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Robust quantum data locking from phase modulationCosmo Lupo,¹ Mark M. Wilde,² and Seth Lloyd^{1,3}¹*Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*²*Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA*³*Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

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Quantum data locking is a uniquely quantum phenomenon that allows a relatively short key of constant size to (un)lock an arbitrarily long message encoded in a quantum state, in such a way that an eavesdropper who measures the state but does not know the key has essentially no information about the message. The application of quantum data locking in cryptography would allow one to overcome the limitations of the one-time pad encryption, which requires the key to have the same length as the message. However, it is known that the strength of quantum data locking is also its Achilles heel, as the leakage of a few bits of the key or the message may in principle allow the eavesdropper to unlock a disproportionate amount of information. In this paper we show that there exist quantum data locking schemes that can be made robust against information leakage by increasing the length of the key by a proportionate amount. This implies that a constant size key can still lock an arbitrarily long message as long as a fraction of it remains secret to the eavesdropper. Moreover, we greatly simplify the structure of the protocol by proving that phase modulation suffices to generate strong locking schemes, paving the way to optical experimental realizations. Also, we show that successful data locking protocols can be constructed using random code words, which very well could be helpful in discovering random codes for data locking over noisy quantum channels.

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I. INTRODUCTION

Quantum data locking (QDL) is a uniquely quantum protocol that provides one of the strongest violations of classical information theory in the quantum setting [1]. In QDL the knowledge of a relatively short key of constant size allows one to (un)lock an arbitrarily long message encoded in a quantum system. Otherwise, without knowledge of the key, only a negligible amount of information about the message can be accessed [2–6]. Such an exponential disproportion between the length of the key and that of the message is impossible in classical information theory, according to which the bits of secret key should be at least as many as the bits of encrypted information [7]. Although cryptographic applications may seem natural, it was recognized early that the security provided by QDL is in general not robust under the leakage of a small fraction of the key or the message. Indeed, as a relatively short key is sufficient to lock a much longer message, it may very well happen that the leakage to the eavesdropper of a few bits is sufficient to unlock a disproportionate amount of information. Here we analyze this issue and show that there exist QDL schemes that can be made resilient against information leakage at the cost of increasing the length of the secret key by a proportional amount. We show that to protect QDL from the leakage of n bits of the key or the message, it is sufficient to add an overhead of about n bits to the secret key.

The security of QDL is based on the *accessible information* criterion. It is well known that such a criterion does not guarantee composable security [8]. That is, security is in general not granted if QDL is used as a subroutine of another protocol. To avoid this problem, one should make an assumption on Eve's technological capability, and require that the message exchanged by QDL is not used for the next

protocol until Eve's quantum memory expires. For instance, composable security is granted if the eavesdropper has no quantum memory and is hence forced to measure her share of the quantum system as soon as she obtains it. Such additional assumptions make the accessible information criterion weaker than the commonly accepted security criterion for quantum private communication [8], which is instead related to the *Holevo information*. Interestingly enough, a large gap exists between these two security criteria that may allow for high rate QDL through quantum channels with poor privacy [9,10]. As a matter of fact, explicit examples of channels with low or even zero privacy that allow for QDL at high rates have been recently provided [11,12]. It is hence of fundamental interest to seek a deeper understanding of the phenomenon of QDL, especially in the presence of noise.

A number of QDL methods exist. In some of the best known, the code words are created by applying uniformly distributed random unitary operations to the vectors of a given orthonormal basis [2,3,5,6]. The role of the random unitaries is to scramble the code words in such a way that an eavesdropper has essentially no information about the message. The crucial feature of strong QDL schemes is that the number of scrambling unitaries is much smaller (in fact exponentially smaller) than the number of different messages [13]. Although a scheme that can be implemented efficiently on a quantum computer exists [5], the realization of QDL with currently available technologies still presents severe experimental difficulties. Moreover, all known QDL protocols are defined for a noiseless quantum system. Hence, a problem of fundamental importance is to design protocols capable of performing QDL through noisy quantum channels. (Explicit protocols for QDL through noisy channels have been recently introduced in [12].)

Here we present two QDL protocols based on random phase shifts. The first one is based on random vectors (instead of random bases) of the form [17]

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{\omega=1}^d e^{i\theta(\omega)} |\omega\rangle, \quad (1)$$

where $\{|\omega\rangle\}_{\omega=1,\dots,d}$ is a given orthonormal basis in a d -dimensional Hilbert space, and $e^{i\theta(\omega)}$ are independent and identically distributed (i.i.d.) random phases with zero mean, $\mathbb{E}[e^{i\theta(\omega)}] = 0$. Even the choice $e^{i\theta(\omega)} = (-1)^{b(\omega)}$ is sufficient, where $b(\omega)$ are random binary variables. We show that codewords sampled from this “phase ensemble” of vectors suffice to build strong QDL schemes. It is worth noticing that, compared to previously known QDL protocols that require the preparation of uniform (Haar distributed) basis vectors, our scheme greatly simplifies the structure of codewords for QDL and represents a major simplification for optical experimental implementations.

Moreover, the expansion of the set of QDL codewords given in this paper paves the way to the application of random coding techniques to lock classical information into noisy quantum channels [9,10], which requires the codewords used to hide information to coincide with the codewords used to protect information from noise. Codewords randomly selected from an ensemble that attains the coherent information bound suffice to protect information from noise [20–25]. This paper shows that such codewords also suffice to keep information secure from an eavesdropper.

