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THE PERTURBATION THEORY OF SOME VOLTERRA OPERATORS

by

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The Perturbation Theory of Volterra Operators

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ABSTRACT

A general procedure is derived for obtaining sufficient conditions for the similarity of operators T and T + P. This is applied to obtain sharp conditions for the similarity of the Volterra operators J: $f(x) \rightarrow \int_0^x f(y)dy$ and J + P where P: $f(x) \rightarrow \int_0^x p(x,y)f(y)dy$. By the same methods perturbations of the one sided shift operator S on $\ell^p(0,\infty)$ by certain trace class operators P are shown to be similar to S.

In the last chapter solvability conditions are obtained for the operator equation

$$TX - XS = A$$

where T and S are normal operators.

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BIOGRAPHY

John Markham Freeman was born in Sarasota, Florida and attended high school there. He graduated ØBK from the University of Florida and attended graduate school in mathematics at the Massachusetts Institute of Technology.

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TABLE OF CONTENTS

INTRODUCTION

In [3] Friedrichs studies perturbations of the self-adjoint operator $T: f(s) \to sf(s)$ on $L^2(a,b)$ by Fredholm integral operators P with regular kernels. In order to determine conditions on the perturbation P sufficient to ensure the similarity of $T + P$ and the unperturbed operator T, Friedrichs used a method which has since been abstractly formulated by Schwartz [21] and applied to the perturbation theory of a number of self-adjoint operators.

In Chapters II and III of this thesis the perturbation theory of certain non self-adjoint operators will be approached in a manner similar in broad outline to these methods of Friedrichs-Schwartz. Chapter II will be concerned with the quasi-nilpotent Volterra operator "indefinite integration" on $L^p(0,1)$, and Chapter III with the discrete Volterra operator "shift right" on $\ell^p(0, \infty)$.

In the paper of Friedrichs mentioned above it is assumed that the kernel $p(s, t)$ of the perturbing Fredholm operator

(1) $$P: \quad f(s) \to \int_a^b p(s, t)\, f(t)dt$$

be regular in the sense that Hölder conditions of order α $(0 < \alpha < 1)$ be satisfied;

(2) $$|p(s_1,t) - p(s_2,t)| \leq K|s_1 - s_2|^\alpha$$
$$|p(s,t_1) - p(s,t_2)| \leq K|t_1 - t_2|^\alpha .$$

It is then proved that $T: \; f(s) \to sf(s)$ is similar to the perturbed operator $T + P$ provided that $|P|$ is small enough, where

$$(3) \quad |P| = \sup|p(s,t)| + \sup\frac{|p(s,t_1)-p(s,t_2)|}{|t_1-t_2|^{\alpha}} + \sup\frac{|p(s_1,t)-p(s_2,t)|}{|s_1-s_2|^{\alpha}}$$

The crux of the method used in proving this result lies in the observation that for a regular Fredholm integral operator A, the commutator equation

$$(4) \qquad T\Gamma(A) - \Gamma(A)T = A$$

is solved by the singular integral operator.

$$(5) \qquad \Gamma(A)\, f(s) = (c) \int_a^b \frac{a(s,t)}{s-t}\, f(t)dt$$

(where (c) denotes the Cauchy principal value).

Chapter IV will deal with the solvability of (4) when T is any normal operator--without restrictions as to type and multiplicity of spectrum. A singular integral analogous to (5) will be defined which solves (4) for operators which are "regular" with respect to T.

CHAPTER I

SPACES OF REGULAR PERTURBATIONS

Let T and P be fixed bounded operators on a Banach space. The operators T and T + P are said to be similar provided that there exists a bounded invertible operator S such that

$$T = S^{-1}(T + P)S \ .$$

In terms of the notion of a "regular perturbation of T" to be formulated in this chapter, it will be possible to state sufficient conditions for the similarity of T + P and the unperturbed operator T.

The basic observation leading to the abstract notion of regularity with respect to an operator T is the following. If X simultaneously solves the two operator equations

$$TX - XT = A \tag{1}$$

$$A + PX = -P, \tag{2}$$

then $(I + X)T = (T + P)(I + X)$. (This is seen by multiplying out both sides and collecting terms according to (1) and (2).) Hence T + P is similar to T provided that I + X is invertible (e.g. if $||X|| < 1$ or merely $\lim_{n\to\infty} ||X^n||^{1/n} < 1$).

In order to apply this observation to the perturbation theory of T, one first determines a class $\mathcal{A}$ of "regular" operators A for which the commutator equation (1) is explicitly solvable by a bounded operator $X = \Gamma(A)$. In the following chapters it will be seen that the operator $\Gamma(A)$ is, as a rule, "singular", i.e. does not belong to .

Now, having determined $\mathcal{A}$ and a map Γ from $\mathcal{A}$ into the bounded operators such that

$$T\Gamma(A) - \Gamma(A)T = A, \tag{3}$$

the equations (1) and (2) then reduce to

$$A + P\Gamma(A) = -P\ ; \tag{4}$$

any solution $A \in \mathcal{A}$ of this equation also satisfies

$$[I + \Gamma(A)]T = (T + P)[I + \Gamma(A)] \tag{5}$$

and hence T and $T + P$ are similar provided that $[I + \Gamma(A)]^{-1}$ exists.

In terms of the map

$$\Gamma_P: \quad A \to P\Gamma(A) \tag{6}$$

equation (4) becomes

$$(I + \Gamma_P)A = -P$$

which is solved formally by the Neumann series

$$A = \sum_{n=0}^{\infty} (-1)^n \, \Gamma_P^n(-P) \ .$$

However, in order to make even the individual terms of the series meaningful one must assume first that $P \in \mathcal{A}$

(so that $\Gamma_P(P)$ is defined) and also that the "singular" operator $\Gamma(A)$ be "smoothed" by left multiplication by $P \in \mathcal{A}$, i.e. if P and $A \in \mathcal{A}$, then $\Gamma_P(A) = P\Gamma(A) \in$. These considerations suggest the definition (below) of a space of regular perturbations of an operator T.

Let T be a fixed (bounded) linear operator on a Banach space $\mathcal{X}$, and denote by $\mathcal{B}(\mathcal{X})$ the Banach space of bounded linear operators on $\mathcal{X}$. Throughout $||\cdot||$ will denote the norm on $\mathcal{B}(\mathcal{X})$.

<u>Definition 1.1</u>. A linear set $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ is called <u>a space of regular perturbations</u> (s.r.p.) of T if there exists a norm $|\cdot|$ on $\mathcal{A}$ and a linear map $\Gamma: \mathcal{A} \to \mathcal{B}(\mathcal{X})$ such that

(a) $\mathcal{A}$ is a Banach space under $|\cdot|$

(b) $T\Gamma(A) - \Gamma(A)T = A$

(c) $||\Gamma(A)|| \leq K|A|$

(d) if $P, A \in \mathcal{A}$, then $P\Gamma(A) \in \mathcal{A}$ and

$$|P\Gamma(A)| \leq K_1|P||A| .$$

In what follows $\mathcal{A}$ is assumed to be an s.r.p. of T and $P \in \mathcal{A}$. The map Γ_P given by (5) is then a bounded operator on $\mathcal{A}$. Its norm and those of its iterates will be denoted by $|\Gamma_P^n|$, $n = 1,2,\ldots$.

<u>Proposition 1.2</u>. A sufficient condition for the (unique) solvability of

$$(7) \qquad (I + \Gamma_P)(A) = -P$$

for $A \in \mathcal{A}$ is that

$$\lim_{n\to\infty} \|\Gamma_P^n\|^{1/n} < 1$$

Proof: If this condition is satisfied, then the series $\sum_{n=0}^{\infty} (-1)^n \Gamma_P^n$ converges (absolutely) in the operator norm. Its sum is $(I + \Gamma_P)^{-1}$.

The following lemma (cf. [2], page 518) will be needed several times in the next chapters.

<u>Lemma 1.3</u>. Let (S, Σ, μ) be a positive measure space and k a measurable function on $S\times S$ with

$$\operatorname{ess-sup}_s \int_S |k(s,t)|\mu(dt) \leq M < \infty \quad \text{and}$$

$$\operatorname{ess-sup}_t \int_S |k(s,t)|\mu(ds) \leq M.$$

Then $Kf(s) = \int_S k(s,t)\, f(t)dt$ defines a bounded linear operator on $L^p(S, \Sigma, \mu)(1 \leq p \leq \infty)$ and $\|K\|_p \leq M$.

CHAPTER II

THE OPERATOR $J: f(x) \to \int_0^x f(y)dy$

In this chapter perturbations of the Volterra operator $J: f(x) \to \int_0^x f(y)dy$ on $L^p(0,1)$ will be treated. Sufficient conditions which are in a precise sense sharp will be obtained for the similarity of J and J + P, where P is also a Volterra operator $P: f(x) \to \int_0^x p(x,y) f(y)dy$.

§1. Preliminaries

Given two Volterra operators $K: f(x) \to \int_0^x k(x,y) f(y)dy$ and $L: f(x) \to \int_0^x \ell(x,y) f(y)dy$ then (under restrictions to be stated below on the kernels k and ℓ) KL is the Volterra operator

$$KL: f(x) \to \int_0^x (k*\ell)(x,y) f(y)dy$$

where

$$(1) \qquad k*\ell(x,y) = \int_y^x k(x,\eta)\, \ell(\eta,y)d\eta \,.$$

To begin with we prove several facts concerning the composition $k*\ell$. By 'kernel' we will mean simply a (measurable) real or complex valued function $k(x,y)$ on $0 \le y < x \le 1$. For $\alpha > 0$, let

$$(2) \quad ||k||_{\alpha,\infty} = \sup_{0\le y<x\le 1} |k(x,y)(x-y)^{1-\alpha}|$$

Lemma 2.1. If $||k||_{\alpha,\infty} < \infty$, then $K: f(x) \to \int_0^x k(x,y) f(y)dy$ is a bounded operator on $L^p(0,1)$ $(1 \le p \le \infty)$ and $||K||_p \le \frac{1}{\alpha} ||k||_{\alpha,\infty}$.

Proof: We have, immediately from (2),

$$\operatorname*{ess-sup}_{0\le x\le 1} \int_0^x |k(x,y)|\,dy \le ||k||_{\alpha,\infty} \sup_{0\le x\le 1} \int_0^x \frac{dy}{(x-y)^{1-\alpha}}$$

and

$$\operatorname*{ess-sup}_{0\le y\le 1} \int_y^1 |k(x,y)|\,dx \le ||k||_{\alpha,\infty} \sup_{0\le y\le 1} \int_y^1 \frac{dx}{(x-y)^{1-\alpha}}$$

and hence by 1.4 we get $||K||_p \le C\ ||k||_{\alpha,\infty}$ with $C = \int_0^1 \frac{dx}{x^{1-\alpha}} = \frac{1}{\alpha}$.

Lemma 2.2. If k and ℓ are kernels for which $||k||_{\alpha,\infty}$ and $||\ell||_{\beta,\infty} < \infty$, then

$$||k*\ell||_{\alpha+\beta,\infty} \le B(\alpha,\beta)||k||_{\alpha,\infty}||\ell||_{\beta,\infty}$$

(where $B(\alpha,\beta)$ is the beta function).

