Obstructions to slicing knots and splitting links

by

Joshua Batson

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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Abstract

In this thesis, we use invariants inspired by quantum field theory to study the smooth topology of links in space and surfaces in space-time.

In the first half, we use Khovanov homology to the study the relationship between links in $\mathbb{R}^3$ and their components. We construct a new spectral sequence beginning at the Khovanov homology of a link and converging to the Khovanov homology of the split union of its components. The page at which the sequence collapses gives a lower bound on the splitting number of the link, the minimum number of times its components must be passed through one another in order to completely separate them. In addition, we build on work of Kronheimer–Mrowka and Hedden–Ni to show that Khovanov homology detects the unlink.

In the second half, we consider knots as potential cross-sections of surfaces in $\mathbb{R}^4$. We use Heegaard Floer homology to show that certain knots never occur as cross-sections of surfaces with small first Betti number. (It was previously thought possible that every knot was a cross-section of a connect sum of three Klein bottles.) In particular, we show that any smooth surface in $\mathbb{R}^4$ with cross-section the $(2k, 2k-1)$ torus knot has first Betti number at least $2k-2$.

Thesis Supervisor: Peter Ozsváth
Title: Professor

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Chapter 1

Introduction

In the 1860s, fluid dynamics was cutting edge physics. Helmholtz had just shown that in addition to propagating waves, fluids propagate vortices [16]. While friction eventually dissipates waves in water and smoke rings in air, a frictionless fluid like ether—thought to carry electromagnetic waves—would maintain a vortex forever. In 1867, Lord Kelvin made a bold proposal: that atoms themselves are knotted vortices in the ether [46]. Chemical elements would correspond to knot types: hydrogen to the unknot ○, carbon to the trefoil, sodium to the Hopf link. For about 20 years, Kelvin's theory of vortex atoms was taken quite seriously, and mathematicians—notably Tait—began to attack the problem of enumerating, classifying, and analyzing knots [43]. As his (aperiodic) table of knots grew, Tait encountered a persistent problem: how to read the identity and interesting three-dimensional topology of a knot from a diagram drawn on a page.

More than a century after its birth as a theory of matter, knot theory was revolutionized by the new theory of quantum fields. In the late 1970s, the strong force was explained by a Yang-Mills theory of $SU(3)$-connections on flat space-time ($\mathbb{R}^4$). In 1983, Donaldson studied the behavior of this gauge theory on arbitrary simply-connected smooth four-manifolds. Specifically, he used the moduli space of anti-self-dual $SU(2)$-connections on a manifold X^4 to constrain its intersection form [7]. Signed counts of certain parts of the moduli space also give numerical invariants, called Donaldson polynomials, which can be used to distinguish exotic smooth structures on closed four-manifolds [8]. Kronheimer and Mrowka subsequently used Donaldson polynomials to prove the Milnor conjecture on the unknotting numbers of torus knots [21]. A movie of an unknotting sweeps out a singular orientable surface in space-time, and the behavior of the anti-self-duality equations near the smoothing of that surface can be used to bound its genus.

If a compact four-manifold X has boundary Y, its Donaldson invariants are no longer integers. They properly take values in a vector space $I(Y)$, called the instanton Floer homology of Y [10]. If X has two boundary components, i.e., X is a cobordism from Y_1 to Y_2, then it induces a linear map $F_X : I(Y_1) \to I(Y_2)$ [3]. This was the first instance of a "topological quantum field theory," or TQFT: one can think of $I(Y)$ as the Hilbert space of ground states for a quantum system on Y and the map F_X as time evolution [47].

In 1994, Seiberg and Witten introduced a different set of partial differential equations designed to see similar topology to the anti-self-duality equations and to be easier to manage analytically [39]. Within months, two teams independently used the SW equations to prove the Thom conjecture: algebraic curves in $\mathbb{CP}^2$ minimize genus in their homology class [22, 25]. A corresponding Floer homology for 3-manifolds—monopole Floer homology—was soon constructed [23]. In 2001, Ozsváth and Szabó defined a more computable analogue called Heegaard Floer homology: a 3-manifold Y is assigned a suite of abelian groups $HF^\circ(Y)$, and a cobordism $W : Y_1 \to Y_2$ is assigned a linear map $F_W^\circ : HF^\circ(Y_1) \to HF^\circ(Y_2)$ [33]. In Chapter 3, we will use Heegaard Floer homology—especially its rational grading—to study which knots can appear as cross-sections of which nonorientable surfaces in $\mathbb{R}^4$.

At the same time that Donaldson applied gauge theory to the study of four-manifolds, Jones applied two-dimensional statistical physics to the study of knots. He associated to a braid an element of a von Neumann algebra; the Jones polynomial of a knot given as the closure of a braid is the trace of that element [17]. The Jones polynomial of a knot $K \subset S^3$ is a Laurent polynomial $J(K) \in \mathbb{Z}[q, q^{-1}]$. Urged by Atiyah, Witten realized the Jones polynomial as a three-dimensional quantum phenomenon: a Wilson line observable in Chern-Simons field theory [48]. What sort of topology that theory can see remains unclear. For example, it is still unknown whether the Jones polynomial can detect the unknot.

1.1 Skein relations, surgery triangles, and spectral sequences

The Jones polynomial can also be defined in terms of an unoriented skein relation. If three links admit planar diagrams differing from each other at only one crossing, their Jones polynomials $J(\times), J(\asymp)$, and $J(\rangle\langle)$ are linearly related. Since $J(\bigcirc) = q + q^{-1}$, the Jones polynomial of any knot can be computed recursively by resolving crossings.

Khovanov categorified this construction. Given a diagram D of a link $L \subset S^3$, one can write down an explicit bigraded chain complex $(C^{i,j}(D), d_0)$ with homology $Kh^{i,j}(K)$. Cobordisms of links induce maps of complexes. The Jones polynomial can be recovered by counting the generators with weight $(-1)^i q^j$, and the skein relation is promoted to an exact triangle:

$$C(\times) = \text{Cone}\left(F_{\text{saddle}} : C(\asymp) \to C(\rangle\langle)\right),$$

where the map F_{saddle} is induced by a saddle cobordism between the two resolved diagrams. In fact, Khovanov homology can be defined by giving the theory for unlinks, then iteratively resolving all of the crossings in a link diagram D to build a cube of resolutions.

This skein relation is reminiscent of the surgery exact triangle in Floer theory. Let Y be a three-manifold containing a knot K, and write $Y_p(K)$ for the manifold given by p-surgery along K. Then

$$CF^\circ(Y) = \text{Cone}\left(F_W : CF^\circ(Y_0(K)) \to CF^\circ(Y_1(K))\right),$$

where CF° is a chain complex computing HF° and W is a cobordism given by attaching a single 2-handle to Y_0. If Y is a branched double cover of a link $L \subset S^3$, then the branched covers of the two resolutions of L are related to Y by 0- and 1-surgery. Iterating this process gives a cube whose vertices and edges, but one with longer diagonals as well as edges (essentially because one must take an A_∞-cone). The Khovanov homology of an unlink is isomorphic to the Heegaard-Floer homology of the branched double cover of an unlink, so the vertices of the two cubes for a fully resolved link are identical. This gives a spectral sequence from the Khovanov homology of L to the Heegaard-Floer homology of the branched double cover of L. In Chapter 2 will use a refinement of this spectral sequence due to Hedden and Ni to lift geometric information from Heegaard-Floer homology up to Khovanov homology.

Kronheimer and Mrowka constructed a similar spectral sequence, from the Khovanov homology of a knot K to its singular instanton knot Floer homology, a Floer theory built from $SU(2)$-connections with prescribed singularities along K [24]. The latter group is nontrivial for nontrivial knots, so:

Theorem 1.1 (Kronheimer-Mrowka). *Let K be a knot, and U the unknot. If*

$$\operatorname{rank} Kh(K) = \operatorname{rank} Kh(U),$$

then K is the unknot.

1.2 Splitting links

In Chapter 2, we construct a deformation of Khovanov homology for links. We use it to show that Khovanov homology detects the unlink:

Theorem 1.2. *Let L be an m-component link, and U^m the m-component unlink. If*

$$\operatorname{rank} Kh^{i,j}(L; \mathbb{F}_2) = \operatorname{rank} Kh^{i,j}(U^m; \mathbb{F}_2)$$

for all i, j, then L is the unlink.

In contrast, there are infinite families of links with the same Jones polynomial as the unlink [9, 45].

We also address the question, "How far is a link from being split?" Since any link can be converted to the split union of its component knots by a sequence of crossing changes between different components, it is sensible to define the *splitting number* of a link as the minimum number of crossing changes needed in such a sequence. (It is analogous to the unknotting number of a knot.) For example, the two-component link in Figure 1.1 can be split into an unknot and a trefoil by changing three crossings. We show that this is best possible.

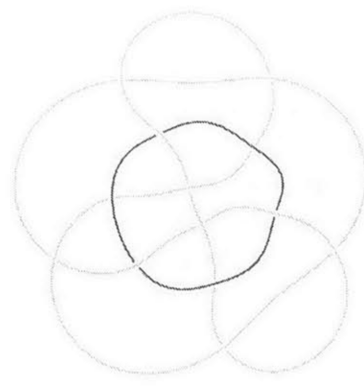


Figure 1.1: The link ${}^2L13n3752$ has splitting number 3.

Our deformation of Khovanov homology (illustrated for 2-component links in Figure 1.2) gives rise to a spectral sequence.

Theorem 1.3. *Let L be a 2-component link, with components K_1 and K_2. Let $\mathbb{F}$ be a field. Then there is a spectral sequence with pages $E_*(L)$, satisfying*

$$E_1(L) \cong Kh(L;\mathbb{F}) \text{ and } E_\infty(L) \cong Kh(K_1;\mathbb{F}) \otimes Kh(K_2;\mathbb{F}).$$

If the spectral sequence has yet to collapse by page k, then L has splitting number at least k.

This gave the first nontrivial lower bounds on the splitting number. Cha, Friedl, and Powell have recently given an entirely different construction of lower bounds using Alexander invariants and covering links [4].

For links with more than two components, one must choose for each component a weight $w_i \in \mathbb{F}$. The corresponding spectral sequence will be described in detail at the beginning of Chapter 2, as will the strategy of proof.

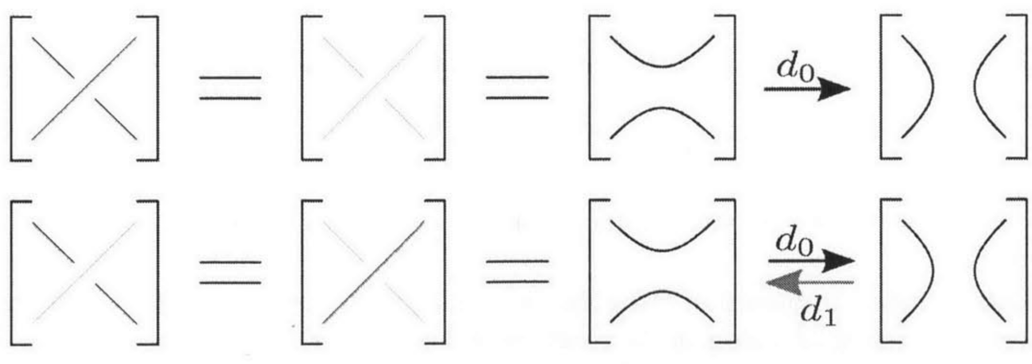


Figure 1.2: The Khovanov complex of a diagram D is given by replacing each crossing with a 2-step complex. If the link has two components, then our deformation of the differential d_0 to $d = d_0 + d_1$ is given by adding in backward maps at crossings mixing the components.

1.3 Slicing surfaces

Let F be a surface smoothly embedded in $\mathbb{R}^4$. Viewing the first coordinate of $\mathbb{R}^4$ as time makes F into a movie[1] in which almost every frame is a link in $\mathbb{R}^3$. Figure 1.3 shows an embedded Klein bottle with the torus knot $T_{2,5}$ and the unknot as cross-sections.

Every knot $K \subset \mathbb{R}^3$ can be realized as a cross-section of some surface in four-space. For example, one may take two Seifert surfaces for K placed in $\{\pm 1\} \times \mathbb{R}^3$ together with the cylinder $[-1, 1] \times K$; smooth the corners to get an embedded $F \subset \mathbb{R}^4$ with $K = F \cap \{0\} \times \mathbb{R}^3$. (All maps and manifolds we discuss will be smooth.) However, most knots cannot be realized as a cross-section of an embedded sphere—Milnor and Fox [11] showed that the Alexander polynomial of such a "slice" knot must factor as $\Delta_K(t) = \pm p(t)p(t^{-1})$ for some polynomial p with integer coefficients.

Some knots never appear as cross-sections of *orientable* surfaces with low genus. This was first shown by Murasugi, who proved that if K is a cross-sectional slice of an orientable surface F, then $b_1(F)$ is at least twice the knot signature $\sigma(K)$ [28]. The signature of $T_{2,n}$ is $n-1$, for example, so any orientable surface with $T_{2,n}$ as a cross-section must have first Betti number at least $2n-2$.

Techniques from gauge theory, Floer homology, and Khovanov homology have given many additional obstructions to realizing certain knots as slices of low-genus orientable surfaces [21] [32] [37]. Since these bounds are tight for $T_{2,n}$, which is a slice of a Klein bottle, and $2n-2 > 2$ for $n > 2$, the bounds do not hold for nonorientable surfaces. The global property of orientability, perhaps recast as the existence of a top homology class, a complex structure, or an infinite cyclic branched covering space, is used in a crucial way. Some obstructions have been found to particular knots appearing as slices of a Klein bottle Kl or $Kl\#Kl$ in $\mathbb{R}^4$ ([49, 27], see [13] especially for a comprehensive survey). But the possibility remained that every knot could be realized as a cross-section of, say, $\#^3 Kl$. We show that this is not the case.

Theorem 1.4. *If the torus knot $T_{2k,2k-1}$ is a cross-section of a smoothly embedded surface $F \subset \mathbb{R}^4$, then $b_1(F) \geq 2k-2$.*

Suppose that a knot K is the intersection of a surface $F \subset \mathbb{R}^4$ with a hyperplane. That hyperplane cuts F into two pieces F_1 and F_2, each of which has boundary K. If we add the point at infinity to $\mathbb{R}^4$ to form the four-sphere, then the knot K lives in an 'equatorial' S^3 and each F_i lives in a 'hemisphere' B^4. Doubling either half (B^4, F_i) across the boundary would produce a new closed surface with cross-section K. Since $b_1(F) = b_1(F_1) + b_1(F_2) \geq 2\min(b_1(F_i))$, a surface of minimal b_1 with cross-section K can always be found by doubling. (Both surfaces in Figure 1.3 are doubles, as the movies are symmetric.) So it is enough to consider surfaces in B^4 with boundary the knot.

Definition 1.5. *The (non)orientable slice genus of a knot $K \subset S^3$ is the smallest first Betti number of any smoothly embedded (non)orientable surface $F \subset B^4$ with boundary K.*

[1] Ayumu Inoue has rendered an excellent movie of a sphere with cross-section the stevedore knot (`http://www.youtube.com/watch?v=61IM9p6XOKo`)

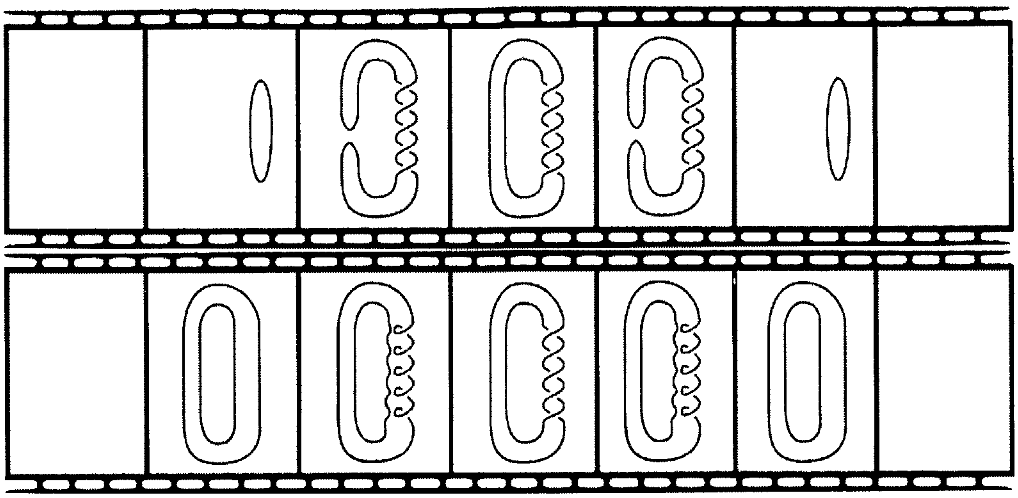


Figure 1.3: A Klein bottle and a genus four orientable surface in four-space, each with cross-section $T_{2,5}$. The topology of each surface can be deduced from its Euler characteristic (count births, deaths, saddles) and orientability (try to consistently orient the cross-sections).

To show that a knot is not the boundary of a surface with small first Betti number, we must bound both its orientable and nonorientable slice genus. We give a new bound for the latter.

Theorem 1.6. *Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then*

$$b_1(F) \geq \frac{\sigma(K)}{2} - d\left(S^3_{-1}(K)\right),$$

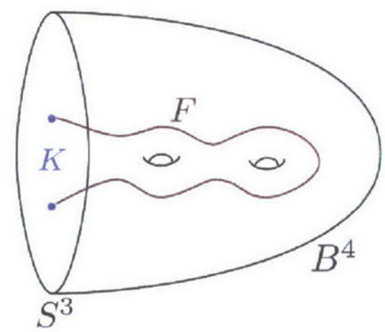
where σ denotes the Murasugi signature and d the Heegaard-Floer d-invariant of the integer homology sphere given by -1 surgery on K.

The strategy of the proof is as follows. (For a pictorial outline, see Figure 1.4). First, we replace our nonorientable surface in B^4 with an orientable surface in another manifold:

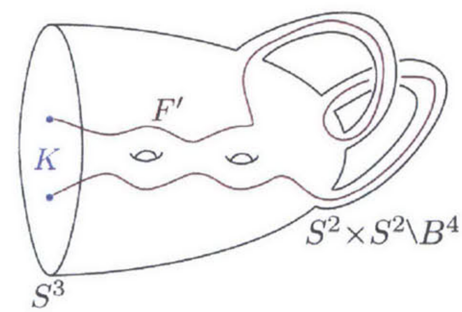
Proposition 1.7. *Let $F \subset B^4$ be a smoothly embedded nonorientable surface with odd b_1 and boundary a knot $K \subset S^3$. Then there exists a smoothly embedded orientable surface $F' \subset S^2 \times S^2 \backslash B^4$, also with boundary K, and with $b_1(F') = b_1(F) - 1$ and $e(F') = e(F) + 2$.*

The expression $e(F)$ denotes the (relative) normal Euler number of F, a measure of the twisting of the normal bundle of F. (While orientable surfaces in B^4 always have trivial normal bundle, nonorientable ones may not.) This construction is similar to one in [49].