The paper proceeds as follows. In Sec. II we describe a QDL protocol where the codewords are obtained by applying a random phase modulation. Section III provides a sketch of the proof of the QDL property of our protocol, while details are provided in Appendix A. Then, Sec. IV proves the robustness of QDL to loss of information to the eavesdropper. Section V discusses possible applications and experimental realizations of our protocol in quantum optics. Finally, conclusions are presented in Sec. VI.

II. QUANTUM DATA LOCKING FROM PHASE MODULATION

In a typical QDL protocol, the legitimate parties, Alice and Bob, publicly agree on a set of $n = MK$ codewords in a d -dimensional quantum system. From this set, they then use a short shared private key of $\log K$ bits to select a set of M codewords that they will use for sending information. If an eavesdropper Eve does not know the private key, then the number of bits—as quantified by the *accessible information* I_{acc} , which is defined as the maximum mutual information between the message and Eve’s measurement result—that she can obtain about the message by intercepting and measuring the state sent by Alice is essentially equal to zero for certain choices of codewords.

We consider here two QDL protocols where Alice and Bob are able to communicate via a d -dimensional noiseless quantum channel. In the first QDL protocol, to encrypt a message $m \in \{1, \dots, M\}$, Alice prepares one of the vectors

$$|\psi_{mk}\rangle = \frac{1}{\sqrt{d}} \sum_{\omega=1}^d e^{i\theta_{mk}(\omega)} |\omega\rangle, \quad (2)$$

where the value of $k \in \{1, \dots, K\}$ is determined by the secret key, and the vectors are sampled i.i.d. from the phase ensemble defined above.

We require that Bob, knowing the key, is able to decode with a probability of success close to 1. That is, for any k there exists a positive operator-valued measurement (POVM) with elements $\{\Lambda_m^{(k)}\}$ such that

$$\bar{p}_{\text{succ}} = \frac{1}{M} \sum_{m=1}^M \text{Tr}(\Lambda_m^{(k)} |\psi_{mk}\rangle \langle \psi_{mk}|) \geq 1 - \epsilon. \quad (3)$$

On the other hand, we require that if Eve intercepts and measures the state $|\psi_{mk}\rangle$, then she will only be able to access a negligible amount of information about the message m , as quantified by the accessible information I_{acc} . We require that [26]

$$I_{\text{acc}} \lesssim \delta \log M. \quad (4)$$

Furthermore, for a key-efficient QDL scheme we demand that $K \ll M$. In particular, we require that $\log K = O(\log \log M)$ and that $\delta \rightarrow 0$ as K increases.

Here we show that a set of MK codewords randomly selected from the phase ensemble will define a QDL protocol with probability arbitrarily close to 1 for d large enough. To show that, we make repeated applications of the following bound on the largest and smallest eigenvalues of a random matrix.

Theorem 1 [29]. Consider a $d \times n$ matrix W , whose entries are independent and identically distributed random variables with zero mean, variance σ^2 , and finite fourth moment. Define $X = (1/n)WW^\dagger$. Then almost surely as $d \rightarrow \infty$, the largest eigenvalue of $X \rightarrow (1 + \sqrt{y})^2 \sigma^2$, where $y = d/n$. In addition, when $d \leq n$, the smallest eigenvalue of $X \rightarrow (1 - \sqrt{y})^2 \sigma^2$.

To apply this theorem to our case, let

$$W = \sum_{j=1}^n |\psi_j\rangle \langle e_j|, \quad (5)$$

where $|\psi_j\rangle$ are n random vectors from the phase ensemble, and the set $\{|e_j\rangle\}$ is an orthonormal basis for an auxiliary n -dimensional Hilbert space. Notice that the elements of W are just the components of the randomly selected codewords $|\psi_j\rangle$ in the basis $\{|\omega\rangle\}_{\omega=1,\dots,d}$. We have

$$X = \frac{1}{n} WW^\dagger = \frac{1}{n} \sum_{j=1}^n |\psi_j\rangle \langle \psi_j|, \quad (6)$$

and for the phase ensemble $\sigma^2 = 1/d$.

For finite d , we use another result from random matrix theory.

Theorem 2 [30]. The probability that the largest eigenvalue of X is larger than $[(1 + \sqrt{y})^2 + \delta]\sigma^2$ is no greater than $C \exp(-d\delta^{3/2}/C)$, where C is a constant. Similarly, if $d < n$, the probability that the smallest eigenvalue is less than $[(1 - \sqrt{y})^2 - \delta]\sigma^2$ is less than $C/(1 - \sqrt{y}) \exp(-d\delta^{3/2}/C)$.

This theorem states that the probability that the bounds of Theorem 1 are violated by more than δ vanishes exponentially with d and δ .

First of all, using these results, we can easily show that for $M \ll d$ Bob's average probability of successful decoding is $\bar{p}_{\text{succ}} \gtrsim 1 - 2\sqrt{M/d}$. To see that, assume that Bob applies the "pretty good measurement" POVM with elements

$$\Lambda_m^{(k)} = \Sigma_k^{-1/2} |\psi_{mk}\rangle \langle \psi_{mk}| \Sigma_k^{-1/2}, \quad (7)$$

where

$$\Sigma_k = \sum_{m=1}^M |\psi_{mk}\rangle \langle \psi_{mk}|. \quad (8)$$

Then we have, assuming $\delta \ll 1$,

$$\bar{p}_{\text{succ}} = \frac{1}{M} \sum_{m=1}^M |\langle \psi_{mk}| \Sigma_k^{-1/2} |\psi_{mk}\rangle|^2 \quad (9)$$

$$\geq \frac{d}{M} \left[\left(1 + \sqrt{\frac{d}{M}}\right)^2 + \delta \right]^{-1} \quad (10)$$

$$\simeq 1 - 2\sqrt{\frac{M}{d}}, \quad (11)$$

where in (10) we have applied Theorems 1 and 2 to bound

$$\Sigma_k^{-1/2} \geq \sqrt{\frac{d}{M}} \left[\left(1 + \sqrt{\frac{d}{M}}\right)^2 + \delta \right]^{-1/2}. \quad (12)$$

On the other hand, the bound on Eve's accessible information is given by the following.