Proof: Since $|k(x,\eta)\ \ell(\eta,y)| \le \frac{||k||_{\alpha,\infty}||\ell||_{\beta,\infty}}{(x-\eta)^{1-\alpha}(\eta-y)^{1-\beta}}$

it follows that

$$|k*\ell(x,y)| \le ||k||_{\alpha,\infty}||\ell||_{\beta,\infty} \int_y^x \frac{d\eta}{(x-\eta)^{1-\alpha}\ (\eta-y)^{1-\beta}}$$

$$= ||k||_{\alpha,\infty}||\ell||_{\beta,\infty}(x-y)^{\alpha+\beta-1} \int_0^1 \frac{dt}{t^{1-\alpha}(1-t)^{1-\beta}}$$

Since this last integral is $B(\alpha,\beta)$, this is equivalent to the asserted inequality.

The following (known) facts will be used freely and without explicit mention.

(A) If k, ℓ, and m are kernels with $||k||_{\alpha,\infty}$ $||\ell||_{\beta,\infty}$ and $||m||_{\gamma,\infty}$ all finite for some α, β, $\gamma > 0$, then

$$(k*\ell)*m = k*(\ell*m)$$

(B) If $||k||_{\alpha}$ and $||\ell||_{\beta}$ are finite, and K and L are the Volterra operators defined by k and ℓ respectively, then KL is a Volterra operator and its kernel is $k*\ell$.

For a kernel k with $||k||_{\alpha,\infty} < \infty$ we define

(3) $$k^{(n)} = k*k*\ldots*k \quad (n \text{ factors}).$$

For example, the iterates of J: $f(x) \to \int_0^x f(y)dy$ are

(4) $$J^n: \quad f(x) \to \int_0^x 1^{(n)}(x,y)\, f(y)dy$$

where

$$1^{(n)}(x,y) = \frac{(x-y)^{n-1}}{(n-1)!} \quad .$$

<u>Lemma 2.3</u>. If $||k||_{\alpha,\infty} < \infty$ then

$$||k^{(n)}||_{n\alpha,\infty} \leq \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} ||k||^n_{\alpha,\infty}$$

(where Γ denotes the gamma function).

Proof: This holds for $n = 1$. Assuming inductively that it holds for n, we have by 2.2 $||k^{(n+1)}||_{(n+1)\alpha,\infty} =$

$$||k^{(n)}*k||_{(n+1)\alpha,\infty} \leq B(n\alpha,\alpha)\, ||k^{(n)}||_{n\alpha,\infty} ||k||_{\alpha,\infty}$$

$$\leq B(n\alpha,\alpha)\, \frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)} ||k||^{n+1}_{\alpha,\infty} = \frac{\Gamma(\alpha)^{n+1}}{\Gamma((n+1)\alpha)} ||k||^{n+1}_{\alpha,\infty} .$$

Lemma 2.4. If $||k||_{\alpha,\infty}$, then the norms of the operators

$$K^n: \quad f(x) \to \int_0^x k^{(n)}(x,y)\ f(y)dy$$

satisfy $\quad ||K^n||_p \leq \frac{\Gamma(\alpha)^n}{\Gamma[(n+1)\alpha]}\ ||k||^n_{\alpha,\infty}$.

Thus $\lim_{n\to\infty} ||K^n||_p^{1/n} = 0$, i.e., K is a quasi-nilpotent operator on $L^p(0,1)$.

Proof: By 2.1, $||K^n||_p \leq \frac{1}{\alpha}\ ||k^{(n)}||_{n\alpha,\infty}$. By the preceding lemma, this in turn is majorized by $\frac{\Gamma(\alpha)^n}{\alpha\ \Gamma(n\alpha)}\ ||k||^n_{\alpha,\infty}$. That $\lim_{n\to\infty} ||K^n||_p^{1/n} = 0$ now follows since $\lim_{n\to\infty} \Gamma(n\alpha)^{1/n} = \infty$ when $\alpha > 0$.

Lemma 2.5. If kernels k and ℓ are continuous on $0 \leq y < x \leq 1$ and $||k||_{\alpha,\infty}$, $||\ell||_{\beta,\infty} < \infty$, then $k*\ell$ is continuous on $0 \leq y < x \leq 1$. If $\alpha + \beta > 1$, then $k*\ell$ is continuous on $0 \leq y \leq x \leq 1$ with $k*\ell(x,x) = 0$.

Proof: By the assumptions, $k(x,y) = \frac{m(x,y)}{(x-y)^{1-\alpha}}$ and $\ell(x,y) = \frac{n(x,y)}{(x-y)^{1-\beta}}$ where m and n are continuous and bounded on $0 \leq y < x \leq 1$. When $y < x$ the variable change $\eta = y + t(x-y)$ gives

$$k*\ell(x,y) = (x-y)^{\alpha+\beta-1} \int_0^1 \frac{m[x,y+t(x-y)]n\ [y+t(x-y),y]}{(1-t)^{1-\alpha}\quad t^{1-\beta}}dt$$

$$= (x-y)^{\alpha+\beta-1} \int_0^1 f_{(x,y)}(t)\ dt$$

Thus, if $0 \leq y_0 < x_0 \leq 1$ and (x,y) converges to (x_0,y_0), then the number $k*\ell(x,y)$ converges to $k*\ell(x_0,y_0)$, by the dominated convergence theorem.

For we have

$$\lim_{(x,y)\to(x_0,y_0)} f_{(x,y)}(t) = f_{(x_0,y_0)}(t) \quad \text{when} \quad 0 < t < 1$$

and $|f_{(x,y)}(t)| \leq \text{constant} \,/(1-t)^{1-\alpha}\, t^{1-\beta}$.

That $k*\ell$ converges to 0 as (x,y) converges to (x_0,x_0) follows also from the above expression for $k*\ell$ providing $\alpha + \beta > 1$.

Lemma 2.6. If $k(x,y)$ is continuous on $0 \leq x \leq y \leq 1$, $k_1(x,y) = \frac{\partial}{\partial x} k(x,y)$ and $\ell(x,y)$ are continuous on $0 \leq y < x \leq 1$, and $||k_1||_{\alpha,\infty}$, $||\ell||_{\beta,\infty} < \infty$, then

$$\frac{\partial}{\partial x} k*\ell(x,y) = k_1*\ell(x,y) + k(x,x)\ell(x,y).$$

Proof: For $0 \leq y < x \leq 1$,

$$\frac{k*\ell(x+h,y) - k*\ell(x,y)}{h} =$$

$$\int_y^x \frac{k(x+h,\eta) - k(x,\eta)}{h}\ell(\eta,y)d\eta + \frac{1}{h}\int_x^{x+h} k(x,\eta)\ell(\eta,y)d\eta$$

$$+ \int_x^{x+h} \frac{k(x+h,\eta) - k(x,n)}{h}\, \ell(\eta,y)dn$$

As $h \to 0$, the first integral converges to $k_1*\ell(x,y)$ by dominated convergence, the second to $k(x,x)\ell(x,y)$ by continuity of the integrand (recalling that $y < x$), and the third to 0 since the integrand is integrable, uniformly in h, in an interval about x.

§2. Solution of the Commutator Equation.

Let $\mathcal{A}_\alpha$ $(\alpha > 0)$ be the class of kernels a satisfying

(i) a and a_1 are continuous on $0 \leq y \leq x \leq 1$

(ii) $a(x,x) = a_1(x,x) = 0$

(iii) a_{11} exists and is continuous on $0 \leq y < x \leq 1$ and

$$||a_{11}||_{\alpha,\infty} < \infty$$

(The subscript 1 continues to denote differentiation with respect to x.)

By (ii) it follows that $a_1 = 1 * a_{11}$ and $a = 1^{(2)} * a_{11}$ and hence, by 2.2,

$$||a||_{\alpha+2,\infty} \leq \frac{||a_{11}||_{\alpha,\infty}}{\alpha(\alpha+1)}$$

(5)

$$||a_1||_{\alpha+1,\infty} \leq \frac{||a_{11}||_{\alpha,\infty}}{\alpha} \quad .$$

From this it is clear that $|a| = ||a_{11}||_{\alpha,\infty}$ is a norm (and not just a pseudo-norm) on $\mathcal{A}_\alpha$ and that $|\cdot|$ is equivalent on $\mathcal{A}_\alpha$ to the norm

(6) $$|a|_\alpha = ||a||_{\alpha+2,\infty} + ||a_1||_{\alpha+1,\infty} + ||a_{11}||_{\alpha,\infty} \, .$$

Proposition 2.7. $\mathcal{A}_\alpha$ is a Banach space under $|\cdot|_\alpha$.

Proof: By the remark made above, it suffices to show that $\mathcal{A}_\alpha$ is complete in the norm $|\cdot|$. So let $a^n \in \mathcal{A}_\alpha$ be a $|\cdot|$-Cauchy sequence;

$$|a^n - a^m| = ||a^n_{11} - a^m_{11}||_{\alpha,\infty} \to 0 \quad \text{as} \quad m, n \to 0 .$$

By the definition of $||\cdot||_{\alpha,\infty}$ this means that $[a^n_{11}(x,y) - a^m_{11}(x,y)](x-y)^{1-\alpha}$ converges uniformly to 0 on $0 \leq y < x \leq 1$. Hence $a^n_{11}(x,y)(x-y)^{1-\alpha}$ converges uniformly on $0 \leq y < x \leq 1$ to a function $c(x,y)(x-y)^{1-\alpha}$, continuous and bounded there. Now setting $a = 1^{(2)} * c$, we have $a \in {}_\alpha$, $a_{11} = c$ and

$$|a^n - a| = ||a^n_{11} - c||_{\alpha,\infty} \to 0 \quad \text{as} \quad n \to \infty .$$

We now solve the commutator equation

$$(7) \qquad J\Gamma(A) - \Gamma(A)J = A$$

when A is a Volterra operator with kernel $a \in \mathcal{A}_\alpha$. By the general remarks made earlier (7) becomes

$$(8) \qquad 1*\Gamma(a) - \Gamma(a)*1 = a$$

if one assumes a solution to (7) of the form

$$(9) \qquad \Gamma(A)\colon\ f(x) \to \int_0^x \Gamma(a)(x,y)\, f(y)dy \quad .$$

<u>Proposition 2.8</u>. If $a \in \mathcal{A}_\alpha$, then the kernel $\Gamma(a)$ defined by

(10) $$\Gamma(a)(x,y) = \frac{\partial^2}{\partial x \partial y} \int_0^y a(\xi+x-y,\xi)d\xi \qquad 0 \le y < x \le 1$$

satisfies (8), is continuous on $0 \le y < x \le 1$, and $||\Gamma(a)||_{\alpha,\infty} \le |a|_\alpha$. Thus $\Gamma(a)$ represents a bounded quasi-nilpotent operator $\Gamma(A)$ on $L^p(0,1)$ with $||\Gamma(A)||_p \le \frac{1}{\alpha}|a|_\alpha$.

Proof: Since a_1 and a_{11} are continuous on $0 \le y + \epsilon \le x \le 1$ ($\epsilon > 0$) the Leibniz rule for differentiating an integral with parameter can be applied twice to $\int_0^y a(\xi+x-y,\xi)d\xi$. This gives (applying either $\frac{\partial^2}{\partial x \partial y}$ or $\frac{\partial^2}{\partial y \partial x}$)

$$\Gamma(a)(x,y) = -\int_0^y a_{11}(\xi+x-y,\xi)d\xi + a_1(x,y).$$

From this follows the continuity of $\Gamma(a)$ on $0 \le y < x \le 1$ and

$$|\Gamma(a)(x,y)| \le \int_0^y \frac{||a_{11}||_{\alpha,\infty}}{(x-y)^{1-\alpha}} d\xi + ||a_1||_{\alpha+1,\infty}(x-y)^\alpha$$

$$\le \frac{||a_{11}||_{\alpha,\infty} + ||a_1||_{\alpha+1,\infty}}{(x-y)^{1-\alpha}} \le \frac{|a|_\alpha}{(x-y)^{1-\alpha}}$$

and hence $||\Gamma(a)||_{\alpha,\infty} \le |a|_\alpha$.