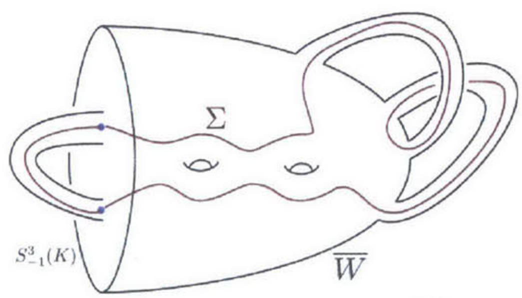
We then attach a -1-framed 2-handle along K to get a four-manifold $\overline{W}$ with boundary $S^3_{-1}(K)$. (We write $S^3_p(K)$ for the product of p-surgery along $K \subset S^3$.) There is a closed, orientable surface Σ in $\overline{W}$ formed by the union of F' and the core of the 2-handle. Excizing a neighborhood of Σ from $\overline{W}$ produces a negative semi-definite cobordism W from a circle



Start with a nonorientable surface $F \subset B^4$



Find an orientable replacement



Build a closed surface in $\overline{W}$

Figure 1.4: The topological steps in the proof of Theorem 1.8

bundle over Σ to $S^3_{-1}(K)$. The definiteness of W gives us an inequality between the Heegaard-Floer d-invariants of its two boundaries, ultimately yielding:

Theorem 1.8. *Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then*

$$b_1(F) \geq \frac{e(F)}{2} - 2d\left(S^3_{-1}(K)\right).$$

To prove Theorem 1.6, we cancel the Euler number using:

Theorem 1.9 (Gordon-Litherland, [14]). *Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then*

$$b_1(F) \geq \sigma(K) - \frac{e(F)}{2}.$$

The d-invariants in Theorem 1.6 are determined by the Alexander polynomial of the knot K if it admits a lens space surgery. For a general knot, they can be computed using an algorithm beginning with the filtered Heegaard Floer knot complex $CFK^\infty(K)$ [36]. Using a recursive formula of Murasugi for the signatures of torus knots—which do admit lens space surgeries—we can compute our lower bound for all torus knots.

There is a simple construction of nonorientable cobordisms between torus knots: pinch parallel strands, then pull taut. Beginning with $T_{p,q}$ and composing these cobordisms produces a surface $F_{p,q} \subset B^4$. (See Section 3.4.)

Conjecture 1.10. *Suppose $T_{p,q}$ bounds a smoothly embedded surface $F \subset B^4$. Then $b_1(F) \geq b_1(F_{p,q})$.*

We show that this conjecture holds for infinitely many torus knots, including $T_{2k,2k-1}$.

Chapter 2

A link splitting spectral sequence

In this chapter, we construct the link splitting spectral sequence. It begins at the Khovanov homology of a link and converges to the Khovanov homology of the split union of its components.

Theorem 2.1. *Let L be a link and R a ring. Choose a weight $w_c \in R$ for each component c of L. Then there is a spectral sequence with pages $E_k(L, w)$, and*

$$E_1(L, w) \cong Kh(L; R).$$

If the difference $w_c - w_d$ is invertible in R for each pair of components c and d with distinct weights, then the spectral sequence converges to

$$Kh\left(\coprod_{r \in R} L^{(r)}; R\right),$$

where $L^{(r)}$ denotes the sub-link of L consisting of those components with weight r.

Corollary 2.2. *Let $\mathbb{F}$ be any field, and let L be a link with components $K_1, \ldots, K_m$. Then*

$$\operatorname{rank} Kh(L; \mathbb{F}) \geq \operatorname{rank} \otimes_{c=1}^{m} Kh(K_c; \mathbb{F}).$$

Each choice of weights for a link L gives a lower bound on the splitting number.

Theorem 2.3. *Let L be a link and let $w_c \in R$ be a set of component weights such that $w_c - w_d$ is invertible for each pair of components c and d. Let $b(L, w)$ be largest k such that $E_k(L, w) \neq E_\infty(L, w)$. Then $b(L, w) \leq \operatorname{sp}(L)$.*

The spectral sequence only depends on the differences of the weights, $\{w_c - w_d\}$. For a two-component link, there is a just one difference $w = w_1 - w_2 \in R$. It turns out that different choices of nonzero w produce isomorphic spectral sequences.

Proposition 2.4. *Let L be a link with 2 components, and $w_1, w_2, w_1', w_2' \in R$ choices of weights such $w = w_1 - w_2$ and $w' = w_1' - w_2'$ are invertible. Then $E_*(L, w) \cong E_*(L, w')$.*

Theorem 1.3 follows by applying the above to a two-component link L, with coefficients in a field $\mathbb{F}$ and weights $w_1 = 0$ and $w_2 = 1$.

The proof of Theorem 1.2, that the Poincaré polynomial of Khovanov homology detects the unlink, depends on two earlier spectral sequences that relate Khovanov homology to more manifestly geometric invariants coming from Floer homology. As discussed in Section 1.1, Ozsváth and Szabó constructed a spectral sequence beginning at the Khovanov homology of a link and converging to the Heegaard Floer homology of its branched double cover [34]. The second, constructed by Kronheimer and Mrowka, begins at the Khovanov homology of a knot and converges to its instanton knot Floer homology [24]. The latter was used to prove that the only knot K with $\operatorname{rank} Kh(K) = 2$ is the unknot.

The Khovanov homology groups contain more information than their ranks alone—there is a natural action of the algebra

$$A_m = \mathbb{F}_2[X_1, \dots, X_m]/(X_1^2, X_2^2, \dots, X_m^2)$$

on the homology of an m-component link. Hedden and Ni [15] showed that the entire spectral sequence of Ozsváth and Szabó admits a compatible A_m action. They then used Floer homology to detect $S^1 \times S^2$ summands in the branched double cover of the link, and showed:

Theorem 2.5 (Hedden-Ni). *Let L be an m-component link, and U^m the m-component unlink. If there is an isomorphism of A_m modules*

$$Kh(L; \mathbb{F}_2) \cong Kh(U^m; \mathbb{F}_2),$$

then L is the unlink.

To prove Theorem 1.2, we apply our spectral sequence with component weights in a suitably large finite field $\mathbb{F}$ of characteristic 2. We lift the A_m-module structure from the abutment of our spectral sequence, which turns out to be isomorphic to $Kh(U^m; \mathbb{F})$, to the first page, $Kh(L; \mathbb{F})$, and then to $Kh(L; \mathbb{F}_2)$, where we apply Theorem 2.5.

In §2.1, we recall the construction of the Khovanov complex, define a filtered chain complex $C(D, w)$ which induces the spectral sequence E_k, and compute the E_1 and E_∞ pages. In §2.2, we give an alternative construction of the deformation as a higher homotopy coming from the chain-level ambiguity in the definition of the action by a marked point. In §2.3, we verify that our spectral sequence is independent of the choice of link diagram by checking invariance under the Reidemeister moves. In §2.4, we review how endomorphisms of a filtered complex act on the associated spectral sequence and discuss the effect of changing the filtration. In §2.5, we prove Theorem 2.3 on the splitting number. In §2.6, we give the proof of unlink detection. In §2.7, we discuss computations illustrating the strength of this spectral sequence and the splitting number bound.

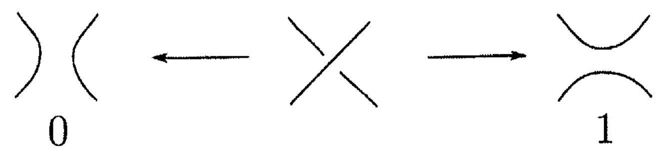


Figure 2.1: The 0 and 1 resolutions associated to a crossing.

2.1 Our construction

Khovanov's construction begins with a diagram D for a link L. He builds a cube of resolutions for D and applies a $(1 + 1)$-dimensional TQFT $\mathcal{A}$ to produce a cube-graded complex. A sprinkling of signs yields a chain complex $(C(D), d_0)$ with homology $Kh(L)$. We will give another differential d on the same chain complex, but first we must set some notation.

A review of Khovanov homology

(Following [18] and [1].)

A crossing in a link diagram can be resolved in two ways, called the 0-resolution and 1-resolution in Figure 2.1. A (complete) resolution of D is a choice of resolution at each crossing. Number the crossings of D from 1 to n so we can index complete resolutions by vertices in the hypercube $\{0, 1\}^n$. An edge in the cube connects a pair of resolutions (I, J), where J is obtained from I by changing the i^{th} digit from 0 to 1. A complete resolution I yields a finite collection of circles in the plane, which we may also call I. An edge (I, J) yields a cobordism from I to J, given by the natural saddle cobordism from the 0- to the 1-resolution in a neighborhood of the changing crossing and the product cobordism elsewhere.

A $(1 + 1)$-dimensional TQFT is determined by a commutative Frobenius algebra [20]. We fix a ring of coefficients R, and let $\mathcal{A}$ be the TQFT associated to the Frobenius algebra $V = H^*(S^2; R) = R[x]/(x^2)$. The diagonal map $i : S^2 \hookrightarrow S^2 \times S^2$ induces the multiplication $i^* : H^*(S^2 \times S^2) \to H^*(S^2)$. The comultiplication comes from Poincaré duality, $PD \circ i_* \circ PD : H^*(S^2) \to H^*(S^2 \times S^2)$. More explicitly, the multiplication $m : V \otimes V \to V$ is given by

$$m(1 \otimes 1) = 1 \qquad m(x \otimes 1) = x$$
$$m(1 \otimes x) = x \qquad m(x \otimes x) = 0,$$

and the comultiplication $\Delta : V \to V \otimes V$ is given by

$$\Delta(1) = 1 \qquad \Delta(x) = 1 \otimes x + x \otimes 1.$$

The TQFT $\mathcal{A}$ associates to a circle the R-module V and takes disjoint unions to tensor products. The pair of pants cobordism that merges two circles into one induces the multiplication map m, and the pair of pants cobordism that splits one circle into two induces the comultiplication map Δ.

Let $S = (x_1, \dots, x_p)$ be a collection of circles. To simplify notation, we note that

$$\begin{aligned}\mathcal{A}(S) &= \bigotimes_{i=1}^{p} V \\ &= R[x_1, \dots, x_t]/(x_1^2, \dots, x_p^2).\end{aligned}$$

We will write elements of $V(S)$ as (commutative) products of the circles x_i rather elements of the tensor product. Such a product of circles is called a monomial of S.

Applying the TQFT $\mathcal{A}$ to the cube of resolutions, we obtain a cube-graded complex of R-modules. For each resolution I, we have an R-module $\mathcal{A}(I)$, and for each edge (I, J), we have a homomorphism $\mathcal{A}(I, J) : \mathcal{A}(I) \to \mathcal{A}(J)$. Khovanov's complex is obtained by collapsing the cube-graded complex. We set

$$C(D) = \bigoplus_{\text{resolutions } I} V(I).$$

The differential $d_0 : C(D) \to C(D)$ is given by

$$d_0 = \sum_{\text{edges } (I,J)} (-1)^{n(I,J)} \mathcal{A}(I, J),$$

where, if (I, J) differ at i,

$$n(I, J) = \#\{I(k) = 1 \,|\, 1 \le k < i\}.$$

We define four related gradings on $C(D)$ as follows. Let $x \in V(I)$. The homological or h grading is given by

$$h(x) = |I| - n_-(D),$$

where $|I|$ is the number of 1 digits in I and $n_-(D)$ is the number of negative crossings in D. Monomials in $V^{\otimes p}$ have a natural degree induced by

$$\deg(1) = 0 \text{ and } \deg(x_i) = 2.$$

The internal or ℓ grading is given by

$$\ell(x) = \deg(x) - p(I) - \text{writhe}(D),$$

where $p(I)$ is the number of circles in the resolution I. The quantum or q grading is given by

$$q(x) = h(x) - \ell(x)$$

Finally, we define the g grading, a normalization of the q grading, by

$$g(x) = \frac{q(x) - m}{2},$$

where m is the number of components of L. (It turns out that g is always an integer [18, §6.1].) The g grading will induce the filtration on $C(D)$ in the definition of our spectral sequence.

Khovanov's differential d_0 increases both h and ℓ by 1, so it preserves q and g. Khovanov homology is

$$Kh(L) = H^*(C(D), d_0),$$

and has a bigrading given by (h, q).

A choice of marked point on the diagram D induces an endomorphism of Khovanov homology [19]: Let p be a marked point on D away from the double points. For a resolution I, let $x_p = x_p(I)$ denote the circle of I meeting p. Define a map $X_p : C(D) \to C(D)$ by

$$X_p(x) = x_p x$$

for $x \in V(I)$. The map X_p is a chain map and shifts the (h, q) bigrading by $(0, -2)$. The map induced on homology, which we also call X_p, depends only on the marked component and not on the choice of marked point.

The deformation

We begin by describing our construction in the case of a two-component link L with coefficients in $\mathbb{F}_2$. Khovanov's construction assigns a bigraded chain complex $(C(D), d_0)$ to a planar diagram D for L. We will give an endomorphism d_1 of $C(D)$ with the following properties:

(P1) $d := d_0 + d_1$ is a differential, which increases the ℓ-grading by 1.

(P2) d_1 lowers the g-grading by 1, making $(C(D), d)$ a g-filtered complex.

(P3) If i is a crossing in D involving strands from different components of L (a *mixed* crossing), and D' is the diagram for a link L' produced by changing over-strand to under-strand at i, then $(C(D), d)$ and $(C(D'), d')$ are isomorphic chain complexes (with different g-filtrations).

The new endomorphism is

$$d_1 = \sum_{\text{mixed edges } (I,J)} \mathcal{A}(J, I), \tag{2.1.1}$$

where an edge in the cube of resolutions is mixed if the I and J differ at a mixed crossing, and (J, I) denotes the saddle cobordism (I, J) viewed backwards as a cobordism from J to

I. The total differential

$$d = \sum_{\text{non-mixed edges } (I,J)} \mathcal{A}(I,J) + \sum_{\text{mixed edges } (I,J)} \mathcal{A}(I,J) + \mathcal{A}(J,I)$$

is manifestly unchanged if we swap a mixed crossing. The square d^2 can have a component from $V(I)$ to $V(J)$ only when I and J differ at 2 crossings or when $I = J$. The former vanish because they come in commuting squares (all maps are induced by cobordisms, and those commute due to the TQFT). The latter will vanish too, essentially because each circle in a complete resolution must have an even number of mixed crossings. We will establish that $d^2 = 0$ more carefully in Proposition 2.7, where we also handle multi-component links and other rings of coefficients.

To define the endomorphism d_1 when there are more than two components, or over bigger rings, we need some additional data. First, we must weight each component by an element of the coefficient ring R: component c has weight w_c. Then we must construct a sign assignment so that d^2 will be zero, not just even. As usual, different choices of sign assignment will produce isomorphic complexes.

We now define a sign assignment. The shadow of the diagram D in the plane gives a CW decomposition X of S^2: the 0-cells are the double points of the diagram, the 1-cells are the $2n$ edges between the crossings (oriented by the orientation of the link), and the 2-cells are the remaining regions (with the natural orientation induced from S^2). For each 1-cell e, let $e(0)$ denote the initial vertex and $e(1)$ denote the final vertex.

Let

$$h(e,i) = \begin{cases} 1 & e \text{ is an upper strand at } e(i) \\ -1 & e \text{ is a lower strand at } e(i), \end{cases}$$

where $i \in \{0,1\}$. There is a natural 1-cochain $\beta : X^1 \to \mathbb{Z}/2$, where $\mathbb{Z}/2 = \{\pm 1\}$ is written multiplicatively, given by

$$\beta(e) = \begin{cases} -1 & h(e,0) = h(e,1) \\ 1 & \text{otherwise.} \end{cases}$$

A *sign assignment* is a 0-cochain $s : X^0 \to \{\pm 1\}$ such that

$$s(e(0))s(e(1)) = \beta(e), \tag{2.1.2}$$

for all 1-cells e. This is equivalent to $\delta s = \beta$. Note that if D is an alternating diagram, then $s \equiv 1$ is a legal sign assignment. In the definition of d_1, we will use s to sign the weight of the top strand at each crossing; the bottom strand will get the opposite sign. The condition $\delta s = \beta$ means that at adjacent crossings, connected by a strand in component c of the link, the weight w_c will appear with opposite signs in the contributions from each.

We now define the endomorphism d_1 of $C(D)$ as

$$d_1 = \sum_{\text{edges (I, J)}} (-1)^{n(I,J)} s(i)(w^i_{\text{over}} - w^i_{\text{under}})\mathcal{A}(J,I), \tag{2.1.3}$$

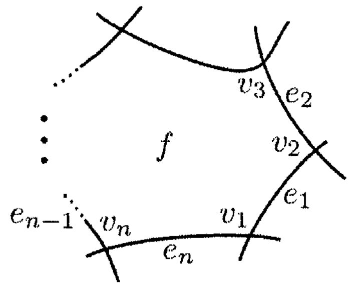


Figure 2.2:

where I and J differ at the i^{th} crossing, and w^i_{over} and w^i_{under} are the weights of the over- and under-strands at the i^{th} crossing. Only the differences of weights appear, so shifting all the weights by some $r \in R$ leaves the complex invariant.

In particular, the complex for a two-component link is determined by the choice of a single value $w_1 - w_2 \in R$. If that difference is $1 \in \mathbb{F}_2$, then this definition of d_1 reduces to (2.1.1).

The complex $(C(D), d = d_0 + d_1)$ now satisfies properties (P1) and (P2) from the beginning of this section. Both d_0 and d_1 increase the (internal) l-grading by 1. The differential d_0 preserves the g grading and d_1 decreases the g grading by 1. So we have a g-filtration on $(C(D), d)$ given by

$$\mathcal{F}^p C(D) := \{x \mid x \in C(D), g(x) \leq p\}.$$

Moreover, the spectral sequence associated to this filtration has E_1 page given by $H^*(C(D), d_0) \cong Kh(L)$.

We now show it is always possible to choose a sign assignment.

Proposition 2.6. *Let D be a connected diagram. There are precisely two sign assignments s_1 and s_2 for D, and $s_1 = -s_2$.*

Proof. By (2.1.2), a choice of sign at one crossing determines the sign assignment for a connected diagram, if one exists. Existence is a simple cohomological argument. Since a sign assigment is just a cochain $s \in C^0(S^2)$ with $\delta s = \beta$, such an s exists if and only if $\beta \in C^1(S^2)$ is exact, and is unique up to multiplication by an element of $H^0(S^2) = \{\pm 1\}$. Since $H^1(S^2) = 0$, β is exact if and only if it is closed.

