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Constructing symmetric topological phases of bosons in three dimensions via fermionic projective construction and dyon condensation

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Recently, there is a considerable study on gapped symmetric phases of bosons that do not break any symmetry. Even without symmetry breaking, the bosons can still be in many exotic new states of matter, such as symmetry-protected topological (SPT) phases, which are short-range entangled and symmetry-enriched topological (SET) phases, which are long-range entangled. It is well known that *noninteracting fermionic topological insulators* are SPT states protected by time-reversal symmetry and U(1) fermion number conservation symmetry. In this paper, we construct three-dimensional exotic phases of bosons with time-reversal symmetry and boson number conservation U(1) symmetry by means of *fermionic projective construction*. We first construct an *algebraic bosonic insulator,* which is a symmetric bosonic state with an emergent U(1) gapless gauge field. We then obtain many gapped bosonic states that do not break the time-reversal symmetry and boson number conservation via proper *dyon condensations*. We identify the constructed states by calculating the allowed electric and magnetic charges of their excitations, as well as the statistics and the symmetric transformation properties of those excitations. This allows us to show that our constructed states can be trivial SPT states (i.e., trivial Mott insulators of bosons with symmetry), nontrivial SPT states (i.e., bosonic topological insulators), and SET states (i.e., fractional bosonic topological insulators). In nontrivial SPT states, the elementary monopole (carrying zero electric charge but unit magnetic charge) and elementary dyon (carrying both unit electric charge and unit magnetic charge) are fermionic and bosonic, respectively. In SET states, intrinsic excitations may carry fractional charge.

I. INTRODUCTION

A quantum ground state of a many-boson system can be in a spontaneous-symmetry-breaking state, or a topologically ordered (TO) state [1–3]. A TO state is defined by the following features: ground-state degeneracy in a topologically nontrivial closed manifold [1–3], emergent fermionic/anyonic excitations [4,5], or chiral gapless edge excitations [6,7]. If, in addition to a TO, the ground state also has a symmetry, such a state will be referred as a "symmetry-enriched topological (SET) phase."

Recently, it was predicted that even if the bosonic ground state does not break any symmetry and has a trivial TO, it can still be in a nontrivial phase called bosonic symmetry-protected topological phase (SPT) [10–12]. Since the bosonic SPT phases do not have TOs, a systematic description/construction of those SPT phases was obtained via group cohomology theory [10–12]. Many new SPT phases were predicted/constructed with all possible symmetries and in any dimensions. In the following, we also refer all *gapped* phases of bosons that do not break the symmetry (including SPT and SET) as "*topological phases.*"

To realize bosonic TO phases or SPT phases, the interaction is crucial, since without interactions, bosons always tend to condense trivially. Weak interactions in most cases only lead to superfluid states. This fact hinders the conventional perturbation approach if we want to realize TO or SPT phases. One useful approach is via the exactly soluble models, as in the string-net approach [8,9] and the group cohomology approach [10–12]. Recently, many other approaches were proposed, which are based on field theory, topological invariants, critical theory of surface, topological response theory, etc [13–21,23–25]. A quite effective approach for strongly interacting systems

is the "projective construction" [26–37]. Some appealing advantages of the projective construction are that (i) it can generate many useful trial wave functions for many-body systems and (ii) fractional charge/statistics and emergent gauge fields can be constructed effectively. In other words, it is quite easy to obtain TO states by using the projective construction. However, it has been recently realized that the projective construction is also helpful in constructing bosonic SPT states [22,38–41].

Roughly speaking, in the so-called "fermionic projective construction," each bosonic operator of a given boson system is split into a product of fermionic parton operators. Different kinds of partons can individually form different mean-field ground states. The Hilbert space of partons is larger than the physical Hilbert space $H_{phys.}$ of the initial boson model. The physical ground state of the boson system is realized by doing Gutzwiller projection. In other words, the direct product of multiple mean-field ground states is projected into $H_{phys.}$ in which the multiple partons are glued back into a physical boson on each site. In terms of path integral formulation, such a gluing process is done by introducing fluctuating internal gauge fields that couple to partons. The gauge degrees of freedom can be in Coulomb phase, Higgs phase (e.g., residual Z_N gauge symmetry), or confined phase. In Coulomb phase, gapless photon excitations close the bulk gap, while discrete gauge symmetry in Higgs phase generates TO.

To obtain SPT states (gapped, symmetric, without TO), we need to consider the confined phase of the internal gauge fields, where the gauge fluctuations are very strong and solitonic excitations (e.g., monopoles and dyons in 3D and instantons in 2D) are allowed. In 2D SPT construction, we have the well-known K-matrix Chern-Simons formalism, which is a "controllable program" to avoid the discussion of the

confinement problem [41] during the fermionic projection construction. For example, one can compute the determinant of K matrix to probe TO and even classify SPT states in 2D with Abelian symmetry group [14]. However, K-matrix Chern-Simons formalism is only applicable in 2D. The question is this: in the 3D fermionic projective construction, is there an "efficient program" that can lead to SPT states in a controllable way? In the present work, we will handle this problem by focusing on 3D SPT states with U(1) symmetry (boson particle number conservation) and time-reversal symmetry (Z_2^T). Here, $Z_2^T = \{\mathbb{I}, \mathcal{T}\}$ where $\mathbb{I}$ is identity while $\mathcal{T}^2 = \mathbb{I}$. We will refer those phases as bosonic topological insulators (BTI). If a SET state also contains these two symmetry groups, we call it fractional BTI (fBTI).

From our proposed algorithm, we can even learn more than the SPT construction itself. It is known that two fermions near a Fermi surface can have BCS superconducting instability under certain conditions. Various pairing symmetries and other pairing dynamical details may generically lead to various gapped superconducting states. In the fermionic projective construction, we may ask a similar question: by driving a gapless Coulomb phase into a gapped confinement phase and keeping some global symmetries, are there more than one type of confinement phases that the quantum many-body system eventually enters? In this paper, we will see that there are indeed many different kinds of confinement phases that are featured by different choices of monopole (or dyon) condensations without breaking symmetry. These different phases are finally classified into trivial SPT, nontrivial SPT, and SET states. We note that recently symmetry-breaking patterns induced by dyon condensations are discussed in the context of topological Mott insulators [42].

We specially choose the symmetry group U(1)$\rtimes Z_2^T$ in this paper. One direct motivation is that three-dimensional *noninteracting fermionic* topological insulators (TI) [43] are well-understood and also have U(1)$\rtimes Z_2^T$ symmetry group. TI is classified by Z_2, i.e., only one type of nontrivial state. Trivial and nontrivial TI states can be further elegantly labeled by the so-called "axionic Θ angle" in the electromagnetic response action "$S_{EM} = \frac{\Theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu \partial_\lambda A_\rho$" ($A_\mu$ is the external electromagnetic gauge field). $\Theta = 0\,(\pi)$ corresponds to the trivial (nontrivial) phase [44]. It is interesting to ask whether there exists a bosonic version of TI, i.e., BTI and fBTI via the fermionic projective construction and how about the physical properties? Reference [45] applied the fermionic projective construction approach in which the boson creation operator is split into a singlet pair of spin-1/2 fermions. It is assumed that the fermions are described by a nontrivial TI mean-field ansatz that explicitly breaks the internal SU(2) gauge symmetry down to Z_2. The resultant physical ground state is a SET state admitting a fractional Θ angle and emergent Z_2 TO. By definition, this bosonic insulator is an fBTI, following Ref. [46] where a fermionic version in the presence of strong interactions is proposed.

In the present work, the underlying boson model contains *four* kinds of charge-1 bosons with U(1)$\rtimes Z_2^T$ symmetry in three dimensions [i.e., Eq. (1)]. In the fermionic projective construction, each boson is split into two different fermions (f_1, f_2) carrying "spin-1/2." f_1 and f_2 carry α and $(1 - \alpha)$ electric charge, respectively (see Table I).

TABLE I. Assignment of EM electric charge and a_μ-gauge charge.

Particle	EM electric charge	a_μ-gauge charge
f_1	α	$+1$
f_2	$1 - \alpha$	-1
b	$+1$	0

To ensure that the mean-field *Ansätze* of fermions respect symmetry before projection, we assume that mean-field *Ansätze* of the fermions describe a fermionic gapped phase with θ-angle θ_1 for f_1 fermions and θ_2 for f_2 fermions. We assume $\theta_1 = \theta_2 = 0$ or $\theta_1 = \theta_2 = \pi$ where the two fermions form the same trivial band insulator state or the same topological insulator states. We will use $(\theta_1, \theta_2, \alpha)$ to label those mean-field *Ansätze*. Due to the projective construction, an internal U(1) gauge field a_μ exists and is gapless. So our construction (at this first step) leads to gapless insulating states of the bosons after the projection. We call such a state algebraic bosonic insulator.

To obtain gapped insulating states of the bosons, we shall push the internal gauge field into its confined phases, where quantum fluctuations are very strong leading to a proliferation of a certain dyon. There are many different kinds of confined phases that correspond to proliferation of different dyons. These dyons may carry many quantum numbers including: fermion numbers of f_1, f_2, magnetic charge and gauge charge of the internal gauge field, magnetic charge and electric charge of an external electromagnetic gauge field. The latter is assumed to be compact such that a magnetic monopole is naturally allowed although the electromagnetism in our world is noncompact so far. We will use (l, s) to label those different proliferated (or condensed) dyons that do not break the U(1) and time reversal symmetries.

After a symmetric dyon condensate (l, s) is selected and the charge assignment α is fixed, we may construct a gapped topological phase with U(1)$\rtimes Z_2^T$ symmetry. We find that the dyon condensation breaks the "gauge symmetry" of shifting α (dubbed "α-gauge symmetry"), so that different α will generally lead to different bosonic states (cf. Sec. II C). Thus those topological phases are eventually labeled by $(\theta_1, \theta_2, \alpha)$ as

TABLE II. Some concrete examples of nontrivial BTI phases in three dimensions. Each BTI state is labeled by five numbers $(\theta_1, \theta_2, \alpha, l, s)$. θ_1 (θ_2) equals to 0 or π, denoting the trivial or nontrivial TI phases of the fermion f_1 (f_2). f_1 and f_2 carry α and $(1 - \alpha)$ electric charge of external electromagnetic field, respectively. In the mean-field *Ansatz* (0,0), the condensed dyon is composed by s magnetic charge of internal U(1) gauge field and l physical bosons. In the mean-field *Ansatz* (π, π), the condensed dyon is composed by s magnetic charge of internal U(1) gauge field, l physical bosons, and, in addition, s f_2 fermions. Each physical boson is composed by one f_1 and one f_2.

$(\theta_1, \theta_2, \alpha)$	Dyon condensate (l, s)
$(0, 0, 1)$	$(1, 1)$
$(0, 0, 1)$	$(3, 1)$
$(\pi, \pi, -\frac{1}{2})$	$(1, 1)$
$(\pi, \pi, \frac{3}{2})$	$(1, 1)$

TABLE III. Topological phases of bosons with $U(1) \rtimes Z_2^T$ symmetry in three dimensions [labeled by $(\theta_1, \theta_2, \alpha = 1/2, l, s)$]. All symmetric dyons are labeled by two integers (l,s). In each mean-field *Ansatz*, different kinds of dyon condensations lead to, generally, different topological phases (trivial SPT, nontrivial SPT, or SET). If $s = l = 0$, i.e., there is no dyon condensation, the resultant symmetric state is the algebraic bosonic insulator (ABI) state, which is gapless and can be viewed as a parent state of all topological phases before condensing some dyons. If $s = 0, l \neq 0$, the condensed dyon will break U(1) symmetry, rendering a symmetry-breaking phase. To get symmetric gapped phases, $s \neq 0$ has been required in the table. The physical interpretation of s and l is the following. In the mean-field *Ansatz* $(\theta_1, \theta_2) = (0,0)$, the condensed dyon with $U(1) \rtimes Z_2^T$ symmetry is a composite of s monopoles of internal gauge field and l physical bosons. In the mean-field *Ansatz* $(\theta_1, \theta_2) = (\pi,\pi)$, the condensed dyon with $U(1) \rtimes Z_2^T$ symmetry is a composite of s monopoles of internal gauge field, l physical bosons, and s f_2 fermions. One "physical boson" is equal to one f_1 fermion plus one f_2 fermion. "$Z_{|s|}$ TO" denotes the TO of $Z_{|s|}$ gauge theory, which arises from the gauge sector of the ground state. "None" in a given entry means that the topological phase does not exist in the corresponding mean-field *Ansatz*. All trivial SPT have Witten effect with $\Theta = 0 \mod(4\pi)$ and all nontrivial SPT have Witten effect with $\Theta = 2\pi \mod(4\pi)$. The discussion on Witten effect of SET will be presented in Sec. V where trivial fBTI and nontrivial fBTI are defined and classified. The mean-field *Ansatz* $(0,\pi)$ always breaks time-reversal symmetry.

Mean-field *Ansatz* (θ_1, θ_2)	Trivial SPT(trivial Mott insulator of bosons)	Nontrivial SPT(BTI: bosonic topological insulator)	SET(fBTI: fractional bosonic topological insulator)
$(0,0)$	$\{l/s \in \mathbb{Z}, l \neq 0\} \cup \{l = 0, s = \pm 1\}$	None	$\{l/s \notin \mathbb{Z}\}$; Especially, $\{l = 0, |s| \geq 2\}$ is a pure $Z_{|s|}$ TO state.
(π,π)	$\{l/s \in \mathbb{Z}\}$	None	$\{l/s \notin \mathbb{Z}\}$ (if $l/s = -1/2$, an additional $Z_{|s|}$ TO emerges.)

well as (l,s) (see Tables II–IV), and we have to choose certain special values of α to preserve the U(1) and time reversal symmetries (see Fig. 1). The excitation spectrum above a topological phase is formed by the so-called "deconfined dyons" that have trivial *mutual statistics* with the condensed dyon.

