

MIT Open Access Articles

Markov dynamics on the Thoma cone: a model of time-dependent determinantal processes with infinitely many particles

The MIT Faculty has made this article openly available. **Please share** how this access benefits you. Your story matters.

Citation: Borodin, Alexei, and Grigori Olshanski. "Markov Dynamics on the Thoma Cone: a Model of Time-Dependent Determinantal Processes with Infinitely Many Particles." *Electronic Journal of Probability* 18, no. 0 (January 4, 2013).

As Published: <http://dx.doi.org/10.1214/EJP.v18-2729>

Publisher: Institute of Mathematical Statistics

Persistent URL: <http://hdl.handle.net/1721.1/89530>

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

Terms of use: Creative Commons Attribution



Markov dynamics on the Thoma cone: a model of time-dependent determinantal processes with infinitely many particles

Alexei Borodin* Grigori Olshanski†

Abstract

The Thoma cone is an infinite-dimensional locally compact space, which is closely related to the space of extremal characters of the infinite symmetric group S_∞ . In another context, the Thoma cone appears as the set of parameters for totally positive, upper triangular Toeplitz matrices of infinite size.

The purpose of the paper is to construct a family $\{X^{(z,z')}\}$ of continuous time Markov processes on the Thoma cone, depending on two continuous parameters z and z' . Our construction largely exploits specific properties of the Thoma cone related to its representation-theoretic origin, although we do not use representations directly. On the other hand, we were inspired by analogies with random matrix theory coming from models of Markov dynamics related to orthogonal polynomial ensembles.

We show that processes $X^{(z,z')}$ possess a number of nice properties, namely: (1) every $X^{(z,z')}$ is a Feller process; (2) the infinitesimal generator of $X^{(z,z')}$, its spectrum, and the eigenfunctions admit an explicit description; (3) in the equilibrium regime, the finite-dimensional distributions of $X^{(z,z')}$ can be interpreted as (the laws of) infinite-particle systems with determinantal correlations; (4) the corresponding time-dependent correlation kernel admits an explicit expression, and its structure is similar to that of time-dependent correlation kernels appearing in random matrix theory.

Keywords: determinantal processes; Feller processes; Thoma simplex; Thoma cone; Markov intertwiners; Meixner polynomials; Laguerre polynomials.

AMS MSC 2010: Primary 60J25; 60J27, Secondary 60G55; 60C05; 05E05.

Submitted to EJP on April 9, 2013, final version accepted on July 13, 2013.

1 Introduction

The first two subsections of the introduction contain short preliminary remarks and a few necessary definitions. Next we state the main results of the paper, Theorems 1.2 and 1.3. Then we describe the method of proof and make a comparison with some related works.

*Department of Mathematics, Massachusetts Institute of Technology, USA, and Institute for Information Transmission Problems, Moscow, Russia. E-mail: borodin@math.mit.edu.

†Institute for Information Transmission Problems, Independent University of Moscow, National Research University Higher School of Economics, Moscow, Russia. E-mail: olsh2007@gmail.com.

1.1 Preliminaries: Markov processes related to orthogonal polynomials

It is well known that for each family of classical orthogonal polynomials p_0, p_1, p_2, \dots , there exists a second order differential operator D , which preserves the space of polynomials and is diagonalized in the basis $\{p_n\}$:

$$Dp_n = m_n p_n, \quad n = 0, 1, 2, \dots,$$

where $0 = m_0 > m_1 > m_2 > \dots$ are the eigenvalues. Let $W(x)$ be the weight function of $\{p_n\}$ and $\text{supp } W$ be its support. Operator D determines a diffusion Markov process X on $\text{supp } W$ with $W(x)dx$ being a symmetrizing measure, hence also a stationary distribution.

All these objects, family $\{p_n\}$, operator D , and Markov process X , have multidimensional analogs:

Namely, fix $N = 2, 3, \dots$. From $\{p_n\}$ on can construct a family of symmetric polynomials in N variables indexed by partitions ν of length at most N , as follows:

$$p_\nu(x_1, \dots, x_N) := \frac{\det[p_{\nu_i+N-i}(x_j)]}{V(x_1, \dots, x_N)},$$

where the determinant in the numerator is of order N and

$$V(x_1, \dots, x_N) := \prod_{1 \leq i < j \leq N} (x_i - x_j).$$

These polynomials form a basis in the space of symmetric polynomials. Next, the role of D is played by the second order partial differential operator

$$D_N := \frac{1}{V(x_1, \dots, x_N)} (D_{x_1} + \dots + D_{x_N}) V(x_1, \dots, x_N) - \text{const}_N,$$

where D_{x_i} denotes a copy of D acting on variable x_i and

$$\text{const}_N = m_0 + \dots + m_{N-1}.$$

Although the coefficients of D_N in front of the first order derivatives have singularities on the diagonals $x_i = x_j$, the operator is well defined on the space of symmetric polynomials and is diagonalized in the basis $\{p_\nu\}$:

$$D_N p_\nu = m_\nu p_\nu, \quad m_\nu := \sum_{i=1}^N (m_{\nu_i+N-i} - m_{N-i}).$$

Finally, one can use D_N to define a diffusion process X_N on the space of N -point configurations contained in $\text{supp } W \subseteq R$. Again, this process has a symmetrizing measure, with density

$$\prod_{i=1}^N W(x_i) \cdot V^2(x_1, \dots, x_N).$$

This construction is well known in random matrix literature. The case of Hermite polynomials arises from Dyson's Brownian motion model [16]. Some other examples can be found in König [25]. The construction also works for some families of discrete orthogonal polynomials, only then X_N is a jump process.

In the present paper, we make a further step of generalization leading to a two-parameter family of infinite-dimensional, continuous time Markov processes $X^{(z,z')}$, which are related to the Laguerre polynomials. These words can bring the reader to believe that the processes $X^{(z,z')}$ are obtained from the finite-dimensional Laguerre processes X_N by a large- N limit transition, but this is not true. Actually, the connection between $X^{(z,z')}$'s and X_N 's is of a different kind: informally, one can say that the former are related to the later by analytic continuation in two parameters, dimension N and the continuous parameter entering the definition of the classical Laguerre polynomials.

1.2 The infinite-dimensional Laguerre differential operator and the z-measures

The operator in question, denoted by $\mathfrak{D}^{(z,z')}$, serves as the pre-generator of process $X^{(z,z')}$. Initially, $\mathfrak{D}^{(z,z')}$ is defined in the algebra of symmetric functions, Sym , which replaces the algebra of N -variate symmetric polynomials. The elementary symmetric functions e_1, e_2, \dots are algebraically independent generators of Sym ; we use them as independent variables and define $\mathfrak{D}^{(z,z')} : \text{Sym} \rightarrow \text{Sym}$ as a second order differential operator

$$\begin{aligned} \mathfrak{D}^{(z,z')} = & \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} (2n-1-2k)e_{2n-1-k}e_k \right) \frac{\partial^2}{\partial e_n^2} \\ & + 2 \sum_{n' > n \geq 1} \left(\sum_{k=0}^{n-1} (n'+n-1-2k)e_{n'+n-1-k}e_k \right) \frac{\partial^2}{\partial e_{n'} \partial e_n} \\ & + \sum_{n=1}^{\infty} (-ne_n + (z-n+1)(z'-n+1)e_{n-1}) \frac{\partial}{\partial e_n} \end{aligned} \tag{1.1}$$

depending symmetrically on two complex parameters z and z' . Recall that the classical Laguerre polynomials depend on a continuous parameter (the ‘‘Laguerre parameter’’) and so does the N -variate Laguerre operator D_N . The origin of operator $\mathfrak{D}^{(z,z')}$ is explained in Olshanski [32]: it is obtained from D_N by formal analytic continuation with respect to N and the Laguerre parameter.

Operator $\mathfrak{D}^{(z,z')}$ is diagonalized in a special basis of Sym formed by the so-called *Laguerre symmetric functions*. These functions, denoted by $\mathfrak{L}_\nu^{(z,z')}$, depend on parameters (z, z') and are indexed by arbitrary partitions $\nu = (\nu_1, \nu_2, \dots)$. One has

$$\mathfrak{D}^{(z,z')} \mathfrak{L}_\nu^{(z,z')} = -|\nu| \mathfrak{L}_\nu^{(z,z')}, \quad |\nu| := \nu_1 + \nu_2 + \dots \tag{1.2}$$

As shown in [32], the Laguerre symmetric functions form an orthogonal basis in a Hilbert L^2 space. Let us explain briefly this point (for more detail, see [32] and Section 8.4 below).

So far we treated Sym as an abstract commutative algebra, freely generated by elements e_1, e_2, \dots , but now we embed it into the algebra of continuous functions on a topological space, called the *Thoma cone* and denoted by $\tilde{\Omega}$:

$$\begin{aligned} \tilde{\Omega} := & \left\{ (\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots; \delta) \in \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} : \right. \\ & \left. \alpha_1 \geq \alpha_2 \geq \dots \geq 0, \quad \beta_1 \geq \beta_2 \geq \dots \geq 0, \quad \sum \alpha_i + \sum \beta_i \leq \delta \right\}. \end{aligned}$$

Note that the space $\tilde{\Omega}$ is locally compact and has infinite dimension in the sense that its points depend on countably many continuous parameters. The way of converting elements $F \in \text{Sym}$ into continuous functions $F(\omega)$ on $\tilde{\Omega}$ is described in Section 7.4.

Next, we impose the following condition on the parameters:

Condition 1.1. *Either both parameters z and z' are complex numbers with nonzero imaginary part and $z' = \bar{z}$, or both parameters are real and contained in an open unit interval of the form $(m, m + 1)$ for some $m \in \mathbb{Z}$.*

This is equivalent to requiring that $(z+k)(z'+k) > 0$ for every $k \in \mathbb{Z}$. In particular, Condition 1.1 implies that zz' and $z+z'$ are real, so that the coefficients of operator $\mathfrak{D}^{(z,z')}$ are real.

It was shown in [32] that for every (z, z') satisfying Condition 1.1, there exists a unique probability distribution $M^{(z,z')}$ on $\tilde{\Omega}$ such that all elements of Sym produce square integrable functions on $\tilde{\Omega}$ with respect to measure $M^{(z,z')}$, and the Laguerre

functions $\mathfrak{L}_\lambda^{(z,z')}$ are pairwise orthogonal with respect to the inner product of the Hilbert space $L^2(\tilde{\Omega}, M^{(z,z')})$. In other words, $M^{(z,z')}$ serves as the *orthogonality measure* for the Laguerre symmetric functions. The measures $M^{(z,z')}$ appeared even earlier in connection with the problem of harmonic analysis on the infinite symmetric group; we call them the *z-measures* on the Thoma cone.

A difficulty of working with the Laguerre operator $\mathfrak{D}^{(z,z')}$ is that its domain as defined above consists of unbounded functions (more precisely, all the nonconstant functions from Sym are unbounded functions on $\tilde{\Omega}$). To overcome this difficulty we modify the domain of definition of the operator in the following way.

For a triple $\omega = (\alpha, \beta, \delta) \in \tilde{\Omega}$, write $|\omega| := \delta$. Let \mathcal{F} stand for the space of functions on $\tilde{\Omega}$ spanned by the functions of the form

$$e^{-r|\omega|}F(\omega), \quad F \in \text{Sym}, \quad r > 0.$$

Such functions are bounded; even more, they vanish at infinity. On the other hand, $\mathfrak{D}^{(z,z')}$ operates on \mathcal{F} in a natural way: here we use the fact that $|\omega| = e_1(\omega)$, so that each function from \mathcal{F} is expressed through variables e_1, e_2, \dots .

1.3 Main results

Given a locally compact separable metrizable space E , denote by $C_0(E)$ the Banach space of real continuous functions on E , vanishing at infinity, with the supremum norm. A *Feller semigroup* is a strongly continuous operator semigroup $T(t)$ on $C_0(E)$ afforded by a transition function $P(t; x, dy)$ (such that $P(t; x, \cdot)$ is a probability measure),

$$(T(t)f)(x) = \int_{y \in E} P(t; x, dy)f(y), \quad x \in E, \quad f \in C_0(E).$$

A Feller semigroup gives rise to a Markov process on E with càdlàg sample trajectories, called a *Feller process*.

Throughout the paper we assume that (z, z') satisfies Condition 1.1.

Theorem 1.2. (i) *The differential operator $\mathfrak{D}^{(z,z')}$, viewed as an operator on $C_0(\tilde{\Omega})$ with domain \mathcal{F} , is dissipative, and its closure serves as the generator of a Feller semigroup on $C_0(\tilde{\Omega})$, which we denote by $T^{(z,z')}(t)$.*

(ii) *The corresponding Feller Markov process $X^{(z,z')}$ has a unique stationary distribution, which is the z-measure $M^{(z,z')}$.*

Proof is given in Section 8.

Claim (ii) shows that the z-measures $M^{(z,z')}$ can be characterized as the stationary distributions of Markov processes $X^{(z,z')}$.

Taking as the initial distribution for Markov process $X^{(z,z')}$ its stationary distribution we get a stationary in time stochastic process, which we denote by $\tilde{X}^{(z,z')}$. Theorem 1.2 is complemented by the following result, established in Section 9:

Theorem 1.3. *$\tilde{X}^{(z,z')}$ can be interpreted as a time-dependent determinantal point process whose correlation kernel can be explicitly computed.*

Let us explain this claim. Consider the punctured real line $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and the space $\text{Conf}(\mathbb{R}^*)$ of locally finite point configurations on \mathbb{R}^* . The stationary distribution $M^{(z,z')}$ can be interpreted as a probability measure on $\text{Conf}(\mathbb{R}^*)$. More generally, for any finite collection $t_1 < \dots < t_n$ of time moments, the corresponding finite-dimensional distribution $M^{(z,z')}(t_1, \dots, t_n)$ of stochastic process $\tilde{X}^{(z,z')}$ can be interpreted as a probability measure on the space $\text{Conf}(\underbrace{\mathbb{R}^* \sqcup \dots \sqcup \mathbb{R}^*}_n)$. This makes it possible to describe

$M^{(z,z')}(t_1, \dots, t_n)$ in the language of correlation functions. The determinantal property claimed in the theorem means that the correlations functions are given by $n \times n$ minors extracted from a certain kernel. The kernel in question, denoted by $K^{(z,z')}(x, s; y, t)$, has as arguments two space-time variables, (x, s) and (y, t) , where $s \in \mathbb{R}$ and $t \in \mathbb{R}$ are time moments, while $x \in \mathbb{R}^*$ and $y \in \mathbb{R}^*$ are space positions.

The kernel $K^{(z,z')}(x, s; y, t)$ appeared first in our paper [11], but there it was derived as the result of a formal limit transition, without reference to an infinite-dimensional Markov process. We called $K^{(z,z')}(x, s; y, t)$ the *extended Whittaker kernel* to emphasize a similarity with the well-known dynamical kernels from random matrix theory, the “extended” versions of the classical sine, Airy, and Bessel kernels (see Tracy-Widom [40]).

1.4 Method of Markov intertwiners

The results stated above, together with those of [32], were announced without proofs in the note Olshanski [31]. The scheme of the initial proof of Theorem 1.2 was the following:

- Start with the semigroup $\tilde{T}^{(z,z')}(t)$ in the Hilbert space $L^2(\tilde{\Omega}, M^{(z,z')})$ generated by the closure of operator $\mathfrak{D}^{(z,z')}$ and show that $\tilde{T}^{(z,z')}(t)$ is positivity preserving.
- Show that $\tilde{T}^{(z,z')}(t)$ preserves functions from $C_0(\tilde{\Omega})$.
- Show that the topological support of $M^{(z,z')}$ is the whole space $\tilde{\Omega}$.

The third claim means that the natural map $C_0(\tilde{\Omega}) \rightarrow L^2(\tilde{\Omega}, M^{(z,z')})$ is injective, so that restricting $\tilde{T}^{(z,z')}(t)$ to $C_0(\tilde{\Omega})$ gives the desired Feller semigroup $T^{(z,z')}(t)$.

In the present paper, we use a different approach, based on the *method of Markov intertwiners* proposed in Borodin–Olshanski [13], combined with the main idea of another recent paper, Borodin–Olshanski [14]. To explain this approach, we have first to briefly review what we did in [13].

That paper deals with the Gelfand–Tsetlin graph \mathbb{GT} describing the branching rule for the irreducible characters of unitary groups $U(N)$. The graph is graded, and its N th level \mathbb{GT}_N is a countable set, identified with the dual object to the unitary group $U(N)$. The graph structure determines a sequence of stochastic matrices $\Lambda_1^2, \Lambda_2^3, \dots$, where the N th matrix Λ_N^{N+1} has format $\mathbb{GT}_{N+1} \times \mathbb{GT}_N$ and is viewed as a “link” connecting the $(N + 1)$ th and N th levels of graph \mathbb{GT} . The *boundary* of graph \mathbb{GT} is defined as the entrance boundary for the inhomogeneous Markov chain with varying state spaces \mathbb{GT}_N , discrete time parameter ranging over $\{\dots, 3, 2, 1\}$, and transition function given by the links. The boundary serves as the space of parameters for the extremal characters of the infinite-symmetric group $U(\infty)$; this space is a connected, infinite-dimensional locally compact space. Now, the idea is to find a family $\{T_N(t) : N = 1, 2, \dots\}$ of Feller semigroups, acting on the spaces $C_0(\mathbb{GT}_N)$ and compatible with the links in the sense that

$$T_{N+1}(t)\Lambda_N^{N+1} = \Lambda_N^{N+1}T_N(t), \quad N = 1, 2, \dots, \quad t \geq 0$$

(here the operators $T_{N+1}(t)$ and $T_N(t)$ are viewed as matrices of format $\mathbb{GT}_{N+1} \times \mathbb{GT}_{N+1}$ and $\mathbb{GT}_N \times \mathbb{GT}_N$, respectively). One can say that the links serve as *Markov intertwiners* for the semigroups $T_N(t)$. Given such a family of semigroups, a simple (essentially formal) argument shows that it gives rise to a “limit” Feller semigroup $T_\infty(t)$ generating a Feller process on the boundary. We showed in [13] that there is quite a natural way to construct requiring pre-limit semigroups $T_N(t)$ depending on four additional continuous parameters, and so we obtain a four-parameter family of limit Feller processes on the boundary.

In the present paper we show that a similar approach works for the Thoma cone $\tilde{\Omega}$. A nontrivial point is what is a suitable substitute of the Gelfand–Tsetlin graph. As is well known, a natural analog of the Gelfand–Tsetlin graph is the Young graph, which is the branching graph of the symmetric group characters. The boundary of the Young graph is an infinite-dimensional compact space Ω , called the *Thoma simplex*, and $\tilde{\Omega}$ appears as the cone built over Ω . Although harmonic analysis on the infinite symmetric group deals with the Thoma simplex and probability measures thereof, things go simpler when objects living on Ω are “lifted” to $\tilde{\Omega}$; this was the main reason for working with the Thoma cone. However, $\tilde{\Omega}$ itself is not a boundary of a branching graph, which was an evident obstacle for extending the method of [13].

A solution was found due to the results of [14], where we showed that $\tilde{\Omega}$ can be identified with the entrance boundary of a continuous time Markov chain on the set \mathbb{Y} of all Young diagrams. This fact enabled us to apply the formalism of Markov intertwiners with appropriate modifications; in particular, the discrete index $N = 1, 2, \dots$ is replaced by continuous index r ranging over the half-line $\mathbb{R}_{>0}$.

In one direction, the present work goes further than [13], because for the processes related to the Gelfand–Tsetlin graph, a result similar to Theorem 1.3 is yet unknown.

1.5 Comments

It is natural to compare the results of the present paper to those of Borodin–Olshanski [12], [13], and Borodin–Gorin [5]. In all four papers the authors construct a Feller Markov process on an infinite-dimensional boundary of a “projective system”.

The process of [12] can be obtained by a normalization of the one we construct here, much similar to the way the Brownian Motion on the sphere can be obtained from that in the Euclidian space. However, the stationary distribution of the normalized process does not define a determinantal point process. Also, in that case the state space is compact, which is much easier to deal with from the analytic viewpoint.

On the other hand, the process of the present paper is a certain scaling limit of that from [13], but in the case of [13] the situation is more complicated and we were not able to prove there that the time-dependent correlation functions of the equilibrium process are determinantal (we prove such a statement in this work). We also do not dispose of an explicit eigenbasis for the generator there, in contrast to (1.2) above.

The process considered in [5] was proven to have time-dependent determinantal structure but it does not possess a stationary distribution, unlike the three other ones. Also, the underlying state space is quite different as its coordinates live on a lattice, not on the real line.

