The Blowup Formula for Higher Rank Donaldson
Invariants
by
Lucas Howard Culler
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Abstract

In this thesis, I study the relationship between the higher rank Donaldson invariants of a smooth 4-manifold $X$ and the invariants of its blowup $X \# \mathbb{CP}^2$. This relationship can be expressed in terms of a formal power series in several variables, called the blowup function. I compute the restriction of the blowup function to one of its variables, by solving a special system of ordinary differential equations. I also compute the $SU(3)$ blowup function completely, and show that it is a theta function on a family of genus 2 hyperelliptic Jacobians. Finally, I give a formal argument to explain the appearance of Abelian varieties and theta functions in four dimensional topological field theories.

Thesis Supervisor: Tomasz Mrowka
Title: Singer Professor of Mathematics

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Chapter 1

Introduction

Donaldson's invariants of smooth 4-manifolds were originally defined using moduli spaces of anti-self dual connections on principal bundles with structure group $SU(2)$ or $SO(3) = PU(2)$. These invariants and their corresponding generating functions have many interesting structural features which have been described in a mathematically rigorous way. For instance, a paper of Fintushel and Stern [7] shows that the invariants of a 4-manifold $X$ and its blowup $\hat{X} = X \# \mathbb{CP}^2$ are related in a predictable way, and that the relationship can be expressed in terms of a theta function on the elliptic curve

$$y^2 = x^3 + ax^2 + x$$

over the ring $\mathbb{Q}[a]$. More generally, the structure theorem of Kronheimer-Mrowka expresses the Donaldson invariants of a 4-manifold of simple type in terms of exponential functions (theta functions on a degenerate genus 1 curve), and there is a conjectural structure theorem for 4-manifolds not of simple type, which expresses the invariants of such a manifold in terms of Jacobi elliptic functions.

It is natural to ask if Donaldson invariants can be defined for higher rank bundles, and if so, whether or not these structural features have some natural generalization. Indeed, physicists have achieved results in this direction (see for example [3] and [12]), which indicate that the blow-up formula and structure theorem for $SU(N)$ invariants can be expressed in terms of function theory on the hyperelliptic curve

$$\Sigma_N : y^2 = (x^N + a_2 x^{N-2} + \cdots + a_{N-1} x + a_N)^2 + 4$$

over the ring

$$\Lambda = H^*(BPU(N); \mathbb{Q}) = \mathbb{Q}[a_2, \ldots, a_N].$$

In particular it is shown in [3] that the blow-up function is a theta function on the Jacobian of this curve, generalizing the result of Fintushel and Stern. Unfortunately, their derivation relies on the use of quantum field theory techniques, which do not appear to be mathematically rigorous.

One route to overcoming these difficulties, pursued by Nakajima and Yoshioka in [14], is through Nekrasov's partition function. Using the localization formula in equivariant cohomology, they rigorously define certain integrals which are analogous to the Donaldson
invariants of the plane blown up at a single point. They compute these integrals using techniques of algebraic geometry, and thereby arrive at a blow-up formula which agrees with the one from the physics literature.

It is not clear, however, what either of the methods above has to do with moduli spaces of instantons on general 4-manifolds. In this paper we take a more direct route. In [8], Peter Kronheimer has given a rigorous definition of $PU(N)$ invariants for general $N$. Using his definition we directly attempt to compute the blowup function, using the method of Fintushel and Stern. For small values of $N$ we compute the blowup function recursively, and for $N = 3$ we explicitly identify it with a theta function on the Jacobian of the curve $\Sigma_3$. For general $N$ we can partially compute the blowup function. More precisely, we restrict the blowup function to one of its variables and characterize the resulting single variable function by a system of ordinary differential equations. After a change of variables, this system of equations reduces to the Toda chain, an integrable system well-known to physicists, whose relation with the blowup formula was predicted in [12].

Theta functions also arise in Seiberg-Witten theory, where the analogous blowup formula [9] is encoded by an infinite Laurent series over the formal power series ring $\mathbb{Z}[[U]]$:

$$\Theta(T) = \sum_{n \in \mathbb{Z}} (-1)^n U^{\frac{2}{T^2}} T^n$$

This formal sum can be viewed as a theta function on the Tate curve, a special family of elliptic curves defined over a formal disk. Going further in this direction, a paper of Lekili and Perutz [10] uses mirror symmetry to relate Heegaard Floer theory to the derived category of the Tate curve. Their paper can be viewed either as an explanation or generalization of the blowup formula, depending on one's point of view.

One might hope for a conceptual explanation of the appearance of theta functions in topological field theory. We have taken some partial steps in this direction. Given a Floer homology theory satisfying certain formal properties, we construct a graded ring $A$, which in the case of Seiberg-Witten theory is the homogeneous coordinate ring of the Tate curve. From $A$ we build a projective variety $\mathcal{A}$, and explain how to interpret the blowup function as a section of a line bundle on $\mathcal{A}$. In favorable circumstances, we can use the geometry of $\mathcal{A}$ to give a formal proof that the blowup function is necessarily a theta function.

This thesis is divided into three chapters. The first chapter deals with general background on Donaldson invariants, and the definition of higher rank invariants due to Kronheimer. It mostly follows standard references on Donaldson invariants, specifically [4], [5], and [13], as well as Kronheimer's paper [8] on higher rank invariants. The only original material in this chapter is the definition of extra cohomology classes not considered by Kronheimer, which were implicit in the physics literature on the blowup formula, but had not been defined explicitly in the mathematical literature.

The second chapter contains a proof of the partial blowup formula for $SU(N)$ invariants and the complete blowup formula for $SU(3)$ invariants. It begins with a general discussion of instantons on a tubular neighborhood of an embedded sphere of negative self-intersection. To verify transversality of the higher rank instanton moduli spaces on these manifolds we produce special metrics with anti-self-dual Weyl curvature and positive scalar curvature. We give a detailed analysis of reducible connections in these moduli spaces, and use some special
cases to derive certain relations that imply the existence of a blowup formula. In the case $N = 3$ we use classical results from the function theory of genus two Jacobians to verify that the blowup function is a theta function on the curve 1.1.

Finally, in the third chapter we axiomatically define a notion of formal Floer theory, which assigns vector spaces to 3-manifolds and maps to suitably decorated cobordisms. Such a theory has a tautologically defined blowup function, which spans the first graded piece of the ring referred to above. Given some nondegeneracy conditions on the theory, we show that the blowup function is entire and has a natural set of quasiperiods, such that the automorphy factor corresponding to any nontrivial quasiperiod is the exponential of a linear function. If the quasiperiods span a full lattice, we conclude that the formal blowup function is a theta function on a complex torus.
Chapter 2

Higher Rank Invariants

2.1 ASD Connections

In this section we set up the basic analytic framework needed to study the moduli space of anti-self-dual connections on a 4-manifold.

2.1.1 The Configuration Space

Let \( X \) be a compact, connected, Riemannian 4-manifold, let \( E \) be a rank \( N \) vector bundle on \( X \), and let \( P = P(E) \) be the associated principal \( PU(N) \) bundle. Fix a smooth connection \( A_0 \) on \( P \) and an integer \( l > 2 \).

**Definition 1.** We define the configuration space of connections on \( P \) to be

\[
\mathcal{A}(P) = \{ A_0 + a \mid a \in L^2_t(X, \Lambda^1 \otimes \text{ad} P) \}
\]

Since the difference of any two smooth connections is a smooth section of \( \Lambda^1(X) \otimes \text{ad} P \), the definition does not depend on the choice of \( A_0 \). To explain the condition \( l > 2 \), recall that in dimension 4 there is a Sobolev embedding \( L^2_{2+\epsilon} \hookrightarrow C^{0,\epsilon} \), for \( 0 < \epsilon < 1 \). Thus our definition guarantees that every element of \( \mathcal{A}(P) \) is continuous.

**Definition 2.** We define the determinant one gauge group of \( P \) to be

\[
\mathcal{G}(P) = \{ \gamma \in L^2_{l+1}(X, G(P)) \}
\]

where \( G(P) = P \times_{PU(N)} SU(N) \) is the bundle of groups associated to the conjugation action of \( PU(N) \) on \( SU(N) \).

To explain the use of the index \( l+1 \) rather than \( l \), consider the explicit formula for the action of \( \mathcal{G}(P) \) on \( \mathcal{A}(P) \),

\[
\gamma \cdot (A_0 + a) = A_0 + \gamma d_{A_0} \gamma^{-1} + \gamma a \gamma^{-1}.
\]

From this formula it is clear that elements of \( \mathcal{G}(P) \) must have at least one more derivative than do elements of \( \mathcal{A}(P) \), in order for the action to extend over both completions.
Definition 3. We define the quotient configuration space to be $B(P) = A(P)/G(P)$, with the natural quotient topology.

We would like to say that $B(P)$ is a smooth Banach manifold. However, this is not the case, due to the existence of connections with nontrivial stabilizers.

**Proposition 4.** For any connection $A \in A(P)$, its stabilizer $\Gamma_A$ is a finite dimensional Lie group. In fact, evaluation at any point $x \in X$ defines an isomorphism from $\Gamma_A$ to the centralizer of the holonomy group

$$H_A = \text{Hol}_x(A) \subset G(P)_x$$

consisting of holonomies around all piecewise smooth curves based at $x$.

**Proof.** Suppose that $A$ is stabilized by $\gamma \in G(P)$. Then we have:

$$A = \gamma^{-1} \cdot A = A + \gamma^{-1}d_A\gamma$$

Hence $\gamma^{-1}d_A\gamma = 0$, so $\gamma$ is parallel when considered as a section of $G^{ad}(P)$. Since $Z$ is discrete, this implies that $\gamma$ is a parallel section of $G(P)$. Conversely, any parallel section of $G(P)$ defines a gauge transformation that stabilizes $A$.

Now, for connected $X$ a parallel section $\gamma$ is determined by its value at any point $x \in X$, so $\Gamma_A$ injects into $G(P)_x = PU(N)$. Moreover, a given $\gamma_0 \in G(P)_x$ will extend to a parallel section if and only if it returns to itself under parallel transport around any closed loop based at $x$, which is equivalent to saying that it centralizes $H_A$. $\square$

**Definition 5.** We say that a connection $A \in A(P)$ is reducible if its stabilizer $\Gamma_A$ is strictly larger than $Z$. Otherwise we say that $A$ is irreducible. We write $A^*(P)$ for the set of irreducible connections, and we write $B^*(P)$ for the quotient $A^*(P)/G(P)$.

To determine the local structure of $B(P)$ we need to construct equivariant tubular neighborhoods for every gauge orbit in $A(P)$. The key input for this construction is the following proposition:

**Proposition 6.** Let $A \in A(P)$, let $d_A : L^2_{i+1}(X, \text{ad}(P)) \to L^2_i(X, \Lambda^1 \otimes \text{ad}(P))$ be the associated covariant derivative, and let $d_A^*$ be its formal adjoint. Then there is a direct sum decomposition into $L^2$-orthogonal subspaces:

$$L^2_i(X, \Lambda^1 \otimes \text{ad}(P)) = \ker d_A^* \oplus \text{im } d_A$$

**Proof.** By definition $\text{im } d_A$ and $\ker d_A^*$ are $L^2$-orthogonal. To show that they span, we must solve the elliptic equation

$$d_A^*d_A\xi = d_A^*a$$

for any given $a \in L^2_i(X, \Lambda^1 \otimes \text{ad}(P))$. To produce a solution, consider the inner product

$$Q_A(\xi, \psi) = \langle d_A\xi, d_A\psi \rangle.$$
on $L^2_t(X, \text{ad}(P))$. Modulo the finite dimensional kernel of $d_A$ the corresponding quadratic form is nondegenerate and equivalent to the $L^2$ norm. Hence we can find $\xi \in L^2_t(X, \text{ad}(P))$ such that

$$\langle a, d_A \psi \rangle = Q_A(\xi, \psi) = \langle d_A \xi, d_A \psi \rangle$$

for all $\eta \in L^2_t$. In other words, $\xi$ is a weak solution to (2.1.1). By elliptic regularity we know that $\xi \in L^2_{t+1}$ as desired.

We therefore define

$$T_{A,\epsilon} = \left\{ A + a \big| d^*_A a = 0, |a|_{L^2_t} < \epsilon \right\}.$$ 

The stabilizer $\Gamma_A$ acts on $T_{A,\epsilon}$, so we can form a $G(P)$-equivariant vector bundle over the orbit $G(P) \cdot A = G(P)/\Gamma_A$:

$$N_{A,\epsilon} = G(P) \times_{\Gamma_A} T_{A,\epsilon}.$$ 

$N_{A,\epsilon}$ provides a local model for $\mathcal{A}(P)$ in a neighborhood of $G(P) \cdot A$, and allows us to determine the local structure of $\mathcal{B}(P)$.

**Proposition 7.** Let $A$ be an arbitrary connection in $\mathcal{A}(P)$. Then for some $\epsilon > 0$ there is a neighborhood of $[A]$ in $\mathcal{B}(P)$ homeomorphic to $T_{A,\epsilon}/\Gamma_A$.

**Proof.** There is an evident map $\alpha : N_{A,\epsilon} \to \mathcal{A}(P)$, whose differential is invertible along the zero section by virtue of Proposition 6. In fact, $\alpha$ is a diffeomorphism onto its image for $\epsilon$ sufficiently small, and its image is an open neighborhood of the orbit of $A$. The quotient $N_{A,\epsilon}/G(P)$ is homeomorphic to $T_{A,\epsilon}/\Gamma_A$, which proves the claim.

**Corollary 8.** $\mathcal{B}^*(P)$ is a smooth Banach manifold.

**Proof.** If $\Gamma_A = Z$ then it acts trivially on $T_{A,\epsilon}$, hence Corollary 7 produces topological charts on $\mathcal{B}^*(P)$. The fact that these charts overlap smoothly follows from the fact that $\alpha : N_{A,\epsilon} \to \mathcal{A}(P)$ is smooth for any $A$ and $\epsilon$.

Note that most of the above discussion goes unchanged when $X$ is a manifold with boundary. However, we must modify our construction of slices by adding suitable boundary conditions. A natural choice is:

$$T_A = \left\{ A + a \big| d^*_A a = 0, *a\big|_{\partial X} = 0 \right\}.$$ 

To check that this is a slice we must solve

$$d^*_A(a - d_A \xi) = 0$$

subject to the boundary condition $*d_A \xi|_{\partial X} = 0$. We can do this by simply reiterating the argument of Proposition 6, and being careful about boundary conditions whenever we integrate by parts.

### 2.1.2 The Moduli Space

Let $X$ be a smooth oriented 4-manifold with a Riemannian metric. The hodge star $*$ acts on the bundle $\Lambda^2(X)$, and satisfies $*^2 = 1$. Therefore we can write $\Lambda^2(X) = \Lambda^+(X) \oplus \Lambda^-(X)$,
where $\ast$ acts on the bundles $\Lambda^\pm(X)$ by $\pm 1$. In particular, any 2-form $\omega$ on $X$ can be expressed uniquely as $\omega^+ + \omega^-$, where $\omega^\pm$ is a section of $\Lambda^\pm(X)$.

**Definition 9.** We say that $\omega$ is anti-self-dual if $\ast \omega = -\omega$, or equivalently, if $\omega^+ = 0$.

Now let $P \to X$ be a principal $G$-bundle, and let $A$ be a connection on $P$. The curvature of $A$ is a section of $\Lambda^2(X) \otimes \text{ad}P$, hence it can be written uniquely as

$$F_A = F_A^+ + F_A^-$$

where $F_A^\pm$ is a section of $\Lambda^\pm(X) \otimes \text{ad}P$.

**Definition 10.** We say that $A$ is anti-self-dual if $\ast F_A = -F_A$, or equivalently if it satisfies the anti-self-duality (ASD) equations:

$$F_A^+ = 0$$

Note that the ASD equations are invariant under gauge transformations, so each solution gives rise to an infinite dimensional space of gauge equivalent solutions. In particular, anti-self-duality can be viewed as a property of the gauge equivalence class $[A] \in \mathcal{B}(P)$.

**Definition 11.** We define the moduli space $\mathcal{M}(P)$ to be the set of all equivalence classes of ASD connections $[A] \in \mathcal{B}(P)$.

To find the dimension of $\mathcal{M}(P)$ we need to compute the linearization of the ASD equations. Let $A$ be an ASD connection, and let $A + a$ be a nearby connection. Then we have

$$F_{A+a}^+ = F_A^+ + d_A^+ a + (a \wedge a)^+ = 0,$$

where $d_A^+ a$ is the self-dual part of $d_A$. Dropping quadratic terms and setting $F_A^+ = 0$, we see that the linearization is

$$d_A^+ a = 0.$$ 

Solutions of this equation are tangent vectors to the space of all ASD connections, which sits inside in the configuration space $\mathcal{A}(P)$. By definition, the moduli space $\mathcal{M}(P)$ is the image of this infinite dimensional space in $\mathcal{B}(P)$. To compute the tangent space to $\mathcal{M}(P)$ we restrict our attention to the slice

$$d_A^+ a = 0,$$

so as to select a single ASD connection from each gauge equivalence class. Combining the two conditions, we can identify the tangent space of $\mathcal{M}(P)$ with the kernel of the Dirac operator,

$$D_A = d_A^+ + d_A^\ast : \Lambda^1(X, \text{ad}(P)) \to \Lambda^0(X, \text{ad}(P)) \oplus \Lambda^2(X, \text{ad}(P)).$$

We will show in the next section that $D_A$ is an elliptic operator, and therefore has a finite index. Modifying $A$ amounts to a compact perturbation, and does not change the index.

**Definition 12.** We define the virtual dimension of $\mathcal{M}(P)$ to be the index of the Dirac operator $D_A$, for any $A \in \mathcal{M}(P)$.
If $D_A$ is surjective, then the moduli space will be cut out transversely, and will therefore be a smooth manifold of dimension equal to the index of $D_A$, in which case the notions of dimension and virtual dimension coincide. On the other hand, if $D_A$ has nontrivial cokernel, then $\mathcal{M}(P)$ may fail to be a smooth manifold near $[A]$. The cokernel can be understood precisely in terms of the "deformation complex",

$$
\begin{array}{c}
0 \longrightarrow \Lambda^0(X, \text{ad}(P)) \xrightarrow{d_A} \Lambda^1(X, \text{ad}(P)) \xrightarrow{d_A^+} \Lambda^+(X, \text{ad}(P)) \longrightarrow 0,
\end{array}
$$

where $d_A^+ \circ d_A = F^+_A = 0$.

General theory of elliptic complexes shows that the cokernel of $D_A$ is the direct sum of the two even dimensional cohomology groups $H^0_A$ and $H^2_A$. Thus there are two separate ways that $\mathcal{M}(P)$ can fail to be regular, and we will eventually need to explain how to avoid or deal with both kinds of failure.

2.1.3 Index Theory

We now compute the virtual dimension of ASD moduli spaces on compact manifolds, using the Atiyah-Singer index theorem for Dirac operators.

**Proposition 13.** Let $X$ be a compact Riemannian 4-manifold and let $V \to X$ be a unitary complex vector bundle of rank $r$. If $A$ is any unitary connection on $V$, then

$$
D_A = d_A^* + d_A^+ : \Omega^1(X, V) \to \Omega^0(X, V) \oplus \Omega^+(X, V)
$$

is an elliptic operator, whose index (as a complex linear operator) is given by

$$
\text{ind}(D_A) = c_1(V)^2 - 2c_2(V) - \frac{r}{2}(\chi + \sigma).
$$

Here $d_A^*$ denotes the adjoint of the operator $d_A$, and $\chi$, $\sigma$ are the Euler characteristic and signature of $X$.

**Proof.** Ellipticity is a local condition, and only depends on the highest order part of the operator, hence it suffices to show that the operator $D = d^* + d^+$ is elliptic, for $V$ a trivial bundle with the trivial connection. To see this consider the operator

$$
D^* = d \oplus 2d^* : \Omega^0 \oplus \Omega^+(X) \to \Omega^1(X).
$$

Composing the two operators,

$$
D^*D = dd^* + 2d^* \left( \frac{1 + *}{2} \right) d
$$

$$
= d^*d + d^*(1 + *)d
$$

we get the Laplace operator, which has an invertible symbol. We conclude that $D$ has an invertible symbol as well, hence it is elliptic.

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To calculate the index, we first treat the case where $V$ is the trivial complex line bundle. Modifying $A$ does not change the index, so in this case $\ker D$ can be identified with the harmonic 1-forms on $X$, and $\text{coker} D$ can be identified with the sum of the harmonic 0 forms and the self-dual harmonic 2-forms. Hence

$$\text{ind}(D) = -1 + b_1 - b_2^+ = \frac{x + \sigma}{2}$$

(2.4)

In general, the Atiyah-Singer index theorem for Dirac operators allows us to express the index of $D_A$ as

$$\text{ind}(D_A) = - \int_X (2 + \alpha)\text{Ch}(V)$$

(2.5)

where $\alpha$ is a degree 4 cohomology class that is independent of $V$, and

$$\text{Ch}(V) = r + c_1(V) + \frac{1}{2} c_1(V)^2 - c_2(V) + \cdots$$

is the Chern character of $V$. Some explanation of the factor $2 + \alpha$ is required. The reason for the 2 is that $D$ is a Dirac operator on a rank 2 Clifford module. The absence of a degree 2 cohomology class is due to the fact that this Clifford module is real. The exact nature of $\alpha$ is not important to us, because we only need to compute its integral over $X$, which can be done by comparing equations (2.4) and (2.5). Hence

$$\text{ind}(D_A) = - \int_X (2 + \alpha)(r + c_1(V) + \frac{1}{2} c_1(V)^2 + c_2(V))$$

$$= - \int_X c_1(V)^2 - 2c_2(V) + r\alpha$$

$$= (2c_2(V) - c_1(V)^2) [X] - \frac{r}{2}(x + \sigma)$$

as desired. \qed

**Corollary 14.** Let $E$ be a $U(N)$ bundle on a compact 4-manifold $X$ and let $P$ be the associated $PU(N)$ bundle. Then the moduli space $\mathcal{M}(P)$ has virtual dimension

$$\dim \mathcal{M}(P) = 4Nc_2(E) + (2 - 2N)c_1(E)^2 + (1 - N^2)\frac{x + \sigma}{2}$$

**Proof.** We apply Proposition 13 to the complexified adjoint bundle

$$\mathfrak{sl}(E) = \mathbb{C} \otimes \text{ad}(P).$$

Note that $\mathfrak{sl}(E) \oplus \mathbb{C} = E \oplus E^*$. Therefore,

$$2c_2(\mathfrak{sl}(E)) - c_1(\mathfrak{sl}(E))^2 = -2\text{Ch}_2(E \otimes E^*)$$

$$= -2N\text{Ch}_2(E) - 2\text{Ch}_1(E)\text{Ch}_1(E^*) - 2N\text{Ch}_2(E^*)$$

$$= 4Nc_2(E) + (2 - 2N)c_1(E)^2,$$

which completes the proof. \qed
2.1.4 Reducibles

Let \( E \) be a rank \( N \) vector bundle on a connected, oriented 4-manifold \( X \). Let \( P = P(E) \) be the corresponding \( PU(N) \) bundle. Observe that if \( \tilde{A} \) is any connection on \( E \), we can uniquely write it as

\[
\tilde{A} = A + \frac{1}{N} \delta
\]

where \( A \in \mathcal{A}(P) \) is a \( PU(N) \) connection and \( \delta = \text{tr}(\tilde{A}) \) is the connection induced by \( \tilde{A} \) on \( \det(E) \). Keeping this in mind, we can fix a connection \( \delta \), and view elements of \( \mathcal{A}(P) \) as connections on \( E \) rather than on \( P \).

