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Exceptional field theory. II. E[subscript 7(7)]

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Exceptional field theory. II. E$_{7(7)}$

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We introduce the exceptional field theory for the group E$_{7(7)}$, based on a $(4 + 56)$-dimensional spacetime subject to a covariant section condition. The "internal" generalized diffeomorphisms of the coordinates in the fundamental representation of E$_{7(7)}$ are governed by a covariant "E-bracket," which is gauged by 56 vector fields. We construct the complete and unique set of field equations that is gauge invariant under generalized diffeomorphisms in the internal and external coordinates. Among them are featured the non-Abelian twisted self-duality equations for the 56 gauge vectors. We discuss the explicit solutions of the section condition describing the embedding of the full, untruncated 11-dimensional and type IIB supergravity, respectively. As a new feature compared to the previously constructed $E_{6(6)}$ formulation, some components among the 56 gauge vectors descend from the 11-dimensional dual graviton but nevertheless allow for a consistent coupling by virtue of a covariantly constrained compensating 2-form gauge field.

I. INTRODUCTION

In this paper we present the details of the recently announced "exceptional field theory" (EFT) [1] for the group E$_{7(7)}$, complementing the E$_{6(6)}$ covariant construction given in Ref. [2]. The approach is a generalization of double field theory (DFT) [3–8],[1] with the goal being to render the dynamics of the complete $D = 11$ supergravity [10] covariant under the exceptional groups that are known to appear under dimensional reduction [11]. We refer to the Introduction of Ref. [2] for a more detailed outline of the general ideas, previous attempts, and extensive references. Here we will mainly present and discuss the novel aspects relevant for the larger group E$_{7(7)}$.

The E$_{7(7)}$ EFT is based on a generalized $4 + 56$-dimensional spacetime, with the "external" spacetime coordinates x^μ and "internal" coordinates Y^M in the fundamental representation 56 of E$_{7(7)}$, with dual derivatives ∂_M.[2] Correspondingly, the field content incorporates an external frame field ("vierbein") $e_\mu{}^a$ and an internal generalized metric $\mathcal{M}_{MN}$, parametrizing the coset space $E_{7(7)}/\text{SU}(8)$. Crucially, the theory also requires the presence of generalized gauge connections $A_\mu{}^M$ and a set of 2-forms $\{B_{\mu\nu\alpha}, B_{\mu\nu M}\}$, in order to consistently describe the complete degrees of freedom of $D = 11$ supergravity (and

necessarily including also some of their duals). The 2-forms $B_{\mu\nu\alpha}$ in the adjoint representation of E$_{7(7)}$ are known from the dimensionally reduced theory where they show up as the on-shell duals of the four-dimensional scalar fields. The significance of the additional 2-forms $B_{\mu\nu M}$ in the fundamental representation will become apparent shortly. The presence of these fields that go beyond the field content of the dimensionally reduced theory, is required for gauge invariance (under generalized diffeomorphisms) and at the same time are crucial in order to reproduce the full dynamics of $D = 11$ supergravity. All fields are subject to a covariant section constraint which implies that only a subset of the 56 internal coordinates is physical. The constraint can be written in terms of the E$_{7(7)}$ generators $(t_\alpha)^{MN}$ in the fundamental representation, and the invariant symplectic form Ω_{MN} of $E_{7(7)} \subset \text{Sp}(56)$, as

$$
\begin{aligned}
(t_\alpha)^{MN}\partial_M\partial_N A &= 0, \\
(t_\alpha)^{MN}\partial_M A\partial_N B &= 0, \\
\Omega^{MN}\partial_M A\partial_N B &= 0
\end{aligned}
\tag{1.1}
$$

for any fields or gauge parameters A, B.

Our main result is the construction of the gauge-invariant E$_{7(7)}$ EFT with the field content described above,

$$
\{e_\mu{}^a, \mathcal{M}_{MN}, A_\mu{}^M, B_{\mu\nu\alpha}, B_{\mu\nu M}\}.
\tag{1.2}
$$

The 56 gauge fields $A_\mu{}^M$ are subject to the first-order twisted self-duality equations

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[1]See Ref. [9] for a review and further references.
[2]Such generalized spacetimes also appear in the proposal of Ref. [12].

$$\mathcal{F}_{\mu\nu}{}^M = -\frac{1}{2}e\varepsilon_{\mu\nu\rho\sigma}\Omega^{MN}\mathcal{M}_{NK}\mathcal{F}^{\rho\sigma K}, \qquad (1.3)$$

with properly covariantized non-Abelian field strengths $\mathcal{F}_{\mu\nu}{}^M$ that we will introduce below. In the Abelian limit and upon dropping the dependence on all internal coordinates Y^M, these duality equations are known from the dimensional reduction of $D = 11$ supergravity to four spacetime dimensions [11]. In that case, they provide a duality covariant description of the dynamics of the gauge field sector. In particular, after the choice of a symplectic frame, these equations readily encode the standard second-order field equations for the 28 electric vector fields. On the other hand, the full non-Abelian self-duality equations (1.3) that we present in this paper reproduce the dynamics of the full (untruncated) 11-dimensional supergravity for these fields.

In addition to Eq. (1.3), the dynamics of the remaining fields is described by second-order field equations, which are most conveniently derived from an action,

$$S_{\mathrm{EFT}} = \int d^4x\, d^{56}Y\, e\Bigl(\hat{R} + \frac{1}{48}g^{\mu\nu}\mathcal{D}_\mu\mathcal{M}^{MN}\mathcal{D}_\nu\mathcal{M}_{MN}$$
$$-\frac{1}{8}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N + e^{-1}\mathcal{L}_{\mathrm{top}} - V(\mathcal{M}_{MN}, g_{\mu\nu})\Bigr). \qquad (1.4)$$

The theory takes the same structural form as gauged $\mathcal{N} = 8$ supergravity in $D = 4$ [13,14], with a (covariantized) Einstein-Hilbert term for the vierbein $e_\mu{}^a$, a kinetic term for $\mathcal{M}$ given by a nonlinear (gauged) sigma model with target space $E_{7(7)}/\mathrm{SU}(8)$, a Yang-Mills-type kinetic term for the gauge vectors and a "potential" $V(\mathcal{M}, g)$ that is a manifestly $E_{7(7)}$-covariant expression based only on internal derivatives ∂_M. In addition, there is a topological Chern-Simons-like term, which is required for consistency with the duality relations (1.3). We stress that here all fields depend on the $4 + 56$ coordinates, with the internal derivatives entering the non-Abelian gauge structure of covariant derivatives and field strengths, and that the theory encodes in particular $D = 11$ supergravity for a particular solution of the constraints (1.1). The detailed construction of all terms in the action will be given below.

The EFT is uniquely determined by its bosonic gauge symmetries, which are the generalized diffeomorphisms in the external and internal coordinates. In the rest of the Introduction we will briefly explain the novel features of its gauge structure. As in DFT, the generalized internal diffeomorphisms take the form of generalized Lie derivatives $\mathbb{L}_\Lambda$ with respect to a vector parameter Λ^M, e.g., $\delta_\Lambda \mathcal{M}_{MN} = \mathbb{L}_\Lambda \mathcal{M}_{MN}$. These generalized Lie derivatives, which preserve the $E_{7(7)}$ group properties of $\mathcal{M}_{MN}$, form an algebra according to

$$[\mathbb{L}_{\Lambda_1}, \mathbb{L}_{\Lambda_2}] = \mathbb{L}_{[\Lambda_1,\Lambda_2]_E}, \qquad (1.5)$$

modulo the constraints (1.1), and with the $E_{7(7)}$ E-bracket $[\Lambda_1, \Lambda_2]_E$ defined by

$$[\Lambda_1, \Lambda_2]_E^M = 2\Lambda_{[1}^K \partial_K \Lambda_{2]}^M + 12(t_\alpha)^{MN}(t^\alpha)_{KL}\Lambda_{[1}^K \partial_N \Lambda_{2]}^L$$
$$-\frac{1}{4}\Omega^{MN}\Omega_{KL}\partial_N(\Lambda_1^K \Lambda_2^L). \qquad (1.6)$$

This is the $E_{7(7)}$-covariant extension of the usual Lie bracket in differential geometry. However, it does not define a proper Lie algebra in that the Jacobi identity is violated. In order to resolve the apparent contradiction with the fact that the Lie derivatives define symmetry variations δ_Λ of the theory (which do satisfy the Jacobi identities), the usual explanation is common to DFT and the higher-dimensional versions of EFT: the section constraints (1.1) imply the existence of gauge parameters that are trivial in the sense that their action on an arbitrary field vanishes on the "constraint surface" of Eq. (1.1). Specifically, this is the case for gauge parameters given by total (internal) derivatives according to

$$\Lambda^M \equiv (t^\alpha)^{MN}\partial_N\chi_\alpha \quad \text{or} \quad \Lambda^M \equiv \Omega^{MN}\partial_N\chi, \qquad (1.7)$$

with arbitrary χ_α and χ. As will become important shortly, however, for the $E_{7(7)}$ generalized Lie derivative there is actually a more general class of trivial parameters, for which there is no direct analogue in DFT or the $E_{6(6)}$ EFT. These are of the form

$$\Lambda^M \equiv \Omega^{MN}\chi_N, \quad \text{with} \quad \chi_N \text{ covariantly constrained,} \qquad (1.8)$$

where by "covariantly constrained" we denote a field χ_M that satisfies the same covariant constraints (1.1) as the internal derivative ∂_M, i.e.,

$$(t_\alpha)^{MN}\chi_M\partial_N = (t_\alpha)^{MN}\chi_M\chi_N = 0,$$
$$\Omega^{MN}\chi_M\partial_N = 0, \quad \text{etc.,} \qquad (1.9)$$

in arbitrary combinations and acting on arbitrary functions. It is straightforward to see that with $\chi_M = \partial_M\chi$ the class of trivial gauge parameters (1.8) contains the last term in Eq. (1.7) as a special case, but in general this constitutes a larger class which will prove important in the following. In particular, the Jacobiator associated with Eq. (1.6) can be shown to be of the form

$$J^M(\Lambda_1, \Lambda_2, \Lambda_3) \equiv 3[[\Lambda_{[1}, \Lambda_2]_E, \Lambda_{3]}]_E^M$$
$$= (t^\alpha)^{MN}\partial_N\chi_\alpha(\Lambda) + \Omega^{MN}\chi_N(\Lambda), \qquad (1.10)$$

where

$$\chi_\alpha(\Lambda) = -\frac{1}{2}(t_\alpha)_{PQ}\Lambda_1^P[\Lambda_2, \Lambda_3]_E^Q + \text{cycl},$$

$$\chi_N(\Lambda) = \frac{1}{12}\Omega_{PQ}(\Lambda_1^P \partial_N[\Lambda_2, \Lambda_3]_E^Q + [\Lambda_2, \Lambda_3]_E^P \partial_N \Lambda_1^Q + \text{cycl})$$

$$(1.11)$$

constitute trivial gauge parameters of the type (1.7) and (1.8). Thus the Jacobiator has trivial action on all fields and becomes consistent with the Jacobi identity for the symmetry variations. Let us stress that the general class (1.8) of trivial gauge parameters is crucial in order to establish the consistency of the gauge transformations with the Jacobi identity. This seemingly innocent generalization of Eq. (1.7) has direct consequences for the required field content and couplings of the theory.

