Choice of Discount Rate for Cost Levelization

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The manuscript which follows is in the nature of a working paper. It deals with the mathematically and conceptually correct discount rate and procedure for the computation of a lifetime-levelized cost-of-product for enterprises which are funded by a mixture of debt and equity, and taxed according to representative US practices. It shows that methods of long standing contain a subtle but important, albeit generally small, error. Thus the paper is open to two interpretations: the first is a call for the use of a correct approach in calculations which are represented to be at a state-of-the-art level of complexity - which is also a simple matter when such computations are incorporated in computer programs; the second is the demonstration and quantification of what may be an acceptable degree of inaccuracy for simpler, more approximate back-of-the-envelope estimates, and for more transparent pedagogical demonstrations of general principles.

Michael J. Driscoll
Professor Emeritus
Cambridge, MA, September 20, 1991
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INTRODUCTION

Computation of a levelized revenue expectation or requirement, or unit cost of product, is a central task of the engineering economist. Despite preference for the use of the tax-sheltered cost of corporate capital in authoritative texts such as Ref. [1], backed by detailed analyses, as in Refs. [2] - [4], we show here that there is an abundance of comparable choices, only one of which, the cost of composite capital, properly discounts the periodic cash flow of taxes. Although levelized costs usually differ only slightly with the choice of rate (if applied to properly weighted cash flows), this should be sufficient reason to prefer this particular discount rate in economic analyses.

NOTATION

\[ I_i = \text{capital investment made at the end of the } i^{th} \text{ year, } \$ \]

\[ C_i = \text{annual expensed costs at the end of the } i^{th} \text{ year, } \$ \]

\[ D_i, D'_i = \text{annual book and tax depreciation at the end of the } i^{th} \text{ year, } \$ \]

\[ V_i = \text{undepreciated book investment remaining at the start of year } i, \$ \]

\[ T_i = \text{annual income taxes paid at the end of year } i, \$ \]
\[ T_{iy} = \text{annual income taxes paid at the end of year } i \text{ for the virtual cash flow, } \$ \]

\[ R_i = \text{annual revenue requirement at the end of year } i, \$ \]

\[ \tau = \text{income tax rate, } \%/100 \]

\[ r = \text{weighted total rate of return } = b + s, \% \text{ per yr/100} \]

\[ b = \text{weighted bond rate of return } = f_b r_b, \% \text{ per yr/100} \]

\[ s = \text{weighted stock rate of return } = f_s r_s, \% \text{ per yr/100} \]

\[ r_b = \text{bond interest rate, } \% \text{ per yr/100} \]

\[ r_s = \text{stock rate of return, } \% \text{ per yr/100} \]

\[ f_b = \text{bond fraction of investment} \]

\[ f_s = \text{stock fraction of investment} \]

\[ x = \text{rate of return to the stockholders if the investment is 100% composed of stock in the virtual cash flow, } \% \text{ per yr/100} \]

\[ y = \text{total discount rate for the virtual cash flow, } \% \text{ per yr/100} \]

\[ w = \text{interest rate which could be paid to the bondholders if the investment is 100% composed of bonds in the virtual cash flow, } \% \text{ per yr/100} \]

\[ \alpha = \text{ratio between weighted bond to total return rate for the virtual cash flow} \]

\[ n = \text{economic lifetime, yr.} \]

\[ R_r = \text{levelized revenue for the original cash flow pattern} \]

\[ R_y = \text{levelized revenue for the virtual cash flow pattern} \]
\[ R_x = \text{levelized revenue for the virtual cash flow pattern with } \alpha = 0 (y = x) \]

\[ R_w = \text{levelized revenue for the virtual cash flow pattern with } \alpha = 1 (y = w) \]

**Note:** The superscript symbol * identifies levelized cash flows.

**THEORY OF LEVELIZED COSTS - PART I**

The basic equations and nomenclature of Ref. [5], will be followed with some minor exceptions. The required revenue is the sum of costs, return on capital, depreciation and taxes:

\[ R_i = C_i + rV_i + D_i + T_i \]  \hspace{1cm} (1)

while taxes are some fraction of revenue less deductions, which include bond payments and depreciation which is not necessarily the same as book depreciation:

\[ T_i = \tau \left( R_i - C_i - bV_i - D_i' \right) \]  \hspace{1cm} (2)

where

\[ V_i = \sum_{j=0}^{i-1} I_j - \sum_{j=1}^{i-1} D_j \]  \hspace{1cm} (3)

\[ \sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D_i' = \sum_{i=0}^{n} I_i \]  \hspace{1cm} (4)
\[ r = b + s = f_b r_b + f_s r_s \]  \hspace{1cm} (5)

and

\[ f_b + f_s = 1 \]  \hspace{1cm} (6)

Substituting Eq. (2) into Eq. (1),

\[ R_i = C_i + \frac{1}{1-\tau}[(r - b\tau)V_i + D_i] - \frac{\tau}{1-\tau}D_i' \]  \hspace{1cm} (7)

Let us add to Eq. (7):

\[ -\frac{\alpha\tau}{1-\tau}D_i + \frac{\alpha\tau}{1-\tau}D_i \]

where \( \alpha \) is a free parameter, then:
\[
R_i = C_i + \frac{1 - \alpha \tau}{1 - \tau} \left[ \frac{r - b \tau}{1 - \alpha \tau} V_i + D_i \right] - \frac{\tau}{1 - \tau} \left( D'_i - \alpha D_i \right) \tag{8}
\]

Defining:
\[
y = \frac{r - b \tau}{1 - \alpha \tau} \tag{9}
\]

and multiplying Eq. (8) by the factor \(1/(1+y)^i\) and adding Eqs. (8) from \(i = 1\) to \(i = n\), one finds:

\[
\sum_{i=1}^{n} \frac{R_i}{(1+y)^i} = \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} + \frac{1 - \alpha \tau}{1 - \tau} \sum_{i=1}^{n} \frac{y V_i + D_i}{(1+y)^i} - \frac{\tau}{1 - \tau} \sum_{i=1}^{n} \frac{D'_i - \alpha D_i}{(1+y)^i} \tag{10}
\]