The moduli space \( \mathcal{M}(P) \) can fail to be smooth at a connection \( A \) if the stabilizer \( \Gamma_A \) is nontrivial. In particular, \( H^2_A = \ker d_A \) is the Lie algebra of the stabilizer \( \Gamma_A \), so \( D_A \) fails to be surjective if \( \Gamma_A \) has positive dimension. It is therefore worth understanding the stabilizers in more detail.

**Proposition 15.** Let \( H \subset U(N) \) be any subgroup and let \( \Gamma \subset U(N) \) be its centralizer. Then \( \Gamma \) is isomorphic to a group of the form \( \prod_{i=1}^{r} U(n_i) \), and it acts on \( \mathbb{C}^N \) through a decomposition

\[
\mathbb{C}^N = \bigoplus_{i=1}^{r} \mathbb{C}^{n_i} \otimes V_i
\]

**Proof.** By basic representation theory, the action of \( H \) on \( \mathbb{C}^N \) decomposes as a sum of irreducible representations

\[
\mathbb{C}^N = \bigoplus_{i=1}^{r} V_i^{n_i} = \bigoplus_{i=1}^{r} \mathbb{C}^{n_i} \otimes V_i.
\]

By Schur's lemma any \( \gamma \in \text{End}(\mathbb{C}^N) \) that commutes with \( H \) lies in the subspace

\[
\bigoplus_{i=1}^{r} \text{End}(\mathbb{C}^{n_i}) \otimes 1_{W_i} \subset \text{End}(\mathbb{C}^N).
\]

Intersecting this with \( U(N) \) gives the desired decomposition \( \Gamma = \prod_{i=1}^{r} U(n_i) \).

**Corollary 16.** If \( A \in \mathcal{A}(P) \) is reducible, then \( \Gamma_A \) has positive dimension.

**Proof.** Let \( H \) be the holonomy group of \( \tilde{A} \) and let \( \Gamma \) be its centralizer. Then \( \Gamma_A = \Gamma \cap SU(N) \). If there are two nontrivial factors \( U(n_1) \times U(n_2) \subset \Gamma \), then \( \Gamma \) has dimension at least two. Since \( SU(N) \) has codimension one, the intersection \( \Gamma \cap SU(N) \) has positive dimension. The only other possibility is that \( \Gamma = U(N) \), but then \( \Gamma_A = SU(N) \) still has positive dimension.

In many cases it is not possible to rule out reducible connections. In particular, if \( E \) is the trivial bundle then it always admits the product connection. However, there is a simple condition we can impose on \( E \) to avoid reducibles.

**Definition 17.** We say that a class \( c \in H^2(X; \mathbb{Z}) \) is coprime to \( N \) if there is a homology class \( \sigma \in H_2(X; \mathbb{Z}) \) such that \( c(\sigma) \equiv 1 \mod N \).
Proposition 18. If \( E \) is a rank \( N \) bundle admitting a reducible ASD connection, then there is an integral class \( c \) such that \( Nc - nc_1(E) \) is represented by an anti-self-dual 2-form.

Proof. Suppose \( A \) is a reducible connection on \( P(E) \) and let \( \bar{A} \) be a lift to \( E \) with trace \( \delta \). Then there is a splitting \( E = E_1 \oplus E_2 \) compatible with \( \bar{A} \). Write \( \bar{A} = \bar{A}_1 \oplus \bar{A}_2 \) and let \( \text{tr}\bar{A}_i = \delta_i \). Then we have

\[
\frac{1}{N} \delta = \frac{1}{n_1} \delta_1 = \frac{1}{n_2} \delta_2.
\]

Since the curvature of \( \bar{A} - \frac{1}{N} \delta \) is anti-self-dual, we have

\[
F^+_A = \frac{1}{n_1} F^+_{\delta_1} = \frac{1}{N} F^+_{\delta_2}
\]

In particular the class \( Nc_1(E_1) - n_1c_1(E) \) is represented by an anti-self-dual 2-form. \( \square \)

Corollary 19. If \( c_1(E) \) is coprime to \( N \) then the set of metrics for which there exists a reducible ASD connection on \( E \) is a countable union of submanifolds of codimension \( b_2^+(X) \).

Proof. If \( c_1(E) \) is coprime to \( N \) then evaluating on \( \sigma \) shows that \( Nc - nc_1(E) \) is nonzero for any integral class \( c \). The corollary then follows from the fact that the space of metrics for which a given nonzero cohomology class is anti-self-dual is a submanifold of codimension \( b_2^+(X) \) in the space of metrics. \( \square \)

In particular, if \( b_2^+(X) > 1 \) then we do not find reducible connections in generic 1-parameter families of metrics.

2.1.5 Perturbations

The second way that \( \mathcal{M}(P) \) can fail to be smooth is if \( H_2^X = \text{coker}d_A^r \) is nonzero. To rule out this possibility, we need to artificially perturb the ASD equations. Acceptable perturbations take the form

\[
F_A^+ = V(A)
\]

where \( V(A) \) is a section of the bundle \( \Lambda^+ \otimes \text{ad}(P) \), depending on the connection \( A \). To be gauge invariant, the perturbated equations must transform like the curvature under gauge transformations:

\[
V(\gamma \cdot A) = \gamma V(A) \gamma^{-1}.
\]

Note that if \( q : S^1 \to X \) is a loop based at a point \( x_0 \), then the holonomy \( \text{Hol}_q(A) \) transforms in this way:

\[
\text{Hol}_q(\gamma \cdot A) = \gamma \text{Hol}_q(A) \gamma^{-1}.
\]

However, \( \text{Hol}_q(A) \) lies in \( G(P)_{x_0} \) rather than \( \text{ad}(P)_{x_0} \). To remedy this, we choose a \( G \)-equivariant map \( \phi : G \to g \) with the property that \( \phi \) inverts the exponential map in a neighborhood of the identity. Then \( \phi(\text{Hol}_q(A)) \) is an element of \( \text{ad}(P)_{x_0} \) that transforms appropriately under gauge transformations.

To produce a section \( V(A) \), we choose a map \( q : S^1 \times B^4 \to X \), with the property that each map \( q(t, -) \) is a diffeomorphism onto its image, and a section \( \omega \) of \( \Lambda^+ \) whose support
is contained in the open set $U = q(0, B^4)$. We then define

$$V_{q, \omega}(A) = \omega \otimes \phi(\text{Hol}_q(A))$$

where $\text{Hol}_q(A)$ is regarded as a section of $\text{ad}(P)$ on $U$. Given a countable collection $\pi = \{(q_i, \omega_i)\}_{i \in I}$ of loops and anti-self-dual 2-forms, we can define a perturbation

$$V_{\pi}(A) = \sum_{i \in I} V_{q_i, \omega_i}(A).$$

**Definition 20.** Let $X$ be a compact 4-manifold, let $E$ be a rank $N$ bundle, and let $V_{\pi}$ be a perturbation of the ASD equations on $P = P(E)$. We define the perturbed moduli space to be

$$\mathcal{M}_{\pi}(P) = \{[A] \in \mathcal{B}(P) \mid F_A^+ + V_{\pi}(A) = 0\}$$

Because we have chosen $V_{\pi}(A)$ to be gauge invariant, its derivative $DV_{\pi}$ vanishes on the tangent space to any gauge orbit. Hence we obtain a deformation complex

$$0 \rightarrow L^2_{l+1}(X, \text{ad}(P)) \xrightarrow{d_A} L^2(X, \Lambda^1 \otimes \text{ad}(P)) \xrightarrow{d^+_{A,\pi}} L^2_{l-1}(X, \Lambda^+ \otimes \text{ad}(P)) \rightarrow 0$$

whose cohomology groups we denote by $H^j_{A,\pi}$. The operator $d^+_{A,\pi}$ is defined to be

$$d^+_{A,\pi} = d^+_A + DV_{\pi}(A),$$

the linearization of the perturbed ASD equations.

**Definition 21.** We say that $A$ is a regular solution of the perturbed equations if $H^2_{A,\pi} = 0$

**Proposition 22.** Suppose that $A$ is regular and irreducible. Then in a neighborhood of $A$, the moduli space $\mathcal{M}(P)$ is a smooth manifold whose tangent space is isomorphic to $H^1_{A,\pi}$.

**Proof.** In a neighborhood of $A$, every connection in the moduli space is equivalent to one in the slice

$$T_{A,\epsilon} = A + \text{ker} d^*_A,$$

which is a chart for $\mathcal{B}(P)$. Thus it suffices to show that the set of ASD connections in this slice is a smooth manifold. If $A$ is regular then the map

$$d^+_A : \text{ker} d^*_A \rightarrow L^2_{l+1}(X, \Lambda^+ \otimes \text{ad}(P))$$

is surjective, hence by the implicit function theorem $\mathcal{M}(P)$ is a smooth manifold near $A$ and its tangent space is $\text{ker} d^+_{A,\pi} \cap \text{ker} d^*_A = H^1_{A,\pi}$. $\square$

The use of holonomy perturbations is justified by the fact that generic perturbations yield regular moduli spaces.

**Proposition 23.** Let $E$ be a rank $N$ bundle over a compact Riemannian 4-manifold $X$. Then there exists a residual set of small perturbations $\pi$, such that every $A \in \bar{\mathcal{M}}_{\pi}(P)$ is regular.
The ASD moduli spaces $\mathcal{M}(P)$ are not compact in general, due to bubbling. However, we do have the following compactness theorem for the perturbed equations

**Proposition 24.** Let $P = P(E)$ and let $[A_n]$ be a sequence in $\mathcal{M}(P(E))$. Then there are points $x_1, \ldots, x_k \in X$, a bundle $F$ with $c_2(F) = c_2(E) - k$, a point $[A_\infty] \in \mathcal{M}(P(F))$, a subsequence of the $A_n$, and gauge transformations $\gamma_n : P(E) \to P(F)$ such that $\gamma_n A_n \to A_\infty$ on every compact $K \subset X \setminus \{x_1, \ldots, x_k\}$ in the $L^1_\|\|_1$ topology.

For the proofs of Propositions 23 and 24, we refer to Kronheimer’s paper [8].

### 2.1.6 Orientations

It is important for the definition of Donaldson invariants that we give the moduli space of ASD connections an orientation.

**Proposition 25.** Let $E$ be a rank $N$ bundle and let $P = P(E)$. Then the moduli space $\mathcal{M}(P)$ is orientable.

**Proof.** We refer to Donaldson [6], but we outline his proof. Orientations are induced by sections of the determinant line bundle of the family of Dirac operators $D_A$ over $\mathcal{M}(P)$. This is the restriction of a universal determinant line bundle $\Lambda$ over $B(P)$. Therefore it suffices to show that $\Lambda$ is trivial, or equivalently that $w_1(\Lambda)$ pairs trivially with any loop in $B(E)$. Given a class $[\gamma] \in H_1(B(E))$, Donaldson writes down an explicit loop of connections representing $[\gamma]$ and checks that its determinant line is trivial. \hfill $\square$

To define invariants in $\mathbb{Z}$ rather than $\mathbb{Z}/\{\pm 1\}$, one must single out a preferred trivialization of $\Lambda$. To do this it suffices to trivialize the determinant line of any particular connection $A \in \mathcal{A}(E)$, and we briefly describe how to do this.

For a bundle with $c_2(E) = 0$ one has $E = L \oplus \mathbb{C}^{N-1}$ for a unique complex line bundle $L$. Choosing a connection $\lambda$ on $L$ and taking $A = \lambda \oplus 1 \cdots \oplus 1$, one sees that the adjoint bundle $\text{ad}(P)$ splits as a sum of $\frac{N^2 - N}{2}$ complex line bundles and $N - 1$ trivial real line bundles. Using the complex orientation of the complex line bundles, ordering the real line bundles, and trivializing the determinant line of the operator $d^* + d^+$, one obtains a trivialization of the determinant of $D_A$. The result is clearly independent of the choice of $\lambda$, hence only depends on a homology orientation of $X$ (which is equivalent to a trivialization of the determinant of $d^* + d^+$). Reversing the homology orientation of $X$ changes the orientation when $N$ is even but preserves it when $N$ is odd.

For a bundle with $c_2(E) = k$, one chooses $k$ points $x_1, \ldots, x_k \in X$. Starting with a connection of the form $\lambda \oplus \mathbb{C}^N$ one can glue in a pointlike instanton at each $x_i$, obtaining a connection $A$. Trivializing the determinant of $D_A$ is equivalent (by excision, see [6]) to orienting the moduli space of ASD $SU(2)$ connections on $S^4$ with $c_2(E) = 1$. This reduces us to an arbitrary choice, which we make according to the conventions in [8].

### 2.2 Donaldson Invariants

In this section we define various cohomology classes on the quotient configuration space $B(X)$, and study when these classes can be localized on compact subsets of the moduli
space of ASD connections. In this section $X$ will always be a compact, connected, oriented
4-manifold.

2.2.1 Cohomology Classes

We now begin to study the topology of the quotient configuration space $\mathcal{B}(P)$. Because
the concepts in this section are fairly general, we write $G$ for $SU(N)$ and $G^{\text{ad}}$ for $PU(N)$.
We begin by constructing a principal $G^{\text{ad}}$ bundle over $X \times \mathcal{B}(P)$.

**Definition 26.** We define the universal bundle to be

$$\mathcal{P} = P \times_{g(P)} A^*(P)$$

where $g(P)$ acts from the left on $A(P)$ by pullback and from the right on $P$ by automor-
phisms.

Note that there is a natural quotient map $\mathcal{P} \rightarrow X \times \mathcal{B}(P)$, whose fibers admit an action
of $G^{\text{ad}}$.

**Proposition 27.** $\mathcal{P} = P \times_{g(P)} A^*(P)$ is a principal $G^{\text{ad}}$ bundle over $B^*(P) \times X$.

**Proof.** We only need to produce local trivializations of $\mathcal{P}$. If $A$ is a connection with slice $T_{A,\epsilon}$
and $\phi : U \times G^{\text{ad}} \rightarrow P|_{U}$ is a local trivialization on $X$, then the map

$$\phi \times \text{id} : (U \times G^{\text{ad}}) \times T_{A,\epsilon} \rightarrow \mathcal{P}$$

defines a trivialization near $([A], x)$.

The universal bundle allows us to construct an abundance of rational cohomology classes
on $B^*(P)$. In particular, its classifying map $B^*(P) \times X \rightarrow BG$ induces a map on rational
cohomology

$$\mu^* : H^*(BG^{\text{ad}}; \mathbb{Q}) \rightarrow H^*(B^*(P); \mathbb{Q}) \otimes H^*(X; \mathbb{Q})$$

and hence by duality a map on rational (co)homology

$$\mu : H^*(BG^{\text{ad}}; \mathbb{Q}) \otimes H_*(X; \mathbb{Q}) \rightarrow H^*(\mathcal{B}(P); \mathbb{Q})$$

In place of $\mu(a \otimes \sigma)$ we will usually write $\mu_a(\sigma)$. For the special case where $\sigma$ is a point we
will write $\mu_a(pt) = a$. This notation is unambiguous when $X$ is connected.

Note that the classes $\mu_a(\sigma)$ necessarily satisfy some relations, due to the fact that $\mu^*$ is an
algebra homomorphism. In general, given a graded-commutative algebra $\Lambda$ and a topological
space $X$ there is an algebra $A(X; \Lambda)$ that is universal among all pairs $(A, \mu)$, where $A$ is a
graded-commutative algebra and

$$\mu : A \rightarrow H^*(X; \mathbb{Q}) \otimes \Lambda$$

is a grading-preserving homomorphism. We can construct $A(X; \Lambda)$ explicitly, given genera-
tors and relations for the algebras $\Lambda$ and $H^*(X; \mathbb{Q})$. 

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Proposition 28. If $\Lambda$ is a polynomial algebra on generators $a_1, \ldots, a_n$ of even degree, and $H_*(X)$ has basis $\sigma_1, \ldots, \sigma_n$, then $A(X; \Lambda)$ is a free graded-commutative algebra on the elements $\mu_{a_i}(\sigma_j)$.

Proof. Let $m^* : H_*(X) \to H_*(X) \otimes H_*(X)$ be the comultiplication induced by the cup product on $H^*(X)$. Then the relations that must be satisfied by the classes $\mu_{a_i}(\sigma)$ take the form

$$\mu_{a_i \cdot a_j}(\sigma) = \mu_{a_i} \otimes \mu_{a_j}(m^*(\sigma)) = \sum_{i,j} m_{ij} \mu_{a_i}(\sigma_i) \mu_{a_j}(\sigma_j)$$

(2.7)

for constants $m_{ij}$. These relations merely say that any $\mu_{a_i}(\sigma)$ can be expressed as a polynomial in the $\mu_{a_i}(\sigma)$.

\[\square\]

Definition 29. We write $A(X; G) = A(X; H^*(BG^\text{ad}; \mathbb{Q}))$. When context is clear we omit $G$.

In general $A(X; G)$ can be described quite concretely, due to the following description of $H^*(BG^\text{ad}; \mathbb{Q})$.

Proposition 30. Let $G$ be a semisimple compact Lie group. Then $H^*(BG; \mathbb{Q})$ is a polynomial algebra on finitely many even generators, and the map $BG \to BG^\text{ad}$ induces an isomorphism on rational cohomology.

Proof. It is well known that the map $BG \to BT$ induces an isomorphism $H^*(BG; \mathbb{Q}) \to H^*(BT; \mathbb{Q})^W$, for any compact Lie group $G$, and the latter is a polynomial algebra on finitely many generators. To prove the second part, note that the map $BG \to BG^\text{ad}$ has homotopy fiber $BZ$, where $Z$ is the center of $G$. Since $G$ is semisimple, $Z$ is finite, so $BZ$ has the rational cohomology of a point. Hence the Serre spectral sequence

$$H^*(BG^\text{ad}; H^*(BZ; \mathbb{Q})) \Rightarrow H^*(BG; \mathbb{Q})$$

collapses on the $E^2$ page, which means that $H^*(BG^\text{ad}; \mathbb{Q}) \to H^*(BG; \mathbb{Q})$ is an isomorphism.

\[\square\]

For example, if $G = U(N)$ we might take as generators the Chern classes $c_2, \ldots, c_N$. On the other hand, Newton’s theorem on symmetric polynomials implies that we could equally well use the first $N$ components of the Chern character, $Ch_1, \ldots, Ch_N$. In the context of higher rank invariants we need a set of generators for the group $PU(N)$.

Definition 31. We denote by $a_2, \ldots, a_N$ the rational cohomology classes corresponding to $c_2, \ldots, c_N$ under the isomorphism $H^*(BP^U(N), \mathbb{Q}) \to H^*(BSU(N); \mathbb{Q})$. Given a homology class $\sigma \in H_*(X)$ we write $\mu_\sigma(\sigma)$ for the element $\mu_{a_i}(\sigma) \in A(X; PU(N))$. We also make the abuse of notation $\mu_\sigma(\text{pt}) = a_i$.

Note that classes $a_i$ are not the Chern classes of any bundle over $BP^U(N)$, and in fact they are not integral classes in general. However, we can construct them as rational linear combinations of Chern classes, using the following general principle.
Proposition 32. Let $G$ be a compact Lie group, and let $P \to BG$ be the universal $G$-bundle. Then $H^*(BG; \mathbb{Q})$ is generated by classes of the form $c_i(V(P))$, where $V$ is a finite dimensional representation of $G$.

Proof. It is useful to think in terms of the Chern character $\text{Ch}(V(P))$. This gives us a ring homomorphism

$$\text{Ch}_G : R(G) \to \prod_n H^n(BG; \mathbb{Q}),$$

where $R(G)$ is the virtual representation ring with rational coefficients, and $\prod_n H^n(BG; \mathbb{Q})$ is the completion of the usual cohomology ring, with the product topology. When $G = T$ is a torus, the map $\text{Ch}_G$ is injective and has dense image (in the product of the discrete topologies). In general, restricting to a maximal torus $T \subset G$ gives us a commutative diagram:

$$\begin{array}{ccc}
R(G; \mathbb{Q}) & \xrightarrow{\text{Ch}_G} & \prod_n H^n(BG; \mathbb{Q}) \\
\text{res} & & \\
R(T; \mathbb{Q})^W & \xrightarrow{\text{Ch}_T} & \prod_n H^n(BT; \mathbb{Q})^W
\end{array}$$

Both the left and right arrows are isomorphisms, and since $\text{Ch}_T$ is $W$-equivariant, the bottom arrow is an injection with dense image. Hence so is the top arrow. Since the Chern character can be explicitly written as a formal power series in the Chern classes, this shows that Chern classes of associated bundles generate $H^*(BG; \mathbb{Q})$. \hfill \Box

In the case $G = SU(N)$ we can be even more explicit. Let $V = \mathbb{C}^N$ be the defining representation of $SU(N)$, and let $W = V^\otimes N$. The center of $SU(N)$ acts trivially on $W$, hence it descends to a representation of $PU(N)$. Viewing $V$ and $W$ as bundles over $BSU(N)$, we have $\text{Ch}(W) = (\text{Ch}(V))^N$, hence taking an appropriate branch of the $N$-th root power series we can write $\text{Ch}(V) = \sqrt[N]{\text{Ch}(W)}$. Since the Chern classes $c_k(V)$ can be expressed as rational polynomials in the components of $\text{Ch}(V)$ and vice versa, we see that the classes $a_k \in H^*(BPU(N); \mathbb{Q})$ can be expressed as rational polynomials in the Chern classes $c_k(W)$, with $2 \leq k \leq N$. Likewise, the classes $\mu_i(\sigma)$ can be represented as rational polynomials in the classes $\mu_{c_k(W)}(\tau_j)$, where $\tau_1, \ldots, \tau_n$ form a basis for $H_*(X)$.

2.2.2 Invariants

In the previous section, we defined a map $\mu : A(X; G) \to H^*(B(X); \mathbb{Q})$. To produce this map, however, we needed to use the existence of a classifying map for the universal bundle. Strictly speaking, we only know that such a map exists because $B(P)$ is a separable Banach manifold and therefore has the homotopy type of a CW complex. If we only wish to evaluate cohomology classes unambiguously on compact submanifolds $M \subset B(P)$, such a construction is valid but quite inexplicit.

On the other hand, the ASD moduli space $\mathcal{M}(P)$ is noncompact in general. It does have a fundamental class in compactly supported cohomology, but in order to make use of this we must have some notion of support for classes in $A(X; G)$. Therefore, we now give a more explicit recipe for computing these classes. Our starting point is an explicit description of the Chern classes of a complex vector bundle.
Proposition 33. Let $X$ be a finite dimensional compact manifold, let $E \to X$ be a vector bundle of rank $N$, and let $s : X \to \text{Hom}(\mathbb{C}^{n-k+1}, E)$ be a section that is transverse to the stratification of $\text{Hom}(\mathbb{C}^{n-k+1}, E)$ by rank. Define

$$V(s) = \{ x \in X \mid \text{rk } s(x) \leq N - k \}$$

Then $V$ is a stratified subspace of $X$, with strata of even codimension. It therefore has a fundamental class $[V]$, which is Poincaré dual to $c_k(E)$.