Theorem 3. Select MK i.i.d. random codewords $|\psi_{mk}\rangle$ ($m = 1, \dots, M$ and $k = 1, \dots, K$) from the phase ensemble, with $MK \gg d$ and $M \ll d$. Then, for any $\delta > 0$ and

$$K > \frac{4}{\delta^2} \left(\ln M + \frac{2d}{\delta M} \ln \frac{5}{\delta} \right), \quad (13)$$

Eve's accessible information satisfies

$$I_{\text{acc}} = O(\delta \log d) \quad (14)$$

up to a probability

$$p_{\text{fail}} \leq \exp \left[-M \left(\frac{\delta^3 K}{4} - \delta \ln M - \frac{2d}{M} \ln \frac{5}{\delta} \right) \right] \quad (15)$$

that vanishes exponentially in M .

The sketch of the proof is provided in Sec. III, while details are in Appendix A.

This theorem states that a set of MK random codewords from the phase ensemble defines a strong QDL protocol. For instance, if we put $\delta \simeq 1/\log M$, then I_{acc} is smaller than a constant with $\log K = O(\log \log M)$. Otherwise, for $\delta \ll 1/\log d$, a secret key of size $\log K = O(\log 1/\delta)$ is sufficient to guarantee $I_{\text{acc}} = O(\delta \log d)$.

Quantum data locking from random unitaries

The second QDL protocol is defined by a set of random unitaries of a particular form. We define the "phase ensemble" of unitaries of the form

$$U = \sum_{\omega=1}^d e^{i\theta(\omega)} |\omega\rangle \langle \omega|, \quad (16)$$

where $\theta(\omega)$ are i.i.d. random phases with $\mathbb{E}[e^{i\theta(\omega)}] = 0$. Given the set of Fourier-transformed basis vectors

$$|m\rangle = \frac{1}{\sqrt{d}} \sum_{\omega=1}^d e^{i2\pi m\omega/d} |\omega\rangle, \quad (17)$$

for $m = 1, \dots, d$, we define a set of dK codewords as

$$|\psi_{mk}\rangle = U_k |m\rangle = \frac{1}{\sqrt{d}} \sum_{\omega=1}^d e^{i2\pi m\omega/d + i\theta_k(\omega)} |\omega\rangle. \quad (18)$$

Notice that for any k , codewords with different m are mutually orthogonal. This implies that Bob can decode with $\bar{p}_{\text{succ}} = 1$. Furthermore, each codeword in (17) has the same distribution of the codewords selected from the phase ensemble of vectors. The only difference is that $|\psi_{mk}\rangle$ and $|\psi_{m'k}\rangle$ are no longer statistically independent. We thus obtain the following.

Theorem 4. Select K i.i.d. random unitaries U_k ($k = 1, \dots, K$) from the phase ensemble and define the codewords $|\psi_{mk}\rangle = U_k |m\rangle$ ($m = 1, \dots, d$). Then, for any $\delta > 0$ and

$$K > \frac{4}{\delta^2} \left(\ln d + \frac{2}{\delta} \ln \frac{5}{\delta} \right), \quad (19)$$

Eve's accessible information satisfies

$$I_{\text{acc}} = O(\delta \log d) \quad (20)$$

up to a probability that vanishes exponentially in d .

The proof of this theorem can be obtained by a straightforward modification of that of Theorem 3 and is not reported here.

It is worth noticing that these unitaries form an Abelian group. It is hence somehow surprising that they yield strong QDL properties.

III. SKETCH OF THE PROOF OF THEOREM 3

To compute Eve's accessible information about the message, we associate to the sender Alice a dummy M -dimensional quantum system carrying the classical variable m as a set of basis vectors $\{|m\rangle\}_{m=1, \dots, M}$. We then suppose that Eve intercepts the quantum system that has the encrypted message. Since Eve does not know the secret key, the correlations between Alice and Eve are described by the following classical-quantum state:

$$\rho_{AE} = \frac{1}{M} \sum_{m=1}^M |m\rangle \langle m|_A \otimes \frac{1}{K} \sum_{k=1}^K |\psi_{mk}\rangle \langle \psi_{mk}|_E, \quad (21)$$

where the codewords $|\psi_{mk}\rangle$ are as in Eq. (2).

The accessible information is by definition the maximum classical mutual information that can be achieved by local measurements on Alice's and Eve's subsystems:

$$\begin{aligned} I_{\text{acc}} &:= I_{\text{acc}}(A; E)_\rho \\ &= \max_{\mathcal{M}_A, \mathcal{M}_E} H(X) + H(Y) - H(X, Y), \end{aligned} \quad (22)$$

where $\mathcal{M}_A : A \rightarrow X$, $\mathcal{M}_E : E \rightarrow Y$ are local measurements with output variables X and Y , respectively, and $H(\cdot)$ denotes the Shannon entropy of the measurement results.