Overall, the Markov process we consider in the present paper is the only one so far that is proven to have all the nice properties one would like to carry over from the well-known finite dimensional analogs, i.e. Feller property, existence of a stationary distribution, an explicit description of the (pre)generator and its eigenbasis, and determinantal formulas for the time-dependent correlations.

To the best of our knowledge, such completeness of the picture was not achieved in the study of infinite-particle versions of Dyson’s Brownian Motion Model that are also expected to have determinantal time-dependent correlations, see Jones [20], Katori–Tanemura [21], [22], [23], Osada [34], [35], Spohn [38].

1.6 Covering Markov process

Informally, both the Markov process $X^{(z, z')}$ on the Thoma cone and its relative, the Markov process on the boundary of the Gelfand–Tsetlin graph \mathbb{GT} , studied in our paper [13], may be viewed as interacting particle processes with *nonlocal* (or *long-range*)

interaction. On the other hand, as shown in [13], the process on the boundary of \mathbb{GT} is “covered” by a certain Markov process with local interaction, living on the path space of \mathbb{GT} . In the companion note [15] we describe a curious model which conjecturally provides a similar “covering” process for $X^{(z, z')}$. If the conjectural claims stated in [15] hold true, this model leads to an alternative approach to our processes $X^{(z, z')}$, which looks simple and intuitively appealing.

1.7 Organization of the paper

In Section 2 we recall basic facts about Feller semigroups and their generators, and state a remarkable general theorem from Ethier–Kurtz [18], which gives a convenient sufficient condition on a matrix of jump rates ensuring that it generates a Feller Markov chain.

In Section 3 we review necessary definitions and facts concerning convergence of Markov semigroups, taken again from Ethier–Kurtz [18].

Sections 4 and 5 are devoted to the formalism of Markov intertwiners (here we present a minimal necessary material and refer to [13] for more details).

In Section 6 we apply the method of Markov intertwiners to constructing a concrete one-dimensional diffusion process; our goal here is to present all the steps of the main construction in a simplified situation.

Short Section 7 introduces the Thoma cone and some related objects.

Long Section 8 is devoted to the proof of Theorem 1.2; the argument is developed in strict parallelism with that of Section 6.

Section 9 contains the proof of Theorem 1.3.

Finally, in Section 10 we briefly describe a Plancherel-type degeneration of our main construction.

2 Feller semigroups

Let E be a locally compact, noncompact, metrizable separable space. Denote by $C(E)$ the Banach space of real-valued continuous functions on E with the uniform norm

$$\|f\| = \sup_{x \in E} |f(x)|.$$

Let $C_0(E) \subset C(E)$ denote its closed subspace formed by the functions vanishing at infinity, and let $C_c(E)$ be the dense subspace of $C_0(E)$ consisting of compactly supported functions.

If E is a discrete countable space, then the continuity requirement disappears, $C_0(E)$ becomes the space of arbitrary real functions on E vanishing at infinity, and $C_c(E)$ becomes the subspace of finitely supported functions.

Definition 2.1. A Feller semigroup $\{T(t) : t \geq 0\}$ is a strongly continuous, positive, conservative contraction semigroup on $C_0(E)$, see [18, p. 166].

Note that in [18], the conservativeness condition is stated in terms of the semigroup generator. Here are two equivalent reformulations of this property (see also Liggett [27, Chapter 3]):

- For any fixed $x \in E$ and $t \geq 0$, one has

$$\sup\{(T(t)f)(x) : f \in C_0(E), 0 \leq f \leq 1\} = 1$$

(if E is compact, then this simply means that $T(t)$ preserves the constant function 1).

- The semigroup admits a transition function, where we mean that a transition function $P(t \mid x, \cdot)$ is a probability measure (not a sub-probability one!) for all $t \geq 0$ and $x \in E$.

Assume now that E is a countably infinite set and $Q = [Q(a, b)]$ is a matrix of format $E \times E$ such that

$$Q(a, b) \geq 0 \text{ for all } a \neq b \text{ and } -Q(a, a) = \sum_{b: b \neq a} Q(a, b) < +\infty \text{ for all } a \in E. \quad (2.1)$$

Then there is a constructive way to define a semigroup $\{P_{\min}(t) : t \geq 0\}$ of *substochastic* matrices, which provides the *minimal solution* to Kolmogorov's backward and forward equations,

$$\frac{d}{dt} P(t) = QP(t), \quad \frac{d}{dt} P(t) = P(t)Q,$$

see Feller [19] and Liggett [27, Chapter 2].

Definition 2.2. *One says that Q is regular if the matrices $P_{\min}(t)$ from the minimal solution are stochastic.*

If the Q -matrix is regular, then $P_{\min}(t)$ is a unique solution to both the backward and forward Kolmogorov equations. Qualitatively, regularity of the Q -matrix means that the Markov chain is non-exploding: one cannot escape to infinity in finite time.

Recall a few general notions (see Ethier–Kurtz [18, Chapter 1, Sections 1–3]). Any strongly continuous contractive semigroup on a Banach space is uniquely determined by its *generator*, which is a densely defined closed dissipative operator. We will denote generators by symbol A (possibly with additional indices), and $\text{Dom } A$ will denote the domain of A . A *core* of a generator A is a subspace $\mathcal{F} \subseteq \text{Dom } A$ such that the closure of the operator $A|_{\mathcal{F}}$ (the restriction of A to \mathcal{F}) coincides with A itself; thus A is uniquely determined by its restriction to a core. It often happens that an explicit description of $\text{Dom}(A)$ is unavailable but one can write down the action of A on a core \mathcal{F} , and then the *pre-generator* $A|_{\mathcal{F}}$ serves as a substitute of A .

We will need a result from Ethier–Kurtz [18] which provides a convenient sufficient condition of regularity together with important additional information:

Theorem 2.3. *Let E be a countably infinite set and $Q = [Q(a, b)]$ be a matrix of format $E \times E$ satisfying (2.1). Assume additionally that Q has finitely many nonzero entries in every row and every column, and there exist strictly positive functions $\gamma(a)$ and $\eta(a)$ on E that tend to $+\infty$ at infinity and are such that*

$$-Q(a, a) \leq C\gamma(a), \quad \forall a \in E, \quad (2.2)$$

$$Q \frac{1}{\gamma} \leq \frac{C}{\gamma} \text{ pointwise} \quad (2.3)$$

$$Q\eta \leq C\eta \text{ pointwise} \quad (2.4)$$

where C is a positive constant and, for an arbitrary function $f(a)$ on E , the notation Qf means the function

$$(Qf)(a) = \sum_{b \in E} Q(a, b)f(b) = \sum_{b \in E, b \neq a} Q(a, b)(f(b) - f(a)),$$

the sum being finite because of the row finiteness condition.

Under these hypotheses we have:

- (i) Q is regular and so determines a Markov semigroup $P(t)$.

- (ii) This semigroup induces a Feller semigroup $\{T(t) : t \geq 0\}$ on $C_0(E)$.
- (iii) Let A denote the generator of $T(t)$; its domain $\text{Dom}(A)$ consists of those functions $f \in C_0(E)$ for which $Qf \in C_0(E)$. Moreover, $A = Q$ on $\text{Dom } A$.
- (iv) The subspace $C_c(E) \subset C_0(E)$ of compactly supported functions is a core for A .

Proof. This is an adaptation of Theorem 3.1 in [18, Chapter 8], which actually holds under less restrictive assumptions. □

3 Convergence of semigroups and Markov processes

3.1 Convergence of semigroups

Let I be one of the sets $\mathbb{R}_{>0}$ (strictly positive real numbers) or $\mathbb{Z}_{>0}$ (strictly positive integers). Assume that $\{\mathcal{L}_r : r \in I\}$ is a family of real Banach spaces, \mathcal{L}_∞ is one more real Banach space, and for every $r \in I$ we are given a contractive linear operator $\pi_r : \mathcal{L}_\infty \rightarrow \mathcal{L}_r$. If f is a vector of one of these spaces, then $\|f\|$ denotes its norm.

Definition 3.1. We say that vectors $f_r \in \mathcal{L}_r$ approximate a vector $f \in \mathcal{L}_\infty$ and write $f_r \rightarrow f$ if

$$\lim_{r \rightarrow \infty} \|f_r - \pi_r f\| = 0.$$

Definition 3.2. Let $\{T_\infty(t) : t \geq 0\}$ and $\{T_r(t) : t \geq 0\}$ be strongly continuous contraction semigroups on \mathcal{L}_∞ and \mathcal{L}_r . We say that the semigroups $T_r(t)$ approximate the semigroup $T_\infty(t)$ and write $T_r(t) \rightarrow T_\infty(t)$ if

$$\lim_{r \rightarrow \infty} \sup_{0 \leq t \leq t_0} \|T_r(t)\pi_r f - \pi_r T_\infty(t)f\| = 0 \quad \text{for all } f \in \mathcal{L}_\infty \text{ and any } t_0 > 0. \quad (3.1)$$

Our aim is to check this condition using an appropriate convergence of semigroup generators. So let A_∞ and A_r denote the generators of the above semigroups and let $\text{Dom}(A_\infty)$, $\text{Dom}(A_r)$ be the domains of the generators.

Definition 3.3. Fix a core $\mathcal{F} \subseteq \text{Dom}(A)$. We say that the operator $A_\infty|_{\mathcal{F}}$ is approximated by the operators A_r if for any vector $f \in \mathcal{F}$ one can find a family of vectors $\{f_r \in \text{Dom}(A_r) : r \in I\}$ such that $f_r \rightarrow f$, and $A_r f_r \rightarrow A_\infty f$ as $r \rightarrow \infty$.

In other words, this kind of operator convergence means that every vector from the graph of $A_\infty|_{\mathcal{F}}$ can be approximated by vectors from the graphs of the operators A_r .

Theorem 3.4. Let $T_\infty(t)$, $T_r(t)$, A_∞ , A_r , and \mathcal{F} be as above. If $A_\infty|_{\mathcal{F}}$ is approximated by the operators A_r , then $T_r(t) \rightarrow T_\infty(t)$ in the sense of Definition 3.2.

Proof. For $I = \mathbb{Z}_{>0}$, this is part of Ethier–Kurtz [18, Chapter 1, Theorem 6.1]. The case $I = \mathbb{R}_{>0}$ is immediately reduced to the case $I = \mathbb{Z}_{>0}$, because condition (3.1) is equivalent to saying that the same limit relation holds along any sequence of positive real numbers tending to $+\infty$. □

3.2 Convergence of Markov processes

Below we use the term *Markov process* as a shorthand for a Markov family which may start from any given point of the state space or from any given initial probability distribution. We are dealing exclusively with processes stationary in time and with infinite life time.

Given an initial distribution $M(0)$ of a Markov process on a space E , one may speak about its finite-dimensional distributions $M(t_1, \dots, t_k)$ corresponding to any prescribed time moments $0 \leq t_1 < \dots < t_k$, $k = 1, 2, \dots$. Every such distribution $M(t_1, \dots, t_k)$ is a probability measure on the k -fold direct product $E^k = E \times \dots \times E$.

Let E be a locally compact metrizable space and $T(t)$ be a Feller semigroup on $C_0(E)$; then $T(t)$ gives rise to a Markov process $X(t)$ on E with càdlàg sample trajectories, see Ethier–Kurtz [18, Chapter 4, Section 2]. The finite-dimensional distributions of $X(t)$ are determined by the semigroup $T(t)$ in the following way: For arbitrary functions $g_1, \dots, g_k \in C_0(E)$, define recursively functions h_k, \dots, h_0 by

$$h_k = g_k, \quad h_{k-1} = g_{k-1} \cdot (T(t_k - t_{k-1})h_k), \dots \\ \dots, h_1 = g_1 \cdot (T(t_2 - t_1)h_2), \quad h_0 = T(t_1)h_1, \quad (3.2)$$

where dots mean pointwise product, so that h_{k-1} is obtained by applying operator $T(t_k - t_{k-1})$ to h_k and then multiplying the resulting function by g_{k-1} , etc. Then

$$\langle g_1 \otimes \dots \otimes g_k, M(t_1, \dots, t_k) \rangle = \langle h_0, M(0) \rangle, \quad (3.3)$$

where the angle brackets denote the canonical pairing between functions and measures, and $(g_1 \otimes \dots \otimes g_k)(x_1, \dots, x_k) = g_1(x_1) \dots g_k(x_k)$ for $(x_1, \dots, x_k) \in E^k$ (this is a function from $C_0(E^k)$).

Let $X_r(t)$ and $X(t)$ be Markov processes with state spaces E_r and E , respectively (as before, r ranges over the index set I , which is either $\mathbb{R}_{>0}$ or $\mathbb{Z}_{>0}$). Assume that E is a locally compact metrizable separable space and each E_r is realized as a discrete locally finite subset of E . Further, assume that as $r \rightarrow \infty$, E_r becomes more and more dense in E ; more precisely, we postulate that any probability measure P on E can be represented as the weak limit $w\text{-}\lim_{r \rightarrow \infty} P_r$, where P_r is a probability measure supported by E_r .

Definition 3.5. *Under these assumptions we say that the processes $X_r(t)$ approximate the process $X(t)$ and write $X_r(t) \rightarrow X(t)$ if whenever an initial distribution $M(0)$ for the process $X(t)$ is represented as a weak limit of a family $\{M_r(0)\}$ of initial distributions of processes $X_r(t)$, we have*

$$w\text{-}\lim_{r \rightarrow \infty} M_r(t_1, \dots, t_k) = M(t_1, \dots, t_k),$$

meaning weak convergence on E^k of the finite-dimensional distributions corresponding to any given time moments $0 < t_1 < \dots < t_k$, $k = 1, 2, \dots$.

Corollary 3.6. *Under the above assumptions, assume additionally that the Markov processes $X_r(t)$ and $X(t)$ come from some Feller semigroups on the Banach spaces $\mathcal{L}_r = C_0(E_r)$ and $\mathcal{L} = C_0(E)$, respectively. Further, let the projection $\pi_r : \mathcal{L} \rightarrow \mathcal{L}_r$ be defined as the restriction map from E to E_r .*

If the hypotheses of Theorem 3.4 are satisfied, then $X_r(t) \rightarrow X(t)$ in the sense of Definition 3.5.

Note that π_r is well defined as a map from $C_0(E)$ to $C_0(E_r)$ because E_r is assumed to be locally finite, so that if a sequence of points goes to infinity along E_r then it also goes to infinity in E .

Proof. It suffices to prove that

$$\lim_{r \rightarrow \infty} \langle g_1 \otimes \dots \otimes g_k, M_r(t_1, \dots, t_k) \rangle = \langle g_1 \otimes \dots \otimes g_k, M(t_1, \dots, t_k) \rangle \quad (3.4)$$

for any collection $g_1, \dots, g_k \in C_0(E)$, because the functions of the form $g_1 \otimes \dots \otimes g_k$ are dense in $C_0(E^k)$.

Let $h_k, \dots, h_0 \in C_0(E)$ be defined as in (3.2) and, for each $r \in I$, let $h_{k;r}, \dots, h_{0;r} \in C_0(E_r)$ be defined in the same way, starting from the collection

$$g_{1;r} := \pi_r(g_1), \quad g_{2;r} := \pi_r(g_2), \quad \dots, \quad g_{k;r} := \pi_r(g_k).$$

By virtue of (3.3), the desired limit relation (3.4) is equivalent to

$$\lim_{r \rightarrow \infty} \langle h_{0;r}, M_r(0) \rangle = \langle h_0, M(0) \rangle$$

Since $w\text{-}\lim_{r \rightarrow \infty} M_r(0) = M(0)$ by assumption, it suffices to prove that

$$\lim_{r \rightarrow \infty} \|h_{0;r} - \pi_r h_0\| = 0.$$

To do this, we prove step by step that

$$\lim_{r \rightarrow \infty} \|h_{i;r} - \pi_r h_i\| = 0,$$

for $i = k, \dots, 0$, where each transition $i \rightarrow i - 1$ is justified by making use of Theorem 3.4. □

This argument is patterned from the proof of Theorem 2.5 in [18, Chapter 4]. Note also that another kind of convergence is established in [18, Chapter 4, Theorem 2.11].

4 Feller projective systems

4.1 Links

Let E' and E be two measurable spaces. Recall that a *Markov kernel* linking E' to E is a function $\Lambda(\cdot, \cdot)$ in two variables, one ranging over E' and the other ranging over measurable subsets of E , such that Λ is measurable with respect to the first argument and is a probability measure relative to the second argument. We use the notation $\Lambda : E' \dashrightarrow E$ and call Λ a *link* between E' and E .

If E is a discrete set, then, setting $\Lambda(x, y) := \Lambda(x, \{y\})$, we may regard Λ as a function on $E' \times E$. If both E' and E are discrete, then Λ is simply a stochastic matrix of format $E' \times E$.

The operation of composition of two links $E'' \dashrightarrow E'$ and $E' \dashrightarrow E$ is defined in a natural way: denoting the first link by $\Lambda_{E'}^{E''}$ and the second one by $\Lambda_E^{E'}$ we have

$$(\Lambda_{E'}^{E''} \Lambda_E^{E'})(x, dz) = \int_{y \in E'} \Lambda_{E'}^{E''}(x, dy) \Lambda_E^{E'}(y, dz).$$

In the discrete case this operation reduces to conventional matrix product.

The possibility of composing links makes it possible to regard them as morphisms in a category whose objects are measurable spaces, see [14]. However, links are not ordinary maps; this is why we denote them by the dash arrow.

A link $\Lambda : E' \dashrightarrow E$ takes a probability measure M on E' to a probability measure $M\Lambda$ on E :

$$(M\Lambda)(dy) = \int_{x \in E'} M'(dx) \Lambda(x, dy).$$

If both spaces are discrete then measures may be viewed as row-vectors and then the product $M\Lambda$ becomes the conventional product of a row-vector by a matrix.

Dually, Λ determines a contractive linear map $B(E) \rightarrow B(E')$ between the Banach spaces of bounded measurable functions, denoted as $F \mapsto \Lambda F$:

$$(\Lambda F)(x) = \int_{y \in E} \Lambda(x, dy) F(y).$$

In the discrete case, functions may be viewed as column-vectors and then ΛF becomes the conventional product of a matrix by a column-vector.

We say that a link $\Lambda : E' \rightarrow E$ between two locally compact spaces is a *Feller link* if the corresponding linear map $B(E) \rightarrow B(E')$ sends $C_0(E) \subset B(E)$ to $C_0(E') \subset B(E')$.

If E is discrete, then this condition means that for any fixed $y \in E$, the function $x \mapsto \Lambda(x, y) := \Lambda(x, \{y\})$ on E' lies in $C_0(E')$.

4.2 Projective systems and boundaries

Let, as above, I denote one of the two sets $\mathbb{R}_{>0}$ or $\mathbb{Z}_{>0}$. By a *projective system* with index set I we mean a family $\{E_r : r \in I\}$ of discrete spaces together with a family of links $\{\Lambda_r^{r'} : E_{r'} \dashrightarrow E_r : r' > r\}$, where every E_r is finite or countably infinite, and for any triple $r'' > r' > r$ of indices one has $\Lambda_r^{r''} \Lambda_r^{r'} = \Lambda_r^{r''}$; see [14]. If $I = \mathbb{Z}_{>0}$, then it suffices to specify the links $\Lambda_r^{r'}$ for neighboring indices $r' = r + 1$ and then set

$$\Lambda_r^{r'} := \Lambda_{r'-1}^{r'} \dots \Lambda_r^{r+1}$$

for arbitrary couples $r' > r$.

(The above definition is applicable to more general ordered index sets but we would like to avoid excessive formalism. For the purpose of the present paper we need the continuous index set $I = \mathbb{R}_+$. Concrete projective systems with discrete index sets are considered in [13] and [14]. In some general considerations (see below) the case $I = \mathbb{R}_{>0}$ is readily reduced to that of $I = \mathbb{Z}_{>0}$.)

Following [14], we define the *boundary* E_∞ of a projective system $\{E_r, \Lambda_r^{r'}\}$ in the following way. Consider the projective limit space $\varprojlim \mathcal{M}(E_r)$, where $\mathcal{M}(E_r)$ stands for the set of probability measures on E_r and the limit is taken with respect to the projections $\mathcal{M}(E_{r'}) \rightarrow \mathcal{M}(E_r)$ induced by the links $\Lambda_r^{r'}$. Assuming that the projective limit space is nonempty, we take as E_∞ the set of its extreme points.