Since

$$1*\Gamma(a)(x,y) = \int_y^x d\eta[\frac{\partial^2}{\partial \eta \partial y} \int_0^y a(\xi+\eta-y,\xi)d\xi]$$

$$= -\int_0^y a_1(\xi+\eta-y,\xi)d\xi + a(\eta,y)\Big|_{\eta=y}^{\eta=x}$$

$$= a(x,y)-\int_0^y a_1(\xi+x-y,\xi)d\xi+\int_0^y a_1(\xi,\xi)d\xi-a(y,y),$$

and

$$\Gamma(a)*1(x,y) = \int_y^x d\eta\left[\frac{\partial^2}{\partial\eta\partial x}\int_0^\eta a(\xi+x-\eta,\xi)d\xi\right]$$

$$= \int_0^\eta a_1(\xi+x-\eta,\xi)d\xi \Big|_{\eta = y}^{\eta = x}$$

$$= \int_0^x a_1(\xi,\xi)d\xi - \int_0^y a_1(\xi+x-y,\xi)d\xi$$

we have

$$1*\Gamma(a) - \Gamma(a)*1 = a(x,y) - \int_y^x a_1(\xi,\xi)d\xi - a(y,y).$$

But the last two terms vanish since $a \in \mathcal{A}_\alpha$ so that (4) is satisfied by $\Gamma(a)$. The last assertion now follows directly from 2.4.

Remark. For a kernel k of the form $k(x,y) = m(y)/m(x)$, it can be shown that the commutator equation

$$k*\Gamma(a) - \Gamma(a)*k = a$$

is formally solved by

$$\Gamma(a)(x,y) = \frac{m(y)}{m(x)}\frac{\partial^2}{\partial x\partial y}\left[\int_0^y a(\xi+x-y,\xi)\,\frac{m(\xi+x-y,\xi)}{m(\xi)}\,d\xi\right]$$

provided $a(x,x) = a_1(x,x) = 0$. By using this, results analogous to those of the present chapter can be obtained for Volterra operators K with kernels k of the above type.

§3. Solution of the Operator Equation $A + P\Gamma(A) = -P$

For Volterra operators P and A with kernels $p \in \mathcal{A}_\sigma$ and $a \in \mathcal{A}_\alpha$ the equation $A + P\Gamma(A) = -P$ is equivalent to

$$\text{(11)} \qquad a + p * \Gamma(a) = -p \ ,$$

i.e. to the integro-differential equation

$$a(x,y) + \int_y^x p(x,\eta)\left[\frac{\partial^2}{\partial\eta\partial y}\int_0^y a(\xi+\eta-y,\xi)\,d\xi\right]d\eta = -p(x,y).$$

Lemma 2.9. If $p \in \mathcal{A}_\sigma$ and $b \in \mathcal{A}_\beta$, then $p * \Gamma(b) \in \mathcal{A}_{\sigma+\beta}$ and

$$|p * \Gamma(b)|_{\sigma+\beta} \leq B(\sigma,\beta)\ |p|_\sigma\ |b|_\beta \ .$$

Proof: Using 2.6 we differentiate $p * \Gamma(b)$ twice with respect to x. This yields first

$$\frac{\partial}{\partial x} p*\Gamma(b)(x,y) = p_1*\Gamma(b)(x,y) + p(x,x)\Gamma(b)(x,y)$$

$$= p_1*\Gamma(b)(x,y) \quad (\text{since } p(x,x) = 0),$$

and then

$$\frac{\partial^2}{\partial x} p*\Gamma(b)(x,y) = p_{11}*\Gamma(b)(x,y) + p_1(x,x)\Gamma(b)(x,y)$$

$$= p_{11}*\Gamma(b)(x,y) \quad (\text{since } p_1(x,x) = 0).$$

Thus by 2.5 and 2.8, $p*\Gamma(b)$ and $(p*\Gamma(b))_1 = p_1*\Gamma(b)$ are continuous on $0 \leq y \leq x \leq 1$ and $(p*\Gamma(b))_{11} = p_{11}*\Gamma(b)$ is continuous on $0 \leq y < x \leq 1$. Three applications of 2.2 now yield

$$||p*\Gamma(b)||_{\sigma+\beta+2,\infty} \leq B(\sigma+2,\beta)\ ||p||_{\sigma+2,\infty}||\Gamma(b)||_{\beta}$$

$$||(p*\Gamma(b))_1||_{\sigma+\beta+1,\infty} \leq B(\sigma+1,\beta)\ ||p_1||_{\sigma+1,\infty}||\Gamma(b)||_{\beta}$$

$$||(p*\Gamma(b))_{11}||_{\sigma+\beta,\infty} \leq B(\sigma,\beta)\ ||p_{11}||_{\sigma,\infty}||\Gamma(b)||_{\beta}\ .$$

Finally, using the fact that $||\Gamma(b)||_{\beta} \leq |b|_{\beta}$ and $B(\gamma,\beta) \leq B(\sigma,\beta)$ when $\gamma \geq \sigma$, we get

$$|p*\Gamma(b)|_{\sigma+\beta} \leq B(\sigma,\beta)\ |p|_{\sigma}\ |b|_{\beta}$$

by adding the three inequalities.

The next lemma will give bounds for the norms of the iterates of the operator

$$(12) \qquad \Gamma_p:\ a \to p * \Gamma(a)\quad .$$

<u>Lemma 2.10</u>. If $p \in \mathcal{A}_{\sigma}$ and $a \in \mathcal{A}_{\alpha}$, then $\Gamma_p^n(a) \in \mathcal{A}_{n\sigma+\alpha}$ and

$$|\Gamma_p^n(a)|_{n\sigma+\alpha} \leq \frac{\Gamma(\sigma)^n\ \Gamma(\alpha)}{\Gamma(n\sigma+\alpha)}\ |p|_{\sigma}^n\ |a|_{\alpha}\quad .$$

Proof: Taking $\beta = \alpha$ and $b = a$ in 2.9 yields

$$|\Gamma_p(a)|_{\sigma+\alpha} \leq \frac{\Gamma(\sigma)\ \Gamma(\alpha)}{\Gamma(\sigma+\alpha)}\ |p|_{\sigma}\ |a|_{\alpha}$$

(since $B(\sigma,\alpha) = \Gamma(\sigma)\ \Gamma(\alpha)/\Gamma(\sigma+\alpha)$).

Now assume inductively that the lemma holds for n and take $\beta = n\sigma + \alpha$ and $b = \Gamma_p^n(a)$ in 2.9.

Then $\Gamma_p^{n+1}(a) = \Gamma_p(b) \in \mathcal{A}_{(n+1)\sigma+\alpha}$ and

$$|\Gamma_p^{n+1}(a)| \leq B(\sigma, n\sigma+\alpha)\ |p|_\sigma\ |\Gamma_p^n(a)|_{n\sigma+\alpha}$$

$$\leq B(\sigma, n\sigma+\alpha)\ |p_\sigma| \left[\frac{\Gamma(\sigma)^n\,\Gamma(\alpha)}{\Gamma(n\sigma+\alpha)}|p_\sigma|^n\ |a|_\alpha\right]$$

$$= \frac{\Gamma(\sigma)^{n+1}\,\Gamma(\alpha)}{\Gamma[(n+1)\sigma+\alpha]}\ |p_\sigma|^{n+1}\ |a|_\alpha\ ,$$

the last inequality following by induction assumption.

Proposition 2.11. If $p \in \mathcal{A}_\alpha$, then Γ_p: $a \to p * \Gamma(a)$ is a bounded operator on $\mathcal{A}_\alpha$ and

$$\lim_{n\to\infty} |\Gamma_p^n|_\alpha^{1/n} = 0$$

Proof: By (2) and (6) it is clear that the norms $|\cdot|_\alpha$ increase with α. Thus for $a \in \mathcal{A}_\alpha$

$$|\Gamma_p^n(a)|_\alpha \leq |\Gamma_p^n(a)|_{(n+1)\alpha} \leq \frac{\Gamma(\alpha)^{n+1}}{\Gamma[(n+1)\alpha]}|p|_\alpha^n\ |a|_\alpha\ ,$$

the last inequality being a special case of 2.10. Hence $[\Gamma_p^n|_\alpha \leq \Gamma(\alpha)^{n+1}/\Gamma[(n+1)\alpha]$ from which 2.11 follows since $\lim_{n\to\infty} \Gamma(n\alpha)^{1/n} = \infty$.

§4. The Similarity of $J + P$ and J.

Having now established the axioms 1.1 for $\mathcal{A}_\alpha$, we pass to the question of similarity of the perturbed and unperturbed operators.

Theorem A. If $p \in \mathcal{A}_\alpha$, then the operators J and $J + P$, where

$$J: \quad f(x) \to \int_0^y f(y)dy$$

and

$$P: \quad f(x) \to \int_0^y p(x,y)\, f(y)dy$$

are similar on $L^p(0,1)$ for any p with $1 \leq p \leq \infty$.

Proof: By 2.7, 2.8 and 2.9, the class of Volterra operators A with kernels $a \in \mathcal{A}_\alpha$ and

$$|A|_\alpha = |a|_\alpha$$

$$\Gamma(A): \quad f(x) \to \int_0^x \Gamma(a)(x,y)\, f(y)dy$$

is a space of regular perturbations of J. By 2.11 and 1.2, $A + P\,\Gamma(A) = -P$ is solvable for A with $a \in \mathcal{A}_\alpha$ given P with $p \in \mathcal{A}_\alpha$. Since $\Gamma(A)$ is quasi-nilpotent, $[I + \Gamma(A)]^{-1}$ exists. Hence by the general considerations of Chapter I, J and $J + P$ are similar.

The preceding theorem can be strengthened by a procedure used by Volterra-Peres [22] and Kalisch [9].

Let $G: f(x) \to \int_0^x g(x,y)\, f(y)dy$ be a Volterra operator whose kernel satisfies

(i) $g(x,y)$ and $g_1(x,y)$ are continuous on $0 \le y \le x \le 1$

(ii) $g(x,x) > 0$ and $\int_0^1 g(x,x)dx = c$

(iii) $\frac{d}{dt}\tilde{g}(t)$ and $\frac{d}{dt}\tilde{g}_1(t)$ are continuous on $0 \le t \le 1$.

where $\tilde{g}(t) = g(t,t)$ and $\tilde{g}_1(t) = g_1(t,t)$.

(iv) $g_{11}(x,y)$ is continuous on $0 \le y < x \le 1$ and $||g_{11}||_{\alpha,\infty} < \infty$ where $0 < \alpha \le 1$.

Corollary A': G is similar to cJ.

This will follow easily from the next lemmas.

Lemma 2.12. Let G be as above with $c = 1$, and set $r(x) = \int_0^x g(t,t)dt$. Then S_r: $f(x) \to f(r(x))$ is a bounded non-singular operator on $L^p(0,1)$. Moreover $H = S_r^{-1}GS_r$ is a Volterra operator whose kernel h satisfies $h(x,x) = 1$ and the conditions (i) to (iv) above.

Proof: Since $g(t,t)$ is continuous and > 0 on $0 \le t \le 1$, and $\int_0^1 g(t,t)dt = 1$, $m = r^{-1}$ exists and both r and m are continuously differentiable:

$$\frac{dr}{dx} = g(x,x) \quad \text{and} \quad \frac{dm}{dx} = \frac{1}{g(m(x),m(x))} .$$

Thus S_r and $S_r^{-1} = S_m$ are bounded operators on $L^p(0,1)$ (bounds $\le ||\frac{dm}{dx}||_\infty^{1/p}$ and $||\frac{dr}{dx}||_\infty^{1/p}$ respectively).