But, from Eqs. (3) and (4), one obtains the following relation (see Appendix I):

\[
\sum_{i=1}^{n} \frac{y V_i + D_i}{(1+y)^i} = \sum_{i=0}^{n-1} \frac{I_i}{(1+y)^i} \quad \text{for } y > -1 \tag{11}
\]

Substituting Eq. (11) into Eq. (10):

\[
\sum_{i=1}^{n} \frac{R_i}{(1+y)^i} = \sum_{i=1}^{n} \frac{C_i}{(1+y)^i} + \frac{1 - \alpha \tau}{1 - \tau} \sum_{i=0}^{n-1} \frac{I_i}{(1+y)^i} - \frac{\tau}{1 - \tau} \sum_{i=1}^{n} \frac{D'_i - \alpha D_i}{(1+y)^i} \tag{12}
\]
Defining:

\[ R_y = \left( \frac{\sum_{i=1}^{n} \frac{R_i}{(1+y)^i}}{\sum_{i=1}^{n} \frac{1}{(1+y)^i}} \right) \]  

(13)

Then:

\[ R_y = \frac{\sum_{i=1}^{n} \frac{C_i}{(1+y)^i} + 1 - \alpha \tau}{\sum_{i=1}^{n} \frac{1}{(1+y)^i}} \left( 1 - \tau \frac{\sum_{i=1}^{n} \frac{1}{(1+y)^i}}{\sum_{i=1}^{n} \frac{1}{(1+y)^i}} \right) \frac{\sum_{i=1}^{n} D_i - \alpha D_i}{(1+y)^i} \]  

(14)

\( R_y \) is the levelized revenue with the discount rate equal to \( y \). Then \( R_r, R_x \) and \( R_w \) are found with \( y = r, y = x \) and \( y = w \) (or \( \alpha = b/r, \alpha = 0 \) and \( \alpha = 1 \)), respectively:

(a) \( y = r = b + s \rightarrow \alpha = b/r \)  

(15)
\[
R_{r} = \sum_{i=1}^{n} \frac{C_{i}}{(1+r)^{i}} + \frac{1-b}{r} \sum_{i=0}^{n} \frac{I_{i}}{(1+r)^{i}} - \tau \sum_{i=1}^{n} \frac{D'_{i} - bD_{i}}{(1+r)^{i}} \\
\sum_{i=1}^{n} \frac{1}{(1+r)^{i}} - 1 - \tau \sum_{i=1}^{n} \frac{1}{(1+r)^{i}} - 1 - \tau \sum_{i=1}^{n} \frac{1}{(1+r)^{i}}
\]

(16)

(b) \quad y = x = r - b\tau \rightarrow \alpha = 0

(17)

\[
R_{x} = \sum_{i=1}^{n} \frac{C_{i}}{(1+x)^{i}} + \frac{1}{(1+x)^{i}} - \tau \sum_{i=1}^{n} \frac{1}{(1+x)^{i}} - 1 - \tau \sum_{i=1}^{n} \frac{1}{(1+x)^{i}}
\]

(18)

\[
y = w = \frac{r - b\tau}{1 - \tau} = \frac{x}{(1 - \tau)} \rightarrow \alpha = 1
\]

(19)

\[
R_{w} = \sum_{i=1}^{n} \frac{C_{i}}{(1+w)^{i}} + \sum_{i=0}^{n} \frac{I_{i}}{(1+w)^{i}} - \tau \sum_{i=1}^{n} \frac{D'_{i} - D_{i}}{(1+w)^{i}} \\
\sum_{i=1}^{n} \frac{1}{(1+w)^{i}} - 1 - \tau \sum_{i=1}^{n} \frac{1}{(1+w)^{i}} - 1 - \tau \sum_{i=1}^{n} \frac{1}{(1+w)^{i}}
\]

(20)
Expression (14) also holds for $y > -1$ or, equivalently, for $\alpha < 1/\tau$ (with $\tau > 0$) or $\alpha > (r - b\tau + 1)/\tau$ (with $\tau > 0$). This can be clearly seen in Figure 1. The condition $\tau = 0$ implies that $y$ has only one value: $y = r$.

![Figure 1. Variation of $y$ with $\alpha$ (for $0 < \tau < 1$).](image)

Consequently, there are an infinite number of levelized revenues $R_y$; some of the more interesting $R_y$ are as follows:

- $\alpha = b/r \quad \rightarrow \quad y = r \quad \rightarrow \quad R_y = R_r$
- $\alpha = 0 \quad \rightarrow \quad y = x = r - b\tau \quad \rightarrow \quad R_y = R_x$
- $\alpha = 1 \quad \rightarrow \quad y = w = \frac{r - b\tau}{1 - \tau} = x \frac{1}{1 - \tau} \quad \rightarrow \quad R_y = R_w$
- $\alpha \rightarrow -\infty \quad \rightarrow \quad y \rightarrow 0 \quad \rightarrow \quad R_y \rightarrow \frac{1}{n} \sum_{i=1}^{n} R_i$
Other $R_y$ could also be derived, such as those for $y = r_b$ or $y = r_s$, for example. But, the central question is now the following: which levelized revenue $R_y$ should be used to compare similar economic alternatives?