To understand whether the state supports TO, we have to examine a subset of excitations, namely *intrinsic excitations* (defined as those excitations that have zero magnetic charge of the external electromagnetic field). Noting that the underlying model is formed by charge-1 bosons, if there are intrinsic

excitations that either carry fractional electric charge of the external electromagnetic field or carry fermionic statistcs, the state must be a TO state. The reason we only consider intrinsic excitations is that the symmetry group U(1) is specific to the conservation of electric charge of the external electromagnetic field. The magnetic charge of the external electromagnetic field is not an intrinsic quantity of the underlying boson system. For example, a nontrivial TI state in a background compact electromagnetic field allows a composite excitation that contains 1/2 electric charge and one magnetic charge of

TABLE IV. Topological phases of bosons with $U(1) \rtimes Z_2^T$ symmetry in three dimensions (for a generic α sequence), labeled by $(\theta_1, \theta_2, \alpha, l, s)$. A state is $U(1) \rtimes Z_2^T$ symmetric if the parameters l, s, α satisfy the conditions in this table. We see that if $\alpha = 1/2$, all allowed SPT states are trivial in any mean-field *Ansatz* consistent with Table III. All SET states can be further classified into trivial fBTI (without Witten effect) and nontrivial fBTI (admitting Witten effect). The mean-field *Ansatz* $(0,\pi)$ always breaks time-reversal symmetry.

| Mean-field *Ansatz* (θ_1, θ_2)	Trivial SPT(trivial Mott insulator of bosons)	Nontrivial SPT(BTI: bosonic topological insulator)	SET(fBTI: fractional bosonic topological insulator)
$(0,0)$	$\{l/s = \text{odd}, \alpha = \text{half-odd}\} \cup \{l/s = \text{even}, l \neq 0, 2\alpha = \text{integer}\} \cup \{l = 0, s = \pm 1, 2\alpha = \text{integer}\}$	$\{l/s = \text{odd}, \alpha = \text{integer}\}$	$\{l = 0, |s| \geq 2, 2\alpha = \text{integer}\}$: $Z_{|s|}$ TO and $\Theta = 0 \mod 4\pi$
			$\{l/s \notin \mathbb{Z}\}$, e.g., $l/s = \frac{1}{3}$: $\alpha = \text{half-odd} \longrightarrow$ trivial fBTI with $\Theta = 0 \mod \frac{4\pi}{9}$ Witten effect; $\alpha = \text{integer} \longrightarrow$ nontrivial fBTI with $\Theta = \frac{2\pi}{9} \mod \frac{4\pi}{9}$ Witten effect
(π,π)	$\{l/s \in \mathbb{Z}, \alpha - \frac{1}{2} = \text{even}\}$	$\{l/s \in \mathbb{Z}, \alpha - \frac{1}{2} = \text{odd}\}$	$\{l/s = -1/2, \alpha = \text{half-odd}\}$: $Z_{|s|}$ TO and $\Theta = \frac{\pi}{2} \mod \pi$ Witten effect $\{l/s \notin \mathbb{Z}, l/s \neq -\frac{1}{2}\}$, e.g., $l/s = \frac{1}{3}$: $\alpha - \frac{1}{2} = \text{even} \longrightarrow$ trivial fBTI with $\Theta = 0 \mod \frac{4\pi}{9}$ Witten effect; $\alpha - \frac{1}{2} = \text{odd} \longrightarrow$ nontrivial fBTI with $\Theta = \frac{2\pi}{9} \mod \frac{4\pi}{9}$ Witten effect
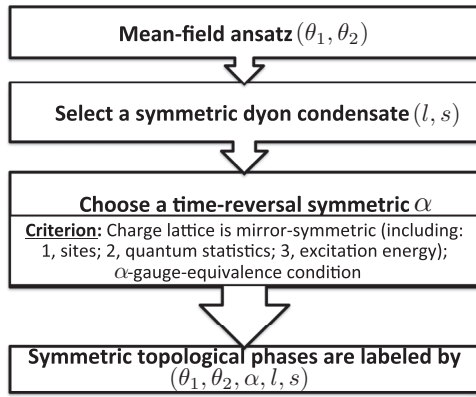
FIG. 1. Illustration of the basic process of constructing symmetric topological phases via fermionic projective construction and dyon condensation.

the external electromagnetic field, but TI cannot be interpreted to a TO state supporting fractional excitations. It is well known that as a noninteracting fermionic gapped state, TI is not a TO state for sure, and, all "intrinsic excitations" do not carry fractional electric charge of the external electromagnetic field. The 1/2 fractional charge is actually induced by nonzero magnetic charge.

In addition to the above analysis of deconfined dyons, if a topological phase contains deconfined discrete gauge symmetry (e.g., Z_2 gauge theory), the topological phase must be a TO state regardless of the properties of deconfined dyons. This state can be viewed as a realization of a Higgs phase of an internal gauge field, which will be discussed later.

Based on the above clarification, we have the following *quantitative criterion* that will be frequently utilized in this work. (1) If a topological phase respects global symmetry [i.e., U(1)$\rtimes Z_2^T$] and contains deconfined discrete gauge symmetry (e.g., Z_2 gauge theory), the topological phase must be a SET state where both TO and symmetry are present. (2) If a topological phase respects global symmetry and there are intrinsic excitations that either carry fractional electric charge of the external electromagnetic field or carry fermionic statistics, the topological phase is a SET state (i.e., fBTI) with both TO and symmetry. (3) If TO is absent and symmetry is still unbroken, the topological phase must be an SPT state. If the excitation spectrum of an SPT state admits a nontrivial Witten effect with $\Theta = 2\pi \mod 4\pi$ [15,21,47–49], the topological phase is a nontrivial SPT (i.e., a BTI). Otherwise, the state is a trivial SPT with $\Theta = 0 \mod 4\pi$, i.e., a trivial Mott insulator of bosons with symmetry.

All topological phases that we constructed are summarized in Tables III ($\alpha = 1/2$) and IV (for a general α sequence). These two tables contain the general results. For reader's convenience, some concrete examples of nontrivial BTI phases are shown in Table II. The basic process of constructing symmetric topological phases is shown in Fig. 1.

The remaining parts of the paper are organized as follows. In Sec. II, the underlying boson degrees of freedom as well as the fermionic projective construction are introduced. Symmetry operations [both U(1) and Z_2^T] on physical bosons and fermonic partons are defined. In Sec. III, the general properties

of dyons are discussed. The main results of topological phases are derived in Sec. IV where topological phases are constructed by setting $\alpha = 1/2$. The general construction of topological phases in the presence of general α sequence is provided in Sec. V. Conclusions and future directions are made in Sec. VI.

II. FERMIONIC PROJECTIVE CONSTRUCTION OF MANY-BOSON STATE WITH U(1)$\rtimes Z_2^T$ SYMMETRY

A. Definition of boson operators

We will use a system with four kinds of electric charge-1 bosons in three dimensions. Those bosons are described by four boson operators. We split the boson operators into two different spin-1/2 fermions:

$$(b_1, b_2, b_3, b_4) = (f_{1\uparrow} f_{2\uparrow}, f_{1\uparrow} f_{2\downarrow}, f_{1\downarrow} f_{2\uparrow}, f_{1\downarrow} f_{2\downarrow}). \quad (1)$$

The fermionic projective construction of the four bosons implies that the underlying bosons are of hard-core nature since the dimension of the bosonic Hilbert space at each lattice site i is truncated to be finite and exchange of two bosons at different sites does not generate a fermionic sign. As a result, at the very beginning, the underlying boson model on the lattice must be a correlated bosonic system. All possible ground states with boson charge conservation symmetry U(1) and time-reversal symmetry Z_2^T are what we shall look for in this paper.

The physical ground-state wave function $|GS\rangle$ in $H_{phys.}$ can be written in terms of direct product of fermions' mean-field *Ansätze*, subject to Gutzwiller projection:

$$|GS\rangle = \hat{P}_G(|\Psi(f_1)\rangle \otimes |\Psi(f_2)\rangle), \quad (2)$$

where $\hat{P}_G$ is the Gutzwiller projection operator, which enforces that the total number of f_1 is equal to that of f_2 at each site in the physical (projected) Hilbert space $H_{phys.}$. $|\Psi(f_1)\rangle$ and $|\Psi(f_2)\rangle$ are mean-field *Ansätze* for the ground states of f_1 and f_2, respectively. In the present work, we assume that both of f_1 and f_2 form *mean-field Ansätze* with band structures that respect U(1)$\rtimes Z_2^T$ symmetry. Such band structures are of TI classified by Z_2, i.e., one trivial and one nontrivial states. It is thus instructive to separately study the physical ground states in two different classes: (1) both are trivial and (2) both are nontrivial. Other mean-field *Ansätze* explicitly break time-reversal symmetry already at mean-field level.

B. Definition of symmetry transformations

1. Time-reversal symmetry ($\mathcal{T}^2 = \mathbb{I}$)

Under time reversal, the above fermions transform as the usual spin-1/2 fermions, but with an additional exchange $f_{1\sigma} \leftrightarrow f_{2\sigma}$, and the bosons transform as

$$b_1 \to -b_4, \quad b_2 \to b_2, \quad b_3 \to b_3, \quad b_4 \to -b_1. \quad (3)$$

For instance, $b_2 = f_{1\uparrow} f_{2\downarrow} \xrightarrow{①} -f_{1\downarrow} f_{2\uparrow} \xrightarrow{②} -f_{2\downarrow} f_{1\uparrow} = f_{1\uparrow} f_{2\downarrow} = b_2$ such that b_2 is unchanged, where ① represents $f_{1\downarrow} \to -f_{1\uparrow}, f_{1\uparrow} \to f_{1\downarrow}, f_{2\downarrow} \to -f_{2\uparrow}, f_{2\uparrow} \to f_{2\downarrow}$ and ② represents exchange of labels: $1 \leftrightarrow 2$. In this projective construction, there is an internal U(1) gauge field, a_μ minimally coupled to f_1, f_2. Let us define a "pseudospinor" $\mathbf{B} \equiv (b_1, b_2, b_3, b_4)^T$. Equation (3) also indicates the time-reversal symmetry matrix that acts on the pseudospinor, which has the

following form:

$$
\mathsf{T} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{4}
$$

which has determinant Det(T) = −1. After two consecutive transformations (T^2 is a unit matrix of four dimensions), the pseudospinor **B** remains invariant, implying that the time reversal we consider here is $\mathcal{T}^2 = \mathbb{I}$, where $\mathcal{T}$ and an identity $\mathbb{I}$ form the time-reversal symmetry group $Z_2^T = \{\mathbb{I}, \mathcal{T}\}$. As a side note, we emphasize that the four-boson model shown in Eq. (1) is the *minimal choice* of $\mathcal{T}^2 = \mathbb{I}$ bosonic system in the present fermionic projective construction. If this choice is not adopted, one has to consider a more complicated choice, e.g., $b_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = f_{1\sigma_1} f_{2\sigma_2} f_{3\sigma_3} f_{4\sigma_4}$ where there are four spin-1/2 partonic fermions (spin indices $\sigma_1 \cdots \sigma_4 = \uparrow, \downarrow$) leading to a pseudospinor with 16 components.

The assignment of gauge charges carried by f_s ($s = 1, 2$) is shown in Table I. f_1 and f_2 carry +1 and −1 gauge charges of a_μ, respectively, such that all physical boson operators are invariant under a_μ gauge transformation.

2. Boson number conservation U(1) and the charge assignement

Each boson carries +1 fundamental electric charge of external electromagnetic (EM) field A_μ such that one can make the following assignment for fermions shown in Table I: f_1 and f_2 carry α and $1 - \alpha$ EM electric charge of A_μ, respectively. Here, α is a real number whose value should not alter the vacuum expectation value of EM gauge-invariant operators. More precisely, α is not a defining parameter of the underlying boson model. Rather, it is introduced in the projective construction at ultraviolet (UV) scale. When we only change α, the projected wave function should not change once the projection is done *exactly at lattice scale*. As a side note, one can check that U(1) symmetry here corresponds to conservation of *total number* of bosons, i.e., $\sum_i (\mathbf{B}_i^\dagger \mathbf{B}_i) = \sum_i (b_{i1}^\dagger b_{i1} + b_{i2}^\dagger b_{i2} + b_{i3}^\dagger b_{i3} + b_{i4}^\dagger b_{i4}) = $ constant.

C. Residual α-gauge symmetry after dyon condensation

The above discussion about α suggests that α is a pure gauge degree of freedom (or more precisely: a gauge redundancy). We conclude that before the dyon condensation, there is an α-gauge symmetry, which is defined as $\alpha \rightarrow \alpha + \lambda$, where λ is any real number. Later, we will see that the dyon condensation can break such an "α-gauge symmetry," just like the Higgs condensation can break the usual "gauge symmetry." However, we believe that dyon condensation does *not* break all the α-gauge symmetry: shifting α by any integer remains to be a "gauge symmetry" even after the dyon condensation. The physical consideration behind this statement is that the EM *charge quantization* is unaffected by any integer shift at all, and, such an integer shift is nothing but redefinition of field variables.

In this paper, we will show that in the mean-field *Ansatz* $(\theta_1, \theta_2) = (0, 0)$ case, all topological phases, including symmetry-protected topological phases (SPT) and symmetry-enriched topological phases (SET), satisfy this statement (cf.

Sec. V). In other words, after $\alpha \rightarrow \alpha + 1$, the calculated properties of the topological phases are unaffected.

However, the statement is invalid in the mean-field *Ansatz* $(\theta_1, \theta_2) = (\pi, \pi)$ via our continuum effective field theory approach (cf. Sec. V). After $\alpha \rightarrow \alpha + 1$, the physical properties are changed. In this case, it appears that α-gauge symmetry with any *odd* integer shift is broken by the dyon condensation. Shifting α by any *even* integer remains to be a "gauge symmetry," after the dyon condensation.

At the moment, we do not understand why dyon condensations for $(\theta_1, \theta_2) = (0, 0)$ and $(\theta_1, \theta_2) = (\pi, \pi)$ lead to different α-gauge symmetry breaking. However, we would like to point out that our field theoretic treatment on dyons is established in the continuum limit of space-time. More rigorous approach, however, should involve the regularization procedures of dyon fields on lattice. For example, dyons are not point particles at all on lattice. Rather, dyons are regularized on a dual hypercubic lattice of space-time where magnetic charge and electric charge are put on dual sublattices. This more careful lattice consideration may allow us to understand how dyons condensation may break the α-gauge symmetry and in which way the α-gauge symmetry is broken. We leave this issue to future work.

III. GENERAL PROPERTIES OF DYONS

A. Quantum numbers of dyons

The projective construction is a very natural way to obtain topological phases with TO since at the very beginning the fermionic degrees of freedom and internal gauge fields are introduced at UV scale. To obtain SPT phases, we must prohibit the emergence of TO, by *at least* considering the confined phase of the internal gauge field, where the dyons of the internal gauge field play a very important role. For the purpose of probing the EM response, the nondynamical EM field is applied and is assumed to be compact. Thus a dyon may carry gauge (electric) charges and magnetic charges of both internal gauge field and EM field. The terms "gauge (electric) charge" and "magnetic charge" are belonging to both gauge fields, while, for the EM field, we specify the charges by adding "EM" to avoid confusion. A dyon can also include f_1 and f_2 fermions, resulting in nonzero "fermion number." Thus a generic dyon is labeled by a set of quantum numbers that describe those gauge charges, magnetic charges, and fermion numbers. Specially, a *monopole* is defined as a special dyon which does not carry any kind of gauge (electric) charges.

To describe those dyon excitations systematically, let us assume that each fermion (f_s) couples to its own gauge field ($A_\mu^{f_s}$) with "+1" gauge charge. In fact, $A_\mu^{f_s}$ are combinations of a_μ and A_μ (cf. Table I):

$$
A_\mu^{f1} \equiv a_\mu + \alpha A_\mu, \quad A_\mu^{f2} \equiv -a_\mu + (1 - \alpha) A_\mu. \tag{5}
$$

A dyon can carry the magnetic charges in A^{fs} gauge groups, which are labeled by $\{N_m^{(s)}\} \in \mathbb{Z}(s = 1, 2)$. These two magnetic charges $\{N_m^{(s)}\}$ are related to the magnetic charge N_m^a in a_μ gauge group and magnetic charge N_M in A_μ gauge group in the following way:

$$
N_m^{(1)} = N_m^a + \alpha N_M, \quad N_m^{(2)} = -N_m^a + (1 - \alpha) N_M, \tag{6}
$$

where the EM magnetic charge N_M is integer-valued as usual: $N_M \in \mathbb{Z}$. For this reason, the quantization of magnetic charge N_m^a of a_μ gauge group is determined by two integers: "$N_m^{(1)}$" and "N_M" via Eq. (6) with a given α. The following relations are useful:

$$N_M = N_m^{(1)} + N_m^{(2)}, \quad N_m^a = (1-\alpha)N_m^{(1)} - \alpha N_m^{(2)}. \quad (7)$$

A dyon can also carry the fermion numbers of f_1, f_2 denoted by $\{N_f^{(s)}\}, s = 1,2$. They are related to magnetic charges in the following way:

$$N_f^{(1)} = n_f^{(1)} + \frac{\theta_1}{2\pi}N_m^{(1)}, \quad N_f^{(2)} = n_f^{(2)} + \frac{\theta_2}{2\pi}N_m^{(2)}, \quad (8)$$

where the θ-related terms are *polarization electric charge clouds* due to Witten effect [47,48] and $n_f^{(s)}$ are integer-valued, indicating that integer numbers of fermions are able to be *trivially* attached to the dyon. The nature of "polarization" is related to the fact that this charge cloud does not contribute *quantum statistics* to dyons [50]. θ_1 and θ_2 determine the topology of fermionic band structures of f_1 and f_2, respectively, if symmetry group U(1)$\rtimes Z_2^T$ is implemented. For example, $\theta_1 = 0$ if f_1 forms a trivial TI *Ansatz* and $\theta_1 = \pi$ if f_1 forms a nontrivial TI *Ansatz*.