Proof. The rank stratification of $\text{Hom}(\mathbb{C}^{N-k+1}, \mathbb{C}^N)$ is a stratification by complex algebraic varieties. In particular, each stratum has even codimension. Since $V$ is (locally) the pullback of the rank stratification under a transverse map, it also has strata of even codimension.

To show that $[V]$ represents $c_k(E)$ it suffices to check the case where $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles. In this case let $\sigma_i$ be transverse sections of $L_i$, whose zero loci intersect transversely. Now define $n - k + 1$ sections by

$$s_i = \sum \sigma_i,$$

where $C = (c_{ij})$ is a constant matrix of size $(N - k + 1) \times n$, whose $(N - k + 1) \times (N - k + 1)$ minors all have full rank. The locus of points $x \in X$ where the vectors $s_i(x)$ become linear dependent is then the same as the locus where $k$ of the $\sigma_j(x)$ vanish simultaneously. In other words, the class represented by $V(s_1, \ldots, s_{n-k+1})$ is

$$\text{PD}[V] = \sum_{i_1, \ldots, i_k} c_1(L_{i_1}) \cup \cdots \cup c_1(L_{i_k}) = c_k(E)$$

Now consider the universal bundle $\mathbb{P} \to B^*(P) \times X$. It is a principal $PU(N)$ bundle, hence we can take the faithful representation $W = (\mathbb{C}^N)^{\otimes N}$ and form its associated bundle $\mathbb{W} \to B^*(P) \times X$.

Proposition 34. Let $M$ be a finite dimensional manifold and let $f : M \to B^*(P) \times X$ be a smooth map. Then for each integer $k$ and every even integer $p \geq 2$ there exists a section $s : B^*(P) \times X \to \text{Hom}(\mathbb{C}^{n-k+1}, \mathbb{W})$, continuous in the $L^1_p$ topology on $B^*(P)$, such that the pullback $s \circ f$ is smooth and transverse to the stratification of $\text{Hom}(\mathbb{C}^{n-k+1}, f^*\mathbb{W})$ by rank.

Proof. We refer to Kronheimer [8].

Let $\sigma \in H_{\leq 2}(X)$ be an integral homology class of dimension at most 2. It is always possible to represent $\sigma$ by an immersed submanifold $\Sigma \subset X$. Let $\nu(\Sigma) \subset X$ be a connected 4-dimensional submanifold with boundary whose interior contains $\Sigma$. Let $P_\Sigma$ be the restriction of $P$ to $\nu(\Sigma)$. Then there is a partially defined restriction map $r : B^4(P) \to B^4(P_\Sigma)$ whose domain of definition consists of those connections which remain irreducible when restricted to $\nu(\Sigma)$. Denote this domain of definition by $B^4(P, \Sigma)$.

Now let $M \subset B^4(P, \Sigma)$ be a finite dimensional submanifold. Let $\sigma_1, \ldots, \sigma_r$ be homology classes on $X$ and let $\alpha_i = c_{k_i}(W) \in H^*(BU(N))$. Suppose that there is a well-defined
restriction map

\[ r = \prod_i r_i : M \to \prod_i B^*(P_{\Sigma_i}), \]

or in other words that every connection in \( M \) remains irreducible when restricted to each \( \nu(\Sigma_i) \). Consider the associated bundles \( W_{\Sigma_i} \) over \( B^*(P_{\Sigma_i}) \times \nu(\Sigma_i) \). Each of these pulls back to a bundle \( W_i \) on the product

\[ B = \prod_i B^*(P_{\Sigma_i}) \times \Sigma_i. \]

As above we can choose sections \( s_i : B \to \text{Hom}(\mathbb{C}^{n-ki+1}, W_i) \) such that the restriction

\[ s = \prod_i s_i \circ r : M \times \Sigma_1 \times \cdots \times \Sigma_r \to \prod_i r^*\text{Hom}(\mathbb{C}^{n-ki+1}, W_i) \]

is transverse to the product of the rank stratifications of \( \text{Hom}(\mathbb{C}^{n-ki+1}, W_i) \). If \( M \) is compact and oriented then the pairing

\[ \langle \mu_{\alpha_1}(\sigma_1) \cdots \mu_{\alpha_r}(\sigma_r), M \rangle \]

is equal to the number of points \( m \in M \), counted with sign, such that all components of \( s(m) \) have strictly less than full rank.

**Proposition 35.** Let \( P = P(E) \), and let \( \alpha_i, \sigma_i \) as above. Then there are:

1. Neighborhoods \( \nu(\Sigma_i) \) such that the restriction maps \( r_i : M_{\nu(P(F))} \to B^*(\nu(\Sigma_i), P(F)) \) are defined for all \( U(N) \) bundles \( F \) with \( c_1(F) = c_1(E) \) and \( c_2(E) < c_2(F) \).

2. Sections \( s_i \) such that \( \prod_i s_i \circ r \) is transverse to \( M_{\nu(P(F))} \times \Sigma_1 \times \cdots \times \Sigma_r \).

**Proof.** We again refer to Kronheimer [8].

**Proposition 36.** Let \( \alpha_i, \sigma_i, \Sigma_i \) as above, and let \( V_k \subset B(\Sigma_i) \times \Sigma_i \) be the representatives of \( z_i = \mu_{\alpha_i}(\sigma_k) \) as in Proposition 35. Let \( M(P) \) have dimension \( d \), and suppose that the sum of the degrees of the classes \( z_k \) is less than or equal to \( d \). Then the intersection

\[ M(P) \times \Sigma_1 \times \cdots \times \Sigma_r \cap V_1 \cap \cdots \cap V_r \quad (2.8) \]

is 0-dimensional and compact. The signed count of points in this intersection is independent of the choice of metric, perturbations, and sections \( s_i \). It only depends on the homology classes \( \sigma_i \) represented by \( \Sigma_i \), and this dependence is multilinear (over \( \mathbb{Z} \)). It is graded-symmetric in its arguments in the sense that swapping two classes \( \mu_{\alpha_i}(\sigma_i) \) and \( \mu_{\alpha_j}(\sigma_j) \), of degrees \( d_i \) and \( d_j \) respectively, changes the sign of the count by a factor of \( (-1)^{d_id_j} \).

**Proof.** This is a standard dimension counting argument. Choose the neighborhoods \( \nu(\Sigma_i) \) such that all triple intersections

\[ \nu(\Sigma_{i_1}) \cap \nu(\Sigma_{i_2}) \cap \nu(\Sigma_{i_3}) \]
are empty, and every double intersection involving a point class is empty as well. Any point \( x \in X \) is contained in at most two neighborhoods \( \nu(\Sigma_i) \), and at most one neighborhood such that \( \Sigma_i \) is a point. Thus for any finite set of points \( \{x_1, \ldots, x_k\} \subset X \), the intersection of all \( V_i \) such that \( \{x_j\} \cap \nu(\Sigma_i) = \emptyset \) has codimension strictly larger than \( d - 4Nk \). By Proposition 35, such an intersection is disjoint from \( \mathcal{M}(P(F)) \), where \( F \) is the bundle with \( c_1(F) = c_1(E) \) and \( c_2(F) = c_2(E) - k \).

Now let \( (a_n, y_n^1, \ldots, y_n^r) \) be a sequence of points in the intersection (2.8). The connections \( A_n \) must limit, after passing to a subsequence and bubbling at a finite set of points \( x_1, \ldots, x_k \), to a connection \( A_\infty \) on \( P(F) \). But the moduli space \( \mathcal{M}(P(F)) \) is transverse to the intersection of all \( V_i \) such that \( \{x_j\} \cap \nu(\Sigma_i) = \emptyset \). In particular the set of possible limiting connections \( A_\infty \) is empty, unless \( k = 0 \), in which case the limit lies in (2.8). This shows that the intersection is compact.

To show graded symmetry, just observe that the only effect of swapping two factors is to change the orientation on \( \Sigma_1 \times \cdots \times \Sigma_r \), and the effect of this change is to modify the orientation by a factor of \((-1)^{d_i d_j}\).

To show independence of metric and perturbation, sections, and representatives of the \( \Sigma_i \), we apply the same argument to a 1-parameter family of moduli spaces obtained by varying the metric and perturbation, restricting the family to neighborhoods of various cobordisms \( \Sigma_i \Rightarrow \Sigma'_i \), and taking transverse sections on the configuration spaces of these neighborhoods. We thereby obtain a compact, oriented cobordism between two sets of signed points, which verifies that the signed counts are equal.

The same argument also shows that the invariant is linear, because whenever we can write \( \sigma = \sigma' + \sigma'' \) there is a homology (more precisely, a stratified cobordism) from \( \Sigma \) to the union of \( \Sigma' \) and \( \Sigma'' \).

**Definition 37.** Given \( \alpha_i \) and \( \sigma_i \) as above, and a class \( c \in H^2(X) \), we define the \( SU(N) \) Donaldson invariant

\[
D_{X,c}(\mu_{\alpha_1}(\sigma_1) \cdots \mu_{\alpha_r}(\sigma_r))
\]

to be the signed count of points defined in Proposition 36, viewed as a linear map

\[
D_{X,c} : A(X; SU(N)) \to \mathbb{Q}
\]
defined on the image of \( \mu : H^*(PU(N)) \otimes H^{<2}(X) \to A(X; SU(N)) \).

There is something a bit arbitrary about our definition of \( D_{X,c} \), in that we chose a particular associated bundle \( W \) in the construction. We could have made the same definition as above, replacing \( \mu_{\alpha_i}(\sigma_i) \) with a combination of classes \( \mu_{\beta_j}(\tau_j) \), where the \( \beta_j \) are Chern classes of a different associated bundle and the \( \tau_j \) are different homology classes on \( X \) (see equation 2.7). We would like to know that such a modification results in the same map \( D_{X,c} : A(X; SU(N)) \to \mathbb{Q} \). We can use the following refinement of our dimension counting argument to handle some cases:

**Proposition 38.** Let \( \alpha_k \) and \( \sigma_k \) as above. Suppose further that all geometric representatives \( \Sigma_1, \ldots, \Sigma_{r-1} \) are mutually disjoint (but not necessarily disjoint from \( \Sigma_r \)). Then the signed count of points in the intersection (2.8) depends only on \( \mu_{\alpha_k}(\Sigma_r) \) as an abstract cohomology class on \( B(P) \).
Proof. For a fixed metric and perturbation, it suffices to show that the intersection

\[ K = \mathcal{M}(P) \times \Sigma_1 \times \cdots \times \Sigma_r \cap V_1 \cap \cdots \cap V_{r-1} \]

is compact, because then the signed count of points in \( V_r \cap K \) is just given by the cohomological pairing \( \langle \mu_\alpha_\tau(\sigma_r), [K] \rangle \), which does not depend on the choice of geometric representative for \( \mu_\alpha_\tau(\sigma_r) \).

As in Proposition 36, consider a finite set of points \( x_1, \ldots, x_k \in X \). Because \( \Sigma_1, \ldots, \Sigma_{r-1} \) are mutually disjoint, each point \( x_j \) can be contained in at most one neighborhood \( \nu(\Sigma_i) \). Hence the intersection of all \( V_i \) such that \( \{x_j\} \cap \nu(\Sigma_i) = \emptyset \) has codimension at least \( d - 2Nk \). In particular, it is disjoint from the moduli space \( \mathcal{M}(P(F)) \) with \( c_2(F) = c_2(E) - k \). To see that \( K \) is compact, we now proceed exactly as in Proposition 36.

Clearly we can do better in cases where there are not “too many” intersections between the \( \Sigma_i \). However, in the simplest cases it is impossible to do better using the methods above. For example, we might try to evaluate classes \( \mu_{a_3}(\sigma_1), \mu_{a_3}(\sigma_2), \) and \( \mu_{a_3}(\sigma_3) \) on a 12-dimension moduli space of \( SU(3) \) connections. If the representatives \( \Sigma_i \) each have nontrivial pairwise intersections, then we cannot show that \( V_2 \cap V_3 \) is compact, because there might be bubbling on \( \Sigma_2 \cap \Sigma_3 \).

One way to resolve the issue is to use the blowup formula. In the next chapter we will prove a vanishing result (Proposition 65) which implies that if \( \Sigma_1 \) is a sphere of self-intersection \(-1\), then the signed count of Proposition 38 automatically vanishes. As a consequence, we can blow up \( X \) at all points where the \( \Sigma_i \) intersect, then replace each \( \Sigma_i \) with its proper transform \( \xi_i = \Sigma_i + \sum_j \pm e_{ij} \), without changing the signed count of points. We can then replace each \( \mu_{a_3}(\sigma_i) \) with a combination of \( \mu_{a_j}(\sigma_i) \) in the blowup. Applying Proposition 65 again, we can eliminate all \( \tau_i \) which intersect the exceptional curves. Blowing back down, we find that the corresponding replacement is valid on \( X \) as well.

## 2.3 Cylindrical Ends and Gluing

### 2.3.1 Cylindrical Ends

Much of our discussion of ASD moduli spaces on compact manifolds carries over to more general noncompact manifolds.

**Definition 39.** We say that an oriented Riemannian 4-manifold \( X \) has asymptotically cylindrical ends if there is a compact subset \( K \subset X \) whose complement \( X \setminus K \) is diffeomorphic to a product \( (0, \infty) \times Y \) for some 3-manifold \( Y \), such that the metric on \( X \setminus K \) is given by

\[ g_X = dt^2 + g_Y + e^{-\lambda t} h(t), \]

where \( g_Y \) is a fixed metric on \( Y \), \( \lambda \) is a positive constant, and \( h(t) \) is a smooth bounded, \( t \)-dependent section of \( \text{Sym}^2(T^*Y) \). If \( h(t) \) is identically zero we say that \( W \) has cylindrical ends.

To set up configuration spaces on manifolds with asymptotically cylindrical ends we need to choose appropriate boundary conditions at infinity.
**Definition 40.** An adapted bundle \((P, \eta)\) on a 4-manifold \(X\) with asymptotically cylindrical ends is a principal bundle \(P \rightarrow X\), together with a choice of a flat connection \(\eta\) on \(P|_Y\).

Given an adapted bundle, we choose connection \(A_0\) that restricts to \(\eta\) on the cylinder \(Y \times [0, \infty)\). We would like to define the configuration space to be the space of all connections on \(X\) that approach \(\eta\) asymptotically at infinity. If \(\eta\) is irreducible we can imitate the compact case exactly and define

\[
\mathcal{A}(P; \eta) = \{ A_0 + a | a \in L^2_t(X, \Lambda^1 \otimes \text{ad}P) \},
\]

However, when \(\eta\) is reducible this definition is inadequate. The issue is that the operators \(d_A^*d_A\) are not Fredholm, hence our construction of slices is invalid.

To remedy the difficulty, we need to introduce weighted norms on the space of sections of \(\Lambda^1 \otimes \text{ad}P\). Fix a small positive real number \(\alpha\), and a smooth nonvanishing function \(W\) on \(X\) such that

\[ W(y, t) = e^{\alpha t} \]
on the cylindrical end \(Y \times (0, \infty)\). Define a weighted norm

\[ |a|_{L^2_t\alpha} = |W|_{L^2_t} \]

and define \(L^2_t\alpha(X, \Lambda^1 \otimes \text{ad}(P))\) to be the completion of the space of smooth sections with respect to this norm.

**Definition 41.** Let \((P, \eta)\) be an adapted bundle on a 4-manifold \(X\) with cylindrical ends. We define

\[
\mathcal{A}^\alpha(P, \eta) = \{ A_0 + a | a \in L^2_t\alpha(X, \Lambda^1 \otimes \text{ad}P) \}.
\]

\[
\mathcal{G}^\alpha(P, \eta) = \{ \gamma : X \rightarrow G(P) | \gamma^{-1}d_{A_0}\gamma \in L^2_t\alpha(X, \Lambda^1 \otimes \text{ad}(P)) \}
\]

\[
\mathcal{B}^\alpha(P, \eta) = \mathcal{A}^\alpha(P, \eta)/\mathcal{G}^\alpha(P, \eta)
\]

When the context is clear, we will drop \(\alpha\) and \(\eta\) from the notation.

There is a homomorphism \(ev_\infty : \mathcal{G}^\alpha(P) \rightarrow \Gamma_\eta\) given by "evaluation at infinity". It is natural to consider the kernel of this homomorphism.

**Definition 42.** We define the framed gauge group to be

\[
\mathcal{G}^\alpha_0(P) = \{ \gamma \in \mathcal{G}^\alpha(P) | \gamma_\infty = 1 \},
\]

and we define the framed configuration space to be

\[
\mathcal{B}^\alpha_0(P) = \mathcal{A}^\alpha(P)/\mathcal{G}^\alpha_0(P)
\]

The framed gauge group is in some ways a more natural group to consider. For example, its tangent space is \(L^2_t\alpha(X, \text{ad}(P))\), which fits naturally into the framework of weighted spaces. It also acts freely on \(\mathcal{A}^\alpha(P)\), because a nontrivial stabilizer would evaluate nontrivially at infinity. Thus \(\mathcal{B}^\alpha(P, \eta)\) has no quotient singularities, unlike the configuration space on a compact manifold.
**Proposition 43.** \( \tilde{B}(P, \eta) \) is a Banach manifold with a smooth action of \( \Gamma_\eta \).

**Proof.** Let \( d^*_A \) denote the formal adjoint of \( d_A \) with respect to the weighted norm. To construct a slice for the action of \( \mathcal{G}_0^\alpha(P) \), we consider its kernel:

\[
T_{A, \varepsilon} = \left\{ A + a \left| d^*_A a = 0, |a|_{L^2} < \varepsilon \right. \right\}.
\]

To check that this is a slice we can imitate the compact case. The key point is that \( d^*_A d_{A} \) is now a Fredholm operator, provided that \( \alpha^2 \) is nonzero and smaller than any eigenvalue of the operator \( d^*_\eta d_{\eta} \). Since \( \mathcal{G}_0^\alpha(P) \) acts freely we see that \( \tilde{B}(P) \) is a Banach manifold with a smooth action of \( \mathcal{G}^\alpha(P)/\mathcal{G}_0^\alpha(P) \).

It remains to be shown that the homomorphism \( \text{ev}_\infty : \mathcal{G}^\alpha(P) \to \Gamma_\eta \) is surjective, so that all of \( \Gamma_\eta \) acts on \( \tilde{B}(P) \). The proof of Proposition 15 shows that \( \Gamma_\eta \) is connected mod \( \mathbb{Z} \). Since elements of \( \mathbb{Z} \) extend over \( X \), it suffices to show any \( \gamma \) in the identity component of \( \Gamma_\eta \) extends as well. But such an extension can be given explicitly, by homotoping \( \gamma \) to the identity along the neck, then extending by the identity over the rest of \( X \).

Our discussion of moduli spaces and perturbations in the compact case carries over to manifolds with asymptotically cylindrical ends.

**Definition 44.** Let \( \pi \) be a perturbation of the ASD equations on an adapted bundle \( (P, \eta) \). We define the framed moduli space of solutions to the perturbed equations to be

\[
\tilde{\mathcal{M}}_\pi(P) = \left\{ A \in \tilde{B}(P) \mid F^+_A = V_\pi(A) \right\}
\]

**Proposition 45.** Let \( P \) be an adapted bundle over a Riemannian 4-manifold \( X \) with asymptotically cylindrical ends. Then there exists a residual set of small perturbations \( \pi \), supported in a compact subset of \( X \), such that every \( A \in \tilde{\mathcal{M}}_\pi(P) \) is regular.

**Proof.** The connection \( A \) is regular if and only if the cokernel of \( d^*_A \) is trivial. The proof of Proposition 23 shows that we can choose compactly supported perturbations such that any element of the cokernel vanishes outside of the cylindrical end. But elements of the cokernel satisfy a unique continuation property on the cylindrical end (because they are solutions of an ODE), and hence must vanish there as well.

**2.3.2 The Equivariant Universal Bundle**

Because connections in \( \tilde{B}(P, \eta) \) all have trivial stabilizers, there is a universal \( PU(N) \) bundle \( \tilde{P} \) defined over all of \( \tilde{B}(P, \eta) \times X \), not just over the space of irreducible connections. This bundle is equivariant with respect to the action of \( \Gamma_\eta \), so it has equivariant characteristic classes \( a_k(\tilde{P}) \in H^*_\Gamma_\eta(\tilde{B}(P, \eta)) \). To compute these characteristic classes we will need a description of the restriction of \( \tilde{P} \) to the orbit of a reducible connection \( A \in \tilde{B}(P, \eta) \).

**Proposition 46.** Suppose that \( A \) admits a reduction of structure to a subgroup \( H \subset PU(N) \), so that \( P = Q \times_H PU(N) \) for some \( H \)-bundle \( Q \) and \( A \) is induced by a connection on \( Q \).
Then the restriction of $\widetilde{P}$ to the orbit $O_A = \Gamma_\eta/\Gamma_A$ in $\tilde{B}(P) \times X$ is isomorphic to

$$(\Gamma_\eta \times Q) \times_{\Gamma_A \times H} PU(N)$$

as a $\Gamma_\eta$-equivariant bundle over $O_A$.

**Proof.** By definition, the universal bundle is

$$\mathbb{P} = \mathcal{G}_0(P) \backslash \tilde{B}(P) \times P = \Gamma_\eta \times_{\mathcal{G}(P)} (\tilde{B}(P) \times P)$$

Restricting to the orbit of $A$, we have an isomorphism

$$\Gamma_\eta \times_{\Gamma_A} P \to \mathbb{P}$$

given by sending $(z,p)$ to $(z,A,p)$ for any $z \in \Gamma_\eta$ and any $p \in P$. Applying $P = Q \times_H PU(N)$ gives

$$\Gamma_\eta \times_{\Gamma_A} P = (\Gamma_\eta \times Q) \times_{\Gamma_A \times G} PU(N)$$

as desired. □

**Corollary 47.** If $\sigma \subset X$ is a cycle of positive dimension then for any $\alpha \in H^*(BU(N))$ the equivariant classes $\bar{\mu}_\alpha(\sigma)$ are trivial in a neighborhood of the trivial connection $\Theta$.

**Proof.** There is a neighborhood $U$ of the orbit $O_\Theta$ that retracts equivariantly onto $O_\Theta$, so it suffices to show that $\bar{\mu}_\alpha(\sigma)$ is trivial on $O_\Theta$. Because $\Gamma_\Theta = PU(N)$ we have $H = \{\text{id}\}$, hence Proposition 46 shows that $\mathbb{P}$ is pulled back from an equivariant bundle on a point. Therefore,

$$\bar{\mu}_\alpha(\sigma) = \alpha(\mathbb{P})/\sigma = 0$$

as desired. □

### 2.3.3 Index Theory

We can compute the virtual dimensions of cylindrical end moduli spaces using the excision principle for indices.