The levelized cost $R_y$ being sought is one that, if charged uniformly throughout the life of the plant, will just pay for all current expenses, taxes, return on and return of investment. The levelized cash flow that satisfies these assumptions is the following:

$$R_i^* = C_i^* + r V_i^* + D_i^* + T_i^*$$  \hspace{1cm} (21)

$$T_i^* = \tau \left( R_i^* - C_i^* - b V_i^* - D_i^* \right)$$  \hspace{1cm} (22)

$$C_i^* = C_i$$  \hspace{1cm} (23)

$$R_i^* = R_y$$  \hspace{1cm} (24)
In this cash flow, there are $6n$ unknowns: $R^*_i$, $C^*_i$, $V^*_i$, $D^*_i$, $T^*_i$, and $D'_i$; and $6n$ equations (Eq. 21 to Eq. 26).

It must be proven now that:

\[ \sum_{i=1}^{n} D^*_i = \sum_{i=1}^{n} D'_i - \alpha I_i \]

Following the steps to obtain Eq. (8) leads to:

\[ R^*_i = C^*_i + \frac{1 - \alpha \tau}{1 - \tau} \left[ y V^*_i + D^*_i \right] - \frac{\tau}{1 - \tau} \left( D^*_i - \alpha D^*_i \right) \]

Subtracting Eq.(8) from Eq.(27), multiplying by the factor $1/(1 + y)^i$ and adding equations from $i = 1$ to $i = n$, gives:
\[
\sum_{i=1}^{n} \frac{R_i^* - R_i}{(1 + y)^i} = \sum_{i=1}^{n} \frac{C_i^* - C_i}{(1 + y)^i} + \frac{1 - \alpha \tau}{1 - \tau} \left[ \sum_{i=1}^{n} \frac{yV_i^* + D_i^*}{(1 + y)^i} - \sum_{i=1}^{n} \frac{yV_i + D_i}{(1 + y)^i} \right] - \frac{\tau}{1 - \tau} \sum_{i=1}^{n} \frac{(D_i^* - \alpha D_i^*) - (D_i - \alpha D_i)}{(1 + y)^i}
\]

From Eqs. (23), (24) and (26):

\[
\sum_{i=1}^{n} \frac{yV_i^* + D_i^*}{(1 + y)^i} = \sum_{i=1}^{n} \frac{yV_i + D_i}{(1 + y)^i}
\]

From Eq. (11):

\[
\sum_{i=1}^{n} \frac{yV_i^* + D_i^*}{(1 + y)^i} = \sum_{i=0}^{n} \frac{I_i}{(1 + y)^i}
\]

From Eqs. (30) and (25), it can be shown that (see Appendix II)

\[
\sum_{i=1}^{n} D_i^* = \sum_{i=0}^{n} I_i
\]
From Eqs. (31) and (26) and knowing that \[ \sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D'_i = \sum_{i=0}^{n} I_i \], it follows that

\[ \sum_{i=1}^{n} D'_i = \sum_{i=0}^{n} I_i \]  

(32)

The "levelized cash flow" defined as having a constant revenue \( R_y \), pays for all current expenses \( C_i \) and return on and return of investment. On the other hand, it is impossible to have \( T'_i = T_i \) without disobeying other equations, such as (32) for example.

For \( \alpha = 0 \), from Eq. (26), it follows that \( D'_i = D'_i \). This is the reason why the approach (choice of \( x \) and cash flow weighting) embodied in Eq. (18) is said to be the one which pays for all current expenses \( C_i \), return on and of investment, while keeping \( D'_i = D'_i \).

But, if it is impossible to keep \( T'_i = T_i \), at least one can look for an \( R_y \) such that

\[ \sum_{i=1}^{n} \frac{T'_i}{(1+y)^i} = \sum_{i=1}^{n} \frac{T_i}{(1+y)^i} \]

From Eqs. (22) and (2):

\[ T'_i - T_i = \tau (R'_i - R_i) - \tau (C'_i - C_i) - \tau (bV'_i - bV_i) - \tau (D'_i - D_i) \]  

(33)
Multiplying by $1/(1 + y)^i$ and adding terms:

$$
\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1 + y)^i} = \tau \sum_{i=1}^{n} \frac{R_i^* - R_i}{(1 + y)^i} - \tau \sum_{i=1}^{n} \frac{C_i^* - C_i}{(1 + y)^i} - \tau \sum_{i=1}^{n} \frac{(bV_i^* - bV_i) + (D_i^* - D_i)}{(1 + y)^i}
$$

(34)

Using Eqs. (23), (24) and (26)

$$
\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1 + y)^i} = -\tau \sum_{i=1}^{n} \frac{bV_i^* - bV_i + \alpha D_i^* - \alpha D_i}{(1 + y)^i}
$$

(35)

$$
\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1 + y)^i} = -\alpha \tau \left[ \sum_{i=1}^{n} \frac{bV_i + D_i^*}{\alpha (1 + y)^i} - \sum_{i=1}^{n} \frac{bV_i + D_i}{(1 + y)^i} \right]
$$

(36)

in order to have:

$$
\sum_{i=1}^{n} \frac{T_i^* - T_i}{(1 + y)^i} = 0,
$$

then:

- either (a) $\tau = 0$
- or (b) $\alpha = 0$ and $b = 0$
- or (c) $\alpha = 1$ and $s = 0$
- or (d) $y = \frac{b}{\alpha} \rightarrow y = r$ and $\alpha = \frac{b}{r}$ (from Eq. 9).
(for $\alpha \to -\infty$ ($y \to 0$) this relation is also satisfied, but the practical range of $\alpha$ is $0 \leq \alpha \leq 1$ as will be shown in Part II of this article).

Note that alternative (d) always includes the other alternatives (a), (b) and (c).

One must conclude then that the levelized cost (revenue) $R_r$ with $y = r$ should be used to compare similar alternatives.