We now show that β is closed. Let f be a 2-cell with the incident 0- and 1-cells numbered counterclockwise $v_1, \ldots, v_n$ and $e_1, \ldots, e_n$, respectively; see Figure 2.2. Each vertex v_i is incident to two edges, e_{i-1} and e_i (where we set $e_0 = e_n$). For one of those edges, v_i is an over-crossing, and for the other v_i is an under-crossing. More formally, if $v_i = e_{i-1}(a_i) = e_i(b_i)$ for some $a_i, b_i \in \{0, 1\}$, then

$$h(e_{i-1}, a_i) h(e_i, b_i) = -1.$$

By definition, $\beta(e_i) = -h(e_i, 0) h(e_i, 1)$. We then have

Figure 2.3: We choose marked points p_i and q_i on the understrands at each crossing i (left) and a marked point p_e on each edge e (right).

$$
\begin{aligned}
(\delta\beta)(f) &= \prod_{i=1}^{n} \beta(e_i) \\
&= \prod_{i=1}^{n} -h(e_i,0)h(e_i,1) \\
&= \prod_{i=1}^{n} -h(e_{i-1},a_i)h(e_i,b_i) \\
&= 1.
\end{aligned}
$$

□

For a split diagram, sign assignments can be chosen on each connected component independently.

Property (P3) does not hold on the nose. If D and D' are related by changing a crossing, then the associated differentials d and d' are not identical—they differ by elements of R. We will investigate this in Subsection 2.1 after verifying that our new differential squares to zero and checking independence of sign assignment.

Proposition 2.7. *We have that* $d^2 = 0$.

Proof. Fix a resolution I and let $x \in V(I)$. The terms of $d^2(x)$ lie in $V(K)$ where K differs from I in exactly two positions or $K = I$ itself. We study these two cases.

Case 1. Let K be a resolution that differs from I in exactly two positions i, j with $i < j$. Let J differ from I at i, and J' differ from I at j. Then I, J, J' and K are the four vertices of a face of the hypercube of resolutions. By functoriality of $\mathcal{A}$, we have that $\mathcal{A}(J,K)\mathcal{A}(I,J) = \mathcal{A}(J',K)\mathcal{A}(I,J')$. The endomorphism d_1 uses the usual Khovanov sign assignments, so the two paths around the face have different signs. Namely, we have that $n(I,J) = n(J',K)$ and $n(J,K) = -n(I,J')$. The weights on the cobordism maps in d_0 and d_1 depend only on which crossing is changed, not the edge of the cube. Denote the weights involved by $c(k)$, where

$$
c(k) = \begin{cases} 1 & I(k) = 0 \\ s(k)(w^k_{\text{over}} - w^k_{\text{under}}) & I(k) = 1. \end{cases}
$$

The terms of $d^2(x)$ in $V(K)$ are

$$\begin{aligned}
&c(i)c(j)((-1)^{n(I,J)+n(J,K)}\mathcal{A}(J,K)\mathcal{A}(I,J)(x) \\
&\qquad + (-1)^{n(I,J')+n(J',K)}\mathcal{A}(J',K)\mathcal{A}(I,J')(x) \\
&= c(i)c(j)(-1)^{n(I,J)+n(J,K)}(\mathcal{A}(J,K)\mathcal{A}(I,J)(x) - \mathcal{A}(J',K)\mathcal{A}(I,J')(x)) \\
&= 0.
\end{aligned}$$

Case 2. The terms of $d^2(x)$ in $V(I)$ are

$$\sum_{i=1}^{n} s(i)(w_{\text{over}} - w_{\text{under}})\mathcal{A}(J_i, I)\mathcal{A}(I, J_i)(x),$$

where J_i is the resolution which differs from I solely at the position i. We choose marked points on the under-strands at each crossing and on each edge, see Figure 2.3. Straightforward computation shows that $\mathcal{A}(J_i, I)\mathcal{A}(I, J_i) = X_{p_i} + X_{q_i}$. We can rewrite the above sum as

$$\begin{aligned}
&\sum_{i=1}^{n} s(i)(w^i_{\text{over}} - w^i_{\text{under}})(X_{p_i} + X_{q_i}) \\
&= \sum_{e \in X^1} s(e(0))h(e,0)w_e X_{p_e} + s(e(1))h(e,1)w_e X_{p_e} \\
&= \sum_{e \in X^1} (s(e(0))h(e,0) + s(e(1))h(e,1))w_e X_{p_e} \\
&= 0,
\end{aligned}$$

where w_e denotes the weight of the component containing the edge e, the first equality follows from indexing the sum by edges, and the second equality follows from the definition of a sign assignment. □

Change of sign assignment

While finding a sign assignment s is crucial for defining the complex over rings where $2 \neq 0$, different choices produce isomorphic complexes. Indeed, consider a connected diagram D, weight w, and sign assignment s producing the complex $(C(D), d = d_0 + d_1)$. Then taking the other sign assignment, $-s$, yields the differential $d' = d_0 - d_1$ on the same group of chains $C(D)$. Since d_0 fixes g-grading, and d_1 lowers it by 1, the endomorphism

$$\begin{aligned}
\phi : C(D) &\to C(D) \\
x &\mapsto (-1)^{g(x)}x
\end{aligned}$$

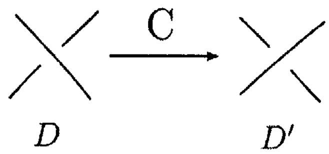


Figure 2.4: The crossing change move C.

has the property that $d\phi = \phi d'$. That is, ϕ is an invertible chain map between $(C(D), d)$ and $(C(D), d')$.

Next, consider the case when D is possibly split and s and s' are two sign assignments. Then, since $\mathcal{A}$ is a monoidal functor, the complexes $(C(D), d)$ and $(C(D), d)$ each decomposes into a tensor product of complexes indexed over the components of D. The above analysis gives a chain equivalence ϕ for each component, and their tensor product gives an invertible chain map between $(C(D), d)$ and $(C(D), d')$.

Henceforth, we will often suppress the choice of a sign assignment, writing $C(D, w)$ to indicate one of the two possible complexes.

Total homology

We now show that changing a crossing doesn't affect the total homology of $(C(D), d)$, so long as the relevant weight $w_{\text{over}} - w_{\text{under}}$ is invertible. Of course, changing the crossing does *not* preserve the g-filtration on C.

Proposition 2.8. *Let D and D' be diagrams for links L and L' related by changing a crossing i between components c and d. Let w be a weighting for L, and write w' for the induced weighting on L'. Then if $w_c - w_d$ is invertible in R, the complexes $C(D, w)$ and $C(D', w')$ are isomorphic as relatively ℓ-graded chain complexes.*

Proof. Let s be a sign assignment for D. A sign assigment s' for D' is given by $s'(j) = s(j)$ for $j \neq i$ and $s'(i) = -s(i)$. Let (C, d) be the complex $C(D, w, s)$, and let (C', d') be the complex $C(D', w, s')$. Let C_0 be the summand of C consisting of complete resolutions which include the 0 resolution at crossing i, and let C_1, C_0' and C_1' be defined analgously. Note that C_0 and C_1' are identical as relatively ℓ-graded complexes; similarly for C_1 and C_0'. (The writhes of the diagrams differ by 2, which will contribute a global shift between their ℓ-gradings.)

The crossing change exchanges over-strand for under-strand, so $(w'^i_{\text{over}} - w'^i_{\text{under}}) = -(w^i_{\text{over}} - w^i_{\text{under}})$. This means that

$$s(i)(w^i_{\text{over}} - w^i_{\text{under}}) = s'(i)(w'^i_{\text{over}} - w'^i_{\text{under}}).$$

Before giving the chain map $f : C \to C'$, we must first introduce some notation. Let I be a resolution of D. We write I' to denote the same element of $\{0, 1\}^n$ interpreted as a resolution of D'. We write I_i for the resolution of D that differs with I solely at crossing i.

Note that I and I'_i are canonically isomorphic resolutions. Let J denote a resolution of D that differs from I at some crossing $j \neq i$. Finally, let

$$a(I,i) = \#\{I(k) = 1 \mid i < k \leq n\}$$

be the number of one digits in I above i.

We define the map $f : C \to C'$ as follows.

$$x \mapsto \begin{cases} (-1)^{a(I,i)} x \in C'_1 \text{ if } x \in V(I) \subset C_0 \\ (-1)^{a(I_i,i)} s(i)(w^i_{\text{over}} - w^i_{\text{under}}) x \in C'_0 \text{ if } x \in V(J) \subset C_1, \end{cases}$$

To verify f is a chain map, we use two easily verifiable facts about the signs:

$$(-1)^{a(I,i)} = (-1)^{a(I_i,i)}$$

and

$$(-1)^{n(I,J)}(-1)^{a(J,i)} = (-1)^{n(I'_i,J'_i)}(-1)^{a(I,i)}.$$

Consider $x \in V(I) \subset C_0$. The image of x under fd or $d'f$ has components in $V(I')$ and $V(J'_i)$, for the resolutions J differing from I at one crossing.

First, consider the $V(I')$-component of the image. We have

$$\begin{aligned} fd(x)\big|_{V(I')} &= f((-1)^{n(I,I_i)} \mathcal{A}(I, I_i)(x)) \\ &= (-1)^{a(I_i,i)}(-1)^{n(I,I_i)} s(i)(w^i_{\text{over}} - w^i_{\text{under}}) \mathcal{A}(I, I_i)(x) \\ &= (-1)^{a(I,i)}(-1)^{n(I'_i,I')} s'(i)(w'^i_{\text{over}} - w'^i_{\text{under}}) \mathcal{A}(I'_i, I')(x) \\ &= d'((-1)^{a(I,i)} x)\big|_{V(I')} \\ &= d'f(x)\big|_{V(I')}. \end{aligned}$$

Next, consider the image in $V(J'_i)$ for some J which differs from I at crossing j. Let

$$c(j) = \begin{cases} 1 & I(j) = 0 \\ s(i)(w^j_{\text{over}} - w^j_{\text{under}}) & I(j) = 1 \end{cases}$$

denote the coefficient of $\mathcal{A}(I, J)$ in d. It is the same as the coeffient of $\mathcal{A}(I'_i, J'_i)$ in d'. We have

$$\begin{aligned} fd(x)\big|_{V(J'_i)} &= f((-1)^{n(I,J)} c(j) \mathcal{A}(I, J)(x)) \\ &= (-1)^{a(J,i)}(-1)^{n(I,J)} c(j) \mathcal{A}(I, J) \\ &= (-1)^{a(I,i)}(-1)^{n(I'_i,J'_i)} c(j) \mathcal{A}(I'_i, J'_i) \\ &= d'((-1)^{a(I,i)} x)\big|_{V(J'_i)} \\ &= d'f(x)\big|_{V(J'_i)}. \end{aligned}$$

A similar analysis shows that $fd(x) = d'f(x)$ for $x \in C_1$.

Let $f' : C' \to C$ be the chain map produced by reversing the roles of D and D'. The composition

$$ff' = f'f = s(i)(w^i_{\text{over}} - w^i_{\text{under}})$$

is an isomorphism if $(w^i_{\text{over}} - w^i_{\text{under}})$ is invertible for all i. Then f an isomorphism too. □

Dependence on weights

Let L be a link with two components, and $w_1, w_2, w'_1, w'_2 \in R$ choices of weights such $w_1 - w_2$ and $w'_1 - w'_2$ are invertible. Let D be a diagram for L. Consider the map $f : C(D, w) \to C(D', w)$ defined by

$$x \mapsto \left(\frac{w_1 - w_2}{w'_1 - w'_2}\right)^{g(x)} x$$

for $x \in C(D, w)$ of homogeneous g-degree. Since d_0 preserves g-degree, we have $fd_0 = d_0 f$. Since d_1 lowers g degree by 1, we have

$$\begin{aligned} fd_1 x &= f \sum_{\text{mixed edges (I, J)}} (-1)^{n(I,J)} s(i)(w^i_{\text{over}} - w^i_{\text{under}})\mathcal{A}(J, I)x \\ &= \sum_{\text{mixed edges (I, J)}} (-1)^{n(I,J)} s(i)(w^i_{\text{over}} - w^i_{\text{under}}) \left(\frac{w_{\text{over}} - w_{\text{under}}}{w'_{\text{over}} - w'_{\text{under}}}\right)^{g(x)-1} \mathcal{A}(J, I)x \\ &= \sum_{\text{mixed edges (I, J)}} (-1)^{n(I,J)} s(i)(w'^i_{\text{over}} - w'^i_{\text{under}})\mathcal{A}(J, I) \left(\frac{w_{\text{over}} - w_{\text{under}}}{w'_{\text{over}} - w'_{\text{under}}}\right)^{g(x)} x \\ &= d_1 f x. \end{aligned}$$

Since f is an invertible, grading-preserving chain map, it is an isomorphism of filtered complexes and induces an isomorphism of spectral sequences. This establishes Proposition 2.4.

2.2 Sliding the marked point

In this section, we give an alternative origin myth for the endomorphism d_1. Let L be a link with diagram D. As described in Section 2.1, a choice of marked point p on D defines an endomorphism X_p of the Khovanov chain complex $C(D)$. Points on the same arc of the diagram D will obviously give the same action, but if we slide p under or over a crossing to another position q we get a manifestly different endomorphism X_q. Nevertheless, X_p and X_q induce the same action on homology.

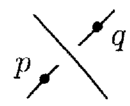


Figure 2.5: Moving a marked point across a crossing.

Proposition 2.9. *Let p and q be marked points on either side of crossing i, as shown in Figure 2.5. Then X_p and X_q are chain homotopic.*

Proof. We recall the chain homotopy H used by Hedden and Ni [15]:

$$H_i = \sum_{\substack{\text{resolutions } I \\ I(i)=1}} (-1)^{n(J_i,I)} \mathcal{A}(I, J_i),$$

where J_i differs from I solely at i. The signs are chosen so that H_i will anticommute with the components of d_0 which change crossings other than i. The only nonzero component of $H_i d_0 + d_0 H_i$ on a resolution I is the cobordism merging together the circles adjacent to i then splitting them apart, or vice versa. It is straightforward to check—using the TQFT $\mathcal{A}$—that this acts by $X_p + X_q$. □

If we pick a point p_1 on a component c of a link, and slide it all the way around (Figure 2.6), we get a sequence of chain homotopies:

$$\begin{aligned} dH_1 + H_1 d &= X_{p_1} + X_{p_2} \\ dH_2 + H_2 d &= X_{p_2} + X_{p_3} \\ dH_3 + H_3 d &= X_{p_3} + X_{p_4} \\ &\cdots \\ dH_n + H_n d &= X_{p_n} + X_{p_1} \end{aligned}$$

The alternating sum of the homotopies $H_c := \sum_{i=1}^{n} (-1)^{i-1} H_i$ satisfies

$$dH_c + H_c d = (X_{p_1} + X_{p_2}) - (X_{p_2} + X_{p_3}) \pm \cdots - (X_{p_n} + X_{p_1}) = 0.$$

In other words, H_c is an endomorphism of the Khovanov complex.

If we choose a weight w_c for each component c, then the sum

$$H := \sum_{\text{components } c} w_c H_c$$

is also an endomorphism of $C(D)$. In fact, there is a choice of sign assignment for which H precisely matches our endomorphism d_1.

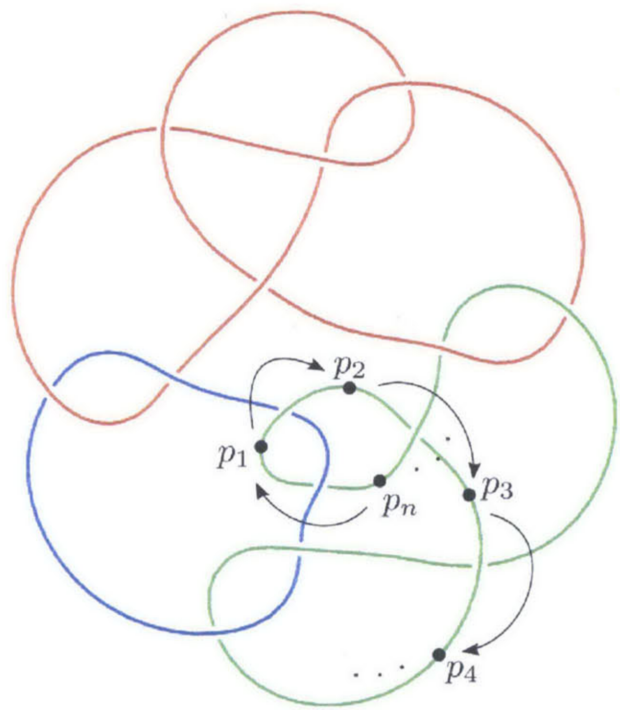


Figure 2.6: Sliding the marked point all the way around a component gives a loop of homotopies and a new endomorphism of Khovanov homology.

To summarize: the X-action of a component on Khovanov homology does not lift to a canonical action on chains. Different choices of marked point give different actions, but neighboring points are related by canonical homotopies H_i. By adding up a "loop" of these homotopies, we get a "higher" endomorphism. (The fact that H_c happens to square to zero, making $d_0 + H_c$ a differential, is not guaranteed by this approach.)

This sort of phenomenon exists elsewhere in topology.

Example: Steenrod Squares

In singular homology, cup product is not commutative on the chain level. However, there is a canonical homotopy $\cup_1$ such that if a and b are cochains [26], then

$$a \cup b - (-1)^{|a||b|} b \cup a = d(a \cup_1 b) + (da) \cup_1 b + a \cup_1 db.$$

If we take $a = b$ and work over $\mathbb{F}_2$, then the left-hand side vanishes. This defines a Steenrod square: $Sq^{n-1}([a]) = [a \cup_1 a]$.

Example: Monopole Floer homology

In Monopole Floer homology [23], a circle $\eta \in Y^3$ gives an action A_η on the chain groups $\hat{C}(Y)$. If η and η' are homologous via some surface θ with $\partial\theta = \eta - \eta'$, then there is a homotopy h_θ satisfying

$$A_\eta - A_{\eta'} = dh_\theta + h_\theta d.$$

If we view a torus Σ in Y as a homology between some circle η and itself, we get that h_Σ is a chain map. On the subcomplex $\hat{C}(Y, \mathfrak{s})$ defined by a Spin^c-structure $\mathfrak{s}$, the map h_Σ is multiplication by $c_1(\mathfrak{s})[\Sigma]$.