The optimal measurement on Alice's subsystem is obviously a projective measurement on the basis $\{|m\rangle\}_{m=1, \dots, M}$.

Concerning Eve's measurement, it is sufficient to consider only rank-one POVM, which are described by measurement operators of the form $\{\mu_j |\phi_j\rangle\langle\phi_j|\}_{j=1,\dots,J}$, with $\mu_j \geq 0$, and satisfying the normalization condition $\sum_j \mu_j |\phi_j\rangle\langle\phi_j| = \mathbb{I}$. A straightforward calculation then yields

$$I_{\text{acc}} = \log M - \min_{\{\mu_j |\phi_j\rangle\langle\phi_j|\}} \sum_j \frac{\mu_j}{M} \left\{ H[Q(\phi_j)] - \eta \left[\sum_{m=1}^M Q_m(\phi_j) \right] \right\}, \quad (23)$$

where $\eta(x) = -x \log x$, $Q(\phi)$ is the M -dimensional real vector of non-negative components

$$Q_m(\phi) = \frac{1}{K} \sum_{k=1}^K |\langle\phi|\psi_{mk}\rangle|^2, \quad (24)$$

and $H[Q(\phi)] = -\sum_{m=1}^M Q_m(\phi) \log Q_m(\phi)$ denotes the Shannon entropy of $Q(\phi)$.

We now apply a standard convexity argument, first used in [1]. To do that, notice that assuming $\langle\phi_j|\phi_j\rangle = 1$, then $\sum_j \mu_j/d = 1$. This implies that the positive quantities μ_j/d can be interpreted as probability weights. An upper bound on the accessible information is then obtained by using the fact that the average cannot exceed the maximum. We thus obtain

$$I_{\text{acc}} \leq \log M - \frac{d}{M} \min_{|\phi\rangle} \left\{ H[Q(\phi)] - \eta \left[\sum_{m=1}^M Q_m(\phi) \right] \right\}. \quad (25)$$

Notice that now the maximization is no longer over a POVM with elements $\{\mu_j |\phi_j\rangle\langle\phi_j|\}$ but over a single normalized vector $|\phi\rangle$.

Then the proof proceeds along three main conceptual steps:

(1) (See Appendix A1 for details.) Theorems 1 and 2 imply that the random variable $\sum_{m=1}^M Q_m(\phi)$ is close to M/d with arbitrarily high probability for all vectors $|\phi\rangle$ if d is large enough and $MK \gg d$. In particular, the inequality

$$\sum_{m=1}^M Q_m(\phi) \leq \frac{M}{d} \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right] \quad (26)$$

applied to Eq. (25) yields

$$I_{\text{acc}} \leq \alpha \log d + \eta(\alpha) - \frac{d}{M} \min_{|\phi\rangle} H[Q(\phi)], \quad (27)$$

where

$$\alpha = \left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta. \quad (28)$$

(2) (See Appendix A2 for details.) We show that for any given $|\phi\rangle$ and $\delta > 0$,

$$\eta[Q_m(\phi)] \geq -\frac{1-\delta}{d} \log \left(\frac{1-\delta}{d} \right) \quad (29)$$

for at least $(1-\delta)M$ values of m (up to a probability exponentially small in M). To do that, we show that there is a negligible probability that $Q_m(\phi) < (1-\delta)/d$ for more

than $\ell = \delta M$ values of m . This result implies

$$H[Q(\phi)] \geq \frac{M}{d} (1-2\delta) \log d. \quad (30)$$

(3) (See Appendix A3 for details.) Finally, to account for the minimum over all unit vectors $|\phi\rangle$, we introduce a discrete set \mathcal{N}_δ of vectors with the property that for any $|\phi\rangle$ there exists $|\phi_i\rangle \in \mathcal{N}_\delta$ such that

$$\| |\phi\rangle\langle\phi| - |\phi_i\rangle\langle\phi_i| \|_1 \leq \delta. \quad (31)$$

A set with this property is called a δ -net. The δ -net is used to approximate the value of $H[Q(\phi)]$ up to an error of the order of $\delta \log d$. We show that the inequality in (30) holds true with high probability for all unit vectors in the δ -net.

In conclusion we obtain that for any $\delta > 0$ Eve's accessible information satisfies

$$I_{\text{acc}} = O(\delta \log d), \quad (32)$$

up to a probability which is exponentially small in d provided

$$K > \frac{4}{\delta^2} \left(\ln M + \frac{2d}{\delta M} \ln \frac{5}{\delta} \right). \quad (33)$$

IV. ROBUSTNESS AGAINST INFORMATION LEAKAGE

We consider our protocols based on the phase ensemble of random vectors to assess the robustness of QDL against the leakage to Eve of part of the key or the message.