To find all unknowns of each levelized cash flow given by Eqs. (21) to (26), it is convenient to proceed in the following way:

$$V_i^* = I_0$$

(37)

From Eqs. (27) and (26):

$$D_i^* = \frac{1 - \tau}{1 - \alpha \tau} \left[ R_i^* - C_i^* + \frac{\tau}{1 - \tau} (D_i'^* - \alpha D_i) \right] - y V_i^*$$

(38)

From Eq. (26):

$$D_i'^* = D_i' + \alpha (D_i^* - D_i)$$

(39)
Using Eqs. (3), (37), (38) and (39), each levelized cash flow can be developed in turn, as defined for each value of y (or x). The example in Table I shows that all levelized cash flows for y = r, y = x and y = w, keep $C_i^* = C_i$ and that for y = x, $D_i^* = D_i'$; but only for y = r does

$$\sum_{i=1}^{n} \frac{T_i^*}{(1+y)^i} = \sum_{i=1}^{n} \frac{T_i}{(1+y)^i}.$$ 

While the differences in $R_y$ are small, case to case, other examples can be constructed in which larger discrepancies are displayed. Hence we believe that the use of $y = r$ is not a trivial requirement.

**THEORY OF LEVELIZED COSTS - PART II**

The Meaning of $R_y$

The true meaning of $R_y$ is the following: $R_y$ is the levelized revenue of a virtual cash flow given by Equations (40) and (41), which is very similar to the original cash flow given by Equations (1) and (2). In this virtual cash flow, the total rate of return of the project is y instead of r and the weighted bond rate of return is $\alpha y$ instead of $b$. Comparing both cash flows, it is seen that $R_i$, $C_i$, $V_i$, $D_i$, $D_i'$ and $I_i$ are the same, while differences in income taxes are given by Eq. (42).

$$R_i = C_i + y V_i + D_i + T_{iy}$$  \hspace{1cm} (40)

$$T_{iy} = \tau \left( R_i - C_i - \alpha y V_i - D_i' \right)$$  \hspace{1cm} (41)

$$T_{iy} - T_i = \tau \left( b - \alpha y \right) V_i = (r - y) V_i$$  \hspace{1cm} (42)
TABLE I. Numerical Example Showing an Original Cash Flow and Some Levelized Cash Flows (for $y=r$, $y=x$ and $y=w$).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual cash flow</td>
<td>$R_i$</td>
<td>0</td>
<td>6.6</td>
<td>7.3</td>
<td>7.2</td>
</tr>
<tr>
<td>flow with $r=0.12$</td>
<td>$V_i$</td>
<td>0</td>
<td>8</td>
<td>6.5</td>
<td>3.5</td>
</tr>
<tr>
<td>$b=0.04$</td>
<td>$D_i$</td>
<td>0</td>
<td>3.5</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>$\tau=0.5$</td>
<td>$T_i$</td>
<td>0</td>
<td>0.14</td>
<td>0.52</td>
<td>0.78</td>
</tr>
</tbody>
</table>

Levelized cash flow for $y=r$:

| $R_i^*$ | 0   | 6.631 | 6.631 | 6.631 | 6.631 |
| $V_i^*$ | 0   | 8     | 6.481 | 3.880 | 1.767 |
| $D_i^*$ | 0   | 3.519 | 2.601 | 2.113 | 1.767 |
| $T_i^*$ | 0   | 0.153 | 0.253 | 0.553 | 0.652 |

Levelized cash flow for $y=x$:

| $V_i^*$ | 0   | 8     | 6.488 | 3.826 | 1.647 |
| $D_i^*$ | 0   | 3.511 | 2.663 | 2.179 | 1.647 |
| $T_i^*$ | 0   | 0.152 | 0.182 | 0.485 | 0.779 |

Levelized cash flow for $y=w$:

| $V_i^*$ | 0   | 8     | 6.488 | 3.826 | 1.647 |
| $D_i^*$ | 0   | 3.511 | 2.663 | 2.179 | 1.647 |
| $T_i^*$ | 0   | 0.152 | 0.182 | 0.485 | 0.779 |

$D_i^*$
TABLE I. (CONTINUED) Numerical Example Showing an Original Cash Flow and Some Levelized Cash Flows (for y=r, y=x and y=w).

<table>
<thead>
<tr>
<th></th>
<th>i</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Levelized</td>
<td>R_i^*</td>
<td>R_w</td>
<td>0</td>
<td>6.661</td>
<td>6.661</td>
<td>6.661</td>
</tr>
<tr>
<td>cash</td>
<td>V_i^*</td>
<td>0</td>
<td>8</td>
<td>6.439</td>
<td>4.066</td>
<td>2.218</td>
</tr>
<tr>
<td>flow</td>
<td>D_i^*</td>
<td>0</td>
<td>3.561</td>
<td>2.373</td>
<td>1.848</td>
<td>2.218</td>
</tr>
<tr>
<td>for</td>
<td>T_i^*</td>
<td>0</td>
<td>0.140</td>
<td>0.515</td>
<td>0.825</td>
<td>0.177</td>
</tr>
<tr>
<td>y = w</td>
<td>D_i^*</td>
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<td>4.061</td>
<td>2.373</td>
<td>1.348</td>
<td>2.218</td>
</tr>
</tbody>
</table>

Notes:

a) Note that in the levelized cash flows above: C_i^* = C_i and I_i^* = I_i

b) Small differences in cash flow balances are due to roundoff.

c) Tax cash flows have present worths as follows:

<table>
<thead>
<tr>
<th></th>
<th>y = r</th>
<th>y = x</th>
<th>y = w</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1.146</td>
<td>1.198</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.146</td>
<td>1.185</td>
</tr>
</tbody>
</table>
Note that when $\alpha = \frac{b}{r}$, $y = r$, and the original cash flow is obtained. Similarly, when $\alpha = 0$, $y = x$, and the following virtual cash flow is obtained:

$$R_i = C_i + x V_i + D_i + T_{ix}$$  \hspace{1cm} (43)$$
$$T_{ix} = \tau \left( R_i - C_i - D_i \right)$$  \hspace{1cm} (44)$$

In the same way, when $\alpha = 1$, $y = w$, another virtual cash flow is obtained:

$$R_i = C_i + w V_i + D_i + T_{iw}$$  \hspace{1cm} (45)$$
$$T_{iw} = \tau \left( R_i - C_i - w V_i - D_i \right)$$  \hspace{1cm} (46)$$

Table II repeats the original cash flow of Table I and gives its virtual cash flows for $x$ ($\alpha = 0$) and for $w$ ($\alpha = 1$). Note the consistency of the virtual cash flows. The similarity between the original and the virtual cash flows has caused great confusion in the choice of the proper discount rate for cost levelization, with many authors [1, 2, 3, 4] incorrectly favoring the use of the so-called "effective discount rate $x$", instead of the total rate of return $r$, to calculate the levelized revenue of the original cash flow given by Eqs. (1) and (2).