We refer to [14] for more details. Note that $\mathcal{M}(E_r)$ may be viewed as a simplex with vertex set E_r , and every projection $\mathcal{M}(E_{r'}) \rightarrow \mathcal{M}(E_r)$ is an affine map of simplices (that is, it preserves barycenters), so our projective limit space is a projective limit of simplices.

By the very definition of projective limit, an element of $\varprojlim \mathcal{M}(E_r)$ is a family $\{M_r \in \mathcal{M}(E_r) : r \in I\}$ of probability measures satisfying the relation $M_{r'} \Lambda_r^{r'} = M_r$ for every couple of indices $r' > r$. Such a family is called a *coherent system* of measures.

As explained in [14], there is a canonical bijection

$$\mathcal{M}(E_\infty) \longleftrightarrow \varprojlim \mathcal{M}(E_r), \tag{4.1}$$

where $\mathcal{M}(E_\infty)$ denotes the space of probability measures on E_∞ . This means that for every $r \in I$ there is a link $\Lambda_r^\infty : E_\infty \rightarrow E_r$ such that the correspondence $M_\infty \mapsto \{M_r : r \in I\}$ given by $M_r := M_\infty \Lambda_r^\infty$ establishes a one-to-one correspondence between probability measures on the boundary and coherent families of probability measures. We say that M_∞ is the *boundary measure* for the coherent system $\{M_r\}$.

Obviously, the links Λ_r^∞ are compatible with the links $\Lambda_r^{r'}$ in the sense that

$$\Lambda_r^\infty \Lambda_r^{r'} = \Lambda_r^\infty \quad \text{for any } r' > r.$$

Observe that in the case of $I = \mathbb{R}_{>0}$ the boundary does not change if in the above construction we will assume that the indices range along an arbitrary fixed sequence of strictly increasing real numbers converging to $+\infty$. This enables one to reduce the case $I = \mathbb{R}_{>0}$ to that of $I = \mathbb{Z}_{>0}$. For further reference, let us call this simple trick *discretization of the index set*.

4.3 Running example: The binomial projective system B

In this illustrative example taken from Borodin–Olshanski [14], the index set I is $\mathbb{R}_{>0}$; for every index $r \in \mathbb{R}_{>0}$ the corresponding discrete set E_r is a copy of $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$; and for every two indices $r' > r$ the corresponding link $\mathbb{Z}_+ \dashrightarrow \mathbb{Z}_+$ is given by

$$\mathbb{B}\Lambda_r^{r'}(l, m) = \frac{l!}{m!(l-m)!} \left(\frac{r}{r'}\right)^m \left(1 - \frac{r}{r'}\right)^{l-m}, \quad l, m \in \mathbb{Z}_+.$$

Note that $\Lambda_r^{r'}(l, \cdot)$ is a binomial distribution on the set $\{m : 0 \leq m \leq l\}$. For this reason we call this system the *binomial projective system*.

As shown in [14], its boundary E_∞ can be identified with the halfline \mathbb{R}_+ (the set of nonnegative real numbers) and the links $\Lambda_r^\infty : \mathbb{R}_+ \rightarrow \mathbb{Z}_+$ are given by Poisson distributions:

$$\mathbb{E}\Lambda_r^\infty(x, m) = e^{-rx} \frac{(rx)^m}{m!}, \quad x \in \mathbb{R}_+, \quad m \in \mathbb{Z}_+.$$

4.4 Feller projective systems

Let $\{E_r, \Lambda_r^{r'}\}$ be a projective system as defined above. Equip the boundary E_∞ with the *intrinsic topology* — the weakest one in which all functions of the form

$$x \mapsto \Lambda_r^\infty(x, y), \quad r \in I, \quad y \in E_r,$$

are continuous. We say that $\{E_r, \Lambda_r^{r'}\}$ is a *Feller system* if the following three conditions are satisfied:

- (1) All links $\Lambda_r^{r'}$ are Feller.
- (2) The boundary E_∞ is a locally compact Hausdorff space with respect to the intrinsic topology.
- (3) In this topology, all links Λ_r^∞ are Feller.

Note that under condition (1), the definition of the intrinsic topology is not affected by discretization of the index set, which entails that the intrinsic topology is automatically metrizable with countable base.

As an illustration, let us check that the binomial projective system from our running example (see Section 4.3 above) is a Feller system.

Indeed, from the very definition of the “binomial” links $\Lambda_r^{r'}$ and “Poissonian” links Λ_r^∞ it is clear that they are Feller links. It remains to check that the intrinsic boundary topology on \mathbb{R}_+ is the conventional topology and so is locally compact.

By the very definition, the intrinsic topology is the weakest one in which all the functions $x \mapsto \Lambda_r^\infty(x, m)$, where parameter r ranges over \mathbb{R}_+ and parameter m ranges over \mathbb{Z}_+ , are continuous. We will prove a stronger claim: even if only m varies but $r > 0$ is chosen arbitrarily and fixed, then the corresponding topology coincides with the conventional one.

To do this, consider the map $\mathbb{R}_+ \rightarrow [0, 1]^\infty$ assigning to $x \in \mathbb{R}_+$ the sequence

$$\{a_m(x) : m \in \mathbb{Z}_+\}, \quad a_m(x) := \Lambda_r^\infty(x, m) = e^{-rx} \frac{(rx)^m}{m!}.$$

This map is injective, for x is recovered from $\{a_m(x)\}$ from the identity

$$\sum_{m=0}^{\infty} s^m a_m(x) = e^{(s-1)rx}.$$

By the very definition, the weakest topology on \mathbb{R}_+ making all the functions $a_m(x)$ continuous is exactly the topology induced by the embedding of \mathbb{R}_+ into the cube $[0, 1]^\infty$ equipped with the product topology.

Observe now that the cube $[0, 1]^\infty$ is compact and the above map extends by continuity to the one-point compactification $\mathbb{R}_+ \cup \{+\infty\}$ of \mathbb{R}_+ by setting $a_m(+\infty) = 0$ for all m . Obviously, the extended map is injective, too. Therefore, it is a homeomorphism onto a closed subset of $[0, 1]^\infty$. This implies the desired claim.

4.5 The density lemma

If $\{E_r, \Lambda_r^{r'}\}$ is a Feller projective system with boundary E_∞ , then the subspace

$$\bigcup_{r \in I} \Lambda_r^\infty C_0(E_r) \subset C_0(E_\infty)$$

is dense in the norm topology; see Borodin–Olshanski [13, Lemma 2.3]. Here $\Lambda_r^\infty C_0(E_r)$ denotes the range of the operator $\Lambda_r^\infty : C_0(E_r) \rightarrow C_0(E_\infty)$.

For further reference we call this assertion the *density lemma*. Its proof is simple; it relies on the fact that for a locally compact space E , the vector space of (signed) measures on E with finite total variation is the Banach dual to $C_0(E)$.

Since $C_c(E_r)$ is dense in $C_0(E_r)$ and the operator $\Lambda_r^\infty : C_0(E_r) \rightarrow C_0(E_\infty)$ is contractive, the density lemma is equivalent to the assertion that the set of functions of the form

$$x \mapsto \Lambda_r^\infty(x, y), \quad r \in I, \quad y \in E_r,$$

is *total* in $C_0(E_\infty)$ meaning that the linear span of these functions is dense.

For our running example, the latter assertion means that the set of functions

$$e^{-rx} x^n, \quad r > 0, \quad n \in \mathbb{Z}_+$$

is total in $C_0(\mathbb{R}_+)$. But here a stronger claim holds: it is not necessary to take all $r > 0$, we may assume that r is *fixed*. In other words, for any fixed $r > 0$, the space of polynomials in x multiplied by the exponential e^{-rx} is dense in $C_0(\mathbb{R}_+)$; see [14, Corollary 3.1.6] for a simple proof. Thus, in this situation, $\Lambda_r^\infty C_0(E_r) \subset C_0(E_\infty)$ is dense for any fixed r . However, this is a special property of the projective system under consideration; for instance, it does not hold in the context of [13].

4.6 Approximation of boundary measures

Our definition of the boundary measure M_∞ as a limit of a coherent system of measures M_r was purely formal. Here we show that, under a suitable additional assumption, M_∞ is a limit of $\{M_r\}$ in a conventional sense.

Let, as above, $\{E_r, \Lambda_r^{r'}\}$ be a Feller projective system with boundary E_∞ , and adopt the following assumption:

Condition 4.1. *For every $r \in I$ there exists an embedding $\varphi_r : E_r \hookrightarrow E_\infty$ such that:*

- (i) *The image $\varphi_r(E_r)$ is a discrete subset in E_∞ .*
- (ii) *For any fixed $s \in I$ and any fixed $y \in E_s$*

$$\lim_{r \rightarrow \infty} \sup_{x \in E_r} |\Lambda_s^r(x, y) - \Lambda_s^\infty(\varphi_r(x), y)| = 0.$$

So far our measures lived on varying spaces. Now, using the maps φ_r , we can put all them on one and the same space, the boundary E_∞ . Namely, we simply replace M_r with its pushforward $\varphi_r(M_r)$, which is a probability measure on E_∞ . A natural question is whether the resulting measures converge to M_∞ , and the next proposition gives an affirmative answer.

Proposition 4.2. *Assume that Condition 4.1 is satisfied. Let $\{M_r : r \in I\}$ be a coherent system of probability distributions and M_∞ be the corresponding boundary measure. As $r \rightarrow \infty$, the measures $\varphi_r(M_r)$ converge to M_∞ in the weak topology.*

Note that for this proposition, part (i) of the condition is not relevant, but it will be used in the sequel (see Section 5.2).

Proof. We have to show that for any bounded continuous function F

$$\langle F, \varphi_r(M_r) \rangle \rightarrow \langle F, M_\infty \rangle.$$

Since all the measures in question are probability measures, we may replace the weak convergence by the vague convergence, that is, we may assume that F lies in the space $C_0(E_\infty)$. Next, we apply the density lemma (see Section 4.5), which enables us to further assume that F has the form $F(x) = \Lambda_s^\infty(x, y)$ for some fixed $s \in I$ and $y \in E_s$. Then we get

$$\langle F, M_\infty \rangle = \int_{x \in E_\infty} M_\infty(dx) \Lambda_s^\infty(x, y) = M_s(y).$$

On the other hand,

$$\langle F, \varphi_r(M_r) \rangle = \langle F \circ \varphi_r, M_r \rangle. \tag{4.2}$$

Here the function $F \circ \varphi_r$ lives on E_r , and for $x \in E_r$ one can write

$$(F \circ \varphi_r)(x) = F(\varphi_r(x)) = \Lambda_r^\infty(\varphi_r(x), y) = \Lambda_s^r(x, y) + \varepsilon(r, x),$$

where, by virtue of Condition 4.1, the remainder term $\varepsilon(r, x)$ tends to 0 uniformly on x , as $r \rightarrow \infty$. Therefore, (4.2) equals

$$\langle \Lambda_s^r(\cdot, y), M_r \rangle + \dots = M_s(y) + \dots,$$

where the dots denote a remainder term converging to 0. This completes the proof. \square

Example 4.3. Consider the projective system \mathbb{B} introduced in Section 4.3. Recall that then the index set I is $\mathbb{R}_{>0}$, $E_r = \mathbb{Z}_+$ for all $r > 0$, and the boundary E_+ is \mathbb{R}_+ . Define the map $\varphi_r : E_r \rightarrow E_\infty$ as

$$\varphi_r(l) = r^{-1}l, \quad l \in \mathbb{Z}_+,$$

and let us check that Condition 4.1 is satisfied.

Indeed, in our situation it means that for fixed $s > 0$ and $m \in \mathbb{Z}_+$

$$\lim_{r \rightarrow \infty} \sup_{l \in \mathbb{Z}_+} |\mathbb{B}\Lambda_s^r(l, m) - \mathbb{B}\Lambda_s^\infty(r^{-1}l, m)| = 0. \tag{4.3}$$

The explicit expressions for the links in question are (see Section 4.3):

$$\begin{aligned} \mathbb{B}\Lambda_s^r(l, m) &= \frac{l!}{m!(l-m)!} \left(\frac{s}{r}\right)^m \left(1 - \frac{s}{r}\right)^{l-m}, \quad l, m \in \mathbb{Z}_+, \\ \mathbb{B}\Lambda_s^\infty(x, m) &= e^{-sx} \frac{(sx)^m}{m!}, \quad x \in \mathbb{R}_+, \quad m \in \mathbb{Z}_+. \end{aligned}$$

In (4.3), set $x = r^{-1}l$ and note that

$$\frac{l!}{(l-m)!r^m} = x^m (1 + O(r^{-1})), \quad \left(1 - \frac{s}{r}\right)^{l-m} = \left(1 - \frac{s}{r}\right)^{rx} (1 + O(r^{-1})).$$

Therefore, (4.3) follows from the fact that (see [14, Lemma 3.1.4])

$$\lim_{r \rightarrow +\infty} \left(1 - \frac{s}{r}\right)^{rx} x^m = e^{-rx} x^m \text{ uniformly on } x \in \mathbb{R}_+.$$

For this example, Proposition 4.2 gives a specific recipe for approximating arbitrary probability measures on \mathbb{R}_+ by atomic measures supported by the grids $r^{-1}\mathbb{Z}_+$.

5 Boundary Feller semigroups: general formalism

In this section, $\{E_r, \Lambda_r^r\}$ is a Feller projective system with index set I equal to $\mathbb{R}_{>0}$ or $\mathbb{Z}_{>0}$, and boundary E_∞ .

5.1 Intertwining of semigroups

Let E' and E be two locally compact metrizable spaces, $T'(t)$ and $T(t)$ be Feller semigroups on $C_0(E')$ and $C_0(E)$, respectively, and $\Lambda : E' \dashrightarrow E$ be a Feller link. Let us say that Λ *intertwines* the semigroups $T'(t)$ and $T(t)$ if

$$T'(t)\Lambda = \Lambda T(t), \quad t \geq 0, \tag{5.1}$$

where both sides are interpreted as operators $C_0(E) \rightarrow C_0(E')$.

Proposition 5.1. *Assume that for every $r \in I$ we are given a Feller semigroup $\{T_r(t) : t \geq 0\}$ on $C_0(E_r)$. Assume further that the links Λ_r^r intertwine the corresponding semigroups, that is, for any two indices $r' > r$*

$$T_{r'}(t)\Lambda_r^{r'} = \Lambda_r^{r'}T_r(t). \tag{5.2}$$

Then there exists a unique Feller semigroup $\{T_\infty(t) : t \geq 0\}$ on E_∞ such that Λ_r^∞ intertwines $T_\infty(t)$ and $T_r(t)$ for every $r \in I$,

$$T_\infty(t)\Lambda_r^\infty = \Lambda_r^\infty T_r(t), \quad t \geq 0. \tag{5.3}$$

Proof. In the case $I = \mathbb{Z}_{>0}$ this assertion was established in [13, Proposition 2.4]. The same argument works in the case $I = \mathbb{R}_{>0}$. □

We call the semigroup $T_\infty(t)$ constructed in the above proposition the *boundary semigroup*. Now we are going to describe its generator.

We start with the simple observation that relation (5.1) has an infinitesimal analog: namely, denoting by A' and A the generators of the semigroups $T'(t)$ and $T(t)$ from (5.1), one has

$$\Lambda : \text{Dom}(A) \rightarrow \text{Dom}(A')$$

and

$$A'\Lambda = \Lambda A. \tag{5.4}$$

In words, if a Feller link intertwines two Feller semigroups, then it also intertwines their generators. Indeed, this is an immediate consequence of the very definition of the semigroup generator.

Proposition 5.2. *Let the semigroups $T_r(t)$ be as in the above proposition, $T_\infty(t)$ be the corresponding boundary semigroup, and A_r and A_∞ denote the generators of these semigroups. Take for each $r \in I$ an arbitrary core $\mathcal{F}_r \subseteq \text{Dom}(A_r)$ for the operator A_r ; then the linear span of the vectors of the form $\Lambda_r^\infty f$, where r ranges over I and f ranges over \mathcal{F}_r , is a core for A_∞ .*

Note that the action of A_∞ on such a core is determined according to (5.4), that is

$$A_\infty \Lambda_r^\infty f = \Lambda_r^\infty A_r f, \quad f \in \text{Dom}(A_r). \tag{5.5}$$

Proof. We will apply a well-known characterization of cores based on Hille–Yosida’s theorem: Let A be the generator of a strongly continuous contraction semigroup on a Banach space; a subspace $\mathcal{F} \subseteq \text{Dom}(A)$ is a core for A if and only if, for any constant $c > 0$, the subspace $(c - A)\mathcal{F}$ is dense. The proof is simple (cf. [18, Chapter 1, Proposition 3.1]). Indeed, fix an arbitrary $c > 0$. By Hille–Yosida’s theorem, the operator $(c - A)^{-1}$ is defined on the whole space and bounded. Next, the closure of $A|_{\mathcal{F}}$ coincides with A if and only if the closure of $(c - A|_{\mathcal{F}})^{-1}$ coincides with $(c - A)^{-1}$, and this in turn just means that $(c - A)\mathcal{F}$, which is the domain of $(c - A|_{\mathcal{F}})^{-1}$, is dense.

Take now as \mathcal{F} the linear span of the union of the subspaces $\Lambda_r^\infty \mathcal{F}_r$. We already know that \mathcal{F} is contained in $\text{Dom}(A_\infty)$.

By the criterion above, it suffices to prove that $(c - A_\infty)\mathcal{F}$ is dense in $C_0(E_\infty)$ for any $c > 0$. We have

$$(c - A_\infty)\mathcal{F} = \text{span} \left(\bigcup_{r \in I} (c - A_\infty)\Lambda_r^\infty \mathcal{F}_r \right) = \text{span} \left(\bigcup_{r \in I} \Lambda_r^\infty (c - A_r)\mathcal{F}_r \right),$$

where the last equality follows from (5.5). On the other hand, we know that for every $r \in I$, $(c - A_r)\mathcal{F}_r$ is dense in $C_0(E_r)$, because \mathcal{F}_r is a core for A_r . Therefore, the closure of $(c - A_\infty)\mathcal{F}$ coincides with the closure of the subspace $\bigcup_{r \in I} \Lambda_r^\infty C_0(E_r)$. But the latter subspace is dense by Proposition 5.1. Therefore, $(c - A_\infty)\mathcal{F}$ is dense, too. \square

Let us return to the basic intertwining relation (5.1). Under suitable assumptions, one can check it on the infinitesimal level, as seen from the next proposition.

Proposition 5.3. *Assume that:*

- E' and E are two finite or countably infinite sets;
- $\Lambda : E' \dashrightarrow E$ is a stochastic Feller matrix with finitely many nonzero entries in every row;
- Q' and Q are two matrices of format $E' \times E'$ and $E \times E$, respectively, satisfying the assumptions of Theorem 2.3;
- $\{T'(t)\}$ and $\{T(t)\}$ are the corresponding Feller semigroups afforded by that theorem.

Then $Q'\Lambda = \Lambda Q$ implies that $T'(t)\Lambda = \Lambda T(t)$ for all $t \geq 0$.

Note that the assumptions on Λ , Q' , and Q imply that the products $Q'\Lambda$ and ΛQ are well defined and, moreover, these two matrices have finitely many nonzero entries in every row.

Proof. See [13, Section 6.2]. \square

We will use this result to check condition (5.2) from Proposition 5.1.

5.2 Approximation of semigroups

Here we are going to show that, under suitable additional assumptions, the boundary semigroup $T_\infty(t)$ that is afforded by the construction of Proposition 5.1 is approximated by semigroups $T_r(t)$ in the sense of Definition 3.2.

We keep to the hypotheses of Proposition 5.1. Next, we assume that Condition 4.1 is satisfied and one more condition holds:

Condition 5.4. *For every $r \in I$, the space $C_c(E_r)$ of finitely supported functions is a core for the generator A_r of the semigroup $T_r(t)$. Moreover, this space is invariant under the action of A_r .*

We set $\mathcal{L}_r = C_0(E_r)$, $\mathcal{L}_\infty = C_0(E_\infty)$. Given a function f on E_∞ , we define the function $\pi_r f$ on E_r by

$$(\pi_r f)(x) := f(\varphi_r(x)), \quad x \in E_r.$$

Since $\varphi_r(E_r)$ is assumed to be a locally finite subset of E_∞ (see part (i) of Condition 4.1), π_r maps \mathcal{L}_∞ into \mathcal{L}_r . Obviously, the norm of π_r is less or equal to 1.