B. Time-reversal transformation of dyons, gauge fields, and Lagrangians

To see whether the ground state breaks symmetry or not, it is necessary to understand how the symmetry acts on dyon labels $(N_m^a, N_M, N_f^{(1)}, N_f^{(2)}, N_m^{(1)}, N_m^{(2)})$ as well as gauge fields $(A_\mu, a_\mu, \widetilde{A}_\mu, \widetilde{a}_\mu)$, where $\widetilde{A}_\mu$ and $\widetilde{a}_\mu$ are two dual gauge fields, which are introduced to describe the minimal coupling in the presence of magnetic charge.

The fermion-exchange process defined in Sec. II B implies that the following transformation rules are obeyed by dyon labels (all transformed symbols are marked by "—"):

$$\overline{N_f^{(1)}} = N_f^{(2)}, \quad \overline{N_f^{(2)}} = N_f^{(1)}, \quad (9)$$

$$\overline{N_M} = -N_M, \quad \overline{N_m^a} = N_m^a, \quad (10)$$

where Eq. (9) holds by definition. In Eq. (10), the EM magnetic charge's sign is reversed as usual, which is consistent to reverse the sign of the EM gauge potential **A**, which is a polar vector:

$$\overline{\mathbf{A}} = -\mathbf{A}. \quad (11)$$

The second formula in Eq. (10) can be understood in the following way. A single f_1 fermion couples to **A** and **a** with α and $+1$ coupling constants respectively, as shown in Table I. A single f_2 fermion couples to **A** and **a** with $1 - \alpha$ and -1 coupling constants, respectively. Under Z_2^T, all spatial components of gauge fields will firstly change signs and α is replaced by

$$\overline{\alpha} = 1 - \alpha. \quad (12)$$

At this *intermediate* status, f_1 fermion couples to **A** and **a** with $-\overline{\alpha}$ and -1 coupling constants, respectively, and, f_2 fermion couples to **A** and **a** with $-1 + \overline{\alpha}$ and $+1$ coupling constants, respectively. The second step is to exchange the two fermions as defined in Sec. II B. By definition, $\overline{f_2}$ (i.e., the new f_2

fermion after time-reversal transformation) should couple to $\overline{\mathbf{A}}$ and $\overline{\mathbf{a}}$ with $1 - \alpha$ and -1 coupling constants respectively, which results in $(1-\alpha)\overline{\mathbf{A}} - \overline{\mathbf{a}} = -\overline{\alpha}\mathbf{A} - \mathbf{a}$. Likewise, $\overline{f_1}$ couples to $\overline{\mathbf{A}}$ and $\overline{\mathbf{a}}$ with α and $+1$ coupling constants, such that $\alpha\overline{\mathbf{A}} + \overline{\mathbf{a}} = (-1 + \overline{\alpha})\mathbf{A} + \mathbf{a}$. Overall, we obtain the following rule by using Eq. (11):

$$\overline{\mathbf{a}} = \mathbf{a}, \quad (13)$$

which requires the relation of magnetic charges $\overline{N_m^a} = N_m^a$ in a self-consistent manner as shown in Eq. (10).

Based on the above results, one may directly derive the transformation rules obeyed by other quantum numbers:

$$\overline{N_m^{(1)}} = -N_m^{(2)}, \quad \overline{N_m^{(2)}} = -N_m^{(1)},$$

$$\overline{n_f^{(1)}} = n_f^{(2)} + \frac{\theta_1 + \theta_2}{2\pi}N_m^{(2)},$$

$$\overline{n_f^{(2)}} = n_f^{(1)} + \frac{\theta_1 + \theta_2}{2\pi}N_m^{(1)}. \quad (14)$$

Suppose that N_A and N^a are the bare EM electric charge and a_μ-gauge charge carried by dyons (cf. Table I):

$$N_A = \alpha N_f^{(1)} + (1-\alpha)N_f^{(2)}, \quad N^a = N_f^{(1)} - N_f^{(2)}. \quad (15)$$

We have

$$\overline{N_A} = N_A, \quad \overline{N^a} = -N^a, \quad (16)$$

by noting that $\overline{\alpha} = 1 - \alpha$.

By definition, the curls of dual gauge potentials $(\widetilde{\mathbf{A}}, \widetilde{\mathbf{a}})$ contribute electric fields $(\mathbf{E}, \mathbf{E}^a)$. Therefore the dual gauge potentials should obey the same rules as electric fields under time-reversal transformation, and electric fields should also obey the same rules as electric charges (N_A, N^a) in a consistent manner such that the dual gauge potentials are transformed in the following way:

$$\overline{\widetilde{\mathbf{A}}} = \widetilde{\mathbf{A}}, \quad \overline{\widetilde{\mathbf{a}}} = -\widetilde{\mathbf{a}}. \quad (17)$$

The four formulas in Eqs. (11), (13), and (17) are transformation rules obeyed by the spatial components of the gauge potentials. The time components of the gauge potentials $(A_0, a_0, \widetilde{A}_0, \widetilde{a}_0)$ obey the following rules:

$$\overline{A_0} = A_0, \quad \overline{a_0} = -a_0, \quad (18)$$

$$\overline{\widetilde{A}_0} = -\widetilde{A}_0, \quad \overline{\widetilde{a}_0} = \widetilde{a}_0 \quad (19)$$

by adding an overall minus sign in each of Eqs. (11), (13), and (17).

On the other hand, let us consider the effective Lagrangian that describes the dyon dynamics. Let us start with a general dyon ϕ and try to understand its time-reversal partner $\overline{\phi}$. The effective Lagrangian term $\mathcal{L}_{\text{kin}}$ that describes the kinetic energy of ϕ can be written as (up to a quadratic level)

$$\mathcal{L}_{\text{kin}}[\phi] = \frac{1}{2m}\left|\left(-i\nabla + N^a\mathbf{a} + N_A\mathbf{A} + N_m^a\widetilde{\mathbf{a}} + N_M\widetilde{\mathbf{A}}\right)\phi\right|^2. \quad (20)$$

Here, we are performing time-reversal transformation in field theory action such that we keep all real-valued numerical

coefficients $(N^a, N_A, \dots)$ but change all field variables. By using Eqs. (11), (13), (17), and (19) and noting that $\overline{-i\nabla} = i\nabla$, we obtain the result

$$
\begin{aligned}
\mathcal{L}_{\mathrm{kin}}[\overline{\phi}] &= \frac{1}{2m}\big|\big(i\nabla + N^a\overline{\mathbf{a}} + N_A\overline{\mathbf{A}} + N_m^a\widetilde{\overline{\mathbf{a}}} + N_M\widetilde{\overline{\mathbf{A}}}\big)\overline{\phi}\big|^2 \\
&= \frac{1}{2m}\big|\big(-i\nabla - N^a\mathbf{a} + N_A\mathbf{A} + N_m^a\widetilde{\mathbf{a}} - N_M\widetilde{\mathbf{A}}\big)\overline{\phi}\big|^2.
\end{aligned}
\tag{21}
$$

The time component is similar:

$$
\mathcal{L}_t[\phi] = \frac{1}{2m}\big|\big(i\,\partial_t + N^a a_0 + N_A A_0 + N_m^a\widetilde{a}_0 + N_M\widetilde{A}_0\big)\phi\big|^2.
\tag{22}
$$

After Z_2^T operation,

$$
\mathcal{L}_t[\overline{\phi}] = \frac{1}{2m}\big|\big(i\,\partial_t - N^a a_0 + N_A A_0 + N_m^a\widetilde{a}_0 - N_M\widetilde{A}_0\big)\overline{\phi}\big|^2.
\tag{23}
$$

C. Mutual statistics and quantum statistics

One of the important properties of dyons is their 3D "mutual statistics." Two dyons with different quantum numbers may perceive a nonzero quantum Berry phase mutually. More specifically, let us fix one dyon ("ϕ_1") at the origin and move another dyon "ϕ_2" (labeled by symbol with primes) along a closed trajectory that forms a solid angle Ω with respect to the origin. Under this circumstance, one can calculate the Berry phase that is added into the single-particle wave function of ϕ_2:

$$
\text{Berry phase} = \frac{1}{2}\left[\sum_s N_m^{(s)}N_f^{(s)\prime} - \sum_s N_m^{(s)\prime}N_f^{(s)}\right]\Omega.
$$

If the Berry phase is nonvanishing for *any* given Ω, i.e., $\sum_s N_m^{(s)}N_f^{(s)\prime} \neq \sum_s N_m^{(s)\prime}N_f^{(s)}$, these two dyons then have a nontrivial "mutual statistics." The physical consequence of mutual statistics is the following. If the confined phase of the internal gauge field is formed by a condensate of dyon ϕ_1, all other allowed deconfined particles (i.e., the particles that may form the excitation spectrum with a finite gap) must have trivial mutual statistics with respect to ϕ_1, i.e.,

$$
\sum_s N_m^{(s)}N_f^{(s)\prime} = \sum_s N_m^{(s)\prime}N_f^{(s)}.
\tag{24}
$$

Otherwise, they are confined by infinite energy gap. There are two useful corollaries: (i) it is obvious that a particle has a trivial mutual statistics with respect to itself, and (ii) we also note that $N_f^{(s)}$ and $N_f^{(s)\prime}$ may be replaced by integers $n_f^{(s)}$ and $n_f^{(s)\prime}$, respectively, by taking Eq. (8) into consideration. As a result, the criterion of *trivial mutual statistics* (24) may be equivalently expressed as

$$
\sum_s N_m^{(s)}n_f^{(s)\prime} = \sum_s N_m^{(s)\prime}n_f^{(s)}.
\tag{25}
$$

On the other hand, it is also crucial to determine the quantum statistics of a generic dyon. A generic dyon can be viewed as $N_m^{(s)}$ magnetic charges of A_μ^{fs} gauge field attached by $n_f^{(s)}$ f_s

fermions. The quantum statistics of such a dyon is given by

$$
\mathrm{Sgn} = \prod_s (-1)^{N_m^{(s)}n_f^{(s)}}(-1)^{n_f^{(s)}},
\tag{26}
$$

where $+/-$ represents bosonic/fermionic [50]. The first part, $(-1)^{N_m^{(s)}n_f^{(s)}}$, is due to the interaction between the magnetic charge $N_m^{(s)}$ and the gauge charge $n_f^{(s)}$ of the dyon. The polarization electric charges due to Witten effect do not attend the formation of internal angular momentum of electric-magnetic composite according to the exact proof by Goldhaber *et al.* [50] so that $n_f^{(s)}$ instead of $N_f^{(s)}$ is put in Eq. (26). One may also express $n_f^{(s)}$ in terms of "$N_f^{(s)} - \frac{\theta_s}{2\pi}N_m^{(s)}$." After this replacement, it should be kept in mind that both "$N_f^{(s)} - \frac{\theta_s}{2\pi}N_m^{(s)}$" and "$N_m^{(s)}$" are integer-valued and $N_f^{(s)}$ can be any real number in order to ensure that $n_f^{(s)}$ are integer valued. The second part, $(-1)^{n_f^{(s)}}$, is due to the Fermi statistics from the attachment of $n_f^{(s)}$ f_s fermions. Alternatively, the quantum statistics formula (26) can be reorganized into the following $\mathrm{Sgn} \equiv (-1)^\Gamma$:

$$
\Gamma \doteq \Gamma_1 + \Gamma_2 + \Gamma_3
\tag{27}
$$

with

$$
\begin{aligned}
\Gamma_1 &\doteq N_M\big[\alpha n_f^{(1)} + (1-\alpha)n_f^{(2)}\big], \\
\Gamma_2 &\doteq N_m^a\big[n_f^{(1)} - n_f^{(2)}\big], \\
\Gamma_3 &\doteq n_f^{(1)} + n_f^{(2)},
\end{aligned}
$$

in which Γ_1 and Γ_2 are contributed from the two gauge groups A_μ and a_μ, respectively. Γ_3 is from the fermionic sign carried by the attached fermions. The notation "$\doteq$" here represents that the two sides of the equality can be different up to any even integer.

IV. TOPOLOGICAL PHASES WITH SYMMETRY: $\alpha = 1/2$

A. Algebraic bosonic insulators: parent states of topological phases

Let us use the projective construction to study an exotic gapless bosonic insulator (without the dyon condensation), which is called algebraic bosonic insulator (ABI) and can be viewed as a *parent state* of gapped symmetric phases (i.e., SPT and SET phases).

For keeping time-reversal symmetry at least at mean-field level, we will only focus on $(\theta_1, \theta_2) = (0,0)$ and $(1,1)$. The ABI state does not break the $\mathrm{U}(1) \rtimes Z_2^T$ symmetry since all possible dyons (each dyon and its time-reversal partner) are included without condensation. However, the bulk is gapless since it contains an emergent gapless $\mathrm{U}(1)$ gauge boson described by a_μ. The emergent $\mathrm{U}(1)$ gauge bosons are neutral. In addition to the emergent $\mathrm{U}(1)$ gauge bosons, ABI also contains many dyon excitations, which may carry fractional electric charges [$N_f^{(s)}$ in Eq. (8)] and emergent Fermi statistics [determined by Eq. (26)]. Since all electrically charged excitations are gapped, such a phase is an electric insulator. At mean-field level, if $\theta_1 = \theta_2 = \pi$, a gapless surface state emerges, which is described by Dirac fermions. Beyond mean-field theory, those gapless surface Dirac fermions (possibly with Fermi energies away

from the nodes) will interact with the emergent U(1) gauge fields that live in 3 +1 dimensions.

We note that the internal U(1) gauge field a_μ has strong quantum fluctuations, and its "fine structure constants" are of order 1. It is possible (relying on the physical boson Hamiltonian) that the internal U(1) gauge field is driven into a confined phase of gauge theory due to too strong quantum fluctuations. Due to the strong quantum fluctuations, the internal U(1) gauge-field configuration will contain many monopoles and even more general dyons. The ABI discussed above is realized as an unstable gapless fixed-point residing at the boundary between Coulomb and confined phases. ABI finally flows into a strongly coupled fixed point of a confined phase by energetically condensing a bosonic dyon and thus opening a bulk gap. In this case, it is possible that some nontrivial topological (gapped) phases with a global symmetry (including SPT and SET states) may be constructed in this confined phase, which is featured by dyon condensations. As we have seen, our ABI has many kinds of dyons. Relying on the details of the physical boson Hamiltonian, different dyon condensations may appear. Different dyon condensations will lead to many different confined phases.

In the following, we will set $\alpha = 1/2$ and focus on looking for dyon condensations that generate a bulk spectral gap, and, *most importantly*, respect $U(1) \rtimes Z_2^T$ symmetry. All topological phases are summarized in Table III. We should note that in each mean-field *Ansatz*, only one dyon whose quantum numbers are self-time-reversal invariant is condensed to form a topological phase. As a matter of fact, two time-reversal conjugated dyons can condense simultaneously, still without breaking time-reversal symmetry. However, this situation is trivially back to the single dyon condensate for the reason that the two dyons are exactly the same once the trivial mutual statistics between them is considered. (The details can be found in Appendix A.)

