phi|\psi) = \int_Y <\phi,\psi>dv$ for product in L^2 sense.

Finally $*$ denotes the Hodge star operator (for this and what follows, see for instance [13], Chapter IV).

Def 1

$$D = -\varepsilon_p(*d - (-1)^p d*) \quad \text{on } p\text{-forms on } Y,$$

where

$$\varepsilon_p = (-1)^{k+\frac{1}{2}p(p-1)}$$

then it is immediate that: A is a 1^{st} order, elliptic, formally self-adjoint operator.

Denote by u the normal coordinate to ∂Y, so that du is inner co-normal, $|du|^2 = <du,du> = 1$.

Then we can write near ∂Y:

$$D = \sigma\frac{\partial}{\partial u} + \mathfrak{D} \quad , \quad \text{where}$$

$$\sigma = -\varepsilon_p(*e_{du} - (-1)^p e_{du}*)$$

$$\mathfrak{D} = -\varepsilon_p(*d_t - (-1)^p d_{t*})$$

on p-forms, e_{du} is exterior multiplication by du and d_t is exterior derivation along directions tangential to ∂Y.

Can check easily: $\mathfrak{D}$ acting on $C^\infty(\Omega(Y)|_{\partial Y})$ is also a formally self-adjoint operator.

<u>Prop. 1</u>

$$\mathfrak{D}\,\sigma = -\sigma\,\mathfrak{D}$$

Proof:

an immediate calculation shows

$D^2 = \Delta$ = Laplace operator on Y

$\mathfrak{D}^2 = \Delta_\partial$ = Laplace operator on ∂Y in the sense:

$\Delta_\partial(\phi+\psi\wedge du) = \Delta_\partial\phi + (\Delta_\partial\phi)\wedge du$, where ϕ,ψ are forms tangential to ∂Y. Now:

$$D^2 = \sigma^2\partial^2/\partial u^2 + \sigma\mathfrak{D} + \mathfrak{D}\sigma + \mathfrak{D}^2$$

but $\sigma^2 = -1$, so:

$$D^2 = -\partial^2/\partial u^2 + \sigma\mathfrak{D} + \mathfrak{D}\sigma + \mathfrak{D}^2$$

so as the metric is a product near the boundary, we know $\Delta = -\partial^2/\partial u^2 + \Delta_\partial$, then proposition follows.

Now:

$$D = \sigma(\partial/\partial u - \sigma\mathfrak{D}) = \sigma(\partial/\partial u + \mathfrak{a}) \quad , \qquad \mathfrak{a} = -\sigma\mathfrak{D}$$

and

$$(-\sigma\mathfrak{D})^* = -\mathfrak{D}^*\sigma^* = \mathfrak{D}\sigma = -\sigma\mathfrak{D}$$

so:

$\mathfrak{a}$ is a formally self-adjoint 1^{st} order elliptic operator on $C^\infty(\Omega(Y)|\partial Y)$. Then it has a set $\{\phi_j\}$ of C^∞ eigenfunctions, orthonormal and complete in L^2 sense.

Note:

if $\mu_j > 0$ is an eigenvalue, $\mathfrak{a}\phi_j = \mu_j\phi_j$,

then:

$$\mathfrak{a}(\sigma\phi_j) = -\mu_j(\sigma\phi_j)$$

so we can assume that for $\mu_j \neq 0$, the eigenfunctions are of the form $\{\phi_j, \sigma\phi_j\}$, $\mu_j > 0$.

Write $D = L + M$, where

$L = -\varepsilon_p *d$ on p-forms, and we can check:

i) $L^* = L, \quad L^2 = \delta d$

ii) $M^* = M, \quad M^2 = d\delta$

iii) $LM = ML = 0$

Then near the boundary, with $\sigma_L = \sigma_L(du)$:

$$L = \sigma_L \, \partial/\partial u + \mathcal{L}$$

$$M = \sigma_M \, \partial/\partial u + \mathfrak{M}$$

and from the previous equations for L, M we get a set of relations between $\sigma_L, \ldots, \mathfrak{M}$.

Also define on a p-form ϕ:

$$\nu\phi = \begin{cases} (-1)^q *\phi & \text{if } p = 2q \\ (-1)^q *\phi & \text{if } p = 2q-1 \end{cases}$$

Then by direct computation we obtain

i) $\nu^2 = 1, \quad \nu^* = \nu$

ii) $D\nu = \nu D, \quad \nu L = M\nu$

iii) $\nu\mathcal{L} = \nu\mathfrak{M}$

Let $\mu \neq 0$ be an eigenvalue of $\mathfrak{D}$ (they are the same as those of $\mathfrak{a}$), then have a finite dimensional vector space of C^∞ sections:

$$V = V(\mu) = \{\phi \mid \mathfrak{D}\phi = \mu\phi\}$$

From $ML = LM = 0$, we obtain $\mathcal{L}\mathfrak{M} = \mathfrak{M}\mathcal{L} = 0$. If $\phi \in V$:

$$\mathfrak{D}\mathcal{L}\phi = \mathcal{L}^2\phi = \mathcal{L}\mathfrak{D}\phi = \mu(\mathcal{L}\phi)$$

and similar equations for $\mathfrak{M}$. As a consequence:

$$\mathcal{L}, \mathfrak{M}: V \to V, \text{ and also } \nu : V \to V.$$

Then we have that $\mathcal{L}$, $\mathfrak{M}$ as linear operators on V can be simultaneously diagonalized, and using that $\mathcal{L}^2 + \mathfrak{M}^2 = \mathfrak{D}^2 = \mu^2 I$, and $\nu\mathcal{L} = \mathfrak{M}\nu$, we arrive to:

there is an orthonormal basis for V of the form

$\{\phi_1,\ldots,\phi_N,\nu\phi_1,\ldots,\nu\phi_N\}$ so that:

$$\mathcal{L}\phi_j = \mu\phi_j, \quad \mathfrak{M}\phi_j = 0$$

$$\mathcal{L}\nu\phi_j = 0, \quad \mathfrak{M}\nu\phi_j = \mu(\nu\phi_j)$$

as for the eigenvalue $-\mu$, same is accomplished by

$\{\sigma\nu\phi_j, \sigma\phi_j\}$.

But now:

if $\mathfrak{D}\,\phi = \mu\phi$, then

$$\mathfrak{a}(\phi-\sigma\phi) = (-\sigma\mathfrak{D})(\phi-\sigma\phi)$$
$$= -\mu\sigma\phi + \mu\phi$$
$$= \mu(\phi-\sigma\phi)$$
$$\mathfrak{a}(\phi+\sigma\phi) = -\mu(\phi+\sigma\phi)$$

Then we see:

can find a basis, orthonormal, consisting of C^∞ sections for the space of $\{\phi|\mathfrak{a}\phi = \mu\phi\}$ of the form $\{\phi_1,\ldots,\phi_N, \nu\phi_1,\ldots,\nu\phi_N\}$, and for $-\mu$ can take $\{\sigma\phi_j,\sigma\nu\phi_j\}$.

Now we look at the eigenvalue 0. The solutions of $\mathfrak{a}\phi = 0$ are the same than those of $\mathfrak{D}\,\phi = 0$, which in turn are those of $\mathfrak{D}^2\phi = \Delta_\partial\phi = 0$.

Then we see that $\mathfrak{D}\,\phi = 0$ will be valid for the homogeneous degree components of ϕ, and recalling that as ∂Y is a compact manifold without boundary, we see $\Delta_\partial\phi^p = 0$ if and only if $d\phi^p = \delta\phi^p = 0$. But if we look to the explicit form of $\mathcal{L}$, $\mathfrak{m}$, we see:

$\mathfrak{D}\,\phi = 0$ if and only if $\mathcal{L}\,\phi = \mathfrak{M}\phi = 0$

Now note: $-\mathcal{L}\sigma_L = \sigma_M\mathfrak{M}$, so $\sigma_L, \sigma_M : \ker\mathfrak{D} \to \ker\mathfrak{D}$ and have:

$$\sigma_L{}^* = -\sigma_L,\ \sigma_M{}^* = -\sigma_L,\ \sigma_L{}^2 + \sigma_M{}^2 = -1.$$

Final property that we need is $\sigma_L\nu = \nu\sigma_M$. Then using normal forms for anti-symmetric operators, we get an orthonormal basis for $\ker\mathfrak{D}$, consisting of harmonic forms of the type $\{\phi_j, \nu\phi_j, \sigma\phi_j, \sigma\nu\phi_j\}$, with:

$\sigma\phi_j = \sigma_L\phi_j$, $\sigma\nu\phi_j = \sigma_M\nu\phi_j$.