Proposition 5.5. *Under the above assumptions, $T_r(t) \rightarrow T_\infty(t)$ in the sense of Definition 3.2.*

Proof. Let A_∞ be the generator of the boundary semigroup $T_\infty(t)$. By virtue of Theorem 3.4, it suffices to prove that the restriction of A_∞ to some core \mathcal{F} is approximated by the operators A_r . As \mathcal{F} we take the linear span of the subspaces $\Lambda_r^\infty C_c(E_r) \subset C_0(E_\infty)$, where r ranges over I . The second condition postulated above says that $C_c(E_r)$ is a core of A_r ; consequently, \mathcal{F} is a core for A_∞ , by virtue of Proposition 5.2.

According to Definition 3.3 we have to show that for any vector $f \in \mathcal{F}$ one can find a family of vectors $f_r \in \text{Dom}(A_r)$ such that the following two limit relations hold: $f_r \rightarrow f$ and $A_r f_r \rightarrow A_\infty f$ as $r \rightarrow \infty$.

Without loss of generality we may assume that $f \in \Lambda_s^\infty g$ with $g \in C_c(E_s)$ for some $s \in I$. Next, for $r > s$ we set $f_r := \Lambda_s^r g$ and observe that it suffices to prove the first limit relation only. Indeed, once we know that $f_r \rightarrow f$ with such a choice of $\{f_r\}$, the second limit relation, $A_r f_r \rightarrow A_\infty f$, follows simply by replacing g with $A_s g$, because the links intertwine the generators. We also use the fact that $g \in C_c(E_s)$ implies $A_s g \in C_c(E_s)$ (see the end of the second condition above).

We proceed to the proof of the convergence $f_r \rightarrow f$. By Definition 3.1, it means that

$$\lim_{r \rightarrow \infty} \sup_{x \in E_r} |f_r(x) - f(\varphi_r(x))| = 0.$$

Without loss of generality we may assume that g is the delta-function at a point $y \in E_s$, but then the desired limit relation holds by virtue of Condition 4.1. \square

6 A toy example: the one-dimensional Laguerre diffusion

In this section we apply the abstract formalism described above to a construction of the Laguerre diffusion process on the halfline \mathbb{R}_+ , generated by the differential operator

$$x \frac{d^2}{dx^2} + (c - x) \frac{d}{dx}$$

(here $c > 0$ is a parameter). This process is well known — it is related to the Bessel process in the same way as the Ornstein-Uhlenbeck process is related to the Wiener process, see, e.g. Eie [17]. Thus, the final result is by no means new. However, the detailed exposition presented below will serve us as a preparation and a guiding example for Section 8, where we establish the main results.

6.1 The binomial projective system \mathbb{B}

Recall that \mathbb{B} was introduced in Section 4.3. We will prove two technical propositions concerning the properties of the links of \mathbb{B} .

Observe that every link ${}^{\mathbb{B}}\Lambda_r'$ can be applied to an arbitrary function on \mathbb{Z}_+ (viewed as a column vector), because each row in ${}^{\mathbb{B}}\Lambda_r'$ has finitely many nonzero entries. As for ${}^{\mathbb{B}}\Lambda_r^\infty$, it can be applied to functions on \mathbb{Z}_+ with moderate (say, at most polynomial) growth at infinity. In the next two propositions we provide explicit formulas for the action of the links on functions of some special kind.

Introduce a notation:

$$y^{\downarrow m} = y(y-1)\dots(y-m+1), \quad m \in \mathbb{Z}_+.$$

Here y is assumed to range over \mathbb{R}_+ or \mathbb{Z}_+ , depending on the context. By $\mathbf{1}_m$, where $m \in \mathbb{Z}_+$, we denote the function on \mathbb{Z}_+ equal to 1 at m and to 0 on $\mathbb{Z}_+ \setminus \{m\}$. The letter q always denotes a number from the open interval $(0, 1)$. Note that

$$\lim_{q \rightarrow +0} \frac{1}{m!q^m} q^y y^{\downarrow m} = \mathbf{1}_m \quad \text{on } \mathbb{Z}_+. \tag{6.1}$$

Proposition 6.1. Assume $r' > r > 0$ and $0 < q < 1$, and let l range over \mathbb{Z}_+ . Regard ${}^{\mathbb{B}}\Lambda_r^{r'}$ as an operator in the space of functions on \mathbb{Z}_+ transforming a function $F(l)$ to a function $G(l)$. Under this transformation

$$l \downarrow^m \mapsto \left(\frac{r}{r'}\right)^m l \downarrow^m \tag{6.2}$$

$$\mathbf{1}_m \mapsto \frac{1}{m!} \left(\frac{1-q'}{q'}\right)^m \cdot (q')^l l \downarrow^m, \quad q' := 1 - \frac{r}{r'} \tag{6.3}$$

$$q^l l \downarrow^m \mapsto \left(\frac{qr}{q'r'}\right)^m (q')^l l \downarrow^m, \quad q' := 1 - (1-q)\frac{r}{r'} \tag{6.4}$$

Proof. Let us prove (6.4). The function $F(l) = q^l l \downarrow^m$ vanishes on $\{0, \dots, m-1\}$ and the same holds for ${}^{\mathbb{B}}\Lambda_r^{r'} F$, because the matrix ${}^{\mathbb{B}}\Lambda_r^{r'}$ is lower triangular. Therefore, it suffices to compute $({}^{\mathbb{B}}\Lambda_r^{r'} F)(l)$ for $l \geq m$. We have

$$({}^{\mathbb{B}}\Lambda_r^{r'} F)(l) = \sum_{k=m}^l \left(1 - \frac{r}{r'}\right)^{l-k} \left(\frac{r}{r'}\right)^k \frac{l!}{(l-k)! k!} q^k \frac{k!}{(k-m)!}.$$

Setting $k' = k - m$ and $l' = l - m$ we rewrite the right-hand side as

$$\left(\frac{qr}{r'}\right)^m l \downarrow^m \sum_{k'=0}^{l'} \left(1 - \frac{r}{r'}\right)^{l'-k'} \left(\frac{qr}{r'}\right)^{k'} \frac{l!}{(l'-k')! k'!}.$$

The latter sum equals

$$(q')^{l'} = (q')^{-m} (q')^l,$$

which leads to the desired result.

Formula (6.2) can be checked in exactly the same way. Observe also that (6.2) is a limit case of (6.4) as $q \rightarrow 1$.

Formula (6.3) is immediate from the very definition of ${}^{\mathbb{B}}\Lambda_r^{r'}$. On the other hand, (6.3) can also be obtained from (6.4) as a limit case: to see this, divide by $m!q^m$, let $q \rightarrow 0$ and use (6.9). \square

Proposition 6.2. Assume $r > 0$ and $0 < q < 1$, and let l range over \mathbb{Z}_+ while x ranges over \mathbb{R}_+ . Regard ${}^{\mathbb{B}}\Lambda_r^\infty$ as an operator transforming a function $F(l)$ on \mathbb{Z}_+ to a function $G(x)$ on \mathbb{R}_+ . Under this transformation

$$l \downarrow^m \mapsto r^m x^m \tag{6.5}$$

$$\mathbf{1}_m \mapsto \frac{r^m}{m!} \cdot q_\infty^x x^m, \quad q_\infty := e^{-r} \tag{6.6}$$

$$q^l l \downarrow^m \mapsto q^m r^m \cdot q_\infty^x x^m, \quad q_\infty := e^{-(1-q)r}. \tag{6.7}$$

Proof. We may argue exactly as in the proof of Proposition 6.1, replacing the binomial distribution by the Poisson distribution.

Alternatively, one can use (4.3) and pass to the limit $l \rightarrow \infty$, $r' \rightarrow \infty$, $l/r' \rightarrow x$ in the formulas of Proposition 6.1. \square

6.2 The Meixner and Laguerre semigroups

Introduce a Q -matrix of format $\mathbb{Z}_+ \times \mathbb{Z}_+$, depending on parameters $c > 0$ and $r > 0$, with the entries

$$\begin{aligned} Q_r^{(c)}(k, k+1) &= r(c+k), & Q_r^{(c)}(k, k-1) &= (r+1)k, \\ Q_r^{(c)}(k, k) &= -[r(c+k) + (r+1)k] = -[(2r+1)k + rc], \\ Q_r^{(c)}(k, k') &= 0, & |k - k'| &\geq 2, \end{aligned}$$

where k ranges over \mathbb{Z}_+ . Let us regard $Q_r^{(c)}$ as a difference operator acting on functions on \mathbb{Z}_+ , which are interpreted as column vectors:

$$(Q_r^{(c)}F)(l) = r(c+l)F(l+1) + (r+1)lF(l-1) - [(2r+1)l+rc]F(l), \tag{6.8}$$

where $l \in \mathbb{Z}_+$. As is seen from the next proposition, this difference operator is related to the classical *Meixner orthogonal polynomials*. Recall the definition of these polynomials (see, e.g., Koekoek–Lesky–Swarttouw [24] and references therein):

The Meixner polynomials are orthogonal with respect to the negative binomial distribution on \mathbb{Z}_+ ,

$$\sum_{l \in \mathbb{Z}_+} (1+r)^{-c} \frac{(c)_l}{l!} \left(\frac{r}{1+r}\right)^l \delta_l,$$

where $(c)_l := c(c+1)\dots(c+l-1)$ is the Pochhammer symbol and δ_l denotes the delta measure at l . The explicit expression for the *monic* Meixner polynomial of degree $n = 0, 1, 2, \dots$ is

$$M_n(l; c, r) = (c)_n \sum_{m=0}^n (-r)^{n-m} \frac{n^{\downarrow m}}{(c)_m m!} l^{\downarrow m}. \tag{6.9}$$

Proposition 6.3. *The Meixner difference operator (6.8) preserves the space of polynomials. We have*

$$Q_r^{(c)} : l^{\downarrow m} \rightarrow -ml^{\downarrow m} + rm(m+c-1)l^{\downarrow(m-1)} \tag{6.10}$$

and

$$Q_r^{(c)} : M_n(l; c, r) \rightarrow -nM_n(l; c, r). \tag{6.11}$$

Thus, the Meixner difference operator is diagonalized in the basis of the Meixner polynomials.

Proof. All claims can be verified directly. For (6.11), see also [24]. □

Proposition 6.4. *For arbitrary $r' > r > 0$, we have*

$$Q_{r'}^{(c)} \mathbb{B}\Lambda_r^{r'} = \mathbb{B}\Lambda_r^{r'} Q_r^{(c)}.$$

Proof. Because the Q -matrices in question have a simple tridiagonal form and the entries of $\mathbb{B}\Lambda_r^{r'}$ are given by a simple expression, a direct check is possible. However, we prefer to give another proof, which has the advantage of being more conceptual and well suited for the generalization that we need.

Observe that

$$Q_{r'}^{(c)} \mathbb{B}\Lambda_r^{r'} F = \mathbb{B}\Lambda_r^{r'} Q_r^{(c)} F$$

for any polynomial F . Indeed, it suffices to check this for $F = M_m(\cdot; c, r)$. It follows from (6.2) and (6.9) that

$$\mathbb{B}\Lambda_r^{r'} M_m(\cdot; c, r) = \left(\frac{r}{r'}\right)^m M_m(\cdot; c, r'),$$

and then we use (6.11) to conclude that both $Q_{r'}^{(c)} \mathbb{B}\Lambda_r^{r'}$ and $\mathbb{B}\Lambda_r^{r'} Q_r^{(c)}$ multiply $M_m(\cdot; c, r)$ by $-m(r/r')^m$.

Further, both matrices $Q_{r'}^{(c)} \mathbb{B}\Lambda_r^{r'}$ and $\mathbb{B}\Lambda_r^{r'} Q_r^{(c)}$ have finitely many nonzero entries in every row. Since the polynomials separate points on \mathbb{Z}_+ , these two matrices coincide. □

Proposition 6.5. *For any $c, r > 0$, the matrix $Q_r^{(c)}$ satisfies the assumptions of Theorem 2.3 with functions $\gamma(k) = \eta(k) = k + 1$.*

Proof. Easy direct check. □

This proposition makes it possible to apply Theorem 2.3, which in turn entails the following assertions.

Corollary 6.6. (i) *The Q -matrix $Q_r^{(c)}$ gives rise to a Feller semigroup $T_r^{(c)}(t)$ on $C_0(\mathbb{Z}_+)$ whose generator $A_r^{(c)}$ is implemented by $Q^{(c)}$.*

(ii) *The subspace $C_0(\mathbb{Z}_+)$ is a core for generator $A_r^{(c)}$.*

We call $T_r^{(c)}(t)$ the *Meixner semigroup*. It determines a continuous time Markov chain on \mathbb{Z}_+ which we call the *Meixner chain* and denote by $X^{(c)}(t)$.

Proposition 6.7. *For every $c > 0$ there exists a unique Feller Markov process $X^{(c)}(t)$ on \mathbb{R}_+ such that the corresponding Feller semigroup, denoted by $T^{(c)}(t)$, is consistent with the Meixner semigroups $T_r^{(c)}(t)$, $r > 0$, in the sense that*

$$T^{(c)}(t) \mathbb{B}\Lambda_r^\infty = \mathbb{B}\Lambda_r^\infty T_r^{(c)}(t), \quad t \geq 0, \quad r > 0.$$

Proof. We know that the Q -matrices with various values of parameter r are consistent with the links (Proposition 6.4). It follows, by virtue of Proposition 5.3, that the semigroups are also consistent with the links. Therefore, we may apply Proposition 5.1, which gives the desired result. □

We call $X^{(c)}(t)$ and $T^{(c)}(t)$ the *Laguerre process* and the *Laguerre semigroup*, respectively; this terminology is justified by the results of Section 6.4.

6.3 A family of cores for Markov semigroup generators

For any fixed $q \in (0, 1)$, the functions $q^x x^m$, $m = 0, 1, 2, \dots$, span a dense subspace in $C_0(\mathbb{R}_+)$, see Borodin–Olshanski [14, Corollary 3.1.6]. This also implies that the functions $q^l l^m$, where $m = 0, 1, 2, \dots$ and l ranges over \mathbb{Z}_+ , span a dense subspace in $C_0(\mathbb{Z}_+)$. These facts are used in the next proposition.

Proposition 6.8. (i) *For any $r' > r > 0$, the operator $\mathbb{B}\Lambda_r^{r'} : C_0(\mathbb{Z}_+) \rightarrow C_0(\mathbb{Z}_+)$ has a dense range.*

(ii) *Likewise, for any $r > 0$, the operator $\mathbb{B}\Lambda_r^\infty : C_0(\mathbb{Z}_+) \rightarrow C_0(\mathbb{R}_+)$ has a dense range.*

Proof. (i) Take an arbitrary $q \in (0, 1)$. By (6.4), $\mathbb{B}\Lambda_r^{r'}$ maps the linear span of functions $q^l l^m$, $m = 0, 1, 2, \dots$ onto the linear span of functions $(q')^l l^m$, with some other $q' \in (0, 1)$, see (6.4). Since these spans are dense, we get the desired claim.

(ii) The same argument, with reference to (6.7). □

Recall that $A_r^{(c)}$ denotes the generator of semigroup $T_r^{(c)}(t)$ (Corollary 6.6). Likewise, let $A^{(c)}$ denote the generator of semigroup $T^{(c)}(t)$.

Proposition 6.9. *Fix an arbitrary number $q \in (0, 1)$.*

(i) *For every $r > 0$, the linear span of functions $q^l l^m$, $m = 0, 1, 2, \dots$, where argument l ranges over \mathbb{Z}_+ , is a core for $A_r^{(c)}$.*

(ii) *Likewise, the linear span of functions $q^x x^m$, $m = 0, 1, 2, \dots$, where argument x ranges over \mathbb{R}_+ , is a core for $A^{(c)}$.*

Proof. (i) Observe that if $r_2 > r_1 > 0$ and \mathcal{F}_1 is a core for $A_{r_1}^{(c)}$, then $\mathcal{F}_2 := \mathbb{B}\Lambda_{r_1}^{r_2} \mathcal{F}_1$ is a core for $A_{r_2}^{(c)}$. Indeed, by virtue of claim (i) of Proposition 6.8, we may apply the argument of Proposition 5.2.

Now take $r_2 = r$ and $r_1 = (1 - q)r$. Then, as seen from (6.3), the linear span of functions $q^y y^m$ is just the image under $\mathbb{B}\Lambda_{r_1}^{r_2}$ of the space $C_c(\mathbb{Z}_+)$. By virtue of Proposition

6.5 and claim (iv) of Theorem 2.3, $C_c(\mathbb{Z}_+)$ is a core for $A_{r_1}^{(c)}$. Therefore, its image is a core for $A_{r_2}^{(c)}$.

(ii) We argue as above. First, application of claim (ii) of Proposition 6.8 allows us to conclude that if $\mathcal{F} \subset C_0(\mathbb{Z}_+)$ is a core for $A_r^{(c)}$ for some $r > 0$, then ${}^{\mathbb{B}}\Lambda_r^\infty \mathcal{F}$ is a core for $A^{(c)}$.

Next, given $q \in (0, 1)$ we take $r = -\log q$ and $\mathcal{F} = C_0(\mathbb{Z}_+)$. As mentioned above, \mathcal{F} is a core for $A_r^{(c)}$. On the other hand, (6.6) shows that the linear span of functions $q^x x^m$ coincides with ${}^{\mathbb{B}}\Lambda_r^\infty \mathcal{F}$. □

6.4 The Laguerre differential operator

Proposition 6.9 implies that the generator $A^{(c)}$ is uniquely determined by its action on functions of the form $q^x x^m$, $m = 0, 1, 2, \dots$, with an arbitrary fixed $q \in (0, 1)$. This action can be readily computed from the basic relation $A^{(c)} {}^{\mathbb{B}}\Lambda_r^\infty = {}^{\mathbb{B}}\Lambda_r^\infty Q_r^{(c)}$:

Proposition 6.10. *The action of $A^{(c)}$ on functions of the form $q^x x^m$, $m = 0, 1, 2, \dots$ is implemented by the differential operator*

$$D^{(c)} := x \frac{d^2}{dx^2} + (c - x) \frac{d}{dx}. \tag{6.12}$$

Proof. Let $r > 0$ be related to $q \in (0, 1)$ by $r = -\log q$. Consider the functions

$$f_m(x) := \frac{r^m}{m!} q^x x^m = \frac{r^m}{m!} e^{-rx} x^m, \quad m = 0, 1, 2, \dots, \quad x \in \mathbb{R}_+.$$

By (6.6),

$${}^{\mathbb{B}}\Lambda_r^\infty \mathbf{1}_m = f_m, \quad m = 0, 1, 2, \dots$$

On the other hand, it is directly verified that the difference operator $Q_r^{(c)}$ defined in (6.8) acts on the delta functions $\mathbf{1}_m$ in the same way as the differential operator $D^{(c)}$ acts on the functions f_m :

$$Q_r^{(c)} \mathbf{1}_m = r(c + m - 1) \mathbf{1}_{m-1} + (r + 1)(m + 1) \mathbf{1}_{m+1} - [(2r + 1)m + rc] \mathbf{1}_m \tag{6.13}$$

$$D^{(c)} f_m = r(c + m - 1) f_{m-1} + (r + 1)(m + 1) f_{m+1} - [(2r + 1)m + rc] f_m, \tag{6.14}$$

where

$$\mathbf{1}_{-1} := 0, \quad f_{-1} := 0.$$

This concludes the proof. □

Consider the gamma distribution on \mathbb{R}_+ with parameter c :

$$\frac{1}{\Gamma(c)} x^{c-1} e^{-x} dx, \quad x \in \mathbb{R}_+,$$

and let $L_n(x; c)$ denote the *monic Laguerre polynomials* of degree $n = 0, 1, 2, \dots$, which are orthogonal with respect to this distribution:

$$L_n(x; c) = (c)_n \sum_{m=0}^n (-1)^{n-m} \frac{n \downarrow m}{(c)_m m!} x^m. \tag{6.15}$$

The differential operator $D^{(c)}$ is diagonalized in the basis of the Laguerre polynomials:

$$D^{(c)} L_n(\cdot; c) = -n L_n(\cdot; c), \quad n = 0, 1, 2, \dots \tag{6.16}$$

Note also that

$${}^{\mathbb{B}}\Lambda_r^\infty M_n(\cdot; c, r) = r^n L_n(\cdot; c). \tag{6.17}$$

The proof is immediate: we compare the expansions of the Meixner and Laguerre polynomials in the bases $\{l \downarrow m\}$ and $\{x^m\}$, respectively (see (6.9) and (6.15)), and then apply (6.5), which says that ${}^{\mathbb{B}}\Lambda_r^\infty$ takes the factorial monomial $l \downarrow m$ to $r^m x^m$.