What happens if part of the key is known by Eve? For example, suppose that Eve knows the first $\gamma \log K$ bits of the secret key. Since the remaining $(1-\gamma) \log K$ are still secret and random, it follows that we can still apply Theorem 3 and Eve's accessible information satisfies $I_{\text{acc}} = O(\delta \log d)$ up to a failure probability

$$p'_{\text{fail}} \leq \exp \left[-M \left(\frac{\delta^3 K^{1-\gamma}}{4} - \delta \ln M - \frac{2d}{M} \ln \frac{5}{\delta} \right) \right]. \quad (34)$$

To assess the security of the QDL protocol under the leakage of any fraction of the key, we have to take into account all the possible subsets of $\gamma \log K$ bits. Each of these subsets determines a corresponding subset of $K^{1-\gamma}$ values of the key that remain secret to Eve. The number of ways in which these values can be chosen is $\binom{K}{K^{1-\gamma}}$. Applying the union bound we can hence estimate from (34) the failure probability if any fraction of $\gamma \log K$ bits leaks to Eve:

$$p''_{\text{fail}} \leq \binom{K}{K^{1-\gamma}} p'_{\text{fail}} \leq \exp(K^{1-\gamma} \ln K) p'_{\text{fail}}. \quad (35)$$

Putting $M/d = \delta$, this probability is exponentially small in M under conditions

$$K^{1-\gamma} > \frac{4}{\delta^2} \left(\ln M + \frac{2}{\delta^2} \ln \frac{5}{\delta} \right) \quad (36)$$

and

$$M > K^{1-\gamma} \ln K \left(\frac{\delta^3 K^{1-\gamma}}{4} - \delta \ln M - \frac{2}{\delta} \ln \frac{5}{\delta} \right)^{-1}. \quad (37)$$

In particular, for $\gamma \ll 1$, the condition (36) can be replaced by

$$K \gtrsim \left[\frac{4}{\delta^2} \left(\ln M + \frac{2}{\delta^2} \ln \frac{5}{\delta} \right) \right]^{1+\gamma}. \quad (38)$$

Compared to (13), this condition implies that to make QDL robust against the leakage of a fraction of $\gamma \log K$ bits of the key, one simply needs to use a longer key of about $(1 + \gamma) \log K$ bits. This result shows the existence of QDL schemes that are robust to some loss of key.

What happens if a small fraction of n bits of the message leaks to Eve? Suppose that Eve knows the first n bits of the message, then one has to require that the key is sufficiently long to lock the remaining $\log M - n$ bits of the message. Since the codewords corresponding to the remaining part of the message are still random, we have

$$I_{\text{acc}} = O(\delta(\log M - n)), \quad (39)$$

up to a probability

$$p'_{\text{fail}} \leq \exp \left\{ -M2^{-n} \left[\frac{\delta^3 K}{4} - \delta(\ln M - n) - 2^{n+1} \frac{d}{M} \ln \frac{5}{\delta} \right] \right\}. \quad (40)$$

We apply again the union bound to estimate the failure probability if any subset of n bits of the message leaks to Eve:

$$p''_{\text{fail}} \leq \binom{M}{M2^{-n}} p'_{\text{fail}} \leq \exp(M2^{-n} \ln M) p'_{\text{fail}} \quad (41)$$

$$\leq \exp \left\{ -M2^{-n} \left[\frac{\delta^3 K}{4} - \ln M - \delta(\ln M - n) - 2^{n+1} \frac{d}{M} \ln \frac{5}{\delta} \right] \right\}. \quad (42)$$

For any given n , the latter is exponentially small in M provided

$$K > \frac{4}{\delta^3} \left[\ln M + \delta(\ln M - n) + 2^{n+1} \frac{d}{M} \ln \frac{5}{\delta} \right]. \quad (43)$$

Compared to (13), the last condition implies that to protect QDL against the leakage of n bits of message, the key should be enlarged by $\Delta(\log K) \simeq n$ bits. This result shows the existence of QDL schemes that are robust to plain-text attack. A similar result can be obtained starting from other QDL protocols, e.g., using the results of [6,14] based on the min-entropy of the message.

V. APPLICATIONS

Towards quantum optical realizations. The phase ensemble finds natural applications in the context of linear optics. For instance, codewords belonging to the phase ensemble can be realized by coherently splitting a photon over d modes (e.g., spatial, temporal, orbital angular momentum modes, etc.), then by applying independent random phase shifts to each mode. If information is encoded in the arrival time, the codewords can be prepared by first applying a linear dispersion transformation (see, e.g., [31]) and then random phase shifts at different times.

Concerning Bob decoding, in the case of QDL by the phase ensemble of random unitaries (see Sec. II) it is sufficient to apply the inverse transformation of the one applied by Alice for encoding, then measure by photodetection. Notice that both encoding and decoding operations can be realized by linear passive optical elements and photodetection (for decoding) as discussed in [9]. For the QDL based on random codewords from the phase ensemble of vectors, Bob should in principle decode by applying the pretty good measurement associated to the set of QDL codewords, yet we do not know at the moment an explicit construction.

A crucial requirement to realize our strong QDL protocols in quantum optics is the ability to prepare and manipulate quantum states of light in high dimension. The latter is the goal of cutting edge research and several important milestones have been achieved so far. See, for example, the recent report of entanglement production between two photons in a 100×100 -dimensional Hilbert space [32].

Quantum data locking of noisy channels. Although the phenomenon of QDL has been known for about ten years now [1], only recently has it been reconsidered in the context of noisy quantum channels [9,10]. In particular, the locking capacity of a (noisy) quantum channel has been defined in [10] as the maximum rate at which classical information can be locked through N instances of the channel with the assistance of a secret key shared by Alice and Bob which grows sub-linearly in N . One can indeed define a weak and a strong notion of locking capacity [10]. In the weak case one assumes that Eve has access to the channel environment (the output of the complementary channel), while in the strong case one gives her access to the quantum system being input to the noisy channel. Remarkably, there are examples of quantum channels whose locking capacity is much larger than the private capacity [11,12].