2.3 Reidemeister invariance

The proof that the filtered chain homotopy type of $C(D, w, s)$ is invariant under the Reidemeister moves parallels the standard proof that the Khovanov chain complex is invariant. We divide the complex into the summands corresponding to the 2, 4, or 8 ways of resolving the crossings involved in the move, and cancel isomorphic summands along components of the differential. This is complicated slightly by the d_1 terms which prevent the natural summands of C from being subcomplexes; the post-cancellation differential is not merely a restriction of the original one. The new differential is provided by the following standard cancellation lemma.

Lemma 2.10. *Let (C, d) be a chain complex. Suppose that C, viewed as an R-module, splits as a direct sum $V \oplus W \oplus C'$. Let d_{WV} denote the component of d mapping from V to W, and similarly for other components. If d_{WV} is an isomorphism, then (C, d) is chain homotopy equivalent to (C', d') with*

$$d' = d_{C'C'} - d_{C'V} d_{WV}^{-1} d_{WC'}.$$

Proof. Let $f : C' \to C$, $g : C \to C'$, and $h : C \to C$ be defined by

$$f = \iota_{C'} - d_{WV}^{-1} d_{WC'}, \qquad g = \pi_{C'} - d_{C'V} d_{WV}^{-1}, \quad \text{and} \quad h = d_{WV}^{-1},$$

where ι and π denote inclusion and projection with respect to the direct sum decomposition of C. The map f is an isomorphism onto its image, since the second term in f merely adds a V-component. The image of f turns out to be a subcomplex, and the new differential d' is merely the pullback of d along f.

We claim that f and g are mutually inverse chain homotopy equivalences between (C, d) and (C', d'). Specifically, the following four equations hold:

$$fd' = df \qquad gd = d'g \qquad \mathbb{I}_{C'} = gf \qquad \mathbb{I}_C = fg + hd + dh$$

Verifying these is a routine exercise in applying the identities contained in the equation $d^2 = 0$, such as

$$d_{WV}d_{VV} + d_{WC'}d_{C'V} + d_{WW}d_{WV} = 0. \qquad \square$$

If the complex (C, d) has a filtration induced by a grading g and the cancelled map, d_{WV} above, preserves g-degree, then d' will respect the induced filtration on C' and the maps f and g will be filtered chain homotopy equivalences. This will be our situation in each of the Reidemeister moves below.

Proposition 2.11. *Let D and D' be two diagrams for a link L related by a Reidemeister move of type I, II, or III. Fix an R-weighting w for L and a sign assignment s for the diagram D. Then there exists a sign assignment s' for the diagram D' which agrees with the sign assignment for D at all crossings uninvolved in the Reidemeister move, and the complexes $C(D, w, s)$ and $C(D', w, s')$ are chain homotopy equivalent as ℓ-graded, q-filtered complexes.*

In Section 2.1, we saw that different sign assignments produce isomorphic complexes. Since any two diagrams for a link are related by a sequence of Reidemeister moves, this proposition implies that that the ℓ-graded q-filtered chain homotopy type of the complex $C(D, w, s)$ is also independent of the choice of planar diagram, and hence an invariant of the R-weighted link (L, w). This establishes that the associated spectral sequence, called $E_k(L, w)$ in Theorem 2.1, is an invariant of (L, w).

Proof. The proof for each of the three Reidemeister moves is similar. We first decompose the complex into summands sitting over each of the 2^k different resolutions of the crossings implicated in the k-th move. One of these resolutions contains an isolated circle, and we split the complex over that resolution further according to whether or not the monomial contains that circle. We then identify two summands V and W for which d_{WV} is a q-grading-preserving isomorphism, and apply the cancellation lemma.

R1 Consider two diagrams D and D' for a link L in Figure 2.7. Let s be a sign assignment for D. It can be verified easily that the restriction of s to the vertices of the diagram for D' yields a valid sign assignment s'.

Let (C, d) be the complex $C(D, w, s)$, and let (C', d') be the complex $C(D', w, s')$. Let C_0 be the summand of C corresponding to complete resolutions which include the 0-resolution at the pictured crossing, and let C_1 be the summand of C corresponding to complete resolutions which include the 1-resolution at the pictured crossing. Let C_0^- and C_0^+ be the summands

Figure 2.7: Left is the first Reidemeister move R1. Right is chain complex for the diagram D, split into two summands corresponding to the two resolutions of the pictured crossing.

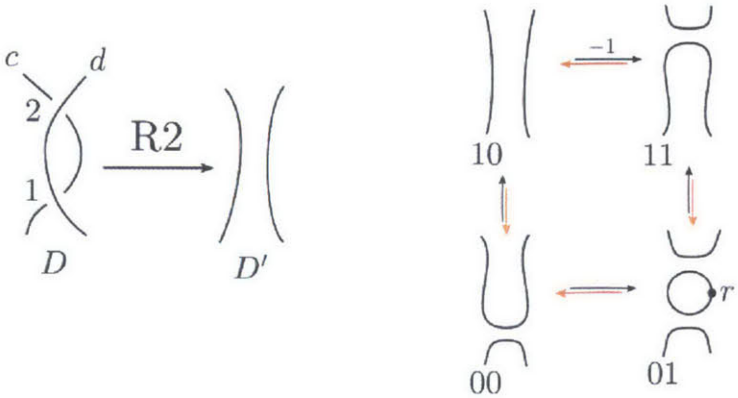


Figure 2.8: Left is the second Reidemeister move R2. Right is chain complex for the diagram D, split into four summands corresponding to the resolutions of the pictured crossings.

of C_0 spanned by monomials divisible and not divisible, respectively, by the variable x_r corresponding to the pictured circle.

Since the component of d mapping from C_0^+ to C_1 is just merging in the 1 on the pictured circle, it is an isomorphism. Hence we may apply the cancellation lemma with with $V = C_0^+$ and $W = C_1$. Since C_0^+ and C_0^- have the same resolution at the pictured crossing, there is no component of d mapping from one to the other. Hence the new complex is just C_0^- with the restriction of the original differential. (This cancellation preserves the filtration, since the cancelled part of the differential is a component of the ordinary Khovanov d_0, which preserves $q-$ and $g-$ degree.) Since the extra circle never interacts with the remainder of the diagram for L, this complex (C_0^-, d) is isomorphic to the post-move complex (C', d). That isomorphism also respects the gradings, as can be verified from $n_+(D) = n_+(D') + 1$, $n_-(D) = n_-(D')$, and the definitions of ℓ and h.

R2 Consider two diagrams D and D' for a link L in Figure 2.8. Let s be a sign assignment for D. It can be verified easily that the restriction of s to the vertices of the diagram for D' yields a valid sign assignment s'.

Let D_{ij} with $i, j \in \{0, 1\}$ denote the diagrams obtained by resolving the crossings involved in the Reidemeister move in D. Let $C_{ij} = C(D_{ij}, w, s)$. Let C_{01}^- and C_{01}^+ be the summands of C_{01} spanned by generators divisible and not divisible, respectively, by the variable x_r corresponding to the pictured circle. The four summands C_{00}, C_{11}, C_{01}^+ and C_{01}^- are all naturally

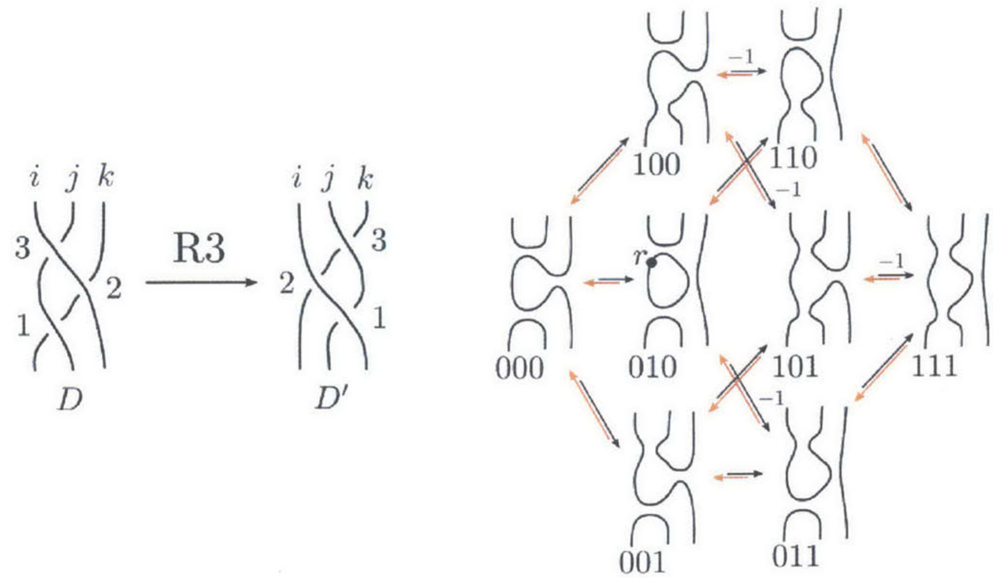


Figure 2.9: Left is the third Reidemeister move R3. Right is chain complex for the diagram D, split into eight summands corresponding to the resolutions of the pictured crossings.

isomorphic, and the summand C_{10} is isomorphic to the post-move complex $C' = C(D', w, s')$.

We will apply the cancellation lemma with $V = C_{00} \oplus C_{01}^{+}$ and $W = C_{01}^{-} \oplus C_{11}$. The component of d from V to W is just the original Khovanov differential d_0, and it is block diagonal: C_{01}^{+} maps to C_{11} isomorphically (merging in a 1) and C_{00} maps to C_{01}^{-} isomorphically (splitting of an x).

The cancelled complex is just C_{10}, with differential

$$d_{C_{10}C_{10}} - d_{C_{10}V} d_{WV}^{-1} d_{WC_{10}}.$$

But $d_{WC_{10}}$ lands on C_{11}, which is carried to C_{01}^{+} by d_{WV}^{-1}, and d has no component from C_{01}^{+} to C_{10}. Hence the new differential is just the restriction of the old, and we have

$$(C, d) \cong (C_{10}, d|_{C_{10}}) \cong (C', d').$$

Again, the cancelled pieces of the differential come from Khovanov's d_0, which preserves g and q, so the first isomorphism preserves the filtration. The second isomorphism also preserves the bigrading, as can be verified from $n_{\pm}(D) = n_{\pm}(D') + 1$.

R3 Reidemeister 3 is more complicated, and we must keep track of the signs in Khovanov's cube, the sign assignment s, and the weights.

Consider the diagrams D and D' in Figure 2.9. Label the strands i, j, and k, from left to right along the top of D. Denote by w_i, w_j, and w_k the weights of their components. Order the crossings up the page 1, 2, and 3. Using Khovanov's sign assignment, the edges in the cube of resolutions for D labeled -1 in the figure have a negative sign in the differential:

$(-1)^{n(I,J)} = -1$. $(100 \leftrightarrow 110, 100 \leftrightarrow 101, 010 \leftrightarrow 011, 101 \leftrightarrow 111.)$

Choose a sign assigment s for D such that

$$s(1) = s(3) = 1 \text{ and } s(2) = -1.$$

A choice of sign at one crossing determines the sign assignment on that component of the diagram by (2.1.2). Take the sign assignment s' for D' which agrees with s on the crossings not involved in the Reidemeister move. Again, (2.1.2) implies

$$s'(1) = s'(3) = -1 \text{ and } s'(2) = 1.$$

Let $(C,d) := C(D,w,s)$ and $(C',d') := C(D',w,s')$. The weights $c(j) = s(j)(w_{\text{over}} - w_{\text{under}})$ of the reverse edge maps in d_1 evaluate to

$$\begin{aligned} c(1) &= w_j - w_k \\ c(2) &= w_k - w_i \\ c(3) &= w_i - w_j, \end{aligned}$$

at the three pictured crossings, and the weights $c'(j)$ in d'_1 are

$$\begin{aligned} c'(1) &= w_j - w_i \\ c'(2) &= w_i - w_k \\ c'(3) &= w_k - w_j. \end{aligned}$$

First, we will simplify the complex (C,d). As in the previous parts, let C^-_{010} and C^+_{010} be the summands of C_{010} spanned by monomials divisible and not divisible, respectively, by the variable x_r corresponding to the pictured circle.

We apply the cancellation lemma with

$$V = C_{000} \oplus C^+_{010} \qquad W = C^-_{010} \oplus C_{011}.$$

The component of d from V to W is just the Khovanov differential d_0, and it is block diagonal: C_{000} maps to C^-_{010} isomorphically (splitting off an x) and C^+_{010} maps to C_{011} isomorphically (merging in a 1, with a minus sign from the cube). The reduced complex will have underlying abelian group

$$C_{\text{red}} = C_{100} \oplus C_{001} \oplus C_{110} \oplus C_{101} \oplus C_{111}.$$

After chasing the diagram to find the maps into V and the maps out of W, you will find that the correction term $d_{C_{\text{red}}V} d^{-1}_{WV} d_{WC_{\text{red}}}$ has four components.

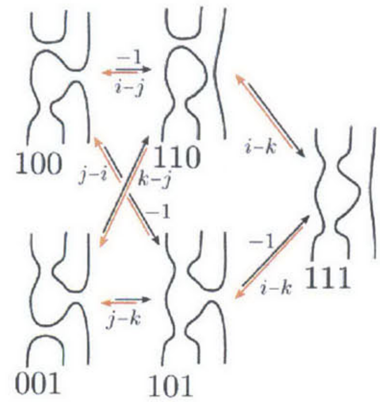


Figure 2.10: A reduced chain complex for the diagram D.

$$
\begin{aligned}
C_{001} &\xrightarrow{-1} C_{110} \\
C_{111} &\xrightarrow{w_k - w_j} C_{110} \\
C_{110} &\xrightarrow{w_j - w_k} C_{100} \\
C_{110} &\xrightarrow{w_j - w_k} C_{001}.
\end{aligned}
$$

Each map is induced by the obvious cobordism relating the resolutions, weighted by some element of R indicated by the label on the arrow. Subtracting these from the restriction of the original differential d to C_{red} yields the complex pictured in Figure 2.10. Here, the edge labels give the total coefficient of the forward or reverse edge maps in d_{red}. The absence of a label on a forward edge maps the coefficient is $+1$. The label $i - j$, for example, denotes the coefficient $w_i - w_j$.

The complex (C', d') can be simplified using a similar cancellation. The relevant resolutions are drawn in Figure 2.11. Apply the cancellation lemma with

$$V = C'_{000} \oplus C'^{+}_{010} \qquad W = C'^{-}_{010} \oplus C'_{011}.$$

The resulting complex, C'_{red} is pictured in Figure 2.12. It contains all the same resolutions as C_{red}; the only difference is that all of the maps between pictured summands have reversed signs. The map $\phi : C_{\text{red}} \to C'_{\text{red}}$, defined by

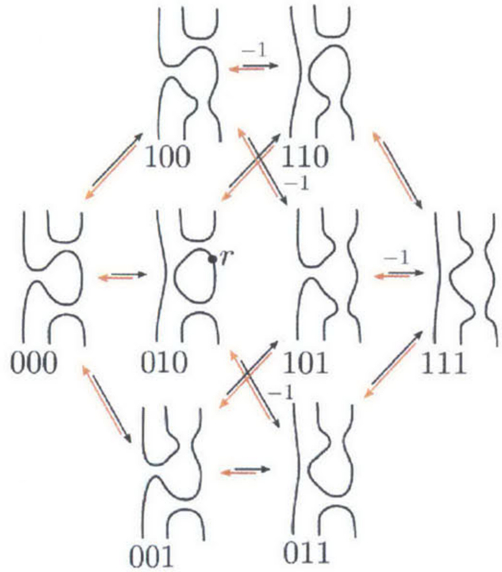


Figure 2.11: The chain complex for the post-R3 diagram D'.

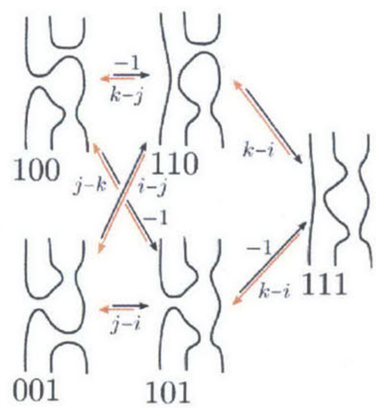


Figure 2.12: A reduced chain complex for the post-R3 diagram D'.

$$\begin{aligned} C_{100} &\xrightarrow{1} C'_{001} \\ C_{001} &\xrightarrow{1} C'_{100} \\ C_{110} &\xrightarrow{-1} C'_{101} \\ C_{101} &\xrightarrow{-1} C'_{110} \\ C_{111} &\xrightarrow{1} C'_{111} \end{aligned}$$

is an invertible chain map. The sequence $C \cong C_{\text{red}} \cong C''_{\text{red}} \cong C'$ yields the desired isomorphism for diagrams related by Reidemeister 3. The first and third isomorphism preserve the filtration, since the cancelled components of the differential all preserved q. The second isomorphism preserves the bigrading, since the diagrams satisfy $n_\pm(D) = n_\pm(D')$ and the map ϕ preserves the norm $|I|$ of each resolution I. □

We can now prove that the total homology of the complex for a link is just the Khovanov homology of the disjoint union of its components. This completes the proof of Theorem 2.1.

Theorem 2.12. *Let (L, w) be an R-weighted link, and suppose that for each pair of components i and j with distinct weights, the difference $w_i - w_j$ is invertible in R. Let D be any diagram for L. Let $L^{(r)}$ denote the sublink of L consisting of those components with weight r. Then the spectral sequence converges to*

$$H^*(C(D, w)) \cong Kh^*\left(\coprod_{r \in R} L^{(r)}; R\right)$$

Proof. Choose an arbitrary ordering $\succ$ on the set $w_1, \ldots, w_n \subset R$ of weights. By Proposition 2.8, changing a crossing between components with distinct weights will produce a chain complex $C(D', w)$ with the same l-graded total homology. So we may change crossings until each component i lies entirely over component j whenever $w_i \succ w_j$. This produces a diagram D' for some link L', whose sublinks are still the $L^{(r)}$, now completely unlinked from one another. By repeated application of Reidemeister moves 1 and 2, we may slide these components off of one another until we get a diagram D'' for L' with no crossings between $L^{(r)}$ and $L^{(r')}$ for $r \neq r'$. The differential for $C(D'', w)$ is the same as Khovanov's differential, since $d_1 = 0$, and L' is just the disjoint union of the sublinks $L^{(r)}$. □

We can now give a stronger version of the rank inequality Corollary 2.2.