Now denote by B the following operator on $L^2(\partial Y)$:

$$B\psi = \sum_j (\psi|\alpha_j\phi_j+\beta_j\sigma\phi_j)(\alpha_j\phi_j+\beta_j\sigma\phi_j)$$

where

i) $\{\phi_j, \sigma\phi_j\}$ is a basis as previously described, the ϕ_j correspond to $\mu_j \geq 0$, and we supress the ν for simplicity

ii) $\alpha_j = \cos\theta_j$, $\beta_j = \sin\theta_j$, and besides they satisfy

a) they are the same inside each $V(\mu_j)$, i.e. for eigenfunctions with same eigenvalue.

b) There is c so that: $|\cos\theta_j| \geq c > 0$ for all j.

c) $\alpha_j^2 - \beta_j^2 = \cos 2\theta_j \geq 0$

We summarize the properties of the operator B in the following

Prop. 2

1) $B^2 = B$ and $B^* = B$

2) $B\psi = 0$ implies $B\nu\psi = 0$

3) $B\psi = 0$ if and only if $(1-B)\sigma\psi = 0$

4) B is bounded from $H^s \to H^s$ (Sobolev spaces)

and its range is closed.

Proof: the first three are easy consequences of the form of our basis, we just do 3) in detail:

$B\psi = 0$ iff $(\psi|\alpha_j\phi_j + \beta_j\sigma\phi_j) = 0$ for all j

iff $(\psi|-\sigma^2(\alpha_j\phi_j + \beta_j\sigma\phi_j)) = 0$ for all j $(\sigma^2 = -1)$

iff $(\sigma\psi|(-\beta_j\phi_j + \alpha_j\sigma\phi_j)) = 0$ for all j $(\sigma^* = -\sigma)$

iff $(1-B)\sigma\psi = 0$.

Finally 4) comes from the fact that:

$$H^s(\partial Y) = \{\Sigma(a_j\phi_j + b_j\sigma\phi_j) \text{ such that: } \Sigma(|a_j|^2 + |b_j|^2)\mu_j^{2s} < \infty\}$$

and $\| \|_s^2$ is equivalent to above sum + $\| \|_{L^2}^2$. So the proposition is proven.

One can try to characterize the above operators B from some general properties. We observe that B satisfies the relations:

$$\alpha^2 B = B\alpha^2 \qquad \text{(i)}$$

$$\sigma = B\sigma + \sigma B \qquad \text{(ii)}$$

If we assume some B, continuous from $H^s \to H^s$ and orthogonal projector when restricted to L^2, satisfies the above equations, the first of these reduces the problem to one of linear algebra on each finite dimensional space $W(\lambda_j) = V(\lambda_j) \oplus V(-\lambda_j) = \{\phi | \alpha^2\phi = \lambda_j^2\phi\}$. Both σ and B leave W invariant. In spite that equation (ii) imposes many conditions on B, it is not enough to determine that B should have the previous form.

A more interesting problem arises if we try to obtain a B using these equations just at the symbol level (in those cases when B is a pseudo-differential operator). This would give a more invariant explanation to B, we note that the important point is the commutation relations satisfied by σ, which in turn is a special case of Clifford multiplication by the conormal du.

In general B is not a pseudo-differential operator, but anyways many important properties of well posed elliptic boundary value problems (as defined in [15], Ch. VI) are valid for the pair (D,B). First we note that we have the inequality:

$$||\phi||_{\ell+1} \leq C_\ell\{||D\phi||_\ell + ||\phi||_\ell + ||B\phi||_{H^{\ell+1/2}(\partial Y)}\}$$

for $\ell \geq 0$, $||\ \ ||_\ell$ denotes norm in $H^\ell(Y)$.
We can see this as follows: call B_0 the operator corresponding to $\beta_j = 0$ for all j. This is a pseudo-differential operator of order zero (as observed in [3], where it is called P). We can calculate its symbol (top order) using Seeley's formulas for fractional powers of pseudo-differential operators, and then we get that it coincides with the symbol of the operators called P_+ in [15] and Q in [10], Chapter II, corresponding to $\partial/\partial u + \mathfrak{a}$. But then Th. 2.2.1 in [10] tells us:

$$||(I-B_0)\phi||_{H^{s+1/2}(\partial Y)} \leq C_s\{||D\phi||_s + ||\phi||_s\}$$

so in order to establish the inequaltiy we only need to estimate $B_0\phi$ by $BB_0\phi$. But it is immediate from the expression for B and the condition b) imposed that there is $C > 0$ so that:

$$||B_0\phi||_{H^t(\partial Y)} \leq ||BB_0\phi||_{H^t(\partial Y)}$$

so the inequality is valid.

Also we can construct a parametrix for D with boundary conditions $B\phi = 0$. This is an operator R, continuous from $H^\ell(Y) \to H^{\ell+1}(Y)$ so that

a) $BR\phi = 0$ on ∂Y

b) R is a two sided inverse of D modulo operators which are continuous from $H^\ell \to H^{\ell+1}$ (note then they are compact).

The construction of such R is carried out in complete detail in [3], so we won't repeat it here. Everything done there is valid in this case (except the assertions about the kernels of the operators), with the obvious modifications as we are using B instead of B_0.

Then recalling Green's formula for 1st order differential operators and Prop. 2, 3) we see that D_B, that is D with domain

$$\text{Dom } D_B = \{\phi \in H^1(Y) \mid B\phi = 0 \text{ on } \partial Y\}$$

is self-adjoint, and as in the usual case we get:

D_B has a complete set of C^∞ eigenfunctions $\{\psi_j\}$, corresponding to a sequence of real eigenvalues $\{\lambda_j\}$, and the ψ_j satisfy $B\phi_j = 0$ on ∂Y.

Now we proceed to make some calculations that will be used in proving a priori estimates.

Def. 2

$$H^1_B(Y) = \{\psi \in H^1(Y) \mid B\psi = 0 \text{ on } \partial Y\}$$

with norm:

$$||\psi||^2_{H^1_B} = ||D\psi||^2 + \alpha||\psi||^2$$

norms on right hand side are L^2, and $\alpha > 0$ is a constant to be chosen later.

Fix now f in $C^\infty([0,+\infty))$ so that:

$$f(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1/2 \\ 0 & \text{if } u \geq 3/4 \\ \geq 0 & \text{otherwise} \end{cases}$$

Consider the following bilinear functional (the spaces where it acts will be precise):

$$\ell(\psi,\phi) = \int_Y <f\sigma\psi, \frac{\partial}{\partial u}(f\phi)>dv(Y)$$

then:

$$|\ell(\psi,\phi)| \leq ||\psi||_{L^2}||\frac{\partial}{\partial u}(f\phi)||_{L^2}$$

Now:

if both ψ and ϕ belong to $H^1_B(Y)$, using the special form of the boundary condition we see that if we integrate by parts the expression for ℓ above, the boundary integral disappears, so in this case:

$$\ell(\psi,\phi) = -\int_Y <\frac{\partial}{\partial u}(f\sigma\psi), f\phi>dv(Y)$$

and then:

$$|\ell(\psi,\phi)| \leq ||\frac{\partial}{\partial u}(f\psi)||_{L^2}||\phi||_{L^2}$$

Lemma 1

there is $C = C(f,f') > 0$ so that for all $\psi \in H^1_B(Y)$ we have:

$$||\frac{\partial}{\partial u}(f\psi)||^2 \leq ||D\psi||^2 + C||\psi||^2,$$ all norms are L^2.

Proof:

$D = \sigma(\partial/\partial u + \mathfrak{a})$, then if $\phi \in H^1_B$ and $\phi \equiv 0$ for $\mu \geq 3/4$ we have:

$$||D\phi||^2 = ||\frac{\partial\phi}{\partial u}||^2 + ||\mathfrak{a}\phi||^2 - \int_{\partial Y} \langle\phi,\mathfrak{a}\phi\rangle$$

As $\phi \in H^{1/2}(\partial Y)$, the boundary integral is well defined. Now: as $B\phi = 0$, this means on ∂Y:

$$\phi = \Sigma a_j(-\beta_j\phi_j + \alpha_j\sigma\phi_j)$$

then the boundary integral is equal to:

$$-\sum_j |a_j|^2\mu_j(\alpha_j^2-\beta_j^2),$$

and as it is preceded by a minus sign, using our condition i), c) for the α_j, β_j, we see

$$||D\phi||^2 = ||\frac{\partial\phi}{\partial u}||^2 + ||\mathfrak{a}\phi||^2 + \text{(positive quantity)}$$

Now set $\phi = f\psi$, then $\frac{\partial\phi}{\partial u} = f\frac{\partial\psi}{\partial u} + f'\psi$ and the lemma follows immediately.

In this moment we are able to fix α in the definition of the norm of $H^1_B(Y)$: we set $\alpha \geq C$, where C is that of the above lemma.