**Proposition 48.** Let $X_{12} = X_1 \cup X_2$ and $X_{34} = X_3 \cup X_4$ be 4-manifolds such that $X_1 \cap X_2 = X_3 \cap X_4 = Y \times (0,1)$ for some 3-manifold $Y$. Write $X_{14} = X_1 \cup X_4$ and $X_{32} = X_3 \cup X_2$. If $D_i$ are elliptic operators on $X_i$ that agree on $Y \times (0,1)$, and $D_{ij}$ are the elliptic operators obtained by identifying $D_i$ and $D_j$ over $Y \times (0,1)$, then we have

$$\text{ind}(D_{12}) + \text{ind}(D_{34}) = \text{ind}(D_{14}) + \text{ind}(D_{32})$$

**Proposition 49.** Let $V_\pm$ be adapted $U(N)$ bundles over 4-manifolds $X_\pm$, with limiting flat connections $\eta_\pm$. Suppose that $X_+$ has a cylindrical end $Y \times (0,\infty)$, that $X_-$ has a cylindrical end $Y \times (-\infty,0)$ and that there is an isomorphism $\gamma : \eta_+ \to \eta_-$. Let $X$ be the closed 4-manifolds obtained by gluing together $X_+$ and $X_-$ and let $V$ be the bundle obtained by gluing $V_+$ and $V_-$. Let $A_\pm$ be connections on $X_\pm$, and let $DA_\pm$ be the corresponded Dirac operator,

$$d_A : L^{2,1}_2(V \otimes \Lambda^1) \to L^{2,1}_2(V \oplus V \otimes \Lambda^2).$$
viewed as an operator on weighted spaces. Then for all sufficiently small $\alpha > 0$ and any connection $A$ on $V$ we have

$$\text{ind}(D_{A+}) + \text{ind}(D_{A-}) = \text{ind}(D_A) + h^0(\eta) - h^1(\eta)$$

where $h^1(\eta)$ are the ranks of the cohomology groups of $Y$ with coefficients twisted by $\eta$.

**Proof.** By the excision principle, we can reduce to computing the index of the operator

$$D^\alpha_\eta = \frac{\partial}{\partial t} + H_\eta + \frac{W'(t)}{W(t)} \epsilon$$

on the cylinder $Y \times \mathbb{R}$, where $H_\eta$ is the self-adjoint operator

$$H_\eta = \begin{pmatrix} 0 & d^*_\eta \\ -d_\eta & *d_\eta \end{pmatrix},$$

$\epsilon$ is the operator which acts as $+1$ on $\Lambda^1$ and as $-1$ on $\Lambda^0$, and the weight function $W$ is equal to $e^{-\alpha t}$ for negative $t$ and $e^{\alpha t}$ for positive $t$. But the index of $D^\alpha_\eta$ is the spectral flow of $H_\eta + \frac{\text{dlog}W}{\text{d}t} \epsilon$, which is equal to $h^0(\eta) - h^1(\eta)$. \qed

**Proposition 50.** Let $X$ be a 4-manifold with cylindrical ends, and let $V$ be an adapted $U(N)$ bundle over $X$, with limiting flat connection $\eta$. Then there is a constant $\rho(\eta)$, depending only on $\eta$, such that

$$\text{ind}(D_A) = c_1(V)^2 - 2c_2(V) + N(\chi + \sigma) + \frac{h^1(\eta) - h^0(\eta)}{2} + \rho(\eta).$$

where $A$ is any connection that restricts to $\eta$ on the ends of $X$ and $c_2(V)$ and $c_1(V)^2$ are defined by Chern-Weil integrals of the curvature of $A$.

**Proof.** Given an adapted bundle $V$ define a quantity $\rho(V)$ by

$$\rho(V) = 2\text{ind}D_{A_0} + c_1(V)^2 - 2c_2(V) + N(\chi + \sigma) + h^0(\eta) - h^1(\eta)$$

Note that this formula is additive under gluing adapted bundles, by virtue of Proposition ???. It is zero for a bundle over a closed 4-manifold, by Proposition ???. Thus if $W$ is an adapted bundle that can be glued to $V$, we have

$$\rho(V) + \rho(W) = 0$$

But then if $V'$ is another adapted bundle with limiting connection $\eta$, we have

$$\rho(V) = -\rho(W) = \rho(V'),$$

so $\rho(V)$ only really depends on the limiting connection $\eta$. \qed

Note that $\rho(\eta)$ changes sign if we reverse the orientation on the underlying 3-manifold $Y$. In particular, $\rho(\eta)$ vanishes if $\eta$ is invariant under an orientation reversing diffeomorphism.
Corollary 51. Let $X$ be a 4-manifold with cylindrical ends, let $P = P(V)$ be an adapted $PU(N)$ bundle over $X$, with limiting flat connection $\eta$. Then the formal dimension of the ASD moduli space on $P$ is

$$\dim \mathcal{M}(P) = 4Nc_2(V) + (2 - 2N)c_1(V)^2 + (1 - N^2)\frac{x + \sigma}{2} + \frac{h^0(\eta) + \rho(\text{ad}(\eta))}{2} \tag{2.9}$$

and the formal dimension of the framed moduli space is

$$\dim \tilde{\mathcal{M}}(P) = 4Nc_2(V) + (2 - 2N)c_1(V)^2 + (1 - N^2)\frac{x + \sigma}{2} + \frac{h^0(\eta) - \rho(\text{ad}(\eta))}{2}$$

Proof. Simply apply Proposition 2.9 to the complexification of $\text{ad}(P)$.

Finally, we observe that the results above remain valid for manifolds with asymptotically cylindrical ends, because the operators in that context are homotopic to the ones used above.

2.3.4 The Gluing Theorem

To prove the blowup formula we need to have a general understanding of ASD moduli spaces on a 4-manifold that is decomposed along separating 3-manifold $Y$. In general suppose that we can write $X = X^+ \cup_Y X^-$, where $X^+$ and $X^-$ are joined along their oriented boundaries $\partial X^\pm = \pm Y$. Let $X^\pm_\infty$ be the 4-manifolds obtained by attaching infinite cylinders $(0, \infty) \times Y$ and $(-\infty, 0) \times Y$ to $X^+$ and $X^-$, and let $X^\pm_T$ be obtained by similarly attaching cylinders of length $2T$. Given asymptotically cylindrical metrics on $X^\pm_\infty$ (with the same limiting metric $g_Y$) and a partition of unity $\chi_1 + \chi_2 = 1$ on the cylinder $Z_T = [-T, T] \times Y$ we can form a Riemannian 4-manifold $X_T = X^+_T \cup_{Z_T} X^-_T$, by joining $X^\pm_T$ along $Z_T$ and superimposing their metrics using $\chi$. The rough idea of gluing is that as $T$ tends to infinity, most ASD connections on $X_T$ can be described as a pair of ASD connections on the manifolds $X^\pm_\infty$, plus an isomorphism of their limiting flat connections. The connections which cannot be described in this way admit an analogous decomposition into a series of ASD connections on infinite cylinders $Z_\infty$ (with the product metric $dt^2 + g_Y$), plus a series of identifications of their limiting flat connections.

Proposition 52. Let $X_T = X^+ \cup_Y Z_T \cup_Y X^-$ as above, with $b^+_2(X_T) \geq 2$, and let $P = P(E)$ be a principal bundle over $X_T$, such that $\mathcal{M}(X_T; P)$ has dimension less than $4N$ (so that it is compact). Suppose that the 3-manifold $Y$ carries only finitely many flat $PU(N)$ connections up to gauge equivalence, each of which is regular (but possibly reducible), and that the (possibly perturbed) cylindrical end moduli spaces $\mathcal{M}(X^+; \alpha)$ are regular. Then for $T$ sufficiently large, the moduli space $\mathcal{M}(X_T; P)$ can be covered by finitely many open sets, each of which is diffeomorphic to an equivariant product of the form

$$\mathcal{M}(X^+_\infty, \alpha_1) \times_{\Gamma_1} \mathcal{M}(Z_\infty, \alpha_1, \alpha_2) \times_{\Gamma_2} \cdots \times_{\Gamma_{k-1}} \mathcal{M}(Z_\infty, \alpha_{k-1}, \alpha_k) \times_{\Gamma_k} \mathcal{M}(X^-_\infty, \alpha_k),$$

where the $\alpha_i$ are flat connections on $Y$, and have stabilizers $\Gamma_i$. Moreover, if $\sigma$ is a subspace of $X^\pm$, then the restriction of the universal bundle over $\mathcal{M}(X_T) \times \alpha$ to such an open set is obtained by pulling back the equivariant universal bundle over $\mathcal{M}(X^\pm_\infty, \alpha_k)$. 

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We will not try to give a full exposition of this theorem here. References include [16], [9], [13].
Chapter 3

Higher Rank Blowup Formulas

3.1 Solutions Near a Negative Sphere

To understand the blowup formula it is useful to have a more general understanding of how the $SU(N)$ invariants of a 4-manifold $X$ behave in the presence of an embedded sphere $\sigma_p \subset X$ of negative self intersection $-p$. Following general principals of gluing, we consider an open tubular neighborhood $N_p$ of such a sphere. This noncompact manifold has a single end diffeomorphic to $L_p \times (0, \infty)$, where $L_p = L(p, 1)$ is the quotient of $S^3 \subset \mathbb{C}^2$ by the group of $p$-th roots of unity. In this section, we will study moduli spaces of ASD connections on the manifolds $N_p$.

3.1.1 Regularity

In general one needs perturbations to achieve regularity of the ASD equations. However, in the case of the manifold $N_p$, for sufficiently small $p$ we can find a metric for which transversality is guaranteed automatically.

Proposition 53. Let $p$ be a sufficiently small integer (for our purposes, $p \leq 2$). Then the manifold $N_p$ admits a metric $g_p$ with the following properties:

1. $g_p$ has positive scalar curvature and anti-self-dual Weyl curvature.
2. $g_p$ decays exponentially to a product metric on $(0, \infty) \times L_p$.

Proof. Let $\sigma_1, \sigma_2$, and $\sigma_3$ be an orthonormal coframe field on $S^3$, invariant under the left action of $SU(2)$, such that $\sigma_3$ is invariant under $U(2)$. Define a metric on $(0, \infty) \times S^3$ by

$$g = dr^2 + \sigma_1^2 + \sigma_2^2 + \lambda^2 \sigma_3^2$$

Because both $\sigma_3$ and $\sigma_1^2 + \sigma_2^2$ are invariant under $U(2)$, and $L_p$ is the quotient of $S^3$ by a cyclic subgroup of $U(2)$, $g$ descends to a metric on $(0, \infty) \times L_p$. If we furthermore demand that $\lambda$ extend to a smooth odd function on all of $\mathbb{R}$, satisfying $\lambda(0) = 0$ and $\lambda'(0) = p$, then $g$ will extend to a smooth metric $g_p$ on $N_p$. 37
We now compute the Weyl curvature of the metric \( g \), using the method of Cartan. We choose as an orthonormal coframe
\[
\theta = (\theta_0, \theta_1, \theta_2, \theta_3)^T = (dr, \sigma_1, \sigma_2, \lambda \sigma_3)^T,
\]
and compute its differential using the identities
\[
\begin{align*}
d\sigma_1 &= 2\sigma_2 \wedge \sigma_3 \\
d\sigma_2 &= 2\sigma_3 \wedge \sigma_1 \\
d\sigma_3 &= 2\sigma_1 \wedge \sigma_2.
\end{align*}
\]
We find the matrix \( \omega \) of the Levi-Civita connection from \( d\theta \) by imposing the condition \( d\theta + \omega \wedge \theta = 0 \). The result is:
\[
\omega = \begin{pmatrix}
0 & 0 & 0 & -\lambda \sigma_3 \\
0 & 0 & (2 - \lambda^2)\sigma_3 & -\lambda \sigma_2 \\
0 & (\lambda^2 - 2)\sigma_3 & 0 & \lambda \sigma_1 \\
\lambda \sigma_3 & \lambda \sigma_2 & -\lambda \sigma_1 & 0
\end{pmatrix}
\]
The curvature matrix \( \Omega = d\omega + \omega \wedge \omega \) is given by
\[
\Omega = \begin{pmatrix}
0 & \lambda \theta_{23} & \lambda \theta_{31} & -\frac{\lambda''}{\lambda} \theta_{03} - 2\lambda \theta_{12} \\
-\lambda \theta_{23} & 0 & -2\lambda \theta_{03} + (4 - 3\lambda^2)\theta_{12} & -\lambda \theta_{02} - \lambda^2 \theta_{31} \\
-\lambda \theta_{31} & 2\lambda \theta_{03} + (3\lambda^2 - 4)\theta_{12} & 0 & \lambda \theta_{01} + \lambda^2 \theta_{23} \\
\frac{\lambda''}{\lambda} \theta_{03} - 2\lambda \theta_{12} & \lambda \theta_{02} + \lambda^2 & -\lambda \theta_{01} - \lambda^2 \theta_{23} & 0
\end{pmatrix}
\]
where to save space we have used the shorthand \( \theta_{ij} = \theta_i \wedge \theta_j \). We then compute the self-dual part of the 2-forms \( \Omega_{01} + \Omega_{23}, \Omega_{02} + \Omega_{31}, \text{ and } \Omega_{03} + \Omega_{12} \). The result can be written as a square matrix,
\[
W^+ + \frac{s}{12} = \begin{pmatrix}
\lambda' + \frac{\lambda}{2} \lambda^2 & 0 & 0 \\
0 & \lambda' + \frac{\lambda}{2} \lambda^2 & 0 \\
0 & 0 & -\frac{\lambda''}{2 \lambda} - 2\lambda' + 2 + \frac{3}{2} \lambda^2
\end{pmatrix},
\]
whose traceless part is the self-dual Weyl curvature \( W^+ \). Demanding that it vanish leads us to the ordinary differential equation
\[
\lambda'' + 6\lambda \lambda' + 4\lambda^3 - 4\lambda = 0,
\]
and the positive scalar curvature condition leads us to the inequality
\[
\lambda' + \frac{\lambda^2}{2} > 0.
\]
The equation (3.1) has two special solutions. Namely, if \( \lambda \) satisfies one of the ordinary
differential equations

\begin{align}
\lambda' &= 1 - \lambda^2 \\
\lambda' &= 2 - 2\lambda^2,
\end{align}

then it automatically satisfies (3.1). We can use these special solutions to find \( \lambda(r) \) in the cases \( p = 1 \) and \( p = 2 \), respectively. When \( p = 1 \) integrating (3.3) gives

\[ \lambda(r) = \tanh(r) \]

and in the case \( p = 2 \), integrating (3.4) gives

\[ \lambda(r) = \tanh(2r). \]

Both of these solutions are increasing, so the corresponding metrics have positive scalar curvature by (3.2). They rise exponentially to the limiting value \( \lambda = 1 \), so the corresponding metrics are asymptotically cylindrical and limit on the round metric on \( L_p \).

I have not tried to solve the equation (3.1) explicitly for larger values of \( p \), but numerical solutions indicate that \( \lambda \) is increasing if and only if \( p \leq 2 \), and satisfies the positive scalar curvature inequality (3.2) if and only if \( p \leq 13 \).

In fact, the computation above can be modified to show that there is a unique \( U(2) \)-invariant metric on \( N_p \) with vanishing anti-self-dual Weyl curvature, up to conformal rescaling and reparameterization of the radial coordinate. In particular, the metrics \( g_p \) are conformally equivalent to those described by LeBrun in [11]. It is conceivable that \( g_p \) is conformally equivalent to an asymptotically cylindrical metric with positive scalar curvature, but I have not investigated this possibility.

**Proposition 54.** Every finite energy ASD connection on \( (N_p, g_p) \) is regular.

**Proof.** Consider the Dirac operator

\[ D_A = d_A^+ + d_A^- : \Omega^1(X, \text{ad}(P)) \to \Omega^0(X, \text{ad}(P)) \oplus \Omega^+(X, \text{ad}(P)). \]

Given any \( \omega \in \Omega^2(X, \text{ad}(P)) \), the Weitzenbock formula for \( D_A \) gives us:

\[ d_A(d_A^+)^*\omega = \nabla_A \nabla_A^*\omega + \langle S, \omega \rangle + \langle W^-, \omega \rangle + \langle F_A^+, \omega \rangle \]

where \( \nabla_A \) is the connection on \( \Lambda^+(X) \otimes \text{ad}(P) \) obtained by combining the Levi-Civita connection on \( \Lambda^+ \) with the connection \( A \) on \( \text{ad}(P) \). Since \( F_A^+ = W^- = 0 \) and \( S > 0 \), we conclude that any \( \omega \) in the cokernel of \( d_A^+ \) must vanish identically. Thus for any such connection \( H_A^2 = 0 \), as desired.

In the future we will implicitly use the metric \( g_p \) when referring to ASD moduli spaces on \( N_p \). Therefore, any time we use regularity of the moduli space we will be assuming implicitly that \( p \) is small enough so that \( g_p \) has positive scalar curvature, or that there is a conformally equivalent metric with positive scalar curvature.

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3.1.2 Completely Reducible Connections

On a negative definite manifold like $N_p$, one cannot avoid the presence of reducible connections, even when the bundles in question are coprime to $N$. In this section we'll describe some reducible connections on $N_p$ and the moduli spaces they inhabit.

First we need to briefly discuss the topology of $N_p$. Since it deformation retracts onto $\sigma_p$, we have $H^2(N_p) = \mathbb{Z}$. Since $L_p$ is the quotient of $S^3$ by a fixed point free action of $\mathbb{Z}/p\mathbb{Z}$, we have $\pi_1(L_p) = H_1(L_p) = \mathbb{Z}/p\mathbb{Z}$. Geometrically, let $D_p$ be a fiber of $N_p \to \sigma_p$, oriented so that $D_p$ and $\sigma_p$ intersect positively, and let $\gamma_p$ be the oriented boundary of $D_p$. Then $\gamma_p$ is a generator of $H_1(L_p)$ and $D_p$ is Poincaré dual to a generator of $H^2(N_p)$.

Now let $L$ be the complex line bundle whose Chern class $c_1(L)$ is Poincaré dual to $-D_p$. Since $N_p$ is negative definite, the class $c_1(L)$ can be represented by a square integrable anti-self-dual 2-form $\omega$, and there is a connection $\lambda$ on $L$ with curvature $-2\pi i \omega$, which is unique up to gauge equivalence. At infinity, the connection $\lambda$ decays exponentially to a flat connection $\chi$ on $L_p$, which we can also think of as a character $\chi : \pi_1(L_p) \to U(1)$. The fact that $c_1(L)$ is Poincaré dual to $-D_p$ means that

$$
\chi(\gamma_p) = \text{Hol}_\chi(\gamma_p) = \exp \left( -2\pi i \int_{D_p} \omega \right) = \exp \left( \frac{2\pi i}{p} \int_{\sigma_p} \omega \right) = e^{\frac{2\pi i}{p}},
$$

hence $\chi$ generates the character group of $\pi_1(L_p)$.

Recall that when talking about ASD connections on a bundle $P(E)$, we fixed a connection $\delta$ on the determinant $\det E$. For the manifold $N_p$ it is useful to take $\delta$ to be the unique (up to gauge equivalence) finite energy ASD connection $\lambda^d$, where $d$ is the degree of $\det E$. Then we can view any ASD connection on $P(E)$ as an ASD connection on $E$ and vice versa.

**Definition 55.** We say that a connection $A$ on a $U(N)$ bundle $E$ is completely reducible if it is gauge equivalent to a connection of the form $\lambda_1 \oplus \cdots \oplus \lambda_n$, where each $\lambda_i$ is a connection on a line bundle $L_i$.

There is a general principle that minimal energy moduli spaces on negative definite manifolds contain completely reducible connections. The remainder of this section is dedicated to making this principle precise for the manifolds $N_p$.

**Proposition 56.** Let $k_1, \ldots, k_n$ be integers such that $|k_i - k_j| \leq p$ for all pairs $i, j$. Then the ASD moduli space containing the completely reducible connection

$$
A_k = \lambda^{k_1} \oplus \cdots \oplus \lambda^{k_N}
$$

over $N_p$ has formal dimension

$$
\dim \mathcal{M}(A_k) = 1 - N^2 + \sum_{i \neq j} |k_i - k_j|
$$
Proof. By the index formula (2.9), we have

\[
\dim \mathcal{M}(A_k) = \left( \sum_{i,j} \frac{(k_i - k_j)^2}{p} \right) + 1 - N^2 + \left( \frac{h^0(1) + \rho(1) + \sum_{i,j} -h^0(\chi^{k_i-k_j}) - \rho(\chi^{k_i-k_j})}{2} \right)
\]

It therefore suffices to show that for any integer \( k \) with \(|k| \leq p\) we have

\[
h^0(\chi^k) + \rho(\chi^k) = 1 - 2|k| + \frac{2k^2}{p}.
\]

This fact was proved by Atiyah, Patodi, and Singer in [1].

Corollary 57. Let \( k_1, \ldots, k_n \) as above, and suppose that \( 2p < N + 1 \). Then the moduli space \( \mathcal{M}(A_k) \) has negative virtual dimension, and therefore contains no irreducible connections. In particular, every irreducible ASD connection on \( E_k = L^{k_1} \oplus \cdots \oplus L^{k_N} \) has energy strictly greater than that of \( A_k \). If \( 2p < N + 1 + \frac{4N}{N-1} \) then we can replace "strictly greater than" with "greater than or equal to".

Proof. Without loss of generality, we have \( k_1 \leq k_2 \leq \cdots \leq k_n \). Then for any fixed \( i \), we have

\[
\sum_{j>i} |k_j - k_i| = |k_n - k_i| \leq p
\]

Therefore,

\[
\dim \mathcal{M}(A_k) = 1 - N^2 + \sum_{i} \sum_{j>i} 2|k_j - k_i| \\
\leq 1 - N^2 + 2p(N - 1) \\
= (2p - N - 1)(N - 1)
\]

The right hand side of this equation is negative if \( 2p < N + 1 \), hence \( \mathcal{M}(A_k) \) has negative virtual dimension. If \( 2p < N + 1 + \frac{4N}{N-1} \), then the right hand side is less than \( 4N \), so any moduli space of strictly smaller energy has negative virtual dimension, hence contains no irreducible connections. \( \square \)

As an application, we can compute all the minimal energy moduli spaces on \( \mathbb{CP}^2 \).

Proposition 58. Let \( 0 \leq k < N \) and let \( E_k \) be the \( U(N) \) bundle over \( N_1 \) with \( \langle c_1(E_k) = kc_1(L) \). Then the completely reducible connection

\[
A_k = k\lambda \oplus (N - k)1
\]

has energy strictly less than any other finite energy ASD connection on \( E_k \). As a consequence, the framed moduli space \( \mathcal{M}(A_k) \) is diffeomorphic to the Grassmannian \( \text{Gr}_k(\mathbb{C}^N) \).

Proof. Let \( B \) be an ASD connection on \( E_k \), and suppose that \( B \) has minimal energy among all ASD connections. Write \( B = \bigoplus_i B_i \) as a sum of irreducible ASD connections on bundles
Write \( c_1(F_i) = (k_i + N d_i) c_1(L) \) with \( 0 \leq k_i \leq N \). Then by Proposition 57, the completely reducible connection

\[
A_i = k_i^4 + 1 \oplus (N - k_i) 4
\]

has strictly less energy than \( B_i \). Substituting \( A_i \) for \( B_i \), we obtain a completely reducible connection \( A \) on \( E_k \) with strictly less energy than \( B \). We conclude that if \( A \) has minimal energy then it is completely reducible, so

\[
A = \lambda^{k_1} \oplus \lambda^{k_2} \oplus \cdots \oplus \lambda^{k_N}
\]

for some integers \( k_i \). If \( k_i - k_j > 1 \) for any \( i \) and \( j \), then the connection \( \lambda_i^4 \oplus \lambda_j^4 \) has strictly greater energy than \( \lambda_i^{k+1} \oplus \lambda_j^{k-1} \). Since \( A \) has minimal energy, this implies that \( |k_i - k_j| \leq 1 \) for all \( i \) and \( j \), hence \( A = A_k \).