In EFT the gauge transformations, given by generalized Lie derivatives [Eq. (1.5)], are local both with respect to the internal and external space, i.e., the gauge parameters are functions of x and Y, $\Lambda^M = \Lambda^M(x, Y)$. All external derivatives ∂_μ thus require covariantization by the introduction of an associated gauge connection $A_\mu{}^M$. We are then faced with the need to construct a gauge-covariant field strength associated to symmetry transformations with nonvanishing Jacobiator [Eq. (1.10)]. This is a standard scenario in the tensor hierarchy of gauged supergravity [15,16] and it is solved by introducing as compensator fields an appropriate set of 2-form potentials with their associated tensor gauge transformations. Applied to our case, the full covariant field strength reads

$$\mathcal{F}_{\mu\nu}{}^M \equiv F_{\mu\nu}{}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MN}B_{\mu\nu N}, \quad (1.12)$$

where $F_{\mu\nu}{}^M$ denotes the standard non-Abelian Yang-Mills field strength associated with Eq. (1.6), and the 2-forms $B_{\mu\nu\alpha}$, $B_{\mu\nu M}$ enter in correspondence with the two terms in the Jacobiator (1.10). The novelty in this field strength, as compared to the corresponding field strength of DFT [17] and the E$_{6(6)}$ EFT [2], is the last term which carries a 2-form $B_{\mu\nu M}$ that itself is a covariantly constrained field in the sense of Eq. (1.9). The form of the Jacobiator (1.11) shows that gauge covariance of the field strength requires this type of coupling, whereas a (more conventional but weaker) compensating term of the form $\Omega^{MN}\partial_N B_{\mu\nu}$ with an unconstrained singlet 2-form $B_{\mu\nu}$ would not be sufficient to absorb all noncovariant terms in the variation.

While the notion of such a constrained compensator field may appear somewhat outlandish, the above discussion shows that its presence is a direct consequence of the properties of the E-bracket Jacobiator for E$_{7(7)}$. In turn, this compensator field will play a crucial role in identifying the dynamics of Eqs. (1.3) and (1.4), with the one of the full $D = 11$ supergravity. It ensures the correct and duality-covariant description of those degrees of freedom that are on-shell dual to the 11-dimensional graviton. More

specifically, after finding an explicit solution of the section constraint (1.1) and upon matching the field content (1.2) with that of $D = 11$ supergravity, seven components among the 56 gauge fields $A_\mu{}^M$ find their origin in the Kaluza-Klein vectors descending from the $D = 11$ metric. The twisted self-duality equations (1.3) thus seem to provide a first-order description of (at least a part of) the higher-dimensional gravitational dynamics by relating the seven Kaluza-Klein vectors to seven vector fields descending from what should be considered the $D = 11$ dual graviton [18–21]. Such a duality is commonly recognized to be restricted to the linearized level on the grounds of the no-go results of Refs. [22,23]. The nonlinear equations (1.3) circumvent this problem precisely by virtue of the covariantly constrained compensator fields $B_{\mu\nu M}$, which can be viewed as a covariantization of the formulation of Ref. [24]. As a result, the E$_{7(7)}$-covariant model (1.3) and (1.4), upon the appropriate solution of the section constraint (1.1), precisely reproduces the complete set of untruncated $D = 11$ field equations while featuring components of the dual graviton. The very same pattern has been observed in the three-dimensional duality-covariant formulation of $D = 4$ Einstein gravity in Ref. [25] where the constrained compensator gauge fields appear among the gauge vectors. In contrast, in the E$_{6(6)}$-covariant construction of Ref. [2], the degrees of freedom from the higher-dimensional dual graviton do not figure among the fields in the EFT action and the constrained compensator fields only enter the p-form hierarchy at the level of the 3-forms.

We finally note that while the above action (1.4) is manifestly invariant under the internal generalized diffeomorphisms with gauge parameter Λ^M (in the sense that each term is separately invariant), it also features a nonmanifest gauge invariance under diffeomorphisms in the external coordinates x^μ (with the parameter ξ^μ depending on coordinates x and Y). In fact, it is this symmetry, to be discussed below in more detail, that determines all relative coefficients in Eq. (1.4).

This paper is organized as follows. In Sec. II we introduce the details of the E$_{7(7)}$ generalized Lie derivatives and their E-bracket algebra, together with the associated covariant derivatives, field strengths and the tensor hierarchy. With these ingredients at hand, we define in Sec. III the full E$_{7(7)}$ EFT, including a discussion of the non-manifest invariance under the $(3 + 1)$-dimensional diffeomorphism of the x^μ. In Sec. IV we discuss the embedding of 11-dimensional supergravity and IIB supergravity upon choosing particular solutions of the section constraint. We conclude in Sec. V, while we collect some important E$_{7(7)}$ relations in the Appendix.

II. E$_{7(7)}$ GENERALIZED DIFFEOMORPHISMS AND THE TENSOR HIERARCHY

In this section, we introduce the E$_{7(7)}$ generalized Lie derivatives that generate the internal (generalized)

diffeomorphisms and the E-bracket and work out the associated tensor hierarchy. Vector fields $A_\mu{}^M$ in the fundamental 56-dimensional representation of $E_{7(7)}$ act as gauge fields in order to covariantize the theory under x-dependent internal (generalized) diffeomorphisms. The nontrivial Jacobiator of the E-bracket further requires the introduction of the 2-form $B_{\mu\nu\alpha}$ in the adjoint of $E_{7(7)}$ in accordance with the general tensor hierarchy of non-Abelian p-forms [15,16]. Up to this point, the construction is completely parallel to the construction of the $E_{6(6)}$-covariant tensor hierarchy, presented in detail in Ref. [2]. We will thus keep the presentation brief and compact. The new ingredient with respect to the $E_{6(6)}$-covariant construction is the appearance of a covariantly constrained compensating gauge field $B_{\mu\nu M}$ among the 2-forms, whose presence is required by closure of the tensor hierarchy. This field takes values in the fundamental representation of $E_{7(7)}$, however, restricted by covariant constraints; see Eq. (2.34) below.

A. Generalized Lie derivative and E-bracket

Let us start by collecting the relevant ingredients of the exceptional Lie group $E_{7(7)}$. Its Lie algebra is of dimension 133, with generators that we denote by t_α with the adjoint index $\alpha = 1, \ldots, 133$. The fundamental representation of $E_{7(7)}$ is of dimension 56 and denoted by indices $M, N = 1, \ldots, 56$. The symplectic embedding $E_{7(7)} \subset \mathrm{Sp}(56)$ implies the existence of an invariant antisymmetric tensor Ω^{MN} which we will use to raise and lower fundamental indices, adopting north-west south-east conventions: $V^M = \Omega^{MN}V_N$, $V_M = V^N\Omega_{NM}$, with $\Omega^{MK}\Omega_{NK} = \delta_N{}^M$. In contrast, adjoint indices are raised and lowered by the (rescaled) symmetric Cartan-Killing form $\kappa_{\alpha\beta} \equiv (t_\alpha)_M{}^N(t_\beta)_N{}^M$. Due to the invariance of Ω^{MN}, the gauge group generator in the fundamental representation with one index lowered, $(t_\alpha)_{MN}$, is *symmetric* in its two fundamental indices. Below we will need the projector onto the adjoint representation,

$$\mathbb{P}^K{}_M{}^L{}_N \equiv (t_\alpha)_M{}^K(t^\alpha)_N{}^L$$
$$= \frac{1}{24}\delta_M^K\delta_N^L + \frac{1}{12}\delta_M^L\delta_N^K + (t_\alpha)_{MN}(t^\alpha)^{KL}$$
$$- \frac{1}{24}\Omega_{MN}\Omega^{KL}, \tag{2.1}$$

which satisfies

$$\mathbb{P}^M{}_N{}^N{}_M = 133. \tag{2.2}$$

Next, we introduce the generalized Lie derivative with respect to the vector parameter Λ^M. Its action on a vector V^M of weight λ is defined as [26,27]

$$\delta V^M = \mathbb{L}_\Lambda V^M \equiv \Lambda^K\partial_K V^M - 12\mathbb{P}^M{}_N{}^K{}_L\partial_K\Lambda^L V^N$$
$$+ \lambda\partial_P\Lambda^P V^M, \tag{2.3}$$

with an appropriate generalization for its action on an $E_{7(7)}$ tensor with an arbitrary number of fundamental indices. Because of the projector in Eq. (2.3), the generalized Lie derivative is compatible with the $E_{7(7)}$ algebra structure: e.g. the Ω-tensor is an invariant tensor of weight 0,

$$\mathbb{L}_\Lambda\Omega^{MN} = 0, \tag{2.4}$$

implying that the definition (2.3) also induces the proper covariant transformation behavior for the covariant vector $V_M \equiv \Omega_{NM}V^N$. Explicitly, writing out the projector (2.1), the Lie derivative (2.3) reads

$$\delta_\Lambda V^M = \Lambda^K\partial_K V^M - \partial_N\Lambda^M V^N + \left(\lambda - \frac{1}{2}\right)\partial_P\Lambda^P V^M$$
$$- 12(t_\alpha)^{MN}(t^\alpha)_{KL}\partial_N\Lambda^K V^L - \frac{1}{2}\Omega^{MN}\Omega_{KL}\partial_N\Lambda^K V^L. \tag{2.5}$$

We now discuss some properties of the generalized Lie derivative. As mentioned in the introduction, there are trivial gauge parameters that do not generate a gauge transformation. They are of the form

$$\Lambda^M \equiv (t^\alpha)^{MN}\partial_N\chi_\alpha, \qquad \Lambda^M = \Omega^{MN}\chi_N, \tag{2.6}$$

with a covariantly constrained co-vector χ_M in the sense of satisfying Eq. (1.9). In order to state the constraints in a more compact form, let us introduce the projector $\mathbb{P}_{1+133}$ onto the $\mathbf{1} \oplus \mathbf{133}$ subrepresentation in the tensor product $\mathbf{56} \otimes \mathbf{56}$. In terms of this projector the constraints (1.9) take the compact form

$$(\mathbb{P}_{1+133})^{MN}\chi_M\partial_N = 0 = (\mathbb{P}_{1+133})^{MN}\chi_M\chi_N. \tag{2.7}$$

The triviality of $\Lambda^M = \Omega^{MN}\chi_N$ follows by a straightforward explicit calculation, using the identity (A1) and making repeated use of the constraints. The triviality of the first parameter in Eq. (2.6) follows similarly by a straightforward but somewhat more involved computation, using the identities in the Appendix.