Then: if ϕ, $\psi \in H^1_B(Y)$, we have

$$|\ell(\phi,\psi)| \leq ||\phi||_{H^1_B} \, ||\psi||_{L^2} \, , \quad \text{and}$$

$$|\ell(\phi,\psi)| \leq ||\psi||_{L^2} ||\psi||_{H^1_B}$$

Now we apply an interpolation theorem, all the notation and norms that appear are as in [5], more precisely we use 10.1 of that reference, with:

$$A_1 = B_2 = H^1_B(Y), \; A_2 = B_1 = L^2(Y), \; A = B = \mathbb{C}$$

so:

$$A_1 \cap B_1 = A_2 \cap B_2 = H^1_B(Y), \text{ then if we set}$$

$$H^{1/2}_B(Y) = [L^2(Y), H^1_B(Y)]_{1/2}$$

$$= [H^1_B(Y), L^2(Y)]_{1/2}, \text{ we have}$$

Prop. 3

ℓ can be extended to a sesqui-linear continuous functional:

$\ell: H_B^{1/2}(Y) \times H_B^{1/2}(Y) \to \mathbb{C}$ and we have

$$|\ell(\phi,\psi)| \leq ||\phi||_{H_B^{1/2}} ||\psi||_{H_B^{1/2}}$$

We define now a second bilinear functional:

$$b(\phi,\psi) = -(\phi|D\psi)$$

which is well defined if for instance ϕ,ψ are in $H_B^1(Y)$. We recall that D with that domain is self-adjoint, then we get:

$$|b(\phi,\psi)| \leq ||\phi||_{L^2} ||\psi||_{H_B^1}$$

$$|b(\phi,\psi)| \leq ||\phi||_{H_B^1} ||\psi||_{L^2}$$

so the same arguments as given for ℓ can be applied to b, and we get:

<u>Prop. 4</u>

b can be extended to a sesqui-linear continuous functional:

$b : H_B^{1/2}(Y) \times H_B^{1/2}(Y) \to \mathbb{C}$ and we have

$$|b(\phi,\psi)| \leq ||\phi||_{H_B^{1/2}} ||\psi||_{H_B^{1/2}}$$

Now:

interpolation spaces can also be defined as domains of fractional powers of self-adjoint operators, and for our case we obtain the same spaces (see for instance [11], Chapter I). Then we see:

$$H_B^{1/2}(Y) = \{\Sigma a_j\psi_j \mid \Sigma|\lambda_j|\ |a_j|^2 < \infty\}$$

where: $\{\psi_j\}$ are eigenfunctions of D_B and $\{\lambda_j\}$ their eigenvalues.

As consequence of above and density of H_B^1 and $H_B^{1/2}$, we get:

if $\phi = \Sigma a_j\psi_j$, $\psi = \Sigma b_j\psi_j$ both in $H_B^{1/2}$,

then:

$$b(\psi,\phi) = -\Sigma\lambda_j b_j\bar{a}_j, \quad \text{and then}$$

$$-(\phi|D\phi) = -\Sigma\lambda_j|a_j|^2$$

We need a last lemma to obtain the estimate we'll need.

<u>Lemma 2</u>

There is $C > 0$ so that for all $\phi \in H_B^{1/2}(Y)$, $\phi = \Sigma a_j\psi_j$ we have:

$$||\phi||^2_{[L^2,H^1_B]_{1/2}} \leq \Sigma|\lambda_j|\ \ |a_j|^2 + c||\phi||^2_{L^2}$$

the norm on the left is as defined in [5].

Proof: consider the L^2 valued function

$$f_\delta(z) = e^{\delta(1/2-z)^2}\Sigma|\lambda_j|^{1/2-z}a_j\psi_j$$

where $\delta > 0$. Then

$f(1/2) = \phi$ and we can calculate $||f_\delta(it)||^2_{L^2}$ and $||f_\delta(1+it)||^2_{H^1_B}$, then it is immediate from its definition that:

$$||\phi||^2_{[L^2,H^1_B]_{1/2}} \leq e^{\delta/2}(\Sigma|\lambda_j|\ \ |a_j|^2 + \alpha||\phi||^2_{L^2})$$

and taking a sequence $\delta \downarrow 0^+$ the lemma is proven.

Finally we combine the previous lemmas and propositions to get our final one, whose proof has already been done:

we observe that if $\phi = \Sigma a_j\psi_j$ in $H^{1/2}_B$ and the $a_j = 0$ for all $\lambda_j \geq 0$, then:

$$-(\phi|D\phi) = -\sum_{\lambda_j<0} |a_j|^2\lambda_j = \sum_{\lambda_j<0} |\lambda_j|\ |a_j|^2 \geq 0, \text{ so:}$$

Prop. 5

there is $c > 0$ so that for all $\phi \in H_B^{1/2}(Y)$ such that $\phi = \sum_{\lambda_j<0} a_j\psi_j$, we have:

$$-(\phi|D\phi) + \int_Y \langle f\sigma\phi, \frac{\partial}{\partial u}(f\phi)\rangle dv(Y) + c||\phi||^2_{L^2} \geq 0.$$

As we will see in the next section, we need above inequality exactly, that is, without multiplying $-(\phi|D\phi)$ by a positive constant, in which case it is a trivial consequence of estimates for elliptic boundary value problems.

We present an example to illustrate how the operator B appears. For notation and the results used, see [4], part 6 and [3], part 4.

Consider two manifolds with boundary, X and Y, of dimensions 4k and 4m respectively. Then we know that the signature of $X \times Y$ is equal to (sign X)·(sign Y). Moreover, as

$\tau_{X\times Y} = \tau_X \otimes \tau_Y$ (we use product metric and orientations), then

$$\Omega_+(X \times Y) = \Omega_+(X)\wedge\Omega_+(Y) + \Omega_-(X)\wedge\Omega_-(Y)$$

$$\Omega_-(X \times Y) = \Omega_+(X)\wedge\Omega_-(Y) + \Omega_-(X)\wedge\Omega_+(Y)$$

so if we have harmonic forms representing the cohomology classes involved in the calculation of the signature of X and Y, we can construct using exterior products those necessary for $X \times Y$ (we need to use Künneth Th as in [18], Chapter 5 and Prop. 4.9 of [3], but can be easily checked).

So consider $\Phi\wedge\psi$ where $\Phi \in \Omega_+(X)$, $\psi \in \Omega_+(Y)$, $d\Phi = \delta\Phi = 0$ and same for ψ on Y, both representing elements in $(H^*(\cdot,\partial\cdot) \to H^*(\cdot))$. We use subscript 1 for objects associated to X or ∂X and 2 for Y.

Then if $\mu_1(\phi) = \frac{1}{2}(\phi+\tau_1\phi) = \Phi$, where μ_1 is the $\tilde{=}$ of $\Omega_+(X)|_{\partial X}$ with $\Omega(\partial X)$, then:

$$\mu(\phi\wedge\psi) = \Phi\wedge\psi$$

where μ corresponds to $\partial X \times Y$.

For the operator D on $\partial X \times Y$ explicit calculation shows:

$$D = D_1 \wedge \tau_2 + (\varepsilon_1 *_1) \wedge \tau_2(d_2+\delta_2)$$

where $*_1$ = Hodge opt. induced by X on ∂X. So:
if ϕ_j is an eigenfunction of D_1, i.e.

$$D_1\phi_j = \mu_j\phi_j, \text{ then}$$

$$D(\phi_j \wedge \psi) = \mu_j(\phi_j \wedge \psi)$$

The tangential part of D on $\partial X \times \partial Y$, that is what we called $\mathcal{D}$ is:

$$\mathcal{D} = D_1 \wedge \tau_2 + \varepsilon_1 *_1 \wedge \tau_2 \sigma_2 A_2$$

where:

$$d_2 + \delta_2 = \sigma_2(du)(\partial/\partial u + A_2)$$

du is inner conormal to ∂Y in Y, $\sigma_2 = \sigma_2(du)$.

Now we have: near ∂Y

$$A_2 : \Omega_\pm(Y) \to \Omega_\pm(Y)$$

$$\sigma_2 : \Omega_\pm(Y) \to \Omega_\mp(Y)$$

and $A_2\sigma_2 = -\sigma_2 A_2$, $A_2{}^* = A_2$

so if:

$$A_2\psi_k = \gamma_k\psi_k, \text{ then}$$

$$A_2\psi_k^\pm = \gamma_k\psi_k^\pm ,$$

denoting by $\pm$ their components in $\Omega_{\pm}$.

As our ψ is in Ω_+, we look for eigenfunctions of related to $\phi_j \wedge \psi_k$, where $\psi_k \in \Omega_+(Y)$. Set:

$$e_1 = \phi_j \wedge \psi_k, \quad e_2 = -\varepsilon_1 *_1 \phi_j \wedge \sigma_2 \psi_k$$

then note:

$$e_1 = \mu_j e_1 + \gamma_k e_2$$

$$e_2 = \gamma_k e_1 - \mu_j e_2$$

Using above equations, we get eigenvalues $\pm(\gamma_k^2 + \mu_j^2)^{1/2}$, and passing to $\mathfrak{a} = -\sigma_D \mathfrak{D}$ we obtain: if

$$h_1 = \frac{1}{\sqrt{2}c}\{(\gamma_k+\mu_j+(\gamma_k^2+\mu_j^2)^{1/2})e_1 + (\gamma_k-\mu_j-(\gamma_k^2+\mu_j^2)^{1/2})e_2\}$$

$$h_2 = \frac{1}{\sqrt{2}c}\{(-\gamma_k+\mu_j+(\gamma_k^2+\mu_j^2)^{1/2})e_1+(\gamma_k+\mu_j+(\gamma_k^2+\mu_j^2)^{1/2})e_2\}$$

where:

$$c^2 = (\mu_j + (\gamma_k^2+\mu_j^2)^{1/2})^2 + \gamma_k^2$$

then they satisfy:

$$\alpha h_1 = (\gamma_k^2+\mu_j^2)^{1/2} h_1$$

$$\alpha h_2 = -(\gamma_k^2+\mu_j^2)^{1/2} h_2$$

$$\sigma h_1 = h_2 \quad \text{and} \quad ||h_j||^2 = 1$$

and if we express e_1 in terms of h_1, h_2:

$$e_1 = \frac{1}{\sqrt{2}c}\{(\gamma_k+\mu_j+(\gamma_k^2+\mu_j^2)^{1/2})h_1$$

$$+ (\mu_j-\gamma_k+(\gamma_k^2+\mu_j^2)^{1/2})h_2\}$$

and:

$$(e_1|\alpha e_1) = \frac{4}{\sqrt{2}c}(\gamma_k^2+\mu_j^2)^{1/2}\gamma_k(\mu_j+(\mu_j^2+\gamma_k^2)^{1/2})$$