Note that, consistent with this definition of the virtual cash flow, for each value of $y$, there can be associated a corresponding (and unique) levelized-virtual-cash flow by exchanging $y$ for $r$, $\alpha y$ for $b$ and $T_{iy}$ for $T_i^*$ in Eqs. (21), (22) and their derivations. Making these modifications to Eq. (36) and using Eqs. (11) and (30), it follows that:

$$\sum_{i=1}^{n} \frac{T_{iy}^*}{(1 + y)^i} = \sum_{i=1}^{n} \frac{T_{iy}}{(1 + y)^i}$$  \hspace{1cm} (47)$$

Table II also gives some examples of levelized virtual cash flows. Note that Eq. (47) is always satisfied for all values of $y$. Observe again the internal coherence of the virtual cash-flow model.
TABLE II. Numerical Example of the Original Cash Flow of Table I and Some of its Virtual Cash Flows and Levelized Virtual Cash Flows.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Original cash flow with</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_i$</td>
<td>0</td>
<td>6.6</td>
<td>7.3</td>
<td>7.2</td>
</tr>
<tr>
<td>$C_i$</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3.5</td>
</tr>
<tr>
<td>$V_i$</td>
<td>0</td>
<td>8</td>
<td>6.5</td>
<td>3.5</td>
</tr>
<tr>
<td>$D_i$</td>
<td>0</td>
<td>3.5</td>
<td>3</td>
<td>2.5</td>
</tr>
<tr>
<td>$D'_i$</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$I_i$</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$rV_i$</td>
<td>0</td>
<td>0.96</td>
<td>0.78</td>
<td>0.42</td>
</tr>
<tr>
<td>$T_i$</td>
<td>0</td>
<td>0.14</td>
<td>0.52</td>
<td>0.78</td>
</tr>
<tr>
<td>$r = 0.12$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = 0.04$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tau = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Virtual cash flow for $y = x$

$x = 0.10$

(R, C, V, D, D' and I are the same as in the original cash flow above).

Virtual cash flow for $y = w$

$w = 0.20$

(R, C, V, D, D' and I are the same as in the original cash flow above).

Note that $\sum_{i=1}^{n} T_{iw} = 0$

\[ rV_i + T_i = x V_i + T_{ix} = wV_i + T_{iw} = y V_i + T_{iy} \]
TABLE II. (CONTINUED) Numerical Example of the Original Cash Flow of Table I and Some of its Virtual Cash Flows and Levelized Virtual Cash Flows.

<table>
<thead>
<tr>
<th>Levelized</th>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow</td>
<td>$V_i^*$</td>
<td>8</td>
<td>6.488</td>
<td>3.826</td>
<td>1.647</td>
</tr>
<tr>
<td>for</td>
<td>$D_i^*$</td>
<td>3.511</td>
<td>2.663</td>
<td>2.179</td>
<td>1.647</td>
</tr>
<tr>
<td>$y = x$</td>
<td>$D_i^{**}$</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$x = 0.10$</td>
<td>$xV_i^*$</td>
<td>0.800</td>
<td>0.649</td>
<td>0.383</td>
<td>0.165</td>
</tr>
<tr>
<td></td>
<td>$T_{ix}$</td>
<td>0.312</td>
<td>0.312</td>
<td>0.562</td>
<td>0.812</td>
</tr>
</tbody>
</table>

| virtual   | $V_i^*$ | 8 | 6.439 | 4.066 | 2.218 |
| cash flow | $D_i^*$ | 3.561 | 2.373 | 1.848 | 2.218 |
| for       | $D_i^{**}$ | 4.061 | 2.373 | 1.348 | 2.218 |
| $y = w$   | $wV_i^*$ | 1.600 | 1.288 | 0.813 | 0.444 |
| $w = 0.20$| $T_{iw}$ | -0.500 | 0.000 | 0.500 | 0.000 |

Note: a) In the levelized cash flows above, $C_i^* = C_i$ and $I_i^* = I_i$, as in the original cash flow.

b) Small differences in cash flow balances of this table are due to roundoff.

c) Tax cash flows have present worths as follows:

<table>
<thead>
<tr>
<th>$y = r$</th>
<th>$y = x$</th>
<th>$y = w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{i=1}^{n} \frac{T_{iy}}{(1 + y)^i}$</td>
<td>1.146</td>
<td>1.517</td>
</tr>
<tr>
<td>$\sum_{i=1}^{n} \frac{T_{iy}^*}{(1 + y)^i}$</td>
<td>1.146</td>
<td>1.517</td>
</tr>
</tbody>
</table>
It should then be repeated here that $R_r$, given by Eq. (16) is the levelized revenue associated with the original cash flow given by Eqs. (1) and (2). $R_x$, given by Eq. (18) is the levelized revenue associated with the virtual cash flow given by Eqs. (43) and (44). $R_w$, given by Eq. (20) is the levelized revenue which corresponds to the virtual cash flow when $y = w (\alpha = 1)$, given by Eqs. (45) and (46). Finally, for each value of $y$, $R_y$, given by Eq. (14) is the levelized revenue corresponding to the virtual cash flow defined by Eqs. (40) and (41).