Let us now discuss the algebra of gauge transformations (2.3). A direct computation making use of the algebraic identities collected in the Appendix shows that modulo the section constraints (1.1), these gauge transformations close [26,27],

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{[\Lambda_2, \Lambda_1]_E}, \tag{2.8}$$

according to the "E-bracket"

$$[\Lambda_2, \Lambda_1]^M_E = 2\Lambda^K_{[2}\partial_K\Lambda^M_{1]} + 12(t_\alpha)^{MN}(t^\alpha)_{KL}\Lambda^K_{[2}\partial_N\Lambda^L_{1]}$$
$$- \frac{1}{4}\Omega^{MK}\Omega_{NL}\partial_K(\Lambda^N_2\Lambda^L_1). \qquad (2.9)$$

Note that the last term in here is actually of the trivial form (2.6) and so does not generate a gauge transformation. This term is therefore ambiguous, and the reason we added it here (with this particular coefficient) is that the associated Jacobiator, i.e. the failure of the E-bracket to satisfy the Jacobi identity, takes a simple form. The appearance of this term is novel compared to the $E_{6(6)}$ case and therefore we go in some detail through the proof of the triviality of the Jacobiator. We first need some notation and define the Dorfman-type product between vectors of weight $\frac{1}{2}$ as

$$(V \circ W)^M \equiv (\mathbb{L}_V W)^M = V^K\partial_K W^M - W^K\partial_K V^M$$
$$- 12(t_\alpha)^{MN}(t^\alpha)_{KL}\partial_N V^K W^L$$
$$- \frac{1}{2}\Omega^{MN}\Omega_{KL}\partial_N V^K W^L. \qquad (2.10)$$

Comparing this with the E-bracket we conclude

$$(V \circ W)^M = [V, W]^M_E - 6(t^\alpha)^{MN}\partial_N((t_\alpha)_{KL}W^K V^L)$$
$$+ \frac{1}{4}\Omega^{MK}\Omega_{NL}(V^N\partial_K W^L + W^N\partial_K V^L)$$
$$\equiv [V, W]^M_E + \{V, W\}^M, \qquad (2.11)$$

introducing for later convenience the short-hand notation in the third line defined by the symmetric pairing in the first equation. In contrast to the situation in DFT and the $E_{6(6)}$ E-bracket, the final term in the first line cannot be written as a total derivative. Rather, it is of a trivial form in the stronger sense of Eq. (2.7). Therefore, both terms generate a trivial action, and we have

$$\mathbb{L}_{[V,W]_E} = \mathbb{L}_{(V \circ W)}. \qquad (2.12)$$

Another important property is that the antisymmetrized Dorfman product coincides with the E-bracket as defined in Eq. (2.9),

$$\frac{1}{2}(V \circ W - W \circ V) = [V, W]_E. \qquad (2.13)$$

It is this property that determines the *a priori* ambiguous coefficient of the $\Omega\Omega$ term in the E-bracket. Finally, the Dorfman product satisfies the Jacobi-like (or Leibniz-type) identity

$$U \circ (V \circ W) = (U \circ V) \circ W + V \circ (U \circ W). \qquad (2.14)$$

This follows from the algebra and the property (2.12) in complete analogy to the discussion in Ref. [2]. It is now straightforward to compute the Jacobiator,

$$J(V_1, V_2, V_3) \equiv 3[[V_{[1}, V_2]_E, V_{3]}]_E = -3[V_{[1}, [V_2, V_{3]}]_E]_E. \qquad (2.15)$$

In the following computation we will assume total antisymmetrization in the three arguments 1,2,3, but not display it explicitly. Keeping this in mind we compute for the term on the right-hand side with Eqs. (2.13) and (2.14),

$$[V_1, [V_2, V_3]_E]_E = [V_1, V_2 \circ V_3]_E = \frac{1}{2}(V_1 \circ (V_2 \circ V_3)$$
$$- (V_2 \circ V_3) \circ V_1)$$
$$= \frac{1}{2}((V_1 \circ V_2) \circ V_3 + V_2 \circ (V_1 \circ V_3)$$
$$- (V_2 \circ V_3) \circ V_1)$$
$$= -\frac{1}{2}V_1 \circ (V_2 \circ V_3), \qquad (2.16)$$

where we recalled the total antisymmetry in the last step. Thus, the E-bracket Jacobiator is proportional to the "Dorfman-Jacobiator." On the other hand, from Eq. (2.11) we also have

$$[V_1, [V_2, V_3]_E]_E = [V_1, V_2 \circ V_3]_E$$
$$= V_1 \circ (V_2 \circ V_3) - \{V_1, [V_2, V_3]_E\}. \qquad (2.17)$$

Using the fact that this equals Eq. (2.16) we can determine the Dorfman-Jacobiator and, via Eq. (2.16) again, the E-bracket Jacobiator (2.15),

$$J(V_1, V_2, V_3) = \frac{1}{3}(\{V_1, [V_2, V_3]_E\} + \{V_2, [V_3, V_1]_E\}$$
$$+ \{V_3, [V_1, V_2]_E\}), \qquad (2.18)$$

writing out the total antisymmetrization. This shows that the Jacobiator is of a trivial form that does not generate a gauge transformation. More explicitly, using the notation introduced in Eq. (2.11), the Jacobiator is given by

$$J^M(V_1, V_2, V_3) = -\frac{1}{2}(t_\alpha)^{MK}\partial_K((t^\alpha)_{PL}(V^P_1[V_2, V_3]^L_E + \text{cycl}))$$
$$+ \frac{1}{12}\Omega^{MK}\Omega_{NL}(V^N_1\partial_K[V_2, V_3]^L_E$$
$$+ [V_1, V_2]^N_E\partial_K V^L_3 + \text{cycl}). \qquad (2.19)$$

So far, we have discussed the action of the generalized Lie derivative on vectors in the fundamental representation of $E_{7(7)}$. From Eq. (2.3), we likewise obtain the action of the Lie derivative on a tensor in the adjoint representation (of weight λ')

$$\delta W_\alpha = \Lambda^K \partial_K W_\alpha + 12 f_{\alpha\beta}{}^\gamma (t^\beta)_L{}^K \partial_K \Lambda^L W_\gamma + \lambda' \partial_K \Lambda^K W_\alpha, \tag{2.20}$$

with the $E_{7(7)}$ structure constants $f_{\alpha\beta}{}^\gamma$. By construction, the $E_{7(7)}$ generators $(t_\alpha)^{MN}$ then are invariant tensors of weight 0 with respect to the generalized Lie derivative. In the following we will be led to consider such adjoint tensors under internal derivatives, more specifically combinations of the type

$$T^M \equiv (t^\alpha)^{MN} \partial_N W_\alpha. \tag{2.21}$$

Some straightforward computation (and the use of some of the algebraic relations collected in the Appendix) shows that under the generalized Lie derivative, the combination (2.21) transforms as

$$\begin{aligned}
\delta_\Lambda T^M &= \Lambda^K \partial_K T^M - 12 \mathbb{P}^M{}_N{}^K{}_L \partial_K \Lambda^L T^N \\
&+ \left(\lambda' - \frac{1}{2}\right) \partial_K \Lambda^K T^M \\
&+ (\lambda' - 1)(t^\alpha)^{MN} W_\alpha \partial_N \partial_K \Lambda^K \\
&+ \Omega^{MN} (t^\alpha)_L{}^K W_\alpha \partial_N \partial_K \Lambda^L.
\end{aligned} \tag{2.22}$$

The first line amounts to the covariant transformation of a vector of weight $\lambda = \lambda' - \frac{1}{2}$, while the second line represents noncovariant terms. The full result (2.22) then shows that for $\lambda' = 1$, T^M transforms like a contravariant vector of weight $\lambda = \frac{1}{2}$ up to a term proportional to $\Omega^{MN} \partial_N$. To correct for the latter, we may introduce a compensating field W_M subject to the same constraints as those discussed in Eq. (2.7), i.e.

$$(\mathbb{P}_{1+133})^{MN} W_M \partial_N = 0 = (\mathbb{P}_{1+133})^{MN} W_M W_N, \tag{2.23}$$

and consider the combination

$$\hat{T}^M \equiv (t^\alpha)^{MN} \partial_N W_\alpha + \frac{1}{24} \Omega^{MN} W_N. \tag{2.24}$$

This combination then transforms as a covariant vector of weight $\lambda = \frac{1}{2}$,

$$\delta_\Lambda \hat{T}^M = \Lambda^K \partial_K \hat{T}^M - 12 \mathbb{P}^M{}_N{}^K{}_L \partial_K \Lambda^L \hat{T}^N + \frac{1}{2} \partial_K \Lambda^K \hat{T}^M, \tag{2.25}$$

provided the compensating field W_M transforms as

$$\begin{aligned}
\delta_\Lambda W_M &= \Lambda^K \partial_K W_M + 12 \mathbb{P}^N{}_M{}^K{}_L \partial_K \Lambda^L W_N + \frac{1}{2} \partial_K \Lambda^K W_M \\
&- 24 (t^\alpha)_L{}^K W_\alpha \partial_M \partial_K \Lambda^L.
\end{aligned} \tag{2.26}$$

A short calculation confirms that the transformation (2.26) indeed preserves the constraints (2.23) on W_M. The tensorial

nature of Eq. (2.24) will prove crucial below for the structure of the tensor hierarchy of non-Abelian p-forms. We note that this crucially hinges on the introduction of the compensating field W_M.