The description of the framed moduli space then follows from the fact the stabilizer of \( A_k \) is \( \Gamma_k = S(U(k) \times U(n - k)) \), hence its orbit in \( \mathcal{M}(A_k) \) is \( \text{Gr}_k(C^N) = SU(N) / S(U(k) \times U(N - k)) \).

Note that the argument above fails if \( p > 1 \), because the inequality \( 2p < N + 1 \) does not hold for all \( N \). However, the weaker inequality \( 2p < N + 1 + \frac{4N}{N-1} \) is valid for all \( N \) as long as \( p \leq 4 \). In this regime we can therefore hope to partially salvage Proposition 58.

Before proceeding, we need to establish some notation. Let \( 0 \leq r_1 \leq \cdots \leq r_N < p \) be integers, and consider the flat connection

\[
\eta_c = \chi^{r_1} \oplus \chi^{r_2} \oplus \cdots \oplus \chi^{r_N}
\]

on \( L_p \). Let \( r = r_1 + \cdots + r_N \), so that \( c_1(\eta_c) = r c_1(L) \). If \( E \) is an adapted bundle on \( N_p \) with limiting flat connection \( \eta_c \), then we necessarily have

\[
\langle c_1(E), \sigma_p \rangle \equiv r \mod p.
\]

If \( E \) satisfies this constraint, we say that it is admissible for \( \eta_c \).

**Proposition 59.** Let \( E \) be a vector bundle over \( N_p \). Then \( E \) admits a completely reducible ASD connection, satisfying the hypotheses of Proposition 57.

**Proof.** Define a connection

\[
A_{c,s} = \chi^{r_s} \oplus \cdots \oplus \chi^{r_N} \oplus \chi^{r_1+p} \oplus \cdots \oplus \chi^{r_s+p}
\]

and call the underlying bundle of this connection \( E_{c,s} \). Then \( c_1(E_{c,s}) = (r + ps) c_1(L) \), so as \( s \) runs from 1 to \( N \) we obtain all admissible values of \( c_1(E) \mod N \). In particular, for some \( s \) and some integer \( d \) we have \( E = L^d \otimes E_{c,s} \), so \( E \) admits the completely reducible ASD connection \( \lambda^d \otimes A_{c,s} \). \( \square \)

**Proposition 60.** Suppose that \( p \leq 4 \). Let \( E \) be a \( U(N) \) bundle on \( N_p \), and let \( B \) be a finite energy ASD connection on \( E \), which limits on a flat connection \( \eta_c \). Then there is a completely reducible ASD connection \( A \) on \( E \), also with limiting value \( \eta_c \), whose energy is less than or equal to that of \( B \).
Proof. Write \( B = \bigoplus_i B_i \), where each \( B_i \) is an irreducible ASD connection on a bundle \( F_i \) of dimension \( n_i \). Each \( F_i \) admits a completely reducible connection \( A_i = A_{r_i,s_i} \), which satisfies the hypotheses of Proposition 57. Because \( p \leq 4 \), the inequality

\[
2p < n + 1 + \frac{4n}{n - 1}
\]

holds for all \( n \geq 2 \). Since \( B_i \) is irreducible, we conclude that it has energy greater than or equal to that of \( A_i \). Therefore the completely reducible connection \( A = \bigoplus_i A_i \) has energy less than or equal to that of \( B \), as desired. \( \square \)

It is unknown to the author whether Proposition 60 holds for all \( p \). As an example of the difficulties involved, let \( E \) be the trivial \( SU(2) \) bundle over \( N_{13} \), and let \( A \) be the connection \( \lambda^3 \oplus \lambda^{-3} \), which has limiting flat connection \( \eta = \chi^{10} \oplus \chi^{10} \). Then \( A \) has minimal energy among all reducibles limiting on \( \eta \), but the moduli space containing \( A \) has virtual dimension 9. It is conceivable that the moduli space of virtual dimension 1 is always empty in this case, but proving it would require more subtle methods and might depend on the choice of metric on \( N_{13} \).

In the case \( p = 2 \) we can explicitly write down the completely reducible connections of minimal energy.

**Proposition 61.** Let \( E_k \) be the \( U(N) \) bundle over \( N_2 \) with \( c_1(E_k) = kc_1(L) \). Then

\[
A_{k,r} = \begin{cases} 
\frac{k+r}{2} \lambda \oplus (N-r)1 \oplus \frac{r-k}{2} \lambda^{-1} & r \geq k \\
\frac{k-r}{2} \lambda^2 \oplus r \lambda \oplus (N - \frac{k+r}{2})1 & k \geq r
\end{cases}
\]

has minimal energy among all ASD connections on \( E_k \) with limiting flat connection \( \eta_r = r \chi \oplus (N-r)1 \), and it is the unique completely reducible connection with this property.

**Proof.** By Proposition 60 we only need to show that this connection has less energy than any other completely reducible connection on \( E_k \) limiting on \( \eta_r \). So, let

\[
A = \lambda^{k_1} \oplus \cdots \lambda^{k_n}
\]

be completely reducible. Write \( r_i = 1 \) for \( i \leq r \) and \( r_i = 0 \) for \( i > r \), and without loss of generality write \( k_i = r_i + 2d_i \). With \( k = \sum_i k_i \) fixed, the energy of \( A \) is proportional to

\[
c_2(A) - 2c_1(A)^2 = \sum_i k_i^2.
\]

(3.5)

In the case \( k \geq r \), we rewrite (3.5) using the identity \( r_i^2 = r_i \):

\[
c_2(A) - 2c_1(A)^2 = r + \sum_i k_i^2 + (r_i - 1)^2 - 1
\]

(3.6)

Because \( k_i \) and \( r_i \) are congruent mod 2, each term on the right hand side of (3.6) is positive, hence

\[
c_2(A) - 2c_1(A)^2 \geq r.
\]
Equality holds precisely when \((r_i, k_i) = (0, 0), (1, 1), \text{or} (1, -1)\) for every \(i\), in which case \(A = A_{k,r}\).

In the case \(r \geq k\), we can rewrite (3.5) in a different way:

\[
c_2(A) - 2c_1(A)^2 = r + \sum_{i} k_i^2 + (r_i - 1)^2 - 1 \geq 2k - r
\]

Again, equality holds precisely when \((r_i, k_i) = (0, 0), (0, 2), \text{or} (1, 1)\) for every \(i\), in which case \(A = A_{k,r}\).

3.2 The Blowup Formula

In this section we prove the existence of a blowup formula for higher rank invariants.

3.2.1 Initial Conditions

Let \(Y\) be a 4-manifold, let \(X = Y \# \mathbb{C}P^2\), and let \(e \subset X\) be the exceptional sphere.

**Proposition 62.** Let \(k\) and \(d\) be integers satisfying \(0 \leq k < N\) and \(d < k(N - k) + 2N\). There is a class \(\beta_{k,d} \in H^*(BPU(N); \mathbb{Q})\) such that for any \(z \in A(Y)\) and any \(c \in H^2(Y; \mathbb{Z})\) we have

\[
D_{X,c+ke}(\mu_2(e)^d z) = D_{Y,c}(\beta_{k,d} z).
\]

Moreover, the class \(\beta_{k,d}\) is homogeneous and has degree \(2d - 2k(N - k)\), so it vanishes if \(d < k(N - k)\).

**Proof.** Suppose that \(z\) has degree \(2r\). As we stretch the neck between \(Y\) and \(\mathbb{C}P^2\), transversality implies that every connection in \(\mathcal{M}_{c+ke}(X) \cap V(z)\) must eventually lie in

\[
\overline{\mathcal{M}}_{c+2r+2d-2k(N-k)}^{2r+2d-2k(N-k)}(Y) \times_{SU(N)} \overline{\mathcal{M}}_{ke}^{1-k^2-(N-k)^2}((N_1),
\]

where \(\overline{\mathcal{M}}_{ke}(N_1) = \text{Gr}_k(\mathbb{C}^N)\) is the minimal energy moduli space on the bundle \(E_k = L^k \oplus \mathbb{C}^{N-k}\). In particular, \(\mathcal{G} = \mathcal{M}_{c+ke}(X) \cap V(z)\) is the bundle of Grassmannians associated to a \(PU(N)\) bundle \(\mathbb{P} = \mathcal{M}_c(Y) \cap V(z)\) over a compact stratified space \(B = \mathcal{M}_c(Y) \cap V(z)\). By definition, the number \(D_{X,c+ke}(\mu_2(e)^d z)\) is obtained by evaluating \(\mu_2(e)^d\) on the fundamental class of \(\mathcal{G}\). We can therefore calculate it in two stages, by first pushing forward to \(B\), then pushing forward to a point.

Observe that \(\mu_2(e)^d\) is the pullback of an equivariant class \(\tilde{\mu}_2(e)^d \in H^*_{PU(N)}(\text{Gr}_k(\mathbb{C}^N))\). Hence we can compute its pushforward to \(B\) by first applying the equivariant pushforward

\[
\pi_* : H^*_{PU(N)}(\text{Gr}_k(\mathbb{C}^N)) \to H^*_{BPU(N)}(pt) = H^*(BPU(N))
\]

to the class \(\tilde{\mu}(e)^l\), then pulling back to \(B\) along the classifying map of \(\mathbb{P}\). Setting \(\beta_{k,d} = \pi_*(\tilde{\mu}_2(e)^d)\), we conclude that \(D_{X,c+ke}(\mu_2(e)^d z) = D_{Y,c}(\beta_{k,d} z)\), as desired. \(\square\)

In fact, we can be much more explicit about the classes \(\beta_{k,d}\). On the Grassmannian \(\text{Gr}_k(\mathbb{C}^N)\) we have a splitting of equivariant bundles \(\mathbb{C}^N = F_k \oplus F_{N-k}\). By Proposition 46,
the universal bundle over $\mathcal{M}_{c+ke}(X) \times e$ is the projectivization of the vector bundle

$$V = (L \otimes F_k) \oplus F_{N-k}.$$ 

We can easily compute the Chern character of this bundle,

$$\text{Ch}(V) = \exp(L)\text{Ch}(F_k) + \text{Ch}(F_{n-k})$$

and from this we can obtain the classes $a_2(\mathbb{P})$, from the formula

$$a_2(\mathbb{P}) = -[\exp\left(\frac{-kL}{N}\right)\text{Ch}(V)]_2.$$ 

Slanting with $e$, we get

$$e_2(e) = \frac{N-k}{N}c_1(F_k) - \frac{k}{N}c_1(F_{n-k}) = c_1(F_k).$$

We are therefore reduced to computing the equivariant pushforward

$$\beta_{k,d} = \int_{\text{Gr}_k(C^N)} c_1(F_k)^d$$

which is straightforward for any particular values of $k$ and $d$. For example, Kronheimer proved that $\beta_{1,N-1} = 1$ and used this to define higher rank invariants in the case where $c$ is not coprime to $N$. In more generality, we have:

**Proposition 63.** The class $\beta_{k,d}$ is a nonzero integer when $d = k(N - k)$. In particular we have $\beta_{0,0} = \beta_{1,N-1} = 1$.

**Proof.** The class $c_1(F_k)$ is represented by a section of an ample line bundle on the complex projective variety $\text{Gr}_k(C^N)$, hence the integral of $c_1(F_k)^{k(N-k)}$ is nonzero. In the case $k = 0$ we have $\beta_{0,0} = \pm 1$ trivially. In the case $k - 1$ we are evaluating the top power of the hyperplane class on $\mathbb{CP}^{N-1}$, hence we get $\beta_{1,N-1} = \pm 1$. For evaluation of the sign we refer Kronheimer [8].

The vanishing criterion in Proposition 62 generalizes as follows, with the same proof:

**Proposition 64.** Let $0 < k < N$ and let $I = (i_2, \ldots, i_N)$ be a multi-index satisfying

$$0 \leq \dim(I) = 2i_2 + 4i_3 + \cdots + (2N - 2)i_N < 2k(N - k) + 4N.$$ 

Then there is a class $\beta_{k,I} \in H^*(BPU(N); \mathbb{Q})$ such that for any $z \in A(Y)$, and any $c \in H^2(Y; \mathbb{Z})$ we have

$$D_{X,c+ke}(\mu(e)^I z) = D_{X,c+ke}(\mu_2(e)^{i_2} \cdots \mu_N(e)^{i_N} z) = D_{Y,c}(\beta_{k,I} z)$$

If $\dim(I) < 2k(N - k)$ then $\beta_{k,I} = 0$.

There is another vanishing condition in the case $k = 0$. 

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Proposition 65. Let \( I = (i_2, \ldots, i_N) \) be a multi-index satisfying

\[
0 < \dim(I) = 2i_2 + 4i_3 + \cdots + (2N - 2)i_N < 4N.
\]

Then \( \beta_{k,I} = 0 \).

Proof. The classes \( \mu_i(e) \) vanish in a neighborhood of the trivial connection by Proposition 46, hence do not intersect the moduli space \( \tilde{M}^{1-N^2}(N_1) \). Thus for a sufficiently long neck they do not intersect the moduli space \( \mathcal{M}^{\dim(I)}(X) \).

3.2.2 A Simple Relation

In this section, we consider a 4-manifold \( X \) that contains a sphere \( \sigma \) of self-intersection \(-2\). Such a 4-manifold can be written as \( X = Y \cup_{L_2} N_2 \) for some \( Y \) with \( \partial Y = L_2 \). In this section we will use our understanding of the minimal energy moduli spaces on \( N_2 \) derive a simple relationship between the invariants \( D_{X,c+\sigma} \) and \( D_{X,c} \).

Proposition 66. Let \( \mathcal{M}_{0,2} \) be the moduli space containing the connection \( A_{0,2} = \lambda \oplus \lambda^{-1} \oplus (N - 2)1 \) of Proposition 61. Then every connection \( A \in \mathcal{M}_{0,2} \) takes the form

\[
A = B \oplus (N - 2)1
\]

for some rank 2 connection \( B \).

Proof. Write \( A = \bigoplus_i m_i A_i \) as a sum of irreducible connections of dimension \( n_i \), each having multiplicity \( m_i \). Then the orbit of \( A \) in \( \tilde{M}_{0,2} \) has dimension

\[
d = 2^2 + (N - 2)^2 - \sum_i m_i^2
\]

On the other hand, by Proposition 56, the moduli space \( \tilde{M}_{0,2} \) has dimension 4. Therefore,

\[
\sum_i m_i^2 \geq (N - 2)^2
\]

If one of the \( A_i \) has dimension at least 3, or if two of the \( A_i \) have dimension at least 2, then \( \sum_i m_i \leq N - 2 \), hence

\[
\sum_i m_i^2 \leq (\sum_i m_i)^2 \leq (N - 2)^2.
\]

The first inequality is strict unless there is a single term in the sum, in which case the second inequality is strict. We conclude that \( A = B \oplus C \), where \( C \) is completely reducible. We know that there is a reducible \( B_0 = \lambda^* \oplus \lambda^* \) with the same energy as \( B \). By Proposition 61, the connection \( \lambda^* \oplus \lambda^* \oplus C \) has to be \( A_{0,2} \), so either \( B_0 = \lambda \oplus \lambda^{-1} \) or \( B_0 = \lambda^{\pm 1} \oplus 1 \). In the latter case, \( B \) cannot be irreducible, because it lives in a moduli space of negative dimension. We conclude that either \( A = A_{2,0} \) or \( C \) is trivial, which completes the proof.

Proposition 67. The \( SU(2) \) moduli space \( \mathcal{M}_{0,2} \) consists of a half-infinite segment \([0, \infty)\), plus a union of finitely many circles and infinite segments \((-\infty, \infty)\). The corresponding
components of the framed moduli space $\mathcal{M}_{0,2}$ are equivariantly diffeomorphic to $T^*(S^2)$, $SO(3) \times S^1$, and $SO(3) \times \mathbb{R}$, respectively.

Proof. The space of irreducible connections $\mathcal{M}_{0,2}^*$ is a smooth 1-manifold with two kinds of ends. First, there is the reducible connection $\lambda \oplus \lambda^{-1}$. Second, there are limits where energy slides off the end of $N_2$. The latter take the form $F \# \Theta$, where $\Theta$ is the trivial connection and $F$ is one of finitely many connections $F_1, \ldots, F_n$ on $L_2 \times \mathbb{R}$. Exactly one component of $\mathcal{M}_{0,2}^*$ joins $\lambda \oplus \lambda^{-1}$ to one of the $F_i \# \Theta$. The other components are either compact (circles) or they join two of the $F_i \# \Theta$.

To show that the component of the framed moduli space lying over $[0, \infty)$ is diffeomorphic to $T^*(S^2)$, it suffices to show that the normal bundle to the orbit of $\lambda \oplus \lambda^{-1}$ has degree $\pm 2$. But this is clear, because the link of the orbit is $SO(3) = L_2$.

We now prove the simple relation promised above. In the case $N = 2$ this result is due to Ruberman [15].

**Proposition 68.** For any $c \in H^2(Y)$ and any $z \in A(Y)$ we have

$$D_{X,c}(\mu_2(\sigma)^2 z) = \epsilon_N D_{X,c+\sigma}(2z),$$

where $\epsilon_N$ is a universal sign given by

$$\epsilon_N = \begin{cases} +1 & N \equiv 2 \mod 4 \\ -1 & N \text{ odd} \end{cases}$$

Proof. First we describe how to compute the left hand side of the equation. From Propositions 61 and 56, we see that the framed moduli space $\tilde{\mathcal{M}}_{0,2r}$ has dimension $4r^2$. Since $\mu_2(\sigma)^2$ has degree 4, the only moduli spaces we encounter as we stretch the neck are $\mathcal{M}_{0,0}$ and $\mathcal{M}_{0,2}$. Since $\mathcal{M}_{0,0}$ only contains the trivial connection, it does not intersect any $V(\mu_4(\sigma))$. Hence all intersections take place on an open subset of $\mathcal{M}_{c,2d+4}(X)$ diffeomorphic to

$$U = \tilde{\mathcal{M}}_{0,2r}[\eta_2] \times \Gamma \tilde{\mathcal{M}}_{0,2},$$

where $\Gamma = S(U(2) \times U(N-2))$ is the stabilizer of $\eta_{0,2}$.

We can compute the universal bundle over $\tilde{\mathcal{M}}_{0,2} \times \sigma$, using Proposition 46. Note that the stabilizer of any connection in $\tilde{\mathcal{M}}_{0,2}$ is a subgroup of $S(U(1) \times U(1) \times U(N-2))$. Hence there are line bundles $P$ and $Q$, and a rank $N-2$ vector bundle $F$ over $\tilde{\mathcal{M}}_{0,2}$, such that the universal bundle is the projectivization of

$$V = (L \otimes P) \oplus (L^{-1} \otimes Q) \oplus F.$$ 

Away from $A_{0,2}$ the stabilizers reduce to $S(U(1)_2 \times U(N))$, hence the line bundles $P$ and $Q$ become isomorphic. Writing $p = c_1(P)$ and $q = c_1(Q)$, we conclude that $p - q$ is supported in a neighborhood of the orbit of $A_{0,2}$. On the orbit of $A_{0,2}$, which is a 2-sphere, $P$ is a degree one line bundle and $Q$ is its inverse. Therefore,

$$(p - q)^2, \tilde{\mathcal{M}}_{0,2} = \pm 2$$
with the sign depending on how we orient $\tilde{M}_{0,2}$.

Now, the Chern character of $V$ is given by

$$Ch(V) = \exp(l) \exp(p) + \exp(-l) \exp(q) + Ch(F).$$

From this formula we can compute $a_2(\mathbb{P})$ and $\mu_2(\sigma)$. Slanting both sides by $\sigma$, we obtain

$$\mu_2(\sigma) = p - q$$

from which it follows that $D_{X,c}(\mu_2(\sigma)^2 z) = \pm 2n$, where $\epsilon$ is a universal sign and $n$ is the signed count of points in $\mathcal{M}_{\mathcal{Y},c}^{2d}[\mathbb{Z}] \cap V(z)$.

The left hand side is easier to compute. Indeed, the moduli spaces $\tilde{M}_{2,2r}$ all have positive dimension, except for $\tilde{M}_{2,2}$, which has dimension 0 and consists of a single connection $A_{2,2}$. Stretching the neck, we conclude that the moduli space $\mathcal{M}_{\mathcal{Y},c}^{2d}(X) \cap V(z)$ consists of a single connection for each point in $\mathcal{M}_{\mathcal{Y},c}^{2d}[\mathbb{Z}] \cap V(z)$. Therefore,

$$2D_{c+\sigma}(z) = \pm 2n = \epsilon D_{X,c}(\mu_2(\sigma)^2 z)$$

for some choice of sign $\epsilon$.

To compute $\epsilon$, let $Y$ be a 4-manifold with $D_{Y,c}(z) = 1$ for some $c$ and $z$. Let $X$ be the manifold obtained by blowing up $Y$ twice. Then we have

$$D_{X,c+k\beta_1+k\beta_2}(\mu_2(e)^{k(N-k)} \mu_2(e)^{l(N-l)} z) = \beta_k \beta_l$$

for some nonzero integers $\beta_k$ and $\beta_l$. The result we have proved so far shows that

$$D_{c+k\beta_1+k\beta_2}(\mu_2(e_1 - e_2)^2 \mu_2(e_1 + e_2)^{2k(N-k)-2}) = 2\epsilon D_{c+(k+1)e_1+(k-1)e_2}(\mu_2(e_1 + e_2)^{k(N-k)-2})$$

$$= 2\epsilon D_{c+(k+1)e_1+(k-1)e_2}(\mu_2(e_1 + e_2)^{(k+1)(N-k-1)+(k-1)(N-k+1)}).$$

Assume that $k$ is strictly between 0 and $N$, and expand out both sides. For dimension reasons there is only one nonzero term on either side. Examining this term shows that

$$-\text{sign}(\beta_k^2) = \epsilon \cdot \text{sign}(\beta_{k+1}\beta_{k-1})$$

Since $\beta_0$ and $\beta_1$ are both positive we conclude that

$$\text{sign}(\beta_k) = (-\epsilon)^{\left\lfloor \frac{k}{2} \right\rfloor + 1}$$

On the other hand, we also have

$$\beta_k = D_{c+k\epsilon}(\mu_2(e)^{k(N-k)} z)$$

$$= D_{c-k\epsilon}(\mu_2(e)^{k(N-k)} z)$$

$$= -\epsilon N D_{c+(N-k)e}(\mu_2(e)^{k(N-k)} z)$$

$$= (-1)^{(k(N-k)+1)} \epsilon N \beta_{N-k}.$$
Combining these two pieces of information we see that

\[ (-\epsilon)\left[ \frac{3}{2} \right] + [\frac{k}{2}] = (-1)^{(k(N-k)+1)}\epsilon_N. \]

If \( N \) is odd then the right hand side is \(+1\) for all \( k \), hence \( \epsilon = 1 \). If \( N \) is congruent to 2 mod 4 then setting \( k = 1 \) gives \( \epsilon = \epsilon_N = -1 \).