**Meaning of $x$ and $R_x$**

From Eqs. (43) and (44), $R_x$ may be interpreted as being the levelized revenue of its corresponding virtual cash flow, for 0% (zero percent) participation of bondholders in the investments. In other words, $x$ is the rate of return to the stockholders if the investment is 100% composed of stock.

Income taxes given by Eq. (48), which was derived from Eqs. (40) and (41), are maximum in this case, where $\alpha = 0$.

$$T_{iy} = \frac{\tau}{1 - \tau} \left[(1 - \alpha) y V_i + D_i - D'_i\right]$$

(48)

**Meaning of $w$ and $R_w$**

From Eqs. (45) and (46), $R_w$ may be interpreted as being the levelized revenue, for its corresponding virtual cash flow, for 100% participation of bondholders in the investments. In other words, $w$ is the interest rate which could be paid to the bondholders if the investment was 100% composed of bonds. Income taxes given by Eq. (48) are minimum in this case, where $\alpha = 1$. In fact, the net income taxes paid throughout the life of the project would be equal to zero (see numerical example in Table II). Then, one would be tempted to interpret $w$ as being
equivalent to the internal rate of return of the project before taxes, although this interpretation is correct only when \( D_i = D'_i \) for all \( i = 1 \) to \( n \).

**Range of significance of \( \alpha \) and \( y \)**

Since \( \alpha \) may be interpreted as being the ratio between the weighted bond to total rate of return of the virtual cash flow, in order to have physical significance, the following relation must be satisfied:

\[
0 \leq \alpha \leq 1
\]  

Consequently, from Eqs. (9), (17), (19) and (49), (see Figure 1):

\[
x \leq y \leq w
\]  

**Relationship between \( r \), \( y \), \( x \) and \( w \)**

As the original and the virtual cash flows were defined, \( r \), \( x \), and \( w \) are fixed parameters while \( y \) is variable. From Eqs. (9) and (17), \( y \) may be expressed as follows:

\[
y = \frac{x}{1 - \alpha \tau}
\]  

\( y \) then increases from \( y = x \), when \( \alpha = 0 \), to \( y = w \), when \( \alpha = 1 \), passing through \( y = r \), when \( \alpha = b/r \) (see Figure 1).

When \( y = r \) (\( \alpha = b/r \)), Eq. (51) transforms into Eq. (52)

\[
r = x + b \tau
\]  

Some authors [3, 4] have used Eq. (52) as an argument in favor of the use of \( x \) instead of \( r \) to calculate the levelized cost of the original cash flow. They have argued that the \( b \tau \) portion of the rate \( r \) can be deducted from income taxes with the difference \( x \) being the "effective after-tax
cost of money”. In fact, it has been shown here that \( r \) and \( x \) are discount rates corresponding to different cash flows. From Eqs. (9) and (19), the following relationship may be derived:

\[
w = y + \frac{\tau}{1-\tau} \left(1 - \alpha\right) y
\]

which may be interpreted in the following way: for each value of \( y \), the maximum rate of return \( w \) (which is fixed), of the set of appropriate virtual cash flows, is divided between the investors through the total rate of return \( y \) (bondholders receive the rate \( \alpha y \) and stockholders receive the rate \( (1 - \alpha) y \)), and the government, through the rate of return for “fictitious income”, \( \frac{\tau}{1-\tau} \left(1 - \alpha\right) y \) (see Equation 48). Equation (53) clearly shows that \( y \) increases as \( \alpha \) increases (and/or \( \tau \) decreases!).

Some practical applications of \( x, r \) and \( w \) in economic analyses

(a) To find the levelized revenue when one is given the cash flows \( \{I_i, C_i, D_i, D_i', i = 1 \) to \( n \), with \( r, b \) and \( \tau \) also known\}, Eq. (16) should be used. The original cash flow of the numerical example in Table I was constructed with the above vectors and parameters given, and using Eq. (3) to find \( V_i \), Eq. (7) to find \( R_i \) and Eq. (2) to find \( T_i \). The levelized cash flow corresponding to this original cash flow is also given in Table I for \( y = r \) with \( R_r \) being the levelized revenue.

(b) To maximize the rate of return \( r_s \) to the stockholders, given the cash flow: \( \{R_i, I_i, C_i, D_i, D_i', i = 1 \) to \( n \), with \( \tau \) and \( r_b \) also known\}, use Eq. (12) with \( \alpha = 0 \) to find \( x \) and then Eq. (19) to find \( w \). Compare the bond rate of return \( r_b \) with \( w \) through Eq. (54) (which was derived from Eqs. (5), (6) and (19)).

\[
r_s = \frac{1-\tau}{1-f_b} \left( w - f_b r_b \right)
\]
If \( r_b < w \), maximize the bond fraction \( f_b \) in the investments in order to maximize the rate of return \( r_s \) to the stockholders. If \( r_b = w \), \( r_s \) does not depend on \( f_b \). If \( r_b > w \) and if the project must be implemented anyway, minimize \( f_b \) to maximize \( r_s \). Figure 2 shows these relationships.

![Figure 2. Dependence of \( r_s \) on \( f_b \), \( r_b \) and \( w \).](image)

To find the vector \( D_i \), use Eqs. (3) and (7) after finding \( x \).

**Numerical Example**

Given the cash flow of a project (Table III) and knowing that the income tax rate is 50%, find the minimum bond fraction \( f_b \) in the investments such that the minimum stockholders rate of return \( r_s \) be 0.22. Assume that the market bond rate is 0.11.