Note:

$$\mu_j + (\mu_j^2+\gamma_k^2)^{1/2} \geq 0 \quad \text{so:}$$

$$\text{sign } (e_1|\alpha e_1) = \text{sign } \gamma_k$$

For simplicity write:

$$e_1 = ah_1 + bh_2, \quad \text{then note:}$$

$$a^2 + b^2 = 1$$

In our case:

from the boundary conditions satisfied by Φ and ψ, we have that $\mu_j < 0$ and $\gamma_k < 0$, so:

$$(e_1|\alpha e_1) < 0$$

This corresponds to our situation for the proof of Lemma 1, and as we'll see to condition ii), c) in the definition of B.

We see that B should be projection on the orthogonal space to e_1, that is:

$$Bf = (f|bh_1-ah_2)(bh_1-ah_2)$$

we already noted $a^2 + b^2 = 1$, and $b^2 - a^2 \geq 0$ as our calculation for $(e_1|ae_1)$ shows, so this corresponds to property ii), c).

Finally we check that b stays away from 0, Note:

$$b = \frac{1}{\sqrt{2}c}(|\gamma_k| - |\mu_j| + (\gamma_k^2+\mu_j^2)^{1/2})$$

as $\gamma_k, \mu_j < 0$, so the numerator of b is always $> |\gamma_k| \geq \min\{|\gamma_k| \mid \gamma_k < 0\} > 0$ and as $|\gamma_k| \to \infty$, b tends to a finite limit > 0, and same for μ_j.

In more precise way:

Consider $b^2 = \frac{1}{2c^2}(2\gamma^2+2\mu^2+2|\gamma|(\quad)^{1/2}-2|\gamma||\mu|-2|\mu|(\quad)^{1/2})$

set $x = |\mu|$, $y = |\gamma|$

$$c^2 = 2(x^2+y^2) - 2x(x^2+y^2)^{1/2}$$

$$= 2\{(x^2+y^2) - x(x^2+y^2)^{1/2}\}$$

Numerator is:

$$2\{(x^2+y^2) - x(x^2+y^2)^{1/2}\} + 2\{y(x^2+y^2)^{1/2} - xy\}$$

so passing to polar coordinates we get

$$b^2 = \frac{1}{2}(1 + \frac{r^2\sin\theta - r^2\sin\theta\cos\theta}{r^2 - r^2\cos\theta})$$

$$= \frac{1}{2}(1 + \sin\theta)$$

and we note: $x \geq |\mu| > 0$, $y \geq |\gamma| > 0$

were $|\mu| = \min\{|\mu_j| \mid \mu_j < 0\}$

$|\gamma| = \min\{|\gamma_k| \mid \gamma_k < 0\}$

II. An a priori estimate

We consider now X a C^∞ manifold of dimension 4k, whose boundary can be written as $\partial X = \overline{Y}_1 \cup \overline{Y}_2$, the Y_j being in turn C^∞ manifolds with C^∞ boundary, and so that $\overline{Y}_1 \cap \overline{Y}_2 = \partial Y_1 \cap \partial Y_2 = \partial Y_1 = \partial Y_2$ is a compact manifold without boundary $\overline{X}$ is also compact, and it is oriented.

On X we have a Riemannian metric which is a product near the Y_j's and near $\overline{Y}_1 \cap \overline{Y}_2$ is isometric to a product of the form $(\overline{Y}_1 \cap \overline{Y}_2) \times [0,3/2) \times [0,3/2)$. Such metrics exist, as is proven in [6], for instance. So we have two coordinates:

x_1 = normal coordinate to Y_2, defined up to 3/2

x_2 = normal coordinate to Y_1, defined up to 3/2

so that:

$$\langle dx_i, dx_j \rangle = \delta_{ij}$$

where we denote by $\langle\ ,\ \rangle$ the inner product induced by the Riemannian metric on differential forms.

Now we take $d+\delta$ acting on differential forms. We can write:

$$d+\delta = \sigma(dx_2)(\partial/\partial x_2 + A_1) \quad \text{near} \quad Y_1, \text{ and}$$

$$d+\delta = \sigma(dx_1)(\partial/\partial x_1 + A_2) \quad \text{near} \quad Y_2,$$

where the A_j contain the derivatives tangential to Y_j, and $\sigma = \sigma_{d+\delta}$, symbol of $d+\delta$. We also have maps:

$$\mu_j : \Omega(Y_j) \to \Omega_+(X) : \phi \to \tfrac{1}{2}(\phi+\tau(\phi))$$

where $\tau = (-1)^{k+\frac{1}{2}p(p-1)} * = \varepsilon_p *$

on p-forms. Then can check that the μ_j, $j = 1,2$ establish isomorphisms of vector bundles, and using induced metrics and orientations on the Y_i by X calculation shows:

$$<\mu_j(\phi),\mu_j(\psi)>_X = \tfrac{1}{2}<\phi,\psi>_{Y_j}$$

If:

$$D_j = -\varepsilon_p(*_j d_j - (-1)^p d_j *_j)$$

acting on p-forms on Y_j, subscripts referring to operators induced on Y_j, we can prove by direct calculation:

$$D_j = \mu_j^{-1} A_j \mu_j$$

We fix again $f \in C^\infty([0,\infty))$ as follows:

$$f(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq \frac{1}{2} \\ 0 & \text{if } u \geq \frac{3}{4} \\ \geq 0 & \text{otherwise} \end{cases}$$

and we fix boundary value problems for each D_j, boundary conditions B_j, on each Y_j, satisfying the properties stated on the first part.

Theorem 1

There is $C > 0$ such that: for all $\psi \in C^\infty(\overline{X},\Omega_+(\overline{X}))$ and $\psi = 0$ on $\overline{Y}_1 \cap \overline{Y}_2$, we have

$$||\psi||_{H^1} \leq C\{||(d+\delta)\psi||_{L^2} + ||\psi||_{L^2} + \Sigma||\phi_+^j||_{H^{1/2}_{B_j}(Y_j)}\}$$

where:

ϕ_+^j denote projection of $\mu_j^{-1}(\psi)$ on the span of eigenfunctions of D_{B_j} corresponding to eigenvalues that are ≥ 0.

The rest of this section consists of the proof of above inequality.

First we note: can replace all above $||\ \ ||$ by

$|| \ \ ||^2$, and $||\psi||_{L^2(X)}$ by any $||\psi||_{H^s(X)}$ for a fixed $s < 1$.

Secondly: if ψ is supported in the interior of X, the inequality is well known. So we see that it is enough to prove:

$$||f(x_2)\psi||_1^2 \leq C_s\{||(d+\delta)(f(x_1)\psi)||^2 + ||(d+\delta)(f(x_2)\psi)||^2 + ||\psi||^2 + \Sigma||\phi_+^j||^2_{H_B^{1/2}(Y_j)}\} \qquad (1)$$

So assume now ϕ is a C^∞ form supported in $Y_1 \times [0,1]$, and equal to 0 on $\overline{Y}_1 \cap \overline{Y}_2$. We then have:
$||\phi||_1^2$ is equivalent to

$$\int_0^1 ||\frac{\partial\phi}{\partial x_2}||^2_{L^2(Y_1\times x_2)} dx_2 + \int_0^1 ||\phi||^2_{H^1(Y_1\times x_2)} dx_2 + ||\phi||^2_{L^2(X)}$$

If on Y_1 we have an elliptic boundary value problem, with a 1^{st} order interior operator A_1 and boundary conditions B_1, as in Part I, then: $||\phi||^2_{H^1(Y_1\times x_2)}$ is equivalent to:

$$||A_1\phi||^2_{L^2(Y_1\times x_2)} + ||B_1\phi||^2_{H^{1/2}(\partial Y_1\times x_2)} + ||\phi||^2_{L^2(Y_1\times x_2)}$$

Then we see: in order to prove (1), we only need to estimate

$$||f(x_2)\frac{\partial\psi}{\partial x_2}||^2_{L^2(X)} + \int_0^1 ||A_1(f\psi)||^2_{L^2(Y_1\times x_2)} dx_2$$

$$+ \int_0^1 ||B_1(f\psi)||^2_{H^{1/2}(\partial Y_1\times x_2)} dx_2 \qquad (2)$$

by the right hand side of (1).

Strictly we should say D_j, etc. and the corresponding isomorphisms, but it is clear what we mean and much simpler to write.

