The unramified principal series of p-adic groups: the Bessel function

by

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Abstract

Let G be a connected reductive group with a split maximal torus defined over a non-archimedean local field. I evaluate a matrix coefficient of the unramified principal series of G known as the "Bessel function" at torus elements of dominant coweight. I show that the Bessel function shares many properties with the Macdonald spherical function of G, in particular the properties described in Casselman's 1980 evaluation of that function. The analogy I demonstrate between the Bessel and Macdonald spherical functions extends to an analogy between the spherical Whittaker function, evaluated by Casselman and Shalika in 1980, and a previously unstudied matrix coefficient.

Thesis Supervisor: Benjamin Brubaker
Title: Associate Professor

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Contents

Chapter 1

Introduction

Let G be a connected reductive group over a non-archimedean local field F with split maximal torus T. Assume that G is defined over $\mathfrak{o}$, the ring of integers of F. Let $B = TU$ be a Borel subgroup for G. The simplest class of representations of this group are the unramified principal series – those arising by parabolic induction of a character χ of $T(F)/T(\mathfrak{o})$. We denote the resulting representation by $\mathrm{Ind}_B^G(\chi)$. In this thesis, we examine two matrix coefficients of the representation $\mathrm{Ind}_B^G(\chi)$; one is known as the "Bessel function" while the other has been previously unstudied.

Recall that a matrix coefficient associated to a representation (ρ, V) is a function on the group G, built from a linear functional $\mathcal{L} : V \longrightarrow \mathbb{C}$ and a vector $v \in V$ as follows:

$$g \longmapsto \mathcal{L}(\rho(g) \cdot v). \tag{1.1}$$

The linear functional is often assumed smooth when working in the category of smooth representations of $G(F)$. We do not assume that here.

Matrix coefficients are key objects in the study of p-adic harmonic analysis and automorphic forms and representations. An important step in gaining a deeper understanding of matrix coefficients is to determine their values at specific elements of G. That is, we seek explicit formulas or closed expressions that are more illuminating than the initial definition (1.1). Two prominent matrix coefficients are the "Macdonald spherical function"and the "spherical Whittaker function"; the terms and constructions are explained below. The Macdonald spherical function (which we abbreviate to "spherical function") was first investigated by Macdonald [Mac] and later Casselman [Cas]. The spherical Whittaker function was studied by Casselman and Shalika [CS], who proved a conjecture of Langlands about the precise formula at torus elements. These authors succeeded in establishing explicit formulas for their matrix coefficients in terms of certain polynomials significant in combinatorics and representation theory.

This thesis continues the pursuit of such explicit formulas. The main result is the evaluation of the Bessel function at torus elements of dominant coweight in terms of representation-theoretic polynomials. To our knowledge, it had not been recognized in the literature that this function should even have an explicit evaluation. This thesis also makes apparent an intimate connection, or analogy, between the Bessel function and the spherical function; this connection had also been unknown. Specifically, the Bessel function shares important properties with the spherical function that were described in [Cas]. We subsequently consider another matrix coefficient, previously unstudied, and show that it is connected to the spherical Whittaker function in such a way that extends the analogy we have demonstrated between the Bessel function and spherical function.

The matrix coefficients mentioned above are constructed by mixing and matching certain linear functionals and vectors of $\mathrm{Ind}_B^G(\chi)$ that are invariant under the action of certain subgroups of G. These subgroups are $K = G(\mathfrak{o})$, a maximal compact subgroup of G, and a unipotent radical U of a Borel subgroup of G.

First, there is the K-invariant functional $\mathcal{S}$, known as the "spherical" functional in comparison with the archimedean setting, where $K = SO_n(\mathbb{R})$ is a maximal compact subgroup of $G = SL_n(\mathbb{R})$. It satisfies

$$\mathcal{S}(\rho(k) \cdot v) = \mathcal{S}(v) \quad \text{for all } k \in K \text{ and } v \in \mathrm{Ind}_B^G(\chi).$$

Second, there is the (U, ψ)-invariant functional $\mathcal{W} = \mathcal{W}(U, \psi)$, where ψ is a non-degenerate character of U. This functional is called the "Whittaker" functional, again in comparison with archimedean case where matrix coefficients made from this functional are associated to a differential equation due to Whittaker. It satisfies

$$\mathcal{W}(\rho(u) \cdot v) = \psi(u)\mathcal{W}(v) \quad \text{for all } u \in U \text{ and } v \in \mathrm{Ind}_B^G(\chi).$$