Moreover, since

$$S_r^{-1} G S_r \; f(x) = \int_0^{m(x)} g(m(x),y) \; f(m(y))dy$$

$$= \int_0^x \frac{g(m(x),m(y))}{g(m(y),m(y))} f(y)dy \quad ,$$

$H = S_r^{-1} G S_r$ is a Volterra operator with kernel

$h(x,y) = \frac{g(m(x),m(y))}{g(m(y),m(y))}$ satisfying $h(x,x) = 1$.

Now

$$h_1(x,y) = \frac{g_1(m(x),m(y))}{\tilde{g}(m(y))\tilde{g}(m(x))} \quad \text{and,}$$

$$h_{11}(x,y) = \frac{1}{\tilde{g}(m(y))}\left[\frac{g_{11}(m(x),m(y))}{\tilde{g}(m(x))^2} - \frac{g_1(m(x),m(y))\frac{d\tilde{g}}{dt}(m(x))}{\tilde{g}(m(x))^3}\right].$$

In view of the above expression for h_1, the continuity of h_1 and $d\tilde{h}_1/dt$ follows from the continuity of g_1 and $d\tilde{g}_1/dt$. Similarly, h_{11} is continuous on $0 \leq y < x \leq 1$ by the assumptions (i) - (iv) on g. To see that h_{11} satisfies the proper growth condition at the diagonal, $h_{11}(x,y) = O[\frac{1}{(x-y)^{1-\alpha}}]$, notice that in the above expression for h_{11}, only the term containing $g_{11}(m(x),m(y))$ can be unbounded near $x = y$. But by the assumption (iv) on g_{11} ,

$$g_{11}(m(x),m(y)) = O[\frac{1}{(m(x)-m(y))^{1-\alpha}}] \quad \text{which in turn}$$

is $O[\frac{1}{(x-y)^{1-\alpha}}]$ since $x-y = r(m(x))-r(m(y)) = \int_{m(y)}^{m(x)} g(t,t)dt$.

Lemma 2.12. Let H be a Volterra operator whose kernel h satisfies $h(x,x) = 1$ and (i) to (iv) above and set $k(x) = \exp \int_0^x h_1(t,t)dt$. Then M_k: $f(x) \to k(x)f(x)$ is a bounded non-singular operator on $L^p(0,1)$. Moreover, $Q = M_k^{-1}HM_k$ is a Volterra operator whose kernel q satisfies (i), (iv) and $q(x,x) = 1$, $q_1(x,x) = 0$.

Proof: Since

$$M_k^{-1}HM_k: \quad f(x) \to \int_0^x \frac{k(y)}{k(x)} h(x,y)\, f(y)dy$$

Q is a Volterra operator with kernel

$$q(x,y) = h(x,y) \exp[-\int_y^x h_1(t,t)dt]$$

so that $q(x,x) = h(x,x) = 1$ and

$$q_1(x,y) = [h_1(x,y) - h_1(x,x)h(x,y)] \exp[-\int_y^x h_1(t,t)dt]$$

$$q_{11}(x,y) = [h_{11}(x,y) - h(x,y)\frac{d\tilde{h}_1}{dt}(x) + \tilde{h}_1(x)^2 h(x,y)]\exp[-\int_y^x h_1(t,t)dt]$$

Thus $q_1(x,x) = h_1(x,x) - h_1(x,x)h(x,x) = 0$. That the properties (i) and (iv) hold for q follows from the above expressions for q, q_1 and q_{11} and the assumptions (i) to (iv) on h.

Proof of A': Multiplying by $1/c$, G can be normalized so that $\int_0^1 g(t,t)dt = 1$. Then by the lemmas, G is similar to a Volterra operator Q whose kernel satisfies $q(x,x) = 1$, $q_1(x,x) = 0$, and (i) and (iv). But then the operator $P = Q - J$ has kernel $p = q-1 \in \mathcal{A}_\alpha$ and hence by Theorem A, $Q = J + P$ is similar to J.

§5. Applications.

The Volterra operator G: $f(x) \to \int_0^x g(x,y)\, f(y)dy$ is similar to J if, say,

$$g(x,y) = e^{\lambda(x-y)} \text{ (where } \lambda \text{ is any complex number)}$$

or if

$$g(x,y) = 1 + \frac{(x-y)^{\beta-1}}{\Gamma(\beta)} \quad \text{where } \beta \geq 2.$$

This latter example shows that J is similar to $J + J^\beta$ when $\beta \geq 2$ where J^β is the fractional integral operator

$$J^\beta: \quad f(x) \to \frac{1}{\Gamma(\beta)} \int_0^x (x-y)^{\beta-1} f(y)dy \quad .$$

By a result of Kalisch [11], J is not similar to $J + J^\beta$ when $\beta < 2$. Thus Theorem A is sharp with respect to the allowable algebraic singularity of p_{11} at the diagonal.

CHAPTER III

THE SHIFT OPERATOR

In the present chapter perturbations of the isometric operator

$$S: \quad (x_0,x_1,x_2,\ldots) \to (0,x_0,x_1,x_2,\ldots)$$

on $\ell^p(0,\infty)$ by certain trace class operators will be shown to be similar to the unperturbed operator S.

§1. Preliminaries

With respect to the basis $[\phi_n: \ n = 0,1,2,\ldots]$ where $\phi_0 = (1,0,0,\ldots)$, $\phi_1 = (0,1,0,\ldots)$, etc., S is represented by the matrix

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & & \cdot \end{bmatrix}.$$

The matrix of the operator "shift left",

$$S^*: \quad (x_0,x_1,x_2,\ldots) \to (x_1,x_2,\ldots)$$

is

$$S^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \\ \cdot & & & & & & \cdot \end{bmatrix}$$

It is well-known that S and S^* both have norm 1 on $\ell^p(0,\infty)$ and satisfy $S^*S = I$ and $SS^* = E_1$ where E_1 is the projection

$$E_1: \ (x_o,x_1,x_2,\ldots) \to (0,x_1,x_2,\ldots) \ .$$

More generally,

$$S^nS^{*n} = E_n$$

where

$$E_n: \ (x_o,x_1,\ldots,x_n,\ldots) \to (0,0,\ldots,x_n,x_{n+1},\ldots) \ .$$

The projections E_n are represented by the matrices

$$e_n = \text{diag}(\overbrace{0,0,\ldots,0}^{\text{n-zeros}},1,1,1,\ldots)$$

For an infinite matrix $a = [a_{nm}]$ we define

(1) $$|a| = \sum_{m,n=0}^{\infty} |a_{nm}|$$

and will denote by M the class of matrices a with $|a| < \infty$.

The matrices $a \in M$ represent bounded operators on $\ell^p(0,\infty)$:

$$A: \ (x_0, x_1, x_2, \ldots) \to (y_0, y_1, y_2, \ldots) ,$$

$$y_n = \sum_{m=0}^{\infty} a_{nm} x_m .$$

As an operator on $\ell^2(0,\infty)$, A is of trace class (see [23]). Its trace is given by

$$\mathrm{tr}(A) = \sum_{n=0}^{\infty} a_{nn} .$$

Lemma 3.1. If $a \in M$, then the series

$$\sum_{k=0}^{\infty} S^{*k} A S^{k}$$

converges (conditionally) in the norm of $\mathcal{B}(\ell^p)$; its sum $\gamma(A)$ satisfies $||\gamma(A)||_p \le |a|$ and is represented by the matrix

$$\gamma(A) = [\mathrm{tr}(S^{*n} A S^{m})] .$$

Proof: We first observe that the operation $a \to as$ shifts a matrix left one column, and $a \to s^* a$ shifts up one row. Hence $a \to s^{*k} a s^{k}$ shifts a matrix k units diagonally upwards. Thus the partial sums $\gamma_N(A) = \sum_{k=0}^{N} S^{*k} A S^{k}$ have matrices

$$\gamma_N(a) = \sum_{k=0}^{N} s^{*k} a s^{k} = [\sum_{k=0}^{N} a_{n+k,m+k}] .$$

Now to establish 3.1, we first show that $\gamma(a)$ represents a bounded operator $\gamma(A)$ on $\ell^p(0,\infty)$ with $||\gamma(A)||_p \le |a|$.

We have

$$\sup_{m\geq 0} \sum_{n=0}^{\infty} |tr(S^{*n}AS^{m})| \leq \sup_{m\geq 0} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{n+k,m+k}| \leq |a|$$

and

$$\sup_{n\geq 0} \sum_{m=0}^{\infty} |tr(S^{*n}AS^{m})| \leq \sup_{n\geq 0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |a_{n+k,m+k}| \leq |a|$$

which, using 1.3, establishes the assertion.

Finally we show that $Y_N(A) \to Y(A)$ in the norm of $\mathcal{B}(\ell_p)$. To do this we observe that $Y(A) - Y_{N-1}(A)$ is represented by the matrix

$$Y(a) - Y_{N-1}(a) = [tr(S^{*N+n}AS^{N+m})] ,$$

and

$$\sup_{n\geq 0} \sum_{m=0}^{\infty} |tr(S^{*N+n}AS^{N+m})| \leq \sup_{n\geq 0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |a_{N+n+k,N+m+k}|$$

$$= \sup_{n\geq N} \sum_{m=N}^{\infty} \sum_{k=0}^{\infty} |a_{n+k,m+k}| \leq |s^{*N}as^{N}|$$

and (similarly)

$$\sup_{m\geq 0} \sum_{n=0}^{\infty} |tr(S^{*N+n}AS^{N+m})| \leq |s^{*N}as^{N}| .$$

Hence (again by 1.3) we have

$$||Y(A) - Y_{N-1}(A)||_p \leq |s^{*N}as^{N}| .$$

But the latter converges to 0 as $N \to \infty$, which proves the lemma.

§2. Spaces of Regular Perturbations of S

For a matrix $a \in M$ we define

$$\Gamma(a) = s^{*}\gamma(a). \tag{2}$$

$\Gamma(a)$ is thus the matrix of $\Gamma(A) = S^{*}\gamma(A)$, i.e. of the operator

$$\Gamma(A) = \sum_{k=0}^{\infty} S^{*k+1} A S^{k}$$

Since $||S^{*}||_p = 1$, we have by 3.1,

$$||\Gamma(A)||_p \le ||\gamma(A)||_p \le |a|.$$

Proposition 3.2. $\Gamma(A)$ satisfies the commutator equation

$$S\Gamma(A) - \Gamma(A)S = A$$

if and only if $tr(AS^n) = 0$ for $n = 0,1,2,\dots$.

Proof: Recalling that $S^{*}S = I$ and $SS^{*} = E_1$ we have

$$\begin{aligned} S\Gamma(A) - \Gamma(A)S &= E_1\gamma(A) - S^{*}\gamma(A)S \\ &= \gamma(A) - S^{*}\gamma(A)S - (I-E_1)\gamma(A) \\ &= A - (I-E_1)\gamma(A). \end{aligned}$$

But $(I-E_1)\gamma(A)$ has as matrix

$$(I-e_1)\gamma(a) = \begin{bmatrix} tr(A) & tr(AS) & tr(AS^2) & \cdots \\ 0 & 0 & 0 & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

from which the result follow immediately.