Now we proceed to develop the terms in the right hand side of (1). Recall that $\sigma^* = -\sigma$ and $\sigma^2 = -1$ (see for instance [13], Chapter IV), then using the decomposition of $d+\delta$ near Y_1:

$$||(d+\delta)\phi||^2 = ||\partial\phi/\partial x_2||^2 + ||A_1\phi||^2$$

$$+ 2\ \mathrm{Re}\ (\partial\phi/\partial x_2 | A_1\phi)$$

(all norms in $L^2(X)$).

Set now $p_1 = \sigma_{A_1}(dx_1)$, then

$$\frac{\partial}{\partial x_2}\int_{Y_1\times x_2}<\phi,A_1\phi>dv(Y_1) = \int_{Y_1\times x_2}<\frac{\partial\phi}{\partial x_2},A_1\phi>dv(Y_1)$$

$$+\int_{Y_1\times x_2}<\phi,A_1\frac{\partial\phi}{\partial x_2}>dv(Y_1)$$

$$= 2\ \mathrm{Re}\int_{Y_1\times x_2}<\frac{\partial\phi}{\partial x_2},A_1\phi>dv(Y_1)$$

$$+\int_{\partial Y_1\times x_2}<p_1\phi,\frac{\partial\phi}{\partial x_2}>dv(\partial Y_1)$$

($dv(\cdot)$ denotes Riemannian volume element of the corresponding manifold).

We integrate above equality with respect to x_2 between zero and one. Recalling that $\phi \equiv 0$ at $x_2 = 1$, we get

$$-\int_{Y_1}<\phi,A_1\phi>dv(Y_1) = 2\ \mathrm{Re}\ (\frac{\partial\phi}{\partial x_2}|A_1\phi)$$

$$+\int_0^1 dx_2\int_{\partial Y_1\times x_2}<p_1\phi,\frac{\partial\phi}{\partial x_2}>dv(\partial Y_1)$$

The last integral above is equal to:

$$\int_{Y_2}<p_1\phi,\frac{\partial\phi}{\partial x_2}>dv(Y_2)$$

so we get:

$$||(d+\delta)\phi||^2 = ||\frac{\partial\phi}{\partial x_2}||^2 + ||A_1\phi||^2 - \int_{Y_1} <\phi, A_1\phi> dv(Y_1)$$

$$- \int_{Y_2} <p_1\phi, \frac{\partial\phi}{\partial x_2}> dv(Y_2)$$

and a similar expression for a form supported on $Y_2 \times [0,1)$.

Applying this to $\phi = f(x_2)\psi$ and $f(x_1)\psi$, and recalling that $f(0) = 1$, we get:

$$||(d+\delta)(f(x_1)\psi)||^2 + ||(d+\delta)(f(x_2)\psi)||^2$$

$$= \Sigma||\frac{\partial}{\partial x_j}(f(x_j)\psi)||^2 + \Sigma||A_j(f\psi)||^2$$

$$- \int_{Y_1} <\psi, A_1\psi> dv(Y_1) - \int_{Y_1} <f(x_1)p_2\psi, \frac{\partial}{\partial x_1}(f(x_1)\psi)> dv(Y_1)$$

$$- \int_{Y_2} <\psi, A_2\psi> dv(Y_2) - \int_{Y_2} <f(x_2)p_1\psi, \frac{\partial}{\partial x_2}(f(x_2)\psi)> dv(Y_2) \qquad (3)$$

Now we add $C_s\{\Sigma||\phi_j^{\pm}||^2_{H_B^{1/2}(Y_j)} + ||\psi||^2_{H^s(X)}\}$ for an $s < 1$. We will prove the following: after adding $C_s\{\ldots\}$ to the last two lines of the right hand side of above equality, the resulting quantity is positive. Assuming this claim, we can finish the proof of (1) as follows: by the remark preceeding (2), we see that the following inequality implies (1):

$$\int_0^1 ||B_1(f\psi)||^2_{H^{1/2}(\partial Y_1 \times x_2)} dx_2 \leq C\{||\frac{\partial}{\partial x_1}(f\psi)||^2_{L^2(X)}$$

$$+ ||A_1(f\psi)||^2_{L^2(X)} + ||\psi||^2_{H^s(X)}\}$$

In order to prove this, we observe: consider $\partial Y_1 \times x_2$ as the boundary of the manifold $Y_1 \times x_2 \cong \partial Y_1 \times [0,1) \times x_2$ (coordinate x_1 on $[0,1)$). Then:

If ξ is any form supported on $\partial Y_1 \times [0,1) \times x_2$, and L is any 1^{st} order elliptic differential operator on ∂Y_1, $||\xi||^2_{H^1(Y_1 \times x_2)}$ is equivalent to:

$$||\frac{\partial \xi}{\partial x_1}||^2_{L^2(Y_1 \times x_2)} + \int_0^1 ||L\xi||^2_{L^2(\partial Y_1 \times x_1 \times x_2)} dx_1$$

$$+ ||\xi||^2_{L^2(Y_1 \times x_2)}$$

and

$$||B\xi||^2_{H^{1/2}(\partial Y_1 \times x_2)} \leq C_1 ||\xi||^2_{H^{1/2}(\partial Y_1 \times x_2)}$$

$$\leq C\{\text{above sum}\}$$

and after we integrate above inequality with $\xi = f(x_1) f(x_2) \psi$.

But now note: the first and third term that we get are already included in (3), so we only need to explicit L. For this set:

$L = L_1 = -p_1 A_1 + \partial/\partial x_1$, then is clear that

$$||L\xi||^2_{L^2(X)} \leq c(||A_1\xi||^2_{L^2(X)} + ||\frac{\partial\xi}{\partial x_1}||^2_{L^2(X)})$$

then our estimate for $\int ||B_1||^2$ follows after adding $c||\psi||^2_{H^s(X)}$, here $1/2 < s < 1$.

So we only need now to prove our assertion about the last two lines in (3).

We **use** our isomorphisms and see that: if ψ near Y_1 comes from ϕ and we decompose ϕ in $\phi^+ + \phi^-$ corresponding to non-negative and negative eigenfunctions of D_1, we get (except for a $1/2$ factor that is irrelevant):

$$\{\int_Y \langle qf\phi_+, \frac{\partial}{\partial x_1}(f\phi_+)\rangle + \int_Y \langle qf\phi_+, \frac{\partial}{\partial x_1}(f\phi_-)\rangle$$

$$+ \int_Y \langle qf\phi_-, \frac{\partial}{\partial x_1}(f\phi_+)\rangle + \int_Y \langle qf\phi_-, \frac{\partial}{\partial x_1}(f\phi_-)\rangle\}$$

$$- \int_Y \langle \phi_-, D\phi_-\rangle - \int_Y \langle \phi_+, D\phi_+\rangle$$

we have suppressed mention to Y_1, term for Y_2 is of the same form, and

$$q = \mu_1^{-1} p_1 \mu_1$$

is precisely the symbol of D in direction dx_1, this can be seen using the equations:

$\sigma_{d+\delta}(dx_2) = \sigma_{d+\delta}(dx_1)p_2$ and the relation between D and A_1.

As ψ was C^∞ on $\overline{X}$ and 0 on $\overline{Y}_1$ $\overline{Y}_2$, we have that $\phi \in H_B^{1/2}(Y)$.

Now: using the same arguments of part I and abstract theorems on interpolation, we obtain the following inequalities:

i) $\left|\int_Y \langle qf\phi_+, \frac{\partial}{\partial x_1}(f\phi_+)\rangle\right| \leq c||\phi_+||^2_{H_B^{1/2}(Y)}$

ii) $\left|\int_Y \{\langle qf\phi_+, \frac{\partial}{\partial x_1}(f\phi_-)\rangle + \langle qf\phi_-, \frac{\partial}{\partial x_1}(f\phi_+)\rangle\}\right|$

$$\leq c||\phi_+||_{H_B^{1/2}(Y)}\,||\phi_-||_{H_B^{1/2}(Y)}$$

$$\leq \varepsilon||\phi_-||^2_{H_B^{1/2}} + c(\varepsilon)||\phi_+||^2_{H_B^{1/2}},$$

$$\varepsilon > 0.$$

(here we use $|a||b| \leq \varepsilon b^2 + c(\varepsilon)a^2$)

iii) $\left|\int_Y \langle\phi_+, D\phi_+\rangle\right| \leq c||\phi_+||^2_{H_B^{1/2}}$

and according to Prop. 5, we have for some $c > 0$:

iv) $-\int_Y \langle\phi_-, D\phi_-\rangle + \int_Y \langle qf\phi_-, \frac{\partial}{\partial x_1}(f\phi_-)\rangle + c||\phi_-||^2_{L^2(Y)} \geq 0$

then by (3), it is clear that we have proven:
the inequality:

$$||\psi||^2_{H^1(X)} \leq C\{||(d+\delta)\psi||^2_{L^2(X)} + ||\psi||^2_{L^2(X)}$$

$$+ \varepsilon\Sigma||\phi_-^j||^2_{H_{B_j}^{1/2}(Y_j)} + C(\varepsilon)\Sigma||\phi_+^j||^2_{H_{B_j}^{1/2}(Y_j)}\}$$

In order to finish the proof of the theorem, we must show that the term with ϕ_-^j can be eliminated. Due to the ε, which can be made as small as desired, we see that it is enough to prove:

$$||\phi_-^1||^2_{H_B^{1/2}(Y_1)} \leq c\{||\psi||^2_{H^1(X)} + ||\phi_+^1||^2_{H_B^{1/2}(Y_1)}\} \quad (6)$$

We suppress subindex 1 and proceed to prove above inequality.