B. Standard labeling and defining properties of topological phases

Before moving on to topological phases of boson systems, we need to quantitatively define trivial SPT, nontrivial SPT, and SET states based on physically detectable properties in EM thought experiments (compactness of EM field is assumed).

Each dyon is sufficiently determined by four independent quantum numbers in ABI state. The total number of independent quantum numbers will be decreased to three in a specific topological phase where the condensed dyon provides a constraint on the four quantum numbers as we will see later. There are many equivalent choices of labeling. In the following, we choose these four integer-valued quantum numbers $[N_M, N_m^{(1)}, n_f^{(1)}, n_f^{(2)}]$ to express the final key results of a given mean-field *Ansatz*, such as quantum statistics and the total EM electric charge of excitations. We call it "standard labeling." Based on these four integers, we can obtain $N_m^{(2)}$, $N_f^{(1)}$, and $N_f^{(2)}$ via Eqs. (7) and (8). As a result, N_A and N^a can be determined by Eq. (15). In each mean-field *Ansatz*, we will unify all key results by using the standard labeling.

A trivial SPT state has the following properties: (1) quantum statistics: $\Gamma_1 = N_M N_E$; (2) quantization condition: (i) $N_M \in \mathbb{Z}$, $N_E \in \mathbb{Z}$ and (ii) at least one excitation exists for any given integer combination (N_M, N_E); and (3) TO does not exist.
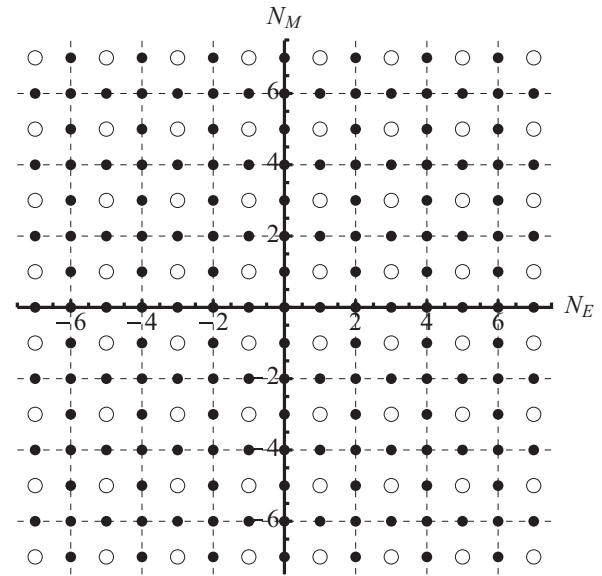

FIG. 2. Illustration of the charge lattice of trivial SPT. Excitations with $-7 \leqslant N_E \leqslant 7$ and $-7 \leqslant N_M \leqslant 7$ are plotted. The solid circles (open circles) denote bosonic (fermionic) statistics. The elementary EM monopole ($N_M = 1, N_E = 0$) is bosonic, while the elementary EM dyon ($N_M = 1, N_E = 1$) is fermionic.

The first two conditions (quantum statistics plus quantization condition) define a "charge lattice" formed by two discrete data points, N_M (y axis) and N_E (x axis). Differing from N_A, which is "bare EM electric charge," N_E is the "total EM electric charge" in which possible dynamical screening arising from the ground state is taken into consideration. In an experiment, N_E is detectable, while N_A is not.

In the trivial SPT state here, the charge lattice corresponds to the "trivial Witten effect" phenomenon. It rules out TO with fractional electric charges for intrinsic excitations and TO with fermionic intrinsic excitations. The trivial Witten effect implies the *elementary EM monopole* ($N_M = 1, N_E = 0$) is bosonic while the *elementary EM dyon* ($N_M = 1, N_E = 1$) is fermionic. This charge lattice is shown in Fig. 2.

A nontrivial SPT state has the following properties: (1) quantum statistics: $\Gamma_2 = N_M(N_E - N_M) \doteq N_M N_E - N_M$; (2) quantization condition: (i) $N_M \in \mathbb{Z}$, $N_E \in \mathbb{Z}$ and (ii) at least one excitation exists for any given integer combination (N_M, N_E); (3) TO does not exist. The first two conditions here correspond to the charge lattice with the "nontrivial Witten effect with $\Theta = 2\pi \mod(4\pi)$" phenomenon. It also rules out TO with fractional electric charges for intrinsic excitations and TO with fermionic intrinsic excitations. The nontrivial Witten effect implies the *elementary EM monopole* ($N_M = 1, N_E = 0$) is fermionic, while the *elementary EM dyon* ($N_M = 1, N_E = 1$) is bosonic. This charge lattice is shown in Fig. 3. This statistical transmutation has been recently discussed in Ref. [21].

If the state *with symmetry* supports a charge lattice that cannot be categorized into both of trivial and nontrivial SPT state, it must be an SET state. A constructed topological phase is a time-reversal symmetric state if the following three conditions are satisfied. (1) *Condition I*: the dyon condensate is Z_2^T-symmetric. The selected condensed dyon is
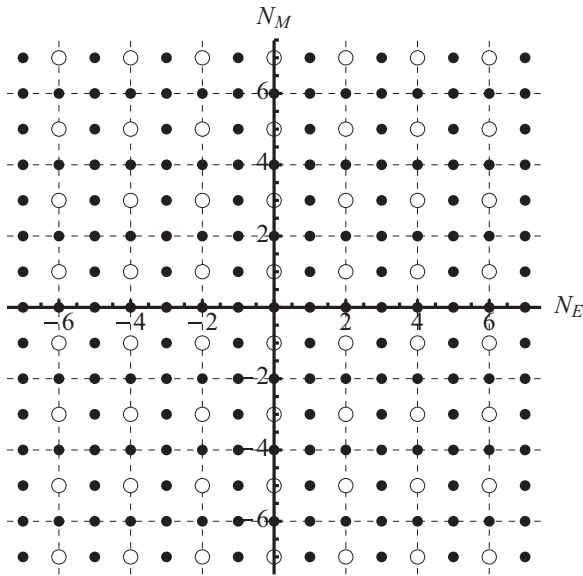
FIG. 3. Illustration of the charge lattice of nontrivial SPT. Excitations with $-7 \leqslant N_E \leqslant 7$ and $-7 \leqslant N_M \leqslant 7$ are plotted. The solid circles (open circles) denote bosonic (fermionic) statistics. The elementary EM monopole ($N_M = 1, N_E = 0$) is fermionic, while the elementary EM dyon ($N_M = 1, N_E = 1$) is bosonic.

self-time-reversal symmetric (time-reversal pair condensates are not possible, see Appendix A). (2) *Condition II*: the charge lattice is mirror-symmetric about x axis. On the charge lattice, the distribution of sites, quantum statistics and excitation energy are mirror-symmetric about x axis. More specifically, (N_E, N_M) and $(N_E, -N_M)$ are simultaneously two sites of the charge lattice. At each site, there are many excitations that are further labeled by the third quantum number [e.g., $N_m^{(1)}$] in addition to the given N_E and N_M. Each excitation $[N_E, N_M, N_m^{(1)}]$ has a counterpart $[N_E, -N_M, N_m^{(1)'}]$ with the same quantum statistics and the same excitation energy, and vice versa. (3) *Condition III*: α-gauge equivalence condition. 2α = integer in the mean-field *Ansatz* $(0,0)$; α = half-odd in the mean-field *Ansatz* (π, π). This condition and condition II determine α altogether. Details of proof and a related discussion on this condition are presented in Secs. II C and V.

We simply say that *the charge lattice is mirror-symmetric* if condition II is satisfied. These three conditions lead to time-reversal invariance of the whole excitation spectrum.

C. Mean-field *Ansatz* $(\theta_1, \theta_2) = (0,0)$

1. Dyon condensation with symmetry

Let us first consider the simplest starting point: the mean-field *Ansatz* with $(\theta_1, \theta_2) = (0,0)$. In other words, both of fermions (f_1, f_2) are trivial TI. In this case,

$$\left[N_f^{(1)}\right]_c = \left[n_f^{(1)}\right]_c \in \mathbb{Z}, \quad \left[N_f^{(2)}\right]_c = \left[n_f^{(2)}\right]_c \in \mathbb{Z} \quad (28)$$

according to Eq. (8). Hereafter, we use the subscript "c" to specify all symbols related to the condensed dyon ϕ_c. Excitations ϕ are labeled by symbols without subscript c. Thus, in the present mean-field *Ansatz* $(\theta_1, \theta_2) = (0,0)$, the quantum numbers of excitations take values in the following

domains:

$$N_f^{(1)} = n_f^{(1)} \in \mathbb{Z}, \quad N_f^{(2)} = n_f^{(2)} \in \mathbb{Z}, \quad (29)$$

$$N^a \equiv N_f^{(1)} - N_f^{(2)} = n_f^{(1)} - n_f^{(2)} \in \mathbb{Z}. \quad (30)$$

Condition I further restricts $[N_f^{(1)}]_c = [N_f^{(2)}]_c$. Therefore a general dyon with time-reversal symmetry is labeled by two integers l and s, i.e.,

$$\left(N_m^a\right)_c = s, \quad \left[N_f^{(1)}\right]_c = l, \quad \left[N_f^{(2)}\right]_c = l, \quad (N_M)_c = 0. \quad (31)$$

According to Eq. (6), where $[N_m^{(1)}]_c \in \mathbb{Z}$ and $(N_M)_c = 0$, $(N_m^a)_c$ is also an integer. Such a time-reversal symmetric dyon is always bosonic since $(\Gamma)_c$ = even integer, according to Eq. (27). Most importantly, by definition, the EM field here is a probe field such that once the EM field is switched off, the physical ground state (formed by the dyon considered here) should not carry EM magnetic charge. Therefore, $(N_M)_c$ vanishes, which is also required by time-reversal symmetry. Other quantum numbers of the condensed dyon are straightforward:

$$(N_A)_c = l, \quad \left[N_m^{(1)}\right]_c = s, \quad \left[N_m^{(2)}\right]_c = -s,$$

$$(N^a)_c = 0, \quad \left[n_f^{(1)}\right]_c = l, \quad \left[n_f^{(2)}\right]_c = l. \quad (32)$$

But will such a dyon condensed state respect the EM electric U(1) symmetry and behave like an insulator? To answer this question, according to Sec. III B, let us write down the effective Lagrangian of the condensed dyon ϕ_c in real time (only spatial components are written here for simplicity—the time components are similar):

$$\mathcal{L}_{\text{kin}}[\phi_c] = \frac{1}{2m}|(-i\nabla + s\widetilde{\mathbf{a}} + l\mathbf{A})\phi_c|^2 - V(\phi_c), \quad (33)$$

where $V(\phi_c)$ is a symmetric potential energy term, which energetically stabilizes the bosonic condensate.

In the dyon condensed state $\phi_c \neq 0$, the internal gauge field a_μ is gapped and satisfies

$$s\widetilde{\mathbf{a}} = -l\mathbf{A}, \quad (34)$$

which indicates that the internal gauge field cannot fluctuate freely and is locked to the nondynamical EM background. We see that the dyon condensation does not generate the $\mathbf{A}^2$ term if

$$s \neq 0. \quad (35)$$

This requirement may be understood in the following way. If $s = l = 0$, there is no dyon condensation, which is nothing but the ABI state discussed in Sec. IV A. If $s = 0$ and $l \neq 0$, Eq. (34) reduces to $\mathbf{A} = 0$, which is the consequence of mass term $\mathbf{A}^2$, a fingerprint of superconductor/superfluid with broken U(1). This case is nothing but condensation of l bosons (carrying EM electric charge l), which breaks U(1) symmetry down to $Z_{|l|}$ discrete symmetry spontaneously [Z_1 represents complete breaking of U(1)]. In the following, in order to preserve U(1) symmetry and consider dyon condensation, we will restrict our attention to $s \neq 0$. Thus the dyon condensed state indeed respects the EM electric U(1) symmetry and represents a fully gapped insulator.

To construct excitations "ϕ" (including intrinsic excitations and test particles), one must trivialize the mutual statistics between the excitation considered and ϕ_c such that the excitation is a deconfined particle, which is observable in the excitation spectrum. According to Eq. (25), the mutual statistics between ϕ_c and an excitation ϕ (labeled by quantum numbers without subscript "c") is trivialized by the following formula:

$$N_m^{(1)}l + [N_M - N_m^{(1)}]l = sn_f^{(1)} - sn_f^{(2)}, \qquad (36)$$

which constrains the quantum numbers of excitations leading to three independent labels instead of four. It may be equivalently expressed as

$$\frac{l}{s}N_M = n_f^{(1)} - n_f^{(2)} = N_f^{(1)} - N_f^{(2)} \equiv N^a, \qquad (37)$$

where the definition (15) is applied, and, the condition (35) is implicit. Therefore all excitations in the present mean-field *Ansatz* can be uniquely labeled by $[N_M, N_m^{(1)}, n_f^{(2)}]$ in the standard labeling, while $n_f^{(1)}$ is determined by Eq. (37).

Meanwhile, due to the screening effect shown in Eq. (34), the total EM electric charge N_E is the sum of N_A [which is equal to $\alpha N_f^{(1)} + (1-\alpha)N_f^{(2)}$ according to Eq. (15)] and an additional screening part:

$$N_E = N_A - \frac{l}{s}N_m^a. \qquad (38)$$

In the standard labeling, N_E is expressed as (details of derivation are present in Appendix B)

$$N_E = n_f^{(2)} - \frac{l}{s}N_m^{(1)} + 2\alpha\frac{l}{s}N_M. \qquad (39)$$

In fact, the condensed dyon has a trivial mutual statistics with itself. Thus the total EM electric charge of the condensed dyon can also be calculated via Eq. (38):

$$(N_E)_c = (N_A)_c - \frac{l}{s}(N_m^a)_c = l - s\frac{l}{s} = l - l = 0, \qquad (40)$$

which indicates that the condensation indeed does not carry total EM electric charge and U(1) symmetry is exactly unbroken.

Under time-reversal symmetry transformation, N_E has the following property:

$$\overline{N_E} \equiv \overline{n_f^{(2)}} - \frac{l}{s}\overline{N_m^{(1)}} + 2\overline{\alpha}\frac{l}{s}\overline{N_M} = N_E, \qquad (41)$$

where Eqs. (10) and (14) are applied and $\overline{\alpha} = 1 - \alpha$ due to the exchange of f_1 and f_2. Since $[N_M, n_f^{(2)}, N_m^{(1)}]$ is an excitation [and thus (N_E, N_M) is a site on the charge lattice], we can prove that $[\overline{N_M}, n_f^{(2)}, \overline{N_m^{(1)}}]$ is also an excitation [and thus $(\overline{N_E}, -N_M)$ is a site on the charge lattice] by justifying that $[\overline{N_M}, \overline{n_f^{(2)}}, \overline{N_m^{(1)}}]$ satisfies the trivial mutual statistics condition (37). Required by the time-reversal invariant mean-field *Ansatz* we considered, the two dyons have the same excitation energy, such that Eq. (41) indicates that both the excitation energy and site distribution are mirror-symmetric at arbitrary α.

We have selected a time-reversal invariant dyon condensate ϕ_c but will the excitation spectrum (i.e., charge lattice) respect time-reversal symmetry? According to Sec. IV B, in order to

preserve time-reversal symmetry, one must also require that the charge lattice is mirror-symmetric about x axis (including site distribution, quantum statistics, and excitation energy). As we have proved that the site distribution and excitation energy are already mirror-symmetric, shown in Eq. (41), the subsequent task is to examine whether the quantum statistics is mirror-symmetric.