6.5 Approximation

We use the embedding $\varphi_r : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ introduced in Example 4.3 and define the projection $\pi_r : C_0(\mathbb{R}_+) \rightarrow C_0(\mathbb{Z}_+)$ as in Section 5.2.

Proposition 6.11. *Let $c > 0$ be fixed. As $r \rightarrow +\infty$, the Meixner semigroups $T_r^{(c)}(t)$ approximate the Laguerre semigroup $T^{(c)}(t)$ in the sense of Definition 3.2.*

Proof. Let us check all the hypotheses of Proposition 5.5. Then the desired result will follow from that proposition.

In fact, the assumptions stated in Section 5.1 are satisfied: we know that \mathbb{B} is a Feller system, the Meixner semigroups are consistent with the links of \mathbb{B} , those are Feller links, and, by the very definition, the Laguerre semigroup is the boundary semigroup determined by the Meixner semigroups.

Next, the fulfilment of Condition 4.1 was established in Example 4.3.

It remains to check Condition 5.4. In our situation, it consists in the requirement that $C_c(\mathbb{Z}_+)$ is a core for generator $A_r^{(c)}$ and, moreover, is invariant under its action. The fact that $C_c(\mathbb{Z}_+)$ is a core follows from Corollary 6.6, item (ii). Its invariance follows from item (i), because $C_c(\mathbb{Z}_+)$ is obviously invariant under the action of $Q_r^{(c)}$.

This completes the proof. □

7 A few definitions

Here we collect some basic definitions that will be needed in the next section. For a more detailed information we refer to Sagan [36] and Stanley [39] (generalities on Young diagrams, Young tableaux, and symmetric functions); Olshanski–Regev–Vershik [33] (Frobenius–Schur symmetric functions); Borodin–Olshanski [14], [12] (Thoma’s simplex and Thoma’s cone).

7.1 Young diagrams

Recall that the *Young poset* is the set \mathbb{Y} of all Young diagrams (including the empty diagram \emptyset) with the partial order determined by containment of one Young diagram in another. For $\lambda \in \mathbb{Y}$ we denote by $|\lambda|$ the number of boxes of λ and we set

$$\mathbb{Y}_n = \{\lambda \in \mathbb{Y} : |\lambda| = n\}, \quad n = 0, 1, 2, \dots$$

This makes \mathbb{Y} a *graded poset*. It is actually a lattice, so it is often called the *Young lattice*.

The *dimension* of a diagram $\lambda \in \mathbb{Y}$, denoted by $\dim \lambda$, is the number of standard Young tableaux of shape λ , which is the same as the number of saturated chains

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(n)} = \lambda, \quad n := |\lambda|,$$

in the poset \mathbb{Y} .

More generally, for arbitrary two diagrams $\mu, \lambda \in \mathbb{Y}$ we define $\dim(\mu, \lambda)$ as the number of standard Young tableaux of skew shape λ/μ provided that $\mu \subseteq \lambda$; otherwise $\dim(\mu, \lambda) = 0$ (let us agree that $\dim(\lambda, \lambda) = 1$). Obviously, $\dim \lambda = \dim(\emptyset, \lambda)$. If $\mu \subset \lambda$, then $\dim(\mu, \lambda)$ equals the number of saturated chains with ends μ and λ .

7.2 Symmetric functions

By Sym we denote the graded algebra of symmetric functions over the base field \mathbb{R} . We will need two bases in Sym , both indexed by arbitrary diagrams $\mu \in \mathbb{Y}$: the *Schur functions* S_μ and the *Frobenius–Schur functions* FS_μ . The relationship between

S_μ 's and FS_μ 's is similar to the relationship between the one-variate monomials x^m and their factorial counterparts $x^{\downarrow m}$. Observe that $x^{\downarrow m}$ can be characterized as a unique polynomial in x with highest degree term x^m and such that it vanishes at the integer points $0, 1, \dots, m - 1$. Likewise, one can realize Sym as a subalgebra in $\text{Fun}(\mathbb{Y})$, the algebra of real-valued functions on \mathbb{Y} with all operations defined pointwise; see the next two paragraphs. Then FS_μ can be characterized as a unique element of Sym that has top degree term S_μ and vanishes at all diagrams strictly contained in μ .

Let p_1, p_2, \dots denote the power-sum symmetric functions. We turn them into functions on \mathbb{Y} by setting

$$p_k(\lambda) := \sum_{i=1}^{\infty} ((\lambda_r - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k) = \sum_{i=1}^d (a_r^k + (-1)^{k-1} b_r^k),$$

where λ ranges over \mathbb{Y} , $(\lambda_1, \lambda_2, \dots)$ is the partition corresponding to λ , d is the number of boxes on the main diagonal of λ , and $(a_1, \dots, a_d; b_1, \dots, b_d)$ is the collection of the *modified Frobenius coordinates* of λ :

$$a_r = \lambda_r - i + \frac{1}{2}, \quad b_r = \lambda'_r - i + \frac{1}{2}, \quad i = 1, \dots, d$$

(here λ' is the transposed diagram). One can easily prove that the resulting functions remain algebraically independent.

Next, every element $F \in \text{Sym}$ is uniquely written as a polynomial in p_1, p_2, \dots ; then we define $F(\lambda)$ as the same polynomial in numeric variables $p_1(\lambda), p_2(\lambda), \dots$. In this way we get the desired embedding of Sym into $\text{Fun}(\mathbb{Y})$.

A fundamental property of the Frobenius–Schur functions is the following identity (see [33, Section 2]) relating them to the dimension function in the poset \mathbb{Y} :

$$l^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \lambda} = FS_\mu(\lambda), \quad l := |\lambda|, \quad m := |\mu|. \tag{7.1}$$

7.3 The Thoma simplex and the Thoma cone

The *Thoma simplex* is the subspace Ω of the infinite product space $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$ formed by all couples (α, β) , where $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are two infinite sequences such that

$$\alpha_1 \geq \alpha_2 \geq \dots \geq 0, \quad \beta_1 \geq \beta_2 \geq \dots \geq 0 \tag{7.2}$$

and

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq 1. \tag{7.3}$$

We equip Ω with the product topology inherited from $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$. Note that in this topology, Ω is a compact metrizable space.

The *Thoma cone* $\tilde{\Omega}$ is the subspace of the infinite product space $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$ formed by all triples $\omega = (\alpha, \beta, \delta)$, where $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ are two infinite sequences and δ is a nonnegative real number, such that the couple (α, β) satisfies (7.2) and the modification of the inequality (7.3) of the form

$$\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq \delta.$$

We set $|\omega| = \delta$.

Note that $\tilde{\Omega}$ is a locally compact space in the product topology inherited from $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$. The space $\tilde{\Omega}$ is also metrizable and has countable base. Every subset of the

form $\{\omega \in \tilde{\Omega} : |\omega| \leq \text{const}\}$ is compact. Therefore, a sequence of points ω_n goes to infinity in $\tilde{\Omega}$ if and only if $|\omega_n| \rightarrow \infty$.

We will identify Ω with the subset of $\tilde{\Omega}$ formed by triples $\omega = (\alpha, \beta, \delta)$ with $\delta = 1$. The name ‘‘Thoma cone’’ given to $\tilde{\Omega}$ is justified by the fact that $\tilde{\Omega}$ may be viewed as the cone with the base Ω : the ray of the cone passing through a base point $(\alpha, \beta) \in \Omega$ consists of the triples $\omega = (r\alpha, r\beta, r)$, $r \geq 0$.

More generally, for $\omega = (\alpha, \beta, \delta) \in \tilde{\Omega}$ and $r > 0$ we set $r\omega = (r\alpha, r\beta, r\delta)$.

7.4 Two maps

We embed Sym into the algebra of (non necessarily bounded) continuous functions on the Thoma cone by setting

$$p_k(\omega) = \begin{cases} \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{i=1}^{\infty} \beta_i^k, & k = 2, 3, \dots \\ |\omega|, & k = 1, \end{cases}$$

where ω ranges over $\tilde{\Omega}$.

We embed the set \mathbb{Y} into $\tilde{\Omega}$ through the map

$$\lambda \mapsto \omega_\lambda := ((a_1, \dots, a_d, 0, 0, \dots), (b_1, \dots, b_d, 0, 0, \dots), |\lambda|),$$

where, as above, $(a_1, \dots, a_d; b_1, \dots, b_d)$ is the collection of the modified Frobenius coordinates of a diagram $\lambda \in \mathbb{Y}$. Note that $|\omega_\lambda| = |\lambda|$.

For any $F \in \text{Sym}$, the restriction of the function $F(\omega)$ to the subset $\mathbb{Y} \subset \tilde{\Omega}$ agrees with the previous definition of the function $F(\lambda)$:

$$F(\omega_\lambda) = F(\lambda), \quad \lambda \in \mathbb{Y}.$$

8 Construction of Feller processes on the Thoma cone

8.1 The projective system associated with the Young bouquet

The representation theory of inductive limit groups provides two fundamental examples of projective systems. One is related to the infinite symmetric group $S(\infty)$ and comes from the Young graph \mathbb{Y} , and the other one is related to the infinite-dimensional unitary group $U(\infty)$ and comes from the Gelfand–Tsetlin graph \mathbb{GT} . The boundaries of these two projective systems can be viewed as dual objects to $S(\infty)$ and $U(\infty)$, respectively. In attempt to explain a surprising similarity between the two boundaries, we introduced in [14] a new object which serves as a ‘‘mediator’’ between \mathbb{Y} and \mathbb{GT} . We called it the *Young bouquet*; it is a close relative of \mathbb{Y} and at the same time it can be obtained as a degeneration of \mathbb{GT} . Associated with the Young bouquet is a new projective system denoted by \mathbb{YB} . Because \mathbb{GT} is graded by discrete set \mathbb{Z}_+ , the associated projective system has \mathbb{Z}_+ as its index set, but under degeneration the index set becomes continuous. Here is a formal definition of \mathbb{YB} :

The index set of the projective system \mathbb{YB} is the set $\mathbb{R}_{>0}$ and each set E_r is a copy of the set \mathbb{Y} . For every couple $r' > r$ of positive real numbers, the corresponding link $\mathbb{Y} \dashrightarrow \mathbb{Y}$ is the following stochastic matrix of format $\mathbb{Y} \times \mathbb{Y}$:

$${}^{\mathbb{YB}}\Lambda_r^{r'}(\lambda, \mu) = \left(1 - \frac{r}{r'}\right)^{l-m} \left(\frac{r}{r'}\right)^m \frac{l!}{(l-m)!m!} \frac{\dim \mu \dim(\mu, \lambda)}{\dim \lambda}, \tag{8.1}$$

where $l := |\lambda|$ and $m := |\mu|$.

Note that (8.1) factorizes into a product of two links, which refer to two projective systems, the binomial system \mathbb{B} and the Young graph \mathbb{Y} :

$${}^{\mathbb{YB}}\Lambda_r^{r'}(\lambda, \mu) = {}^{\mathbb{B}}\Lambda_r^{r'}(l, m) {}^{\mathbb{Y}}\Lambda_m^l(\lambda, \mu), \tag{8.2}$$

where

$$\mathbb{Y}\Lambda_m^l(\lambda, \mu) := \frac{\dim \mu \dim(\mu, \lambda)}{\dim \lambda}. \tag{8.3}$$

The links (8.1) satisfy the relation

$$\mathbb{Y}\mathbb{B}\Lambda_{r'}^{r''} \mathbb{Y}\mathbb{B}\Lambda_r^{r'} = \mathbb{Y}\mathbb{B}\Lambda_r^{r''}, \quad r'' > r' > r,$$

so that they do determine a projective system. We refer to [14] for more details.

By [14, Theorem 3.4.7], the boundary of $\mathbb{Y}\mathbb{B}$ is the Thoma cone $\tilde{\Omega}$ together with a family of links $\tilde{\Omega} \dashrightarrow \mathbb{Z}_+$ indexed by positive real numbers r and given by

$$\mathbb{Y}\mathbb{B}\Lambda_r^\infty(\omega, \mu) = e^{-r|\omega|} \frac{r^m}{m!} \dim \mu \cdot S_\mu(\omega), \quad \omega \in \tilde{\Omega}, \quad \mu \in \mathbb{Y}. \tag{8.4}$$

Recall that S_μ is the Schur symmetric function and its value at $\omega \in \tilde{\Omega}$ is understood in accordance with the definition given in Section 7.4.

Proposition 8.1. *The projective system $\mathbb{Y}\mathbb{B}$ is Feller in the sense of the definition given in Section 4.3.*

Proof. The links $\mathbb{Y}\mathbb{B}\Lambda_r^{r'}$ and $\mathbb{Y}\mathbb{B}\Lambda_r^\infty$ are Feller: this immediately follows from the Feller property of the links $\mathbb{B}\Lambda_r^{r'}$ and $\mathbb{B}\Lambda_r^\infty$. It remains to show that the product topology of the space $\tilde{\Omega}$ coincides with that defined by all the maps $\omega \mapsto \Lambda_r^\infty(\omega, \mu)$, where r ranges over $\mathbb{R}_{>0}$ and μ ranges over \mathbb{Y} . Actually, this holds even if r is any fixed number > 0 , and the argument is similar to that given in Section 4.3.

Namely, we extend the above maps to the one-point compactification $\tilde{\Omega} \cup \infty$ of $\tilde{\Omega}$ in a natural way: the value at infinity is equal to 0 for any μ , which agrees with the Feller property of the links. Then we only have to check that any point of $\tilde{\Omega} \cup \infty$ is uniquely determined by its images under the (extended) maps $\mathbb{Y}\mathbb{B}\Lambda_r^\infty(\cdot, \mu)$, where μ ranges over \mathbb{Y} .

To do this, assume first that $\omega \in \tilde{\Omega}$ and recall (8.4). Keeping m fixed and summing the quantity in the right-hand side over $\mu \in \mathbb{Y}_m$ we get

$$e^{-rx} \frac{(rx)^m}{m!}, \quad x := |\omega|,$$

because

$$\sum_{\mu \in \mathbb{Y}_m} \dim \mu S_\mu = (p_1)^m$$

and $p_1(\omega) = |\omega| = x$.

Observe that for $r > 0$ fixed, the quantities $e^{-rx} \frac{(rx)^m}{m!}$, where m ranges over \mathbb{Z}_+ , determine x uniquely. It follows, in particular, that we can recognize whether we are dealing with an element of the Thoma cone $\tilde{\Omega}$ or the added point ∞ , because the latter case corresponds to $x = +\infty$.

Therefore, it suffices to check that an element $\omega \in \tilde{\Omega}$ is uniquely determined by the quantities $S_\mu(\omega)$, where μ ranges over \mathbb{Y} . But this follows from the fact that the functions $p_1(\omega), p_2(\omega), \dots$ separate the points of the Thoma cone. \square

Note that $\mathbb{Y}\mathbb{B}\Lambda_r^{r'}(\nu, \mu)$ vanishes unless $m \leq n$ and $\mu \subseteq \lambda$. This implies that each row of the matrix $\mathbb{Y}\mathbb{B}\Lambda_r^{r'}$ has finitely many nonzero entries, so that the link can be applied to an arbitrary function on \mathbb{Y} .

Below we denote by $\mathbf{1}_\mu$ the delta function on \mathbb{Y} concentrated at the point μ , that is,

$$\mathbf{1}_\mu(\lambda) = \begin{cases} 1, & \lambda = \mu, \\ 0, & \lambda \neq \mu. \end{cases}$$

Proposition 8.2 (cf. Proposition 6.1). *Assume that:*

- $r' > r > 0$ and $0 < q < 1$;
- λ range over \mathbb{Y} and $l := |\lambda|$;
- $\mu \in \mathbb{Y}$ is fixed and $m = |\mu|$.

Regard ${}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'}$ as linear map $F \mapsto G$ transforming a function $F(\lambda)$ on \mathbb{Y} to another function $G(\lambda)$. Under this transformation

$$FS_\mu(\lambda) \mapsto \left(\frac{r}{r'}\right)^m FS_\mu(\lambda), \tag{8.5}$$

$$(\dim \mu)^{-1} \mathbf{1}_\mu \mapsto \frac{1}{m!} \left(\frac{1-q'}{q'}\right)^m \cdot (q')^l FS_\mu(\lambda), \quad q' := 1 - \frac{r}{r'}, \tag{8.6}$$

$$q^l FS_\mu(\lambda) \mapsto \left(\frac{qr}{q'r'}\right)^m (q')^l FS_\mu(\lambda), \quad q' := 1 - (1-q)\frac{r}{r'}. \tag{8.7}$$

Proof. Let us prove (8.7). The function $F(\lambda) := q^l FS_\mu(\lambda)$ vanishes unless $\lambda \supseteq \mu$, and the same holds for ${}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'} F$, because the matrix ${}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'}$ is lower triangular with respect to the partial order on \mathbb{Y} determined by the inclusion relation. Therefore, it suffices to compute $({}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'} F)(\lambda)$ for $\lambda \supseteq \mu$; in particular, $l \geq m$. We have

$$({}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'} F)(\lambda) = \sum_{k=m}^l \mathbb{B}\Lambda_r^{r'}(l, k) \sum_{\varkappa \in \mathbb{Y}_k} {}^{\mathbb{Y}}\Lambda_k^l(\lambda, \varkappa) q^k FS_\mu(\varkappa).$$

For fixed k ,

$$\begin{aligned} {}^{\mathbb{Y}}\Lambda_k^l(\lambda, \varkappa) q^k FS_\mu(\varkappa) &= q^k \frac{\dim \varkappa \dim(\varkappa, \lambda)}{\dim \lambda} k^{\downarrow m} \frac{\dim(\mu, \varkappa)}{\dim \varkappa} \quad \text{by virtue of (7.1)} \\ &= q^k k^{\downarrow m} \frac{\dim(\mu, \varkappa) \dim(\varkappa, \lambda)}{\dim \lambda}, \end{aligned}$$

and summing the latter quantity over $\varkappa \in \mathbb{Y}_k$ gives

$$q^k k^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \lambda}.$$

Therefore,

$$\begin{aligned} ({}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'} F)(\lambda) &= k^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \lambda} \cdot \sum_{k=m}^l \mathbb{B}\Lambda_r^{r'}(l, k) q^k \\ &= k^{\downarrow m} \frac{\dim(\mu, \lambda)}{\dim \lambda} \cdot \left(\frac{qr}{q'r'}\right)^m (q')^l l^{\downarrow m} \quad \text{by (6.4)} \\ &= \left(\frac{qr}{q'r'}\right)^m (q')^l FS_\mu(\lambda), \end{aligned}$$

as desired.

Formula (8.5) can be checked in exactly the same way. Alternatively, it can be obtained a limit case of (8.7) as $q \rightarrow 1$.

Formula (8.6) is immediate from the very definition of ${}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'}$ and FS_μ . Alternatively, (8.6) can also be obtained from (8.7) by a degeneration, like the derivation of (6.3) from (6.4), see the proof of Proposition 6.1. \square

Proposition 8.3 (cf. Proposition 6.2). *Assume $r > 0$ and $0 < q < 1$; let λ range over \mathbb{Y} and $l = |\lambda|$; let ω range over $\tilde{\Omega}$ and $x = |\omega|$; let $\mu \in \mathbb{Y}$ be fixed and $m = |\mu|$. Regard*

$\mathbb{Y}^B \Lambda_r^\infty$ as an operator transforming a function $F(\lambda)$ on \mathbb{Y} to a function $G(\omega)$ on $\tilde{\Omega}$. Under this transformation

$$FS_\mu(\lambda) \mapsto r^m S_\mu(\omega), \tag{8.8}$$

$$(\dim \mu)^{-1} \mathbf{1}_\mu \mapsto \frac{r^m}{m!} \cdot q_\infty^x S_\mu(\omega), \quad q_\infty := e^{-r}, \tag{8.9}$$

$$q^l FS_\mu(\lambda) \mapsto q^m r^m \cdot q_\infty^x S_\mu(\omega), \quad q_\infty := e^{-(1-q)r}. \tag{8.10}$$

Proof. We may argue exactly as in the proof of the previous proposition. □

8.2 Markov semigroups on \mathbb{Y} and $\tilde{\Omega}$

We are going to introduce a Q -matrix of format $\mathbb{Y} \times \mathbb{Y}$ depending on the triple (z, z', r) of parameters, where $r > 0$ and (z, z') , as usual, is subject to Condition 1.1. For this we need some notation. Given $\lambda \in \mathbb{Y}$, let λ^+ and λ^- stand for the collections of boxes that can be appended to, respectively, removed from λ . For a box \square , its *content* is defined as the difference $c(\square) := j - i$, where i and j are the row and column numbers of \square . The Q -matrix in question is denoted by $Q_r^{(z, z')}$ and its non-diagonal entries $Q_r^{(z, z')}(\lambda, \varkappa)$, $\varkappa \neq \lambda$, vanish unless either $\varkappa = \lambda + \square$ or $\varkappa = \lambda - \square$, meaning that \varkappa is obtained from λ by appending a box $\square \in \lambda^+$ or by removing a box $\square \in \lambda^-$. In this notation, the entries are given by

$$\begin{aligned} Q_r^{(z, z')}(\lambda, \lambda + \square) &= r(z + c(\square))(z' + c(\square)) \frac{\dim(\lambda + \square)}{(|\lambda| + 1) \dim \lambda}, \quad \square \in \lambda^+, \\ Q_r^{(z, z')}(\lambda, \lambda - \square) &= (r + 1) \frac{|\lambda| \dim(\lambda - \square)}{\dim \lambda}, \quad \square \in \lambda^-, \\ -Q_r^{(z, z')}(\lambda, \lambda) &= (2r + 1)|\lambda| + rz z'. \end{aligned} \tag{8.11}$$

Note that each row of $Q_r^{(z, z')}$ has finitely many nonzero entries which sum to 0, and the constraints on the parameters imply that all off-diagonal entries are nonnegative (in particular, $Q_r^{(z, z')}(\lambda, \lambda + \square) > 0$ because of Condition 1.1).