For an arbitrary irreducible smooth representation, it is not clear that non-zero functionals with these properties exist. However, it is true for unramified principle series. We note that $(K, 1)$ and (U, ψ) are examples of generalized Gelfand pairs: given a a subgroup H of G and a character ψ of H, recall that (H, ψ) is a generalized Gelfand pair for G if the induced representation $\mathrm{Ind}_H^G(\psi)$ is multiplicity free. This ensures that any G-module homomorphism from V to $\mathrm{Ind}_H^G(\psi)$ is unique up to scalar and this fact may be exploited in applications.

The invariant vectors in $\mathrm{Ind}_B^G(\chi)$ used in the construction of the matrix coefficients are the K-fixed vector ϕ_K, known as the "spherical vector", and the (U, ψ)-invariant vector $\phi_{U,\psi}$, known as the "Whittaker vector". They satisfy

$$\rho(k) \cdot \phi_K = \phi_K \quad \text{for all } k \in K = G(\mathfrak{o})$$

and

$$\rho(u) \cdot \phi_{U,\psi} = \psi(u)\phi_{U,\psi} \quad \text{for all } u \in U.$$

For an arbitrary smooth representation of $G(F)$, it is of course possible that such invariant vectors do not exist in the representation. However, they do exist for unramified principle series.

Thus we arrive at four possible matrix coefficients for $\mathrm{Ind}_B^G(\chi)$ formed from these choices of functionals and vectors. The spherical function is the matrix coefficient constructed from $\mathcal{S}$ and ϕ_K, and the Whittaker spherical function from $\mathcal{W}$ and ϕ_K. As stated above, this thesis considers the Bessel function, constructed from $\mathcal{W}$ and $\phi_{U,\psi}$; it gets its name function in the archimedean case which satisfies a differential equation due to Bessel. We also evaluate the matrix coefficient from $\mathcal{S}$ and $\phi_{U,\psi}$ which has been unstudied. In the remainder of this introduction, we discuss known results and methods of proof.

We summarize some appearances of the Bessel function in the literature. The asymptotics of Bessel function on p-adic groups are studied in the thesis of Averbuch [Ave]. Cogdell, Kim, Piatetski-Shapiro, and Shahidi use the Bessel function to obtain stability of local γ-factors in [CKP-SS] and [CP-SS]. They also use "approximate Whittaker vectors" as a way to define the Bessel function. In addition, Gelbart and Piatetski-Shapiro use the Bessel function to study ϵ-factors. Baruch [Bar] and Lapid and Mao [LM] study general properties of the Bessel function integrals. The matrix coefficient construction using a vector in $\mathrm{Ind})B^G(\chi)$ appears in Baruch and Mao [BM] and Soudry [Sou] but they do not name the vector they use as the "Whittaker vector". The authors of [BM] restrict their attention to rank 1 groups and their double covers; we note that the the limiting process used there to define the integral of the matrix coefficient is generalized in this thesis to arbitrary G. These authors do not seek explicit formulas for these functions on p-adic groups. Finally, in [P-S], Piatetski-Shapiro employs a vector similar to $\phi_{U,\psi}$ in the finite field setting, but there it is called the "Bessel vector".

I first encountered the notion of the Whittaker vector while examining [GKMMMO] which deals with constructions of archimedean Whittaker functions. After computing the corresponding matrix coefficient in the p-adic setting for certain groups, I conjectured the explicit formula for arbitrary G. Upon learning the similarity with the formula for the spherical function, I examined Casselman's paper [Cas] to see if there was a connection. This led me to seek analogues of the objects in [Cas]. We briefly compare [Cas] and this thesis below.