B. Covariant derivatives and tensor hierarchy

We will now introduce gauge connections $A_\mu{}^M$ which manifestly render the model invariant under generalized Lie derivatives [Eq. (2.3)] with x-dependent gauge parameters Λ^M, covariantizing the derivatives in the usual fashion,

$$\partial_\mu \to \mathcal{D}_\mu \equiv \partial_\mu - \mathbb{L}_{A_\mu}. \tag{2.27}$$

Explicitly, from Eq. (2.5) we infer the form of the covariant derivative of a vector of weight λ,

$$\begin{aligned}
\mathcal{D}_\mu V^M &\equiv D_\mu V^M - \lambda \partial_K A_\mu{}^K V^M \\
&\equiv \partial_\mu V^M - A_\mu{}^K \partial_K V^M \\
&+ V^K \partial_K A_\mu{}^M + \frac{1-2\lambda}{2} \partial_K A_\mu{}^K V^M \\
&+ 12 (t_\alpha)^{MN} (t^\alpha)_{KL} \partial_N A_\mu{}^K V^L \\
&+ \frac{1}{2} \Omega^{MN} \Omega_{KL} \partial_N A_\mu{}^K V^L.
\end{aligned} \tag{2.28}$$

The gauge variation of the vector field $A_\mu{}^M$ is obtained by requiring that the covariant derivative transforms covariantly, which imposes

$$\begin{aligned}
\delta A_\mu{}^M &= \partial_\mu \Lambda^M - A_\mu{}^K \partial_K \Lambda^M + \Lambda^K \partial_K A_\mu{}^M \\
&+ 12 (t_\alpha)^{MN} (t^\alpha)_{KL} \Lambda^L \partial_N A_\mu{}^K \\
&+ \frac{1}{2} \Omega^{MN} \Omega_{KL} \Lambda^L \partial_N A_\mu{}^K \\
&= D_\mu \Lambda^M - \frac{1}{2} (\partial_K A_\mu{}^K) \Lambda^M \equiv \mathcal{D}_\mu \Lambda^M,
\end{aligned} \tag{2.29}$$

showing that the gauge parameter Λ^M is a tensor of weight $\lambda = \frac{1}{2}$. The associated Yang-Mills field strength,

$$\begin{aligned}
F_{\mu\nu}{}^M &\equiv 2 \partial_{[\mu} A_{\nu]}{}^M - [A_\mu, A_\nu]_E^M \\
&= 2 \partial_{[\mu} A_{\nu]}{}^M - 2 A_{[\mu}{}^K \partial_K A_{\nu]}{}^M \\
&- \frac{1}{2} (24 (t_\alpha)^{MK} (t^\alpha)_{NL} - \Omega^{MK} \Omega_{NL}) A_{[\mu}{}^N \partial_K A_{\nu]}{}^L,
\end{aligned} \tag{2.30}$$

has a general variation given by

$$\begin{aligned}
\delta F_{\mu\nu}{}^M &= 2 D_{[\mu} \delta A_{\nu]}{}^M - \partial_K A_{[\mu}{}^K \delta A_{\nu]}{}^M \\
&- 12 (t_\alpha)^{MK} (t^\alpha)_{NL} \partial_K (A_{[\mu}{}^N \delta A_{\nu]}{}^L) \\
&- \frac{1}{2} \Omega^{MK} \Omega_{LN} (A_{[\mu}{}^N \partial_K \delta A_{\nu]}{}^L - \partial_K A_{[\mu}{}^N \delta A_{\nu]}{}^L),
\end{aligned} \tag{2.31}$$

and is not covariant with respect to the vector gauge transformations (2.29). This is a consequence of the non-vanishing Jacobiator (2.19). In order to define a covariant field strength, it is natural in the spirit of the tensor hierarchy [15,16] to extend the field strength (2.31) by further Stückelberg-type couplings according to

$$\mathcal{F}^{\circ}_{\mu\nu}{}^M \equiv F_{\mu\nu}{}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha}, \qquad (2.32)$$

to 2-form tensors $B_{\mu\nu\alpha}$ in the adjoint representation of E$_{7(7)}$, whose transformations may absorb some of the noncovariant terms in Eq. (2.31). However, unlike the E$_{6(6)}$-covariant construction of Ref. [2], this modification is not sufficient in order to obtain fully gauge-covariant field strengths. In particular, the last line of Eq. (2.31) continues to spoil the proper transformation behavior of the field strength and cannot be absorbed into a transformation of $B_{\mu\nu\alpha}$. This indicates that in the E$_{7(7)}$-covariant construction new fields are required at the level of the 2-form tensors, as discussed in the Introduction. We recall that with five external dimensions, these additional fields only enter at the level of the 3-forms and remain invisible in the action [2], whereas in the three-dimensional case they are already present among the vector fields [25]. The fully covariantized field strength is given by the expression

$$\mathcal{F}_{\mu\nu}{}^M \equiv F_{\mu\nu}{}^M - 12(t^\alpha)^{MN}\partial_N B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MK}B_{\mu\nu K}, \qquad (2.33)$$

where the 2-form $B_{\mu\nu K}$ is a covariantly constrained compensating gauge field, i.e. a field subject to the same section constraints as the internal derivatives,

$$(\mathbb{P}_{\mathbf{1+133}})^{MN}B_M\partial_N = 0, \quad (\mathbb{P}_{\mathbf{1+133}})^{MN}B_M B_N = 0. \quad (2.34)$$

The general variation of $\mathcal{F}_{\mu\nu}{}^M$ is given by

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\mathcal{D}_{[\mu}\delta A_{\nu]}{}^M - 12(t^\alpha)^{MN}\partial_N\Delta B_{\mu\nu\alpha} - \frac{1}{2}\Omega^{MK}\Delta B_{\mu\nu K},$$
$$(2.35)$$

with the E$_{7(7)}$ tensor $\delta A_\mu{}^M$ of weight $\lambda = \frac{1}{2}$, and

$$\Delta B_{\mu\nu\alpha} \equiv \delta B_{\mu\nu\alpha} + (t_\alpha)_{KL}A_{[\mu}{}^K\delta A_{\nu]}{}^L,$$
$$\Delta B_{\mu\nu K} \equiv \delta B_{\mu\nu K} + \Omega_{LN}(A_{[\mu}{}^N\partial_K\delta A_{\nu]}{}^L - \partial_K A_{[\mu}{}^N\delta A_{\nu]}{}^L).$$
$$(2.36)$$

In particular, we may define vector gauge variations,

$$\delta_\Lambda A_\mu{}^M = \mathcal{D}_\mu\Lambda^M,$$
$$\Delta_\Lambda B_{\mu\nu\alpha} = (t_\alpha)_{KL}\Lambda^K\mathcal{F}_{\mu\nu}{}^L,$$
$$\Delta_\Lambda B_{\mu\nu M} = -\Omega_{KL}(\mathcal{F}_{\mu\nu}{}^K\partial_M\Lambda^L - \Lambda^L\partial_M\mathcal{F}_{\mu\nu}{}^K), \quad (2.37)$$

under which the field strength $\mathcal{F}_{\mu\nu}{}^M$ transforms covariantly,

$$\delta_\Lambda\mathcal{F}_{\mu\nu}{}^M = \Lambda^K\partial_K\mathcal{F}_{\mu\nu}{}^M - 12\mathbb{P}^M{}_N{}^K{}_L\partial_K\Lambda^L\mathcal{F}_{\mu\nu}{}^N$$
$$+ \frac{1}{2}\partial_K\Lambda^K\mathcal{F}_{\mu\nu}{}^M, \qquad (2.38)$$

i.e., as an E$_{7(7)}$ vector of weight $\lambda = \frac{1}{2}$. As part of this calculation, we have used the fact that

$$\mathbb{L}_{F_{\mu\nu}}\Lambda^M = \mathbb{L}_{\mathcal{F}_{\mu\nu}}\Lambda^M, \qquad (2.39)$$

which states that $F_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$ differ by terms that are trivial and so do not generate a generalized Lie derivative, cf. Eq. (2.6). Let us also note that the form of the gauge transformations (2.36) and (2.37) manifestly preserves the constraints (2.34) on the compensating gauge field as a consequence of Eq. (1.1).

The 2-form tensors $B_{\mu\nu\alpha}$ and $B_{\mu\nu M}$ carry their own gauge symmetries which act as

$$\delta A_\mu{}^M = 12(t^\alpha)^{MN}\partial_N\Xi_{\mu\alpha} + \frac{1}{2}\Omega^{MN}\Xi_{\mu N},$$

$$\Delta B_{\mu\nu\alpha} = 2\mathcal{D}_{[\mu}\Xi_{\nu]\alpha},$$

$$\Delta B_{\mu\nu M} = 2\mathcal{D}_{[\mu}\Xi_{\nu]M} + 48(t^\alpha)_L{}^K(\partial_K\partial_M A_{[\mu}{}^L)\Xi_{\nu]\alpha}, \qquad (2.40)$$

and leave the field strength (2.33) invariant. The tensor gauge parameters $\Xi_{\mu\alpha}$ and $\Xi_{\mu M}$ are of weight $\lambda' = 1$ and $\lambda = \frac{1}{2}$, respectively, with their covariant derivatives defined according to Eqs. (2.5) and (2.20), respectively. Note that the seemingly noncovariant term in $\Delta B_{\mu\nu M}$ has its origin in the final term in Eq. (2.26), which reflects the fact that the constrained field B_M does not have a separate tensor character, but only in combinations of the type (2.24). In particular, the computation of the invariance of the field strength $\mathcal{F}_{\mu\nu}{}^M$ under Eq. (2.40) crucially depends on the observation that a tensor combination according to Eq. (2.24) is again of tensorial nature.

We close this presentation of the tensor fields by stating the Bianchi identities,

$$3\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M = -12(t^\alpha)^{MN}\partial_N\mathcal{H}_{\mu\nu\rho\alpha} - \frac{1}{2}\Omega^{MN}\mathcal{H}_{\mu\nu\rho N},$$
$$(2.41)$$

with the 3-form field strengths $\mathcal{H}_{\mu\nu\rho\alpha}$ and $\mathcal{H}_{\mu\nu\rho N}$ defined by this equation up to terms that vanish under the projection with $(t^\alpha)^{MN}\partial_N$. This identity again is a nice illustration of tensorial structures of the type (2.24), with the field strength $\mathcal{H}_{\mu\nu\rho M}$ transforming according to Eq. (2.26) under generalized Lie derivatives.

III. COVARIANT E$_{7(7)}$ THEORY

With the tensor hierarchy associated to generalized diffeomorphisms set up, we are now in the position to define the various terms in the action (1.4) and the duality equation (1.3). We then verify that the complete set of

equations of motion is invariant under generalized internal and external diffeomorphisms, which in turn fixes all the couplings.

