

ON THE ROUTH APPROXIMATION TECHNIQUE  
AND LEAST SQUARES ERRORS

by

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ABSTRACT

A new method for calculating the coefficients of the numerator polynomial of the direct Routh approximation method (DRAM) using the least square error criterion is formulated. The necessary conditions have been obtained in terms of algebraic equations. The method is useful for low frequency as well as high frequency reduced-order models.

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Introduction:

A method for calculating the coefficients of the numerator polynomial of a reduced order model has been presented by RAO et al [1]. The method obtains the denominator polynomial of the transfer function by DRAM [2] and the numerator polynomial is obtained by using Pade type approximations. This was an "improvement" on the DRAM presented in [2] which is based on the Routh approximation technique of Hutton & Friedland [3]. This paper presents a method for obtaining the coefficients of the numerator polynomial of the reduced order model for a similar improved matching of the transfer function and it is shown that it is also a least squares estimate. The method is applicable to single input single output systems and is easily extended to multi-input multi-output systems. The denominator polynomial is obtained by the DRAM or Routh approximation method.

Consider a single-input single output linear-time-invariant system represented by its transfer function  $H(s)$ .

$$H(s) = \frac{c_0 + c_1s + \dots + c_{n+p-1}s^{n+p-1}}{s^p(a_0 + a_1s + \dots + a_ns^n)} \quad (1)$$

The system is unstable due to poles of  $H(s)$  located at the origin. If the system of (1) is approximated by the method of [1] as

$$\hat{H}(s) = \frac{f_0 + f_1s + \dots + f_{k+p-1}s^{k+p-1}}{s^p(d_0 + d_1s + \dots + d_k s^k)} \quad (2)$$



$$\begin{aligned}
 & [(c_{n+p-k} d_k + c_{n+p-k+1} d_{k-1} + \dots + c_{n+p-1} d_1) - \\
 & \quad (a_n f_p + a_{n-1} f_{p+1} + \dots + a_{n-k+1} f_{k+p-1})] s^{n+p} + \\
 & \quad \cdot \\
 & \quad \cdot \\
 & [(c_{n+p-2} d_k + c_{n+p-1} d_{k-1}) - (a_n f_{k+p-2} + a_{n-1} f_{k+p-1})] s^{n+p+k-2} + \\
 & [(c_{n+p-1} d_k) - (a_n f_{k+p-1})] s^{n+p+k-1} \\
 & \hline
 & s^p (a_0 + a_1 s + \dots + a_n s^n) (d_0 + d_1 s + \dots + d_k s^k)
 \end{aligned}$$

In order to minimize the least squares error  $E^2(s)$  with respect to the unknown coefficients  $f_0, f_1, \dots, f_{k+p-1}$  we obtain the following necessary conditions

$$\frac{\partial E^2(s)}{\partial \underline{f}} = 2E(s) \frac{\partial E(s)}{\partial \underline{f}} = 0 \quad \underline{f} = [f_0, f_1, \dots, f_{k+p-1}] \quad (5)$$

since

$$\frac{\partial E}{\partial f_i} = - \frac{s^i}{s^p (d_0 + d_1 s + \dots + d_k s^k)} \neq 0 \quad \text{for } s \neq 0, i = 0, 1, 2, \dots, k+p-1$$

Hence  $E(s) \equiv 0$ . Since expression (4) is a polynomial in  $s$  we can equate the numerator coefficients to zero.

This yields  $n+k+p$  equations with  $k+p$  unknowns  $f_0, f_1, \dots, f_{k+p-1}$ . Therefore  $E(s)$  cannot be made zero for all values of  $s$ . In general any  $k+p$  of these equations can be satisfied. There are  $\binom{n+k+p}{k+p}$  ways of choosing the  $k+p$  equations for  $s \neq 0$ . If the first  $k+p$  equations are selected we obtained the following result.

**Theorem 1:** For low frequency model approximation, the unknown coefficients  $f_0, f_1, \dots, f_{k+p-1}$  can be determined from the following equations.

$$\begin{aligned}
 c_0 d_0 - a_0 f_0 &= 0 \\
 c_0 d_1 + c_1 d_0 - (a_1 f_0 + a_0 f_1) &= 0 \\
 &\vdots \\
 c_0 d_k + c_1 d_{k-1} + \dots + c_k d_0 - (a_k f_0 + a_{k-1} f_1 + \dots + a_0 f_k) &= 0 \\
 c_1 d_k + c_2 d_{k-1} + \dots + c_{k+1} d_0 - (a_{k+1} f_0 + a_k f_1 + \dots + a_0 f_{k+1}) &= 0 \\
 &\vdots \\
 c_{n-k} d_k + c_{n-k+1} d_{k-1} + \dots + c_n d_0 - (a_n f_0 + a_{n-1} f_1 + \dots + a_0 f_n) &= 0 \\
 &\vdots \\
 c_{p-1} d_k + c_p d_{k-1} + \dots + c_{k+p-1} d_0 - (a_{k+p-n-1} f_0 + a_{n-1} f_{k+p-n} + \dots + a_0 f_{k+p-1}) &= 0
 \end{aligned} \tag{6}$$

where  $n \leq k+p$  and  $p = 1, 2, \dots, k$ .

It can readily be shown that these conditions (6) match the first  $k+p$  coefficients of the Taylor series expansion of the  $H(s)$  around  $s=0$  [4]. These conditions satisfy the results of RAO et al [1]. This gives an improved low frequency response.