Corollary 2.13. *Let $\mathbb{F}$ be any field, and let L be a link with components $K_1, \ldots, K_m$. Then*

$$\operatorname{rank}^\ell Kh^*(L; \mathbb{F}) \geq \operatorname{rank}^{\ell+t} \otimes_{c=1}^m Kh^*(K_c; \mathbb{F}),$$

where each side is ℓ-graded and the shift t is given by

$$t = \sum_{c<d} 2\text{lk}(L_c, L_d).$$

Proof. Assume for the moment that the field $\mathbb{F}$ has more elements than L has components, so we can weight each component by a different element $w_c \in \mathbb{F}$. Then all differences will be invertible, so the above theorem characterizing the abutment of the spectral sequence applies. That would give an inequality of total ranks. To see the ℓ-gradings, we need to compute the grading shift in the isomorphism relating $C(D, w)$ and $C(D', w)$ (we use the same notation as in the above proof). Recall the formula for the ℓ grading:

$$\ell(x) = \deg(x) - p(I) - \text{writhe}(D).$$

For a fixed monomial x over a fixed resolution I, the terms $\deg(x)$ and $p(I)$ are the same before and after a crossing change; only the writhe differs. Each time we change a crossing between components c and d, the writhe will shift by ± 2 and the linking number $\text{lk}(L_c, L_d)$ will shift by ± 1. (The linking numbers with other components remain unchanged.) Thus

$$\ell(x) + \sum_{c<d} 2\text{lk}(L_c, L_d) = \ell(x') + \sum_{c<d} 2\text{lk}(L'_c, L'_d),$$

where x' is the same monomial viewed as a generator of $C(D', w)$, and L'_c is the component of L' which L_d turns into. But the components of L' are unlinked, so we ultimately have

$$\ell(x') = \ell(x) + \sum_{c<d} 2\text{lk}(L_c, L_d).$$

Now we address the size of $\mathbb{F}$. Since the differential in the chain complex computing $Kh(L)$ uses only ± 1 coefficients, its rank is the same after a field extension. We may take a suitably large extension $\mathbb{F}'$ of $\mathbb{F}$, run the above argument for some choice of weights, and then note that $\text{rank}'_{\mathbb{F}} Kh(L; \mathbb{F}') = \text{rank}_{\mathbb{F}} Kh(L; \mathbb{F})$. $\square$

2.4 Properties of spectral sequences

We offer a quick review of spectral sequences, following Serre [40]. Let (C, d) be a finitely generated chain complex. A filtration $\mathcal{F}$ on C is an assignment to each element $x \in C$ a filtration degree $p(x) \in \mathbb{Z} \cup \{-\infty\}$ such that $p(x - y) \leq \max(p(x), p(y))$ and $p(dx) \leq p(x)$. (Only 0 is permitted to have filtration degree $-\infty$.) We will occasionally write C^k for the k^{th} piece of the filtration $\mathcal{F}^k C = \{x \in C | p(x) \leq k\}$. Homological algebra usually concerns cycles and boundaries. The filtration provides notions of approximate cycles and early boundaries:

$$\begin{aligned} Z_r^k &= \left\{x \in C^k | dx \in C^{k-r}\right\} \\ B_r^k &= \left\{dy \in C^k | y \in C^{k+r}\right\}. \end{aligned}$$

The spectral sequence corresponding to the filtration is a sequence of chain complexes (E_r^k, d_r), called *pages*, defined by

$$E_r^k = Z_r^k / (Z_{r-1}^{k-1} + B_{r-1}^k) \ .$$

If x is in Z_r^k, then dx is in Z_r^{k-r}: by definition $dx \in C^{k-r}$, and $d(dx) = 0$. The differential on E_r^k is then given by taking the equivalence class: $d_r[x] := [dx]$. The remarkable property of this sequence is that each page is the homology of the previous one: $E_{r+1}^k = H_*(E_r^k, d_r)$.

A spectral sequence is said to *collapse on page* l if $d_r = 0$ for all $r \geq l$.

Since C is finitely generated, there is some integer N such that, for all $r > N$, Z_r^k just consists of all cycles in degree $\leq k$ and B_r^k consists of all boundaries in degree $\leq k$ (that is, $Z_r^k = Z^k$ and $B_r^k = B^k$). The quotient Z^k/B^k is not the homology of the k^{th} filtered piece C^k, because B^k consists of elements of C^k which are boundaries in C, not just boundaries of elements in C^k. In fact, the quotient is

$$Z^k / B^k \cong i_* H_*(C)$$

where $i : C^k \hookrightarrow C$ denotes the inclusion of the k^{th} filtered piece into the total complex. For all $r > N$, we have

$$\begin{aligned} E_r^k &= Z^k / (Z^{k-1} + B^k) \\ &= (Z^k/B^k) \,/\, (Z^{k-1}/B^{k-1}) \\ &= i_* H_*(C^k) / i_* H_*(C^{k-1}) \ . \end{aligned}$$

We denote this stable page by E_∞^k, and observe that it is the associated graded group of the total homology $H_*(C)$ by the filtration

$$\mathcal{G}^k H_*(C) = i_* H_*(C^k).$$

In particular, the total rank of the E_∞ page is independent of the choice of filtration:

$$\sum_k \operatorname{rank} E_\infty^k = \operatorname{rank} H_*(C).$$

In contrast, the time of collapse does depend on the choice of filtration, though in a controlled way. (We doubt that the following proposition is original, but were unable to find it in the literature.)

Proposition 2.14. *Let (C, d) be a finitely generated chain complex, with two different filtrations $\mathcal{F}$ and $\mathcal{F}'$ which are close in the following sense: for any $x \in C$, the difference in filtration degree $p'(x) - p(x)$ is either* 0 *or* 1*. Then the p-spectral sequence collapses at most one page after the p'-spectral sequence does.*

Proof. Say that the p'-spectral sequence has collapsed by the $(r-1)^{st}$ page. We want to show that any class $[x] \in E_r^k$ must have $d_r[x] = 0 \in E_r^{k-r}$, for then the p-spectral sequence will have collapsed on page r.

Suppose for the sake of contradiction that there is some $x \in Z_r$ such that $[x] \in E_r$ has nonzero differential. Without loss of generality, we may take the chain x with minimal $p'(x) + p'(dx)$. Let k be the degree $p(x)$, so $x \in Z_r^k$. If $p(dx) < k - r$, then $dx \in Z_{r-1}^{k-r-1}$ and $[dx]_r$ would represent 0 in E_r^{k-r}. Since $d_r[x] = [dx]$ is nontrivial, we must have $p(dx) = k - r$.

We now consider the p'-degrees of all the elements. Let $k' = p'(x)$ and $r' = p'(x) - p'(dx)$. Note that

$$\begin{aligned} r' &= p'(x) - p'(dx) \\ &= p'(x) - p(x) - (p'(dx) - p(dx)) + p(x) - p(dx) \\ &\in \{0,1\} - \{0,1\} + r \\ &\geq r - 1 \end{aligned}$$

Since the p'-spectral sequence has collapsed by page $r-1$, it has also collapsed by page r'. We will denote the pages, boundaries, cycles, and differential for the p'-spectral sequence with acute accents. By construction, x represents a class in $\acute{E}_{r'}^{k'}$. Post-collapse, the differential is identically zero, so $\acute{d}_{r'}[x]$ must represent zero in $\acute{E}_{r'}^{k'-r'}$. In terms of chains, this means that

$$dx = w + dz$$

for some $w \in \acute{Z}_{r'-1}^{k'-r'-1}$ with $p'(w) \leq k' - r' - 1$ and some $dz \in \acute{B}_{r'-1}^{k'-r'}$ with $p'(z) \leq k' - 1$. Since p-gradings are at most one less than p'-gradings, $p(x) \geq k' - 1$ and $p(dx) \geq k' - r' - 1$. Consequently, $p(z) \leq p(x)$ and $p(w) \leq p(dx)$.

Since $dw = ddx - ddz = 0$, we have that $w \in Z_r^{k-r}$. Since $dz = dx - w$, we have $p(dz) \leq \max(p(dx), p(w))$, and $z \in Z_r^k$.

We break into two cases.

Case 1: $[w] = 0 \in E_r^{k-r}$.

Set $\overline{x} = z$. Then $[\overline{x}]$ is a class in E_r^k with

$$d_r[\overline{x}] = [dz] = [dz] + [w] = [dx] \neq 0.$$

But $p'(\overline{x}) = p'(z) < p'(x)$ and $p'(d\overline{x}) = p'(dx - w) = p'(dx)$, violating minimality.

Case 2: $[w] \neq 0 \in E_r^{k-r}$.

Set $\overline{x} = x - z$. Then $[\overline{x}]$ is a class in E_r^k with

$$d_r[\overline{x}] = [dx - dz] = [w] \neq 0.$$

But $p'(\overline{x}) = p'(x - z) = p'(x)$ and $p'(d\overline{x}) = p'(w) < p'(dx)$, violating minimality. $\square$

Endomorphisms of spectral sequences

Suppose that f is an endomorphism of the filtered chain complex C which shifts filtration degree by l,

$$p(fx) = p(x) - l \ \ \forall x \in C.$$

Then f *acts* on the spectral sequence the following sense

1. There is an endomorphism f_r of the r^{th} page given by

$$\begin{aligned} f_r : E_r^k &\to E_r^{k-l} \\ [x] &\mapsto [fx] \end{aligned}$$

   This is well-defined: since f shifts $p(dx)$ by the same amount that it shifts $p(x)$, it takes Z_r^k into Z_r^{k-l} and B_r^k into B_r^{k-l}.

2. The action of f_{r+1} on E_{r+1} is the same as the one induced by f_r on the homology of (E_r, d_r).

3. The action of f_∞ on E_∞ is the associated graded action of

$$f_* : H_*(C) \to H_*(C)$$

   with respect to the filtration $\mathcal{G}$ above. That is, if $[x] \in \mathcal{G}^k = i_* H_*(C^k)$ is represented by $x \in Z^k$, then $fx \in Z^{k-l}$ and the image $f_*[x] = [fx]$ lies in $\mathcal{G}^{k-l}$. Moreover, x also serves as a representative of the equivalence class of $[x]_\infty \in E_\infty^k = \mathcal{G}^k / \mathcal{G}^{k-1}$ and $f_\infty[x]_\infty = [fx]_\infty$.

We will later encounter a spectral sequence where we know the action of an endomorphism X on $H_*(C)$ and investigate the possible associated graded actions on the E_∞ page.

2.5 The splitting number

The unknotting number of a knot is the minimum number of times the knot must be passed through itself to untie it. It is an intuitive measure of the complexity of a knot, though strikingly difficult to compute. We would like to suggest a similar number measuring the complexity of the linking *between* the components of a link, unrelated to the knotting of the individual components.

Definition 2.15. *The splitting number of a link L, written* $\mathrm{sp}(L)$*, is the minimum number of times the different components of the link must be passed through one another to completely split the link. Equivalently,* $\mathrm{sp}(L)$ *is the minimum over all diagrams for L of the number of between-component crossings changes required to produce a completely split link.*

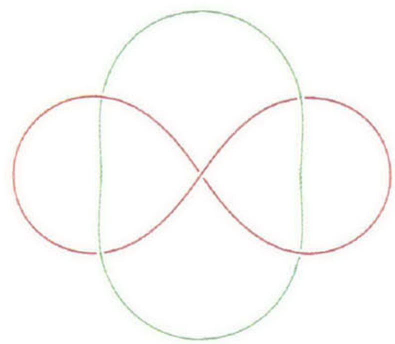


Figure 2.13: The Whitehead link has splitting number 2.

A completely split link has splitting number 0. The Hopf link has splitting number 1, as demonstrated by the standard diagram. In general, any diagram for a link L gives an upper bound on $\mathrm{sp}(L)$, as one may change crossings until the components of the link are layered one atop the next.

The Whitehead link L_W has splitting number 2—change two diagonally opposite crossings in the standard diagram (Figure 2.13). While changing the crossing in the center would split the link, that crossing is internal to one component so not allowed. To see that $\mathrm{sp}(L_W) \neq 1$, note that a crossing change between components K_c and K_d of a link L changes the linking number $\mathrm{lk}(K_c, K_d)$ by ± 1. Since the Whitehead link has linking number 0, an even number of crossing changes will be required.

If L is a two-component link with components K_1 and K_2, then the quantity

$$b_{lk}(L) := \begin{cases} |\mathrm{lk}(K_1, K_2)| & \text{if } L \text{ is non-split and } \mathrm{lk}(K_1, K_2) > 0 \\ 2 & \text{if } L \text{ is non-split and } \mathrm{lk}(K_1, K_2) = 0 \\ 0 & \text{if } L \text{ is split} \end{cases}$$

provides a lower bound on $\mathrm{sp}(L)$. If L has many components, we define

$$b_{lk}(L) := \sum_{c<d} b_{lk}(L_{cd}),$$

where L_{cd} denotes the sublink consisting of the c^{th} and d^{th} components. Since splitting a link certainly requires that one change enough crossings to split each pair of components (and each crossing implicates only one two-component sublink), we conclude that

$$\mathrm{sp}(L) \geq b_{lk}(L).$$

Our spectral sequence provides less obvious lower bound for the splitting number: the splitting number plus one is at least the index of the page on which the spectral sequence collapses.

Theorem 2.3. *Let L be a link and let $w_c \in R$ be a set of component weights such that $w_c - w_d$ is invertible for each pair of components c and d. Let $b(L, w)$ be the largest integer k such that $E_k(L, w) \neq E_\infty(L, w)$. Then $b(L, w) \leq \mathrm{sp}(L)$.*

$$\mathrm{sp}(L) \geq b(L, w)$$

Proof. We proceed by induction on splitting number. If L is a split link, then there is a diagram in which $d_1 = 0$ so the spectral sequence collapses immediately: $E_1 = E_\infty$ and $b(L) = 0$.

If L is non-split, then there is a diagram D in which changing exactly $k = \mathrm{sp}(L)$ crossings produces a diagram for a split link. Consider the diagram D' resulting by changing just one of those crossings, say i; the link L' depicted will have splitting number $k - 1$.

In the proof of Proposition 2.8, we constructed an isomorphism $f : C(D, w) \to C(D', w)$ of ℓ-graded chain complexes. To compare the filtrations and the spectral sequences, we pull back the g-grading on $C(D', w)$ to a grading on $C(D, w)$, which we write g'. In an abuse of notation, we will write g for the original g-grading on $C(D, w)$. The two corresponding filtrations on $C(D, w)$ differ in a controlled way.

Recall that for a generator x of $C(D, w)$, the relevant gradings are

$$q(x) = \deg(x) + p(I) + |I| + n_+(D) - 2n_-(D) \text{ and } g(x) = \frac{q(x) - |L|}{2}.$$

The monomial degree $\deg(x)$ and circle count $p(I)$ are the same in both D and D'. If x sits over the 0-resolution of the crossing i in D, then it sits over the 1-resolution of i in D', and vice versa. So the value of $|I|$ differs by ± 1 between the two complexes. Finally, the difference $n_+(D) - 2n_-(D)$ decreases (increases) by 3 if i is a positive (negative) crossing in D. Thus the difference in filtration degree $g'(x) - g(x)$ is in $\{-1, -2\}$ if the crossing is positive and $\{1, 2\}$ if the crossing is negative.

Since a global shift in filtration degree does not affect the page at which the corresponding spectral sequences collapses, Proposition 2.14 applies. We conclude that the spectral sequence for L collapses at most one page after the spectral sequence for L', so

$$b(L, w) \leq b(L', w) + 1 \leq \mathrm{sp}(L') + 1 = \mathrm{sp}(L). \qquad \square$$

An interesting example is the link $L = {}^2L13n3752$ shown in Figure 1.1. The two components are a trefoil and the unknot, and they have linking number 1. There is an obvious way to split the L by changing three crossings, say, pulling the red component on top of the green one. The spectral sequence with the nontrivial $\mathbb{F}_2$ weighting w, shown in Table 2.1, collapses on the E_3 page, so $\mathrm{sp}(L) \geq b(L, w) = 2$. Since $\mathrm{sp}(L)$ must have the same parity as the linking number, we have that $\mathrm{sp}(L) = 3$.

The calculation of the spectral sequence for ${}^2L13n3752$ and many other links is discussed in Section 2.7.

Table 2.1: $E_k(^2L13n3752, w)$ over $\mathbb{F}_2$ with non-trivial weight function w. $E_1(L,w) = Kh(L)$ omitted.

Link L	E_k	rank E_k	$P_k(q,t) = \sum_{i,j}(\text{rank}\, E_k^{ij})t^i q^j$
$^2L13n3752$	E_2	20	$t^{-2}q^{-2} + t^{-2} + q^2 + q^4 + t^1q^2 + t^1q^4 + t^2q^4 + t^2q^6 + t^3q^6 + t^3q^8 + 2t^4q^8 + 2t^4q^{10} + t^5q^{10} + t^5q^{12} + t^6q^{12} + t^6q^{14} + t^7q^{14} + t^7q^{16}$
	E_3	12	$t^2q^4 + t^2q^6 + 2t^4q^8 + 2t^4q^{10} + t^5q^{10} + t^5q^{12} + t^6q^{12} + t^6q^{14} + t^7q^{14} + t^7q^{16}$

2.6 Detecting unlinks

In this section, we work over a field $\mathbb{F}$ of characteristic 2. Since our construction relies on choosing different weights for different components, $\mathbb{F}_2$ itself is not large enough to accommodate many-component links. The specific choice of a larger field is unimportant, so we will write $\mathbb{F}$ for some finite field of characteristic 2 with more elements than there are components of the link under consideration. Since $Kh(L;\mathbb{F}) \cong Kh(L;\mathbb{F}_2) \otimes \mathbb{F}$, the rank of Khovanov homology is independent of the choice of $\mathbb{F}$. For this reason, we will often write $Kh(L)$ for $Kh(L;\mathbb{F})$.

Kronheimer and Mrowka have shown that Khovanov homology detects the unknot. That is, if a knot K has $Kh(K)$ of rank 2, then K is the unknot.

Corollary 2.2 provides an immediate upgrade.

Proposition 2.16. *Let L be an m-component link, and suppose that the rank of $Kh(L)$ is 2^m. Then each component of L is an unknot.*

Proof. Let $K_1, \ldots, K_m$ be the components of L. By Corollary 2.2, we have a rank inequality

$$\text{rank}\, Kh(L) \geq \text{rank}\, Kh(K_1) \times \text{rank}\, Kh(K_2) \times \cdots \times \text{rank}\, Kh(K_m).$$

The left-hand-side is 2^m. Since every knot has Khovanov homology of rank at least two, the right-hand side is at least 2^m. Hence every one of the components K_i must have $\text{rank}(Kh(K_i)) = 2$. By Kronheimer and Mrowka's result, each of those components is an unknot. $\square$

Equality is possible: the Hopf link has rank four, just like the two-component unlink. This generates a family of such examples: iterated connect-sums and disjoint unions of Hopf links and unknots. The resulting links can be described as forests of unknots: given a (planar) forest F, form a link L_F by placing an unknot at each vertex then clasping them together along each edge (Figure 2.14). By [41], we have $\text{rank}\, Kh(L_F) = \text{rank}\, Kh(U^m)$.