Lemma 3.3. If p and $a \in M$, then $p\Gamma(a) \in M$ and

$$|p\Gamma(a)| \leq |p||a|$$

Proof: By definition (1),

$$|p\Gamma(a)| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\sum_{k=0}^{\infty} p_{nk}\, tr(S^{*kH} A S^{m})|$$

$$\leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |p_{nk}| \sum_{j=0}^{\infty} |a_{k+1+j,m+j}|$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |p_{nk}| \left[\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} |a_{k+1+j,m+j}| \right]$$

$$\leq |p||a| .$$

We are now in a position to determine some spaces of regular perturbations of S. Let $\mathcal{A}_o$ denote the space of matrices $a \in M$ whose entries vanish on and above the main diagonal. More generally, $\mathcal{A}_\alpha$ $(\alpha \geq 0)$ will be the space of $a \in M$ whose entries vanish on and above the α^{th} sub-diagonal.

First, it is clear that the $\mathcal{A}_\alpha$ are Banach spaces under the norm $|\cdot|$ of M. Moreover, since $tr(as^n)$, $n = 0,1,2,\ldots$, are the diagonal and super-diagonal sums of the entries of a, it follows from 3.2 that for $a \in \mathcal{A}_\alpha$, $\Gamma(a)$ surely solves

$$s\Gamma(a) - \Gamma(a)s = a.$$

Lemma 3.4. If $p \in \mathcal{A}_\sigma$ and $b \in \mathcal{A}_\beta$ then $p\Gamma(b) \in \mathcal{A}_{\sigma+\beta}$ and

$$|p\Gamma(b)| \leq |e_{\sigma+\alpha}p|\ |b|.$$

Proof: If $b \in \mathcal{A}_\alpha$, then the entries of $\Gamma(b)$ vanish on and above the $(\beta-1)^{th}$ sub-diagonal. For $\Gamma(b) = s^*\gamma(b)$ and $\gamma(b)$ has only 0's on and above the β^{th} sub-diagonal when $b \in \mathcal{A}_\beta$ (see 3.1). Thus $p\Gamma(b)$ has entries vanishing on and above the $(\sigma+\beta)^{th}$ sub-diagonal. In particular $p\Gamma(b)$ has only 0's on and above the $(\sigma+\beta)^{th}$ row. Hence $p\Gamma(b) = e_{\sigma+\beta}\, p\Gamma(b)$ so the result now follows by 3.3.

We now investigate the bounds of the iterates of the operator

$$\Gamma_p: \quad a \to p\Gamma(a)$$

on $\mathcal{A}_\alpha$.

<u>Lemma 3.5</u>. If $p \in \mathcal{A}_\sigma$ and $a \in \mathcal{A}_\alpha$, the $\Gamma_p^n(a) \in \mathcal{A}_{n\sigma+\alpha}$ and

$$|\Gamma_p^n(a)| \leq [\prod_{k=1}^{n} |e_{k\sigma+\alpha}p|]\ |a| \ .$$

Proof: By 3.4 this is true for $n = 1$. Assuming validity for n and taking $b = \Gamma_p^n(a)$ and $\beta = n\sigma+\alpha$ in 3.4 gives $\Gamma_p^{n+1}(a) = \Gamma_p(b) \in \mathcal{A}_{(n+1)\sigma+\alpha}$ and

$$|\Gamma_p^{n+1}(a)| \leq |e_{(n+1)\sigma+\alpha}p|\ |\Gamma_p^n(a)|$$

$$\leq [\prod_{k=1}^{n+1} |e_{k\sigma+\alpha}p|]\ |a| \ .$$

As an immediate consequence of 3.5 we have

Proposition 3.6. Let $p \in \mathcal{A}_\alpha$ and $|\Gamma_p^n|_\alpha$ denote the norms of the powers of Γ_p as operators on the Banach space $\mathcal{A}_\alpha$. Then

$$|\Gamma_p^n|_\alpha \leq \prod_{k=1}^{n} |e_{(k+1)\alpha}p| \ .$$

Hence $a+p\Gamma(a) = -p$ is (uniquely) solvable for $a \in \mathcal{A}_\alpha$ provided that

$$\varlimsup_{n\to\infty} [\prod_{k=1}^{n} |e_{(k+1)\alpha}p|]^{1/n} < 1 \ .$$

§3. Similarity of $S + P$ and S

We can now easily deduce some sufficient conditions for the similarity of $S + P$ and S.

Theorem B. If $p \in \mathcal{A}_0$ and $|p| < \frac{1}{2}$ then $S + P$ and S are similar.

Proof: $|p| < \frac{1}{2}$ we have by 3.6 that $|\Gamma_p^n|_0 \leq |p|^n < (\frac{1}{2})^n$. Hence $a+p\Gamma(a) = -p$ is solvable for $a \in \mathcal{A}_0$. Moreover,

$$|a| \leq |p|+|p\Gamma(a)| \leq (1+|a|)|p| \leq (1+|a|)\tfrac{1}{2}$$

so that $|a| < 1$. But then $[I + \Gamma(A)]^{-1}$ exists, since $||\Gamma(A)||_p \leq |a|$. Hence by the considerations of Chapter I, $S + P$ and S are similar.

Proposition B'. If $p \in \mathcal{A}_1$ and p has only 0 entries below a certain row, then $S + P$ and S are similar.

Proof: By assumption $e_n p = 0$ for n large enough. Hence, by 3.5, $\Gamma_p^n(p) = 0$ for large n. Thus $a = \sum_{n=0}^{\infty} (-1)^n \Gamma_p^n(-p)$ is a finite sum and satisfies $a + p\Gamma(a) = -p$. Moreover, since p vanishes on and above the first sub-diagonal and below a certain row, the same is true of a. Thus $\Gamma(a)$ vanishes on and above the main diagonal and below some row. Such a matrix is nilpotent and hence $[I + \Gamma(A)]^{-1}$ exists. Thus by Chapter I, S and $S + P$ are similar.

Remark. Theorem B refers to perturbations of s of the form

$$s+p = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ p_{10} & 0 & 0 & 0 & \\ p_{20} & p_{21} & 0 & 0 & \\ p_{30} & p_{32} & p_{32} & 0 & \cdots \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \end{bmatrix}$$

Stronger results can be obtained from 3.6 when the first sub-diagonal of s is not perturbed, i.e. when $p \in \mathcal{a}_\alpha$ with $\alpha \geq 1$. For then, instead of the estimate

$$|\Gamma_p^n|_o \leq |p|^n$$

given by 3.6 when $p \in \mathcal{a}_o$ (since $e_o = I$) we have

$$|\Gamma_p^n|_\alpha \leq \prod_{k=1}^{n} |e_{(n+1)\alpha} p| \;.$$

But, since left multiplication by the projections e_m replace the first m rows of p by rows containing only 0's, we have $|e_{n\alpha} p| \to 0$ as $n \to \infty$ when $\alpha > 0$.

CHAPTER IV

THE OPERATOR EQUATION SX-XT = A

Introduction: In this chapter we will obtain solvability conditions for the commutator equation

(1) $$TX - XT = A$$

when T is a normal operator on a Hilbert space H. The results will apply equally well to

(2) $$SX - XT = A$$

when S and T are both normal.

For two bounded (not necessarily normal) operators S and T on H we define

(3) $$\Box X = SX - XT$$

for $X \in \mathcal{B}(H)$. Then $\Box$ is a bounded operator on the Banach space $\mathcal{B}(H)$ and, by a result of Kleinecke (see [17]), has as spectrum

(4) $$\sigma(\Box) = \sigma(S) - \sigma(T).$$

For $\Box$ one has the Dunford operational calculus $f \to f(\Box)$ defined by

(5) $$f(\Box) = -\frac{1}{2\pi i}\int_{\partial D} f(z)(\Box - z)^{-1}dz$$

for functions f holomorphic on a neighborhood D of $\sigma(\Box)$. A more useful representation of $f(\Box)$ is obtained by Rosenblum [17];

(6) $$f(\Box)A = \frac{1}{2\pi i}\int_{\partial G} f(S-z)A(z-T)^{-1}dz$$

where G is a certain neighborhood of $\sigma(T)$. In particular, when $0 \notin \sigma(\Box)$, (6) gives the explicit inversion formula for $\Box X = A$,

$$\Box^{-1}(A) = \frac{1}{2\pi i}\int_{\partial G} (S-z)^{-1}A(z-T)^{-1}dz, \tag{7}$$

In [8] Heinz shows that if $T + T^* \leq b < a \leq S + S^*$, then $\Box^{-1}$ exists as a bounded operator on $\mathcal{B}(H)$ and is given by

$$\Box^{-1}(A) = -\int_0^\infty e^{tS}Ae^{-tT}dt \tag{8}$$

where the integral is absolutely convergent:

$$\int_0^\infty ||e^{tS}Ae^{-tT}||dt \leq \tfrac{1}{2}(a-b)^{-1}||A|| \ . \tag{9}$$

We, on the other hand, are principally interested in solving (1), i.e. $\Box X = A$ when $S = T$. By Kleinecke's result (4), 0 then belongs to the spectrum of $\Box$, so that $\Box^{-1}$ will not exist as a bounded operator on $\mathcal{B}(H)$. Thus the above formulas do not apply to the problem of inverting the commutator equation (1).

It is the object of this chapter to construct an operational calculus for $\Box$ when $S = T$ is normal and, from this, deduce sufficient conditions for the solvability of $\Box X = A$. The explicit solution will have the form of a singular integral operator $\Gamma(A)$ analogous to (5) of page 0.2.

By carrying through the analysis to include the case $S \neq T$, we will also be able to formulate the question of existence of the wave operators

$$(10) \qquad U_{\pm} = w - \lim_{t \to \pm\infty} e^{itS} e^{-itT}$$

in a new way. These are the unitary [or partial-isometric] operators which implement the unitary equivalence of S and T [or of their absolutely continous parts] in the Kato-Cook-Rosenblum treatment of self-adjoint perturbations $S = T + P$ of a self-adjoint operator T.

In what follows operators $X \to SX$ and $X \to XT$ will be denoted by S_+ and T_-. Using this notation we have $\square = S_+ - T_-$. From (ii), IV.10 below it follows that, as operators on the Hilbert space of Schmidt class operators, S_+ and T_- have as their adjoints $(S_+)^* = (S^*)_+$ and $(T_-)^* = (T^*)_-$ and hence

$$\square^* = (S^*)_+ - (T^*)_- .$$

This implies that <u>if S and T are normal operators on H, then $\square$ is normal as an operator on Schmidt class</u>.

The goal of the next few sections is to calculate explicitly the spectral resolution of $\square$ in terms of those of S and T.

§1. Preliminaries on Rectangles

Given two sets S_1 and S_2, the subsets of $S_1 \times S_2$ which are of the form $\delta \times \gamma$ with $\delta \subset S_1$ and $\gamma \subset S_2$ will be called rectangles. The symbol '$\dot{\cup}$' will denote a union whose summands are pairwise disjoint.

Lemma 4.1. If $\delta_1 \times \gamma_1$ and $\delta_2 \times \gamma_2$ are non-void, then a third rectangle $\delta \times \gamma$ is their disjoint union, $\delta \times \gamma = (\delta_1 \times \gamma_1) \dot{\cup} (\delta_2 \times \gamma_2)$ if and only if

either $\delta = \delta_1 \dot{\cup} \delta_2$ and $\gamma = \gamma_1 = \gamma_2$

or $\delta = \delta_1 = \delta_2$ and $\gamma = \gamma_1 \dot{\cup} \gamma_2$.

Lemma 4.2. $(\delta \times \gamma)-(\delta_1 \times \gamma_1) = [(\delta \cap \delta_1) \times (\gamma - \gamma_1)] \dot{\cup} [(\delta - \delta_1) \times \gamma]$

Proofs: (See Halmos [5].)