Our result on Eve's accessible information applies directly to the strong notion of locking capacity and can be extended (by a simple application of a data processing inequality) to the weak case. It hence remains to show how and at which rate Bob can decode reliably. One way to do that is by concatenating the QDL protocol with an error correcting code [5,10,14]. Consider $N \gg 1$ uses of a quantum channel. If the quantum capacity of the channel is Q , then one can lock information by choosing codewords in an error correcting subspace of dimension $d \simeq 2^{NQ}$. Another approach may consist in designing a code which allows for both QDL and error correction at the same time. Our results indicate that random codes exist that can be applied both for protecting against decoherence and for locking classical information. (An explicit example has been recently presented in [12].)

Locking a quantum memory. The QDL properties of the phase ensemble can be used to lock information into a quantum memory by applying a local random gauge field. Consider an ideal (noiseless) semiclassical model for a quantum memory consisting of d charged particles on a ring of length L . For recording locked information in the quantum memory, Alice applies a random i.i.d. magnetic field to each particle and encodes a classical message into one of the momentum eigenstates $|p\rangle = d^{-1/2} \sum_{k=1}^d \exp[i2\pi px/L + i \int_0^x A(x') dx'] |x\rangle$, where $A(x')$ is the vector

potential in natural units. Notice that the application of the random local fields corresponds to a random phase in the momentum eigenstates. Then, a legitimate receiver who knows the pattern of the magnetic field applied by Alice, can decode the message by simply measuring the momentum. On the other hand, Eve’s accessible information can be made negligibly small by the QDL effect.

VI. CONCLUSION

It is well known that the security provided by QDL can be severely hampered if even a small fraction of the key or the message leaks to Eve. Here we show that, although this is true in general, there exist QDL protocols that are instead robust against the leakage of a small part of the message or the key.

Until now, the codewords used in QDL have been restricted to either Haar-distributed random bases [2,5,6] or approximate mutually unbiased bases [5] (the role of mutually unbiased bases being not yet completely understood [1,33]). This paper showed that codewords modulated by random phase shifts in a given basis suffice to guarantee strong and robust QDL properties.

The fact that random phases suffice to ensure strong QDL properties yields a major simplification for the experimental realization of a QDL protocol. To produce states from the phase ensemble, one only requires to generate d binary phase shifts, instead of d^2 random variables sampling Haar distribution of unitary transformations. The phase ensemble is well adapted for use in quantum optical channels, where a single photon may be coherently split across different modes (e.g., path or time-bin modes), to which i.i.d. random phase shifts are applied. Alternatively, one can employ random unitaries from a set of dispersive transformations [31]. To decode the message, the legitimate receiver can first apply the inverse transformation of the encoding one (both are linear passive transformations), then measure by photodetection [9].

Our results suggest that random codes of the type defined here can be used to perform direct QDL over noisy quantum channels [9,10], which requires that the codewords for QDL (encoding for security) also be appropriate codewords for combating noise on the channel (encoding for error correction). A straightforward way for doing that is to concatenate the QDL protocol with a quantum error correcting code [14], allowing for a rate of locked communication at least equal to the quantum capacity of the channel. A fundamental question is whether one can lock information at a rate higher than the quantum (and private) capacity. Such a question has indeed a positive answer, as shown by the results recently presented in [11,12], providing examples of quantum channels with low or zero privacy that instead allow for high rate QDL. In particular, the phase ensemble was originally proposed in [20] as a code for attaining the coherent information rate for reliable quantum communication over a quantum channel (see also [23–25]). The results proved here suggest that there exist random codes defined from the phase ensemble that allow for robust QDL over a noisy, lossy bosonic channel at a rate equal to and possibly larger than the coherent information.

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APPENDIX A: PROOF OF THEOREM 3

We now proceed by describing in detail the three main steps of the proof of Theorem 3 sketched in Sec. III.

1. Random matrix theory

We apply Theorems 1 and 2 with $n = MK$ and

$$W = \sum_{m=1}^M \sum_{k=1}^K |\psi_{mk}\rangle \langle e_{mk}| \tag{A1}$$

to obtain that

$$\sum_{m=1}^M Q_m(\phi) \leq \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right] \frac{M}{d} \tag{A2}$$

for all unit vectors $|\phi\rangle$, up to a probability exponentially small in d and δ .

Since $\eta(x) = -x \log x$ is a monotonically increasing function near zero and $M/d \ll 1$, Eq. (A2) yields

$$\begin{aligned} \frac{d}{M} \eta \left[\sum_{m=1}^M Q_m(\phi) \right] &\leq \eta \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right] \\ &\quad - \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right] \log \frac{M}{d}, \end{aligned} \tag{A3}$$

which, applied to (25), implies

$$I_{\text{acc}} \leq \alpha \log d + \eta(\alpha) - \frac{d}{M} \min_{|\phi\rangle} H[Q(\phi)], \tag{A4}$$

where

$$\alpha = \left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta. \tag{A5}$$

2. Concentration of the Q_m

For any given $|\phi\rangle$ and $\delta > 0$, we bound the probability that there exist $\ell = \delta M \ll M$ values of m , say m_1, m_2, \dots, m_ℓ , such that

$$\eta[Q_{m_i}(\phi)] < -\frac{1-\delta}{d} \log \left(\frac{1-\delta}{d} \right) \tag{A6}$$

for all $i = 1, 2, \dots, \ell$. This accounts for bounding the probability that $Q_m(\phi)$ takes small values [smaller than $(1-\delta)/d$] as well as the probability of large values [larger than $1 - \eta[(1-\delta)/d] + O(\eta[(1-\delta)/d])$].