Using Eq. (12) with \( \alpha = 0 \), one finds \( x = 0.183 \). Consequently, using Eq. (19), \( w = \frac{x}{1-\tau} = 0.367 \). In order to have \( r_s = 0.22 \), through Eq. (54) one finds the required \( f_b = 0.222 = 22\% \). Note that \( r_s \) may be increased by increasing the bond fraction \( f_b \) to its maximum allowed value. The total rate of return of the project is then \( r = r_b f_b + r_s f_s = 0.196 \). The
TABLE III. Cash Flow Used as Input for Numerical Example of Item (b).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0</td>
<td>7</td>
<td>7</td>
<td>7.5</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>3.5</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

levelized revenue is found using Eq. (16), or Eq. (13) with \( y = r \), giving \( R_f = 7.300 \). The vector \( D_i \) is found using Eqs. (3) and (7):

\[
D_i = [2.850/2.439/2.387/2.324]
\]

(c) To maximize the rate of return, \( r_s \), to the stockholders, given the cash flow \( \{R_i, C_i, I_i, D_i\} \), but without knowing numerical values of \( D_i \) and \( D_i' \), \( i = 1 \) to \( n \), find \( w \) using Eq. (12) with \( \alpha = 1 \). In this case, Eq. (12) becomes:

\[
\sum_{i=1}^{n} \frac{R_i}{(1 + w)^i} = \sum_{i=1}^{n} \frac{C_i}{(1 + w)^i} + \sum_{i=0}^{n} \frac{I_i}{(1 + w)^i}
\]

Given \( \tau \) and \( r_b \), proceed as in part (b) to choose \( f_b \) and \( r_s \).

Numerical Example

Given the estimated cash flow of a project (Table IV) and \( D_i = D_i' \), determines the feasibility of the project assuming \( r_b = 0.12 \) and the minimum \( r_s = 0.18 \). Assume an income tax rate equal to 50%.
TABLE IV. Cash Flow Used as Input for Numerical Example of Item (c).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ri</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Ci</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2.5</td>
<td>2.5</td>
<td>3</td>
</tr>
<tr>
<td>Ii</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Using Eq. (55) one finds \( w = 0.101 \). As \( r_b = 0.12 > w = 0.101 \), capital borrowed should be minimized. As \( x = w (1 - \tau) = 0.051 < r_s = 0.18 \), this project is not feasible and should be abandoned (see Figure 2). Note that \( D_i = D_i' \) may be calculated using Eqs. (3) and (7).

\[
D_i = D_i' = \{1.985/2.187/1.909/2.102/1.816\}
\]

(d) Although the range of physical significance of \( \alpha \) is \( 0 \leq \alpha \leq 1 \), other parts of Figure 1 may be studied in detail to derive useful relations, such as the one given by Eq. (56) which was derived from Eq. (12) as \( \alpha \to -\infty (y \to 0) \) (see Appendix III). Note that while Eq. (12) with \( \alpha = 0 (y = x) \), requires the knowledge of \( D_i' \) in order to find \( x \), Eq. (56) requires \( D_i \) to be given to find \( w \) (Note also that \( w \) given by Eq. (56) does not assume \( D_i = D_i' \), as Eq. (55) does!)

\[
w = \frac{\sum_{i=1}^{n} R_i - \sum_{i=1}^{n} C_i - \sum_{i=0}^{n} I_i}{\sum_{i=1}^{n} i \{D_i - I_i\}}
\]

A practical use of Eq. (56) is the following. Given a cash flow \( \{R_i, I_i, C_i, D_i, i = 1 \text{ to } n\} \) find \( w \) using Eq. (56). Now, given \( \tau \) and \( r_b \), choose \( f_b \) (and then \( r_s \)) in the same way as in (b) above.
Numerical example

Given the following cash flow of a project (Table V), find the bond fraction $f_b$ in the investments such that the stockholders rate of return $r_s$ is 0.22. Assume that the market bond rate $r_b$ is 0.12 and that the income tax rate is 50%; and also if the income tax rate is 30%.

TABLE V. Cash Flow Used as Input for Numerical Example of Item (d).

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_i$</td>
<td>0</td>
<td>6</td>
<td>5.8</td>
<td>5.4</td>
<td>5.2</td>
</tr>
<tr>
<td>$C_i$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$D_i$</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$I_i$</td>
<td>8</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Using Eq. (56), one finds $w = 0.133$. Through Eq. (54), $f_b = 0.958$. With $\tau = 0.3$ , using Eq. (54) again, $f_b = 0.931$.

The vector $D_i$ may be calculated using Eqs. (3) and (7). $D_i = (4.067/3.000/2.000/0.933)$.

(One can check the consistency of $D_i$ by finding $x$ from Eq. (12) with $\alpha = 0$).

CONCLUSION

There is a general formula from which a wide variety of levelized cost conventions can be derived. (In fact, there are an infinite number of types of levelized costs.) But there is only one levelized revenue, which, if charged uniformly throughout the life of the plant, will just pay for all current expenses, return on and return of investment and keep coherence on taxes. This
preferred discount rate and cash flow weighting are embodied in Eq. (16) of the present exposition.

**APPENDIX I**

The objective is to prove relation (11) given Eqs. (3) and (4). Equation (3) can be used to generate the following development:

$$\sum_{i=1}^{n} \frac{y V_{i} + D_{i}}{(1+y)^{i}} = \sum_{i=1}^{n} \frac{y}{(1+y)^{i}} \left[ \sum_{j=0}^{i-1} I_{j} - \sum_{j=1}^{i-1} D_{j} \right] + \sum_{i=1}^{n} \frac{D_{i}}{(1+y)^{i}}$$

$$= \sum_{j=0}^{n-1} \frac{y}{j + 1} \frac{1}{(1+y)^{j}} \sum_{i=j+1}^{n} I_{j} - \sum_{j=1}^{n-1} \frac{y}{j + 1} \frac{1}{(1+y)^{j}} \sum_{i=1}^{n} D_{i} + \sum_{i=1}^{n} \frac{D_{i}}{(1+y)^{i}}$$