Lemma 4.3. If $(\delta \times \gamma) \cap (\delta_1 \times \gamma_1)$ is non-void, then $(\delta \times \gamma)-(\delta_1 \times \gamma_1)$ is a rectangle if and only if either $\delta \subset \delta_1$ or $\gamma \subset \gamma_1$.

Proof: If $\delta \subset \delta_1$, then $(\delta \times \gamma)-(\delta_1 \times \gamma_1) = \delta \times (\gamma - \gamma_1)$ by 4.2. The argument is the same if $\gamma \subset \gamma_1$.

Conversely, suppose $(\delta \times \gamma)-(\delta_1 \times \gamma_1)$ is a rectangle. If $\delta - \delta_1 \neq \emptyset$ and $\gamma - \gamma_1 \neq \emptyset$, then 4.2 expresses this rectangle as the disjoint union of two non-void rectangles. Hence by 4.1 we must then have either $\gamma = \gamma - \gamma_1$ or $\delta - \delta_1 = \delta \cap \delta_1$. But this is impossible since $\gamma \cap \gamma_1 \neq \emptyset$ and $\delta \cap \delta_1 \neq \emptyset$ (recall that the rectangles $\delta \times \gamma$ and $\delta_1 \times \gamma_1$ were assumed to have a non-void intersection). Thus either $\delta - \delta_1 = \emptyset$ or $\gamma - \gamma_1 = \emptyset$, and this proves the lemma.

Lemma 4.4. The smallest rectangle containing a union $\bigcup_i (\delta_i \times \gamma_i)$ of non-void rectangles is the rectangle $(\bigcup_i \delta_i) \times (\bigcup_i \gamma_i)$.

Proof: (Straightforward.)

<u>Lemma 4.5</u>. If $\delta_i x \gamma_i$, $i = 1,2,3$, are non-void pairwise disjoint rectangles whose union is a rectangle $\delta x \gamma$, then $(\delta x \gamma)-(\delta_i x \gamma_i) = \bigcup_{j \neq i} \delta_j x \gamma_j$ is a rectangle for some $i = 1,2$, or 3.

Proof: By 4.3 it suffices to show that either $\delta = \delta_i$ for some i or $\gamma = \gamma_i$ for some i.

We first show: if $\delta_i \cap \delta_j \neq \emptyset$, then either $\delta_i \subset \delta_j$ or $\delta_j \subset \delta_i$. For example: if $\delta_1 \cap \delta_2 \neq \emptyset$, then either $\delta_1 \subset \delta_2$ or $\delta_2 \subset \delta_1$. Suppose, on the contrary, that $\delta_1 \not\subset \delta_2$ and $\delta_2 \not\subset \delta_1$ and take $x_1 \in \delta_1 - \delta_2$ and $x_2 \in \delta_2 - \delta_1$. Since $\delta_1 \cap \delta_2 \neq \emptyset$, we must have $\gamma_1 \cap \gamma_2 = \emptyset$ (by the assumption that $\delta_1 x \gamma_1$ and $\delta_2 x \gamma_2$ are disjoint). Take $y_1 \in \gamma_1$ and $y_2 \in \gamma_2$. Now by 4.4, $\delta x \gamma = (\bigcup_{i=1}^{3} \delta_i) x (\bigcup_{i=1}^{3} \gamma_i)$ and hence (x_1,y_2) and (x_2,y_1) belong to $\delta x \gamma$. But then, since $x_1 \notin \delta_1$, (x_1,y_2) and (x_2,y_1) must belong to $\delta_3 x \gamma_3$. But this is also impossible since, by assumption

$$\emptyset = (\delta_3 x \gamma_3) \cap (\delta_1 x \gamma_1) = (\delta_3 \cap \delta_1) x (\gamma_3 \cap \gamma_1),$$

and hence either $\delta_3 \cap \delta_1 = \emptyset$ or $\gamma_3 \cap \gamma_1 = \emptyset$ so that either $(x_1,y_2) \notin \delta_3 x \gamma_3$ or $(x_2,y_1) \notin \delta_3 x \gamma_3$. This contradiction establishes the assertion made at the beginning of the paragraph. The analogous assertion holds for the γ_i.

<u>Case I</u>: $\delta_i \cap \delta_j = \emptyset$ for $i \neq j$. Then $\gamma = \gamma_1$. For suppose $y \in \gamma - \gamma_1$. Then $(x,y) \in \delta x \delta$ for all $x \in \delta = \bigcup_{i=1}^{3} \delta_i$ (see 4.4). In particular $(x,y) \in \delta x \gamma$ for $x \in \delta_1$. But this is impossible since $x \in \delta_1$ implies that $(x,y) \notin \delta_2 x \gamma_2$ and $(x,y) \notin \delta_3 x \gamma_3$, and $y \notin \gamma_1$ implies that $(x,y) \notin \delta_1 x \gamma_1$.

<u>Case II</u>: $\delta_i \subset \delta_j$ for some i and j with $i \neq j$, say $\delta_2 \subset \delta_1$. Then either $\delta = \delta_1$ or $\delta = \delta_3$ or $\gamma = \gamma_3$. From the above we know that if $\delta_1 \cap \delta_3 \neq \emptyset$, then either $\delta_1 \subset \delta_3$ or $\delta_3 \subset \delta_1$. Since $\delta_2 \subset \delta_1$ and $\delta = \bigcup_{i=1}^{3} \delta_i$, $\delta_1 \subset \delta_3$ implies that $\delta = \delta_3$, and $\delta_3 \subset \delta_1$ implies that $\delta = \delta_1$. Hence we can assume that $\delta_1 \cap \delta_3 = \emptyset$. Then $\delta_2 \cap \delta_3 = \emptyset$ also, since $\delta_2 \subset \delta_1$. This implies that $\gamma = \gamma_3$. For otherwise take $y \in \gamma - \gamma_3$. Then $(x,y) \in \delta x \gamma$ for all $x \in \delta$ and, in particular, for $x \in \delta_3$. But this is a contradiction, since $x \in \delta_3$ implies that $x \notin (\delta_1 x \gamma_1) \cup (\delta_2 x \gamma_2)$, and $x \notin \gamma_3$ implies that $(x,y) \notin \delta_3 x \gamma_3$.

It follows from the first part of the proof that Cases I and II are exhaustive, so the lemma is proved.

§2. <u>The Tensor Product $E \otimes F(\cdot)$</u>.

Let $\mathfrak{B}$ denote the σ-ring of Borel subsets of the complex plane $\mathbb{C}$. A resolution of the identity on a Hilbert space H is a function $E(\cdot)$ on $\mathfrak{B}$ whose values are (orthogonal) projections on H and which satisfies

(i) $E(\emptyset) = 0$, $E(\mathbb{C}) = I$

(ii) $E(\delta \cap \delta') = E(\delta)E(\delta')$ for all $\delta, \delta' \in \mathfrak{B}$ i.e. if $\delta_n \in \mathfrak{B}$ and $\delta = \bigcup_{n=1}^{\infty} \delta_n$ then

$$E(\delta)f = \sum_{n=1}^{\infty} E(\delta_n)f \quad \text{for every } f \in H.$$

Let $\mathfrak{R}$ be the ring generated by the rectangles $\delta x \gamma$ with $\delta, \gamma \in \mathfrak{B}$, i.e. $\mathfrak{R}$ is the set of finite disjoint unions of Borel rectangles. If $E(\cdot)$ and $F(\cdot)$ are resolutions of the identity we define the (bounded) operator $E \otimes F(\delta x \gamma)$ on $\mathcal{B}(H)$ by

(1) $$E \otimes F(\delta x \gamma)A = E(\delta)AF(\gamma).$$

<u>Lemma 4.6</u>. If $\delta x \gamma = \bigcup_{i=1}^{n} (\delta_i x \gamma_i)$ then

$$E \otimes F(\delta x \gamma) = \sum_{i=1}^{n} E \otimes F(\delta_i x \gamma_i).$$

Proof: If $\delta x \gamma = (\delta_1 x \gamma_1) \cup (\delta_2 x \gamma_2)$ then by 4.1 we can assume that, say, $\delta = \delta_1 \cup \delta_2$ and $\gamma = \gamma_1 = \gamma_2$. We then calculate directly

$$E(\delta_1)AF(\gamma)+E(\delta_2)AF(\gamma) = [E(\delta_1)+E(\delta_2)]AF(\gamma) = E(\delta)AF(\gamma)$$

If $\delta x \gamma = \bigcup_{i=1}^{3} (\delta_i x \gamma_i)$, then by 4.5 we can assume that, say, $(\delta_2 x \gamma_2) \cup (\delta_3 x \gamma_3)$ is a rectangle. Thus the case n = 3 follows by applying twice the case n = 2.

Now assume the result for $k \leq n - 1$. Since

$$\delta x \gamma = (\delta_1 x \gamma_1) \cup \bigcup_{i=2}^{n} (\delta_i x \gamma_i) = (\delta_1 x \gamma_1) \cup \Delta_1 \cup \Delta_2$$

where $\Delta_1 = \delta_1 x (\gamma - \gamma_1)$ and $\Delta_2 = (\delta - \delta_1) x \gamma$, we have by an application of the case of three disjoint summands

$$E \otimes F(\delta x \gamma) = E \otimes F(\delta_1 x \gamma_1) + E \otimes F(\Delta_1) + E \otimes F(\Delta_2).$$

Now $\Delta_1 = \bigcup_{i=2}^{n} [\delta_i x \gamma_i) \cap \Delta_1]$ and $\Delta_2 = \bigcup_{i=2}^{n} [(\delta_i x \gamma_i) \cap \Delta_2]$. Thus, since the intersection of two (Borel) rectangles is again a (Borel) rectangle, we can apply the inductive assumption to get

$$E \otimes F(\Delta_1)+E\otimes F(\Delta_2) = \sum_{i=2}^{n} E \otimes F[(\delta_i x Y_i)\cap\Delta_1]+\sum_{i=2}^{n} E \otimes F[(\delta_i x Y_i)\cap\Delta_2]$$

and this is equal to $\sum_{i=2}^{n} E \otimes F(\delta_i x Y_i)$ since for $i \geq 2$,

$$\delta_i x Y_i = [(\delta_i x Y_i)\cap\Delta_1]\cup[(\delta_i x Y_i)\cap\Delta_2]$$

Hence $E \otimes F(\delta x Y) = \sum_{i=1}^{n} E \otimes F(\delta_i x Y_i)$.

Lemma 4.7. If Δ_i $(i = 1,\ldots,n)$ and Δ_j' $(j = 1,2,\ldots,m)$ are two families of pairwise disjoint Borel rectangles and $\bigcup_{i=1}^{n} \Delta_i = \bigcup_{j=1}^{m} \Delta_j'$, then $\sum_{i=1}^{n} E \otimes F(\Delta_i) = \sum_{j=1}^{m} E \otimes F(\Delta_j')$.

Proof: $\Delta_{ij}'' = \Delta_i \cap \Delta_j'$ $(i = 1,\ldots,n;\ j = 1,2,\ldots,m)$ is a family of disjoint rectangles and $\Delta_i = \bigcup_j \Delta_{ij}''$, $\Delta_j' = \bigcup_i \Delta_{ij}''$. Moreover that $E \otimes F(\Delta_i) = \sum_j E \otimes F(\Delta_{ij}'')$ and $E \otimes F(\Delta_j') = \sum_i E \otimes F(\Delta_{ij}'')$ follows from 4.5. Hence both $\sum_i E \otimes F(\Delta_i)$ and $\sum_j E \otimes F(\Delta_j')$ are equal to $\sum_{i,j} E \otimes F(\Delta_{ij}'')$.