3.2.3 Partial Blowup Functions

Using the simple relation of Proposition 68, we can extend Proposition 62 to arbitrary values of \( d \).

**Proposition 69.** Let \( Y \) be a 4-manifold with \( b_2^+ > 1 \) and let \( X \) be its blowup. Then for every integer \( k \) there exists a formal power series \( B_k(t) = \sum_d b_{k,d} t^d \in H^*(BPU(N); \mathbb{Q}[t]) \) such that for any \( c \in H^2(Y) \) and any \( z \in A(Y) \) we have

\[ D_{X,c+ke}(\exp(\mu_2(e)t)z) = D_{X,c}(B_k(t)z). \]

**Proof.** Let \( \tilde{X} \) be the manifold obtained by blowing up \( Y \) twice, and let \( e_1, e_2 \) be the exceptional spheres. Then the simple relation tells us

\[ D_{X,c+ke_1+ke_2}(\mu_2(e_1-e_2)^2 \exp(t\mu_2(e_1)+t\mu_2(e_2))) = 2\epsilon_N D_{X,c+(k+1)e_1+(k-1)e_2}(\exp(t\mu_2(e_1)+t\mu_2(e_2))) \]

Expanding in powers of \( t \) gives us an infinite series of relations between the invariants \( D_{X,c+ke}(\mu_2(e)^d) \). We want to see that these relations can be solved recursively, by induction on \( d \).

To see why a solution exists, it is convenient to think of the relations as a system of differential equations to be satisfied by the functions \( B_k(t) \). Solving these differential equations order by order in \( t \), given suitable initial conditions, is equivalent to using (3.9) to inductively find acceptable values for \( b_{k,d} \). Explicitly, the differential equations we must solve read as follows:

\[ B_k''B_k - B_k'B_k = \epsilon_N B_{k+1}B_{k-1}. \]

When \( k \) is a multiple of \( N \) we have \( B_k(0) = \pm 1 \), so examining the coefficient of \( t^d \) always allows us to solve for \( b_{0,d} \) and \( b_{N,d} \) in terms of coefficients \( b_{k,e} \) with \( e < d \). It therefore remains to solve for the rest of the \( b_{k,d} \) in terms of \( b_{0,d} \) and \( b_{N,d} \) and lower order terms.

Fortunately, for \( 0 < k < N \) the equations are degenerate, because \( B_k(0) = 0 \). To remove the degeneracy, we can write

\[ B_k(t) = b_k t^{k(N-k)} f_k(t) \]

for some power series \( f_k \) satisfying \( f_k(0) = 1 \). By Proposition 63, we can take the coefficients \( b_k \) to be nonzero integers. Rewriting our differential equations in terms of the \( f_k \) we see that

\[ b_k^2 f_k^2 (f_k'' f_k - f_k' f_k') = k(N-k) b_k f_k^2 - \epsilon_N b_{k+1} b_{k-1} f_{k+1} f_{k-1}. \]
Setting \( t = 0 \) in this equation gives the identity

\[
k(N - k)b_k^2 = \epsilon_N b_{k+1} b_{k-1}
\]

so we can eliminate \( b_k^2 \) and \( \epsilon_N \) from the equations, obtaining

\[
t^2(f'_k f_k - f'_k f_k') = k(N - k)(f'_k - f_{k+1} f_{k-1}).
\]

This equation is still degenerate, but we can understand the degeneracy in a precise way. Collecting terms of order \( t^d \), we get a system of equations

\[
\frac{f_{k,d}}{(d-2)!} = k(N - k) \frac{2f_{k,d} - f_{k+1,d} - f_{k-1,d}}{d!} + \text{lower order terms}
\]

which can be rewritten as follows:

\[
\frac{2f_{k,d} - f_{k+1,d} - f_{k-1,d}}{2} - \frac{d(d - 1)}{2k(N - k)} f_{k,d} = g_k.
\]

The left hand side can be viewed as a linear operator \( L : \mathbb{R}^{N+1} \to \mathbb{R}^{N-1} \), and we must show that this operator is surjective for fixed values of \( f_0 \) and \( f_N \). To do so it is sufficient to restrict to \( f_0 = f_N = 0 \), in which case \( L \) becomes a symmetric operator with corresponding quadratic form

\[
Q(f) = \sum_{k=0}^{N-1} (f_{k+1} - f_k)^2 - \sum_{k=1}^{N-1} \frac{d(d - 1)}{k(N - k)} f_k^2.
\]

It therefore suffices to show that \( Q \) is negative definite for sufficiently large \( d \). But

\[
Q(f) \leq \sum_{k=0}^{n-1} 2(f_{k+1}^2 + f_k^2) - \sum_{k=1}^{N-1} \frac{d(d - 1)}{k(N - k)} f_k^2
\]

\[
= \sum_{k=1}^{N-1} \left( 4 - \frac{d(d - 1)}{2k(N - k)} \right) f_k^2,
\]

so \( Q \) will be negative definite provided

\[
\frac{d(d - 1)}{2k(N - k)} > 4
\]

for all \( k \) between 0 and \( N \). This inequality is automatically satisfied if \( d > N \). In the range \( d \leq N \), Proposition 62 already provides us with values for \( f_{k,d} = b_{k,k(N-k)+d} \). Using these values as initial conditions, we can solve for a blowup function \( B_k(t) \) to all orders in \( t \). \( \square \)

**Definition 70.** Any function \( B_k(t) \) which correctly computes the invariants \( D_{X,c}(\mu_2(e)dz) \) as in Proposition 69 is called a partial blowup function.

Note that we make no claims about the uniqueness of the functions \( B_k(t) \). However, the difference of two partial blowup functions vanishes inside the Donaldson invariant of any 4-manifold, so one could in principle prove uniqueness by finding 4-manifolds that distinguish
all classes in \( H^*(BPU(N)) \), or by viewing the blowup function as taking values in a suitable Floer homology theory.

Finally, it is worth remarking on the differential equations satisfied by the partial blowup functions,\[
B''_k B_k - B'_k B'_k = \epsilon_N B_{k+1} B_k.
\]
If we make the substitution
\[
u_k = \log(\sqrt{\epsilon_N}) \frac{B_{k+1}}{B_k},
\]
then this system of equations becomes
\[
\frac{d^2 \nu_k}{dt^2} = e^{\nu_{k+1} - \nu_k} - e^{\nu_k - \nu_{k-1}},
\]
which is known in mathematical physics as the Toda lattice. It describes a periodic chain of particles, each of which interacts with its neighbors via an exponential potential function. A relationship between the Toda lattice and the \( SU(N) \) blowup function was predicted by Maríñio and Moore in [12], so Proposition 69 can be seen as a confirmation of physicists' expectations about the behavior of higher rank invariants.

3.2.4 Full Blowup Functions

We call the \( B_k(t) \) partial blowup functions because they only allow us to calculate invariants of the form \( D x, c(p_2(e) dz) \). The full \( SU(N) \) blowup function would be a power series in \( N - 1 \) variables \( t_2, \ldots, t_N \) such that
\[
D x, c(\exp(t_2 \mu_2(e) + \cdots + t_n \mu_n(e))) = D x, c(B(t_2, \ldots, t_N))
\]
In this section we will prove that a full blowup function exists for \( N \leq 4 \), and study the full blowup function in detail when \( N = 3 \) and \( k = 0 \).

To extend the recursion of Proposition 68 to arbitrary classes \( \mu_i(\sigma) \), we again consider a 4-manifold \( X \) containing a sphere \( \sigma \) of self-intersection \(-2\), and we write \( X \) as \( Y \cup L_2 N_2 \).

**Proposition 71.** Let \( I = (i_2, \ldots, i_N) \) be a multi-index with \( \dim(I) = 2p < \min(16, 4N) \). Then there exist universal \( \alpha_{i,j} \in H^*(BPU(N); \mathbb{Q}) \) such that for any \( c \in H^2(Y, \partial Y) \) and any \( z \in \mathfrak{A}(Y) \) we have
\[
D x, c(\mu(\sigma)^I z) = \sum_{i+j \leq p} D x, c(\alpha_{i,j} \mu_i(\sigma) \mu_j(\sigma) z)).
\]

**Proof.** We proceed as in 68. In the moduli space \( \mathcal{M}^{2d+2p}(X) \) all intersections with \( V(z) \) take place in
\[
\mathcal{U} = \mathcal{M}^{2d+2p}(Y)[\eta_{0,2}] \times_\Gamma \mathcal{M}_{0,2},
\]
where \( \Gamma = S(U(2) \times U(N - 2)) \) is the stabilizer of \( \eta_{0,2} \). As before, the universal bundle is the projectivization of the vector bundle
\[
\mathcal{V} = LP \oplus L^{-1}Q \oplus F,
\]
where $P$ and $Q$ are line bundles that are isomorphic away from $A_{0,2}$.

The Chern character of $V$ is given by

$$\text{Ch}(LP \oplus L^{-1}Q \oplus F) = \exp(l) \exp(p) + \exp(-l) \exp(q) + \text{Ch}(F).$$

From this formula we can extract concrete expressions for $a_k(P)$ and $\mu_k(\sigma)$, by writing the classes $a_k(P)$ in terms of $\text{Ch}(V)$. A nice way to organize the computation is to introduce classes

$$\nu_k = \mu_{(k-1)!} \text{Ch}_k(\sigma) = (k-1)! \text{Ch}_k(V)/\sigma.$$

Because the classes $\text{Ch}_k$ generate $H^*_\text{SU}(N)$ we know that each $\nu_k$ is a linear combination of the classes $\mu_l(\sigma)$, with coefficients in $H^*_\text{SU}(N)$. Hence we can replace $\mu_l(\sigma)$ by $\nu_l$ in the statement of the theorem.

Explicitly, $\nu_k$ is given by

$$(k-1)! \text{Ch}_k(V)/\sigma = p^{k-1} - q^{k-1} = (p-q)s_{k-2},$$

where $s_{k-2}$ is the sum of all monomials of degree $k-2$. In particular, each product $\nu_k \nu_l$ is divisible by $(p-q)^2$, hence is a multiple (over $H^F_\Gamma$) of the equivariant fundamental class of $T^*(S^2)$.

We must therefore show that the classes $s_is_j$ form a basis for $H^F_\Gamma$ over $H^*_\text{SU}(N)$. To prove this, observe that $B\Gamma$ fibers over $BSU(N)$, with fiber $G_{r_2}(\mathbb{C}^N)$. It therefore suffices to show that the $s_is_j$ restrict to a basis for the cohomology of $G_{r_2}(\mathbb{C}^N)$. But it is well known that the Schur polynomials

$$s_{k,l} = \frac{p^k q^l - p^l q^k}{p-q}$$

form a basis, and these can be expressed in terms of the $s_is_j$ by the identity

$$s_{k,l} = s_k s_{l+1} - s_{k+1} s_l$$

Because of the restriction on $\dim(I)$, Proposition 71 only produces 9 identities, which can be derived from the following relations between the $\nu_k$:

<table>
<thead>
<tr>
<th>Degree</th>
<th></th>
<th>$\nu_2^4 = 4\nu_2\nu_4 - 3\nu_3^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree 10</td>
<td>$\nu_2^2 \nu_3 = 3\nu_2 \nu_5 - 2\nu_3 \nu_4$</td>
<td></td>
</tr>
<tr>
<td>Degree 12</td>
<td>$\nu_2^2 \nu_4 = 3\nu_2 \nu_6 - 3\nu_3 \nu_5 + \nu_4^2$</td>
<td>$\nu_2^2 \nu_3^2 = 2\nu_2 \nu_6 + \nu_3 \nu_5 + 2\nu_4^2$</td>
</tr>
<tr>
<td>Degree 14</td>
<td>$\nu_2^4 \nu_5 = 3\nu_2 \nu_7 - 3\nu_3 \nu_6 + \nu_4 \nu_5$</td>
<td>$\nu_2^2 \nu_3 \nu_4 = 2\nu_2 \nu_7 - \nu_4 \nu_5$</td>
</tr>
</tbody>
</table>
We can replace each $\nu_i$ above with a combination of $\mu_i(\sigma)$ using the following table:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$(k-1)!\text{Ch}_k$</th>
<th>$\nu_k = \mu_{(k-1)!\text{Ch}_k}(\sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$-a_2$</td>
<td>$-\mu_2(\sigma)$</td>
</tr>
<tr>
<td>3</td>
<td>$a_3$</td>
<td>$\mu_3(\sigma)$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{1}{2}a_2^2 - a_4$</td>
<td>$a_2\mu_2(\sigma) - \mu_4(\sigma)$</td>
</tr>
<tr>
<td>5</td>
<td>$-a_2a_3 + a_5$</td>
<td>$a_3\mu_2(\sigma) - a_2\mu_3(\sigma) + \mu_5(\sigma)$</td>
</tr>
<tr>
<td>6</td>
<td>$-\frac{1}{2}a_2^3 + \frac{1}{3}a_3^2 + a_2a_4 - a_6$</td>
<td>$(a_4 - a_2^2)\mu_2(\sigma) + a_3\mu_3(\sigma) + a_2\mu_4(\sigma) - \mu_6(\sigma)$</td>
</tr>
</tbody>
</table>

In the case $N = 3$ there only 2 identities, which we write out for convenience.

**Corollary 72.** Let $N = 3$. Then for all $c \in H^2(Y)$ and all $z \in A(Y)$ we have

$$D_{X,c}(\mu_2(\sigma)^4 z) = -4D_{X,c}(a_2\mu_2(\sigma)z) - 3D_{X,c}(\mu_3(\sigma)z)$$
$$D_{X,c}(\mu_2(\sigma)^3 \mu_5(\sigma)z) = -3D_{X,c}(a_3\mu_2(\sigma)^2 z) - D_{X,c}(a_2\mu_3(\sigma)\mu_3(\sigma)z)$$

We can now prove the existence of a full blowup function when $N \leq 4$.

**Proposition 73.** Let $N$ be 3 or 4. Then there is a formal power series $B = B(t_2, \ldots, t_N) \in H^*(BPU(N); \mathbb{Q}[t_2, \ldots, t_N])$ such that for all $c \in H^2(Y)$ and $z \in A(Y)$ we have

$$D_{X,c}(\exp(t_2 \mu_2(e) + \cdots + t_N \mu_N(e))z) = D_{X,c}(B(t_2, \ldots, t_N)z)$$

**Proof.** We proceed as in Proposition 69, by blowing up twice and using the identities produced in Proposition 71. The strategy of the induction is to eliminate all powers of $\mu_3(e)$ and $\mu_4(e)$, thereby expressing the full blowup function in terms of the partial blowup functions.

For example, suppose $N = 3$. Then $B(0) = 1$, and $B$ must satisfy two differential equations:

$$3(BB_3 - B_3B) = -B_{2222}B + 4B_{222}B_2 - 3B_2B_{22} - 4a_2(B_{22}B - B_2B_2)$$
$$B_{2222}B - 2B_{2222}B_2 - 2B_{222}B_2 + 3B_{22}B_{22} = -3a_3(B_{22}B - B_2B_2) - a_2(B_{22}B - B_2B_2)$$

Expanding these equations to all orders in $t$, it becomes clear that we can solve for all partial derivatives of $B$ in terms of $B_{222}, B_{222}, B_2, B_3$, and the coefficients of the partial blowup function that are already known. In fact, the vanishing theorem of Proposition 65 shows that $B_{222} = B_{22} = B_3 = 0$, so knowledge of the partial blowup function is sufficient.

The case $N = 4$ is similar, but involves using 8 of the 9 equations from Section 3.2.4. □

We have not attempted to prove the existence of a full blowup function $B_k(t_2, \ldots, t_N)$ here because the vanishing of $B_k(0)$ means the equations are degenerate, and a precise analysis of the degeneracy is quite complicated.

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3.3 The Case N=3

3.3.1 Theta Functions on a Genus 2 Jacobian

Definition 74. Let \( \Lambda \subset \mathbb{C}^g \) be a full lattice. We say that an entire function \( \theta : \mathbb{C}^g \to \mathbb{C} \) is quasiperiodic with respect to \( \Lambda \) if for every \( \lambda \in \Lambda \) there exist constants \( a_\lambda \in \text{Hom}(\mathbb{C}^g, \mathbb{C}) \) and \( b_\lambda \in \mathbb{C} \) such that

\[
\theta(x + \lambda) = e^{a_\lambda x + b_\lambda} \theta(x)
\]

for all \( x \in \mathbb{C}^g \).

For example, any Gaussian \( f(x) = e^{x^T A x + b^T x + c} \) is periodic. So is the Riemann theta function

\[
\theta(z) = \sum_{n \in \mathbb{Z}^g} e^{\pi i T n} e^{2\pi i T z},
\]

if the lattice takes the form \( \Lambda = \mathbb{Z}^g \oplus \Omega \mathbb{Z}^g \) for \( \Omega \) a symmetric matrix with positive definite imaginary part. If two quasiperiodic functions have the same automorphy factors \( a_\lambda, b_\lambda \), then any linear combination of them is quasiperiodic, with the same automorphy factors. The product of any two quasiperiodic functions is quasiperiodic, with the product of the two automorphy factors. A linear map followed by a quasiperiodic function is quasiperiodic, with respect to a different lattice.

Note that we can regard any quasiperiodic function as a section of a certain line bundle over \( \mathbb{C}^g/\Lambda \), whose transition maps can be derived from the automorphy factors. We will use capital letters to denote the line bundle corresponding to a set of automorphy factors, so if \( \theta \) is quasiperiodic, we will say that it is a section of the line bundle \( \Theta \).

Definition 75. Let \( \theta(x) \) be an analytic function on \( \mathbb{C}^g \). We define the Hirota differentials \( \theta[I](x) \) to be the unique functions such that

\[
\theta(x + t)\theta(x - t) = \sum_I \theta[I](x) \frac{t^I}{I!}
\]

where the sum ranges over all multi-indices \( I = (i_1, \ldots, i_g) \). In other words,

\[
\theta[I](x) = \frac{\partial^{\left| I \right|}}{\partial t^I} \theta(x + t)\theta(x - t)|_{t=0}
\]

Proposition 76. The Hirota differentials of an arbitrary analytic (or \( C^\infty \)) function \( \theta \) satisfy the following properties:

1. \( \theta[I] = 0 \) if \( I \) has odd degree.
2. \( \theta[ij] = 2\partial_{ij} \log \theta \)
3. \( \theta[ijkl] = 2\partial_{ijkl} \log \theta + 4 (\partial_{ij} \log \theta \cdot \partial_{kl} \log \theta + \partial_{ik} \log \theta \cdot \partial_{jl} \log \theta + \partial_{il} \log \theta \cdot \partial_{jk} \log \theta) \)

Proof. Straightforward computation.
Proposition 77. Let \( \theta \) be a quasiperiodic function with respect to a full lattice \( \Lambda \subset \mathbb{C}^g \), so that it defines a section of a line bundle \( \Theta \). Then for any fixed \( t \in \mathbb{C}^g \) the function

\[
\phi_t(x) = \theta(x + t)\theta(x - t)
\]

is a section of \( 2\Theta \).

Proof. By definition, sections of \( \Theta \) correspond to quasiperiodic functions \( \psi \) satisfying identities

\[
\psi(x + \lambda) = e^{a_\lambda x + b_\lambda} \psi(x)
\]

for every \( \lambda \in \Lambda \). Thus sections of \( 2\Theta \) correspond to those satisfying

\[
\phi(x + \lambda) = e^{2a_\lambda x + 2b_\lambda} \phi(x)
\]

It is then straightforward to check that this identity is satisfied by \( \phi_t \).

Corollary 78. If \( \theta \) is a quasiperiodic function, then its Hirota differentials \( \theta[I] \) are all sections of \( 2\Theta \).

Proof. Since \( \theta(x + t)\theta(x - t) \) is quasiperiodic in \( x \), so are all its derivatives with respect to \( t \).

Proposition 79. Let \( \Sigma \) be a curve of genus 2, and let \( \theta \) be the Riemann theta function on its Jacobian \( J\Sigma \). Then the Hirota differentials \( \theta[I] \) with \( \deg I = 0,2 \) form a basis of \( H^0(J\Sigma, 2\Theta) \).

Proof. Translating and multiplying by a Gaussian, we can assume \( \theta \) is an even function whose Hessian vanishes at the origin. In this case it suffices to prove that the 3 components of the logarithmic Hessian of \( \theta \) are linearly independent. If there were a linear relation, then we would have \( \partial_v \partial_w \log \theta = 0 \) for some vectors \( v \) and \( w \). But this implies that an irreducible component of the zero locus of \( \theta \) is invariant under a 1-parameter group of translations, which is absurd since the zero locus is a genus 2 curve.

Corollary 80. The Riemann theta function on \( J\Sigma \) satisfies a system of differential equations of the form:

\[
\theta[ijkl] = 2A_{ijkl}\theta^2 + \sum_{pq} C_{ijkl}^{pq} \theta[pq]
\]

or equivalently,

\[
P_{ijkl} + 2(P_{ij}P_{kl} + P_{ik}P_{jl} + P_{il}P_{jk}) = A_{ijkl} + \sum_{pq} C_{ijkl}^{pq} P_{pq}
\]

where \( P_I = \partial_I \log \theta \).