(For more detail about the definition of $Q_r^{(z, z')}$, we refer to Borodin–Olshanski [11] and Olshanski [32]. Formula (8.11) coincides with that of [32, Proposition 4.25] and is a particular case of [11, (2.19)]. Note that parameter $\xi \in (0, 1)$ from those two papers is related to our parameter $r > 0$ by $\xi = r(r + 1)^{-1}$. In [11, (2.19)], parameter ξ may vary with time; our setup corresponds to the particular case when ξ is fixed, so that the time derivative $\dot{\xi}$ equals 0. Then formula [11, (2.19)] simplifies and reduces to (8.11).)

We can interpret $Q_r^{(z, z')}$ as an operator in the vector space $\text{Fun}(\mathbb{Y})$ formed by arbitrary real-valued functions on \mathbb{Y} :

$$(Q_r^{(z, z')} F)(\lambda) = \sum_{\varkappa \in \mathbb{Y}} Q_r^{(z, z')}(\lambda, \varkappa) F(\varkappa), \quad F \in \text{Fun}(\mathbb{Y}). \tag{8.12}$$

As explained in [32], this operator should be viewed as a counterpart of the Meixner difference operator on \mathbb{Z}_+ . The next step is to introduce counterparts of the Meixner polynomials. According to [32, Definition 4.21], these are elements of Sym called the *Meixner symmetric functions* and denoted by \mathfrak{M}_ν , where the index ν ranges over \mathbb{Y} . They depend on the triple (z, z', r) and are given by the following expansion in the basis of the Frobenius–Schur symmetric functions (cf. (6.9)):

$$\begin{aligned} \mathfrak{M}_\nu^{(z, z', r)} &= \sum_{\mu: \mu \subseteq \nu} (-1)^{|\nu| - |\mu|} r^{|\nu| - |\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} \\ &\times \prod_{\square \in \nu / \mu} (z + c(\square))(z' + c(\square)) \cdot FS_\mu. \end{aligned} \tag{8.13}$$

Proposition 8.4 (cf. Proposition 6.3). *Under the action of $Q_r^{(z,z')}$ in $\text{Fun}(\mathbb{Y})$,*

$$FS_\mu \rightarrow -|\mu|FS_\mu + r \sum_{\square \in \mu^-} (z + c(\square))(z' + c(\square))FS_{\mu \setminus \square}, \quad (8.14)$$

$$\mathfrak{M}_\mu^{(z,z',r)} \rightarrow -|\mu|\mathfrak{M}_\mu. \quad (8.15)$$

Proof. See [32, Section 4.8]. □

Proposition 8.5 (cf. Proposition 6.4). *For arbitrary $r' > r > 0$, we have*

$$Q_r^{z,z'} \mathbb{YB}\Lambda_r^{r'} = \mathbb{YB}\Lambda_r^{r'} Q_r^{z,z'}.$$

Proof. We literally follow the argument in the proof of Proposition 6.4. From the definition of the Meixner symmetric functions and (8.5) it is readily seen that

$$\mathbb{YB}\Lambda_r^{r'} \mathfrak{M}_\nu^{(z,z',r)} = \left(\frac{r}{r'}\right)^{|\nu|} \mathfrak{M}_\nu^{(z,z',r')}$$

and then (8.15) implies that the both sides of the operator equality in question give the same result when applied to $\mathfrak{M}_\nu^{(z,z',r)}$. Therefore, the equality holds on all elements of Sym . As these elements separate points of \mathbb{Y} , this concludes the proof. □

Proposition 8.6 (cf. Proposition 6.5). *The matrix $Q_r^{(z,z')}$ satisfies the assumptions of Theorem 2.3 with functions $\gamma(\lambda) = \eta(\lambda) = |\lambda| + 1$, $\lambda \in \mathbb{Y}$.*

Proof. As seen from the description of the Q -matrix given in [11, Section 2.5] (see the sentence just before [11, Proposition 2.11]), for any $\lambda \in \mathbb{Y}$ one has

$$\sum_{\square \in \lambda^\pm} Q_r^{(z,z')}(\lambda, \lambda \pm \square) = Q_r^{(c)}(|\lambda|, |\lambda| \pm 1), \quad c := zz'$$

(note that $c > 0$ because of Condition 1.1). This implies that the action of $Q_r^{(z,z')}$ preserves the subspace in $\text{Fun}(\mathbb{Y})$ formed by those functions in variable $\lambda \in \mathbb{Y}$ that depend only on $|\lambda|$, and in that subspace, the action reduces to that of the difference operator $Q_r^{(c)}$ with $c = zz'$.

Therefore, the claim of the proposition reduces to that of Proposition 6.5. □

Combining this proposition with Theorem 2.3 we get

Corollary 8.7 (cf. Corollary 6.6). (i) *The Q -matrix $Q_r^{(z,z')}$ gives rise to a Feller semigroup $T_r^{(z,z')}(t)$ on $C_0(\mathbb{Y})$ whose generator $A_r^{(z,z')}$ is implemented by $Q_r^{(z,z')}$.*

(ii) *The subspace $C_0(\mathbb{Y})$ is a core for generator $A_r^{(z,z')}$.*

Proposition 8.8 (cf. Proposition 6.7). *For every couple (z, z') of parameters subject to Condition 1.1 there exists a unique Feller semigroup $T^{(z,z')}(t)$ such that for every $r > 0$, $T^{(z,z')}(t)$ is consistent with the $T_r^{(z,z')}(t)$, $r > 0$, in the sense that*

$$T^{(z,z')}(t) \mathbb{YB}\Lambda_r^\infty = \mathbb{YB}\Lambda_r^\infty T_r^{(z,z')}(t), \quad t \geq 0, \quad r > 0.$$

Proof. The argument is exactly the same as in Proposition 6.7: We know that the Q -matrices with various values of parameter r are consistent with the links (Proposition 8.5). It follows, by virtue of Proposition 5.3, that the semigroups are consistent with the links, too. Therefore, we may apply Proposition 5.1, which gives the desired result. □

Definition 8.9. *For $r > 0$, we denote by $X_r^{(z,z')}$ the Feller Markov process on \mathbb{Y} determined by the semigroup $T_r^{(z,z')}(t)$. Likewise, we denote by $X^{(z,z')}$ the Feller Markov process on $\tilde{\Omega}$ determined by the semigroup $T^{(z,z')}(t)$.*

8.3 A family of cores for Markov semigroup generators

The following two claims are used in the proposition below.

First, let ω range over the Thoma cone $\tilde{\Omega}$ and μ range over \mathbb{Y} . For any fixed $q \in (0, 1)$, the functions $q^{|\omega|} S_\mu(\omega)$ span a dense subspace in $C_0(\tilde{\Omega})$, see [14, Corollary 3.4.6].

Second, let λ range over \mathbb{Y} . Recall that in Section 7.4 we defined an embedding $\mathbb{Y} \hookrightarrow \tilde{\Omega}$ via the map $\lambda \mapsto \omega_\lambda$. Observe that $|\lambda| = |\omega_\lambda|$; this implies that a sequence $\{\lambda\}$ of diagrams goes to infinity in the discrete set \mathbb{Y} if and only if its image $\{\omega_\lambda\}$ goes to infinity in the locally compact space $\tilde{\Omega}$. Combining this with the first claim we conclude that for any fixed $q \in (0, 1)$, the functions $q^{|\lambda|} F_\mu(\lambda)$ span a dense subspace in $C_0(\mathbb{Y})$.

Proposition 8.10 (cf. Proposition 6.8). (i) For any $r' > r > 0$, the operator ${}^{\mathbb{YB}}\Lambda_r^{r'} : C_0(\mathbb{Y}) \rightarrow C_0(\mathbb{Y})$ has a dense range.

(ii) Likewise, for any $r > 0$, the operator ${}^{\mathbb{YB}}\Lambda_r^\infty : C_0(\mathbb{Y}) \rightarrow C_0(\tilde{\Omega})$ has a dense range.

Proof. (i) Fix an arbitrary $q \in (0, 1)$ and let μ range over \mathbb{Y} . By (8.7), ${}^{\mathbb{YB}}\Lambda_r^{r'}$ maps the linear span of functions $q^{|\lambda|} S_\mu(\lambda)$ onto the linear span of functions $(q')^{|\lambda|} S_\mu(\lambda)$ with some other $q' \in (0, 1)$. Since these spans are dense, we get the desired claim.

(ii) The same argument, with reference to (8.10). □

Denote by $A_r^{(z, z')}$ and $A^{(z, z')}$ the generators of the semigroups $T_r^{(z, z')}(t)$ and $T^{(z, z')}(t)$, respectively.

Proposition 8.11 (cf. Proposition 6.9). Fix an arbitrary number $q \in (0, 1)$ and let μ range over \mathbb{Y} .

(i) For every $r > 0$, the linear span of functions $q^{|\lambda|} S_\mu(\lambda)$, where argument λ ranges over \mathbb{Y} , is a core for $A_r^{(z, z')}$.

(ii) Likewise, the linear span of functions $q^{|\omega|} S_\mu(\omega)$, where argument ω ranges over $\tilde{\Omega}$, is a core for $A^{(z, z')}$.

Proof. (i) Observe that if $r_2 > r_1 > 0$ and \mathcal{F}_1 is a core for $A_{r_1}^{(z, z')}$, then $\mathcal{F}_2 := {}^{\mathbb{YB}}\Lambda_{r_1}^{r_2} \mathcal{F}_1$ is a core for $A_{r_2}^{(z, z')}$. Indeed, by virtue of claim (i) of Proposition 8.10, we may apply the argument of Proposition 5.2.

Now take $r_2 = r$ and $r_1 = (1 - q)r$. Then, as is seen from (8.6), the linear span of functions $q^{|\lambda|} S_\mu(\lambda)$ is just the image under ${}^{\mathbb{YB}}\Lambda_{r_1}^{r_2}$ of the space $C_c(\mathbb{Y})$. By virtue of Proposition 8.6 and claim (iv) of Theorem 2.3, $C_c(\mathbb{Y})$ is a core for $A_{r_1}^{(z, z')}$. Therefore, its image is a core for $A_{r_2}^{(z, z')}$.

(ii) We argue as above. First, application of claim (ii) of Proposition 8.10 allows us to conclude that if $\mathcal{F} \subset C_0(\mathbb{Y})$ is a core for $A_r^{(z, z')}$ for some $r > 0$, then ${}^{\mathbb{YB}}\Lambda_r^\infty \mathcal{F}$ is a core for $A^{(z, z')}$.

Next, given $q \in (0, 1)$ we take $r = -\log q$ and $\mathcal{F} = C_0(\mathbb{Y})$. As pointed above, \mathcal{F} is a core for $A_r^{(z, z')}$. On the other hand, (8.9) shows that the linear span of functions $q^{|\omega|} S_\mu(\omega)$ coincides with ${}^{\mathbb{YB}}\Lambda_r^\infty \mathcal{F}$. □

8.4 The infinite-variate Laguerre differential operator

Following [32, Theorem 4.10], we introduce the following partial differential operator in countably many formal variables e_1, e_2, \dots :

$$\begin{aligned} \mathfrak{D}^{(z, z')} &= \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} (2n-1-2k)e_{2n-1-k}e_k \right) \frac{\partial^2}{\partial e_n^2} \\ &+ 2 \sum_{n' > n \geq 1} \left(\sum_{k=0}^{n-1} (n'+n-1-2k)e_{n'+n-1-k}e_k \right) \frac{\partial^2}{\partial e_{n'} \partial e_n} \\ &+ \sum_{n=1}^{\infty} (-ne_n + (z-n+1)(z'-n+1)e_{n-1}) \frac{\partial}{\partial e_n} \end{aligned} \tag{8.16}$$

with the agreement that $e_0 = 1$. We call it the *infinite-variate Laguerre differential operator*.

Since all coefficients of $\mathfrak{D}^{(z, z')}$ are given by finite sums, $\mathfrak{D}^{(z, z')}$ is applicable to any polynomial in e_1, e_2, \dots . This means that it is well defined on Sym provided that we interpret our formal variables as the elementary symmetric functions (here we use the fact that $\{e_1, e_2, \dots\}$ is a system of algebraically independent generators of Sym). But $\mathfrak{D}^{(z, z')}$ is also applicable to more general cylinder functions, in particular, to the functions of the form $q^{e_1}F$, where $q \in (0, 1)$ and $F \in \text{Sym}$. Note that $e_1(\omega) = |\omega|$, so that these are just the functions considered in claim (ii) of Proposition 8.11. By virtue of this claim, for any fixed $q = e^{-r} \in (0, 1)$, the functions of the form $q^{e_1}F$ with $F \in \text{Sym}$ enter the domain of the generator $A^{(z, z')}$, and $A^{(z, z')}$ is uniquely determined by its action on these functions.

Proposition 8.12 (cf. Proposition 6.10). *For any $r > 0$, the action of the generator $A^{(z, z')}$ on the functions of the form $\exp(-re_1)F$ with F ranging over the algebra $\text{Sym} = \mathbb{R}[e_1, e_2, \dots]$ is implemented by the infinite-variate Laguerre differential operator $\mathfrak{D}^{(z, z')}$ defined by (8.16).*

Proof. Let μ range over \mathbb{Y} and $m := |\mu|$. Recall that $\mathbf{1}_\mu$ denotes the delta function on \mathbb{Y} concentrated at the point μ . We also set

$$\tilde{\mathbf{1}}_\mu = (\dim \mu)^{-1} \mathbf{1}_\mu$$

and

$$f_\mu = \frac{r^m}{m!} \exp(-re_1) S_\mu.$$

By virtue of (8.9),

$$\mathbb{YB}\Lambda_r^\infty \tilde{\mathbf{1}}_\mu = f_\mu.$$

Recall also that μ^+ and μ^- denote the sets of boxes that can be appended to or removed from μ , respectively.

We are going to prove the following analogs of formulas (6.13) and (6.14):

$$\begin{aligned} Q_r^{(c)} \tilde{\mathbf{1}}_\mu &= -[(2r+1)m + rzz'] \tilde{\mathbf{1}}_\mu + (r+1)(m+1) \sum_{\square \in \mu^+} \tilde{\mathbf{1}}_{\mu+\square} \\ &+ \frac{r}{m} \sum_{\square \in \mu^-} (z+c(\square))(z'+c(\square)) \tilde{\mathbf{1}}_{\mu-\square} \end{aligned} \tag{8.17}$$

and

$$\begin{aligned} \mathfrak{D}^{(z,z')} f_\mu = & -[(2r+1)m + rzz']f_\mu + (r+1)(m+1) \sum_{\square \in \mu^+} f_{\mu+\square} \\ & + \frac{r}{m} \sum_{\square \in \mu^-} (z + c(\square))(z' + c(\square))f_{\mu-\square} \end{aligned} \quad (8.18)$$

(for the empty diagram μ , the set μ^- is empty and the corresponding sum disappears). These formulas show that the operator $Q_r^{(z,z')}$ acts on the functions $\tilde{\mathbf{I}}_\mu$ in exactly the same way as the operator $\mathfrak{D}^{(z,z')}$ acts on the functions f_μ , which implies the claim of the proposition.

The proof of (8.17) is trivial: this formula directly follows from the very definition of $Q_r^{(z,z')}$, see (8.11).

The proof of (8.18) is a bit more complicated. Observe that if a second order partial differential operator D , symbolically written as

$$D = \sum_{i,i} c_{ij} \partial_i \partial_j + \text{first order terms,}$$

is applied to a product of two functions, GF , then the result can be written as the sum of three expressions:

$$D(GF) = \underbrace{(DF)G}_1 + \underbrace{G(DF)}_2 + \underbrace{\sum_{i,j} c_{ij} [(\partial_i G)(\partial_j F) + (\partial_j G)(\partial_i F)]}_3. \quad (8.19)$$

Let us apply this general formula to

$$D := \mathfrak{D}^{(z,z')}, \quad G := \frac{r^m}{m!} e^{-re_1}, \quad F := S_\mu$$

and examine the corresponding three expressions arising from (8.19).

1. The first expression is equal to

$$\begin{aligned} \frac{r^m}{m!} \left(\mathfrak{D}^{(z,z')} e^{-re_1} \right) S_\mu &= \frac{r^m}{m!} \left\{ \left[e_1 \frac{d^2}{de_1^2} + (-e_1 + zz') \frac{d}{de_1} \right] e^{-re_1} \right\} S_\mu \\ &= \frac{r^m}{m!} (r^2 + r) e^{-re_1} e_1 S_\mu - \frac{r^m}{m!} rzz' e^{-re_1} S_\mu. \end{aligned}$$

It is well known that

$$e_1 S_\mu = \sum_{\square \in \mu^+} S_{\mu+\square}.$$

It follows that the first expression in question is equal to

$$\begin{aligned} (r+1)(m+1) \frac{r^{m+1}}{(m+1)!} e^{-re_1} \sum_{\square \in \mu^+} S_{\mu+\square} - rzz' \frac{r^m}{m!} e^{-re_1} S_\mu \\ = (r+1)(m+1) \sum_{\square \in \mu^+} f_{\mu+\square} - rzz' f_\mu. \end{aligned} \quad (8.20)$$

2. The second expression in (8.19) takes the form

$$\frac{r^m}{m!} e^{-re_1} (\mathfrak{D}^{(z,z')} S_\mu).$$

It follows from [32, Theorem 4.1 and Definition 4.7] that

$$\mathfrak{D}^{(z,z')}S_\mu = -mS_\mu + \sum_{\square \in \mu^-} (z + c(\square))(z' + c(\square))S_{\mu-\square}, \quad m = |\mu|.$$

This implies that the second expression is equal to

$$-mf_\mu + \frac{r}{m} \sum_{\square \in \mu^-} (z + c(\square))(z' + c(\square))f_{\mu-\square}. \tag{8.21}$$

3. The only relevant part of our differential operator $D = \mathfrak{D}^{(z,z')}$ that contributes to the third expression in (8.19) is

$$e_1 \frac{\partial^2}{\partial e_1^2} + 2 \sum_{n' > 1} n' e_{n'} \frac{\partial^2}{\partial e_{n'} \partial e_1},$$

because the remaining terms in $\mathfrak{D}^{(z,z')}$ are either of the first order or do not contain the partial derivative in variable e_1 while our function G depends on e_1 only. It follows that the third expression has the form

$$\begin{aligned} 2 \frac{r^m}{m!} \left(\frac{d}{de_1} e^{-re_1} \right) e_1 \frac{\partial}{\partial e_1} S_\mu + 2 \frac{r^m}{m!} \left(\frac{d}{de_1} e^{-re_1} \right) \sum_{n' > 1} n' e_{n'} \frac{\partial}{\partial e_{n'}} S_\mu \\ = -2r \frac{r^m}{m!} e^{-re_1} \sum_{n \geq 1} n e_n \frac{\partial}{\partial e_n} S_\mu. \end{aligned}$$

Observe that the operator

$$\sum_{n \geq 1} n e_n \frac{\partial}{\partial e_n}$$

is the ‘‘Euler operator’’; its action on the homogeneous function S_μ amounts to multiplication by its degree m . Using this fact we see that the third expression is equal to

$$-2rmf_\mu. \tag{8.22}$$

Finally, summing up (8.20), (8.21), and (8.22) we get the desired formula (8.18) □

The *Laguerre symmetric functions*, introduced in Olshanski [32], are elements of Sym depending on parameters z and z' , and indexed by Young diagrams $\nu \in \mathbb{Y}$:

$$\mathfrak{L}_\nu^{(z,z')} = \sum_{\mu: \mu \subseteq \nu} (-1)^{|\nu|-|\mu|} \frac{\dim \nu / \mu}{(|\nu| - |\mu|)!} (z)_{\nu/\mu} (z')_{\nu/\mu} S_\mu. \tag{8.23}$$

As shown in [32], they form a basis in Sym diagonalizing operator $\mathfrak{D}^{(z,z')}$:

$$\mathfrak{D}^{(z,z')} \mathfrak{L}_\nu^{(z,z')} = -|\nu| \mathfrak{L}_\nu^{(z,z')}, \quad \nu \in \mathbb{Y}. \tag{8.24}$$

The above formula is similar to (6.16), and the next formula is an analog of (6.17):

$${}^{\mathbb{Y}\text{B}}\Lambda_r^\infty \mathfrak{M}_\nu^{(z,z',r)} = r^{|\nu|} \mathfrak{L}_\nu^{(z,z')}, \quad r > 0, \quad \nu \in \mathbb{Y}. \tag{8.25}$$

The proof of (8.25) is easy and analogous to that of (6.17). Namely, we compare the expansions of the Meixner and Laguerre symmetric functions in the bases $\{FS_\mu\}$ and $\{S_\mu\}$, respectively (see (8.13) and (8.23)), and then apply (8.8), which says that ${}^{\mathbb{Y}\text{B}}\Lambda_r^\infty$ takes FS_μ to $r^{|\mu|}S_\mu$.