In virtue of this lemma we can define the operator $E \otimes F(\Delta)$ on $\mathcal{B}(H)$ for any $\Delta \in \mathcal{R}$ by

$$(2) \qquad E \otimes F(\Delta)A = \sum_{i=1}^{n} E(\delta_i)AF(Y_i)$$

where $\delta_i x Y_i$ $(i = 1,\ldots,n)$ is any finite family of disjoint Borel rectangles with $\Delta = \bigcup_{i=1}^{n} \delta_i x Y_i$. Then as a consequence of 4.7 we have

Lemma 4.8. The operator $E \otimes F(\Delta)$ on $\mathcal{B}(H)$ is a finitely additive function of $\Delta \in \mathcal{R}$.

Lemma 4.9. (i) $E \otimes F(\emptyset) = 0$, $E \otimes F(\mathbb{C} \times \mathbb{C}) = I$.

(ii) $E \otimes F(\Delta \cap \Delta') = E \otimes F(\Delta) E \otimes F(\Delta')$ for all $\Delta, \Delta' \in \mathcal{R}$.

Proof: (i) is clear from (1). If $\Delta = \bigcup_{i=1}^{m} \delta_j' \times \gamma_j'$, then for $A \in \mathcal{B}(H)$

$$E \otimes F(\Delta) E \otimes F(\Delta') A = \sum_{i,j} E(\delta_i \cap \delta_j') A E(\gamma_i \cap \gamma_j') = E \otimes F(\Delta \cap \Delta_1) A,$$

since $\bigcup_{i,j} (\delta_i \cap \delta_j') \times (\gamma_i \cap \gamma_j') = \bigcup_{i,j} (\delta_i \times \gamma_i) \cap (\delta_i' \times \gamma_j') = \Delta_1 \cap \Delta_2$

§3. Complete Additivity of $E \otimes F(\cdot)$ on Schmidt Class

Let H now be a separable Hilbert space and $[\phi_n]$ a complete orthonormal set in H. An operator A on H is said to be of Schmidt class if

$$||A||_s^2 = \sum_{n=1}^{\infty} ||A\phi_n||^2 < \infty .$$

The Schmidt class of operators forms a Hilbert space with

$$(A,B)_s = \sum_{n=1}^{\infty} (A\phi_n, B\phi_n) .$$

This Hilbert space is independent of $[\phi_n]$ and is (unitarily equivalent to) the tensor product $H \otimes H^*$. If we denote the linear functional $(\cdot, g)$ by $\bar{g}$, then the elements of $H \otimes H^*$ of the form $f \otimes \bar{g}$ are identified with the operators $h \to (h,g)f$ on H. More generally, $\sum_{i=1}^{n} c_i f_i \otimes \bar{g}_i$ is the operator on H

of finite rank given by

$$\left(\sum_{i=1}^{n} c_i f_i \otimes \overline{g}_i\right)h = \sum_{i=1}^{n} c_i (h,g_i) f_i \ .$$

$H \otimes H^*$ is the closure of the set of operators of finite rank in the norm $||\cdot||_s$. For these remarks and the facts which we list next the reference is Schatten [20].

(i) If $X \in \mathcal{B}(H)$ and $A \in H \otimes H^*$, the AX and XA $\in H \otimes H^*$. In particular $X(f \otimes \overline{g}) = (Xf) \otimes \overline{g}$ and $(f \otimes \overline{g})X = f \otimes \overline{X^* g}$.

(ii) If $X \in \mathcal{B}(H)$ and $A, B \in H \otimes H^*$, then

$$(XA,B)_s = (A,X^*B)_s \quad \text{and} \quad (AX,B)_s = (A,BX^*)_s \ .$$

(iii) For $A \in H \otimes H^*$, $(A, f \otimes \overline{g})_s = (Ag, f)$. In particular, $(\phi \otimes \overline{\psi}, f \otimes \overline{g})_s = (\phi,f)\overline{(\psi,g)}$.

<u>Lemma 4.10</u>. For each $\Delta \in \mathcal{R}$, $E \otimes F(\Delta)$ is an orthogonal projection on $H \otimes H^*$. Moreover, if $\Delta, \Delta' \in \mathcal{R}$ and $\Delta \subset \Delta'$, then $E \otimes F(\Delta) \leq E \otimes F(\Delta')$.

Proof: From 4.9 it follows that $E \otimes F(\Delta)^2 = E \otimes F(\Delta)$. If $\Delta = \bigcup_{i=1}^{n} \delta_i x \gamma_i$ and $A, B \in H \otimes H^*$, then, by (ii) above,

$$\begin{aligned}(E \otimes F(\Delta)A,B)_s &= \sum_{i=1}^{n} (E(\delta_i)AF(\gamma_i),B)_s = \\ &= \sum_{i=1}^{n} (A,E(\delta_i)BF(\gamma_i))_s = (A,E \otimes F(\Delta)B)_s\end{aligned}$$

and hence $E \otimes F(\Delta)^* = E \otimes F(\Delta)$. Thus $E \otimes F(\Delta)$ is an orthogonal projection.

If $\Delta \subset \Delta'$, then $\Delta' = \Delta \mathbin{\dot{\cup}} (\Delta'-\Delta)$ and hence, by the finite additivity of $E \otimes F(\cdot)$ on $\mathcal{R}$,

$$E \otimes F(\Delta') = E \otimes F(\Delta) + E \otimes F(\Delta'-\Delta)$$

from which the last assertion of the lemma follows.

From now on the $E \otimes F(\Delta)$ with $\Delta \in \mathcal{R}$ will be interpreted exclusively as operators on $H \otimes H^*$ (and not on $\mathcal{B}(H)$).

Lemma 4.11. $E \otimes F(\cdot)$ is strongly completely additive on $\mathcal{R}$.

Proof: If $\Delta_n \in \mathcal{R}$ and $\Delta_n \nearrow$, then by 4.10, $E \otimes F(\Delta_n)$ is an increasing sequence of projections on the Hilbert space $H \otimes H^*$. Thus $E \otimes F(\Delta_n)$ converges strongly to a projection P on $H \otimes H^*$. We now show that if $\Delta = \bigcup_{n=1}^{\infty} \Delta_n \in \mathcal{R}$, then $E \otimes F(\Delta) = P$. Since $E \otimes F(\cdot)$ is, by 4.8, finitely additive on $\mathcal{R}$, this will prove the lemma.

From $(i)_n$ above we have $E \otimes F(\delta x \gamma) f \otimes \overline{g} = [E(\delta)f] \otimes \overline{F(\gamma)g}$. Thus, if $\Delta = \dot{\bigcup}_{i=1} \delta_i x \gamma_i$, then

$$||E \otimes F(\Delta) f \otimes \overline{g}||_s^2 = \sum_{i=1}^{n} ||E(\delta_i)f \otimes \overline{F(\gamma_i)g}||_s^2$$

$$= \sum_{i=1}^{n} ||E(\delta_i)f||^2 ||F(\gamma_i)g||^2$$

$$= (\mu \times \nu)(\Delta),$$

the cartesian product of the two measures

$$\mu(\cdot) = ||E(\cdot)f||^2 \quad \text{and} \quad \nu(\cdot) = ||F(\cdot)g||^2 .$$

Hence, if $\Delta_n, \Delta \in \mathcal{R}$ and $\Delta_n \nearrow \Delta$, then

$$|(A, E \otimes F(\Delta-\Delta_n)f \otimes \bar{g})_s|^2 \leq ||A||_s^2 ||E \otimes F(\Delta-\Delta_n)f \otimes \bar{g}||_s^2$$

$$= ||A||_s^2 (\mu \times \nu)(\Delta-\Delta_n) \to 0 \text{ as } n \to \infty,$$

since $\mu \times \nu$ is a finite measure and $\Delta-\Delta_n \searrow \emptyset$. Thus $\lim_{n\to\infty} E \otimes F(\Delta_n)f \otimes \bar{g} = E \otimes F(\Delta)f \otimes \bar{g}$ so that $E \otimes F(\Delta) = P$ at all elements of $H \otimes H^*$ of the form $f \otimes \bar{g}$. But then, since $H \otimes H^*$ is the closed linear span of elements of this form, we have $E \otimes F(\Delta) = P$ throughout $H \otimes H^*$.

<u>Proposition 4.12</u>. $E \otimes F(\cdot)$ has a unique extension to a resolution of the identity on $H \otimes H^*$, i.e.

(i) $E \otimes F(\cdot)$ is defined and strongly completely additive on the σ-ring $\mathcal{B} \times \mathcal{B}$ of Borel subsets of $\mathbb{C} \times \mathbb{C}$

(ii) $E \otimes F(\Delta \cap \Delta') = E \otimes F(\Delta) E \otimes F(\Delta')$ and

$E \otimes F(\Delta)^* = E \otimes F(\Delta)$ for all $\Delta, \Delta' \in \mathcal{B} \times \mathcal{B}$.

(iii) $E \otimes F(\emptyset) = 0$ and $E \otimes F(\mathbb{C} \times \mathbb{C}) = I$.

Proof: The set functions $\mu_{A,B}(\cdot) = (E \otimes F(\cdot)A, B)_s$ are completely additive on the ring $\mathcal{R}$ of Borel rectangles and by Schwartz's inequality $|\mu_{A,B}(\Delta)| \leq ||A||_s ||B||_s$ for all $\Delta \in \mathcal{R}$. Hence by standard theorems (see e.g. [2], p. 136) on the extension of measures $\mu_{A,B}(\cdot)$ can be uniquely extended to a measure on $\mathcal{B} \times \mathcal{B}$. The extended measure also has the bound $||A||_s ||B||_s$. That $\mu_{A,B}(\Delta)$ is linear in A and conjugate linear in B for fixed $\Delta \in \mathcal{B} \times \mathcal{B}$ follows from the uniqueness of the extended measure and the fact that $\mu_{A,B}(\Delta)$ has this property for $\Delta \in \mathcal{R}$. Hence for each $\Delta \in \mathcal{B} \times \mathcal{B}$ there exists a unique bounded operator $E \otimes F(\Delta)$ on $H \otimes H^*$ such that $\mu_{A,B}(\Delta) = (E \otimes F(\Delta)A,B)$. That the $E \otimes F(\cdot)$ thus extended satisfies (ii) follows again from the uniqueness of the measures $\mu_{A,B}(\cdot)$ and the fact that $E \otimes F(\cdot)$ satisfies (ii) on $\mathfrak{R}$. Thus the extended $E \otimes F(\cdot)$ is a weak resolution of the identity. However, the strong complete additivity now follows, since (ii) implies that $||E \otimes F(\cdot)A||_s^2 = (E \otimes F(\cdot)A,A)_s$ and hence for $\Delta_n \in \mathfrak{B} \times \mathfrak{B}$ with $\Delta_n \nearrow \Delta$ we have

$$||E \otimes F(\Delta-\Delta_n)A||_s^2 \to 0 \quad \text{as} \quad n \to \infty .$$

Remark 4.13. In the case of Schmidt class operators of rank one, $A = \phi \otimes \overline{\psi}$ and $B = f \otimes \overline{g}$, we have

$$(E \otimes F(\Delta)A,B)_s = (\mu \times \nu)(\Delta)$$

where $\mu(\cdot) = (E(\cdot)\phi,f)$ and $\nu(\cdot) = (\overline{F(\cdot)\psi,g})$. This follows immediately for $\Delta \in \mathfrak{R}$ using that $(E \otimes F(\delta \times \gamma)\phi \otimes \overline{\psi},\ f \otimes \overline{g}) = (E(\delta)\phi,f)\overline{(F(\gamma)\psi,g)}$. Then by uniqueness the two measures are equal on $\mathfrak{B} \times \mathfrak{B}$.