Generally, we expect that only a sequence of α is allowed. $\alpha = 1/2$ satisfies condition III. In the remaining discussion of Sec. IV, we will only focus on $\alpha = 1/2$, which is the simplest choice in every mean-field *Ansatz*. We will leave the discussion on the general α sequence to Sec. V.

Now we turn to the discussion of quantum statistics of excitations. According to Eq. (26), the statistics sign in the present mean-field *Ansatz* ($\theta_1 = \theta_2 = 0$, $\alpha = 1/2$) can be obtained (details of derivation are present in Appendix C):

$$\Gamma \doteq N_M N_E + [2N_m^{(1)} - N_M + 1]\frac{l}{s}N_M, \qquad (42)$$

where N_E can be expressed as

$$N_E = n_f^{(2)} + \frac{l}{s}[N_M - N_m^{(1)}] \qquad (43)$$

by plugging $\alpha = 1/2$ into Eq. (39). In Eq. (42), N_E is explicitly written in order to compare Γ with the trivial Witten and nontrivial Witten effects defined in Sec. IV B. One may also replace N_E in Eq. (42) by Eq. (43), rendering an equivalent expression of Eq. (42):

$$\Gamma \doteq N_M\left\{n_f^{(2)} + \frac{l}{s}[N_m^{(1)} + 1]\right\}. \qquad (44)$$

2. Different topological phases via different condensed dyons

a. Bosonic intrinsic excitations. With the above preparation, let us study the nature of the $U(1)\rtimes Z_2^T$-symmetric topological (gapped) phases constructed via the condensed dyon ϕ_c, mainly based on Eqs. (37), (38), (42), and (43). From Eq. (42), all intrinsic excitations (carrying zero N_M) are bosonic, which rules out all fermionic intrinsic excitations in the underlying boson system. We also note that the f_s fermions all have nontrivial "mutual statistics" with the ϕ_c dyon, and thus those fermionic excitations are confined. Up to now, the only requirement on topological phases with symmetry is $s \neq 0$. To understand different topological phases via different condensed dyons, one needs to study the physical properties (quantum statistics, total EM electric charge) of all possible excitations constrained by Eq. (37).

b. $\{l/s \in \mathbb{Z}\}$. Let us first focus on the parameter regime defined by $\frac{l}{s} \in \mathbb{Z}$. In this case, Eq. (37) allows excitations carrying arbitrary integer N_M and arbitrary integer $N_m^{(1)}$. In other words, an arbitrarily given integer N_M can ensure that N^a in the right-hand side of Eq. (37) is integer-valued required by Eq. (30). The other two quantum numbers $N_m^{(1)}, n_f^{(2)}$ are still unconstrained and thus can take arbitrary integer values and N_E can also take arbitrary integer values due to Eq. (43). To conclude, if $l/s \in \mathbb{Z}$, for any given integer combination $[N_M, N_m^{(1)}, n_f^{(2)}]$, there exists at least one excitation. Thus, for any given integer combination (N_E, N_M), there exists at least one excitation on the charge lattice. A useful corollary is that for all intrinsic excitations ($N_M = 0$), N_E is always

integer-valued, which rules out intrinsic excitations carrying fractional EM electric charge (namely, fractional intrinsic excitation).

Then we will look for the general solutions of (l,s), which admit a trivial or a nontrivial Witten effect. To look for the general solutions, which admit Γ_2, we solve the equation $\Gamma - \Gamma_2 \doteq 0$, i.e.,

$$\left[2N_m^{(1)} - N_M + 1\right]\frac{l}{s}N_M - N_M \doteq 0, \qquad (45)$$

where $N_m^{(1)}$ and N_M are arbitrary integers if $l/s \in \mathbb{Z}$. More precisely, if N_M is an arbitrary even integer, we require that $N_M \frac{l}{s}[2N_m^{(1)} - N_M + 1]$ is even integer. In other words, $\frac{l}{s}[2N_m^{(1)} - N_M + 1]$ must be an integer for arbitrary integer $N_m^{(1)}$ and even integer N_M, which renders a requirement for the solution: $\frac{l}{s}$ must be integer-valued. This requirement is not new and is nothing but our starting point. On the other hand, if N_M is an arbitrary odd integer, i.e., $N_M = 2k + 1$, where k is an arbitrary integer, $\frac{l}{s}[2N_m^{(1)} - N_M + 1]$ must be odd. In other words, $\frac{l}{s}[2N_m^{(1)} - 2k] = \frac{2l}{s}[N_m^{(1)} - 1]$ must be odd for an arbitrary integer $N_m^{(1)}$. However, it is obviously even. Therefore there is no solution admitting a nontrivial SPT state.

Likewise, to look for a solution that gives Γ_0, we solve the equation $\Gamma - \Gamma_0 \doteq 0$. It is easily obtained that the requirement $\frac{l}{s}$ is an integer, which is our starting point here. Therefore, if the two integers l and s satisfy that $\frac{l}{s}$ is an integer, such a choice (l,s) is a solution that admits a state with a *trivial Witten effect*. The state is a trivial SPT state by definition if TO does not exist. One can check that the quantum statistics and sites are mirror-symmetric about x axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (41). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II.

In fact, we can also directly derive the trivial Witten effect result by reformulating Γ in Eq. (42) to $\Gamma \doteq N_M N_E + [2N_m^{(1)} - N_M + 1]\frac{l}{s}N_M \doteq N_M N_E$ where the last term in Γ is always even once $l/s \in \mathbb{Z}$.

The trivial Witten effect only rules out TO with fractional intrinsic excitations and TO with fermionic intrinsic excitations. Other TO patterns are still possible. Meanwhile, we note that actually the excitations can also come from a pure gauge sector, in addition to the matter field sector (i.e., dyons) considered above. If $l = 0, |s| \geq 2, \frac{l}{s} = 0$, the internal gauge symmetry U(1) of *dynamical* gauge field a_μ is not fully broken but broken down to $\mathbb{Z}_{|s|}$ gauge symmetry according to Eq. (33), which renders $\mathbb{Z}_{|s|}$ TO in three dimensions in the presence of global symmetry U(1)$\rtimes \mathbb{Z}_2^T$. The low-energy field theory of this TO pattern is the *topological BF theory* of level s. The ground-state degeneracy (GSD) on a three-torus is $|s|^3$. Therefore, to get a trivial SPT state, which does not admit any TO by definition, one should further restrict the two integers (l,s) satisfying $\{\frac{l}{s} \in \mathbb{Z}, l \neq 0, s \neq 0\} \cup \{l = 0, s = \pm 1\}$.

c. $\{l/s \notin \mathbb{Z}\}$. If $l/s \notin \mathbb{Z}$, by noting that N^a must be integer-valued in the present mean-field *Ansatz* $\theta_1 = \theta_2 = 0$, Eq. (37) shows that the allowed EM magnetic charge N_M of excitations cannot take arbitrary integer. A direct example is that all particles with $N_M = 1$ must be permanently confined since Eq. (37) cannot be satisfied. Despite that the other two independent quantum numbers of excitations $N_f^{(1)}, N_m^{(1)}$ can still take arbitrary integer values.

For instance, if $l = 1, s = 3$, the allowed value of N_M should take $N_M = 3k$ with $k \in \mathbb{Z}$, i.e., $N_M = 0, \pm 3, \pm 6, \pm 9, \ldots$ in order to ensure the right-hand side of Eq. (37) is integer-valued. This quantization sequence is different from the sequence $(0, \pm 1, \pm 2, \pm 3, \ldots)$ we are familiar with in the vacuum. By recovering full units (each boson carries a fundamental charge unit e), the EM magnetic charge $\frac{h}{e}3k$ can be reexpressed as $\frac{h}{e^*}k$, where h is the Planck constant and the effective fundamental EM electric charge unit e^* of intrinsic excitations is fractional: $e^* \equiv \frac{e}{3}$. This fractional fundamental EM electric charge unit implies that the U(1)$\rtimes \mathbb{Z}_2^T$-symmetric ground state constructed via condensing the dyon ϕ_c labeled by $(l,s) = (1,3)$ in the mean-field *Ansatz* $\theta_1 = \theta_2 = 0$ admits *fractional intrinsic excitations* (which carry fractional EM electric charge), a typical signature of TO. Interestingly, from Eq. (43), we find that N_E of excitations with $N_M = 0$ indeed can take a fractional value. Therefore the dyon excitations have self-consistently included fractional intrinsic excitations in response to the new quantization sequence of the EM magnetic charge N_M. In this sense, the topological phase labeled by $(l,s) = (1,3)$ contains TO (emergence of fractional intrinsic excitations) with global symmetry, i.e., a SET state.

Generally, we may parametrize $l/s = k' + \frac{p}{q}$, where $k',p,q \in \mathbb{Z}, q > p > 0, \gcd(p,q) = 1$ (gcd: greatest common divisor). The allowed excitations constrained by Eq. (37) enforce N_M quantization as

$$N_M = qk, \qquad (46)$$

where $k \in \mathbb{Z}$, i.e., $N_M = 0, \pm q, \pm 2q, \pm 3q, \ldots$ Thus the effective fundamental EM electric charge unit e^* of intrinsic excitations should be consistently fractional, i.e., $e^* = \frac{1}{q}e$. On the other hand, Eq. (43) shows that N_E of all excitations may be fractional as a multiple of $1/q$:

$$N_E = k_1 - pk_2/q, \qquad (47)$$

where the two integer variables k_1,k_2 are introduced,

$$k_1 \equiv n_f^{(2)} + k'\left[qk - N_m^{(1)}\right] + pk, \quad k_2 \equiv N_m^{(1)}, \qquad (48)$$

to replace $N_f^{(1)}$ and $N_m^{(1)}$. The first term in Eq. (47) is always integer-valued, but the second term may be fractional as a multiple of $1/q$ by noting that $N_m^{(1)}$ is an arbitrary integer and $q > p > 0$. By setting $k = 0$, we find that N_E may be fractional with unit $e^* = 1/q$. In short, the state constructed here must be a SET state with fractional intrinsic excitations.

Next, we will focus on the quantum statistics of all excitations defined in Eq. (42). In Eq. (42), the term including $N_m^{(1)}$ can be removed since $2N_m^{(1)}\frac{l}{s}N_M = 2N_m^{(1)}(qk' + p)k \doteq 0$. Therefore Γ is uniquely determined by N_E, N_M in the following formula up to the quantization conditions (46) and (47):

$$\Gamma = N_M N_E - N_M(N_M - 1)\frac{l}{s}. \qquad (49)$$

In short, by giving three arbitrary integers (k,k_1,k_2), one can determine N_E and N_M via Eqs. (47) and (46), respectively, and further determine the quantum statistics of the excitations labeled by (k,k_1,k_2). One can plot the quantum statistics in the charge lattice formed by discrete data N_M and N_E. We call this charge lattice with TO the "charge lattice I." One can check that
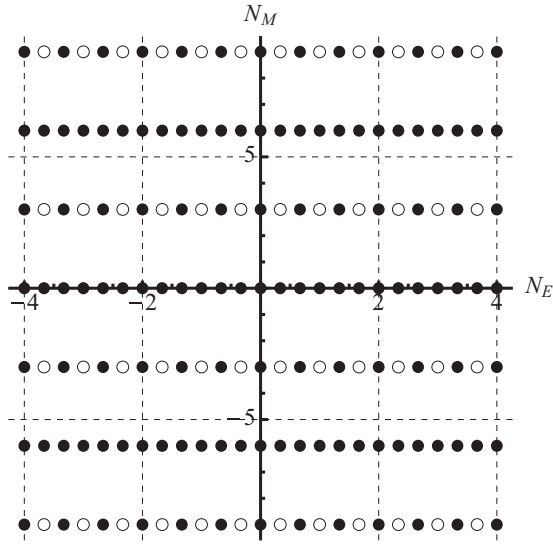
FIG. 4. Illustration of the charge lattice I (any l/s satisfying $l/s = k' + 1/3, k' \in \mathbb{Z}$) with fractional intrinsic excitations (in unit of 1/3). Excitations with $-4 \leqslant N_E \leqslant 4$ and $-9 \leqslant N_M \leqslant 9$ are plotted. The solid circles (open circles) denote bosonic (fermionic) statistics.

the quantum statistics and sites are mirror-symmetric about x axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (41). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II. For example, if $l/s = k' + 1/3$, the charge lattice I is shown in Fig. 4.

In short, this new SET labeled by $l/s = k' + p/q$ with $k', p, q \in \mathbb{Z}, q > p > 0, \gcd(q, p) = 1$ (coined "dyonic TO" in order to distinguish it from TO which arises from the gauge sector, e.g., $Z_{|s|}$ TO) has the following key properties. (1) All excitations are uniquely labeled by three arbitrary integers (k, k_1, k_2), which are related to the standard labeling via Eq. (48). (2) The total EM electric charge N_E of intrinsic excitations is fractional with unit $e^* = 1/q$ given by $N_E = k_1 - \frac{p}{q} k_2$. The allowed EM magnetic charge N_M is quantized at q. (3) The quantum statistics of excitations is uniquely determined by N_M and N_E [cf. Eq. (49)]. (4) All intrinsic excitations are bosonic.

Before closing the analysis of the present mean-field *Ansatz*, a possible confusion should be clarified. The TI state of free fermions admits $\Theta = \pi$ Witten effect and thus $N_E = n + \frac{1}{2} N_M$ (n is integer number of attached fermions) may be fractional. However, this fractional EM electric charge is due to the presence of external EM magnetic monopole. To diagnose the TO of the ground state, we should restrict our attention to "intrinsic excitations," which requires $N_M = 0$ by definition, as we are doing in the present work. Therefore, indeed, the total EM electric charge of any intrinsic excitation in the TI is nonfractional and no TO exist.

In summary, in the mean-field *Ansatz* with $(\theta_1, \theta_2) = (0, 0)$, all symmetric gapped phases (condensed dyons with symmetry) are labeled by two integers (s, l). Physically, a condensed dyon with symmetry labeled here is a composite of l physical bosons (formed by l f_1 fermions and l f_2 fermions) and s unit magnetic monopoles of internal gauge field a_μ. The

state is a trivial SPT state, i.e., a trivial Mott insulator of bosons with $\mathrm{U}(1) \rtimes Z_2^T$ symmetry if the two integers (l, s) satisfy $\{\frac{l}{s} \in \mathbb{Z}, l \neq 0, s \neq 0\} \cup \{l = 0, s = \pm 1\}$. The state is a SET state if the two integers (l, s) either (i) satisfy $\{l = 0, |s| \geq 2\}$, which corresponds to the SET state with $Z_{|s|}$ TO, or (ii) satisfy $\{l/s \notin \mathbb{Z}\}$, which corresponds to a new SET—dyonic TO.

We note that if the fermions f_1 and f_2 in our construction are replaced by two bosons both of which are in trivial Mott insulator states of bosons (i.e., $\theta_1 = \theta_2 = 0$ and $\alpha = 1/2$), there exists a solution for a nontrivial SPT state. For example, $l = s = 1$ is a candidate of nontrivial SPT state, i.e., a nontrivial BTI state, which corresponds to condensation of one physical boson attached to a magnetic monopole of internal gauge field a_μ [22]. The charge lattice is shown in Fig. 3. This can be verified easily with the same technique shown above. The only difference is that in this bosonic projective construction, the terms $\prod_s (-1)^{n_f^{(s)}}$ in the quantum statistics formula (26) should be removed for the reason that attached particles are bosons.