8.5 Approximation

Recall that in Section 7.4 we introduced an embedding $\lambda \mapsto \omega_\lambda$ of the set \mathbb{Y} into the Thoma cone $\tilde{\Omega}$. Now let us introduce a family of embeddings $\varphi_r : \mathbb{Y} \hookrightarrow \tilde{\Omega}$ depending on parameter $r > 0$:

$$\varphi_r(\lambda) = r^{-1}\omega_\lambda, \quad \lambda \in \mathbb{Y},$$

where multiplication by constant factor r^{-1} in the right-hand side means that all coordinates of ω_λ are multiplied by that constant — a natural homothety on the cone. Obviously, φ_1 is the map $\lambda \mapsto \omega_\lambda$.

The latter map should be viewed as a counterpart of the inclusion map $\mathbb{Z}_+ \hookrightarrow \mathbb{R}_+$, while φ_r is a counterpart of the scaled embedding $\mathbb{Z}_+ \ni l \mapsto r^{-1}l \in \mathbb{R}_+$.

Note that $\varphi_r(\mathbb{Y})$ is a discrete, locally finite subset of $\tilde{\Omega}$. Therefore, we may define the projection $\pi_r : C_0(\tilde{\Omega}) \rightarrow C_0(\mathbb{Y})$ as in Section 5.2:

$$(\pi_r f)(\lambda) = f(\varphi_r(\lambda)), \quad \lambda \in \mathbb{Y}.$$

It is used in the proposition below to define the approximation procedure.

Recall that in Corollary 8.7 and Proposition 8.8 we defined Feller semigroups $T_r^{(z,z')}(t)$ and $T^{(z,z')}(t)$ acting on the Banach spaces $C_0(\mathbb{Y})$ and $C_0(\tilde{\Omega})$, respectively.

Proposition 8.13 (cf. Proposition 6.11). *Let (z, z') be fixed. As $r \rightarrow +\infty$, the semigroups $T_r^{(z,z')}(t)$ approximate the semigroup $T^{(z,z')}(t)$ in the sense of Definition 3.2.*

Proof. As in the proof of Proposition 6.11, we only need to check all the hypotheses of Proposition 5.5.

Again, the assumptions stated in Section 5.1 are satisfied: we know that \mathbb{YB} is a Feller system, the semigroups $T_r^{(z,z')}(t)$ with varying parameter $r > 0$ are consistent with the links of \mathbb{YB} , they are Feller links, and, by the very definition, the semigroup $T^{(z,z')}(t)$ is the boundary semigroup determined by the pre-limit semigroups $T_r^{(z,z')}(t)$.

Next, we have to check Conditions 4.1 and 5.4.

The second condition consists in the requirement that $C_c(\mathbb{Y})$ is a core for generator $A_r^{(z,z')}$ and, moreover, is invariant under its action. The fact that $C_c(\mathbb{Y})$ is a core follows from Corollary 8.7, item (ii). Its invariance follows from item (i), because $C_c(\mathbb{Y})$ is obviously invariant under the action of $Q_r^{(z,z')}$.

Finally, the first condition means that for fixed $s > 0$ and $\mu \in \mathbb{Y}$

$$\lim_{r \rightarrow \infty} \sup_{\lambda \in \mathbb{Y}} \left| \mathbb{YB}\Lambda_s^r(\lambda, \mu) - \mathbb{YB}\Lambda_s^\infty(\varphi_r(\lambda), \mu) \right| = 0, \tag{8.26}$$

and this was established in the proof of [14, Theorem 3.4.7] (note only a slight divergence of notation: in [14], we wrote r' and r instead of r and s , respectively). \square

8.6 The stationary distribution

The so-called *mixed z -measure* on \mathbb{Y} with parameters (z, z') and r is defined by

$$M_r^{(z,z')}(\lambda) = (r + 1)^{-zz'} \left(\frac{r}{r + 1} \right)^{|\lambda|} \cdot \prod_{\square \in \lambda} (z + c(\square))(z' + c(\square)) \cdot \left(\frac{\dim \lambda}{|\lambda|!} \right)^2, \tag{8.27}$$

where λ ranges over \mathbb{Y} . As before, we assume that $r > 0$ and (z, z') satisfies Condition 1.1. Then the weights $M_r^{(z,z')}(\lambda)$ are strictly positive and sum to 1, so that $M_r^{(z,z')}$ is a probability measure on \mathbb{Y} whose support is the whole set \mathbb{Y} . Measures $M_r^{(z,z')}$ first appeared in Borodin–Olshanski [7]; additional information can be found in Okounkov [29], Borodin–Olshanski [9], and Olshanski [32]. These measures are a particular case of Okounkov’s *Schur measures* introduced in [28]. (As mentioned above, in those papers, the third parameter, denoted by ξ , is related to our parameter r by $\xi = r(1 + r)^{-1}$.)

Proposition 8.14. $M_r^{(z,z')}$ serves as a unique stationary distribution for the Markov process $X_r^{(z,z')}$ determined by the Feller semigroup $T_r^{(z,z')}(t)$.

Proof. The fact that $M_r^{(z,z')}$ is a stationary measure is a particular case of [11, Proposition 2.12].

Next, from the structure of matrix $Q_r^{(z,z')}$ and the construction of $X_r^{(z,z')}$ it follows that $X_r^{(z,z')}$ is an irreducible Markov chain: all states $\lambda \in \mathbb{Y}$ are communicating. According to a general theorem (see Anderson [1, Chapter 5, Theorem 1.6]) this implies the uniqueness claim. \square

As shown in [14, Proposition 3.5.3], the measures $M_r^{(z,z')}$ with varying parameter r are compatible with the links ${}^{\mathbb{Y}\mathbb{B}}\Lambda_r^{r'}$, that is, they form a coherent family. Therefore, they give rise to a boundary measure on $\tilde{\Omega}$, which we denote by $M^{(z,z')}$ and call the z -measure on the Thoma cone.

Proposition 8.15. $M^{(z,z')}$ serves as a unique stationary distribution for the Markov process $X^{(z,z')}$ determined by the Feller semigroup $T^{(z,z')}(t)$.

Proof. We have to prove that $M^{(z,z')}$ satisfies the relation $M^{(z,z')}T^{(z,z')}(t) = M^{(z,z')}$ and is a unique probability measure on $\tilde{\Omega}$ with this property. By Proposition 8.14, a similar claim holds for measures $M_r^{(z,z')}$. Because $\{M_r^{(z,z')} : r > 0\}$ is a coherent family, this immediately implies the desired claim: an easy formal argument can be found in [13, Section 2.8]. \square

Proposition 8.16. $M^{(z,z')}$ is the weak limit of measures $\varphi_r(M_r^{(z,z')})$ as $r \rightarrow +\infty$.

Proof. Indeed, as mentioned above (see the proof of Proposition 8.13), in our situation Condition 4.1 is satisfied. Therefore, we may apply Proposition 4.2 which gives the desired result. \square

Remark 8.17. In Olshanski [32], the z -measures on the Thoma cone were defined in a different way, see [32, Theorem 5.18]. However, the two definitions are equivalent, as can be seen from the comparison of Proposition 8.16 with [32, Theorem 5.28].

9 Determinantal structure

9.1 Generalities on correlation functions

Let \mathfrak{X} be a locally compact metrizable separable space (we will actually take for \mathfrak{X} the punctured real line $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ or the one-dimensional lattice). A finite or countably infinite collection of points in \mathfrak{X} without accumulation points is called a *configuration*. We say “collection” and not “subset” because, in principle, multiple points are permitted; one could also use the term “multiset”. To a configuration ω we assign the Radon measure

$$\Delta(\omega) := \sum_{x \in \omega} \Delta_x,$$

where Δ_x denotes the delta-measure at x . This assignment establishes a one-to-one correspondence between all possible configurations in \mathfrak{X} and all sigma-finite Radon measures on \mathfrak{X} with the property that the mass of any compact subset is a nonnegative integer. The space of configurations will be denoted by $\text{Conf}(\mathfrak{X})$. We equip it with the topology inherited from the vague topology on the space of Radon measures. In particular, $\text{Conf}(\mathfrak{X})$ has a natural Borel structure. This structure is generated by the integer-valued functions \mathcal{N}_B , where $B \subset \mathfrak{X}$ is an arbitrary relatively compact Borel subset and

$$\mathcal{N}_B(\omega) := |\omega \cap B|, \quad \omega \in \text{Conf}(\mathfrak{X}).$$

Let M be a probability Borel measure on $\text{Conf}(\mathfrak{X})$. Then the functions \mathcal{N}_B become random variables. We will assume that every such function has finite moments of any order,

$$\mathbb{E}_M((\mathcal{N}_B)^k) < +\infty, \quad \forall k = 1, 2, \dots, \quad \forall B,$$

where \mathbb{E}_M means expectation relative to M . Under this assumption one assigns to M an infinite collection $\{\rho_k : k = 1, 2, \dots\}$ of measures, where ρ_k is a (usually infinite) measure on the k -fold product space \mathfrak{X}^k , defined as follows.

First, given $\omega \in \text{Conf}(\mathfrak{X})$, we form a purely atomic measure $\Delta^k(\omega)$ on \mathfrak{X}^k by setting

$$\Delta^k(\omega) := \sum_{x_1, \dots, x_k} \Delta_{x_1} \otimes \dots \otimes \Delta_{x_k},$$

where the sum is taken on arbitrary ordered k -tuples of distinct points extracted from ω .

Second, we interpret $\Delta^k(\omega)$ as a random measure driven by the probability distribution M and average over M ,

$$\rho_k = \rho_k^M := \mathbb{E}_M(\Delta^k(\cdot)).$$

The measure ρ_k^M is called the k th *correlation measure* of M , and the first correlation measure ρ_1^M is also called the *density measure*. Under mild hypotheses on the correlation measures, they determine the initial measure M uniquely, see Lenard [26].

M is said to be a *determinantal measure* if the following condition holds. Choose a “reference” measure σ on \mathfrak{X} , equivalent to the density measure (the condition stated below does not depend on the choice of σ). There should exist a function $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$ such that, for every $k \geq 1$, the k th correlation measure ρ_k^M is absolutely continuous with respect to $\sigma^{\otimes k}$, and the corresponding Radon-Nikodým density is given by a $k \times k$ principal minor extracted from kernel K :

$$\frac{\rho_k^M}{\sigma^{\otimes k}}(x_1, \dots, x_k) = \det[K(x_i, x_j)].$$

The quantity in the left-hand side is called the k th *correlation function*, and $K(x, y)$ is called the *correlation kernel* of M . In contrast to correlation functions, the correlation kernel, if it exists, is not a canonical object: there are ways to modify it without affecting the correlation functions. On the other hand, any determinantal measure is uniquely determined by its correlation functions and hence by the correlation kernel.

“Determinantal measure” is another name for “determinantal point process” (more precisely, for the *law* of such a point process). A standard reference is Soshnikov’s expository paper [37]. See also the more recent survey Borodin [4] and references therein.

9.2 Determinantal structure of the stationary distributions

Set $\mathfrak{X} = \mathbb{R}^*$ and define a map $\tilde{\Omega} \rightarrow \text{Conf}(\mathbb{R}^*)$ as follows:

$$\tilde{\Omega} \ni \omega \mapsto \bar{\omega} := \{\alpha_i : \alpha_i \neq 0\} \cup \{-\beta_i : \beta_i \neq 0\} \in \text{Conf}(\mathbb{R}^*).$$

Because of the constraint $\sum \alpha_i + \sum \beta_i \leq \delta < +\infty$, $\bar{\omega}$ is indeed a configuration on \mathbb{R}^* . Clearly, the map is continuous and hence Borel. So it converts every probability Borel measure M on $\tilde{\Omega}$ to a probability Borel measure \bar{M} on $\text{Conf}(\mathbb{R}^*)$. This makes it possible to speak about the correlation functions of M , referring to those of \bar{M} .

We fix a pair of parameters (z, z') satisfying Condition 1.1 and denote by $\bar{M}^{(z, z')}$ the measure on $\text{Conf}(\mathbb{R}^*)$ coming from the z -measure $M^{(z, z')}$. In the next theorem, $K^{(z, z')}(x, y)$ denotes the Whittaker kernel on $\mathbb{R}^* \times \mathbb{R}^*$ studied in Borodin [3] and Borodin-Olshanski [7], [11]. We will not use its exact form here.

Theorem 9.1. $\bar{M}^{(z,z')}$ is a determinantal measure whose correlation kernel is the Whittaker kernel $K^{(z,z')}(x, y)$.

This result was first proved in [3]. Below we give in detail a different derivation, because it is well suited for the extension to the case of finite-dimensional distributions of processes $X^{(z,z')}$. Note that a similar argument is contained in [8, Proposition 4.2].

Proof. Step 1. Let M be a probability measure on $\tilde{\Omega}$ and \bar{M} be the corresponding measure on $\text{Conf}(\mathbb{R}^*)$. We will establish a simple estimate which, in particular, provides a convenient sufficient condition for the existence of the correlation measures.

For $\varepsilon > 0$, set

$$B_\varepsilon := \mathbb{R} \setminus (-\varepsilon, \varepsilon) \subset \mathbb{R}^*.$$

Recall the notation $|\omega| = |(\alpha, \beta, \delta)| = \delta$. The basic constraint $\sum(\alpha_i + \beta_i) \leq |\omega|$ implies the inequality

$$|\bar{\omega} \cap B_\varepsilon| \leq \varepsilon^{-1}|\omega|, \tag{9.1}$$

which in turn implies that

$$\mathbb{E}_{\bar{M}}((\mathcal{N}_{B_\varepsilon})^k) \leq \varepsilon^{-k} \int_{\tilde{\Omega}} |\omega|^k M(d\omega), \quad k = 1, 2, \dots$$

Denote by $|M|$ the measure on \mathbb{R}_+ that is the pushforward of M under the projection $\omega \mapsto |\omega|$. The above inequality can be rewritten as

$$\mathbb{E}_{\bar{M}}((\mathcal{N}_{B_\varepsilon})^k) \leq \varepsilon^{-k} \int_{\mathbb{R}_+} s^k |M|(ds) \quad k = 1, 2, \dots$$

This shows that if $|M|$ has finite moments of all orders, then the left-hand side is finite for all k and hence the correlation measures of \bar{M} are well defined. (Here we tacitly used the evident fact that any compact subset of \mathbb{R}^* is contained in subset B_ε with ε small enough.)

Step 2. For $r > 0$, set $M_r := M\Lambda_r^\infty$. It is initially defined as a probability distribution on \mathbb{Y} , but it is convenient to transfer it to $\tilde{\Omega}$ using the embedding $\varphi_r : \mathbb{Y} \rightarrow \tilde{\Omega}$. So, we will regard each M_r as a probability distribution on $\tilde{\Omega}$.

By Proposition 4.2, M_r converges to M in the weak topology as $r \rightarrow +\infty$, meaning that

$$\lim_{r \rightarrow +\infty} \langle \Psi, M_r \rangle = \langle \Psi, M \rangle \tag{9.2}$$

for any continuous bounded function Ψ on $\tilde{\Omega}$.

Assume now that we dispose of the following uniform bound on the tails of measures $|M_r|$:

$$\text{For every } k = 1, 2, \dots, \text{ one has } \int_{\mathbb{R}_+} s^k |M_r|(ds) \leq C_k \tag{9.3}$$

with a constant C_k independent on r .

Then, evidently, (9.2) holds under weaker assumptions on Ψ : it suffices to require that Ψ is continuous and has *moderate growth at infinity*, meaning that $|\Psi(\omega)| \leq \text{const}(1 + |\omega|)^k$ for some k .

Step 3. Assume that condition (9.3) is satisfied. We claim that then the correlation measures of M_r vaguely converge to the respective correlation measures of M .

Indeed, first of all, by virtue of step 1, our assumption guarantees the very existence of the correlation measures for measures M_r . Moreover, the inequalities (9.3) are inherited by the limit measure M , so that its correlation measures exist, too.

Fix $k = 1, 2, \dots$. By definition, the vague convergence of the k th correlation measures, $\rho_k^{M_r} \rightarrow \rho_k^M$, means that

$$\lim_{r \rightarrow +\infty} \langle F, \rho_k^{M_r} \rangle = \langle F, \rho_k^M \rangle$$

for any continuous, compactly supported function F on $(\mathbb{R}^*)^k$. By the very definition of the correlation measures, the latter relation is equivalent to fulfillment of relation (9.2), where $\Psi = \Psi_F$ has the following form

$$\Psi_F(\omega) = \langle F, \Delta^k(\bar{\omega}) \rangle = \sum_{x_1, \dots, x_k} F(x_1, \dots, x_k), \tag{9.4}$$

where the sum is taken over ordered k -tuples of distinct points extracted from configuration $\bar{\omega}$.

Now, by virtue of step 2, it suffices to check that Ψ_F is continuous and has moderate growth at infinity.

Choose ε so small that the support of F is contained in B_ε^k . By virtue of bound (9.1),

$$|\Psi_F(\omega)| \leq \varepsilon^{-k} \|F\| |\omega|^k.$$

Therefore, Ψ_F has moderate growth at infinity.

To see that Ψ_F is continuous look at the right-hand side of (9.4) and observe that $F(x_1, \dots, x_k)$ vanishes unless all quantities $|x_1|, \dots, |x_k|$ are bounded from below by ε , which in turn entails that the k -tuple $\{x_1, \dots, x_k\}$ is contained in the subset

$$\{\alpha_1, \dots, \alpha_m, -\beta_1, \dots, -\beta_m\}, \quad m := [\varepsilon^{-1} |\omega|].$$

That is, only coordinates of ω with a few first indices really contribute, and this finite set of possible indices depends only on $|\omega|$. Together with the continuity of F this gives the desired claim.

Step 4. Now we apply the above general arguments to $M := M^{(z, z')}$ and the corresponding pre-limit measures $M_r := M_r^{(z, z')}$. Recall that, according to our convention, $M_r^{(z, z')}$ lives on $\varphi_r(\mathbb{Y}_r) \subset \tilde{\Omega}$. Then we know exactly what is $|M_r^{(z, z')}|$: it is a scaled negative binomial distribution living on the subset $r^{-1}\mathbb{Z}_+ \subset \mathbb{R}_+$:

$$|M_r^{(z, z')}|(r^{-1}l) = (r+1)^{-zz'} \frac{(zz')^{\downarrow l}}{l!} \left(\frac{r}{r+1}\right)^l, \quad l \in \mathbb{Z}_+.$$

Condition (9.3) on the tails is readily checked (note that the limiting measure $|M^{(z, z')}|$ is the Γ -distribution with parameter zz'). Therefore, all correlation functions exist, and we have the limit relation

$$\lim_{r \rightarrow +\infty} \langle F, \rho_k^{M_r} \rangle = \langle F, \rho_k^M \rangle, \quad M_r := M_r^{(z, z')}, \quad M := M^{(z, z')}$$

for any continuous compactly supported function F on $(\mathbb{R}^*)^k$.