Set $\phi_+ = \sum_{\lambda_j \geq 0} a_j\phi_j$, the ϕ_j being the eigenfunctions of D_B. Then define for $x_2 \geq 0$:

$$\xi = f(x_2) \sum_{\lambda_j \geq 0} e^{-\lambda_j x_2} a_j \phi_j$$

where f is as before, then as $||\phi_+||^2_{H^1(Y)}$ is equivalent to $\Sigma|\lambda_j|^2|a_j|^2 + ||\phi_+||^2_{L^2}$, we easily get:

1) ξ is in $H^1(X)$

2) $\xi|Y = \phi_+$

3) $||\xi||_{H^1(X)} \leq c||\phi_+||_{H_B^{1/2}(Y)}$

Consider now the difference $\phi-\xi$, we get then $(\phi-\xi)|Y = \phi_-$.

In order to prove (6), due to the presence of $||\psi||_{H^1(X)}$ on the right hand side, it is necessary only to estimate $\Sigma|\lambda_j||a_j|^2$ if $\phi_- = \Sigma a_j\phi_j$. But:

$$-\sum_{\lambda_j<0} \lambda_j|a_j|^2 = -\int_{Y_1} \langle\phi_-,D\phi_-\rangle$$

$$= -\int_{Y_1} \langle\phi-\xi,D(\phi-\xi)\rangle \tag{7}$$

Using our development of $d+\delta$:

$$||(d+\delta)(f\psi-\mu(\xi))||^2_{L^2} = ||\frac{\partial}{\partial x_1}(f\psi-\mu(\xi))||^2 + ||f\psi-\mu(\xi)||^2$$

$$-2\int_{Y_2} \langle\phi-\xi,D(\phi-\xi)\rangle$$

$$-2\int_{Y_2} \langle\mu_1(\phi-\xi),\frac{\partial}{\partial x_2}(\phi-\xi)\rangle \tag{8}$$

Now we need the following lemma

<u>Lemma 3</u>

Let $v \in H^1(X)$ be such that $v_1 = v|Y_1$ is in $H^{1/2}_{B_1}(Y_1)$. Then:

$v_2 = v|Y_2$ is in $H^{1/2}_{B_1}(Y_2)$ and there is c independent of v such that

$$||v_2||^2_{H^{1/2}_{B_1}(Y_2)} \leq c\{||v||^2_{H^1(X)} + ||v_1||^2_{H^{1/2}_{B_1}(Y_1)}\}$$

Proof:

Denote by t a variable equal to x_1 when it appears on v_1 and x_2 in v_2. Then from results of [7], last section, on compatibility conditions of restrictions of elements of $H^1(R^\mu \times R_+ \times R_+)$ to $R^\mu \times R_+ \times \{0\}$ and $R^\mu \times \{0\} \times R_+$, we get due to local nature of restrictions that they apply to our case, and have:

$$\int_0^1 \frac{1}{t} dt \int_{\partial Y_1 \times t} |v_1 - v_2|^2 \leq c||v||^2_{H^1(X)}$$

Noting that B_1 is an orthogonal projection on $L^2(\partial Y_1)$, have:

$$\int_{\partial X_2 \times t} |v_1-v_2|^2 = \int_{\partial Y_1 \times t} |B_1(v_1-v_2)|^2 + \int_{\partial Y_1 \times t} |(I-B_1)(v_1-v_2)|^2$$

Now combining results from [17], section 4 and [11], Chapter I, we see that the norms $H^{1/2}_B(Y)$ are equivalent to:

$$\text{norm } H^{1/2}(Y) + \left(\int_0^1 \frac{1}{t}dt \int_{\partial Y\times t} |B_1|^2\right)^{1/2}$$

so hypothesis about v_1 implies:

$$\int_0^1 \frac{1}{t}dt \int_{\partial Y_1\times t} |B_1 v_1|^2 \leq c||v_1||^2_{H^{1/2}_{B_1}(Y_1)}$$

but then by the previous lines, we see writing $|B_1 v_2|^2 \leq |B_1(v_1-v_2)|^2 + |B_1 v_1|^2$ that statement of lemma follows.

We apply Lemma 3 to ξ, and recalling observation 3) about ξ, we get:

$$||\xi||^2_{H^{1/2}_{B_1}(Y_2)} \leq c||\phi_+||^2_{H^{1/2}_{B_1}(Y_1)}$$

and also to $\phi-\xi$. As

$$\left|\int_{Y_2} \langle p_1(\phi-\xi), \frac{\partial}{\partial x_2}(\phi-\xi)\rangle\right| \leq c||\phi-\xi||_{H^{1/2}_{B_1}(Y_2)}$$

isolating quantity (7) from the expression (8), we see that all terms are bounded by $||\psi||^2_{H^1(X)}$ and

$||\phi_+||^2_{H^{1/2}_{B_1}(Y_1)}$ multiplied by constants, that is we have proven (6).

But then we have completed the proof of the Theorem.

III. Consequences

Now we proceed to define an extension of $d+\delta$. For this

Def. 3

Set

$$V = \{\psi \in H^1(X,\Omega_+) \mid \text{there is } \{\psi_n\} \subseteq C^\infty(\overline{X},\Omega_+)$$

so that

a) $\psi_n \to \psi$ in H^1 sense

b) $\psi_n = 0$ on $\overline{Y}_1 \cap \overline{Y}_2$

c) $\phi^j_{n,+} \to 0$ in $H^{1/2}_{B_j}(Y_j)$ }

and define:

$$A_+\psi = (d+\delta)\psi \quad \text{for} \quad \psi \in V.$$

We observe:

$A_+\psi = \lim (d+\delta)\psi_n$, the limit taken in L^2 sense, and the result is independent of the approximating sequence. Also we see:

a) $C^\infty_0(X,\Omega_+) \subseteq V$, obviously, and

b) If $\psi \in V$, and $\psi = \mu(\phi)$ on Y_j, then

$\phi \in H^{1/2}(Y)$ and $\phi = \sum_{\lambda_j<0} a_j\phi_j$, the series

converges in $H^s(Y)$ sense for $s < \frac{1}{2}$ where the ϕ_j are the eigenfunctions of the D_B.

The first assertion is consequence of restriction theorems for Sobolev spaces and for the second see for instance [2] or [17].

We make the following remark: it is well known that $C^\infty(\overline{X})$ is dense in $H^1(X)$, as is proved for instance in [1], Section 2. One can prove that the subset of $C^\infty(\overline{X})$ satisfying $\psi = 0$ on $\overline{Y}_1 \cap \overline{Y}_2$ is also dense in $H^1(X)$, the proof can be adapted from related results in [11], Chapter I. So we see that c) is the only condition that we are imposing in the definition of B.

<u>Prop. 6</u>

A_+, considered as an unbounded operator from $L^2(X,\Omega_+) \to L^2(X,\Omega_-)$, is a closed, densely defined operator. Also: there is $c > 0$ so that for all $\psi \in V$:

$$||\psi||_{H^1} \leq c\{||A_+\psi||_{L^2} + ||\psi||_{L^2}\}$$

Proof:

it is clear that A_+ is densely defined, as $C_0^\infty - V$. We now prove the inequality: if $\psi_n \in C^\infty(\overline{X},\Omega_+)$ tends to ψ in the definition of V, then we can apply our theorem to the ψ_n's, so:

$$||\psi_n||_{H^1} \leq c\{||(d+\delta)\psi_n||_{L^2} + ||\psi_n||_{L^2} + \sum_j ||\psi^j_{n,+}||_{H^{1/2}_{B_j}(Y_j)}\}$$

but by the properties of our sequence, taking the limit as $n \to \infty$ we obtain precisely the claimed inequality.

Now we prove that A_+ is closed: let $\psi_n \to \psi$, $A_+\psi_n \to \xi$, $\{\psi_n\} \subseteq V$ and limits in L^2 sense.

Then using the inequality we have just proven, we see $\psi_n \to \psi$ in H^1 sense, and then $A_+\psi_n \to A_+\psi = \xi$ in L^2 sense, as $d+\delta$ is a first order differential operator. To see that ψ can be approximated by a sequence $\{\phi_n\} \subseteq C^\infty(\overline{X},\Omega_+)$ as described in the definition of V, choose sequences $\{\phi_{n,m}\} \subseteq C^\infty$ so that $\phi_{n,m} \to \psi_n$ as $m \to \infty$ with properties required, and choose an appropriate diagonal subsequence from the "square" of the $\{\phi_{n,m}\}$. So Prop. 6 is proven.