D. Mean-field *Ansatz* $(\theta_1, \theta_2) = (\pi, \pi)$

1. Dyon condensation with symmetry

Following the same strategy, we can also consider the mean-field *Ansatz* with $(\theta_1, \theta_2) = (\pi, \pi)$ where both fermions are in nontrivial TI states. In this case, condition I restricts $[N_f^{(1)}]_c = [N_f^{(2)}]_c$. Therefore a general dyon with time-reversal symmetry is labeled by two integers l and s:

$$
\begin{aligned}
(N_m^a)_c = s, \quad [N_f^{(1)}]_c = l + \frac{s}{2}, \\
[N_f^{(2)}]_c = l + \frac{s}{2}, \quad (N_M)_c = 0.
\end{aligned}
\tag{50}
$$

One can check that the dyon condensate is time-reversal symmetric and $(N_m^a)_c$ is quantized at integer with any given α in the present mean-field *Ansatz*, and such a general dyon is always bosonic since $(\Gamma)_c =$ even integer. Most importantly, by definition, the EM field here is a probe field such that by switching off EM field, the physical ground state should not carry EM magnetic charge. So, $(N_M)_c$ must be vanishing, which is also required by time-reversal symmetry. Other quantum numbers of the condensed dyon are straightforward:

$$
\begin{aligned}
(N_A)_c = l + \frac{s}{2}, \quad [N_m^{(1)}]_c = s, \quad [N_m^{(2)}]_c = -s, \\
(N^a)_c = 0, \quad [n_f^{(1)}]_c = l, \quad [n_f^{(2)}]_c = l + s.
\end{aligned}
\tag{51}
$$

But will such a dyon condensed state respect the EM electric U(1) symmetry and behave like an insulator? To answer this question, according to Sec. III B, let us write down the effective Lagrangian of the condensed dyon ϕ_c in real time (only spatial components are written here for simplicity and time component is similar):

$$
\mathcal{L}_{\text{kin}} = \frac{1}{2m} \left| \left(-i\nabla + s\tilde{\mathbf{a}} + \left(l + \frac{s}{2} \right) \mathbf{A} \right) \phi_c \right|^2 - V(\phi_c), \tag{52}
$$

where $V(\phi_c)$ is a symmetric potential energy term that energetically stabilizes the bosonic condensate.

In the dyon condensed state $\phi_c \neq 0$, the internal gauge field a_μ is gapped and satisfies

$$s\widetilde{\mathbf{a}} = -\left(l + \frac{s}{2}\right)\mathbf{A}, \tag{53}$$

which indicates that the internal gauge field cannot fluctuate freely and is locked to the nondynamical EM background. We also require that

$$s \neq 0 \tag{54}$$

in the following in order that the dyon condensation does not generate the $\mathbf{A}^2$ term. This requirement may be understood in the following way. If $s = l = 0$, there is no dyon condensation, which is nothing but the ABI state discussed in Sec. IV A. If $s = 0$ and $l \neq 0$, Eq. (53) reduces to $\mathbf{A} = 0$, which is the consequence of mass term $\mathbf{A}^2$, a fingerprint of superconductor/superfluid with broken U(1). This case is nothing but a condensation of l bosons (carrying EM electric charge l), which breaks U(1) symmetry down to $Z_{|l|}$ discrete symmetry spontaneously [Z_1 represents complete breaking of U(1)]. In the following, in order to preserve U(1) symmetry and consider dyon condensation, we will restrict our attention to $s \neq 0$. Thus the dyon condensed state indeed respects the EM electric U(1) symmetry and represents a fully gapped insulator.

It should be noted that the surface fermionic gapless excitations described by two f_s Dirac fermions are also confined by the ϕ_c condensation. The confinement behaves like a strong attraction between f_1 and f_2 fermions, which may turn the surface into a superconducting state.

To construct excitations "ϕ" (including intrinsic excitations and test particles), one must trivialize the mutual statistics between the excitation ϕ and condensed dyon ϕ_c such that the excitation is a deconfined particle that is observable in the excitation spectrum. According to Eq. (25), the mutual statistics between an excitation ϕ labeled by quantum numbers without subscript "c" and ϕ_c is trivialized by the following formula:

$$N_m^{(1)}l + \left[N_M - N_m^{(1)}\right](l + s) = sn_f^{(1)} - sn_f^{(2)}, \tag{55}$$

which leads to

$$n_f^{(1)} - n_f^{(2)} = \left(\frac{l}{s} + 1\right)N_M - N_m^{(1)}. \tag{56}$$

Note that by considering Eqs. (8), (15), and (56), we obtain an alternative expression of Eq. (56):

$$N^a = \left(\frac{l}{s} + \frac{1}{2}\right)N_M. \tag{57}$$

Equation (56) is an important and unique constraint on the quantum numbers of excitations ϕ constructed above the condensed dyon ϕ_c. In other words, a dyon is served as a deconfined particle (i.e., an excitation with a finite gap) above the condensed dyon ϕ_c if its four quantum numbers are constrained by Eq. (56). Therefore all excitations in the present mean-field *Ansatz* can be uniquely labeled by $[N_M, N_m^{(1)}, n_f^{(2)}]$ in the standard labeling, while $n_f^{(1)}$ is determined by Eq. (56).

Meanwhile, due to the screening effect shown in Eq. (53), the total EM electric charge N_E is sum of N_A and a screening part:

$$N_E = N_A - \frac{l + \frac{s}{2}}{s}N_m^a. \tag{58}$$

In the standard labeling, N_E is expressed as (details of derivation are present in Appendix D)

$$N_E = -\left(\frac{l}{s} + 1\right)N_m^{(1)} + n_f^{(2)} + \left(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}\right)N_M. \tag{59}$$

In fact, the condensed dyon has a trivial mutual statistics with itself. Thus the total EM electric charge of the condensed dyon can also be calculated via Eq. (58):

$$(N_E)_c = (N_A)_c - \left(N_m^a\right)_c\frac{l + \frac{s}{2}}{s}$$
$$= l + \frac{s}{2} - s\left(l + \frac{s}{2}\right)\bigg/ s = 0, \tag{60}$$

which indicates that the condensation indeed does not carry total EM electric charge such that U(1) symmetry is exactly unbroken.

Under time-reversal symmetry transformation, N_E has the following property:

$$\overline{N_E} = -\left(\frac{l}{s} + 1\right)\overline{N_m^{(1)}} + \overline{n_f^{(2)}} + \left(2\overline{\alpha}\frac{l}{s} + \overline{\alpha} + \frac{1}{2}\right)\overline{N_M} = N_E. \tag{61}$$

Since $[N_M, n_f^{(2)}, N_m^{(1)}]$ is an excitation [and thus (N_E, N_M) is a site on the charge lattice], we can prove that $[\overline{N_M}, \overline{n_f^{(2)}}, \overline{N_m^{(1)}}]$ is also an excitation [and thus $(\overline{N_E}, -N_M)$ is a site on the charge lattice] by justifying that $[\overline{N_M}, \overline{n_f^{(2)}}, \overline{N_m^{(1)}}]$ satisfies the trivial mutual statistics condition (37). Required by the time-reversal invariant mean-field *Ansatz* we considered, the two dyons have the same excitation energy, such that Eq. (41) indicates that both the excitation energy and site distribution are mirror-symmetric at arbitrary α. Likewise, we choose the simplest case: $\alpha = 1/2$ in this section. Plugging $\alpha = 1/2$ into Eq. (59), we obtain

$$N_E = -\left(\frac{l}{s} + 1\right)N_m^{(1)} + n_f^{(2)} + \left(\frac{l}{s} + 1\right)N_M. \tag{62}$$

Now we turn to the discussion of quantum statistics of excitations. According to Eq. (26), the statistics sign in the present mean-field *Ansatz* ($\theta_1 = \theta_2 = \pi$, $\alpha = 1/2$) is (details of the derivation are present in Appendix E)

$$\Gamma \doteq N_M\left[N_m^{(1)} + 1\right]\left(\frac{l}{s} + 1\right) + N_M n_f^{(2)}. \tag{63}$$

2. Different topological phases via different condensed dyons

With the above preparation, let us study the nature of the $U(1) \rtimes Z_2^T$-symmetric topological (gapped) phases constructed above the condensed dyon ϕ_c, mainly based on Eqs. (56), (62), and (63).

a. Bosonic intrinsic excitations. From Eq. (63), all intrinsic excitations with zero N_M are bosonic [excitations with $N_M = 0$ always exist in Eq. (56)], which rules out all fermionic intrinsic excitations in the underlying boson system. We also note that the f_s fermions all have nontrivial mutual statistics with the

ϕ_c dyon, and thus those fermionic excitations are confined. Up to now, the only requirement on topological phases with symmetry is $s \neq 0$. To understand different topological phases via different condensed dyons, one needs to study the physical properties (quantum statistics, total EM electric charge) of all possible excitations constrained by Eq. (56).

b. $\{l/s \in \mathbb{Z}\}$. Let us first focus on the parameter regime defined by $\frac{l}{s} \in \mathbb{Z}$. In this case, Eq. (56) allows excitations carrying arbitrary integers N_M, $N_m^{(1)}$, and $n_f^{(2)}$. $n_f^{(1)}$ is uniquely fixed by Eq. (56), and N_E in Eq. (62) is also fixed and can also take arbitrary integer values.

We may reformulate Γ in Eq. (63) by means of $l/s \in \mathbb{Z}$ (details of derivation are present in Appendix F):

$$\Gamma \doteq N_M N_E. \tag{64}$$

Due to Eq. (56), $n_f^{(2)}$ and $N_m^{(1)}$ can be still arbitrarily integer-valued, and $n_f^{(1)}$ is fixed once $N_m^{(1)}$, N_M, and $n_f^{(2)}$ are given. Thus N_E in Eq. (62) can take any integer. There exists at least one excitation for any given integer combination (N_E, N_M). In short, if the two integers l and s satisfy that $\frac{l}{s}$ is an integer, such a choice (l,s) is a solution that admits a state with a trivial Witten effect. The state is a trivial SPT state. One can check that the quantum statistics and sites are mirror-symmetric about x axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (61). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II.

c. $\{l/s \notin \mathbb{Z}\}$. We note that $l/s = -1/2$ is a special point where the internal gauge symmetry U(1) is broken down to $Z_{|s|}$ gauge symmetry according to Eq. (52). It leads to $Z_{|s|}$ TO in three dimensions in the presence of global symmetry $U(1) \rtimes Z_2^T$. In the following, we will not consider this point.

For a general parameter choice in $l/s \notin \mathbb{Z}$, we will see that there is a dyonic TO, which is defined as TO arising from dyons. Generally, we may parametrize $l/s = k' + \frac{p}{q}$, where $k', p, q \in \mathbb{Z}, q > p > 0$, $\gcd(p,q) = 1$ (gcd: greatest common divisor). Plugging $l/s = k' + \frac{p}{q}$ into Eq. (56), we find that N_M must be quantized at q in all allowed excitations constrained by Eq. (56). That is,

$$N_M = qk, \tag{65}$$

where $k \in \mathbb{Z}$, i.e., $N_M = 0, \pm q, \pm 2q, \pm 3q, \ldots$. On the other hand, Eq. (62) shows that N_E of all excitations may be fractional as a multiple of $1/q$:

$$N_E = k_1 - pk_2/q, \tag{66}$$

where the two integer variables k_1, k_2 are introduced and related to the standard labeling in the following way:

$$k_1 \equiv (k' + 1)qk + pk + n_f^{(2)} - (k' + 1)N_m^{(1)}, \tag{67}$$

$$k_2 \equiv N_m^{(1)}. \tag{68}$$

Due to Eq. (56), $n_f^{(2)}$ and $N_m^{(1)}$ can be still arbitrarily integer-valued, and $n_f^{(1)}$ is fixed once $N_m^{(1)}$, N_M, and $n_f^{(2)}$ are given. Thus the new variables k_1 and k_2 can be any integers. Hereafter, all excitations are labeled by the three independent integers (k, k_1, k_2). Using these new labels, we see that N_E of intrinsic excitations ($k = 0$) can be still fractional according to Eq. (66), which does not depend on k. It indicates that the

state constructed here is a SET state with fractional intrinsic excitations. A useful observation from Eq. (66) is that N_E can also take any integer once $k_2 = q$.

Next, we will focus on the quantum statistics of all excitations defined in Eq. (63). In the present parameter regime, Γ can be expressed as (details of derivation are present in Appendix G)

$$\Gamma \doteq N_M N_E - \frac{l}{s} N_M (N_M - 1). \tag{69}$$

Therefore Γ is uniquely determined by N_E, N_M in the following formula up to the quantization conditions (65) and (66). In short, by giving three arbitrary integers (k, k_1, k_2), one can determine N_E and N_M via Eqs. (66) and (65), respectively, and further determine the quantum statistics of the excitations labeled by (k, k_1, k_2). One can plot the quantum statistics in the charge lattice expanded by discrete variables N_M and N_E, which is same as charge lattice I discussed in Sec. IV C 2 and shown in Fig. 4 ($l/s = 1/3$). One can check that the quantum statistics and sites are mirror-symmetric about x axis. In addition, the distribution of excitation energy is also mirror-symmetric due to Eq. (61). Overall, the charge lattice is indeed mirror-symmetric and thus satisfies condition II.

In summary, in the mean-field *Ansatz* with $(\theta_1, \theta_2) = (\pi, \pi)$, all symmetric gapped phases (condensed dyons with symmetry) are labeled by two integers (s, l). Physically, a condensed dyon with symmetry labeled here is a composite of l physical bosons (formed by l f_1 fermions and l f_2 fermions), s f_2 fermions, and s unit magnetic monopoles of internal gauge field a_μ. If $l/s \in \mathbb{Z}$, the ground state is a trivial SPT state, i.e., trivial Mott insulator of bosons. In the parameter regime $\{l/s \notin \mathbb{Z}, l/s \neq -1/2\}$, the ground state is a SET state with Dyonic TO (fractional intrinsic excitations). If $l/s = -1/2$, the ground state is a SET state with $Z_{|s|}$ TO.

V. TOPOLOGICAL PHASES WITH SYMMETRY: GENERAL α-SEQUENCE

A. Main results

In Sec. IV, we have obtained many topological phases based on dyon condensations (see Table III). The value of α is chosen to be $\alpha = 1/2$ such that both f_1 and f_2 carry $1/2$ EM electric charge (see Table I). Such a choice preserves the time-reversal symmetry. If we choose α to be some other values, the time-reversal symmetry may be broken. However, $\alpha = 1/2$ is not the only value that potentially preserves time-reversal symmetry. In the following, we shall study the general α sequence, which respects time-reversal symmetry. Since the mean-field *Ansatz* $(0, \pi)$ always breaks time-reversal symmetry, we only consider the other two *Ansätze*. The main results are summarized in Table IV. We note that the results in Table III can be obtained by taking $\alpha = 1/2$ in Table IV.

B. Mean-field *Ansatz* $(\theta_1, \theta_2) = (0, 0)$

In this mean-field *Ansatz*, according to the general statement in Sec. II C, two allowed values of α must be differed from each other by any integer, which is required by the charge quantization argument. Let us consider $\bar{\alpha} = 1 - \alpha$ and α, where $\bar{\alpha}$ is the time-reversal transformed α shown in Eq. (12).

The requirement $\bar{\alpha} - \alpha =$ any integer is equivalent to the constraint $2\alpha =$ integer, which is nothing but condition III. This is the first constraint we obtained on the domain value of α.

We have obtained the total EM electric charge N_E in the standard labeling and in the presence of α [cf. Eq. (39)]. To approach a mirror-symmetric charge lattice, we require that the site distribution, excitation energy, and quantum statistics are mirror-symmetric (cf. condition II).