First we describe the setup of the main result, Theorem 1.0.2. The root datum $(X, R, X^\vee, R^\vee)$ of the pair (G, T) determines G as an abstract group whose presentation is given by chapters 9 and 10 of [Spr]. Here X and $X^\vee$ are lattices in duality by the pairing $\langle , \rangle$, and $R \subset X$ and $R^\vee \subset X^\vee$ are dual root systems. Let $\mathfrak{o}$ denote the ring of integers in F and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let π be a fixed generator of $\mathfrak{p}$ and q be the order of the residue field $\mathfrak{o}/\mathfrak{p}$. Let K be the subgroup $G(\mathfrak{o})$ of G; K is maximally compact. Fix a Borel subgroup B of G with unipotent radical U. Let R^+ is the choice of positive roots in R that corresponds to this choice of Borel of group with R^- denoting $-R^+$. In general, for any positive system S of R, we write the corresponding unipotent group as $U(S)$ and set $U^- = U(R^-)$. Then $U(S)$ is generated by unipotent elements $u_\gamma(t)$, where $\gamma \in S$ and $t \in F$. Let D be the simple roots in R^+ with $D^- = -D$. The Haar measure m on G is normalized so that $u_\gamma(\mathfrak{o})$ has measure 1 for each $\gamma \in R$. Let ψ denote a $\mathbb{C}$-valued character of U that is trivial on $u_\alpha(\mathfrak{o})$ but nontrivial on $u_\alpha(\mathfrak{p}^{-1})$ for each $\alpha \in D$. We also use ψ to denote a character of U^- that is trivial on $u_\alpha(\mathfrak{o})$ but nontrivial on $u_\alpha(\mathfrak{p}^{-1})$ for each $\alpha \in D^-$; this should not cause confusion.

Let T be the maximal torus in B. T is generated by elements $\mu(t)$ for $\mu \in X^\vee$ and $t \in F$. Say that $\mu \in X^\vee$ is *dominant* if $\langle \alpha, \mu \rangle < 0$ for all $\alpha \in D$. We denote $\lambda(\pi)$ by π^λ. Now, we let $\mathbb{C}(X^\vee)$ denote the vector space of rational functions in the indeterminates $\pi^{\alpha^\vee}, \alpha \in D$. In $\mathbb{C}(X^\vee)$, we denote $\pi^{\alpha^\vee}\pi^{\beta^\vee}$ by $\pi^{\alpha^\vee+\beta^\vee}$ for $\alpha, \beta \in R$. We give $\mathbb{C}(X^\vee)$ a topology by completing with respect to the ideal generated by the $\pi^{-\alpha^\vee}, \alpha \in D$ (that is, geometric series in $\pi^{-\alpha^\vee}$ are convergent). Let $\chi : B \to \mathbb{C}(X^\vee)$ be the tautological character on B defined by $\chi(\pi^\lambda) = \pi^\lambda$ (caution: we take this notation for $\mathbb{C}(X^\vee)$ from [HKP] where it is used slightly differently). Let ρ be the half-sum of roots in R^+. Let δ_B the modular character of B. We have $\delta_B^{1/2}(\pi^\mu) = q^{-\langle \rho, \mu \rangle}$. Let W be the Weyl group $N_G(T)/T$ with w_ℓ the long element of W with respect to R^+. The Weyl group acts on X and $X^\vee$, and for $w \in W$ we let χ^w be the unramified character of B defined by $\chi^w(\pi^\mu)) = \pi^{w^{-1}(\mu)}$. W also acts on $\mathbb{C}(X^\vee)$ by $w(\pi^\mu) = \pi^{w(\mu)}$ and we extend this action by linearity. In addition, for a positive system, S set $w(U(S)) = U(w(S))$. We let w_ℓ denote the long element of W with respect to R^+.

Let $\mathrm{Ind}_B^G(\chi)$ denote the principal series representation of G constructed by normalized induction:

$$\mathrm{Ind}_B^G(\chi) = \{f : G \to \mathbb{C}(X^\vee) | f(bg) = \delta_B^{1/2}\chi(b)f(g) \, \forall \, b \in B, g \in G\}.$$

The action ρ of G on $\mathrm{Ind}_B^G(\chi)$ is right-translation. Note that we do not require the functions f to be locally constant, for we will need to consider functions that are not locally constant everywhere on G.