$$= \sum_{j=0}^{n-1} \frac{y}{(1+y)^{j}} \sum_{j=0}^{n} I_{j} - \sum_{j=0}^{n-1} \frac{1}{(1+y)^{j}} \sum_{j=0}^{n} D_{j} + \sum_{j=0}^{n} \frac{n}{(1+y)^{j}} \sum_{j=1}^{n} D_{i} + \sum_{i=1}^{n} \frac{D_{i}}{(1+y)^{i}}$$

$$= \sum_{i=0}^{n} \frac{I_{i}}{(1+y)^{i}} - \sum_{i=0}^{n} \frac{D_{i}}{(1+y)^{i}} + \frac{1}{n} \left[ \sum_{i=1}^{n} D_{i} - \sum_{i=1}^{n} I_{i} \right] + \sum_{i=1}^{n} \frac{D_{i}}{(1+y)^{i}}$$

and hence, by Eq. (4)
\[
= \sum_{i=0}^{n} \frac{I_i}{(1 + y)^i}
\]

The resulting relation:

\[
\sum_{i=1}^{n} \frac{y V_i + D_i}{(1 + y)^i} = \sum_{i=0}^{n} \frac{I_i}{(1 + y)^i}
\]

holds for \( y > -1 \).

**APPENDIX II**

Starting with Eq. (25)

\[
V_i^* = \sum_{j=0}^{i-1} I_j^* - \sum_{j=1}^{i-1} D_j^*
\]

define:

\[
\Theta = \frac{1}{1 + y} \text{ and } y = \frac{1 - \Theta}{\Theta}
\]

\[
\sum_{i=1}^{n} V_i^* \Theta^i = \sum_{i=1}^{n} \sum_{j=0}^{i-1} I_j \Theta^i - \sum_{i=1}^{n} \sum_{j=1}^{i-1} D_j^* \Theta^i \Rightarrow
\]

\[
\sum_{i=1}^{n} V_i^* \Theta^i = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} I_j \Theta^i - \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} D_j^* \Theta^i \Rightarrow
\]
\[
\sum_{i=1}^{n} v_i^* \theta^i = I_0 \frac{\theta^{n+1} \theta}{\theta - 1} + \sum_{j=1}^{n-1} \left( I_j - D_j^* \right) \frac{\theta^{n+1} \theta}{\theta - 1}
\]

But, from Eq. (30):

\[
\sum_{i=1}^{n} \left[ y v_i^* + D_i^* \right] \theta^i = \sum_{i=0}^{n} I_i \theta^i
\]

Then:

\[
\frac{1 - \theta}{\theta} I_0 \frac{\theta^{n} - 1}{\theta - 1} + \frac{1 - \theta}{\theta} \sum_{i=1}^{n-1} \left( I_i - D_i^* \right) \frac{\theta^{n-i} \theta^i}{\theta - 1} + \sum_{i=0}^{n} D_i^* \theta^i = \sum_{i=0}^{n} I_i \theta^i \Rightarrow
\]

\[
- \sum_{i=0}^{n-1} I_i \left( \theta^n - \theta^i \right) + \sum_{i=1}^{n-1} D_i^* \left( \theta^n - \theta^i \right) + \sum_{i=0}^{n} D_i^* \theta^i = \sum_{i=0}^{n} I_i \theta^i \Rightarrow
\]

\[
- \sum_{i=0}^{n} I_i \theta^n + \sum_{i=1}^{n} D_i^* \theta^n = 0
\]

hence

\[
\sum_{i=1}^{n} D_i^* = \sum_{i=0}^{n} I_i
\]

**APPENDIX III**

From Eq. (9), as \( \alpha \rightarrow -\infty \), \( y \rightarrow 0 \) (see Figure 1). Taking the limit of Eq. (12) as \( \alpha \rightarrow -\infty \) (\( y \rightarrow 0 \)), one has:
\[
\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \frac{1}{1-\tau} \sum_{i=0}^{n} I_i - \frac{\tau}{1-\tau} \sum_{i=1}^{n} D_i.
\]

\[
\lim_{\alpha \to -\infty} \frac{\alpha\tau}{(y \to 0)} \left[ \sum_{i=0}^{n} \frac{I_i}{(1+y)^i} - \sum_{i=1}^{n} \frac{D_i}{(1+y)^i} \right]
\]

Since from Eq. (4): \( \sum_{i=1}^{n} D_i = \sum_{i=1}^{n} D_i' = \sum_{i=0}^{n} I_i \)

and using Eq. (9) to obtain \( \alpha\tau = 1 - \frac{r-b\tau}{y} \),

\[
\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \sum_{i=0}^{n} I_i - \lim_{y \to 0} \left( \frac{1}{1-\tau} \left(1 - \frac{r-b\tau}{y}\right) \right).
\]

\[\Rightarrow \ (I_0 + (I_1 - D_1)(1-y) + (I_2 - D_2)(1-2y) +
+ (I_3 - D_3)(1-3y) + ... + (I_n - D_n)(1-ny)) \]

\[
\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \sum_{i=0}^{n} I_i - \lim_{y \to 0} \left( \frac{1}{1-\tau} \left(1 - \frac{r-b\tau}{y}\right) \right).
\]

\[
\left[ \sum_{i=0}^{n} I_i - \sum_{i=1}^{n} D_i - \sum_{i=1}^{n} i y (I_i - D_i) \right] \Rightarrow
\]
\[
\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i + \sum_{i=0}^{n} I_i - \frac{r - b\tau}{1 - \tau} \sum_{i=1}^{n} i(I_i - D_i)
\]

Then:

\[
w = \frac{r - b\tau}{1 - \tau} = \frac{\sum_{i=1}^{n} R_i - \sum_{i=1}^{n} C_i - \sum_{i=0}^{n} I_i}{\sum_{i=1}^{n} i(D_i - I_i)}
\]

REFERENCES