1. $l/s \in \mathbb{Z}$

Let us first consider $l/s \in \mathbb{Z}$ such that N_M is arbitrarily integer-valued due to the constraint (37). We assume that the mirror site of $[N_M, n_f^{(2)}, N_m^{(1)}]$ is labeled by $[-N_M, n_f^{(2)'}, N_m^{(1)'}]$. In order that the mirror site does exist on the charge lattice, the integer solutions $[n_f^{(2)'}, N_m^{(1)'}]$ of the following equation must exist for any given integer N_M [see Eq. (39)]:

$$N_E = n_f^{(2)'} - \frac{l}{s} N_m^{(1)'} - 2\alpha \frac{l}{s} N_M, \tag{70}$$

which is equivalent to

$$\left[n_f^{(2)'} - n_f^{(2)}\right] - \frac{l}{s}\left[N_m^{(1)'} - N_m^{(1)}\right] = 4\alpha\frac{l}{s}N_M \tag{71}$$

by means of Eq. (39). Therefore the mirror-symmetric site distribution requires that $4\alpha\frac{l}{s} =$ integer. To construct symmetric topological phases, we need to further check the quantum statistics in the presence of α (details of derivation are present in Appendix H):

$$\Gamma = N_M\left(N_E - 2\alpha\frac{l}{s}N_M + \frac{l}{s}\right). \tag{72}$$

Therefore the quantum statistics is mirror-symmetric if

$$N_M\left(N_E - 2\alpha\frac{l}{s}N_M + \frac{l}{s}\right)$$
$$\doteq (-N_M)\left[N_E - 2\alpha\frac{l}{s}(-N_M) + \frac{l}{s}\right], \tag{73}$$
$$\text{i.e., } 2N_M N_E \doteq 0.$$

In other words, $2N_M N_E$ must be always even.

If $4\alpha\frac{l}{s} =$ odd, Eq. (39) indicates that N_E is half-odd integer if we take $N_M = 1$. As a result, Eq. (73) is not satisfied. Therefore, in order to guarantee mirror-symmetric distribution of quantum statistics, we need to consider a stronger condition: $4\alpha\frac{l}{s} =$ even, i.e., $2\alpha\frac{l}{s} =$ integer. In Eq. (41), we have already proved that energy is mirror-symmetric for any α, so that we conclude that to obtain a mirror-symmetric charge lattice (i.e., condition II), we need $2\alpha\frac{l}{s} =$ integer. Under this condition as well as $2\alpha =$ integer, we may obtain trivial SPT states and nontrivial SPT states summarized in Table IV by comparing Γ with the standard trivial Witten effect and nontrivial Witten effect defined in Sec. IV B. As usual, one should pay attention to the emergence of $Z_{|s|}$ TO if $l = 0$ and $|s| \geq 2$ although charge lattice formed by deconfined dyons is same as a trivial SPT state. Strikingly, we obtain BTI states, which are completely absent in Table III where $\alpha = 1/2$ is fixed.

Since $N_M \in \mathbb{Z}$, $\frac{l}{s} \in \mathbb{Z}$, and $\frac{l}{s}N_M \doteq \frac{l}{s}(N_M)^2$, one may rewrite Eq. (72) as $\Gamma \doteq N_M(N_E - 2\alpha\frac{l}{s}N_M + \frac{l}{s}N_M) \equiv$

$N_M(N_E - \frac{\Theta}{2\pi}N_M)$. The minimal periodicity of Θ is 4π because Γ is invariant after 4π shift. As a result, a Θ angle can be formally defined as

$$\Theta \equiv -2\pi\frac{l}{s} + 4\pi\frac{l}{s}\alpha \mod(4\pi), \tag{74}$$

from which we see that the Θ angle is linearly related to α.

2. $l/s \notin \mathbb{Z}$

Generally, we may parametrize $l/s = k' + \frac{p}{q}$, where $k', p, q \in \mathbb{Z}, q > p > 0, \gcd(p,q) = 1$ (gcd: greatest common divisor). In this case, N_M is quantized at qk as shown in Eq. (46). To guarantee mirror-symmetric site distribution, the integer solutions $(n_f^{(2)'}, N_m^{(1)'})$ of Eq. (71) must exist for any given $N_M = qk$:

$$\left[n_f^{(2)'} - n_f^{(2)}\right] - \left(k' + \frac{p}{q}\right)\left[N_m^{(1)'} - N_m^{(1)}\right]$$
$$= 4\alpha(qk' + p)k \tag{75}$$

by means of Eq. (39). Equation (72) is also valid when $l/s \notin \mathbb{Z}$ by noting that $-2N_M\frac{l}{s}N_m^{(1)}$ is still even integer in deriving the fourth line of Appendix H. Therefore Eq. (73) is also valid when $l/s \notin \mathbb{Z}$.

A general discussion on Eqs. (75) and (73) is intricate. Let us take a simple example: $l/s = 1/3$, i.e., $k' = 0, q = 3, p = 1$. The right-hand side of Eq. (75) becomes $4\alpha k$. To obtain the integer solutions $[n_f^{(2)'}, N_m^{(1)'}]$ for any given integers $[k, N_m^{(1)}, n_f^{(2)}]$, a constraint on α is necessary: $\alpha =$ integer/12. Under this condition, Eq. (73) leads to a stronger condition: $6\alpha = k_0$ where k_0 is an integer. It guarantees mirror-symmetric distribution of both sites and quantum statistics. As we have proved, energy is already mirror-symmetric due to Eq. (41). Overall, to obtain a mirror-symmetric charge lattice (i.e., condition II), we need $\alpha =$ integer/6. Keeping in mind that $2\alpha =$ integer is required by condition III, the two conditions altogether still give $2\alpha =$ integer.

Since $N_M/3 \in \mathbb{Z}$, $\frac{l}{s} = 1/3$, and $\frac{l}{s}N_M = N_M/3 \doteq (N_M)^2/9$, one may rewrite Eq. (72) as $\Gamma \doteq N_M(N_E - \frac{2\alpha}{3}N_M + \frac{1}{9}N_M) \equiv N_M(N_E - \frac{\Theta}{2\pi}N_M)$. The minimal periodicity of Θ is $\frac{4\pi}{9}$ because Γ is invariant after $\frac{4\pi}{9}$ shift. As a result, a Θ angle can be formally defined:

$$\Theta \equiv -\frac{2\pi}{9} + \frac{4\pi}{3}\alpha \mod\left(\frac{4\pi}{9}\right), \tag{76}$$

from which we see that the Θ angle is linearly related to α. All SET states have fractional intrinsic excitations. We can further classify these states into two categories: one is $\Theta = 0 \mod(\frac{4\pi}{9})$ with $\alpha =$ half-odd, and another is $\Theta = \frac{2\pi}{9} \mod(\frac{4\pi}{9})$ with $\alpha =$ integer. In comparison to the trivial and nontrivial SPT states, we call the former "trivial fBTI" and the latter "nontrivial fBTI" via investigating the Witten effect.

C. Mean-field *Ansatz* $(\theta_1, \theta_2) = (\pi, \pi)$

In this mean-field *Ansatz*, we have obtained the total EM electric charge N_E in the standard labeling and in the presence of α [cf. Eq. (59)]. To approach a mirror-symmetric charge

lattice, we require that the site distribution and quantum statistics are mirror-symmetric.

1. $l/s \in \mathbb{Z}$

Let us first consider $l/s \in \mathbb{Z}$ such that N_M is arbitrarily integer-valued due to the constraint (56). We assume that the mirror site of $[N_M, n_f^{(2)}, N_m^{(1)}]$ is labeled by $[-N_M, n_f^{(2)'}, N_m^{(1)'}]$. In order that the mirror site does exist in the charge lattice, the integer solutions $[n_f^{(2)'}, N_m^{(1)'}]$ of the following equation must exist for any given integer N_M:

$$N_E = -\left(\frac{l}{s} + 1\right)N_m^{(1)'} + n_f^{(2)'} - \left(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}\right)N_M, \tag{77}$$

which is equivalent to

$$\left[n_f^{(2)'} - n_f^{(2)}\right] - \left(\frac{l}{s} + 1\right)\left[N_m^{(1)'} - N_m^{(1)}\right]$$
$$= 2\left(2\alpha\frac{l}{s} + \frac{1}{2}\right)N_M \tag{78}$$

by means of Eq. (59). Therefore the mirror-symmetric site distribution requires that $2 \times (2\alpha\frac{l}{s} + \alpha + \frac{1}{2}) = $ integer. To construct symmetric topological phases, we need to further check the quantum statistics in the presence of α (details of derivation are present in Appendix I):

$$\Gamma = N_M\left[N_E - \left(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}\right)N_M - \left(\frac{l}{s} + 1\right)\right]. \tag{79}$$

Therefore the quantum statistics is mirror-symmetric if

$$N_M\left[N_E - \left(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}\right)N_M - \left(\frac{l}{s} + 1\right)\right]$$
$$\doteq -N_M\left[N_E + \left(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}\right)N_M - \left(\frac{l}{s} + 1\right)\right],$$
$$\text{i.e., } 2N_M N_E \doteq 0. \tag{80}$$

In other words, $2N_M N_E$ must be always even.

If $2 \times (2\alpha\frac{l}{s} + \alpha + \frac{1}{2}) = $ odd, Eq. (59) indicates that N_E is half-odd integer if we take $N_M = 1$. As a result, Eq. (80) is not satisfied. Therefore, in order to guarantee mirror-symmetric distribution of quantum statistics, we need to consider a stronger condition: $2 \times (2\alpha\frac{l}{s} + \alpha + \frac{1}{2}) = $ even, i.e., $(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}) = $ integer. In Eq. (61), we have already proved that energy is mirror-symmetric for any α, so that we conclude that the charge lattice is mirror-symmetric if the stronger condition $2\alpha\frac{l}{s} + \alpha + \frac{1}{2} = $ integer is satisfied.

Since $N_M \in \mathbb{Z}$, $\frac{l}{s} \in \mathbb{Z}$ and $(\frac{l}{s} + 1)N_M \doteq (\frac{l}{s} + 1)(N_M)^2$, one may rewrite Eq. (79) as $\Gamma \doteq N_M[N_E - (2\alpha\frac{l}{s} + \alpha + \frac{1}{2})N_M - (\frac{l}{s} + 1)N_M] \equiv N_M(N_E - \frac{\Theta}{2\pi}N_M)$. The minimal periodicity of Θ is 4π because Γ is invariant after 4π shift. As a result, a Θ angle can be formally defined:

$$\Theta \equiv 2\pi\left(\frac{l}{s} + \frac{3}{2}\right) + 4\pi\left(\frac{l}{s} + \frac{1}{2}\right)\alpha \mod(4\pi), \tag{81}$$

from which we see that the Θ angle is linearly related to α.

From this Θ formula, we realize that shifting α by an odd integer will change trivial (nontrivial) SPT to nontrivial

(trivial) SPT. Therefore we arrive at the statement in Sec. II C. Thus two allowed values of α must be differed from each other by an even integer. Let us consider $\bar{\alpha} = 1 - \alpha$ and α, where $\bar{\alpha}$ is the time-reversal transformed α shown in Eq. (12). The requirement $\bar{\alpha} - \alpha = $ any even integer is equivalent to the constraint $\alpha = $ half-odd, which is nothing but condition III. Under this condition as well as the conditions obtained from mirror symmetric charge lattice, we may obtain trivial SPT states and nontrivial SPT states summarized in Table IV by comparing Γ with the standard trivial Witten effect and nontrivial Witten effect defined in Sec. IV B. Strikingly, we obtain BTI states, which are completely absent in Table III where $\alpha = 1/2$ is fixed.

2. $l/s \notin \mathbb{Z}$

Generally, we may parametrize $l/s = k' + \frac{p}{q}$, where $k', p, q \in \mathbb{Z}$, $q > p > 0$, $\gcd(p, q) = 1$ (gcd: greatest common divisor). In this case, N_M is quantized at qk as shown in Eq. (65). To guarantee mirror-symmetric site distribution, the integer solutions $[n_f^{(2)'}, N_m^{(1)'}]$ of Eq. (78) must exist for any given $N_M = qk$:

$$\left[n_f^{(2)'} - n_f^{(2)}\right] - \left(k' + 1 + \frac{p}{q}\right)\left[N_m^{(1)'} - N_m^{(1)}\right]$$
$$= [4\alpha(qk' + p) + 2\alpha q + q]k \tag{82}$$

by means of Eq. (59). Equation (79) is also valid when $l/s \notin \mathbb{Z}$ by noting that $-2N_M(\frac{l}{s} + 1)[N_m^{(1)} + 1]$ is still even integer in deriving the fourth line of Appendix I. Therefore Eq. (80) is also valid when $l/s \notin \mathbb{Z}$.

A general discussion on Eqs. (82) and (80) is intricate. Let us take a simple example: $l/s = 1/3$, i.e., $k' = 0, q = 3, p = 1$. The right-hand side of Eq. (82) becomes $(10\alpha + 3)k$. To obtain the integer solutions $[n_f^{(2)'}, N_m^{(1)'}]$ for any given integers $[k, N_m^{(1)}, n_f^{(2)}]$, a constraint on α is necessary: $10\alpha + 3 = $ integer/3, i.e., $\alpha = \frac{k_0 - 9}{30}$ where k_0 is an integer. Under this condition, Eq. (80) leads to a stronger condition: $\alpha = \frac{2k_0 - 9}{30}$, which guarantees mirror-symmetric distributions of both sites and quantum statistics. As we have proved, excitation energy is already mirror-symmetric due to Eq. (61). By further considering condition III, α is finally restricted to $\alpha = $ half-odd.

Since $N_M/3 \in \mathbb{Z}$, $\frac{l}{s} = 1/3$, and $(\frac{l}{s} + 1)N_M = 4N_M/3 \doteq 0$, one may rewrite Eq. (79) as $\Gamma \doteq N_M[N_E - (\frac{5\alpha}{3} + \frac{1}{2})N_M] \equiv N_M(N_E - \frac{\Theta}{2\pi}N_M)$. The minimal periodicity of Θ is $\frac{4\pi}{9}$ because Γ is invariant after $\frac{4\pi}{9}$ shift. As a result, a Θ angle can be formally defined:

$$\Theta \equiv \pi + \frac{10\pi}{3}\alpha \mod\left(\frac{4\pi}{9}\right), \tag{83}$$

from which we see that the Θ angle is linearly related to α. All SET states have fractional intrinsic excitations. We can further classify these states into two categories: one is $\Theta = 0 \mod(\frac{4\pi}{9})$ with $\alpha - \frac{1}{2} = $ even and another is $\Theta = \frac{2\pi}{9} \mod(\frac{4\pi}{9})$ with $\alpha - \frac{1}{2} = $ odd. In comparison to the trivial and nontrivial SPT states, we call the former "trivial fBTI" and the latter "nontrivial fBTI" via investigating the Witten effect.

As usual, one should pay attention to the emergence of $Z_{|s|}$ TO if $l/s = -1/2$. One may also examine whether there is a Witten effect if $l/s = -1/2$ in addition to $Z_{|s|}$ TO. Following the same procedure, we obtain that

$$\Theta \equiv \frac{\pi}{2} \mod(\pi) \tag{84}$$

and α is restricted to $\alpha =$ half-odd. At this special point, $l/s = -1/2$, we find that the Witten effect is independent on α and the state is a nontrivial f BTI in the presence of $Z_{|s|}$ TO.

VI. CONCLUSION

In conclusion, we used fermionic projective construction and dyon condensation to construct many three-dimensional SPT and SET states with time-reversal symmetry and U(1) boson number conservation symmetry. Without dyon condensation, we obtained an algebraic bosonic insulator, which contains an emergent U(1) gapless photon excitation. Then we assumed the internal U(1) gauge field to fluctuate strongly and form one of many confined phases characterized by different dyon condensations. After a dyon condensate that preserves the $U(1) \rtimes Z_2^T$ symmetry is selected properly, the excitation spectrum (formed by deconfined dyons) above this dyon condensate is entirely determined. The symmetric dyon condensate determines the quantization conditions of EM magnetic charge and EM electric charge of excitations. It also determines the quantum statistics (boson/fermion) and excitation energy. By calculating these properties, we then obtained SPT and SET states summarized in Tables II–IV. The basic process of this construction approach is shown in Fig. 1. In short, we presented an "efficient program" for constructing SPT states in 3D symmetric topological phases; we also found that confined phases of the internal gauge field with global symmetry can be further classified into many different phases. There are some interesting and direct directions for future work.