The Whittaker vector $\phi_{U,\psi}$ is defined via the Bruhat decomposition

$$G = \bigcup_{w \in W} Bn_wU$$

where the element n_w is defined as follows. For $\alpha \in D$, the element n_α is defined by

$$n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1).$$

Given a reduced decomposition $w = s_{\alpha_1}...s_{\alpha_m}$ with $\alpha_i \in D$ possibly non-distinct, n_w is the element

$$n_w = n_{\alpha_1}...n_{\alpha_m}$$

where n_w is independent of the reduced decomposition.

Definition 1.0.1 In $\mathrm{Ind}_B^G(\chi^v)$ there is a vector called the "Whittaker vector" which we denote by $\phi_{U,\psi}^v$. We define it as a $\mathbb{C}(X^\vee)$-valued function on G by

$$\phi_{U,\psi}^v(bn_w u) = \delta_B^{1/2}\chi^v(b)\phi_{U,\psi}^v(n_w)\psi(u) \tag{1.2}$$

where

$$\phi_{U,\psi}^v(n_w) = \delta_{w,w_\ell}$$

and δ is the Kronecker delta. We let $\phi_{U,\psi}$ denote $\phi_{U,\psi}^1$.

The notation is meant to convey that the Whittaker vector transforms under right-translation by $u \in U$ as

$$\phi_{U,\psi}(gu) = \phi_{U,\psi}(g)\psi(u).$$

We define the Whittaker functional $\mathcal{W} = \mathcal{W}(U^-, \psi)$ by integration over U^-:

$$\mathcal{W}(f) = \int_{U^-} f(u)\psi^{-1}(u)\, du.$$

When $f = \phi_{U,\psi}$, the above integral is not absolutely convergent. We define $\mathcal{W}(\rho(g) \cdot \phi_{U,\psi})$ to be a limit of absolutely convergent integrals as explained in chapter 3. We now can state the main theorem.

Theorem 1.0.2 *Let $a_0 \in T(\mathfrak{o})$ and $\mu \in X^\vee$ be dominant. Then*

$$\mathcal{W}(\rho(a_0\pi^\mu)\cdot\phi_{U,\psi}) = \delta_B^{-1/2}(\pi^\mu)\sum_{w\in W} w\left(\prod_{\gamma\in R^+}\frac{1-q^{-1}\pi^{-\gamma^\vee}}{1-\pi^{\gamma^\vee}}\right)\pi^{w(\mu)}.$$

This document contains the proof of this theorem when R is a direct sum of root systems of types A, B, C, and D with some discussion of type E. The proof for exceptional types is forthcoming. The type dependence is solely contained in chapter 7.6.

As mentioned above, this document will describe the similarities between the Bessel function and spherical function. We now summarize the proof in [Cas] of the formula for the spherical function. Recall that K is the subgroup $G(\mathfrak{o})$ of G.

Definition 1.0.3 For $v \in W$, let $\phi_K^v \in \mathrm{Ind}_B^G(\chi^v)$ denote the normalized K-fixed (spherical) vector defined by

$$\phi_K^v(bk) = \delta_B^{1/2}\chi^v(b).$$

We let ϕ_K denote ϕ_K^1.

The Iwasawa decomposition

$$G = BK$$

shows that ϕ_K is well-defined. The spherical function $\Gamma(g)$ is then defined by

$$\Gamma(g) = \int_K \phi_K(kg)\,dk.$$

Compare our Theorem 1.0.2 to Theorem 4.2 of [Cas]. To establish this formula, Casselman makes use of the Iwahori subgroup I of K.

Definition 1.0.4 Let $S \subset R$ be a choice of positive roots. Define $I(S)$ to be the subgroup of G generated by $T(\mathfrak{o})$ and the elements $u_\gamma(x)$ subject to the following conditions:

$$\begin{aligned} x &\in \mathfrak{p} && \text{if } \gamma \in -S \\ x &\in \mathfrak{o} && \text{if } \gamma \in S. \end{aligned} \tag{1.3}$$

We call such $u_\gamma(x)$ the **unipotent generators** of $I(S)$.

For our fixed positive system R^+, we let I denote $I(R^+)$.

The space of Iwahori-fixed vectors $\mathrm{Ind}_B^G(\chi)^I \subset \mathrm{Ind}_B^G(\chi)$ has a basis (the "standard Iwahori-fixed basis") given by the set $\{\phi_{K;w}\}_w$.