A. Kinetic terms

The metric, the scalar fields and the vector gauge fields come with second-order kinetic terms in the action (1.4). As in Refs. [2,17], the Einstein-Hilbert term is built from the improved Riemann tensor,

$$\hat{R}_{\mu\nu}{}^{ab} \equiv R_{\mu\nu}{}^{ab}[\omega] + \mathcal{F}_{\mu\nu}{}^M e^{a\rho}\partial_M e_{\rho}{}^b, \qquad (3.1)$$

where $R_{\mu\nu}{}^{ab}[\omega]$ denotes the curvature of the spin connection which in turn is given by the standard expression in terms of the vierbein with all derivatives covariantized according to

$$\mathcal{D}_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - A_\mu{}^M\partial_M e_\nu{}^a - \frac{1}{2}\partial_M A_\mu{}^M e_\nu{}^a. \qquad (3.2)$$

I.e., the vierbein is an $E_{7(7)}$ scalar of weight $\lambda = \frac{1}{2}$. The covariantized Einstein-Hilbert term

$$\mathcal{L}_{EH} = e\hat{R} = ee_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab} \qquad (3.3)$$

then is invariant under Lorentz transformations and correctly transforms as a density under internal generalized diffeomorphisms with the weight 2 of the vierbein determinant and the weights $-\frac{1}{2}$ of the inverse vierbeins adding up to 1. The 70 scalar fields of the theory parametrize the coset space $E_{7(7)}/SU(8)$, which is conveniently described by the symmetric 56×56 matrix $\mathcal{M}_{MN}$, with the kinetic term given by

$$\mathcal{L}_{sc} = \frac{1}{48}eg^{\mu\nu}\mathcal{D}_\mu\mathcal{M}_{MN}\mathcal{D}_\nu\mathcal{M}^{MN}, \qquad (3.4)$$

with the inverse matrix $\mathcal{M}^{MN}$ related by

$$\mathcal{M}^{MN} = \Omega^{MK}\Omega^{NL}\mathcal{M}_{KL}, \qquad (3.5)$$

as a consequence of the symplectic embedding of $E_{7(7)}$. All derivatives in Eq. (3.4) are covariantized as Eq. (2.28) with $\mathcal{M}_{MN}$ transforming as an $E_{7(7)}$ tensor of weight $\lambda = 0$. This is compatible with the group property $\det \mathcal{M}_{MN} = 1$. As for the Einstein-Hilbert term, the total weight of Eq. (3.4) is 1 as required for Λ^M gauge invariance. Finally, also the Yang-Mills kinetic term

$$\mathcal{L}_{YM} = -\frac{1}{8}e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N \qquad (3.6)$$

carries the correct weight of 1, since the field strengths transform as tensors of weight $\lambda = \frac{1}{2}$, cf. Eq. (2.38). As discussed above, this term gives rise to second-order field equations for all 56 vector fields $A_\mu{}^M$ whereas the Lagrangian (1.4) is amended by the covariant first-order duality equations,

$$\mathcal{E}_{\mu\nu}{}^M \equiv \mathcal{F}_{\mu\nu}{}^M + \frac{1}{2}e\varepsilon_{\mu\nu\rho\sigma}\Omega^{MN}\mathcal{M}_{NK}\mathcal{F}^{\rho\sigma K} = 0, \qquad (3.7)$$

which ensures that only 28 of them correspond to independent propagating degrees of freedom. Both terms in this duality equation are $E_{7(7)}$ tensors of weight $\lambda = \frac{1}{2}$.

B. Topological term

The topological term is required in order to ensure that the variation of the 2-form tensors in Eq. (3.6) does not give rise to inconsistent field equations. This term is most conveniently constructed as the boundary term of a manifestly gauge-invariant exact form in five dimensions as

$$S_{top} = -\frac{1}{24}\int_{\Sigma_5} d^5x \int d^{56}Y \varepsilon^{\mu\nu\rho\sigma\tau}\mathcal{F}_{\mu\nu}{}^M\mathcal{D}_\rho\mathcal{F}_{\sigma\tau M}$$

$$\equiv \int_{\partial\Sigma_5} d^4x \int d^{56}Y \mathcal{L}_{top}. \qquad (3.8)$$

The explicit form of the four-dimensional Lagrangian density is not particularly illuminating, since it is not manifestly gauge invariant. What we will need in the following is its variation,

$$\delta\mathcal{L}_{top} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\left(\delta A_\mu{}^M\mathcal{D}_\nu\mathcal{F}_{\rho\sigma M} + \mathcal{F}_{\mu\nu M}\left(6(t^\alpha)^{MN}\partial_N\Delta B_{\rho\sigma\alpha} + \frac{1}{4}\Omega^{MN}\Delta B_{\rho\sigma N}\right)\right), \qquad (3.9)$$

which takes a covariant form in terms of the general variations introduced in Eq. (2.36). From this expression it is straightforward to explicitly verify gauge invariance under the Λ and Ξ transformations (2.37) and (2.40).

Variation of the combined Lagrangian $\mathcal{L}_{YM} + \mathcal{L}_{top}$ with respect to the 2-forms consistently reproduces parts of the

duality equation (3.7). More precisely, variation with respect to $B_{\mu\nu\alpha}$ yields the duality equation under internal derivatives $(t^\alpha)^{MN}\partial_N$ whereas variation with respect to $B_{\mu\nu M}$ formally seems to give all of Eq. (3.7); however, one must take into account that this field itself is constrained by Eq. (2.34), such that the variation of its components is not independent.

Concerning the Lagrangian of the gauge field sector, the sum $\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{top}}$ constitutes an incomplete (or "pseudo"-) action that must be amended by the additional first-order duality equation (3.7). This is in the spirit of the "democratic formulation" of supergravities [28]. In reality we are thus working on the level of the field equations and simply introduce this Lagrangian as a convenient tool to verify symmetries of the field equations in a compact way. Alternatively, one may switch to a true Lagrangian formulation in the standard fashion [11,29] by choosing a symplectic frame that selects 28 electric vector fields $A_\mu{}^\Lambda$, breaking the matrix $\mathcal{M}_{MN}$ into

$$
\mathcal{M}_{MN} = \begin{pmatrix} \mathcal{M}_{\Lambda\Sigma} & \mathcal{M}_\Lambda{}^\Sigma \\ \mathcal{M}^\Lambda{}_\Sigma & \mathcal{M}^{\Lambda\Sigma} \end{pmatrix}
$$
$$
\equiv \begin{pmatrix} (\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & -(\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^\Sigma \\ -(\mathcal{I}^{-1}\mathcal{R})^\Lambda{}_\Sigma & (\mathcal{I}^{-1})^{\Lambda\Sigma} \end{pmatrix}, \quad (3.10)
$$

and replacing the kinetic term (3.6) by

$$
\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} e \mathcal{I}_{MN} \mathcal{F}^{\mu\nu M} \mathcal{F}_{\mu\nu}{}^N - \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} \mathcal{R}_{MN} \mathcal{F}_{\mu\nu}{}^M \mathcal{F}_{\rho\sigma}{}^N.
$$
$$
(3.11)
$$

The topological term then is modified similar to the structure given in Ref. [14] that treats asymmetrically the electric and magnetic vector fields. The resulting Lagrangian carries 28 electric vectors with proper kinetic term (3.11) and 28 magnetic duals that only appear in covariant derivatives and the topological term. Its field equations are equivalent to those we have been discussing above. For this paper, we prefer to work on the level of the field equations [or equivalently with the "pseudo"-action (3.6)] since that formulation retains the manifest E$_{7(7)}$ covariance.

Let us discuss the field equations of the vector/tensor system. Taking the exterior derivative of Eq. (3.7) and using the Bianchi identity (2.41), one obtains second-order field equations for the vector fields,

$$
\mathcal{D}_\nu(e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}) = -2\varepsilon^{\mu\nu\rho\sigma}(t^\alpha)_M{}^N \partial_N \mathcal{H}_{\nu\rho\sigma\alpha}
$$
$$
+ \frac{1}{12}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma M}. \quad (3.12)
$$

We may compare this equation to the field equations obtained from variation of the Lagrangian (3.6) and (3.8),

$$
\mathcal{D}_\nu(e\mathcal{M}_{MN}\mathcal{F}^{\mu\nu N}) = 2e(\hat{J}^\mu{}_M + \mathcal{J}^\mu{}_M) - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{D}_\nu\mathcal{F}_{\rho\sigma M},
$$
$$
(3.13)
$$

with the gravitational and matter currents defined by general variation with respect to the vector fields,

$$
\delta_A \mathcal{L}_{\mathrm{EH}} \equiv e\hat{J}^\mu{}_M \delta A_\mu{}^M, \qquad \delta_A \mathcal{L}_{\mathrm{sc}} \equiv e\mathcal{J}^\mu{}_M \delta A_\mu{}^M, \quad (3.14)
$$

e.g. explicitly

$$
\mathcal{J}^\mu{}_M = e^{-1}\partial_N(e\mathcal{D}^\mu\mathcal{M}^{NP}\mathcal{M}_{MP}) - \frac{1}{24}\mathcal{D}^\mu\mathcal{M}^{KL}\partial_M\mathcal{M}_{KL}.
$$
$$
(3.15)
$$

Combining Eqs. (3.12) and (3.13), we obtain the duality equations between scalar and tensor fields,

$$
e\hat{J}^\mu{}_M + e\mathcal{J}^\mu{}_M = -2\varepsilon^{\mu\nu\rho\sigma}(t^\alpha)_M{}^N\partial_N\mathcal{H}_{\nu\rho\sigma\alpha} + \frac{1}{12}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma M}.
$$
$$
(3.16)
$$

Inserting Eq. (3.15), we can project this equation onto its irreducible parts and obtain

$$
e\hat{J}^\mu{}_M - \frac{1}{24}e\mathcal{D}^\mu\mathcal{M}^{KL}\partial_M\mathcal{M}_{KL} = \frac{1}{12}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma M},
$$
$$
-\frac{1}{2}(t_\alpha)_K{}^L(e\mathcal{D}^\mu\mathcal{M}^{KP}\mathcal{M}_{LP}) = \varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\nu\rho\sigma\alpha}. \quad (3.17)
$$

More precisely, the second equation only arises under projection with the derivatives $(t^\alpha)^{MN}\partial_N$.