(1) *Classification via projective construction and dyon condensation.* The definition of nontrivial SPT states, i.e., bosonic topological insulators (BTI), is only related to the nontrivial Witten effect (i.e., $\Theta = 2\pi$) as shown in Sec. IV B. As shown in Ref. [19], classification of an SPT state with a certain symmetry corresponds to looking for a complete set of "topological invariants." In this paper, we only consider one Z_2 topological invariant, which distinguishes the physical properties of trivial/nontrivial Witten effect, meaning that it is potentially possible some trivial SPT states we found in this paper actually are nontrivial and are characterized by new features instead of the Witten effect, e.g., 2D surface properties. In other words, it is necessary to construct more topological invariants to *completely* distinguish all SPT states.

There are some clues. Firstly, in this paper, we have systematically shown how to construct a charge lattice that respects symmetry by means of fermionic projective construction and dyon condensation. We expect that, in addition to Witten effect, more information (i.e., more topological invariants) can be extracted from more complete analysis of charge lattice. Secondly, we may consider SPT states with merely time-reversal symmetry. In other words, these states are protected sufficiently by time-reversal symmetry while the

boson number conservation symmetry U(1) does not play any role. Literally, these states are also SPT states with $U(1) \rtimes Z_2^T$ although U(1) here is not necessary. Therefore one may consider new mean-field *Ansätze* for fermions and try to find new SPT states.

(2) *Surface theory and bulk topological field theory via projective construction and dyon condensation.* SPT states have a quite trivial bulk but the surface may admit many nontrivial physical properties that are absent in trivial Mott insulator states. It has been recently shown that classifying surface topological order may provide the answer to classifying the BTI bulk [15,51]. Indeed, the surface detectable features may be tightly connected to the complete set of topological invariants that we shall look for. For instance, a nontrivial Witten effect is indeed related to the surface quantum Hall effect (by breaking time-reversal symmetry on the surface) with anomalous quantization of Hall conductance that cannot be realized in 2D U(1) SPT [49,51]. In short, it is interesting for future work on a surface theoretical description via the present fermionic projective construction and dyon construction. Beside the SPT states constructed in this paper, we also constructed many SET states in which fractional intrinsic excitations exist (defined as excitations with zero EM magnetic charge). A full understanding of these topologically ordered states with symmetry is interesting from the perspective of $(3+1)D$ topological quantum field theory (TQFT) descriptions and fixed-point lattice Hamiltonian realizations. In addition, realistic model Hamiltonians that can realize all the topological phases constructed in this paper are quite interesting; we leave the task of constructing such Hamiltonians to future work.

APPENDIX A: CONDENSATION OF TWO TIME-REVERSAL CONJUGATED DYONS LEADS TO A SINGLE DYON CONDENSATE

The most general *Ansatz* for the quantum numbers of two time-reversal conjugated dyons is shown in Table. V. We see that there are four numbers (l_1, l_2, s, t)—enough to label two dyons, one of which is the time-reversal partner of the other.

If both ϕ_1 and ϕ_2 are condensed, the mutual statistics between them must be trivialized, i.e.,

$$l_1[s - (1-\alpha)t] + l_2(-s - \alpha t)$$
$$= l_2(s + \alpha t) + l_1[-s + (1-\alpha)t], \tag{A1}$$

which leads to

$$l_1[s - (1-\alpha)t] + l_2(-s - \alpha t) = 0. \tag{A2}$$

We note that the U(1) symmetry of the original boson system is generated by conserved EM electric charge rather than EM

TABLE V. Quantum numbers of two time-reversal conjugated dyons (ϕ_1 and ϕ_2).

Dyon	N_m^a	$N_f^{(1)}$	$N_f^{(2)}$	$N_m^{(1)}$	$N_m^{(2)}$	N_M	N_A
ϕ_1	s	l_1	l_2	$s + \alpha t$	$-s + (1-\alpha)t$	t	$\alpha l_1 + (1-\alpha)l_2$
ϕ_2	$s + (2\alpha - 1)t$	l_2	l_1	$s - (1-\alpha)t$	$-s - \alpha t$	$-t$	$\alpha l_2 + (1-\alpha)l_1$

magnetic charge. The EM field here is nondynamical so that the dyon condensates that form the physical ground state should have zero EM magnetic charge. Therefore we have $t = 0$, and the above trivial mutual statistics condition becomes

$$(l_1 - l_2)s = 0. \tag{A3}$$

To satisfy this condition, $l_1 = l_2$ or $s = 0$. However, s must be nonzero in order to preserve U(1) symmetry. The reason is that once $s = 0$, the condensation will break U(1) since it is EM electric charged. The reasonable choice is $l_1 = l_2$. This choice leads to the fact that $\phi_1 = \phi_2$, i.e., a condensate of one kind of dyon.

APPENDIX B: DERIVATION OF EQ. (39)

$$
\begin{aligned}
N_E &= \left[\alpha N_f^{(1)} + (1-\alpha)N_f^{(2)}\right] - \frac{l}{s}N_m^a \\
&= \alpha n_f^{(1)} + (1-\alpha)n_f^{(2)} - \frac{l}{s}\left[N_m^{(1)} - \alpha N_M\right] \\
&= \alpha\left[\frac{l}{s}N_M + n_f^{(2)}\right] + (1-\alpha)n_f^{(2)} + \frac{l}{s}\alpha N_M - \frac{l}{s}N_m^{(1)} \\
&= n_f^{(2)} + 2\alpha\frac{l}{s}N_M - \frac{l}{s}N_m^{(1)}. \tag{B1}
\end{aligned}
$$

In deriving the first line, Eqs. (15) and (38) are applied. In deriving the second line, Eq. (6) is applied. In deriving the third line, Eq. (37) is applied.

APPENDIX C: DERIVATION OF EQ. (42)

$$
\begin{aligned}
\Gamma &\doteq N_M\left[\frac{1}{2}N_f^{(1)} + \frac{1}{2}N_f^{(2)}\right] + N_m^a\left[N_f^{(1)} - N_f^{(2)}\right] + N_f^{(1)} + N_f^{(2)} \\
&\doteq N_M\left[\frac{1}{2}N_f^{(1)} + \frac{1}{2}N_f^{(2)}\right] + N_m^a\left[N_f^{(1)} - N_f^{(2)}\right] + N_f^{(1)} - N_f^{(2)} \\
&= N_M\left[\frac{1}{2}N_f^{(1)} + \frac{1}{2}N_f^{(2)}\right] + \left(N_m^a + 1\right)\left[N_f^{(1)} - N_f^{(2)}\right] \\
&= N_M N_A + \left(N_m^a + 1\right)N^a \\
&= N_M\left(N_E + \frac{l}{s}N_m^a\right) + \left(N_m^a + 1\right)\frac{l}{s}N_M \\
&= N_M N_E + \left(2N_m^a + 1\right)\frac{l}{s}N_M \\
&= N_M N_E + \left[2N_m^{(1)} - N_M + 1\right]\frac{l}{s}N_M. \tag{C1}
\end{aligned}
$$

In deriving the second line, an even integer "$-2N_f^{(2)}$" is added. In deriving the fifth line, Eqs. (37) and (38) are applied. In deriving the last line, Eq. (6) is applied.

APPENDIX D: DERIVATION OF EQ. (59)

$$
\begin{aligned}
N_E &= N_A - \frac{l + \frac{s}{2}}{s}N_m^a \\
&= \alpha\left[n_f^{(1)} + \frac{1}{2}N_m^{(1)}\right] + (1-\alpha)\left\{n_f^{(2)} + \frac{1}{2}\left[N_M - N_m^{(1)}\right]\right\} \\
&\quad - \left(\frac{l}{s} + \frac{1}{2}\right)\left[N_m^{(1)} - \alpha N_M\right] \\
&= \alpha\left[n_f^{(2)} + \left(\frac{l}{s} + 1\right)N_M - N_m^{(1)} + \frac{1}{2}N_m^{(1)}\right] \\
&\quad + (1-\alpha)\left\{n_f^{(2)} + \frac{1}{2}\left[N_M - N_m^{(1)}\right]\right\} \\
&\quad - \left(\frac{l}{s} + \frac{1}{2}\right)\left[N_m^{(1)} - \alpha N_M\right] \\
&= -\left(\frac{l}{s} + 1\right)N_m^{(1)} + n_f^{(2)} + \left(2\alpha\frac{l}{s} + \alpha + \frac{1}{2}\right)N_M. \tag{D1}
\end{aligned}
$$

In deriving the third line, Eq. (56) is applied.

APPENDIX E: DERIVATION OF EQ. (63)

$$
\begin{aligned}
\Gamma &\doteq \frac{1}{2}N_M\left[n_f^{(1)} + n_f^{(2)}\right] + N_m^a\left[n_f^{(1)} - n_f^{(2)}\right] + n_f^{(1)} + n_f^{(2)} \\
&\doteq \frac{1}{2}N_M\left[n_f^{(1)} + n_f^{(2)}\right] + N_m^a\left[n_f^{(1)} - n_f^{(2)}\right] + n_f^{(1)} - n_f^{(2)} \\
&= \left(\frac{1}{2}N_M + N_m^a + 1\right)\left[n_f^{(1)} - n_f^{(2)}\right] + N_M n_f^{(2)} \\
&= \left[N_m^{(1)} + 1\right]\left[n_f^{(1)} - n_f^{(2)}\right] + N_M n_f^{(2)} \\
&= \left[N_m^{(1)} + 1\right]\left[\left(\frac{l}{s} + 1\right)N_M - N_m^{(1)}\right] + N_M n_f^{(2)} \\
&\doteq \left[N_m^{(1)} + 1\right]\left(\frac{l}{s} + 1\right)N_M + N_M n_f^{(2)}. \tag{E1}
\end{aligned}
$$

In deriving the first line, Eq. (27) is applied. In deriving the second line, an even integer $-2n_f^{(2)}$ is added. In deriving the fourth line, the first formula in Eq. (6) is applied with $\alpha = 1/2$. In deriving the fifth line, Eq. (56) is applied. In deriving the last line, the even integer $-N_m^{(1)}[N_m^{(1)} + 1]$ is removed.

APPENDIX F: DERIVATION OF EQ. (64)

$$
\begin{aligned}
\Gamma &\doteq N_M\left[N_m^{(1)} + 1\right]\left(\frac{l}{s} + 1\right) + N_M n_f^{(2)} \\
&\doteq -N_M\left[N_m^{(1)} + 1\right]\left(\frac{l}{s} + 1\right) + N_M n_f^{(2)}
\end{aligned}
$$

$$= N_M\left[-\left(\frac{l}{s}+1\right)N_m^{(1)}+n_f^{(2)}-\left(\frac{l}{s}+1\right)\right]$$

$$= N_M\left[N_E-\left(\frac{l}{s}+1\right)N_M-\left(\frac{l}{s}+1\right)\right]$$

$$= N_M\left[N_E-\left(\frac{l}{s}+1\right)(N_M+1)\right]$$

$$\doteq N_M N_E, \tag{F1}$$

where an even integer $-2N_M(N_m^{(1)}+1)(\frac{l}{s}+1)$ is added in the second line. In the fourth line, Eq. (62) is applied. In the last line, $N_M(N_M+1)(\frac{l}{s}+1)$ is removed since it is always even.

APPENDIX G: DERIVAION OF EQ. (69)

$$\Gamma \doteq N_M\left[N_m^{(1)}+1\right]\left(\frac{l}{s}+1\right)+N_M n_f^{(2)}$$

$$\doteq -N_M\left[N_m^{(1)}+1\right]\left(\frac{l}{s}+1\right)+N_M n_f^{(2)}$$

$$= N_M\left[-\left(\frac{l}{s}+1\right)N_m^{(1)}+n_f^{(2)}-\left(\frac{l}{s}+1\right)\right]$$

$$= N_M\left[N_E-\left(\frac{l}{s}+1\right)N_M-\left(\frac{l}{s}+1\right)\right]$$

$$= N_M N_E-\left(\frac{l}{s}+1\right)N_M(N_M+1)$$

$$\doteq N_M N_E-\frac{l}{s}N_M(N_M+1)$$

$$\doteq N_M N_E-\frac{l}{s}N_M(N_M-1), \tag{G1}$$

where an even integer $-2N_M(N_m^{(1)}+1)(\frac{l}{s}+1)=-2qk(N_m^{(1)}+1)(\frac{p}{q}+1)=-2k(N_m^{(1)}+1)(p+q)$ is added in the second line. In deriving the fourth line, Eq. (62) is applied. In deriving the sixth line, the even integer $-N_M(N_M+1)=-qk(qk+1)$ is removed. In deriving the last line, an even integer $2\frac{l}{s}N_M=2(qk'+p)k$ is added.

APPENDIX H: DERIVATION OF EQ. (72)

$$\Gamma = \left[N_m^{(1)}+1\right]n_f^{(1)}+\left[N_m^{(2)}+1\right]n_f^{(2)}$$

$$= \left[N_m^{(1)}+1\right]\left[\frac{l}{s}N_M+n_f^{(2)}\right]+\left[N_M-N_m^{(1)}+1\right]n_f^{(2)}$$

$$\doteq N_M\left[\frac{l}{s}N_m^{(1)}+\frac{l}{s}+n_f^{(2)}\right]$$

$$\doteq N_M\left[-\frac{l}{s}N_m^{(1)}+\frac{l}{s}+n_f^{(2)}\right]$$

$$= N_M\left(N_E-2\alpha\frac{l}{s}N_M+\frac{l}{s}\right). \tag{H1}$$

In deriving the second line, Eqs. (37) and (7) are applied. In deriving the third line, the even integer $2n_f^{(2)}$ is removed. In deriving the fourth line, an even integer $-2N_M\frac{l}{s}N_m^{(1)}$ is added. In deriving the last line, Eq. (39) is applied.

APPENDIX I: DERIVATION OF EQ. (79)

$$\Gamma = \left[N_m^{(1)}+1\right]n_f^{(1)}+\left[N_m^{(2)}+1\right]n_f^{(2)}$$

$$= \left[N_m^{(1)}+1\right]\left[\left(\frac{l}{s}+1\right)N_M-N_m^{(1)}+n_f^{(2)}\right]$$

$$\quad +\left[N_M-N_m^{(1)}+1\right]n_f^{(2)}$$

$$\doteq N_M\left\{\left(\frac{l}{s}+1\right)\left[N_m^{(1)}+1\right]+n_f^{(2)}\right\}$$

$$\doteq N_M\left\{-\left(\frac{l}{s}+1\right)\left[N_m^{(1)}+1\right]+n_f^{(2)}\right\}$$

$$= N_M\left[N_E-\left(2\alpha\frac{l}{s}+\alpha+\frac{1}{2}\right)N_M-\left(\frac{l}{s}+1\right)\right]. \tag{I1}$$

In deriving the second line, Eqs. (56) and (7) are applied. In deriving the third line, the even integers $2n_f^{(2)}$ and $-[N_m^{(1)}+1]N_m^{(1)}$ are removed. In deriving the fourth line, an even integer $-2N_M(\frac{l}{s}+1)[N_m^{(1)}+1]$ is added. In deriving the last line, Eq. (59) is applied.

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