Question 2.17. *Are forests of unknots the only m-component links with Khovanov homology of rank 2^m over $\mathbb{F}_2$?*

None of these nontrivial links have the same bigradings at the unlink. As we will show later, this is no coincidence.

Figure 2.14: A forest F gives rise a link L_F whose Khovanov homology has the same rank as that of the unlink.

The Khovanov module

Khovanov homology is not just an abelian group: $Kh(L)$ a module over the component algebra

$$A_m = \mathbb{F}_2[X_1, \ldots, X_m]/(X_1^2, X_2^2, \ldots, X_m^2),$$

see [15]. The module structure is defined by choosing marked points p_c on each component c of L. Then X_c acts by X_{p_c}.

In fact, this module structure extends to all the pages E_k. The map X_{p_c} shifts the g gradings by -1, so it preserves the filtration $\mathcal{F}$. It remains to show that X_p for a marked point p is a chain map with respect to the total differential d and that the module structure induced on E_k is independent of the choice of marked points.

Proposition 2.18. *Let p be a marked point on D away from the double points. Then we have that $dX_p = X_p d$.*

Proof. X_p commutes the Khovanov edge maps; this is the standard proof that it commutes with d_0. The deformation d_1 is also a sum of edge maps, so the proposition follows. □

Proposition 2.19. *Let p and q be marked points on either side of crossing i as shown in Figure 2.5. Then X_p and X_q are chain homotopic endomorphisms of $C(D, w)$.*

Proof. We use the same chain homotopy H as in the proof that the Khovanov module is well-defined [15]:

$$H = \sum_{\substack{\text{resolutions } I \\ I(i)=1}} \mathcal{A}(I, J_i),$$

where J_i differs from I solely at i. Hedden and Ni show $X_p + X_q = Hd_0 + d_0 H$. It remains for us to show that $Hd_1 + d_1 H = 0$. This is an immediate consequence of the facts that H and d_1 both decrease homological grading and that the reverse edge maps commute. □

Since we use $\mathbb{F}$ coefficients to define the complex $C(D, w)$, we will first prove results regarding the action of $A_m^{\mathbb{F}} := A_m \otimes \mathbb{F}$. The Khovanov module of the unknot is just a copy of $A_1^{\mathbb{F}}$ viewed as a module over itself: $Kh(U) \cong \mathbb{F}[X]/(X^2)$. Disjoint union of links gives tensor products of modules, over the tensor product algebra; in particular, $Kh(U^m) \cong A_m^{\mathbb{F}}$.

Proposition 2.20. *Let L be a m-component link with* $\operatorname{rank} Kh(L) = 2^m$*. If D is any diagram for L and w any set of distinct weights for the components, then the total homology $H_*(C(D,w))$ is a free rank-one module over the algebra $A_m^{\mathbb{F}}$.*

Proof. By Proposition 2.16, the components of L are all unknots. Number the components from 1 to m. Since we are only interested total homology and its module structure, we can ignore the g-filtration on $C(D,w)$. We can produce a diagram D' for U^m by swapping mixed crossings in D so that at each crossing, the under-strand has lower index than the over-strand. As we saw in the proof of Proposition 2.8, $C(D,w)$ and $C(D',w)$ differ only by rescaling generators by elements of $\mathbb{F}$. The action of X_{c_p} commutes with rescaling generators. The total homology and $A_m^{\mathbb{F}}$ action are also invariant under Reidemeister and marked point moves. By such moves, D' can be transformed into D'', the standard diagram for U^m with no crossings: a disjoint collection of circles with marks. The complex for D'' has vanishing differential, and $H_*(C(D'',w))$ is manifestly a free rank-one $A_m^{\mathbb{F}}$-module, as desired. □

Proof of Theorem 1.2

Hedden and Ni have shown that the module structure of Kh detects the unlink [15].

Theorem 2.21 (Hedden-Ni). *Let L be an m-component link. If there is an isomorphism of A_m modules*

$$Kh(L;\mathbb{F}_2) \cong A_m,$$

then L is the unlink.

We can deduce the module structure from the bigradings.

Theorem 1.2. *Let L be an m-component link, and U^m the m-component unlink. If*

$$\operatorname{rank} Kh^{i,j}(L;\mathbb{F}_2) = \operatorname{rank} Kh^{i,j}(U^m;\mathbb{F}_2)$$

for all i,j, then L is the unlink.

Proof. The Khovanov homology of the unlink is supported entirely in homological grading 0, where it has rank $\binom{m}{r}$ in quantum grading $2r-m$. Since our spectral sequence is graded by $g=(q-m)/2$, the group

$$E_1^{-k}(L,w) \cong Kh^{0,m-2k}(L)$$

has rank $\binom{m}{k}$ for $0 \le k \le m$.

As described in Section 2.4, there is a filtration $\mathcal{G}$ on the total homology $H = H_*(C(D,w))$ with respect to which

$$E_\infty^{-k} \cong \mathcal{G}^{-k}H/\mathcal{G}^{-k-1}H\ .$$

Since the spectral sequence collapses with $E_1 = E_\infty$, this determines the rank of each filtered piece,

$$\operatorname{rank}_{\mathbb{F}} \mathcal{G}^{-k}H = \binom{m}{k} + \binom{m}{k+1} + \cdots + \binom{m}{m}$$

Let I be the (maximal) ideal in $A^{\mathbb{F}}_m$ generated by the X_i. The top nonvanishing power of the ideal is I^m, which is spanned by the element $X_1X_2\cdots X_m$, and we have $I^{m+1}=0$. Consider the filtration

$$0\subset I^m\subset I^{m-1}\subset\cdots\subset I\subset A^{\mathbb{F}}_m.$$

By Proposition 2.20, the total homology is a free rank-one module over $A^{\mathbb{F}}_m$, generated by some $e\in\mathcal{G}^0H\cong H$. Moreover, since each endomorphism X_i lowers the g-grading by 1, it takes $\mathcal{G}^{-k}$ into $\mathcal{G}^{-k-1}$. Hence

$$I^ke\subset\mathcal{G}^{-k}H$$

for every $0\le k\le m$.

Since e is the generator of a free $A^{\mathbb{F}}_m$-module, we know that I^ke actually has the same rank as I^k itself, which is the same as the rank of $\mathcal{G}^{-k}H$ computed above. Hence $I^ke=\mathcal{G}^{-k}H$.

The associated graded module is

$$\bigoplus_k I^ke/I^{k+1}e\cong\bigoplus_k A^{\mathbb{F}}_m[k]e,$$

where $A^{\mathbb{F}}_m[k]$ denotes the linear span of the monomials of degree k in the X_i. This is isomorphic to $A^{\mathbb{F}}_m$ itself, viewed as an $A^{\mathbb{F}}_m$-module. But $E_\infty(L,w)\cong E_1(L,w)\cong Kh(L)$. So $Kh(L)$ is a free, rank-one $A^{\mathbb{F}}_m$ module.

More precisely, $Kh(L;\mathbb{F})$ is a free, rank-one $\mathbb{F}[X_1,\ldots,X_n]/X_i^2$-module. To apply Hedden-Ni, and conclude that L is the unlink, we need to show that $Kh(L;\mathbb{F}_2)$ is a free, rank-one $\mathbb{F}_2[X_1,\ldots,X_n]/X_i^2$-module. In general, extending the ground field can make a free module out of a non-free one [2]. This cannot happen for A_m-modules, essentially because A_m is a local ring.

Indeed, suppose that M is a module over A_m such that $M_{\mathbb{F}}=M\otimes\mathbb{F}$ is a free rank-one module over $A_m\otimes\mathbb{F}$. Let $a\in M_{\mathbb{F}}$ be a generator, so the $\mathbb{F}$-span of A_ma is all of $M_{\mathbb{F}}$. Now pick some element b of the original module M such that $b\notin I\cdot M_{\mathbb{F}}$. Then $b=\alpha(1+X)a$ where $\alpha\in\mathbb{F}$ and $X\in I$. Because I is nilpotent of order $m+1$, the coefficient $\alpha(1+X)$ is a unit with inverse $\alpha^{-1}(1-X+X^2+\cdots\pm X^m)$. Thus b is also a generator for $M_{\mathbb{F}}$ as a free $A_m\otimes\mathbb{F}$-module. In particular, b is not annihilated by any element of A_m. This means that

$$\operatorname{rank}_{\mathbb{F}_2}A_mb=2^m=\operatorname{rank}_{\mathbb{F}}M_{\mathbb{F}}=\operatorname{rank}_{\mathbb{F}_2}M.$$

Hence A_mb is all of M, and M is a free A_m-module. □

Example. It is instructive to see where this argument breaks down for the Hopf link. There, $Kh(L)=E_\infty$ has total rank four, with rank-one summands in g-degrees $-1,0,1,2$. Thus the filtration of the rank-1 free A_n-module $H^*(C(D,w))$ has ranks

$$1<2<3<4.$$

Let e be a generator, and write $x_i \dots x_j$ for $X_i \dots X_j e$. The filtration is

$$\langle x_1 x_2 \rangle \subset \langle x_1 x_2, x_1 + x_2 \rangle \subset \langle x_1 x_2, x_1, x_2 \rangle \subset \langle x_1 x_2, x_1, x_2, 1 \rangle$$

The associated graded has a nonstandard module structure:

$$\langle a, b | X_1 a = X_2 a, X_1 b = X_2 b \rangle.$$

In contrast, the two component unlink has a filtration of ranks $1 < 3 < 4$, and the associated graded is isomorphic A_2 itself.

2.7 Sample computations

The combinatorial definition of the spectral sequence makes it amenable to computer calculation. We use `knotkit`, a C++ software package for knot homology computations written by Seed, to compute the spectral sequence for thousands of links[1] [38].

These computations show that the spectral sequence is not determined by the Khovanov homology of the links involved. The links ${}^2L12n817$ and ${}^2L14n38362$ have the same Khovanov homology, and each has two unknot components (see Figure 2.15). Yet the spectral sequences collapse on different pages (E_2 vs E_3).

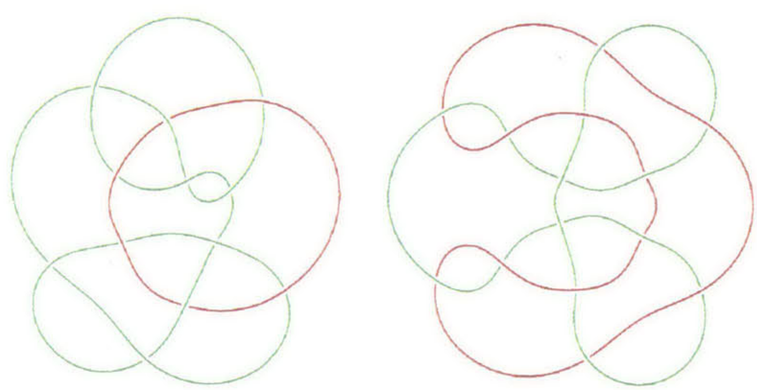


Figure 2.15: Links ${}^2L12n817$ (left) and ${}^2L14n38362$ (right).

Table 2.2: The link splitting spectral sequence $E_k(L)$ over $R = \mathbb{Z}_2(x_1, \dots, x_m)$ with weight function $w_c = x_c$ for examples in this section. $E_1(L) = Kh(L)$ omitted.

Link L	E_k	rank E_k	$P_k(q,t) = \sum_{i,j} (\text{rank } E_k^{ij}) t^i q^j$

[1]Links are named according to the convention used in LinkInfo [5] and SnapPy [6]: ${}^xLy[a/n]z$ denotes the z^{th} alternating/nonalternating link with y crossings; x is the number of components.

${}^2L12n817$	E_2	4	$q^{-2}+2+q^2$
${}^2L14n38362$	E_2	68	$t^{-5}q^{-10}+t^{-5}q^{-8}+2t^{-4}q^{-8}+2t^{-4}q^{-6}+2t^{-3}q^{-6}+2t^{-3}q^{-4}+t^{-2}q^{-6}+4t^{-2}q^{-4}+3t^{-2}q^{-2}+2t^{-1}q^{-4}+5t^{-1}q^{-2}+3t^{-1}+3q^{-2}+6+3q^2+3t^1+5t^1q^2+2t^1q^4+3t^2q^2+4t^2q^4+t^2q^6+2t^3q^4+2t^3q^6+2t^4q^6+2t^4q^8+t^5q^8+t^5q^{10}$
	E_3	4	$q^{-2}+2+q^2$

We have computed splitting number bounds for all links with 12 or fewer crossings in the Morwen hyperbolic link tables from SnapPy [6]. Some choices and approximations must be made, which we describe before giving the results.

We use two coefficient rings, $\mathbb{P}=\mathbb{Z}/2(x)$ and $\mathbb{Q}$. For the former, we weight component c by $w_c=x^c$, and for the latter, we weight component c by the integer c itself.

Since **knotkit** is not currently able to detect split links, we need an approximation to the bound coming from the linking number, b_{lk}. Since the link table contains only non-split links, there is no problem for two-component links. But non-split links with more than two components, such as the Borromean rings, may have split sublinks. We define b'_{lk} as follows: If L has two components and is known to be non-split, we set $b'_{lk}(L)=b_{\text{lk}}(L)$. If $L=K_1\cup K_2$ may be split, then we define

$$b'_{lk}(L)=\begin{cases}|lk(K_1,K_2)| & lk(K_1,K_2)\neq 0\\ 2 & Kh(L;\mathbb{Z}/2)\ncong Kh(K_1\amalg K_2;\mathbb{Z}/2)\\ 0 & \text{otherwise}\end{cases}.$$

If L has more than two components and is non-split, we define

$$b'_{lk}(L)=\max\Big(\sum_{i<j} b'_{lk}(L_{ij}),2\Big),$$

where L_{ij} is the sublink of L consisting of the i^{th} and j^{th} components.

Any diagram D for a link L gives an upper bound on the splitting number. Number the components of L from 1 to m. Let $\sigma\in S_m$ be a permutation of the components. We can produce a diagram D' for a split link by swapping the $u(D,\sigma)$ crossings of D where $\sigma(c_{\text{upper}})<\sigma(c_{\text{lower}})$. Let $u(D)$ be the minimum of $u(D,\sigma)$ over all σ, so $u(D)$ is a upper bound for $\text{sp}(L)$.

We computed

$$b'_{lk}(L),b^{\mathbb{Q}}(L,w^{\mathbb{Q}}),b^{\mathbb{P}}(L,w^{\mathbb{P}}),u(D),$$

for all 5698 links in the Morwen link table with 12 or fewer crossings, where D is the tabulated minimal diagram. Of those links, 4770 (83.7%) have non-trivial lower bounds for $\text{sp}(L)$ and the lower bound is known to be tight for 3587 (63%) links. Our upper bound is very rough, so the lower bound is likely to be tight in many more cases. The bound coming from the spectral sequence is stronger than the linking number bound for 17 of those links and equal to it for 2421. The examples with $b > b'_{lk}$ are shown Table 2.3. For those 17 examples, we verified by hand that $b'_{lk} = b_{lk}$.

Table 2.3: Knots with 12 of fewer crossings for which $b'_{lk}(L) < b(L)$. $u(D)$ gives the upper bound on $\mathrm{sp}(L)$.

Link L	$b_{\mathrm{lk}}(L)$	$b^{\mathbb{P}}(L)$	$b^{\mathbb{Q}}(L)$	$\mathrm{sp}(L)$
$^2L12n1342$	1	0	2	3
$^2L12n1350$	1	1	2	3
$^2L12n1357$	1	1	2	3
$^2L12n1363$	1	1	2	3
$^2L12n1367$	1	1	2	3
$^2L12n1374$	1	1	2	3
$^2L12n1404$	1	1	2	3
$^2L11a372$	1	1	2	3–5
$^2L12a1233$	1	1	2	3–5
$^2L12a1264$	1	1	2	3–5
$^2L12a1384$	1	1	2	3–5
$^2L12n1319$	1	2	2	3–5
$^2L12n1320$	1	2	2	3–5
$^2L12n1321$	1	2	2	3–5
$^2L12n1323$	1	2	2	3–5
$^2L12n1326$	1	2	2	3–5
$^3L12a1622$	1	1	2	3–5

Chapter 3

Nonorientable slice genus

In this chapter, we prove a new lower bound on the nonorientable slice genus. We begin with a knot $K \subset S^3$ bounding a nonorientable surface $F \subset B^4$. In §3.1, we recall the relationship between the first Betti number of F, the Euler number of F, and the signature of K. In §3.2, we give a delicate topological construction of an orientable replacement $F' \subset S^2 \times S^2 \backslash B^4$. In §3.3, we describe the parts of the Heegaard Floer homology package necessary to give a lower bound on the (orientable) genus of F' and prove our main inequality, Theorem 1.6. In §3.4, we compute the requisite signature and d-invariant for torus knots, and precisely compute the nonorientable slice genus of $T_{2k,2k-1}$.

3.1 The signature inequality

One of the major differences between orientable surfaces and nonorientable surfaces in four-space is that the latter can have nontrivial normal bundles. The normal bundle of a closed surface $F \subset B^4$ is determined by its (twisted) Euler number. This can be computed as a geometric self-intersection: take a transverse pushoff of F, and choose arbitrary orientations in the neighborhood of each intersection point. Together with the orientation of $\mathbb{R}^4$, this allows us to assign signs to each intersection; the sum, $e(F)$, turns out to be independent of the choice of pushoff and local orientation. If F is orientable, then it represents an integral homology class and self-intersection can be computed algebraically; since $H_2(B^4; \mathbb{Z})$ vanishes $e(F)$ must be zero. If F is nonorientable, then we may still compute self-intersection algebraically over $\mathbb{Z}/2$. Hence $e(F)$ must be even, though it need not be zero. For example, the $\mathbb{RP}^2$ in Figure 3.4 has normal Euler number 2.

If F has boundary a knot $K \subset S^3$, then it has a relative normal Euler number. One way to compute $e(F)$ is to take an orientable Seifert surface Σ for K and find the Euler number of the closed surface $\Sigma \cup_K F$. Alternatively, one may take a nonvanishing section s of the normal bundle $\nu(F)$ (one exists since F retracts to its 1-skeleton). The restriction of s to the boundary provides a framing of K, and

$$e(F) = -\text{lk}(K, s(K)).$$

For example, the obvious Möbius band with boundary $T_{2,n}$ has normal Euler number $-2n$.