We now examine the kernel of A_+. For this we define a manifold $\tilde{X}$ so that X $\tilde{X}$, explicitly

<u>Def. 4</u>

$$\tilde{X} = X \cup Y_1 \times (-\infty,0] \cup Y_2 \times (-\infty,0]$$

have coordinate x_2 in $Y_1 \times (-\infty,0]$ and x_1 on $Y_2 \times (-\infty,0]$. We extend the Riemannian metric and orientation of X to $\tilde{X}$ in the obvious way, i.e. maintaining the product structure we had near ∂X.

Note that the isomorphisms $\mu_j : \Omega(Y_j) \to \Omega_+(X)|Y_j$ prolong to $\Omega(Y_j \times \{x\}) \to \Omega_+(\tilde{X})|_{Y_j \times \{x\}}$ over $Y_j \times (-\infty,0]$. Now assume we have $\phi = \sum_{\lambda_j < 0} a_j\phi_j$ in $H^{1/2}(Y)$, where Y is Y_1 or Y_2, then define on $Y \times (-\infty,0]$

$$\xi = \Sigma\, e^{-\lambda_j x} a_j\phi_j \qquad (x = x_1 \text{ or } x_2 \text{ accordingly})$$

and $\psi = \mu(\xi)$.

Then is easy to see:

1) ξ is in $C^\infty \cap H^s(Y \times (-\infty,0])$ for $s < 1$

2) $(\frac{\partial}{\partial x} + D)\xi = 0$ on $Y \times (-\infty,0)$ and then $(d+\delta)\psi = 0$ there.

3) $\lim_{x \to 0} \xi = \phi$ in $H^t(Y)$ for $t < \frac{1}{2}$.

4) $||\xi(\cdot,x)||_{L^2(Y)} \leq ce^{-ax}$, where

$a = \min \{|\lambda_j| \text{ so that } \lambda_j < 0\}$

Suppose there is $\eta \in V$ so that:

$A_+\eta = 0$ and $\eta|Y = \mu(\phi)$.

Define then the following distribution on $Y \times (-\infty,\frac{1}{2})$:

$$T = \begin{cases} \eta & \text{on } Y_1 \times (0,\frac{1}{2}) \\ \psi & \text{on } Y_1 \times (-\infty,0) \end{cases}$$

so T is in L^2.

We now calculate $(d+\sigma)T$ in distribution sense, so let $f \in C_0^\infty(Y \times (-\infty,\frac{1}{2}))$, then we have:

$$T[(d+\delta)f] = \int_{Y\times(0,1/2)} \langle\eta,(d+\delta)f\rangle + \int_{Y\times(-\infty,0)} \langle\psi,(d+\delta)f\rangle$$

We use Green's formula to reduce integrals to expressions involving only values over Y.

a) For the first integral: as $\eta \in H^1$ and $(d+\sigma)\eta = 0$ on X, we see as $f \equiv 0$ near ∂Y that it is equal to an expression of the form $\int_Y \ell(\eta,f)$, where ℓ only involves the values of η and f on Y (i.e. there are no derivatives)

b) For the second integral: set $Z_\varepsilon = Y \times (-\infty,-\varepsilon)$, then

$$\int_{Y\times(-\infty,0)} \langle\psi,(d+\delta)f\rangle = \lim_{\varepsilon\to 0^+} \int_{Z_\varepsilon} \langle\psi,(d+\delta)f\rangle$$

Now we use Green's formula for Z_ε, and by remark 3) on previous page we can take the limit as $\varepsilon \downarrow 0^+$, getting the same integral as in a) but with a minus sign due to the opposite orientations induced on Y from the two sides. So we get:

$$(d+\delta)T = 0 \quad \text{on} \quad Y \times (-\infty,\tfrac{1}{2})$$

in distribution sense. Using this we now state the following

<u>Prop. 7</u>

The kernel of A_+ is finite dimensional, and its elements extend to solutions of $(d+\delta)\phi = 0$ on $\tilde{X}$, such ϕ are in $C^\infty \cap H^s(\tilde{X},\Omega_+)$ for $s < 1$.

As a consequence, the elements of $\ker A_+$ are C^∞ on $\overline{X} \smallsetminus (\overline{Y}_1 \cap \overline{Y}_2)$.

Proof:

the first assertion is an immediate consequence of the inequality in Prop. 6 and the compactness of the injection $H^1(X) \to L^2(X)$. The prolongation to $\tilde{X}$ is done as we have just described to $Y_j \times (-\infty,0]$, and such elements are C^∞ as solutions of the elliptic equation $(d+\delta)\phi = 0$, and that they belong to H^s for $s < 1$ was observed in remark 1) about ξ. So Prop. 7 is proven.

For the image of $A_+(V)$ in $L^2(X,\Omega_-)$, we have

<u>Prop. 8</u>

there is $C > 0$ so that for all $\psi \in V \cap (\ker A_+)^\perp$ ($\perp$ in L^2-sense) we have:

$$||\psi||_{H^1(X)} \leq C||A_+\psi||_{L^2(X)}$$

So: $A_+(V)$ is closed in $L^2(X,\Omega_-)$.

Proof:

it is a standard argument, if not there is sequence ψ_n so that $||A_+\psi_n|| \to 0$, $||\psi||_1 = 1$, etc. and this would contradict the compactness of the injection

$H^1(X) \to L^2(X)$.

We can use Dirichlet principle to solve equation $\Delta\phi = \omega$, for $\omega \in L^2(X,\Omega_+) \cap (\ker A_+)^{\perp}$. The arguments are as in [12], Chapter 7 so we don't give many details.

Minimizing the functional on V

$$I(\psi) = ||(d+\delta)\psi||^2 - 2(\omega|\psi)$$

we obtain an element $G\omega$ in $V \cap (\ker A_+)^{\perp}$.

Denote by H the orthogonal projection of $L^2(X,\Omega_+)$ onto $\ker A_+$, and recall that all our operators, boundary conditions, etc. are real.

The properties of G can be summarized in

<u>Prop. 9</u>

$G : L^2(X,\Omega_+) \to L^2(X,\Omega_+)$ satisfies

1) G is a compact, self-adjoint, non-negative operator

2) $\ker G = \ker A_+$

3) $G(L^2) \subseteq V \cap (\ker A_+)^{\perp}$ and G is continuous as map from $L^2 \to V$, V with H^1 topology

4) The following eqt is satisfied:

$\mathrm{Re}((d+\delta)G\omega \mid (d+\delta)\psi) = (\omega - H(\omega) \mid \psi)$

for all $\omega \in L^2$, $\psi \in V$. If all elements are real:

$((d+\delta)G\omega \mid (d+\delta)\psi) = (\omega - H(\omega) \mid \psi)$

In particular:

$$\Delta G\omega = \omega - H(\omega) \quad \text{in distribution sense.}$$

5) G has a basis of eigenfunctions $\{\psi_j\} \subseteq V$ and eigenvalues $\{0, 1/\lambda_j\}$ where $0 < \lambda_j$, $\lambda_j \to +\infty$, complete in $L^2(X,\Omega_+)$.
They also satisfy: $\Delta\psi_j = \lambda_j\psi_j$

6) Any $\omega \in L^2$ can be written as:

$$\omega = H(\omega) \oplus \Delta G\omega$$

Now consider $A_+ : L^2(X,\Omega_+) \to L^2(X,\Omega_-)$ unbounded operator, and we look for:

$$A_+{}^* : L^2(X,\Omega_-) \to L^2(X,\Omega_+)$$

its adjoint in Hilbert space sense.

First we note:

$$A_+GL^2 \subseteq \mathrm{Dom}\ (A_+{}^*)$$

by property 3) of G.

Def. 5

$$E(X,\Omega_-) = \{\psi \in L^2(X,\Omega_-) \mid (d+\delta)\psi \in L^2(X,\Omega_+)$$
$$\text{(in distribution sense)}\}$$

As the formal adjoint of $(d+\delta)$ is again $(d+\delta)$, we see:

$$\text{Dom } (A_+^*) \subseteq E.$$

On E we take: $||\psi||_E^2 = ||\psi||_{L^2}^2 + ||(d+\delta)\psi||_{L^2}^2$

Now we prove an important property of E:

Prop. 10

The set

$$W = \{\phi \in C^\infty(\overline{X},\Omega_-) \mid \phi = 0 \text{ on } \overline{Y}_1 \cap \overline{Y}_2\}$$

is dense in E.

Proof:

note that it is enough to prove that $C^\infty(\overline{X})$ is dense in E, as this implies the density of H^1, and then use the remark following definition 3.

So take $\ell : E \to \mathbb{C}$ continuous linear functional, any such functional can be written as

$$\ell(\psi) = (\xi|\psi) + (\eta|(d+\delta)\psi)$$

with $\xi, \eta \in L^2(X)$. Assuming $\ell \equiv 0$ on $C^\infty(\overline{X})$, we prove $\ell \equiv 0$ on E.

Choose M a manifold such that $X \subset\subset M$, and denote by ξ_1 and η_1 the extensions by 0 of ξ and η to M. Then as $\ell = 0$ on $C^\infty(\overline{X})$, we get:

$$(d+\delta)_{\eta_1} = -\xi_1 \text{ in distribution sense on M.}$$

But then $\eta_1 \in H^1_{\ell oc}(M)$, and as $\eta_1 = 0$ on $M-\overline{X}$, we see: $\eta \in H^1_0(X,\Omega_-)$, and $((d+\delta)\eta|\psi) - (\eta|(d+\delta)\psi) = 0$ for all $\psi \in C^\infty(\overline{X})$. Now choose $\{\eta_j\} \subseteq C^\infty_0(X,\Omega_-)$ so that $\eta_j \to \eta$ in H^1_0 sense.