Bounding the probability of large $Q_m(\phi)$. We first bound the probability that $Q_m(\phi)$ takes large values. We apply

Theorems 1 and 2 with $n = K$ and $y = d/K$. This yields that the inequality

$$Q_m(\phi) \leq \frac{(1 + \sqrt{d/K})^2 + \delta}{d} \quad (\text{A7})$$

is satisfied for all unit vectors $|\phi\rangle$, up to a probability exponentially small in d and δ . In particular, for $d \gg K$ and $\delta \ll 1$ we obtain

$$Q_m(\phi) \lesssim \frac{1}{K}. \quad (\text{A8})$$

Bounding the probability of small $Q_m(\phi)$. To bound the probability that $Q_m(\phi)$ takes small values we apply the following theorem.

Theorem 5 [34]. Given K i.i.d. positive random variables $X_k \sim X$ such that $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ are finite, then

$$\Pr \left\{ \mathbb{E}[X] - \frac{1}{K} \sum_{k=1}^K X_k > \tau \right\} \leq \exp \left(-\frac{K\tau^2}{2\mathbb{E}[X^2]} \right). \quad (\text{A9})$$

Let us put

$$q_{mk}(\phi) = |\langle \phi | \psi_{mk} \rangle|^2 = \frac{1}{d} \left| \sum_{\omega=1}^d \phi_{\omega} e^{-i\theta_{mk}(\omega)} \right|^2. \quad (\text{A10})$$

We have

$$\mathbb{E}[q_{mk}(\phi)] = \frac{1}{d} \sum_{\omega} |\phi_{\omega}|^2 = \frac{1}{d}, \quad (\text{A11})$$

and

$$\mathbb{E}[q_{mk}^2(\phi)] = \frac{1}{d^2} \left[2 \sum_{\omega, \omega'} |\phi_{\omega}|^2 |\phi_{\omega'}|^2 - \sum_{\omega} |\phi_{\omega}|^4 \right] \quad (\text{A12})$$

$$= \frac{1}{d^2} \left[2 - \sum_{\omega} |\phi_{\omega}|^4 \right]. \quad (\text{A13})$$

By noticing that $\sum_{\omega} |\phi_{\omega}|^4 \geq 1/d$, we obtain

$$\mathbb{E}[q_{mk}^2(\phi)] \leq \frac{1}{d^2} \frac{2d-1}{d} \leq \frac{2}{d^2}. \quad (\text{A14})$$

We can thus apply the tail bound in Theorem 5 with $X_k = q_{mk}(\phi)$ and $\tau = \delta/d$ to obtain

$$\Pr \left\{ Q_m(\phi) < \frac{1-\delta}{d} \right\} < \exp \left(-\frac{K\delta^2}{4} \right). \quad (\text{A15})$$

Having bounded the probability of small values of $Q_m(\phi)$, we now consider the probability that there exist ℓ values of m for which $Q_m(\phi) < (1-\delta)/d$. Taking into account all the $\binom{M}{\ell}$ choices of ℓ values of $m = 1, \dots, M$ and applying the union bound we obtain

$$\Pr \left\{ \exists m_1, \dots, m_{\ell} \forall j, Q_{m_j}(\phi) < \frac{1-\delta}{d} \right\} \leq \binom{M}{\ell} \exp \left(-\frac{\ell K \delta^2}{4} \right) \quad (\text{A16})$$

$$\leq \exp \left(-\frac{\ell K \delta^2}{4} + \ell \ln M \right). \quad (\text{A17})$$

3. The δ -net

It remains to prove that (A17) holds true with high probability for all unit vectors $|\phi\rangle$. To show that we follow a standard strategy and introduce a δ -net of vectors [2]. An δ -net \mathcal{N}_{δ} is a discrete set of $|\mathcal{N}_{\delta}|$ vectors such that for any vector $|\phi\rangle$ there exists $|\phi_i\rangle \in \mathcal{N}_{\delta}$ such that

$$\| |\phi\rangle \langle \phi| - |\phi_i\rangle \langle \phi_i| \|_1 \leq \delta, \quad (\text{A18})$$

where $\|X\|_1 = \text{Tr}|X|$ is the trace norm.

As discussed in [2], there exist δ -nets in a d -dimensional Hilbert space with $|\mathcal{N}_{\delta}| \leq (5/\delta)^{2d}$. We can then apply the union bound to estimate

$$\Pr \left\{ \exists |\phi_i\rangle \in \mathcal{N}_{\delta}, m_1, \dots, m_{\ell} \forall j, Q_{m_j}(\phi_i) < \frac{1-\delta}{d} \right\} \leq \left(\frac{5}{\delta} \right)^{2d} \exp \left[-M \left(\frac{\delta^3 K}{4} - \delta \ln M \right) \right] \quad (\text{A19})$$

$$= \exp \left[-M \left(\frac{\delta^3 K}{4} - \delta \ln M - \frac{2d}{M} \ln \frac{5}{\delta} \right) \right]. \quad (\text{A20})$$

Notice that such a probability is exponentially small in M provided that

$$K > \frac{4}{\delta^2} \left(\ln M + \frac{2d}{\delta M} \ln \frac{5}{\delta} \right). \quad (\text{A21})$$