C. The potential

Finally, we discuss the last term in the EFT action (1.4). The potential V is a function of the external metric $g_{\mu\nu}$ and the internal metric $\mathcal{M}_{MN}$ given by

$$
V = -\frac{1}{48}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}
$$
$$
+\frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}
$$
$$
-\frac{1}{2}g^{-1}\partial_M g\partial_N\mathcal{M}^{MN} - \frac{1}{4}\mathcal{M}^{MN}g^{-1}\partial_M g g^{-1}\partial_N g
$$
$$
-\frac{1}{4}\mathcal{M}^{MN}\partial_M g^{\mu\nu}\partial_N g_{\mu\nu}. \quad (3.18)
$$

The relative coefficients in here are determined by Λ^M gauge invariance, in a computation that is analogous to the E$_{6(6)}$ case presented in Ref. [2] and that we briefly sketch in the following. We first note that acting with ∂_M on an E$_{7(7)}$ scalar S adds a density weight of $-\frac{1}{2}$. Consider its variation $\delta_\Lambda S = \Lambda^N\partial_N S$. It can then be easily checked by writing out the projector (2.1) that its partial derivative transforms covariantly as

$$
\delta_\Lambda(\partial_M S) = \mathbb{L}_\Lambda(\partial_M S), \quad \text{where} \quad \lambda(\partial_M S) = -\frac{1}{2}, \quad (3.19)
$$

i.e., as a co-vector density of weight $\lambda = -\frac{1}{2}$. Similarly, while $\mathcal{M}$ is a tensor of weight zero, its partial derivatives $\partial\mathcal{M}$ carry a weight of $-\frac{1}{2}$, which is precisely the right

weight to combine with the weight 2 of the vierbein determinant e to a total weight of 1 for the potential term, as needed for gauge invariance of the action. In contrast to a scalar however, the partial derivative $\partial\mathcal{M}$ receives also various noncovariant terms whose cancellation needs to be verified explicitly. A direct computation gives for the first term in Eq. (3.18), up to boundary terms,

$$\delta_\Lambda\left(-\frac{1}{48}e\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}\right)$$
$$= e\partial_M\partial_R\Lambda^P\mathcal{M}^{MN}\mathcal{M}^{LR}\partial_N\mathcal{M}_{PL}. \quad (3.20)$$

For this computation one has to use the fact that $\mathcal{M}^{-1}\partial\mathcal{M}$ takes values in the Lie algebra of $E_{7(7)}$ so that the adjoint projector acts as the identity,

$$\mathbb{P}^R{}_S{}^K{}_Q\mathcal{M}^{QL}\partial_N\mathcal{M}_{KL} = \mathcal{M}^{RL}\partial_N\mathcal{M}_{SL}. \quad (3.21)$$

For the second term in Eq. (3.18) one finds after a straightforward calculation

$$\delta_\Lambda\left(\frac{1}{2}e\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}\right)$$
$$= -e\partial_M\partial_R\Lambda^P\mathcal{M}^{MN}\mathcal{M}^{LR}\partial_N\mathcal{M}_{PL}$$
$$+ e\partial_M\partial_P\Lambda^L\partial_L\mathcal{M}^{MP} + e\partial_M\partial_P\Lambda^P\partial_L\mathcal{M}^{ML}$$
$$- 12e\partial_M\partial_R\Lambda^P(t_\alpha)^{KR}(t^\alpha)_{PQ}\mathcal{M}^{QL}\mathcal{M}^{MN}\partial_L\mathcal{M}_{NK}$$
$$- \frac{1}{2}e\partial_M\partial_R\Lambda^P\Omega^{KR}\Omega_{PQ}\mathcal{M}^{QL}\mathcal{M}^{MN}\partial_L\mathcal{M}_{NK}$$
$$= -e\partial_M\partial_R\Lambda^P\mathcal{M}^{MN}\mathcal{M}^{LR}\partial_N\mathcal{M}_{PL}$$
$$+ e\partial_M\partial_P\Lambda^L\partial_L\mathcal{M}^{MP} + e\partial_M\partial_P\Lambda^P\partial_L\mathcal{M}^{ML}. \quad (3.22)$$

In the second equality we used again the fact that the current $(J_L)^M{}_K \equiv \mathcal{M}^{MN}\partial_L\mathcal{M}_{NK}$ is Lie-algebra valued, which implies that the terms in the third and fourth line are zero. In order to see this we note that

$$2(J_L)^{(M}{}_K(t_\alpha)^{R)K}$$
$$= 2(J_L)^\beta(t_\beta)^{(M}{}_K(t_\alpha)^{R)K} = (J_L)^\beta f_{\beta\alpha}{}^\gamma(t_\gamma)^{MR}, \quad (3.23)$$

where we expanded the current into the basis t_α and used the invariance of $(t_\alpha)^{MN}$ in the final step. This is precisely the structure in the third line of Eq. (3.22), where this term is contracted with $\partial_M\partial_R\Lambda^P$ and hence is zero by the section constraint. Similarly, in the fourth line in Eq. (3.22) the symplectic form Ω^{KR} raises an index on the current, whose free indices are then contracted with $\partial_M\partial_R\Lambda^P$, giving zero by the section constraint. With the final result in Eq. (3.22) we see that the cubic term in $\mathcal{M}$ cancels the term in Eq. (3.20). It is straightforward to verify that the remaining two terms cancel against the variations coming from the second line in the potential (3.18), up to total derivatives, thus proving full gauge invariance of the potential term.

For comparison of the full result with the truncations that have been given in the literature [26,30,31],[3] we finally note that after the truncation that sets $g_{\mu\nu} = e^{2\Delta}\eta_{\mu\nu}$, the potential term reduces to

$$\mathcal{L}_{\text{pot}} = -eV = e^{4\Delta}\left(\frac{1}{48}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL}\right.$$
$$- \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK}$$
$$\left.+ 4\partial_M\Delta\partial_N\mathcal{M}^{MN} + 12\mathcal{M}^{MN}\partial_M\Delta\partial_N\Delta\right), \quad (3.24)$$

and it can be rewritten in terms of the rescaled matrix $\hat{\mathcal{M}}_{MN} \equiv e^{\gamma\Delta}\mathcal{M}_{MN}$. It is important to note that Eq. (3.24) remains $E_{7(7)}$ invariant only upon keeping Δ as an independent degree of freedom.

D. External diffeomorphisms

The various terms of the EFT action (1.4) have been determined by invariance under generalized internal Λ^M diffeomorphisms. In contrast, the relative coefficients between these terms are determined by invariance of the full action (or equations of motion) under the remaining gauge symmetries, which are a covariantized version of the external $(3 + 1)$-dimensional diffeomorphisms with parameters $\xi^\mu(x, Y)$. For a Y-independent parameter, external diffeomorphism invariance is manifest. On the other hand, gauge invariance for general $\xi^\mu(x, Y)$ determines all equations of motion with no free parameter left. The gauge variations of vielbeins, scalars and the vector fields are given by

$$\delta_\xi e_\mu{}^a = \xi^\nu\mathcal{D}_\nu e_\mu{}^a + \mathcal{D}_\mu\xi^\nu e_\nu{}^a,$$
$$\delta_\xi\mathcal{M}_{MN} = \xi^\mu\mathcal{D}_\mu\mathcal{M}_{MN},$$
$$\delta_\xi A_\mu{}^M = \xi^\nu\mathcal{F}_{\nu\mu}{}^M + \mathcal{M}^{MN}g_{\mu\nu}\partial_N\xi^\nu, \quad (3.25)$$

i.e. take the form of covariantized diffeomorphisms together with an additional $\mathcal{M}$-dependent contribution in δA, that has likewise appeared in Refs. [2,25]. The invariance of Eq. (1.4) can be shown in close analogy to the calculation for the $E_{6(6)}$ case of Ref. [2]. Instead of repeating this discussion, let us say a few words on the particularities of the $E_{7(7)}$ case, i.e. the generalized diffeomorphism invariance of the first-order duality relations (3.7) and the transformation laws for the 2-form tensors. The latter fields transform as

$$\Delta_\xi B_{\mu\nu\alpha} = \xi^\rho\mathcal{H}_{\mu\nu\rho\alpha},$$
$$\Delta_\xi B_{\mu\nu M} = \xi^\rho\mathcal{H}_{\mu\nu\rho M} + 2e\varepsilon_{\mu\nu\rho\sigma}g^{\sigma\tau}\mathcal{D}^\rho(g_{\tau\lambda}\partial_M\xi^\lambda) \quad (3.26)$$

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[3]See also Refs. [32,33] for the geometric interpretation of these terms.

in terms of the covariant variations (2.36). In particular, the variation of the constrained compensating tensor gauge field $B_{\mu\nu M}$ carries an additional noncovariant term that is required for gauge invariance of the equations of motion. We note that a similar term has appeared in the transformation laws of the constrained compensating (vector) gauge fields in the three-dimensional formulation [25]. Moreover, the structure of the transformation rule is manifestly consistent with the constraints (2.34) on this field. From Eqs. (3.25) and (3.26), we find the transformation law of the field strengths,

$$\delta_\xi \mathcal{F}_{\mu\nu}{}^M = \mathcal{L}_\xi \mathcal{F}_{\mu\nu}{}^M + 2(\mathcal{D}_{[\mu}\mathcal{M}^{MN}g_{\nu]\rho} - 6(t^\alpha)^{MN}\mathcal{H}_{\mu\nu\rho\alpha})\partial_N\xi^\rho$$
$$+ 2\mathcal{M}^{MN}\mathcal{D}_{[\mu}(g_{\nu]\rho}\partial_N\xi^\rho)$$
$$- e\varepsilon_{\mu\nu\lambda\sigma}g^{\sigma\tau}\Omega^{MN}\mathcal{D}^\lambda(g_{\tau\rho}\partial_N\xi^\rho), \qquad (3.27)$$

where the first term describes the standard transformation under (covariantized) diffeomorphisms. On-shell, upon using the duality equation (3.17), this transformation may be rewritten in the compact form

$$\delta_\xi \mathcal{F}_{\mu\nu}{}^M = \mathcal{L}_\xi \mathcal{F}_{\mu\nu}{}^M + \mathcal{Z}_{\mu\nu}{}^M$$
$$- \frac{1}{2}e\varepsilon_{\mu\nu\rho\sigma}\Omega^{MN}\mathcal{M}_{NK}\mathcal{Z}^{\rho\sigma K},$$
$$\text{with} \quad \mathcal{Z}_{\mu\nu}{}^M \equiv 2\mathcal{D}_{[\mu}(\mathcal{M}^{MN}g_{\nu]\rho}\partial_N\xi^\rho). \qquad (3.28)$$

From this expression it is evident that the noncovariant terms in the variation of $\mathcal{F}_{\mu\nu}{}^M$ drop out when calculating the variation of the duality equation (3.7),

$$\delta_\xi \mathcal{E}_{\mu\nu}{}^M = \mathcal{L}_\xi \mathcal{E}_{\mu\nu}{}^M, \qquad (3.29)$$

and thus the duality equation is duality covariant. More precisely, since we used Eq. (3.17), it follows that the first-order duality relations transform into each other. The discussion shows that the extra terms in the variation of Eq. (3.26) are crucial for this covariance. Moreover, the calculation requires the precise form (3.17) of the duality equation between scalars and tensors and thereby fixes the corresponding relative coefficients in the action (1.4). Eventually, external diffeomorphism invariance of the complete set of equations of motion fixes all the coefficients in Eq. (1.4) and the equations of motion.