Step 5. Finally, we apply the results of our papers [7] and [11]. As shown in those papers, the pre-limit measures $M_r = M_r^{(z, z')}$ are determinantal, with some correlation kernels $K_r^{(z, z')}(x, y)$, called *discrete hypergeometric kernels*, for which an explicit expression is known.

In accordance with our definition of measure $M_r^{(z, z')}$, it lives on the lattice $r^{-1}\mathbb{Z}' \subset \mathbb{R}^*$, where $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$. As the reference measure σ , we take the counting measure on the lattice. Then one can write

$$\langle F, \rho_k^{M_r} \rangle = \sum_{(x_1, \dots, x_k) \in (r^{-1}\mathbb{Z}')^k} F(x_1, \dots, x_k) \det[K_r^{(z, z')}(x_i, x_j)].$$

On the other hand, the limiting behavior of kernels $K_r^{(z,z')}(x,y)$ was studied in [7, Theorem 5.4]. It follows that, as $r \rightarrow +\infty$, the right-hand side of the above relation converges to

$$\int_{(x_1, \dots, x_k) \in (\mathbb{R}^*)^k} F(x_1, \dots, x_k) \det[K^{(z,z')}(x_i, x_j)] dx_1 \dots dx_k,$$

where $K^{(z,z')}(x,y)$ is the Whittaker kernel. This completes the proof. □

Remark 9.2. The map $M \mapsto \bar{M}$ converting a measure on $\tilde{\Omega}$ to that on $\text{Conf}(\mathbb{R}^*)$ is not injective, because the map $\omega \mapsto \bar{\omega}$ ignores parameter δ . However, M is uniquely determined by its pushforward \bar{M} if it is known a priori that M is supported by the subset

$$\tilde{\Omega}_0 := \{\omega : \sum \alpha_i + \sum \beta_i = \delta\} \subset \tilde{\Omega}.$$

(Note that $\tilde{\Omega}_0$ is a dense Borel subset of type G_δ .)

This is just the case for $M = M^{(z,z')}$, as can be proved using Olshanski [30, Theorem 6.1]. Therefore, $M^{(z,z')}$ is completely specified by the correlation kernel $K^{(z,z')}(x,y)$ of the measure $\bar{M}^{(z,z')}$.

9.3 Determinantal structure of equilibrium finite-dimensional distributions

Starting Markov process $X^{(z,z')}$ at time $t = 0$ from the stationary distribution we get a stationary in time stochastic process $\tilde{X}^{(z,z')}$. Given time moments $0 \leq t_1 < \dots < t_n$, let $M^{(z,z')}(t_1, \dots, t_n)$ stand for the corresponding finite-dimensional distribution of $\tilde{X}^{(z,z')}$. The distributions $M^{(z,z')}(t_1, \dots, t_n)$ are invariant under simultaneous shift of all time moments by a constant; they can be called the *equilibrium* finite-dimensional distributions. For $n = 1$, we have $M^{(z,z')}(t) \equiv M^{(z,z')}$.

Initially $M^{(z,z')}(t_1, \dots, t_n)$ is defined as a probability measure on the n -fold product space $\tilde{\Omega}^n$, but then we convert it to a probability measure $\bar{M}^{(z,z')}(t_1, \dots, t_n)$ on $(\text{Conf}(\mathbb{R}^*))^n$, just as we did above for the case $n = 1$. Observe that $(\text{Conf}(\mathbb{R}^*))^n$ can be identified, in a natural way, with $\text{Conf}(\underbrace{\mathbb{R}^* \sqcup \dots \sqcup \mathbb{R}^*}_n)$. This shows that we can in-

terpret $\bar{M}^{(z,z')}(t_1, \dots, t_n)$ as a probability distribution on configurations, and the next theorem says that it is again in the determinantal class. This means that the correlation functions of $\bar{M}^{(z,z')}(t_1, \dots, t_n)$ are described by a “dynamical” (or “space-time”) kernel $K^{(z,z')}(x,s;y,t)$ on $(\mathbb{R}^* \times \mathbb{R}) \times (\mathbb{R}^* \times \mathbb{R})$ whose two arguments, couples (x,s) and (y,t) , should be viewed as space-time variables ranging over space-time $\mathbb{R}^* \times \mathbb{R}$. Given an arbitrary finite collection $(x_1, t_1), \dots, (x_k, t_k)$, the $k \times k$ determinant

$$\det \left[K^{(z,z')}(x_i, t_i; x_j, t_j) \right]$$

multiplied by $dx_1 \dots dx_k$ gives the probability of the event that at each prescribed moment t_i (where $i = 1, \dots, k$), the configuration $\bar{\omega} \in \text{Conf}(\mathbb{R}^*)$ corresponding to $\omega := X^{(z,z')}(t_i)$ contains a point in the infinitesimal neighborhood dx_i about position x_i , for every $i = 1, \dots, k$.

The kernel $K^{(z,z')}(x,s;y,t)$ in question is the *extended Whittaker kernel*; we refer to Borodin–Olshanski [11] for its description.

Theorem 9.3. The pushforwards $\bar{M}^{(z,z')}(t_1, \dots, t_n)$ of the equilibrium finite-dimensional distributions $M^{(z,z')}(t_1, \dots, t_n)$ are determinantal measures described by the extended Whittaker kernel $K^{(z,z')}(x,s;y,t)$.

This is a generalization of Theorem 9.1, which is a particular case of Theorem 9.3 for $n = 1$, because $M^{(z,z')}(t) \equiv M^{(z,z')}$, and $K^{(z,z')}(x, s; y, t)$ reduces to the Whittaker kernel $K^{(z,z')}(x, y)$ for $s = t$.

Remark 9.4 (cf. Remark 9.2). *Note that measure $M^{(z,z')}(t_1, \dots, t_n)$ is supported by the subset $\tilde{\Omega}_0^n$, because every its one-dimensional marginal coincides with $M^{(z,z')}$ and the latter measure is supported by $\tilde{\Omega}_0$. As in the case $n = 1$, this implies that the equilibrium finite-dimensional distributions are uniquely determined by the extended Whittaker kernel.*

Proof of Theorem 9.3. The argument for Theorem 9.1 extends smoothly, with a few minor evident modifications only. Let $M_r^{(z,z')}(t_1, \dots, t_n)$ stand for the pre-limit equilibrium finite-dimensional distributions. Corollary 3.6 tells us that they approximate the distributions $M^{(z,z')}(t_1, \dots, t_n)$. To bound the tails we use the fact, mentioned above, that the one-dimensional marginals coincide with the stationary distribution. The correlation functions of the pre-limit distributions are described by the extended version of the discrete hypergeometric kernel, which converges to the extended Whittaker kernel as $r \rightarrow +\infty$: this is established in [11]. □

10 Remarks on the Plancherel limit

Let us return to the context of Section 8.2. So far the basic parameters z and z' were fixed, but here we take a limit transition in formulas (8.11) assuming that z and z' go to infinity while the third parameter r goes to 0 in such a way that the product $rz z'$ tends to a fixed real number $\theta > 0$. One may simply assume that r is related to the couple (z, z') by $r = \theta(z z')^{-1}$; recall that because of Condition 1.1, $z z'$ is strictly positive, so that the above relation is compatible with the fact that r should be a positive number. The quantity θ becomes our new parameter.

It is not difficult to verify that in this limit transition, all results of Section 8.2 survive. Namely, the Q -matrix $Q_r^{(z,z')}$ turns into the following matrix:

$$\begin{aligned} Q_\theta(\lambda, \lambda + \square) &= \theta \frac{\dim(\lambda + \square)}{(|\lambda| + 1) \dim \lambda}, \quad \square \in \lambda^+, \\ Q_\theta(\lambda, \lambda - \square) &= \frac{|\lambda| \dim(\lambda - \square)}{\dim \lambda}, \quad \square \in \lambda^-, \\ -Q_\theta(\lambda, \lambda) &= |\lambda| + \theta. \end{aligned} \tag{10.1}$$

An analog of Proposition 8.4 holds, with the Meixner symmetric functions being replaced by the so-called *Charlier symmetric functions*, introduced in [32] (these are obtained from the Meixner functions via the same limit transition). A key observation is that the links ${}^{\text{YB}}\Lambda_r^{r'}$ depend on parameters r and r' through their ratio r/r' , which remains intact under the limit (it translates into the ratio θ/θ'). Because of this fact, all other results of Section 8.2 are smoothly extended, too. We only have to change the notation $r \rightarrow \theta$. Finally, we get a family $\{X_\theta : \theta > 0\}$ of continuous time Feller Markov chains on \mathbb{Y} .

Further, one can prove that X_θ has a unique stationary distribution, which is nothing else than the well-known *Poissonized Plancherel measure*, first introduced in Baik–Deift–Johansson [2]:

$$M_\theta(\lambda) = e^{-\theta|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2.$$

It is a degeneration of the mixed z -measure (8.27), which played an important role in Borodin–Okounkov–Olshanski [6].

The Markov chains X_θ were studied in our paper [10]. As shown in that paper, X_θ admits a nice description in terms of the Poisson process in the quarter-plane and the Robinson–Schensted algorithm.

The formalism of the present paper says that the family $\{X_\theta : \theta > 0\}$ gives rise to a Feller Markov process X on the boundary $\tilde{\Omega}$, and X has a unique stationary distribution $M := \varprojlim M_\theta$, the boundary measure corresponding to the family of the Poissonized Plancherel measures. On the other hand, it is readily seen that this boundary measure is simply the Dirac measure at the point

$$\omega_1 := (\alpha = \underline{0}, \beta = \underline{0}, \delta = 1) \in \tilde{\Omega},$$

where $\underline{0} := (0, 0, \dots)$ is the null sequence.

At first glance, this looks strange, but the key is that X is not a genuine Markov process, but a *deterministic process*. Its transition function $P(t)$ degenerates to a semi-group of continuous maps $\tilde{\Omega} \rightarrow \tilde{\Omega}$ which have the following form:

$$P(t) : (\alpha, \beta, \delta) \mapsto (e^{-t}\alpha, e^{-t}\beta, e^{-t}\delta + (1 - e^{-t})), \quad t \geq 0.$$

From this formula it is seen that, as $t \rightarrow +\infty$, $P(t)$ contracts the whole space $\tilde{\Omega}$ to the point ω_1 . There is no contradiction, because such a deterministic process is formally a Markov process.

On the algebraic level, this phenomenon is clearly seen when we compute the generator of X as an operator in the algebra of symmetric functions: In contrast to the Laguerre operator (8.16) we get a *first order* differential operator. This operator is best written in terms of the generators p_1, p_2, \dots (the power-sum symmetric functions, see Section 7.4), it has the form

$$(1 - p_1) \frac{\partial}{\partial p_1} + \sum_{n \geq 2} n p_n \frac{\partial}{\partial p_n}.$$

The above discussion shows that our abstract formalism of constructing boundary Markov processes via Markov intertwiners conceals a potential danger, as it may happen that the boundary process degenerates to a deterministic process. Therefore, if one is interested in constructing interesting infinite-dimensional Markov processes (as we do), one needs additional arguments guaranteeing that such a degeneration does not occur. We were fortunate that we were able to explicitly compute the generator of our process $X^{(z, z')}$: from the fact that the generator is a second order operator it is easy to conclude that $X^{(z, z')}$ cannot be a deterministic process.

Finally, note that the existence of a nontrivial stationary distribution, $M^{(z, z')}$, makes it possible to prove the non-determinism of the boundary process in a different way.

References

- [1] Anderson, W. J.: Continuous time Markov chains: An applications-oriented approach. *Springer*, 1991. MR-1118840
- [2] Baik, J., Deift, P. and Johansson, K.: On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** (1999), 1119–1178, arXiv:math/9810105. MR-1682248
- [3] Borodin, A. N.: Harmonic analysis on the infinite symmetric group, and the Whittaker kernel. *Algebra i Analiz* **12** (2000), no. 5, 28–63 (Russian); English translation: *St. Petersburg Math. J.* **12** (2001), no. 5, 733–759. MR-1812941
- [4] Borodin, A.: Determinantal point processes. The Oxford Handbook on Random Matrix Theory (Gernot Akemann, Jinho Baik, and Philippe Di Francesco, editors). *Oxford University Press*, 2011, Chapter 11, 231–249, arXiv:0911.1153. MR-2932631

- [5] Borodin, A. and Gorin, V.: Markov processes of infinitely many nonintersecting random walks. *Probab. Theory Rel. Fields*, online publication March 2012, DOI 10.1007/s00440-012-0417-4, arXiv:1106.1299.
- [6] Borodin, A., Okounkov, A. and Olshanski, G.: Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.* **13** (2000), 481–515; arXiv:math/9905032. MR-1758751
- [7] Borodin, A. and Olshanski, G.: Distributions on partitions, point processes, and the hypergeometric kernel. *Comm. Math. Phys.* **211** (2000), no. 2, 335–358, arXiv:math/9904010. MR-1754518
- [8] Borodin, A. and Olshanski, G.: Z-measures on partitions and their scaling limits. *Europ. J. Comb.* **26** (2005), no. 6, 795–834, arXiv:math-ph/0210048. MR-2143199
- [9] Borodin, A. and Olshanski, G.: Random partitions and the Gamma kernel. *Adv. Math.* **194** (2005), 141–202, arXiv:math-ph/0305043. MR-2141857
- [10] Borodin, A. and Olshanski, G.: Stochastic dynamics related to Plancherel measure on partitions. In: Representation Theory, Dynamical Systems, and Asymptotic Combinatorics (V. Kaimanovich and A. Lodkin, editors). *Amer. Math. Soc. Translations – Series 2*, vol. 217, 2006, 9–22, arXiv:math-ph/0402064. MR-2276098
- [11] Borodin, A. and Olshanski, G.: Markov processes on partitions. *Probab. Theory Rel. Fields* **135** (2006), 84–152, arXiv:math-ph/0409075. MR-2214152
- [12] Borodin, A. and Olshanski, G.: Infinite-dimensional diffusions as limits of random walks on partitions. *Probab. Theory Rel. Fields* **144** (2009), 281–318, arXiv:0706.1034. MR-2480792
- [13] Borodin, A. and Olshanski, G.: Markov processes on the path space of the Gelfand-Tsetlin graph and on its boundary. *J. Funct. Anal.* **263** (2012), no. 1, 248–303; arXiv:1009.2029. MR-2920848
- [14] Borodin, A. and Olshanski, G.: The Young bouquet and its boundary. *Moscow Math. J.* **13** (2013), no. 2; arXiv:1110.4458.
- [15] Borodin, A. and Olshanski, G.: An interacting particle process related to Young tableaux, arXiv:1303.2795.
- [16] Dyson, F. J.: A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3** (1962), 1191–1198. MR-0148397
- [17] Eie, B.: The generalized Bessel process corresponding to an Ornstein-Uhlenbeck process. *Scand. J. Stat.* **10** (1983), 247–250. MR-0732920
- [18] Ethier, S. N. and Kurtz, T. G.: Markov processes. Characterization and convergence. *Wiley*, 2005 (first edition 1986). MR-0838085
- [19] Feller, W.: On the integro-differential equations of purely discontinuous Markoff processes. *Trans. Amer. Math. Soc.* **48** (1940), 488–815 and Errata, **58** (1945) p. 474. MR-0002697
- [20] Jones, Liza Anne: Non-colliding diffusions and infinite particle systems. Thesis. University of Oxford, 2008.
- [21] Katori, M. and Tanemura, H.: Zeros of Airy function and relaxation process. *J. Stat. Phys.* **136** (2009), 1177–1204, arXiv:0906.3666. MR-2550400
- [22] Katori, M. and Tanemura, H.: Non-equilibrium dynamics of Dyson’s model with an infinite number of particles. *Commun. Math. Phys.* **293** (2010), 469–497, arXiv:0812.4108. MR-2563791
- [23] Katori, M. and Tanemura, H.: Markov property of determinantal processes with extended sine, Airy, and Bessel kernels. *Markov Processes and Relat. Fields* **17** (2011), 541–580, arXiv:1106.4360. MR-2918121
- [24] Koekoek, R., Lesky, P. A. and Swarttouw, R. F.: Hypergeometric orthogonal polynomials and their q-analogues. *Springer*, 2010. MR-2656096
- [25] König, W.: Orthogonal polynomial ensembles in probability theory. *Probab. Surveys* **2** (2005), 385–447. MR-2203677
- [26] Lenard, A.: Correlation functions and the uniqueness of the state in classical statistical mechanics. *Commun. Math. Phys.* **30** (1973), 35–44. MR-0323270
- [27] Liggett, T. M.: Continuous time Markov processes. Graduate Texts in Math. 113. *Amer. Math. Soc.*, 2010. MR-2574430

- [28] Okounkov, A.: Infinite wedge and measures on partitions. *Selecta Math.* **7** (2001), 1–25, arXiv:math/9907127. MR-1856553
- [29] Okounkov, A.: $SL(2)$ and z -measures. In: Random matrix models and their applications (P. M. Bleher and A. R. Its, editors). Mathematical Sciences Research Institute Publications **40**, Cambridge Univ. Press, 2001, 407–420, arXiv:math/0002136. MR-1842795
- [30] Olshanski, G.: Point processes related to the infinite symmetric group. In: The orbit method in geometry and physics: in honor of A. A. Kirillov (Ch. Duval et al., editors.), Progress in Mathematics **213**, Birkhäuser, 2003, pp. 349–393, arXiv:math.RT/9804086. MR-1995385
- [31] Olshanski, G.: Laguerre and Meixner symmetric functions, and infinite-dimensional diffusion processes. *Zapiski Nauchnykh Seminarov POMI* **378** (2010), 81–110. Reproduced in *J. Math. Sci. (New York)* **174** (2011), no. 1, 41–57, arXiv:1009.2037. MR-2749298
- [32] Olshanski, G.: Laguerre and Meixner orthogonal bases in the algebra of symmetric functions. *Intern. Math. Res. Notices* **2012** (2012), no. 16, 3615–3679, arXiv:1103.5848. MR-2959021
- [33] Olshanski, G., Regev, A. and Vershik, A.: Frobenius-Schur functions. In: Studies in memory of Issai Schur (A. Joseph, A. Melnikov, R. Rentschler, editors). Progress in Mathematics **210**, pp. 251–300. Birkhäuser, 2003, arXiv:math/0110077. MR-1985729
- [34] Osada, H.: Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. *Ann. Prob.* **41** (2013), 1–49, arXiv:0902.3561.
- [35] Osada, H.: Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II: Airy random point field. *Stochastic Process. Appl.* **123** (2013), 813–838, arXiv:1209.0609.
- [36] Sagan, B. E.: The symmetric group. Representations, combinatorial algorithms, and symmetric functions. Second edition, Springer, 2001. MR-1824028
- [37] Soshnikov, A.: Determinantal random point fields. *Russian Math. Surveys* **55**, (2000) 923–975, arXiv: math/0002099. MR-1799012
- [38] Spohn, H.: Interacting Brownian particles: a study of Dyson’s model. In: Hydrodynamic Behavior and Interacting Particle Systems (G. Papanicolaou, editor). IMA Volumes in Mathematics and its Applications **9**, Springer-Verlag, 1987, 151–179. MR-0914993
- [39] Stanley, R. P.: Enumerative combinatorics, vol. 2. Cambridge Univ. Press, 1999. MR-1676282
- [40] Tracy, C. A. and Widom, H.: Differential equations for Dyson processes. *Commun. Math. Phys.* **252** (2004), no. 1-3, 7–41, arXiv:math/0309082. MR-2103903

Acknowledgments. A. B. was partially supported by NSF-grant DMS-1056390. G. O. was partially supported by a grant from Simons Foundation (Simons–IUM Fellowship), the RFBR-CNRS grant 10-01-93114, and the project SFB 701 of Bielefeld University.

Electronic Journal of Probability

Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)

Economical model of EJP-ECP

- Low cost, based on free software (OJS¹)
- Non profit, sponsored by IMS², BS³, PKP⁴
- Purely electronic and secure (LOCKSS⁵)

Help keep the journal free and vigorous

- Donate to the IMS open access fund⁶ (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

¹OJS: Open Journal Systems <http://pkp.sfu.ca/ojs/>

²IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

³BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁴PK: Public Knowledge Project <http://pkp.sfu.ca/>

⁵LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

⁶IMS Open Access Fund: <http://www.imstat.org/publications/open.htm>