**Theorem 2:** For high frequency model approximation, the unknown coefficients  $f_0, f_1, \dots, f_{k+p-1}$  can be obtained by solving the last  $k+p$  equations.

$$\begin{aligned}
 & c_{n-k}^d d_k + c_{n-k+1}^d d_{k-1} + \dots + c_n^d d_0 - (a_n^f f_0 + a_{n-1}^f f_1 + \dots + a_0^f f_n) = 0 \\
 & \quad \vdots \\
 & c_{p-1}^d d_k + c_p^d d_{k-1} + \dots + c_{k+p-1}^d d_0 - (a_n^f f_{k+p-n-1} + a_{n-1}^f f_{k+p-n} + \dots + a_0^f f_{k+p-1}) = 0 \\
 & \quad \vdots \\
 & c_{n+p-1-k}^d d_k + c_{n+p-k}^d d_{k-1} + \dots + c_{n+p-1}^d d_0 - \\
 & \quad (a_{n-p-1}^f f_p + a_{n-1}^f f_p + \dots + a_{n-k}^f f_{k+p-1}) = 0 \\
 & c_{n+p-k}^d d_k + c_{n+p-k+1}^d d_{k-1} + \dots + c_{n+p-1}^d d_1 - \\
 & \quad (a_n^f f_p + a_{n-1}^f f_{p+1} + \dots + a_{n-k+1}^f f_{k+p-1}) = 0 \\
 & \quad \vdots \\
 & c_{n+p-2}^d d_k + c_{n+p-1}^d d_{k-1} - (a_n^f f_{k+p-2} - a_{n-1}^f f_{k+p-1}) = 0 \\
 & c_{n+p-1}^d d_k - a_n^f f_{k+p-1} = 0
 \end{aligned} \tag{7}$$

where  $n \leq k+p$  and  $p = 1, 2, \dots, k$ .

Remarks: a) A similar set of condition to expression (4) be obtained if  $k+p < n$  which would yield similar set of conditions (6) and (7);

b) The conditions (6) improve the low frequency response of the reduced model (2) at the expense of not matching the initial (time) condition of the original model (1).

c) The conditions (7) improve the high frequency response of the reduced model (2) at the expense of not matching the final (time) condition;

d) The initial value and final value conditions can be matched by using the first condition of (6) and the last condition of (7);

e) The optimization problem can be formulated to obtain the optimal selection of  $k+p$  set of equations out of the  $n+k+p$  equations in the frequency range of interest;

f) The applications of the Initial value and Final value theorems to  $E(s)$ , (4), yields the initial time and final time error in the approximation process.

Example

Consider the example given by [1,2,3]

$$H(s) = \frac{2s^4 + 2s^3 + s^2 + 3s + 6}{s^2(s^3 + 7s^2 + 14s + 8)} \quad (8)$$

A fourth order reduced order model is given by

$$\hat{H}(s) = \frac{f_0 + f_1s + f_2s^2 + f_3s^3}{s^2(45s^2 + 98s + 56)} \quad (9)$$

where the denominator polynomial is obtained by DRMA [2] or Routh [3].

The complete least square conditions (5) for this example are as follows:

$$\begin{aligned} c_0d_0 - a_0f_0 \\ c_0d_1 + c_1d_0 - (a_1f_0 + a_0f_1) \\ c_0d_2 + c_1d_1 + c_2d_0 - (a_2f_0 + a_1f_1 + a_0f_2) \\ c_1d_2 + c_2d_1 + c_3d_0 - (a_3f_0 + a_2f_1 + a_1f_2 + a_0f_3) &= 0 \\ c_2d_2 + c_3d_1 + c_4d_0 - (a_3f_1 + a_2f_2 + a_1f_3) \\ c_3d_2 + c_4d_1 - (a_3f_2 + a_2f_3) \\ c_4d_2 - a_3f_3 \end{aligned} \quad (10)$$

Case 1

Using conditions (6) for low frequency operation of the reduced order model we get  $f_0 = 42$ ,  $f_1 = 21$ ,  $f_2 = 4$ ,  $f_3 = 12.5$

$$\hat{H}_{41}(s) = \frac{42 + 21s + 4s^2 + 12.5s^3}{s^2(45s^2 + 98s + 56)} \quad (11)$$

is the same as obtained by [1].

Case 2

Using condition (7) for high frequency operation of the reduced order model we get  $f_0 = -6066$ ,  $f_1 = 1501$ ,  $f_2 = -344$ ,  $f_3 = 90$

$$\hat{H}_{42}(s) = \frac{-6066 + 1501s - 344s^2 + 90s^3}{s^2(45s^2 + 98s + 56)} \quad (12)$$

Case 3

Using the first conditions of (6) for initial value matching and the last 2 conditions of (7) for final value matching we get  $f_0 = 42$ ,  $f_1 = 21$ ,  $f_2 = -344$ ,  $f_3 = 90$

$$\hat{H}_{43}(s) = \frac{42 + 21s - 344s^2 + 90s^3}{s^2(45s^2 + 98s + 56)} \quad (13)$$

Summary:

It has been shown that the coefficients of the numerator polynomial of reduced order model can be evaluated in the least squares error sense. The technique provides a frame work for the choice of the optimal reduced order model for a particular application. The method is easily extended to multi-input-multi output systems.

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