Proof. By Proposition 76, we know that the left hand side is a ratio \( \frac{\phi(x)}{\theta(x)^2} \), where \( \phi \) is a section of \( 2\Theta \). By Proposition 79, \( \phi \) is a linear combination of Hirota differentials of order \( \leq 2 \). Dividing by \( \theta^2 \) and applying Proposition 76 again, the system of equations follows.
Observe that if \( \theta(x) \) satisfies a system of the above form, then so does

\[
e^{x^T A x + b^T x + c} \theta(M x + t)
\]

for any matrices \( A, C \), covector \( b \) and constants \( c, t \). In other words, multiplying by a Gaussian and precomposing with an affine transformation preserves the form of the equations. Ignoring this ambiguity, we can show that the Riemann theta function is characterized by the above system of equations. More precisely:

**Proposition 81.** Suppose that \( \theta \) is a function whose logarithmic derivatives satisfy a system of equations of the form:

\[
P_{1111} + 6P_{11}^2 = A_{1111} + G_2P_{11} - 2G_1P_{12} + G_0P_{22}
\]

\[
P_{1112} + 6P_{11}P_{12} = A_{1112} + G_3P_{11} - 2G_2P_{12} + G_1P_{22}
\]

\[
P_{1122} + 2P_{11}P_{22} + 4P_{12}^2 = A_{1122} + G_4P_{11} - 2G_3P_{12} + G_2P_{22}
\]

\[
P_{1222} + 6P_{12}P_{22} = A_{1222} + G_5P_{11} - 2G_4P_{12} + G_3P_{22}
\]

\[
P_{2222} + 6P_{22}^2 = A_{2222} + G_6P_{11} - 2G_5P_{12} + G_4P_{22}
\]

that the coefficients \( A_{ijkl} \) are given by:

\[
A_{1111} = \frac{1}{3} (G_0G_4 - 4G_1G_3 + 3G_2^2)
\]

\[
A_{1112} = \frac{1}{6} (G_0G_5 - 3G_1G_4 + 2G_2G_3)
\]

\[
A_{1122} = \frac{1}{18} (G_0G_6 - 9G_2G_4 + 8G_3^2)
\]

\[
A_{1222} = \frac{1}{6} (G_1G_6 - 3G_2G_5 + 2G_3G_4)
\]

\[
A_{2222} = \frac{1}{3} (G_2G_6 - 2G_3G_5 + 3G_4^2)
\]

and that \( \theta \) is not itself a Gaussian. Suppose, moreover, that the homogeneous polynomial

\[
G(x, y) = \sum_{i+j=6} G_i x^i y^j / i! j!
\]

has exactly 6 nondegenerate zeroes in \( \mathbb{P}^1 \). Then, up to a multiplying by a Gaussian and precomposing with an affine transformation, \( \theta \) is the Riemann theta function on the Jacobian of the hyperelliptic curve

\[
y^2 = g(x) = G(x, 1)
\]

**Proof.** This is proved in Baker [2]. To give some sense of what is involved in the argument, we will explain how to recover the hyperelliptic curve. The idea is as follows. First differentiate each of the 5 equations above with respect to each of the two variables. This yields a set of equations like:

\[
P_{11112} + 12P_{11}P_{112} = G_2P_{112} - 2G_1P_{122} + G_0P_{222}
\]
If we eliminate all 5th derivatives, we are left with system of 4 equations of the form:

\[ M_{11}P_{111} + M_{12}P_{112} + M_{13}P_{122} + M_{14}P_{222} = 0 \]

where the coefficients \( M_{ij} \) lie in \( H^0(J\Sigma, 2\Theta) \). Since \( \theta \) is not a Gaussian, at least one of the \( P_{ijk} \) is nonzero. Hence the 4 sections of \( H^0(J\Sigma, 2\Theta) \) satisfy a nondegenerate degree 4 equation,

\[ \det M = 0 \]

This is the equation of the Kummer surface in \( \mathbb{P}^3 \). When we intersect it with the plane at infinity (i.e. the image of the theta divisor) we get a doubled conic. The Kummer surface has six singular points along this conic, which an explicit computation shows are precisely the roots of \( G(x, y) \).

We also need to make a remark about initial conditions. Clearly the function \( P \) is determined by \( P_I(0) \) for \( |I| \leq 3 \). But as a consequence of Proposition 81, these initial conditions are subject to the constraint that the second derivatives must lie on the Kummer surface. There is, of course, no constraint on the zeroth or first derivatives, since they can be modified by multiplying by an exponential, which does not change the equations. Finally, it is proved in Baker that the third derivatives can be expressed in terms of the lower derivatives, which gives some actual constraints on the set of valid initial conditions. In the case where the initial conditions are finite and specify a singular point of the Kummer surface, these constraints simply say that the function \( \theta \) is (up to third order) an even function times an exponential.

### 3.3.2 The Blowup Function

**Theorem 82.** The SU(3) blowup function is a quasiperiodic function on the Jacobian of the hyperelliptic curve

\[ y^2 = (x^3 + a_2 x + a_3)^2 - 4. \]

**Proof.** By Corollary 72

\[
\begin{align*}
B_{2222}B - 4B_{222}B_2 + 3B_{22}B_{22} &= -4a_2(B_{22}B - B_2 B_2) - 3(B_{33}B - B_3 B_3) \\
B_{2223}B - 3B_{223}B_2 - B_{222}B_3 + 3B_{22}B_{23} &= -3a_3(B_{22} - B_2 B_2) - a_2(B_{23}B - B_2 B_3)
\end{align*}
\]

and setting \( Q = \log B \) we get

\[
\begin{align*}
Q_{2222} + 6Q_{22} &= -4a_2Q_{22} - 3Q_{33} \\
Q_{2223} + 6Q_{22}Q_{23} &= -3a_3Q_{22} - a_2Q_{23}
\end{align*}
\]

which look similar to the first 2 of the 5 equations satisfied by the Riemann theta function. In fact, if we suppose that \( Q \) satisfies such a system of 5 equations, there is a unique choice of coefficients for the remaining three equations that is consistent with the initial conditions derived in Section 4. In total the five equations read:

\[
\begin{align*}
Q_{2222} + 6Q_{22} &= -4a_2Q_{22} - 3Q_{33}
\end{align*}
\]
These equations are not quite in the form required by Proposition 81. However, we can bring them into that form if we multiply $B(t_2, t_3)$ by a suitable Gaussian. Specifically, we define
\[
\theta(t_1, t_2) = \exp \left( \frac{3}{10} a_2 t_2^2 + \frac{9}{20} a_3 t_2 t_3 - \frac{1}{10} a_2^2 t_3^2 \right) B(t_2, t_3).
\]
Writing $P = \log \theta$ and applying we get the system of equations
\[
\begin{align*}
P_{2222} + 6P_{22}^2 &= \frac{9}{25} a_2^2 - \frac{2}{5} a_2 P_{22} - 3P_{33} \\
P_{2223} + 6P_{22}P_{23} &= \frac{27}{50} a_3 - \frac{3}{10} a_3 P_{22} + \frac{4}{5} a_2 P_{23} \\
P_{2233} + 2P_{23} P_{33} + 4P_{23}^2 &= 2 + \frac{27}{50} a_3^2 - \frac{1}{25} a_3^3 - \frac{1}{5} a_2^2 P_{22} + \frac{3}{5} a_3 P_{23} - \frac{2}{5} a_2 P_{33} \\
P_{2333} + 6P_{23} P_{33} &= -\frac{9}{50} a_2 a_3 - a_2 a_3 P_{22} + \frac{2}{5} a_2^2 P_{23} - \frac{3}{10} a_3 P_{33} \\
P_{3333} + 6P_{33}^2 &= \frac{8}{5} a_2 + \frac{1}{25} a_2^4 - (12 + 3a_3^2) P_{22} + 2a_2 a_3 P_{23} - \frac{1}{5} a_2 P_{33}^2.
\end{align*}
\]

The coefficients $G_i$ now clearly take the form of Theorem 81, and a straightforward but tedious computation shows that the coefficients $A_{ijkl}$ satisfy the necessary conditions as well.

It is also straightforward to check that the initial conditions for $\theta$ correspond to a singular point of the Kummer surface, and since the first derivatives vanish the integrability constraints simply say that the third derivatives vanish as well (which they do). Thus the $SU(3)$ blowup function can be equivalently defined as the unique solution of the complete set of 5 equations above, with the initial conditions derived in Section 4.

From Proposition 81 we now conclude that the $SU(3)$ blowup function is (the Taylor expansion of) a quasiperiodic function on the Jacobian of a certain hyperelliptic curve $\Sigma$. We can compute $\Sigma$ explicitly using Theorem 81. It is given in hyperelliptic form by the equation:
\[
y^2 = g(x) = -\frac{3}{6!} x^6 - \frac{4}{10 a_2^2} x^4 - \frac{3}{10 a_3^3} x^3 - \frac{2}{10 a_2^2} x^2 - a_2 a_3 x - \frac{3a_3^2 + 12}{6!}.
\]
\[
= \frac{-1}{240} (x^6 + 2a_2 x^4 + 2a_3 x^3 + a_2^2 x^2 + 2a_2 a_3 x + a_3^2 + 4)
\]
\[
= \frac{-1}{240} (f(x)^2 + 4),
\]
which is the desired result up to isomorphism. \qed
Chapter 4

The Formal Picture

4.1 Axioms

A formal Floer theory $HF$ over the complex numbers consists of the following data:

D1 A graded-commutative, finitely generated, regular $\mathbb{C}$-algebra $\Lambda$, supported in even degrees.

D2 For each connected, oriented 3-manifold $Y$, a relatively $\mathbb{Z}$-graded, absolutely $\mathbb{Z}/2\mathbb{Z}$-graded $\Lambda$-module $HF(Y)$, on which $\Lambda$ acts by even endomorphisms.

D3 For each connected, oriented, homology oriented 4-manifold $W$ with oriented boundary $\partial W = Y - Y_1 - Y_2 - \cdots - Y_n$, and a graded, $\Lambda$-multilinear map

\[ HF_W : \bigotimes_{j=1}^n HF(Y_j) \otimes \Lambda(W; \Lambda) \to HF(Y). \]

If $n = 0$ we interpret the empty tensor product as the 1-dimensional vector space $k$.

D4 For each connected, oriented, homology oriented 4-manifold $X$ with oriented boundary $\partial X = -Y_1 - Y_2 - \cdots - Y_n$, satisfying $b_+^2(X) \geq 2$, a graded $k$-multilinear map

\[ D_X : \bigotimes_{j=1}^n HF(Y_j) \otimes \Lambda(X; \Lambda) \to k \]

These data should satisfy the following axioms:

A1 If $\phi : HF(Y_1) \to HF(Y_2)$ is a diffeomorphism, then the trivial cobordism from $Y_1$ to $Y_2$ induces an isomorphism $HF_\phi : HF(Y_1) \to HF(Y_2)$. If $\phi : W_1 \to W_2$ is a diffeomorphism of cobordisms, then $HF_{W_1}$ and $HF_{W_2}$ are intertwined by the action of $\phi$ on $HF(\partial W_i)$ and $\Lambda(W_i, \Lambda)$.

A2 Reversing the homology orientation on $W$ multiplies $HF_W$ by a factor of $(-1)^\epsilon$, where $\epsilon$ is the $\mathbb{Z}/2\mathbb{Z}$ grading on $\text{Hom}(\otimes_{j=1}^n HF(Y_j), HF(Y))$. 

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A3 Let $\partial X = -Y - Y_1 - \cdots - Y_n$, $\partial W = Y - Y_{n+1} - \cdots - Y_m$, let $\mu \in A(X; \Lambda)$, and $\nu \in A(W; \Lambda)$. Let $\overline{X}$ be the 4-manifold obtained by gluing $X$ and $W$ along $Y$. Then we have

$$HF_{\overline{X}}(\mu \nu) = HF_X(\mu) \circ HF_W(\nu)$$

and likewise

$$D_{\overline{X}}(\mu \nu) = D_X(\mu) \circ HF_W(\nu)$$

provided that $X$ and $W$ satisfy the conditions necessary to define both sides of these equations.

A4 For every $Y$, $HF(Y)$ is a finitely generated $\Lambda$-module.

A5 The map $HF_{B^4} : \Lambda \to HF(S^3)$ is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded $\Lambda$-modules.

It is worth clarifying some consequences of A5. There is an evident cobordism $Z : S^3 + S^3 \to S^3$ given by removing two 4-balls from $B^4$. This cobordism induces an associative multiplication $HF(S^3) \otimes HF(S^3) \to HF(S^3)$, giving $HF(S^3)$ the structure of an algebra over the ring $\Lambda$. From A3 we conclude that the map $HF_{B^4}$ is not just an isomorphism of $\Lambda$-modules, it is also an algebra isomorphism. The fact that $HF_{B^4}$ respects the $\mathbb{Z}/2\mathbb{Z}$ grading shows that $HF(S^3)$ is supported in even degrees.

4.2 The Formal Blowup Formula

The axioms of a formal Floer theory imply the existence of a blow-up formula for the relative invariants $D_X$. To see how this comes about, we need to recall some general facts from linear algebra.

Definition 83. Let $V$ be a finite dimensional vector space over a field $k$. We define the ring of regular functions on $V$ to be the $k$-algebra

$$k[V] = \text{Sym}^*(V^*) = \bigoplus_{n=0}^{\infty} \text{Sym}^n(V^*)$$

and we define the ring of formal power series on $V$ to be

$$k[[V]] = \prod_{n=0}^{\infty} \text{Sym}^n(V^*).$$

Equivalently, we could define $k[[V]]$ to be the completion of $k[V]$ at the ideal $J$ generated by $V^* = \text{Sym}^1(V^*)$. Note that if $t_1, \ldots, t_g$ are a basis for $V^*$ then the evident map

$$k[t_1, \ldots, t_g] \to k[V]$$

is an isomorphism of $k$-algebras, and likewise for $k[[t_1, \ldots, t_g]]$. This justifies viewing $k[V]$ as a ring of functions on $V$. 

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Proposition 84. If $k$ has characteristic 0, then the formal power series ring $k[[V]]$ is naturally isomorphic to the dual of the vector space $\text{Sym}^*(V)$.

Proof. It suffices to give a duality between $\text{Sym}^n(V^*)$ and $\text{Sym}^n(V)^*$. Define $\epsilon : \text{Sym}^n(V^*) \to \text{Sym}^n(V)^*$ by

$$\langle \epsilon(f_1 \cdots f_n), v_1 \cdots v_n \rangle = \sum_{\sigma \in S_n} f_1(v_{\sigma(1)}) \cdots f_n(v_{\sigma(n)})$$

To show that $\epsilon$ is bijective, let $v_1, \ldots, v_g$ be a basis for $V$ and let $v_1^*, \ldots, v_g^*$ be the dual basis. Then for any pair of multi-indices $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ we have

$$\langle \epsilon(v_I), v_J^* \rangle = |\Gamma_I| \delta_{IJ}$$

where $\Gamma_I \subset S_n$ is the stabilizer of $I$. Since $\epsilon$ can be represented by a diagonal matrix with nonzero entries, hence is invertible. \qed

To understand the meaning of the map $\epsilon$, observe that

$$f(v) = \left\langle \epsilon(f), \frac{v \otimes \cdots \otimes v}{n!} \right\rangle,$$

for any $f \in \text{Sym}^n(V^*)$ and any $v \in V$, where the left hand side is defined by viewing $f$ as a homogeneous polynomial on $V$. More generally, for $f \in k[[V]]$ we have

$$f(v) = \langle \epsilon(f), \exp(v) \rangle,$$  \hspace{1cm} (4.1)

provided we interpret both sides as formally converging infinite sums.

We now generalize the above discussion slightly. Let $\Lambda$ be a commutative Noetherian $k$-algebra, and write $B$ for the corresponding scheme $\text{Spec} \Lambda$. Let $V$ be a finitely generated projective module over $\Lambda$. Viewed as a sheaf of modules over $B = \text{Spec} \Lambda$, $V$ is locally free, hence elements of $V$ can be viewed as global sections of a vector bundle $\mathcal{V}$ over $B$. Explicitly, this vector bundle can be described as the spectrum of the $k$-algebra

$$\Lambda[V] = \bigoplus_{n=0}^{\infty} \text{Sym}^n_{\Lambda}(\text{Hom}_{\Lambda}(V, \Lambda)),$$

and the ideal $J$ generated by terms of positive degree cuts out the zero section $B \to \mathcal{V}$. Completing at $J$ yields the formal power series ring

$$\Lambda[[V]] = \prod_{n=0}^{\infty} \text{Sym}^n_{\Lambda}(\text{Hom}_{\Lambda}(V, \Lambda)),$$

which we can view as the ring of functions defined on a formal neighborhood of the zero section.

Proposition 85. If $k$ has characteristic zero, then the map $\epsilon : \Lambda[[V]] \to \text{Hom}_{\Lambda}(\text{Sym}^*(V), \Lambda)$ is an isomorphism.
Proof. Locally on Spec $\Lambda$, $V$ is free and the map $e$ is an isomorphism by the proof given above. Since $e$ is a local isomorphism, it is an isomorphism of sheaves, so it induces an isomorphism on global sections as well. \qed

For example, let $\Omega^1(\Lambda)$ denote the module of differentials over $\Lambda$. By definition, $\Omega^1(\Lambda)$ is the target of a universal derivation

$$d : \Lambda \to \Omega^1(\Lambda).$$

If $\Lambda$ is regular then $\Omega^1(\Lambda)$ is a finitely generated projective module, and the spectrum of the ring $\Lambda[\Omega^1(\Lambda)]$ is the cotangent bundle $T^*B$. The spectrum of the formal power series ring $\Lambda[[\Omega^1(\Lambda)]]$ is a formal neighborhood of the zero section in $T^*B$, which we will denote by $\widehat{T^*B}$.

We will now show that any formal Floer theory admits a blowup function, which can be naturally viewed as a function on $\widehat{T^*B}$. Consider the 4-manifold $\widehat{B^4}$ obtained by blowing up the origin, and let $e$ denote a generator for $H_2(\widehat{B^4})$. Because $H_1(\widehat{B^4}) = 0$, we have a Liebniz identity in $A(\widehat{B^4}; \Lambda)$:

$$\mu_{ab}(e) = \mu_a(e)b + a\mu_b(e),$$

which induces a map $\Omega^1(\Lambda) \to A(\widehat{B^4}; \Lambda)$. We can therefore interpret $HF_{\widehat{B^4}}$ as a map

$$HF_{\widehat{B^4}} : \text{Sym}^*(\Omega^1(\Lambda)) \to HF(S^3).$$

By assumption $D1$, $\Lambda$ is regular, hence $\Omega^1(\Lambda)$ is a locally free $\Lambda$-module. By $A5$ the natural map $\Lambda \to HF(S^3)$ is an isomorphism, so we can view $HF_{\widehat{B^4}}$ as an element of $\text{Hom}_\Lambda(\text{Sym}^*(\Omega^1(\Lambda)), \Lambda)$. We can then use Proposition 85 to identify it with a formal power series $\beta \in \Lambda[[\Omega^1(\Lambda)]]$.

Definition 86. The blowup function of a formal floer theory $HF$ is the formal power series $\beta$ obtained from $HF_{\widehat{B^4}}$ by the above identifications.

Proposition 87. Let $\tilde{X} = X \# \overline{\mathbb{C}P^2}$ be the 4-manifold obtained by blowing up a 4-manifold $X$, let $\omega \in \Omega^1(\Lambda)$, and let $z \in A(X; \Lambda)$. Then we have a formal blowup formula:

$$D_{\tilde{X}}(\exp(\omega)e) = D_X(\beta(\omega)e).$$

Proof. View $\omega$ as a section of the cotangent bundle $T^*B$, where $B = \text{Spec} \Lambda$. Evaluating $\beta$ on $\omega$ yields an element of $\Lambda$, or more precisely a formal sum with coefficients in $\Lambda$. Using (4.1), we can characterize this formal sum by the property:

$$HF_B(\beta(\omega)) = HF_{\widehat{B^4}}(\exp(\omega)).$$

Now write $\tilde{X} = X_0 \circ \widehat{B^4}$, where $X_0$ is obtained by removing a 4-ball from $X$. Thus

$$D_{\tilde{X}}(\exp(\omega)e) = D_{X_0}(z) \circ HF_{\widehat{B^4}}(\exp(\omega)) = D_{X_0}(z) \circ HF_B(\beta(\omega)) = D_X(\beta(\omega)e).$$

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4.3 The Blowup Algebra

Let \( N_p \) be the disk bundle associated to a complex line bundle of degree \(-p\) over the 2-sphere \( S^2 \). There is a unique orientation on \( N_p \) that is compatible with the outward orientation of \( S^2 \subset \mathbb{R}^3 \) and the orientation of the fibers induced by their complex structures. Let \( L_p \) denote the inwardly oriented boundary of \( N_p \), so that \( L_p \) is diffeomorphic to the lens space \( L(p, 1) \).

We can construct a cobordism \( W_{p,q} : L_p + L_q \rightarrow L_{p+q} \) as follows. Let \( V \) denote the 3-manifold obtained by removing two small 3-balls from the unit 3-ball \( B^3 \). Orient the boundary of \( V \) so as to make it a cobordism from \( S^2 + S^2 \) to \( S^2 \). Then there is a unique disk bundle \( N_{p,q} \) over \( V \) that restricts to \( N_p \) and \( N_q \) on the two incoming spheres. Note that \( N_{p,q} \) necessarily restricts to the disk bundle \( N_{p+q} \) on the outgoing sphere. Let \( W_{p,q} \) denote the inwardly oriented boundary of \( N_{p,q} \).

To be more precise in identifying \( W_{p,q} \) as a cobordism between \( L_p, L_q, \) and \( L_{p+q} \) we actually need to specify a diffeomorphism from each boundary component of \( W_{p,q} \) to the corresponding lens space. However, the space of line bundle isomorphisms from \( L_k \) to itself is just Maps\( (S^2, S^1) \), which is connected, so there is no ambiguity as long as we choose diffeomorphisms which cover the identity map on \( S^2 \) and act complex linearly on the fibers.

We also need a homology orientation on \( W_{p,q} \). This is equivalent to picking an ordering of the incoming 2-spheres in \( M \), which we can do arbitrarily.

**Proposition 88.** The maps \( HF_{W_{p,q}} : HF(L_p) \otimes HF(L_q) \rightarrow HF(L_{p+q}) \) induce the structure of a nonunital graded-commutative \( \Lambda \)-algebra on

\[
A_+ = \bigoplus_{p=1}^{\infty} HF(L_p).
\]

**Proof.** By D3, \( HF_{W_{p,q}} \) is \( \Lambda \)-multilinear, so we only need to check associativity and commutativity of multiplication. To verify associativity, observe that we can construct a cobordism \( W_{p,q,r} : L_p + L_q + L_r \rightarrow L_{p+q+r} \) in perfect analogy with \( W_{p,q} \). Associativity then follows by applying A1 to the evident diffeomorphisms

\[
W_{p+q,r} \circ W_{p,q} \longrightarrow W_{p,q,r} \longrightarrow W_{p,q+r} \circ W_{q,r}.
\]

To verify commutativity, observe that there is a diffeomorphism from \( V \) to itself which swaps the two incoming spheres and restricts to an orientation preserving isometry on either one. This diffeomorphism lifts to a diffeomorphism

\[
W_{p,q} \longrightarrow W_{q,p}.
\]

Because it swaps the role of the incoming spheres, this diffeomorphism reverses the homology orientation on \( W_{p,q} \). By A2 this implies that the multiplication is graded commutative. \( \square \)
Note that we can turn the nonunital algebra $A_+$ into an algebra by formally adjoining a unit in degree 0. Let $A$ denote the graded-commutative $\Lambda$-algebra obtained in this way.

Now let $N_+^* : L_p \to S^3$ be the cobordism obtained by removing a 4-ball from $N_p$. The direct sum of the maps $HF_{N_p}$ induces a $\Lambda$-linear map $i^* : A_+ \to \Lambda$.

**Proposition 89.** The map $i^*$ is an algebra homomorphism.

**Proof.** Consider the composition $N_+^* \circ W_{p,q}$, which corresponds to first multiplying two elements of $A$ and then applying the map $\phi$. We must show that this is diffeomorphic to $Z \circ (N_+^* \sqcup N_+^*)$, where as above $Z : S^3 + S^3 \to S^3$ is the cobordism obtained by removing two 3-balls from $B^4$. Equivalently, we must show that $N_{p+q} \circ W_{p,q}$ is diffeomorphic to the connect sum of $N_p$ and $N_q$.