We now sketch a proof of the signature inequality Theorem 1.9. Let $F \subset B^4$ be a surface with boundary $K \subset S^3$. Let $W(F)$ denote the double cover of B^4 branched over F. Gordon and Litherland [14] use the G-signature theorem to show that the quantity

$$\sigma(K) := \sigma(W(F)) + \frac{e(F)}{2}$$

is independent of the choice of surface F with boundary K. For any such F then,

$$\left|\sigma(K) - \frac{e(F)}{2}\right| = |\sigma(W(F))| \leq b_2(W(F)) = b_1(F).$$

This inequality is tight for the two natural surfaces with boundary $T_{2,n}$, which has signature $-(n-1)$. The Seifert surface has $e(F) = 0$ and $b_1(F) = n - 1$, while the Möbius band has $e(F) = -2n$ and $b_1(F) = 1$. The surfaces in Figure 1.3 can be constructed by taking the Möbius band or Seifert surface in S^3, pushing it into the four-ball, then doubling across the boundary.

In light of the important role played by $e(F)$, it may be clarifying to sort surfaces based on the framing they induce on the knot and try to minimize b_1 separately in each class. The signature inequality can be interpreted as stating that $2\sigma(K)$ is a 'preferred framing' for the knot K, deviation from which requires a more complex surface.

3.2 Constructing an orientable replacement

In this section, we prove

Proposition 1.7. *Let $F \subset B^4$ be a smoothly embedded nonorientable surface with odd b_1 with boundery a knot $K \subset S^3$. Then there exists an orientable surface $F' \subset S^2 \times S^2 \backslash B^4$ which still bounds K, and has $b_1(F') = b_1(F) - 1$ and $e(F') = e(F) + 2$.*

Proof. We break the proof into four steps.

Step 1: There is an embedded disk $D \subset B^4$, with boundary contained in F, such that $F \backslash \partial D$ is orientable.

Since F has odd b_1, it is diffeomorphic to a punctured orientable surface boundary-connect summed with a Möbius band (Figure 3.1). Let $\gamma \subset F$ be the core of the Möbius band, so that $F - \gamma$ is orientable. (The class $[\gamma]$ is Poincaré dual to w_1.)

After an ambient isotopy, we may arrange that γ lies in a small sphere $S^3_\epsilon \subset B^4$. Think of γ a knot, with some projection. Changing a few crossings will convert γ to the unknot U, and this can be realized as an isotopy in B^4 pushing some strands a little outside of S^3_ϵ, over, then back down. The unknot bounds an embedded disk in S^ϵ_3, which we push in towards the center of B^4; that disk together with the trace of the isotopy is our embedded D. After a generic perturbation, D will intersect F transversely on its interior.

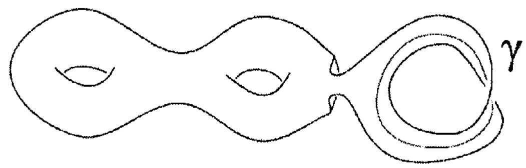


Figure 3.1: The surface F

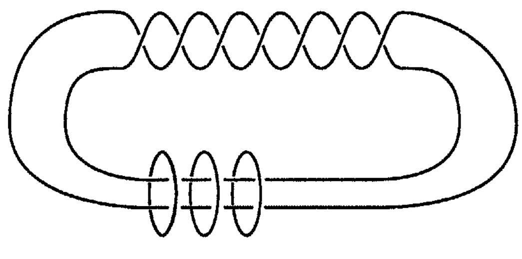


Figure 3.2: Our surface F intersects ∂N in a torus knot and an unlink

Let N be a small regular neighborhood of D.

Step 2: The intersection $\partial N \cap F$ is the link $L \subset S^3$ shown in Figure 3.2.

N is diffeomorphic to $D \times D^2$, and intersects our surface F in a Möbius band (in the neighborhood of $\partial D = C$) and a collection of disks $pt \times D^2$ (neighborhoods of the transverse intersections of F with the interior of D). If we draw $S^3 = \partial N$ with its standard decomposition into solid tori $S^3 \cong S^1 \times D^2 \cup_{T^2} D^2 \times S^1$, we see $F \cap \partial N$ as the link L consisting of a $(2, 2k+1)$-cable of the core of the first factor, together with a collection of l longitudes for the second. By construction, L bounds a Möbius band disjoint union a collection of l disks in $N \cong B^4$.

Step 3: L bounds $l+1$ disjoint embedded disks in $S^2 \times S^2 \backslash B^4$

Suzuki has shown that every link is slice in $S^2 \times S^2 \backslash B^4$ [42]. Since we need control over the Euler numbers, we construct the slice disks explicitly.

A handle decomposition for $S^2 \times S^2 \backslash B^4$ consists of two zero-framed 2-handles H_1 and H_2 attached along a Hopf link in the boundary S^3, together with a 4-handle. To construct the slice disks for L, we begin with $|k| + l$ parallel copies of the core of H_2 and 2 parallel copies of the core of H_1–their boundaries form a multi-Hopf link, with components $U_1, \cdots, U_{|k|+l}$, L_1, L_2, as in the first frame of Figure 3.3. For each $1 \le i \le |k|$, connect U_i to L_1 with a twisted strip, and with one additional twisted strip, connect V_1 to V_2 . Call the surface consisting of the parallel cores and the strips E, and note that the boundary of E is isotopic to L. Since each strip connects a distinct disk to L_1, we see that E is a collection of $l+1$ disjoint embedded disks with boundary L.

Step 4: Construct F', and compute $b_1(F')$ and $e(F')$.

If we excise N from B^4, we are left with an orientable surface $F'' \subset S^3 \times [0,1]$, with

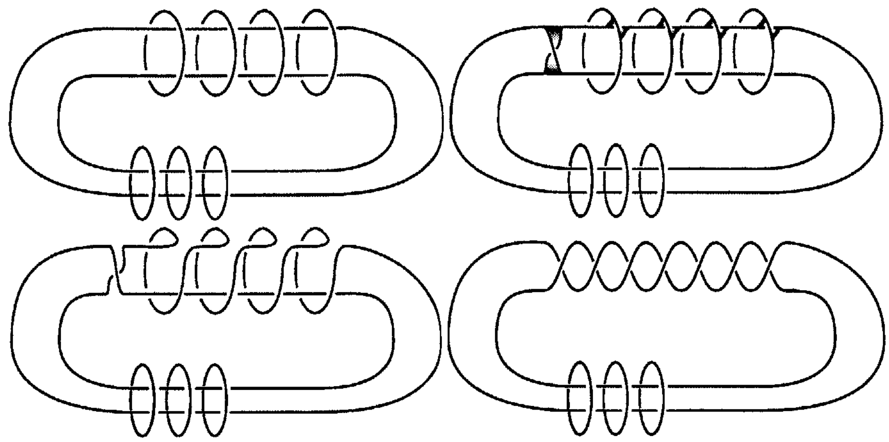


Figure 3.3: For $k = -7$, $l = 3$, we have drawn the multihopf link bounding a collection of parallel disks, the strips which join them to form E, and the boundary of E, which is isotopic to L.

boundary K in $S^3 \times \{0\}$ and L in $S^3 \times \{1\}$. Attach $S^2 \times S^2 \backslash B^4$ along $S^3 \times \{1\}$ to form a larger manifold, still diffeomorphic to $S^2 \times S^2 \backslash B^4$. The slice disks E for L combine with F'' to form an orientable surface F', whose only remaining boundary is the original knot K.

Since we have removed l disks and an Möbius band from F, and replaced them with $l+1$ disks, $b_1(F') = b_1(F) - 1$. It remains to compare the normal Euler numbers. Since F'' is orientable and $H_2(S^3 \times [0,1])$ vanishes, F'' has trivial relative normal bundle. Thus $e(F) = e(F \cap N)$ and $e(F') = e(E)$. The disks in $F \cap N$ with boundary the unknots $U_1, \ldots, U_l$ have trivial self-intersection as well, since they can be added back into F while maintaining orientability. The disks in E with boundary the unknots are attached with framing 0, so they also contribute nothing to the Euler number. The nontrivial contributions come from the Möbius band in $F \cap N$ with boundary the torus knot component of L and the interesting disk in E. We invite the reader to verify that the induced framings differ by 2, due to the difference between the vertical twisted strip connecting V_1 to V_2 in E and the horizontal one in Möbius band. That is,

$$e(F') = e(E) = e(F \cap N) + 2 = e(F) + 2.$$

Since the framing induced by the Möbius band with boundary $T_{2,2k+1}$ is $-2(2k+1)$, the number of twists is not arbitrary: $2k + 1 = -\frac{e(F)}{2}$. □

For future reference, we note that the homology class $[F'] \in H_2(S^2 \times S^2 \backslash B^4)$ is $(2, m)$, in the basis given by H_1 and H_2. Since F' is orientable, its algebraic self-intersection number, $4m$, must be equal to its geometric self-intersection number, $e(F')$.

3.3 d-invariants

Heegaard Floer homology associates to a 3-manifold Y equipped with a Spin^c structure $\mathfrak{t}$ a suite of $\mathbb{Z}[U]$-modules which fit into a long exact sequence:

$$\cdots \to HF^-(Y,\mathfrak{t}) \xrightarrow{i} HF^\infty(Y,\mathfrak{t}) \xrightarrow{\pi} HF^+(Y,\mathfrak{t}) \xrightarrow{\delta} HF^-(Y,\mathfrak{t}) \to \cdots$$

If $c_1(\mathfrak{t})$ is torsion (in which case we also say that $\mathfrak{t}$ is torsion), then there is a $\mathbb{Q}$-grading gr on the each of these groups which is respected by i and π. The action of U decreases grading by 2. If Y is a rational homology sphere, then every Spin^c structure $\mathfrak{t}$ is torsion, and $HF^\infty(Y,\mathfrak{t}) \cong \mathbb{Z}[U,U^{-1}]$. In that case, the d-invariant $d(Y,\mathfrak{t})$ is the minimal grading of a non-$\mathbb{Z}$-torsion element of $HF^+(Y,\mathfrak{t})$ in the image of π.

If $b_1(Y) > 0$, then there is an additional action of $H := H_1(Y)/\text{Tors}$ on the HF° groups, which decreases grading by 1. If for every torsion $\mathfrak{t} \in \text{Spin}^c(Y)$, we have $HF^\infty(Y,\mathfrak{t}) \cong \mathbb{Z}[U,U^{-1}] \otimes_{\mathbb{Z}} \Lambda^* H$, then we say that Y has standard HF^∞. In that case, there are many correction terms, one for each generator of $\Lambda^* H$. We will be concerned with the bottom-most correction term, $d_b(Y,\mathfrak{t})$, defined to be the minimal grading of a nontorsion element of $HF^+(Y,\mathfrak{t})$ in the image of π *and* in the kernel of the H-action. The d-invariants terms will be useful to us because of their relationship to definite cobordisms.

Proposition 3.1. *[30] Let Y be a closed oriented 3-manifold (not necessarily connected) with standard HF^∞, endowed with a torsion Spin^c structure $\mathfrak{t}$. If X is a negative semi-definite four-manifold with boundary Y such that the restriction map $H^1(X;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial, and $\mathfrak{s}$ is a Spin^c structure on X restricting to $\mathfrak{t}$ on Y, then*

$$c_1(\mathfrak{s})^2 + b_2^-(X) \leq 4d_b(Y,\mathfrak{t}) + 2b_1(Y).$$

We are now ready to prove our main inequality.

Theorem 1.6. *Suppose that $K \subset S^3$ bounds a smoothly embedded, nonorientable surface $F \subset B^4$. Then*

$$b_1(F) \geq \frac{e(F)}{2} - 2d\left(S^3_{-1}(K)\right).$$

Proof. Both sides of the inequality change by the same amount under a positive real 'blow-up.' More precisely, if we connect sum $F \subset B^4$ with the embedding of $\mathbb{RP}^2 \subset S^4$ with Euler number $+2$ pictured in Figure 3.4, then both b_1 and $e/2$ increase by 1. After blowing up sufficiently many times, we may assume that $e(F) > 0$ and $b_1(F)$ is odd.

Using Proposition 1.7, we construct an orientable surface $F' \subset S^2 \times S^2 \backslash B^4$ with boundary $K \subset S^3$. Attach a -1-framed 2-handle along K to form a 4-manifold $\overline{W}$ with boundary $S^3_{-1}(K)$ and intersection form

$$Q_{\overline{W}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

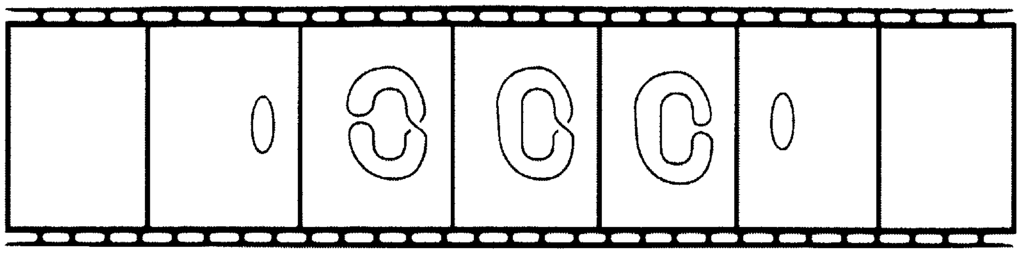


Figure 3.4: An embedded $\mathbb{RP}^2$ with normal Euler number 2.

We may cap off F' with the core of the 2-handle to form a closed surface Σ with genus $g = (b_1(F) - 1)/2$, homology class $(1, 2, m)$, and self-intersection

$$n := 4m - 1 = e(F) + 1 > 0.$$

If we decompose $\overline{W} = \nu(\Sigma) \cup W$, then W will be a negative semi-definite cobordism from $Y_{g,n}$, the Euler number n circle bundle over Σ, to $S^3_{-1}(K)$. Alternatively, we can view W as a negative semi-definite four-manifold with disconnected boundary $Y_{g,-n} \coprod S^3_{-1}(K)$. We need to understand the homology, HF^∞, and d-invariants of $Y_{g,n}$ and $S^3_{-1}(K)$, and the intersection form on W.

The Gysin sequence for the disk bundle $\nu(\Sigma)$ gives

$$\begin{aligned} 0 \to H^1(\nu(\Sigma)) \to H^1(Y_{g,n}) \to H^2(\nu(\Sigma), Y_{g,n}) \\ \xrightarrow{e} H^2(\nu(\Sigma)) \to H^2(Y_{g,n}) \to H^1(\Sigma) \to 0 \end{aligned}$$

where $e \in H^2(\nu(\Sigma)) \cong \mathbb{Z}$ is n times the generator. Thus $H^2(Y_{g,n}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}/n$. Note that the restriction of $H^1(\nu(\Sigma))$ to $H^1(Y_{g,n})$ is an isomorphism. Since $H^1(\overline{W}) = 0$ (no 1-handles were used in its construction), the Mayer-Vietoris sequence

$$\begin{aligned} 0 \to H^1(\overline{W}) \to H^1(\nu(\Sigma)) \oplus H^1(W) \to H^1(Y_{g,n}) \\ \to H^2(\overline{W}) \to H^2(\nu(\Sigma)) \oplus H^2(W) \to H^2(Y_{g,n}) \end{aligned}$$

shows that $H^1(W) = 0$, trivially satisfying the restriction hypothesis of Proposition 3.1. Since $H^2(\overline{W}) \cong \mathbb{Z}^3$ has no 2-torsion, a Spin^c structure on $\overline{W}$ is determined by its first Chern class. Any Spin^c structure on W will give us some inequality between d-invariants, but we will only need to consider a certain Spin^c structure $\mathfrak{s}_t$ with $PD(c_1(\mathfrak{s}_t)) = (\pm 1, 2, 2a)$, where

$$a = \frac{2(m-g) - 1 \pm 1}{4}$$

and the sign is chosen so as to make a an integer. The given vector is characteristic for $Q_{\overline{W}}$, so does correspond to a Spin^c structure. Crucially for our later use, $c_1(\mathfrak{s}_t)$ evaluates to $n - 2g$

on Σ.

To compute the c_1^2 term in the proposition, we decompose the intersection form of $\overline{W}$ in terms of the $\mathbb{Q}$-valued intersection forms on $\nu(\Sigma)$ and W: if $c \in H^2(\overline{W})$, then

$$Q_{\overline{W}}(c) = Q_{\nu(\Sigma)}\left(c|_{\nu(\Sigma)}\right) + Q_W\left(c|_W\right).$$

A generator of $H^2(\nu(\Sigma), Y_{g,n})$ maps to n times the generator of $H^2(\nu(\Sigma))$ in the gysin sequence above, so $Q_{\nu(\Sigma)} = \left(\frac{1}{n}\right)$. The value of $c|_{\nu(\Sigma)} \in H^2(\nu(\Sigma))$ is determined by integrating it over Σ, giving

$$Q_{\overline{W}}(c) = \frac{\langle c, [\Sigma]\rangle^2}{n} + Q_W\left(c|_W\right). \tag{3.3.1}$$

In our case,

$$c_1(\mathfrak{s}_t|_W)^2 = Q_{\overline{W}}(c_1(\mathfrak{s}_t)) - \frac{\langle c_1(\mathfrak{s}_t), [\Sigma]\rangle^2}{n} = -1 + 8a - \frac{(n-2g)^2}{n} = -2 \pm 2 - \frac{4g^2}{n}.$$

The relevant d-invariant of $Y_{g,-n}$ is computed in section 9 of [30], for use in their proof of the Thom conjecture. If $n > 2g$, then

$$d_b\left(Y_{g,-n}, \mathfrak{s}_t|_{Y_{g,-n}}\right) = \frac{1}{4} - \frac{g^2}{n} - \frac{n}{4}.$$

That calculation uses the integer surgeries exact sequence associated to the Borromean knot in $K \subset \#^{2g} S^1 \times S^2$: the $-n$ surgery on K gives $Y_{g,-n}$. Since $\#^{2g} S^1 \times S^2$ has standard HF^∞, so does $Y_{g,-n}$ (cf. Proposition 9.4 of [30]). Finally, since $S^3_{-1}(K)$ is an integer homology sphere, it also has standard HF^∞.