IV. EMBEDDING $D = 11$ AND TYPE IIB SUPERGRAVITY

In the previous sections, we have constructed the unique set of E$_{7(7)}$-covariant equations of motion for the fields (1.2), which is invariant under generalized internal and external diffeomorphisms. It remains to explicitly embed $D = 11$ supergravity. Evaluating the above field equations with an explicit appropriate solution of the section constraints (1.1), one may recover the full dynamics of $D = 11$

supergravity after rearranging the 11-dimensional fields according to a $4 + 7$ Kaluza-Klein split of the coordinates, but retaining the full dependence on all 11 coordinates.

The relevant solution of the section condition is related to the splitting of coordinates according to the decomposition of the fundamental representation of E$_{7(7)}$ under its maximal GL(7) subgroup,

$$\mathbf{56} \longrightarrow 7_{+3} + 21'_{+1} + 21_{-1} + 7'_{-3},$$
$$\{Y^M\} \longrightarrow \{y^m, y_{mn}, y^{mn}, y_m\}. \qquad (4.1)$$

Here subscripts refer to the GL(1) weight, indices $m, n, \dots$ label the vector representation of GL(7), and the coordinates $y^{mn} = y^{[mn]}$, $y_{mn} = y_{[mn]}$ are antisymmetric in their indices. The adjoint representation breaks according to

$$\text{GL}(7) \subset \text{E}_{7(7)}: \mathbf{133} \to 7'_{+4} + 35_{+2} + 1_0 + 48_0 + 35'_{-2} + 7_{-4}. \qquad (4.2)$$

The GL(1) grading of these decompositions shows immediately that

$$(t_\alpha)^{mn} = 0, \qquad (4.3)$$

since there is no generator of charge $+6$ in the adjoint representation. Consequently, the section constraints (1.1) are solved by truncating the coordinate dependence of all fields and gauge parameters to the coordinates in the 7_{+3},

$$\Phi(x^\mu, Y^M) \longrightarrow \Phi(x^\mu, y^m), \qquad \text{i.e.} \quad \partial^{mn} \to 0,$$
$$\partial_{mn} \to 0, \qquad \partial^m \to 0. \qquad (4.4)$$

Accordingly, for the compensating gauge field constrained by Eq. (2.34) we set all but the associated seven components $B_{\mu\nu m}$ to zero,

$$B_{\mu\nu}{}^{mn} \to 0, \qquad B_{\mu\nu mn} \to 0, \qquad B_{\mu\nu}{}^m \to 0. \qquad (4.5)$$

The various fields of $D = 11$ supergravity are recovered by splitting the vector fields $A_\mu{}^M$ and the 2-forms $B_{\mu\nu\alpha}$, $B_{\mu\nu M}$ according to Eqs. (4.1) and (4.2), and parametrizing the scalar matrix $\mathcal{M}_{MN} = (\mathcal{V}\mathcal{V}^T)_{MN}$ in terms of a group-valued vielbein $\mathcal{V}$, defined in triangular gauge according to Ref. [34] as

$$\mathcal{V} \equiv \exp[\phi t_{(0)}]\mathcal{V}_7 \exp[c_{kmn}t^{kmn}_{(+2)}]\exp[\epsilon^{\text{klmnpq}r}c_{\text{klmnpq}}t_{(+4)r}]. \qquad (4.6)$$

Here, $t_{(0)}$ is the E$_{7(7)}$ generator associated to the GL(1) grading, and $\mathcal{V}_7$ denotes a general element of the SL(7) subgroup, whereas the $t_{(+n)}$ refer to the E$_{7(7)}$ generators of positive grading in Eq. (4.2). All generators are evaluated in the fundamental **56** representation (4.1). Upon choosing an

explicit representation of the generators $(t_\alpha)_M{}^N$ in terms of SL(7)-invariant tensors, splitting all tensors according to Eqs. (4.1) and (4.2), and explicitly imposing Eq. (4.4), the above $E_{7(7)}$-covariant field equations can be mapped into those of $D = 11$ supergravity. This requires redefinitions of all the form fields originating from the 11-dimensional 3-form and 6-form in the usual Kaluza-Klein manner, i.e., flattening the world indices with the elfbein and then "unflattening" with the vierbein $e_\mu{}^a$, as well as subsequent further nonlinear field redefinitions and the appropriate dualization of some field components. We have gone through this exercise in detail in the $E_{6(6)}$-covariant construction [2] and reproduced the full and untruncated action of 11-dimensional supergravity. Here, we will restrict the discussion to illustrating the novel features of the $E_{7(7)}$ case.

The scalar fields $c_{mnk} = c_{[mnk]}$ and $c_{mnklpq} = c_{[mnklpq]}$ parametrizing the matrix $\mathcal{M}_{MN}$ according to Eq. (4.6) have an obvious origin in the internal components of the 11-dimensional 3-form and 6-form. Let us consider the 56 vector fields, which split according to Eq. (4.1) above into

$$\{A_\mu{}^M\} \longrightarrow \{A_\mu{}^m, A_{\mu mn}, A_\mu{}^{mn}, A_{\mu m}\}. \qquad (4.7)$$

The first seven vector fields $A_\mu{}^m$ correspond to the $D = 11$ Kaluza-Klein vectors, whereas the $21 + 21$ components $A_{\mu mn}$ and $A_\mu{}^{mn}$ are related to the corresponding components of the 11-dimensional 3-form and 6-form, respectively. The last seven vector fields $A_{\mu m}$ have no direct appearance in $D = 11$ supergravity, but capture some of the degrees of freedom of its dual graviton. Let us consider their role in some more detail. Evaluating a generic covariant derivative [Eq. (2.28)], upon taking the above solution of the section constraint and using the split (4.7), shows that most of the vector fields only appear under internal derivatives ∂_m; more precisely, out of the 56 vectors $A_\mu{}^M$, the full connection only carries the following combinations of gauge fields:

$$\mathcal{D}(\{A^M\}) = \mathcal{D}(\{A^m, \partial_{[k}A_{mn]}, \partial_k A^{km}\}). \qquad (4.8)$$

In particular, the seven vectors $A_{\mu m}$ drop out from all covariant derivatives. Moreover, a quick counting of the independent vector field components in this connection yields

$$A^m : 7, \qquad \partial_{[k}A_{mn]} : 15, \qquad \partial_k A^{km} : 6. \qquad (4.9)$$

E.g. the 21 components A_{mn} enter the connection (4.8) in a way that is invariant under transformations $A_{mn} \to A_{mn} + \partial_{[m}a_{n]}$ which can be used to set six of these components (say the A_{m7}) to zero, etc. This counting shows that in total $7 + 15 + 6 = 28$ out of the 56 vector fields participate in the connections, a counting that is also consistent with Ref. [27]. This is in precise agreement with the general structure of maximal gauged supergravities [14], in which

at most 28 vector fields participate in the gauging of some non-Abelian symmetry. We may perform an analogous counting of the number of 2-form components from $B_{\mu\nu\alpha}$ that actually appear in the covariant field strengths (2.33) and find

$$\partial_{[m}B_{n]} : 6, \qquad \partial_k B^{kmn} : 15. \qquad (4.10)$$

Together with the seven components surviving in $B_{\mu\nu M}$ after imposing Eq. (4.5), this makes a total of 28 2-forms entering the covariant field strengths $\mathcal{F}_{\mu\nu}{}^M$ and thereby the twisted first-order self-duality equation (3.7) and the action (1.4). Again, this counting is in precise agreement with the general structure of maximal gauged supergravities [14]: the existence of non-Abelian self-duality equations requires a compensating 2-form per vector field participating in the gauging.

In order to reproduce the field equations of $D = 11$ supergravity, second-order field equations for the vector fields can be read off from Eq. (3.11), upon first decomposing the matrix $\mathcal{M}_{MN}$ obtained from Eq. (4.6) according to Eq. (3.10), with a specific choice of symplectic frame. Alternatively, 21 of the first-order self-duality equation (3.7) can be mapped directly to the corresponding components of the $D = 11$ duality equations between a 3-form and a 6-form. The seven remaining self-duality equations are those featuring the vector field $A_{\mu m}$ which has no origin in the standard formulation of $D = 11$ supergravity and rather corresponds to components of the $D = 11$ dual graviton. Only their derivatives (such that $A_{\mu m}$ drops from the equations) can be matched to the $D = 11$ second-order field equations. In the $E_{7(7)}$-covariant formulation, these equations exist as first-order duality equations by virtue of the surviving components $B_{\mu\nu m}$ of the covariantly constrained fields $B_{\mu\nu M}$ (4.5), which play the role of compensating tensor gauge fields.