To construct the diffeomorphism, let $D^2 \subset V$ be a disk whose boundary is an equator of the outgoing sphere, and which separates the two incoming spheres from each other. The preimage of $D^2$ in the circle bundle $W_{p,q} \to V$ is the handlebody $D^2 \times S^1$. On the other hand, the preimage of $\partial D^2 = S^1$ in the disk bundle $N_{p+q} \to S^2$ is the complementary handlebody $S^1 \times D^2$. These two handlebodies glue together to form a 3-sphere, which separates $N_{p+q} \circ W_{p,q}$ into two disjoint components. It is clear upon a bit of reflection that these components are diffeomorphic to $N_p^*$ and $N_q^*$. We conclude that $N_{p+q} \circ W_{p,q}$ is diffeomorphic to $N_p \# N_q$, as desired. \[\square\]

Next we will explain a fundamental relationship between the algebra $A$ and the formal blowup formula. Before doing this, we need to make some remarks on the topology of the cobordisms we have been using. Let $\sigma_p \in H_2(N_p)$ be the homology class represented by the zero section of the disk bundle $N_p \to S^2$, and let $\partial_p \in H_2(N_p, \partial N_p)$ be the relative homology class represented by one of its fibers. By virtue of the decomposition $N_p \# N_q = N_{p+q} \circ W_{p,q}$, we can view $\sigma_{p+q}$ as a homology class in $H_2(N_p \# N_q) = H_2(N_p) \oplus H_2(N_q)$. It is geometrically evident that $\partial_p \cdot \sigma_{p+q} = \partial_q \cdot \sigma_p = 1$, from which we conclude that $\sigma_{p+q} = \sigma_p + \sigma_q$.

Imitating the construction of the blowup function, we can use $\sigma_p$ to map $\Omega^1(\Lambda)$ to $A(N_p; \Lambda)$. Hence we can interpret $HF_{N_p}$ as a map

$$HF_{N_p} : HF(L_p) \longrightarrow \text{Hom}(\text{Sym}^*(\Omega^1(\Lambda)), \Lambda) = \Lambda[[\Omega^1(\Lambda)]]$$

Let $\exp^* : A_+ \to \Lambda[[\Omega^1(\Lambda)]]$ be the direct sum of these maps.

**Proposition 90.** $\exp^*$ is an algebra homomorphism.

**Proof.** Let $f_p = \exp^*(a_p)$ and $f_q = \exp^*(a_q)$ where $a_p \in HF(L_p)$ and $a_q \in HF(L_q)$. Then for any $\omega \in \Omega^1(\Lambda)$ we have

$$f_p(\omega) = HF_{N_p}(\exp(\omega \sigma_p) \otimes a_p)$$

By virtue of the identities $N_{p+q}^* \circ W_{p,q} = Z \circ (N_p^* \sqcup N_q^*)$ and $\sigma_{p+q} = \sigma_p + \sigma_q$ we have:

$$\exp^*(a_p) \exp^*(a_q) = f_p(\omega)f_q(\omega) = HF_Z(f_p(\omega) \otimes f_q(\omega)) = HF_{N_p}(\exp(\omega \sigma_p) \otimes a_p) \otimes HF_{N_q}(\exp(\omega \sigma_q) \otimes a_q)$$

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We can now explain the connection to the blowup formula. Note that $N_1$ is diffeomorphic to the blowup $B^4$, and $N_1^*$ is a twice punctured $\mathbb{CP}^2$. Let $\Theta = HF_{B^4}(1)$ generate $HF(S^3) = HF(L_1)$. Then we have

$$\exp^* \Theta(\omega) = HF_{N_1^*}(\exp(\omega \sigma_1) \otimes HF_{B^4}(1)) = HF_{N_1}(\exp(\omega \sigma_1)) = \beta(\omega),$$

so the blowup function is the image of $\Theta$ under $\exp^*$.

### 4.4 Geometry of the Blowup Algebra

In order to proceed any further we will need to make some sort of assumption about the ring $A$.

**Assumption 1.** The map $\exp^*: A \to \Lambda[[\Omega^1(\Lambda)]]$ is injective and has dense image.

**Assumption 2.** The ring $A$ is finitely generated.

One consequence of the first assumption is that each $HF(L_p)$ is supported in a single mod 2 grading. Thus $A$ is a commutative algebra (rather than just graded-commutative). We can therefore give it a geometric interpretation, by viewing $A_p = HF(L_p)$ as the space of global sections of a line bundle $L_{O_p}$ on a variety $E$. More precisely,

**Definition 91.** We define $E = \text{Proj}_A(A)$ to be the scheme obtained by applying the relative Proj construction to the graded $A$-algebra $A$.

Note that injectivity of $\exp^*$ implies that $A$ is an integral domain, hence $E$ is irreducible. By definition $E$ comes with a map $\pi: E \to B = \text{Spec} \Lambda$. The assumption that $A$ is finitely generated tells us that each fiber $E_B = \pi^{-1}(B)$ is actually a projective variety. In fact a basis of $HF(L_p)$ defines a projective embedding, for sufficiently large $p$.

**Proposition 92.** The homomorphism $\exp^*$ induces a map $\exp: \widetilde{T^*B} \to E$ of schemes over $B$, and the map $i^*$ induces a section $i: B \to E$

**Proof.** Since $\exp^*$ has dense image there is an element $f \in A_+$ such that $\exp^* f$ is a unit. Let $A_f$ be the localization of $A$ at any such $f$. Then $E$ has a distinguished open subset $U_f$ isomorphic to $\text{Spec} A_f$. The fact that $i^* f$ is a unit implies that $\exp^* f$ is a unit, hence we obtain a map $\exp^*_f: A_f \to \Lambda[[\Omega^1(\Lambda)]]$. This induces a map of schemes $\exp^*_f: \widetilde{T^*B} \to U_f$. If $i^* g$ is also a unit, then $\exp^*_f$ and $\exp^*_g$ both factor through $\exp^*_{fg}$, hence we get a well-defined map $\exp: \widetilde{T^*B} \to E$, independent of the choice of $f$.

Because $i^*$ is just $\exp^*$ followed by evaluation on the zero section, we see that it similarly induces a map $B \to E$, which is a section because $i^*$ is a map of $\Lambda$-algebras. \qed
Proposition 93. \( \mathcal{E} \) is regular on a Zariski open set containing \( i(B) \).

Proof. Let \( P \) denote the kernel of \( i^* \), let \( A_P \) be the localization at \( P \), and let \( \widetilde{A}_P \) be its completion with respect to \( P \). Because \( \exp^* \) has a dense image, it induces an isomorphism \( \widetilde{A}_P \to \Lambda[[\Omega^1 \Lambda]] \). In particular, \( A_P \) is regular, hence \( \mathcal{E} \) is regular in a Zariski neighborhood of \( V(P) = i(B) \) as desired. \( \square \)

We will now justify our notation by showing that \( \exp \) is obtained by formally integrating certain commuting vector fields on \( \mathcal{E} \).

Proposition 94. Let \( F_\xi : A_p \otimes A_p \to A_{2p} \) be a family of \( \Lambda \)-multilinear maps satisfying

1. (Antisymmetry) \( F_\xi(\phi, \psi) = -F_\xi(\psi, \phi) \)
2. (Liebniz) \( F_\xi(\phi_1 \phi_2, \psi_1 \psi_2) = F_\xi(\phi_1, \psi_1)\phi_2 \psi_2 + \phi_1 \psi_1 F_\xi(\phi_2, \psi_2) \)

Then there is a unique vertical vector field \( \xi \), defined on the smooth locus of \( \mathcal{E} \), such that

\[ \xi(\phi, \psi) = \theta^{-2} F_\xi(\phi, \psi) \]

Proof. The fact that \( \xi \) is a derivation follows from the Liebniz property of \( F_\xi \). However, we must show that \( \xi \) is well-defined, or equivalently that \( \xi(\frac{\phi}{\psi}) = \xi(\frac{\phi}{\psi}) \) for any homogeneous \( s \in A \). But this follows from the identity

\[ F_\xi(s \phi, s \psi) = F_\xi(\phi, \psi) s^2 + \phi \psi F_\xi(s, s) \]

Proposition 95. Let \( a \in \Lambda \) and let \( V_a \) be the vertical vector field on \( T^*B \) whose value at \( (b, \omega) \) is \( da(b) \). Then there is a globally defined vertical vector field \( \xi_a \) on \( \mathcal{E} \) such that \( \exp^*(V_a) = \xi_a \).

Proof. To construct \( \xi_a \) we only need to produce \( F_{\xi_a} : A_p \otimes A_p \to A_{2p} \) satisfying the conditions of Proposition 94. We define it by the formula:

\[ F_{\xi_a}(\phi, \psi) = HF_{W_{p,p}}(\mu_a(\sigma_p - \tau_p) \phi \psi) \]

where \( \sigma_p \) and \( \tau_p \) are a basis for the homology of \( N_{2p} \circ W_{p,p} = N_p \# N_p \), so that \( \sigma_p - \tau_p \) is an integral generator of \( H_2(W_{p,p}) \).

Antisymmetry is clear from the definition, so we only need to prove the Liebniz property. The proof is cumbersome to write down, but only involves applying the identity of cobordisms

\[ W_{p+q,p+q} \circ (W_{p,q} \sqcup W_{p,q}) = W_{p,p+q,q} = W_{2p,2q} \circ (W_{p,p} \sqcup W_{q,q}) \]

and the identity of homology classes

\[ \mu_a(\sigma_{p+q}) = \mu_a(\sigma_p) + \mu_a(\sigma_q) \].
Lastly, we check that $\xi_a$ is compatible with $V_a$. To begin, let $\phi \in HF(L_p)$ and let $f = \exp^* \phi$. Then
\[
V_a(f)(\omega) = \left. \frac{d}{dt} \right|_{t=0} f(\omega + tda) \\
= \left. \frac{d}{dt} \right|_{t=0} HF_N^p(\exp((\omega + tda)\sigma_p)) \\
= HF_N^p(\mu_a(\sigma_p) \exp(\omega \sigma_p) \phi) \\
= HF_N^p(\mu_a(\sigma_p) \exp(\omega \sigma_p) \phi)
\]
Writing $f = \exp^* \phi$ and $g = \exp^* \psi$, and applying the identity of cobordisms
\[
N_p^* \circ W_{p,p} = Z \circ (N_p^* \sqcup N_p^*)
\]
we therefore have
\[
\exp^* F_{\xi_a}(\phi, \psi) = V_a(f)g - fV_a(g) = FV_a(f, g),
\]
from which we conclude that $\exp^*(V_a) = \xi_a$. \hfill $\square$

**Proposition 96.** The vector fields $\xi_a$ are mutually commuting.

**Proof.** Let $a, b \in \Lambda$. In terms of $F_{\xi_a}$ and $F_{\xi_b}$ we must show that
\[
F_{\xi_a}(F_{\xi_b}(\phi, \psi), \psi^2) = F_{\xi_b}(F_{\xi_a}(\phi, \psi), \psi^2)
\]
But the left hand side of this equation is equal to
\[
HF_{W_{p,p,p}}(\mu_a(\sigma_p^{(1)} - \sigma_p^{(2)})\mu_b(\sigma_p^{(1)} - \sigma_p^{(4)})\phi\psi\psi)
\]
which is evidently symmetric in $a$ and $b$. \hfill $\square$

### 4.5 Analytic Continuation

In this section we study the convergence properties of formal power series in the image of $\exp^*$. More precisely, for a fixed point $b \in B$ and any $f \in HF(L_p)$ we consider the function $F = \exp^* f$ as formal power series on $T_b^*B$, and show that it has an infinite radius of convergence.

Since we will be working locally on $B$, we choose a set of functions $a_1, \ldots, a_n \in \Lambda$ such that $da_1(b), \ldots, da_n(b)$ form a basis for $T_b^*(B)$. We can then view each $F$ as a formal power series in $n$ formal variables,
\[
F(t_1da_1 + \cdots + t_n da_n) = \sum_I F_I(0) \frac{t^I}{I!},
\]
where for each multi-index $I = (i_1, \ldots, i_k)$ we have written the $I$-th partial derivative of $F$
as
\[ F_i = \frac{\partial^k F}{\partial t_{i_1} \cdots \partial t_{i_k}}. \]

Note that not all partial derivatives \( F_i \) lie in the image of \( \exp^* \), because \( f \) is not a function on \( \mathcal{E} \). However,

**Proposition 97.** Let \( f \in HF(L_p) \) such that \( F = \exp^* f \) is a unit. Then for any \( i \) and \( j \), the power series
\[
G = F_{ij}F - F_iF_j
\]
is equal to \( \exp^* g \) for some \( g \in HF(L_{2p}) \).

**Proof.** Simply take \( 2g = HF_{W,p} (\mu_i (\sigma_p - \tau_p) \mu_j (\sigma_p - \tau_p)) \). Then
\[
\exp^*(2g) = HF_{N_{2p}} \circ HF_{W,p} (\mu_i (\sigma_p - \tau_p) \mu_j (\sigma_p - \tau_p) \exp(\sigma_{2p} \omega)) = 2HF_{N_p} (\mu_i (\sigma_p) \mu_j (\sigma_p) \exp(\omega \sigma_p)) \exp(\omega \tau_p)) - 2HF_{N_p} (\mu_i (\sigma_p) \exp(\omega \sigma_p)) \exp(\omega \tau_p)) = 2F_{ij}F - 2F_iF_j = 2G
\]
as desired. \( \Box \)

**Proposition 98.** Let \( f \in HF(L_p) \) and let \( F = \exp^* f \). Then \( F \) has a positive radius of convergence.

**Proof.** For fixed \( b \in B \) the formal map \( \exp \) has a positive radius of convergence, because it is given by integrating some holomorphic vector fields. In particular, if \( g, h \in L_q \) and \( \exp^* h \) is a unit then \( G/H = \exp^*(g/h) \) has a positive radius of convergence.

Assume \( F \) is a unit. In this case, for any \( i \) and \( j \) we have
\[
\xi_i \xi_j \log(F) = \frac{F_{ij}F - F_iF_j}{F^2} = \frac{G}{F^2} = \exp^*(g/f^2),
\]
for some \( g \in HF(L_{2p}) \). Therefore \( \log(F) \) has a positive radius of convergence, hence so does \( F \).

Now drop the assumption that \( F \) is a unit. We know that for some \( n > 0 \) there is an \( h \in L_{np} \) such that \( H = \exp^*(h) \) is a unit. But then \( F^n - H = \exp^*(f^n - h) \) has a positive radius of convergence, and \( H \) has a positive radius of convergence, hence so does \( F^n \), hence so does \( F \). \( \Box \)

To show that each \( F \) has an infinite radius of convergence, we use a doubling formula, which takes a form familiar from the theory of theta functions.

**Proposition 99.** Let \( f_1, f_2 \in HF(L_p) \) and let \( F_i = \exp^* f_i \). Let \( G_1, \ldots, G_n \) be a basis for the image of \( HF(L_{2p}) \) in \( \mathbb{C}[T^*_b(B)] \). Then there are constants \( c_{ij} \) such that for all \( t, s \in T^*_b B \) we have
\[
F_1(t + s)F_2(t - s) = \sum_j c_{ij} G_i(t)G_j(s),
\]

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Proof. First let \( g_1, \ldots, g_m \) be a generating set for \( HF(L_{2p}) \) as a \( \Lambda \)-module. We can assume without loss of generality that \( \exp^* g_i = G_i \) for \( i \leq n \), and \( \exp^* g_i = 0 \) for \( i > n \). Then

\[
F(t + s)F(t - s) = HF_{N_p\#N_p}(\exp(t(\sigma_p + \tau_p))\exp(s(\sigma_p - \tau_p))f_1f_2) = HF_{N_p}(\exp(t(\sigma_p + \tau_p)))HF_{W_{p,p}}(\exp(s(\sigma_p - \tau_p))f_1f_2) = \sum_i H_i(s)G_i(t),
\]

where the formal power series \( H_j(s) \) are defined by the identity

\[
HF_{W_{p,p}}(\exp(s(\sigma_p - \tau_p))f_1f_2) = \sum_j H_j(s)g_j.
\]

Switching the role of \( s \) and \( t \), we also see that

\[
F_1(t + s)F_2(t - s) = \sum_i G_i(s)K_i(t)
\]

for some formal power series \( K_j(t) \).

Now, because the functions \( G_i \) are linearly independent, we can find multi-indices \( I_j \) such that the matrix

\[
M_{ij} = \left. \frac{\partial |I_j|}{\partial t^I} \right|_0 G_i
\]

is invertible. Therefore, the identity

\[
\left. \frac{\partial |I_j|}{\partial t^I} \right|_0 F(t + s)F(t - s) = \sum_i H_i(s)M_{ij} = \sum_i G_i(s) \left. \frac{\partial |I_j|}{\partial t^I} \right|_0 K_i
\]

shows that each \( H_i(s) \) is a linear combination of the power series \( G_i(s) \). Therefore,

\[
F(t + s)F(t - s) = \sum_i G_i(t)H_i(s) = \sum_{i,j} c_{ij}G_i(t)G_j(s)
\]

as claimed. \( \square \)

**Proposition 100.** For any \( f \in HF(L_p) \), and any \( b \in B \), the power series \( F = \exp^*(f) \) has an infinite radius of convergence.

Proof. Let \( h_1, \ldots, h_n \) be a generating set for the algebra \( A \), and suppose that not all of the power series \( H_1, \ldots, H_n \in C[[T^*_b(B)]] \) have an infinite radius of convergence. Let \( r \) be the minimal radius of convergence over all the \( H_i \). Then every function in the image of \( \exp^* \) has radius of convergence at least \( r \). But for any \( F \) in the image of \( \exp^* \), the doubling formula shows

\[
F(2t)F(0) = \sum_{i,j} c_{ij}G_i(t)G_j(0),
\]

from which we conclude that \( F \) has radius of convergence at least \( 2r \), provided \( F(0) \neq 0 \). Using the trick from Proposition 98, we conclude that the same is true even if \( F(0) = 0 \). In
particular, each of the generators $H_i$ has radius of convergence at least $2r$. This contradicts
the definition of $r$, and shows that in fact each $H_i$ has an infinite radius of convergence.

**Proposition 101.** $\exp^*$ induces an entire analytic map $\exp: T_bB \to \mathcal{E}_b$

**Proof.** The fact that every function $F = \exp^* f$ is entire shows that we have an entire
analytic map from $T_bB$ to the affine cone $\text{Spec} \, A$. The only issue is that there might be a
point $t_0 \in T_bB$ where all the functions $F \in \Lambda_+$ vanish simultaneously. But then the doubling
formula tells us

$$F(t_0 + s)F(t_0 - s) = 0$$

for all $s$, so one of $F(t_0 \pm s)$ vanishes identically on $T_b^*B$, so $F$ vanishes identically on $T_bB$.
But this contradicts the fact that some $F$ is a unit in $\Lambda[[\Omega^1(\Lambda)]]$. \qed

### 4.6 Periods and Theta Functions

We would like to say that every function in the image of $\exp^*$ is a theta function. However,
this will not generally be the case because some fibers of $\pi : \mathcal{E} \to B$ might be toric varieties,
or even compactifications of an affine space. However, when $\exp : T^*B \to \mathcal{E}$ does have some
nontrivial periods, we can determine the automorphy factors of the functions $\exp^* f$ with
respect to the periods.

**Definition 102.** For any $b \in B$, we define $\Lambda_b$ to be the set of all $\lambda \in T_b^*B$ such that
$\exp(\lambda) = \exp(0)$.

**Proposition 103.** $\Lambda_b$ is a discrete subgroup of $T_b^*B$. Moreover, the fiber $\exp^{-1}(x)$ over any
$x \in \mathcal{E}_b$ is a coset of $\Lambda_b$.

**Proof.** Suppose that $\exp(\lambda) = \exp(\mu)$ for some $\lambda, \mu \in T_b^*B$. Then for all sufficiently large $p$
there is a nonzero constant $m_p$ such that $G(\lambda) = m_pG(\mu)$ for any $G = \exp^* g, g \in H\Phi L_{2p}$.
By the doubling formula we have

$$F_1(t + \lambda)F_2(t - \lambda) = \sum_{i,j} c_{ij}G_i(t)G_j(\lambda) = m_pF_1(t + \mu)F_2(t - \mu)$$

for any $f_1, f_2 \in H\Phi L_{p}$. Therefore,

$$\frac{F_1(t + \lambda)}{F_1(t + \mu)} = \frac{F_2(t - \mu)}{F_2(t - \lambda)} = u_p(t)$$

where $u_p(t)$ is a meromorphic function that is independent of $F_1$. For any $t_0$ there is some
choice of $F_1$ such that $F_1(t_0) \neq 0$, so $u(t)$ is actually holomorphic, and there is also a
choice of $F_1$ such that $F_1(t_0 + \lambda) \neq 0$, so $u_p(t)$ is nonvanishing. We can therefore write
$u_p(t) = \exp(h_{p}(t))$ for some entire function $h_{p}(t)$.

Now drop the assumption that $p$ is large. Then for any $f \in H\Phi L_{p}$ we can still find an
integer $N$ and an entire function $h$ such that

$$F(t + \lambda)^N = \exp(Nh_{p}(t))F(t + \mu)^N$$
and taking $N$-th roots of both sides we get

$$F(t + \lambda) = \exp(h_p(t))F(t + \mu)$$

It is easy to see that the function $h_p(t)$ depends only on $p$ and not on $F$, and that $h_p(t) = ph(t)$ for some $h(t)$.

To show that $\Lambda_b$ is a subgroup, suppose that $\exp(\lambda) = \exp(\mu) = \exp(0)$. Then there is a nonzero constants $m_{\lambda,p}$ and $m_{\mu,p}$ such that for $F = \exp^*(f), f \in HF(L_p)$, we have

$$F(\mu + \lambda) = m_{\lambda,p}F(\mu) = m_{\lambda,p}m_{\mu,p}F(0),$$

hence $\exp(\lambda + \mu) = \exp(0)$.

To show that any fiber of $\exp$ is a coset, observe that

$$F(-\mu + \lambda) = \exp(h_p(-\mu))F(0)$$

for any $F$, so $\lambda - \mu \in \Lambda_b$.

The fact that $\Lambda_b$ is discrete is just a consequence of the fact that some $F(t)$ is nonconstant.

In the course of proving the previous proposition we have also shown that for any $\lambda \in \Lambda_b$ there is an entire function $h_\lambda(t)$ such that

$$F(t + \lambda) = \exp(h_\lambda(t))F(t)$$

for all $F \in \exp(HF(L_p))$.

**Proposition 104.** For every $\lambda \in \Lambda_b$ the function $h_\lambda(t)$ is linear.

**Proof.** For any $F, G \in \exp(HF(L_p))$, the ratio $F(t)/G(t)$ is periodic, because the numerator and denominator transform in the same way when we translate by a period $\lambda$. In particular the second logarithmic derivative of any $F(t)$ is periodic.

Now, taking second logarithmic derivatives of both sides of the equation

$$F(t + \lambda) = \exp(ph_\lambda(t))F(t)$$

we see that

$$\left. \frac{F_{ij}F - F_iF_j}{F^2} \right|_{t+\lambda} = p \left. \frac{\partial^2 h_\lambda}{\partial t_i \partial t_j} + \frac{F_{ij}F - F_iF_j}{F^2} \right|_t$$

Because the second logarithmic derivative of $F$ is periodic, we see that the Hessian of $h_\lambda(t)$ vanishes identically. We conclude that $h_\lambda(t)$ is linear, as claimed.

**Corollary 105.** If $\Lambda_b$ is a lattice of full rank, then the blowup function is a theta function on the complex torus $T^*_b B/\Lambda_b$.

**Proof.** By definition, a theta function is a quasiperiodic function whose automorphy factors are exponentials of linear functions. Since the blowup function is in the image of $\exp^*$, its automorphy factors are of the form $\exp(h_\lambda(t))$ for some linear functions $h_\lambda(t)$.

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Bibliography