It follows that with probability close to 1

$$\eta [Q_m(\phi_i)] \geq \frac{1-\delta}{d} \log d \quad (\text{A22})$$

for all $|\phi_i\rangle \in \mathcal{N}_{\delta}$ and at least $(1-\delta)M$ values of m . This result in turn implies

$$\begin{aligned} H[Q(\phi_i)] &\geq \frac{M}{d} (1-\delta)^2 [\log d - \log(1-\delta)] \\ &\geq \frac{M}{d} (1-\delta)^2 \log d \\ &\geq \frac{M}{d} (1-2\delta) \log d. \end{aligned} \quad (\text{A23})$$

It remains to estimate the error introduced by the δ -net. The Fannes-Audenaert [35] inequality yields

$$\begin{aligned} |H[Q(\phi)] - H[Q(\phi_i)]| &\leq \frac{\|Q(\phi) - Q(\phi_i)\|_1}{2} \log M \\ &\quad + h_2 \left[\frac{\|Q(\phi) - Q(\phi_i)\|_1}{2} \right], \end{aligned}$$

where $\|Q(\phi) - Q(\phi_i)\|_1 = \sum_m |Q_m(\phi) - Q_m(\phi_i)|$ denotes the vector trace norm and $h_2(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy. It is shown in Appendix B that Eq. (A18) implies

$$\|Q(\phi) - Q(\phi_i)\|_1 \leq 2\delta \frac{M}{d}, \quad (\text{A24})$$

up to a probability exponentially small in d and δ .

4. Conclusion

Putting all this together, we obtain that if Eq. (A21) is verified, then

$$I_{\text{acc}} \leq (\alpha - 1 + 2\delta) \log d + \eta(\alpha) + \delta M/d \log M + h_2(\delta M/d) \quad (\text{A25})$$

$$\leq (\alpha - 1 + 3\delta) \log d + \eta(\alpha) + h_2(\delta), \quad (\text{A26})$$

up to a probability exponentially small in M . Equation (A21) implies

$$K > \frac{1}{\delta^3} \frac{d}{M}, \quad (\text{A27})$$

which in turn yields

$$\alpha - 1 < \delta + 2\delta^{3/2} + \delta^3. \quad (\text{A28})$$

In conclusion we have that for any δ and d large enough there exist QDL protocols defined from the phase ensemble such that

$$I_{\text{acc}} = O(\delta \log d), \quad (\text{A29})$$

with a preshared secret key of $\log K$ bits and

$$K > \frac{4}{\delta^2} \left(\ln M + \frac{2d}{\delta M} \ln \frac{5}{\delta} \right). \quad (\text{A30})$$

APPENDIX B: RELATION BETWEEN TRACE NORMS

To find an explicit relation between the trace norms $\|\langle \phi | \langle \phi | - |\phi_i\rangle \langle \phi_i | \|_1$ and $\|Q(\phi) - Q(\phi_i)\|_1$, we introduce the

operators

$$\Gamma_m = \frac{d}{M} \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right]^{-1} \frac{1}{K} \sum_k |\psi_{mk}\rangle \langle \psi_{mk}|. \quad (\text{B1})$$

Notice that the operators $\{\Gamma_m\}_{m=1,\dots,M}$ are positive, and Theorems 1 and 2 imply that $\sum_m \Gamma_m \leq \mathbb{I}$, up to a probability exponentially small in d and δ . They hence define an incomplete POVM.

For any unit vector $|\phi\rangle$, the output of this incomplete POVM is the subnormalized probability vector $\tilde{Q}(\phi)$ with entries

$$\tilde{Q}_m(\phi) = \langle \phi | \Gamma_m | \phi \rangle = \frac{d}{M} \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right]^{-1} Q_m(\phi). \quad (\text{B2})$$

Since an incomplete POVM does not increase the trace distance, we have that if

$$\|\langle \phi | \langle \phi | - |\phi_i\rangle \langle \phi_i | \|_1 \leq \delta, \quad (\text{B3})$$

then

$$\|\tilde{Q}(\phi) - \tilde{Q}(\phi_i)\|_1 \leq \delta, \quad (\text{B4})$$

which in turn implies

$$\|Q(\phi) - Q(\phi_i)\|_1 \leq \delta \frac{M}{d} \left[\left(1 + \sqrt{\frac{d}{MK}} \right)^2 + \delta \right]. \quad (\text{B5})$$

For sufficiently small δ and $d/(MK)$ this implies

$$\|Q(\phi) - Q(\phi_i)\|_1 \leq 2\delta \frac{M}{d}. \quad (\text{B6})$$

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$$D = \sum_{l,m} |p(l|m)p(m) - p(l)p(m)| \quad (\text{B7})$$

to be small for all measurements of Eve, where $p(m)$ is the probability of transmitting message m and $p(l) = \sum_m p(l|m)p(m)$. The interpretation here is that if (B7) is small, then Eve cannot do much better than random guessing to determine the message. As discussed in [5], a straightforward application of the Alicki-Fannes inequality [27] implies that if $D \leq \delta/2$, then $I_{\text{acc}} \leq 4\delta \log d + 2h_2(\delta)$, where $h_2(\delta) = -\delta \log \delta - (1-\delta) \log(1-\delta)$ is the binary entropy. On the other hand, the

Pinsker inequality implies that (see e.g., [28], and references therein)

$$D \leq \sqrt{(2 \ln 2) I_{\text{acc}}}. \quad (\text{B8})$$

That is, if $I_{\text{acc}} \leq \delta \log d$, then $D \leq \sqrt{2\delta \ln d}$.

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