We may now apply Proposition 3.1, to get

$$c_1(\mathfrak{s}_t)^2 + b_2^-(W) \leq 4d_b(Y_{g,-n}, \mathfrak{t}) + 4d(S^3_{-1}(K)) + 2b_1(Y_{g,-n}) + 2b_1(S^3_{-1}(K)).$$

After substituting all the values computed above, this reduces to

$$\left(-2 \pm 2 - \frac{4g^2}{n}\right) + 2 \leq 4\left(\frac{1}{4} - \frac{g^2}{n} - \frac{n}{4}\right) + 4d\left(S^3_{-1}(K)\right) + 2(2g).$$

If we take the unfavorable sign on ± 2, and recall that $b_1(F) = 2g+1$ and $e(F) + 1 = n$, we get the inequality

$$\frac{e(F)}{2} \leq 2d\left(S^3_{-1}(K)\right) + b_1(F). \tag{3.3.2}$$

This argument relied on a value for $d_b(Y_{g,-n})$ only valid if $n > 2g$. But the term $d(S^3_{-1}(K))$ is nonnegative, as can be seen by applying Proposition 3.1, applied to the surgery cobordism $S^3 \to S^3_{-1}(K)$. So if $n \leq 2g$, and consequently $e(F) + 2 \leq b_1(F)$, then Equation 3.3.2 still holds. □

Remark 3.2. Our final lower bound on the nonorientable four-ball genus is the gap $\frac{\sigma(K)}{2}$ −

$d(S^3_{-1}(K))$. For alternating knots, this quantity is nonpositive—in [31], Ozsváth and Szabó show that

$$d\left(S^3_{-1}(K)\right) = \max\left(0, 2\left\lceil\frac{\sigma(K)}{4}\right\rceil\right)$$

For nonalternating knots, $\frac{\sigma(-)}{2}$ and $d(S^3_{-1}(-))$ can diverge widely, though both invariants satisfy a crossing-change inequality [36]:

$$\eta(K_+) \leq \eta(K_-) \leq \eta(K_+) + 2.$$

If K becomes alternating after c crossing changes, then $\frac{\sigma(K)}{2} - d(S^3_{-1}(K))$ can be as large as $2c$.

3.4 Torus knots

In the section, we apply Theorem 1.6 to torus knots, whose signatures and d-invariants are easy to compute. In particular, we will compute the nonorientable slice genus of the family $T_{2k,2k-1}$. Since any surface with boundary $T_{2k,2k-1}$ may be reflected to get a surface with boundary $T_{-2k,2k-1}$, we will focus on the latter (negative) knot.

Signatures of torus knots satisfy a recursion relation [29]. If $\sigma(p,q) := \sigma(T_{-p,q})$, then

$$\sigma(p,q) = \begin{cases} \sigma(q,p) & \text{if } q > p \\ \sigma(p-2q,q) + q^2\ (-1) & \text{if } 2q < p\ (q \text{ odd}) \\ -\sigma(2q-p,p) + q^2 - 2\ (+1) & \text{if } 2q > p\ (q \text{ odd}) \\ p-1 & \text{if } q = 2 \\ 0 & \text{if } q = 1 \end{cases}$$

Let $\sigma_k := \sigma(T_{-2k,2k-1}) = \sigma(2k, 2k-1)$. Applying the first and third conditions twice, we arrive at the recursion

$$\sigma_k = 4k - 2 + \sigma_{k-1},$$

whence $\sigma_k = 2k^2 - 2$.

The d-invariants of torus knots are also simple to compute, since they admit lens space surgeries.

Proposition 3.3. *[30] Let K be a knot admitting a positive lens space surgery. Then*

$$d_{-1/2}(S^3_0(K)) = -\frac{1}{2} \qquad \textit{and} \qquad d_{1/2}(S^3_0(K)) = \frac{1}{2} - 2t_0$$

where if

$$\Delta_K(T) = a_0 + \sum_{j=1}^{d} a_j(T^j + T^{-j})$$

then

$$t_0 = \sum_{j=1}^{d} j a_j.$$

The d-invariants of zero-surgery are related to those of ± 1-surgery via Proposition 4.12 of [30]:

$$d\left(S^3_{-1}(K)\right) = d_{-1/2}\left(S^3_0(K)\right) + \frac{1}{2} \qquad d\left(S^3_1(K)\right) = d_{1/2}\left(S^3_0(K)\right) - \frac{1}{2}.$$

Since $T_{p,q}$ admits a positive lens space surgery, we have

$$d\left(S^3_{-1}\left(T_{-p,q}\right)\right) = -d\left(S^3_1\left(T_{p,q}\right)\right) = -\left(d_{1/2}\left(S^3_0\left(T_{p,q}\right)\right) - \frac{1}{2}\right) = 2t_0.$$

The Alexander polynomial of $T_{p,q}$ is

$$\Delta_{T_{p,q}}(T) = T^{-(p-1)(q-1)/2}\frac{(1-T)(1-T^{pq})}{(1-T^p)(1-T^q)}.$$

For torus knots $T_{2k,2k-1}$, the Alexander polynomial has a simple form:

$$\Delta_{T_{2k,2k-1}} = \sum_{j=1}^{k-1} T^{j(2k-1)} - T^{j(2k-1)-(k-j)} + T^{-j(2k-1)} - T^{-j(2k-1)+(k-j)}$$

so

$$t_0 = \sum_{j=1}^{k-1} j(2k-1) - (j(2k-1) - (k-j)) = \sum_{j=1}^{k-1} k - j = \frac{k^2-k}{2}.$$

Hence

$$d\left(S^3_{-1}\left(T_{-2k,2k-1}\right)\right) = k^2 - k.$$

The relevant difference between signature and d is

$$\frac{\sigma}{2} - d = k^2 - 1 - \left(k^2 - k\right) = k - 1.$$

By Theorem 1.6, we know that

Proposition 3.4. *If $F \subset B^4$ is a smoothly embedded nonorientable surface with boundary $T_{2k,2k-1} \subset S^3$, then $b_1(F) \geq k-1$.*

To prove Theorem 1.4, apply the proposition to both halves of a surface with cross-section $T_{2k,2k-1}$.

The lower bound in the proposition is tight, as demonstrated by the following general construction. View $T_{p,q}$ as lying in a standard torus, as in Figure 3.5. Take any two adjacent strands and join them with a strip, or equivalently perform an index 1 Morse move merging them. The resulting cobordism is nonorientable, since the strands were parallel; it is a

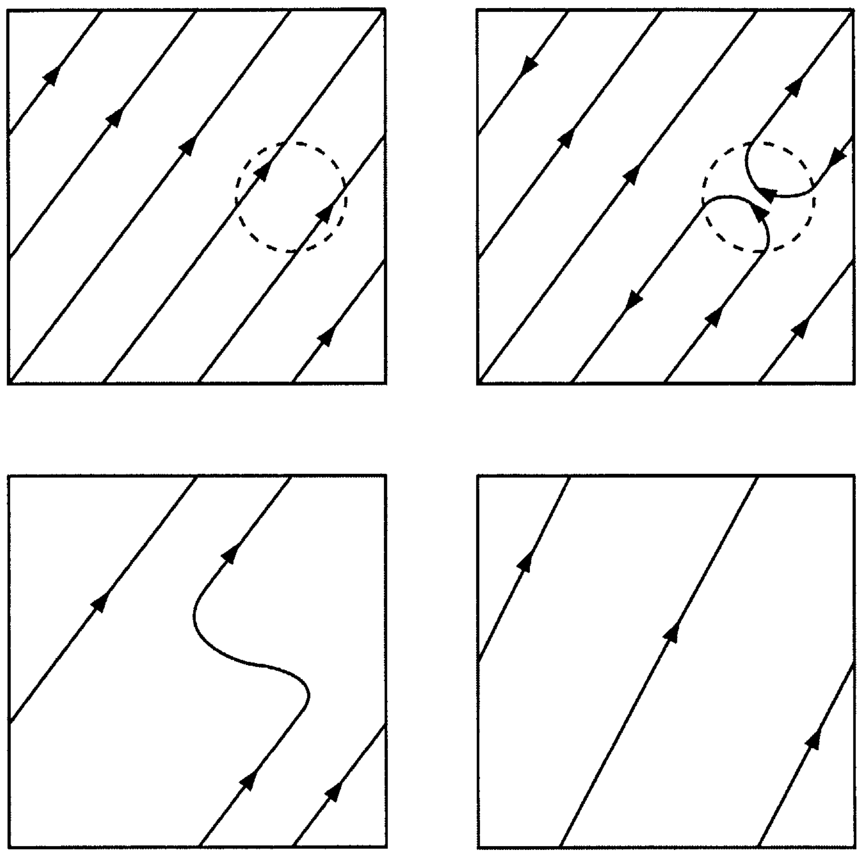


Figure 3.5: A cobordism from $T_{4,3}$ to $T_{2,1}$

punctured Möbius band. Since the resulting knot still lives on the torus, it must be $T_{r,s}$ for some r and s. The values of r and s can be easily computed by orienting the resulting knot and counting the signed intersection with the horizontal and vertical generators of $H_1(T^2)$. A short calculation shows that

$$r = p - 2t \qquad s = q - 2h$$

where $t \equiv -q^{-1} \bmod p$, with $0 \leq t < p$, and $h \equiv p^{-1} \bmod q$, with $0 \leq h < q$. After an isotopy, $T_{r,s}$ will be in standard, taut form on the torus, and we can repeat the process. Eventually, we arrive at $T_{n,1}$ for some n, which is just an unknot. By concatenating all of these cobordisms, then capping off the final unknot with a disk, we produce a surface $F_{p,q}$ in B^4 with boundary $T_{p,q}$.

For example, if $p = 2k$ and $q = 2k - 1$, we have $t = -(-1)^{-1} = 1$ and $h = 1^{-1} = 1$, giving $r = 2k - 2$ and $s = 2k - 3$. Thus $T_{2k,2k-1}$ has a $\chi = -1$ cobordism to $T_{2(k-1),2(k-1)-1}$. Concatenate $k - 1$ of these, then cap off $T_{2,1}$ with a disk to get a closed surface $F_{2k,2k-1} \subset B^4$ with boundary $T_{2k,2k-1}$ and $b_1(F_{2k,2k-1}) = k - 1$.

Since the isotopies and Morse moves take place inside of the torus, we can actually embed each of these cobordisms in a thickened torus $T^2 \times [-\epsilon, \epsilon]$ in S^3, where we view the $[-\epsilon, \epsilon]$ direction as a sort of time. The obstruction to embedding all of $F_{p,q}$ in S^3 is that the final disk with boundary $T_{n,1}$ cuts through all of the previous layers unless $n = 0$. To get a surface in S^3, we must continue with these within-torus cobordisms: $T_{n,1} \mapsto T_{n-2,1} \mapsto \cdots$. If n is even, or equivalently if pq was even to start, then we do get a surface in S^3. Teragaito has computed $\gamma_3(T_{p,q})$, and it agrees with $b_1(F)$ [44]. For example, $\gamma_3(T_{2k,2k-1}) = k$. If n is odd, then this construction fails to give a surface in S^3, though a slight modification (cf. [44] Remark 4.9) will do.

It is interesting to compare with the Milnor conjecture, which states that a minimal genus orientable surface in B^4 with boundary $T_{p,q}$ can be isotoped into S^3. While there is a Möbius band in B^4 with boundary $T_{4,3}$, Teregaito has shown that a punctured Klein bottle is best possible in S^3 [44].

The first Betti numbers of the $F_{p,q}$ do not have a simple closed form: they obey the recursion $b_1(F_{p,q}) = b_1(F_{p-2t,q-2h}) + 1$ where t and h are the minimal nonnegative representatives of $-q^{-1}$ modulo p and p^{-1} modulo q, respectively.

We conjecture that the surfaces $F_{p,q}$ have minimal b_1 among all smoothly embedded surfaces in $\mathbb{R}^4$ with boundary $T_{p,q}$. Many pairs (p, q) for which this conjecture holds can be certified using the d-invariant bounds of this paper. Similar invariants, derived by considering larger surgeries on the knot, give even more examples. The smallest knot for which the conjecture remains open is $T_{7,4}$, which is known to be the boundary of a Klein bottle but may yet be the boundary of a Möbius band.

Remark 3.5. The d-invariant argument works just as well for $-r$-surgery as for -1-surgery. Let $F \subset B^4$ have boundary a knot K. Write $\mathfrak{s}_i$ for the Spin^c-structure on $S^3_{-r}(K)$ indexed as in Lemma 2.2 of [35]. Then

$$b_1(F) \geq \frac{\sigma(K)}{2} - \frac{i^2}{r} - \frac{r-1}{4} - d(S^3_{-r}(K), \mathfrak{s}_i). \tag{3.4.1}$$

We have found experimentally that when K is a torus knot $T_{p,q}$, the lens space surgery $r = pq - 1$ gives the best lower bound, though the optimal choice of $\mathfrak{s}_i$ varies based on the knot. This can give an improvement: if $F \subset B^4$ has boundary $T_{9,5}$, then the inequality for -1-surgery shows that $b_1(F) \geq 0$ while an inequality for the -4-surgery shows that $b_1(F) \geq 2$. Levine, Ruberman, and Strle have recently computed the d-invariants of circle bundles over nonorientable surfaces, which allows one to skip the orientable replacement [12]. The bounds coming from their argument, however, appear to be identical to Equation 3.4.1.

Bibliography

[1] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Algebraic and Geometric Topology **2** (2002), 337–370.

[2] J. Batson, *If M is not a free A-module, can tensoring with a bigger field make it free?*, MathOverflow (version: 2013-01-15), URL: `http://mathoverflow.net/questions/118939`.

[3] P. Braam and S. Donaldson, *Floer's work on instanton homology, knots and surgery*, The Floer memorial volume, Springer, 1995, pp. 195–256.

[4] J Cha, S Friedl, and M Powell, *Splitting numbers of links*, arXiv.org (2013), 5638, arXiv:1308.5638v1.

[5] J. C. Cha and C. Livingston, *Knotinfo: Table of knot invariants*, URL: `http://www.indiana.edu/~knotinfo/`.

[6] M. Culler, N. M. Dunfield, and J. R. Weeks, *Snappy, a computer program for studying the geometry and topology of 3-manifolds*, URL: `http://snappy.computop.org`.

[7] S. Donaldson, *An application of gauge theory to four-dimensional topology*, Journal of Differential Geometry **18** (1983), no. 2, 279–315.

[8] ———, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), no. 3, 257–315.

[9] S. Eliahou, L. H. Kauffman, and M. B. Thistlethwaite, *Infinite families of links with trivial Jones polynomial*, Topology **42** (2003), no. 1, 155–169.

[10] Andreas Floer, *An instanton-invariant for* 3*-manifolds*, Communications in Mathematical Physics **118** (1988), no. 2, 215–240.

[11] R. Fox and J. Milnor, *Singularities of 2-spheres in 4-space and cobordism of knots*, Osaka J. Math **3** (1966), 257–267.

[12] Ira M Gessel, Adam Simon Levine, Daniel Ruberman, and Sašo Strle, *Non-orientable surfaces in homology cobordisms*, arXiv.org (2013), arXiv:1310.8516v1.

[13] P Gilmer and C. Livingston, *The nonorientable 4-genus of knots*, Journal of the London Mathematical Society **84** (2011), no. 3, 559–577.

[14] C.M. Gordon and R.A. Litherland, *On the signature of a link*, Inventiones Mathematicae **47** (1978), no. 1, 53–69.

[15] M. Hedden and Y. Ni, *Khovanov module and the detection of unlinks*, Geometry & Topology **17** (2013), 3027–3076.

[16] H. Helmholtz, *Über integrale der hydrodynamischen gleichungen, welche den wirbelbewegungen entsprechen*, Journal für die reine und angewandte Mathematik (1858), no. 55, 25–55.

[17] V. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111.

[18] M. Khovanov, *A Categorification of the Jones Polynomial*, Duke Math. J. **101** (2000), 359–426.

[19] ———, *Patterns in knot cohomology i*, Experiment. Math. **12** (2003), 365–374.

[20] J. Kock, *Frobenius algebras and 2-d topological quantum field theories*, Cambridge University Press, 2003.

[21] P. Kronheimer and T. Mrowka, *Gauge theory for embedded surfaces. I*, Topology **32** (1993), no. 4, 773–826.

[22] ———, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett **1** (1994), no. 6, 797–808.

[23] ———, *Monopoles and three-manifolds*, Cambridge University Press, 2007.

[24] ———, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci **113** (2011), no. 1, 97–208.

[25] John W Morgan, Zoltán Szabó, Clifford Henry Taubes, et al., *A product formula for the seiberg-witten invariants and the generalized thom conjecture*, Journal of Differential Geometry **44** (1996), no. 4, 706–788.

[26] R. Mosher and M. Tangora, *Cohomology Operations and Applications in Homotopy Theory*, (1968).

[27] H. Murakami and A. Yasuhara, *Four-genus and four-dimensional clasp number of a knot*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 3693–3700.

[28] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc **117** (1965), 387–422.

[29] K. Murasugi and B. Kurpita, *Knot theory and its applications*, Birkhauser, January 2008.

[30] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Advances in Mathematics **173** (2003), no. 2, 179–261.

[31] ———, *Heegaard Floer homology and alternating knots*, Geometry & Topology **7** (2003), no. 1, 225–254.

[32] ———, *Knot Floer homology and the four-ball genus*, Geometry & Topology **7** (2003), no. 2, 615–639.

[33] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, The Annals of Mathematics (2004), 1027–1158.

[34] ———, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33.

[35] ———, *Knot Floer homology and integer surgeries*, Algebraic & Geometric Topology **8** (2008), no. 1, 101–153.

[36] T.D. Peters, *A concordance invariant from the Floer homology of* ± 1 *surgeries*, arXiv:1003.3038 (2010).

[37] J. Rasmussen, *Khovanov homology and the slice genus*, Inventiones Mathematicae (2004), 1–29.

[38] C. Seed, *knotkit*, URL: `http://math.princeton.edu/~cseed/`.

[39] Nathan Seiberg and Edward Witten, *Electric-magnetic duality, monopole condensation, and confinement in* $\mathcal{N} = 2$ *supersymmetric Yang-Mills theory*, Nuclear Physics B **426** (1994), no. 1, 19–52.

[40] J. P. Serre, *Homologie singulière des espaces fibrés, applications*, Ann. of Math. **54** (1951), 425–505.

[41] A. Shumakovitch, *Torsion of the Khovanov homology*, (2002), arXiv:math/1012.3088v2.

[42] S. Suzuki, *Local knots of 2-spheres in 4-manifolds*, Proceedings of the Japan Academy, Series A, Mathematical Sciences **45** (1969), no. 1, 34–38.

[43] P. G. Tait, *On knots I, II, III*, Scientific Papers, Cambridge University Press, 1900.

[44] M. Teragaito, *Crosscap numbers of torus knots*, Topology and its Applications **138** (2004), no. 1-3, 219–238.

[45] M. Thistlethwaite, *Links with trivial Jones polynomial*, J. Knot Theory Ramifications **10** (2001), no. 4, 641–643.

[46] W. Thompson, *On vortex atoms*, Proceedings of the Royal Society of Edinburgh **6** (1867), 94–105.

[47] E. Witten, *Topological quantum field theory*, Communications in Mathematical Physics **117** (1988), no. 3, 353–386.

[48] ———, *Quantum field theory and the Jones polynomial*, Communications in Mathematical Physics **121** (1989), no. 3, 351–399.

[49] A. Yasuhara, *Connecting lemmas and representing homology classes of simply connected 4-manifolds*, Tokyo Journal of Mathematics **19** (1996), no. 1, 245–261.