Let us finally briefly discuss the embedding of IIB supergravity. Just as for the $E_{6(6)}$ EFT [1,2], there is another inequivalent solution to the section conditions (1.1) that describes the embedding of the full ten-dimensional IIB theory [35,36] into the $E_{7(7)}$ EFT.[4] In this case, the relevant maximal subgroup of $E_{7(7)}$ is $GL(6) \times SL(2)$, under which the fundamental and adjoint representation decompose according to

$$\mathbf{56} \to (6, 1)_{+2} + (6', 2)_{+1} + (20, 1)_0 + (6, 2)_{-1}$$
$$+ (6', 1)_{-2},$$

$$\mathbf{133} \to (1, 2)_{+3} + (15', 1)_{+2} + (15, 2)_{+1} + (35 + 1, 1)_0$$
$$+ (15', 2)_{-1} + (15, 1)_{-2} + (1, 2)_{-3}, \qquad (4.11)$$

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[4]An analogous solution of the SL(5)-covariant section condition, corresponding to some three-dimensional truncation of type IIB, was discussed recently in the truncation of the theory to its potential term [37].

with the subscript denoting the GL(1) charge. With the corresponding split of coordinates and vector fields[5]

$$\{Y^M\} \to \{y^m, y_{ma}, y_{kmn}, y^{ma}, y_m\},$$
$$\{A_\mu{}^M\} \to \{A_\mu{}^m, A_{\mu ma}, A_{\mu kmn}, A_\mu{}^{ma}, A_{\mu m}\}, \qquad (4.12)$$

it follows as above that the constraints (1.1) and (2.34) are solved by restricting the coordinate dependence of all fields to the six coordinates y^m [of highest GL(1) charge], and setting all but the associated six components of $B_{\mu\nu M}$ to zero,

$$\partial^{ma} \to 0, \quad \partial^{kmn} \to 0, \quad \partial_{ma} \to 0, \quad \partial^m \to 0,$$
$$B^{ma} \to 0, \quad B^{kmn} \to 0, \quad B_{ma} \to 0, \quad B^m \to 0. \qquad (4.13)$$

The set of IIB fields and equations of motion is recovered upon choosing an explicit representation of the generators $(t_\alpha)_M{}^N$ in terms of SL(6) × SL(2)-invariant tensors, splitting all fields and tensors according to Eq. (4.11), and explicitly imposing Eq. (4.13). As above, this requires the standard Kaluza-Klein redefinitions together with additional nonlinear redefinitions of all the form fields and the appropriate dualization of some field components. The scalar matrix $\mathcal{M}_{MN} = (\mathcal{V}\mathcal{V}^T)_{MN}$ in this case is most conveniently parametrized in terms of a group-valued vielbein $\mathcal{V}$, defined in triangular gauge as

$$\mathcal{V} \equiv \exp[\phi t_{(0)}] \mathcal{V}_6 \mathcal{V}_2 \exp\left[c_{mna} t^{mna}_{(+1)}\right]$$
$$\times \exp\left[\epsilon^{klmnpq} c_{klmn} t_{(+2)\,pq}\right] \exp\left[c_a t^a_{(+3)}\right]. \qquad (4.14)$$

Here, $t_{(0)}$ is the E$_{7(7)}$ generator associated to the GL(1) grading, $\mathcal{V}_6$ and $\mathcal{V}_2$ denote general elements of the SL(6) and SL(2) subgroups, respectively, and the $t_{(+n)}$ refer to the E$_{7(7)}$ generators of positive grading in Eq. (4.11). All generators are evaluated in the fundamental 56 representation. The scalar fields $c_{mna} = c_{[mn]a}$ and c_a in Eq. (4.14) descend from the internal components of the ten-dimensional 2-form doublet and its dual 6-form doublet. In turn, c_{klmn} has its origin in the internal components of the (self-dual) 4-form. From the 56 vector fields, split according to Eq. (4.12), the first six vector fields $A_\mu{}^m$ correspond to the $D = 10$ Kaluza-Klein vectors, whereas the 44 components $A_{\mu ma}$, $A_{\mu kmn}$, and $A_\mu{}^{ma}$ are related to the corresponding components of the ten-dimensional p-forms. Again, the last six vector fields $A_{\mu m}$ have no direct appearance in IIB supergravity, but capture some of the degrees of freedom of its dual graviton. Evaluating a generic covariant derivative [Eq. (2.28)] with Eqs. (4.13) and (4.12) shows that these six vectors drop out from all covariant derivatives. More

precisely, the full connection only carries the following combinations of gauge fields:

$$\mathcal{D}(\{A^M\}) = \mathcal{D}(\{A^m, \partial_{[m}A_{n]a}, \partial_{[k}A_{lmn]}, \partial_m A^{ma}\}). \qquad (4.15)$$

A counting of the independent components similar to Eq. (4.9),

$$A^m : 6, \qquad \partial_{[m}A_{n]a} : 2 \cdot 5, \qquad \partial_{[k}A_{lmn]} : 10, \qquad \partial_m A^{ma} : 2, \qquad (4.16)$$

shows that for IIB there are also precisely 28 out of the 56 vector field components which appear in the connections (4.15). Similarly, an evaluation of the expressions (2.33) shows that in this case from the 133 $B_{\mu\nu\alpha}$, only the combinations

$$\partial_m B_a : 2, \qquad \partial_{[k}B_{mn]} : 10, \qquad \partial_k B^{kma} : 2 \cdot 5 \qquad (4.17)$$

appear in the covariant field strengths (2.33) and the action (1.4). Together with the six components surviving in $B_{\mu\nu M}$ after imposing Eq. (4.13) this again makes a total of 28 2-forms entering the twisted first-order self-duality equation (3.7) as compensating tensor gauge fields.

The first- and second-order field equations of type IIB supergravity are obtained from Eqs. (1.3) and (1.4) with the above split of fields, the constraints (4.13), the field redefinitions and an appropriate dualization. Again, we note that the six self-duality equations from Eq. (1.3) featuring the vector fields $A_{\mu m}$ have no direct origin in IIB, as these vector fields correspond to components of the $D = 10$ dual graviton. Only their derivatives (such that $A_{\mu m}$ drops out from the equations) can be matched to the standard IIB field equations.

V. SUMMARY AND OUTLOOK

In this paper we have spelled out the details of the E$_{7(7)}$ exceptional field theory. The main conceptual novelty of this case as compared to E$_{6(6)}$ is that, from the 11-dimensional perspective, the seven Kaluza-Klein vectors are introduced together with their on-shell duals, satisfying an electric-magnetic twisted self-duality relation. These on-shell duals thus correspond, again from the 11-dimensional perspective, to components of the dual graviton. Despite the no-go theorems of Refs. [22,23], it is possible to consistently include those fields in a nonlinear theory by virtue of the simultaneous inclusion of compensating (2-form) gauge fields. This naturally follows from the structure of the tensor hierarchy, and also gives a duality-covariant form of the mechanism introduced in Ref. [24]. A crucial aspect of this mechanism is that the compensating gauge field itself is covariantly constrained in that it needs to satisfy E$_{7(7)}$-covariant constraints that are of the same structural form as the section constraints.

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[5]Indices $m, n = 1, \ldots, 6$ and $a = 1, 2$, label the fundamental representations of SL(6) and SL(2), respectively. The coordinates $y_{kmn} = y_{[kmn]}$ and vector fields $A_{\mu kmn} = A_\mu$ are antisymmetric in all their internal indices.

Although a deeper conceptual understanding of these constrained fields is certainly desirable, we have seen in the above construction of a fully $E_{7(7)}$-covariant formulation that their presence appears unavoidable. Recall that the need for such constrained 2-forms was an immediate consequence of the algebraic structure of the $E_{7(7)}$ E-bracket Jacobiator. Equivalently, these fields were found indispensable for the definition of a gauge-covariant field strength [Eq. (2.33)] for the vector fields. As we have discussed, this nicely fits into a more general pattern of the tensor hierarchy of exceptional field theories: for the $E_{6(6)}$ theory of Ref. [2] the necessity of introducing additional constrained compensating fields appears at the level of 3-forms (which, however, do not appear explicitly in the action). Similarly, in $E_{8(8)}$ EFT the compensating gauge field appears among the vector fields and can be viewed as an $E_{8(8)}$ gauge potential, again subject to $E_{8(8)}$-covariant constraints as found for the Ehlers $SL(2, \mathbb{R})$ subgroup in Ref. [25]. Its presence also cures the seeming obstacle of nonclosure of the algebra of generalized diffeomorphisms [27].[6] It is intriguing to observe that this purely group-theoretical origin of the constrained compensator fields in the tensor hierarchy precisely matches (and in fact enables) the appearance of components of the 11-dimensional dual graviton field among the physical fields of the exceptional field theories. E.g. we have seen in Eq. (4.5) that in the embedding of 11-dimensional supergravity the constraint (2.34) implies that all but seven components of the compensating 2-form are identically zero. These nonvanishing components are precisely those that couple via Eq. (2.33) to the field strengths associated to the seven vector fields originating from the 11-dimensional dual graviton. One may speculate that eventually the constraints, both the section constraint involving the coordinates and the constraints on the compensator fields, may be relaxed so that in particular the physical significance of the dual graviton may become more transparent.

11-dimensional and type IIB supergravity naturally embed into the $E_{7(7)}$ EFT, as discussed in Sec. 4. We leave a more detailed description of this embedding at the level of the action or the field equations to future work. This should include the formulation of the fermionic sector and the supersymmetry transformations, which in turn will also clarify the relation to the reformulation of de Wit and Nicolai [38]. A natural question, among many, then is which gauged $\mathcal{N} = 8$ supergravities can be embedded, via the $E_{7(7)}$ EFT, into 11-dimensional supergravity and which

may require the extended 56 $E_{7(7)}$ coordinates in a nontrivial fashion (perhaps after a suitable relaxation of the constraints). We leave these and other questions for future work.

APPENDIX: ALGEBRAIC RELATIONS

In this appendix we collect a few important $E_{7(7)}$ relations. First, contracting the adjoint indices of two generators, we have the relation

$$(t_\alpha)_M{}^K(t^\alpha)_N{}^L = \frac{1}{24}\delta_M^K\delta_N^L + \frac{1}{12}\delta_M^L\delta_N^K + (t_\alpha)_{MN}(t^\alpha)^{KL}$$
$$- \frac{1}{24}\Omega_{MN}\Omega^{KL} \tag{A1}$$

for the projector onto the adjoint representation. Contracting two of the fundamental indices, the relation (a1) gives

$$(t_\alpha)_M{}^K(t^\alpha)_K{}^N = \frac{19}{8}\delta_M^N. \tag{A2}$$

There are also various higher-order relations among the generators, which we list as

$$0 = 9(t^\alpha)_M{}^K(t^\beta)_{KN}(t_\alpha)^{(PQ}(t_\beta)^{RS)} + 2(t^\alpha)_{[M}{}^{(P}(t_\alpha)^{QR}\delta_{N]}^{S)}$$
$$- \frac{1}{8}\Omega_{MN}(t^\alpha)^{(PQ}(t_\alpha)^{RS)},$$

$$0 = (t^\alpha)_{NL}(t_\alpha)^{M(K}(t_\beta)^{Q)L} + \frac{1}{12}(t_\beta)^{M(K}\delta_N^{Q)}$$
$$- \frac{1}{24}(t_\beta)_N{}^{(K}\Omega^{Q)M} + \frac{1}{24}(t_\beta)^{KQ}\delta_N^M$$
$$+ \frac{1}{2}(t^\alpha)_{NL}(t_\alpha)^{KQ}(t_\beta)^{ML} - \frac{1}{2}(t^\alpha)^{ML}(t_\alpha)^{KQ}(t_\beta)_{NL}, \tag{A3}$$

and their contraction

$$(t^\alpha)^{ML}(t_\alpha)^{NQ}(t_\beta)_{NL} = -\frac{7}{8}(t_\beta)^{MQ}. \tag{A4}$$

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[6]The details for the $E_{8(8)}$ EFT will be presented in a separate publication.

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