Take a general $\psi \in E$, then:

$$(\eta|(d+\delta)\psi) = \lim (\eta_j|(d+\delta)\psi)$$

but $\psi \in H^1_{\ell oc}(X)$ as $d+\delta$ elliptic, so:

$$(\eta_j|(d+\delta)\psi) = ((d+\delta)\eta_j|\psi)$$

so

$$\lim (\eta_j | (d+\delta)\psi) = ((d+\delta)\eta|\psi)$$

then $\ell(\psi) = 0$, as we wanted to prove.

Now we look in detail to Green formula for $(d+\delta)$. If ξ, η are in $C^\infty(\overline{X})$, we then have:

$$((d+\delta)\xi|\eta) - (\xi|(d+\delta)\eta) = \int_{\partial X} \xi\wedge *\overline{\eta} - \overline{\eta}\wedge *\xi$$

the integral on the right only involves forms of dimension 4k-1. Now if we assume ξ in $C^\infty(\overline{X},\Omega_+)$ and η in $C^\infty(\overline{X},\Omega_-)$, we have for homogeneous degree components:

$$\xi^p = \varepsilon_p *\xi^{4k-p}$$

and $$\eta^q = -\varepsilon_q *\eta^{4k-q}$$

Collecting terms in above integral using these equations, we get:

$$((d+\delta)\xi|\eta) - (\xi|(d+\delta)\eta) = 2\int_{\partial X} \xi\wedge *\overline{\eta} \qquad (9)$$

Call $b(\xi,\eta)$ the right hand side of above equation. Then we see that by definition of V and Prop. 10,

b extends to a continuous sesquilinear form

$$b : V \times E \to \mathbb{C}$$

and clearly

<u>Prop. 11</u>

$$\mathrm{Dom}\,(A_+^*) = \{\psi \in E(X,\Omega_-) \,|\, b(\cdot,\psi) \equiv 0 \text{ on } V\}$$

If we now use the following isomorphisms, where Y is Y_1 or Y_2:

$$\mu_+ : \Omega(Y) \to \Omega_+(X)|_Y : \phi \to \frac{1}{2}(\phi + \tau(\phi))$$

$$\mu_- : \Omega(Y) \to \Omega_-(X)|_Y : \psi \to \frac{1}{2}(\psi - \tau(\phi))$$

take then ϕ, ψ so that

$$\mu_+(\phi) = \xi \;,\quad \mu_-(\psi) = \eta \quad \text{over } Y.$$

Taking now $\mu_+(\phi)\wedge{*}\mu_-(\psi)$, we neglect those terms containing the conormal to Y as they don't contribute to the integral (9), and we get:

$$2\int_Y \mu_+(\phi)\wedge{*}\mu_-(\overline{\psi}) = -\frac{1}{2}\int_Y \phi\wedge{*}\tau(\overline{\psi})$$

$$= -\frac{1}{2}\int_Y \langle\phi,\nu(\phi)\rangle_Y dv(Y) \qquad (10)$$

$$= -\frac{1}{2}(\phi\,|\,\nu(\psi))_Y$$

that is:

$$((d+\delta)\xi|\eta) - (\xi|(d+\delta)\eta) = -\frac{1}{2}\sum_j (\phi^j|\nu_j(\psi^j))_{Y_j}$$

where ν_j is $(-1)^p\varepsilon_p *_j$ on p-forms over Y_j (see definition on B on Part I).

Now recall: near Y_1 (similar for Y_2), we had the decomposition:

$$d+\delta = \sigma(dx_2)(\partial/\partial x_2 + A_1)$$

and

$$D_1 = \mu_+^{-1}A_1\mu_+$$

If we do same calculation for $\Omega_-(X)$ and the isomorphism μ_-, the result is:

$$D_1 = -\mu_-^{-1}A_1\mu_-$$

So we observe: all our results for $\Omega_+(X)$ can be repeated for $\Omega_-(X)$, but this time we have to consider as boundary values for the corresponding space V the span of the eigenfunctions corresponding to the eigenvalues that are ≥ 0.

Equation (10) gives the explanation to conditions for B in order that we have 2) of Prop. 2. From equation (10) and self-explanatory notation, we have

<u>Prop. 12</u>

Let $\phi \in V(X,\Omega_+)$ and $\psi \in V(X,\Omega_-)$, then:

$$(A_+\phi|\psi) = (\phi|A_-\psi)$$

so:

$$A_+^* \text{ is an extension of } A_-$$

We conclude making some remarks on how to prove existence of the index in the case of C^∞ boundary without using a parametrix. As in any case the parametrix is needed to obtain the formula giving it, we won't go into too many details.

The a priori inequalities are valid, we indicate how one proves $A_+^* = A_-$.

We have the corresponding space E, with C^∞ sections dense in it. Looking to (11) we see that, as $H^{-1/2}(\partial X)$ and $H^{1/2}(\partial X)$ are in duality by integration, and the restriction map from $H^1(X) \to H^{1/2}(\partial X)$ is onto and continuous, the elements of E define sections over ∂X in

$H^{-1/2}(\partial X)$. Then, they can be expressed by series of the form $\Sigma a_j\phi_j$, the ϕ_j being the solutions of $D\phi_j = \lambda_j\phi_j$, and satisfying the condition

$$\Sigma\lambda_j^{-1}|a_j|^2 < \infty$$

The bilinear functional b is then:

$$b(\phi,\psi) = \Sigma b_j\bar{a}_{k(j)}$$

where

$$\phi = \Sigma b_j\phi_j \quad \text{in} \quad V, \quad \psi = \Sigma a_j\phi_j \quad \text{in} \quad E$$

and we write $k(j)$ to take ν into account. But now:

if we have any ϕ_j on ∂X such that it corresponds to $\lambda_j < 0$, we can find $\phi \in V$ so that $\phi|\partial X = \phi_j$. It suffices to extend ϕ_j as constant along the normal to ∂X and multiply this by a C^∞ cut-off function equal to 1 on ∂X.

As $b(\phi,\psi) = 0$ for $\phi \in V$ and $\psi \in \text{Dom } A_+^* \subseteq E$, we see: such ψ have expansions

$$\psi = \sum_{\lambda_j \geq 0} a_j\phi_j \quad \text{in} \quad \partial X.$$

Using a variant of Friedrichs' Lemma (with boundary conditions), we can construct a sequence $\psi_n \in H^1(X)$ so that $\psi_n \to \psi$, $(d+\delta)\psi_n \to (d+\delta)\psi$ both in L^2 sense, and the ψ_n have boundary values as ψ above. But then the a priori inequality says the ψ_n are a Cauchy sequence in H^1 sense, and so ψ is H^1 and then it belongs to Dom A_-.

As we see, the crucial point in the argument is that the boundary values of the elements in V give the whole set $\{\phi_j \mid \phi_j$ corresponds to $\lambda_j < 0\}$.

In the C^∞ case, the elements of $\ker A_+$ are closed and co-closed. The proof in [3] uses the extension $\tilde{X}$ of X, but this can be also proven as follows:

$$D = L + M \quad \text{with} \quad LM = ML = 0 \quad \text{(see Part I)},$$

so if $D\phi = \lambda\phi$, we get:

$$DL\phi = L^2\phi = LD\phi = \lambda L\phi$$

same for M, so M and L leave invariant the finite dimensional spaces $V(\lambda) = \{\phi \mid D\phi = \lambda\phi\}$, but then by simultaneous diagonalization we can split $V(\lambda)$ in $V_L \oplus V_M$, where L acts as λI on V_L, 0 on V_M and vice-versa for M.

Assume now we have $\phi \in \Omega_+(X)$, $(d+\delta)\phi = 0$, $\phi = \sum_{\lambda_j<0} a_j\phi_j$ on ∂X (after using the corresponding isomorphisms), then we get using Green's formula:

$$||d\phi||^2 = -(d\phi|\delta\phi)_X = (\phi|L\phi)_{\partial X}$$

but now

$$L\phi = L\Sigma a_j \frac{1}{\lambda_j} L\phi_j, \quad \text{so}$$

$$(\phi|L\phi)_{\partial X} = \Sigma \frac{1}{\lambda_j} |a_j|^2 ||L\phi_j||^2$$

(recall $L^* = L$), but as the $\lambda_j < 0$, above quantity is ≤ 0, but $||d\phi||^2 \geq 0$ so only possibility is 0. So $d\phi = \delta\phi = 0$.

To repeat this argument in our case requires to examine the boundary conditions satisfied by $L\phi$ on ∂Y for the elements $D\phi = \lambda\phi$, $B\phi = 0$ on ∂Y. This involves a long calculation as we must use σ_L besides σ_D, in any case in order to obtain a similar division of the eigenfunctions of D_B one would need to impose more conditions on B and in that case it does not give self-adjointness for D_B.

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