§4. Solvability of $\square X = A$ when A is of Schmidt Class

Let S and T be normal operators on H with resolutions of the identity $E(\cdot)$ and $F(\cdot)$ respectively. Then $E \otimes F(\cdot)$ has support included in $\sigma(S)\times\sigma(T)$. For a function $f(\lambda,\xi)$, bounded and measurable on $\sigma(S)\times\sigma(T)$, we define the operator $f(S_+,T_-)$ on $H \otimes H^*$ by

$$(1) \quad (f(S_+,T_-)A,B)_s = \int_{\sigma(S)\times\sigma(T)} f \, d(E \otimes F(\cdot)A,B)_s$$

For two functions $f(\lambda,\xi)$ and $g(\lambda,\xi)$, bounded and measurable on $\sigma(S)x\sigma(T)$, we have

(2) $$(f \cdot g)(S_+,T_-) = f(S_+,T_-)g(S_+,T_-) \;.$$

This follows from (ii) above by approximating f and g uniformly by simple functions. A similar argument shows that if $h(\lambda)$ is bounded on $\sigma(S)$ and $k(\xi)$ is bounded on $\sigma(T)$ then

(3) $$h(S_+) = h(S)_+ \quad \text{and} \quad k(T_-) = k(T)_-$$

where $h(S) = \int h dE$ and $k(T) = \int k dF$.

<u>Theorem C</u>. Let S and T be bounded normal operators on H with spectral resolutions $E(\cdot)$ and $F(\cdot)$ and let A be a Schmidt class operator on H. If

(*) $\Gamma(A) = (c)\int \frac{1}{\lambda-\xi}\, dE \otimes F(\cdot)A$ exists in the weak operator topology of H, and

(**) $E \otimes F(\delta)A = 0$ where δ is the diagonal of $\sigma(S)x\sigma(T)$,

then $S\Gamma(A) - \Gamma(A)T = A$.

Proof: Let χ_ϵ be the characteristic function of $\Delta_\epsilon = [(\lambda,\xi):|\lambda-\xi| \geq \epsilon]$ and set $f_\epsilon(\lambda,\xi) = \chi_\epsilon(\lambda,\xi)/(\lambda-\xi)$. Then f_ϵ is a bounded function on $\sigma(S)x\sigma(T)$ and the assumption (*) means that the Schmidt class operators $f_\epsilon(S_+,T_-)A$ converge in the weak operator topology of H to $\Gamma(A)$. We have by (2) and (3) above

$$S[f_\epsilon(S_+,T_-)A] - [f_\epsilon(S_+,T_-)A]T =$$

$$= (S_+-T_-)f_\epsilon(S_+,T_-)A = E \otimes F(\Delta_\epsilon)A$$

which converges in $||\cdot||_s$ to $A - E \otimes F(\delta)A$. But then since $\Gamma(A) = w - \lim f_\epsilon(S_+, T_-)A$ we have

$$S\Gamma(A) - \Gamma(A)T = A - E \otimes F(\delta)A,$$

from which the theorem follows.

Examples. Let $H = L^2(-1,1)$ and $S = T$ be the operator $f(s) \to sf(s)$ and

$A:\ f(s) \to \int_{-1}^{+1} a(s,t)f(t)dt$ where $\iint |a(s,t)|^2 dsdt < \infty$.

For a subset Δ of the square, $E \otimes E(\Delta)A$ is the integral operator with kernel $\chi_\Delta a$. More generally, for a bounded function $f(s,t)$ on the square, $f(T_+, T_-)A$ is the operator with kernel $f(s,t)a(s,t)$. Thus the operator $\Gamma(A)$, if it exists, is the weak operator limit of the Schmidt class operators

$$\int_{|s-t|\geq\epsilon} \frac{1}{s-t}\, dE \otimes F(\cdot)A: \quad f(s) \to \int_{|s-t|\geq\epsilon} \frac{a(s,t)}{s-t} f(t)dt,$$

i.e. $\Gamma(A)f(s) = (c) \int_{-1}^{+1} \frac{a(s,t)}{s-t} f(t)dt$.

Here $E \otimes E(\delta)A = 0$, since its kernel is $\chi_\delta a = 0$, so that the condition (**) is vacuously fulfilled.

The situation is reversed if H is finite-dimensional. In this case, $E \otimes F(\delta) = 0$ is necessary and sufficient for the solvability of $SX-XT = A$. For $\sigma(S)x\sigma(T)$ consists of just a finite number of points so that $f(\lambda,\xi) = \frac{\chi_\Delta}{\lambda-\xi}$ where $\Delta = [(\lambda,\xi): \lambda \neq \xi]$ is bounded on $\sigma(S)x\sigma(T)$. Thus $\Gamma(A) = f(S_+,T_-)A$ exists and $S\Gamma(A) - \Gamma(A)T = (S_+-T_-)f(S_+,T_-)A = E \otimes F(\Delta)A = A - E \otimes F(\delta)A$, so that $E \otimes F(\delta)A = 0$ is sufficient.

It is necessary since $E \otimes F(\delta)[SX-XT] = g(S_+,T_-)X$ where $g(\lambda,\xi) = (\lambda-\xi)\ \chi_\delta(\lambda,\xi) \equiv 0$.

Remark. The conditions (*) and (**) are easily shown to be also necessary for the solvability of $SX-XT = A$ for X in Schmidt class where the operators $f(S_+,T_-)$ and $E \otimes F(\Delta)$ are defined. The difficulty in showing the necessity of these conditions for solvability in $\mathcal{B}(H)$ stems from the fact that we may well have solvability in $\mathcal{B}(H)$ but not in Schmidt class. It can be shown that in the first example considered above $TX-XT = A$ possesses a Schmidt class solution if and only if

$$\iint \left|\frac{a(s,t)}{s-t}\right|^2 ds\,dt < \infty ,$$

a property not enjoyed by the regular Fredholm kernels studied by Friedrichs, for which, on the other hand, the commutator equation is solvable in $\mathcal{B}(H)$.

These difficulties will be partially overcome in the last section of the chapter. By other devices, the condition $E \otimes F(\delta)A = 0$ will be shown to be necessary for solvability in $\mathcal{B}(H)$.

§5. The Convolution E * F(•) and Applications

We now assume that S and T are self-adjoint operators on H with resolutions of the identity $E(\cdot)$ and $F(\cdot)$. $E \otimes F(\cdot)$ is then defined on the Borel subsets of $R \times R$ where R is the real line. For a Borel subset δ of R we define

$$E * F(\delta) = E \otimes F(\Delta)$$

where $\Delta = [(\lambda,\xi): \lambda-\xi \in \delta]$. Then $E * F(\cdot)$ is a resolution of the identity on $H \otimes H^*$ defined on R. For functions f(x) bounded and measurable on $\sigma(S)-\sigma(T)$ we have (recalling that $\Box = S_+ - T_-$)

$$f(\Box) = \int_{\sigma(S)\times\sigma(T)} f(\lambda-\xi)d\, E \otimes F(\cdot) = \int_{\sigma(S)-\sigma(T)} f(x)d\, E * F(\cdot)$$

Thus, in particular, we have the expression

$$e^{it\Box} = \int_{-\infty}^{+\infty} e^{itx}\, d_x\, E * F(\cdot)$$

for $e^{it\Box}(A) = e^{itS}Ae^{-itT}$, interpreted as an operator on $H \otimes H^*$. Thus for $A \in H \otimes H^*$ and $f, g \in H$ we have

$$(e^{itS}Ae^{-itT}g,f) = \int_{-\infty}^{+\infty} e^{itx}d_x(E * F(\cdot)A,\ f \otimes \overline{g})_s$$

and hence

<u>Theorem D</u>. If A is of Schmidt class then $(e^{itS}Ae^{-itT}g,f)$ is the Fourier transform of the finite Borel measure

$$(E * F(\cdot)A,\ f \otimes \overline{g})_s \,.$$

<u>Examples</u>. If $A = \phi \otimes \overline{\Psi}$, then

$$(E * F(\cdot)A,\ f \otimes \overline{g})_s = (\mu * \nu)(\cdot)$$

where $\mu(\cdot) = (E(\cdot)\phi,f)$ and $\nu(\cdot) = \overline{(F(\cdot)\Psi,g)}$,

since by 4.13,

$$(E \otimes F(\cdot)A,\ f \otimes \overline{g})_s = (\mu\times\nu)(\cdot).$$

If S and T have absolutely continuous spectrum i.e. $(E(\cdot)\phi,f)$ and $(F(\cdot)\Psi,g)$ are absolutely continuous measures on R, then

$$\mu * \nu(\delta) = \int_\delta \left[\frac{d}{dx}(E(\cdot)\phi,f) * \frac{d}{dx}(\overline{F(\cdot)\Psi,g})\right]dx$$

so that

$$(e^{itS}Ae^{-itT}g,f) = \int_{-\infty}^{+\infty} e^{itx}\left[\frac{d}{dx}(E(\cdot)\phi,f) * \frac{d}{dx}(\overline{F(\cdot)\Psi,g})\right]dx$$

If, finally, S = T is the operator $f(s) \to sf(s)$ on $L^2(-\infty,+\infty)$ then

$$(e^{itT}Ae^{-itT}g,f) = \int_{-\infty}^{+\infty} e^{itx}[(\phi\overline{f})*(\Psi g)]dx \; .$$

Solvability of $\square X = A$

Since $\frac{d}{dt}e^{it\square} = i\square e^{it\square}$ we have

$$-i\square \int_0^t e^{is\square}(A)ds = A - e^{it\square}(A)$$

this suggests, as a solution of $\square X = A$, the integral

$$\Gamma(A) = -i \int_0^\infty e^{it\square}(A)dt \; .$$

If this integral exists in the sense of the weak operator topology of H, then $\square\Gamma(A) = A$. Thus sufficient conditions for solvability can be expressed as integrability conditions on the functions $(e^{it\square}Ag,f)$ which, if A is of Schmidt class, are the Fourier transforms of the finite Borel measures $(E * F(\cdot)A, f \otimes \overline{g})_s$.

Existence of the Wave Operators $U_{\pm}$.

Define $U_t = e^{itS}e^{-itT}$ and set $P = S - T$.

Then $\frac{d}{dt}U_t = ie^{itS}Pe^{-itT} = ie^{it\Box}(P)$ so that

$$U_t = I + i\int_0^t e^{is\Box}(P)ds \ .$$

Thus the existence of $U_+ \equiv w - \lim_{t\to+\infty} U_t$ depends on the existence of $\int_0^\infty e^{it\Box}(P)dt$ as a weak operator integral, for which, if P is of Schmidt class, sufficient conditions are again expressible in terms of the integrability of the Fourier transforms of finite Borel measures $(E * F(\cdot)P, f \otimes g)_s$.

Remark. These considerations suggest how to formulate abstractly the notion of "regularity" or "smoothness" of a Schmidt class operator A with respect to a self-adjoint operator T (or self-adjoint operators T and S) in such a way that regularity of A $\Rightarrow$ solvability of $\Box X = A$ (or the existence of $U_{\pm}$). Namely, the Fourier transforms of certain finite Borel measures on R should be integrable. The Hölder-regularity of the kernels a(s,t) assumed in the Friedrichs example T: $f(s) \to sf(s)$ is clearly expressible in the above terms.

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