Definition 1.0.5 For $v, w \in W$, define $\phi^v_{K;w}$ to be the vector in $\mathrm{Ind}_B^G(\chi^v)$ determined by

$$\phi^v_{K;w}(g) = \phi^v_K(g) 1_{Bn_wI}(g).$$

Let $\phi_{K;w}$ denote $\phi^1_{K,w}$. Define the vector space $V_K(\chi^v) \subset \mathrm{Ind}_B^G(\chi^v)$ to be the space spanned by the set $\{\phi^v_{K;w}\}_w$; thus $V_K(\chi^v) = \mathrm{Ind}_B^G(\chi^v)^I$.

Casselman then introduces another basis of $V_K(\chi)$ which he constructs using the intertwining maps $T_w : \mathrm{Ind}_B^G(\chi^w) \to \mathrm{Ind}_B^G(\chi)$ (see chapter 5). These new basis vectors are $\{T_w(\phi^{w^{-1}}_{K;1})\}_w$. Casselman shows that ϕ_K in this basis is

$$\phi_K = \sum_{w \in W} w\Big(\prod_{\gamma \in R^+} \frac{1 - q\pi^{\gamma^\vee}}{1 - \pi^{\gamma^\vee}}\Big) T_w(\phi^{w^{-1}}_{K;1})$$

(here we use the formulation of [HKP]). It remains to determine

$$\int_K T_w(\phi_{K;w}^{w^{-1}})(k\pi^\mu)\,dk.$$

Casselman proves that this integral evaluates to the simple monomial term, and the formula is proved.

The formula in Theorem 1.0.2 was conjectured by the author after computations for certain G. Motivated by the similarity to the spherical function, we seek a proof of Theorem 1.0.2 analogous to that found in [Cas] for the spherical function. At each step of the proof in [Cas], this thesis constructs corresponding objects in our proof for the Bessel function that play the same role. These objects include a previously unstudied subgroup of G which we call I_{ht} and new vectors in the principal series representation we call $\phi_{U,\psi;w}$ (chapter 2), which take the place of the Iwahori subgroup and the Iwahori-fixed vectors, respectively. Due to the similarities surrounding these matrix coefficients, we call these two frameworks the *K-fixed side* and the *(U,ψ) side.* We do encounter difficulties not present in the K-fixed side; to overcome them we develop new techniques, including novel ways to compute the Whittaker functional (chapter 4) and combinatorial objects we call *paths in root systems* (chapter 7). At this point, we do not have a foundational reason why the analogy between these matrix coefficients exists; searching for such a reason is one of many avenues of future work.

Now we state relations in G used throughout the thesis. These are taken from the presentation of G given in chapter 9 of [Spr]. Fix a total order on R. Then, for linearly-independent α and $\beta \in R$, there exist *structure constants* $c_{\alpha,\beta;i,j} \in F$ such

that

$$u_\alpha(x)u_\beta(y) = u_\beta(y)\left(\prod_{i\alpha+j\beta\in R; i,j>0} u_{i\alpha+j\beta}(c_{\alpha,\beta;i,j}x^i y^j)\right)u_\alpha(x) \tag{1.4}$$

where the product is taken according to our fixed order on R. We call this the **commutator relation**. We will make a slight abuse of notation where we allow the integers i or j to be negative and set

$$c_{\alpha,\beta;i,-j} = c_{\alpha,-\beta;i,j}.$$

There are also structure constants $d_{\alpha,\beta} \in F$ satisfying

$$n_\alpha^{-1}u_\beta(t)n_\alpha = u_{s_\alpha(\beta)}(d_{\alpha,\beta}t).$$

Lemma 1.0.6 *The structure constants may be chosen so that $c_{\alpha,\beta;i,j} \in \mathbb{Z}$ and $d_{\alpha,\beta} \in \{\pm 1\}$.*

Proof 9.5.3 in [Spr].

As $d_{\alpha,\beta} \in \{\pm 1\}$, we will often suppress this constant during integration as it can be removed by a re-scaling of variables.

We use the presentation of G given in chapter 9 of [Spr] where the field is F. For $\mu \in X^\vee$, we have the relations

$$\mu(x)\, u_\gamma(t)\, \mu(x)^{-1} = u_\gamma(x^{\langle\gamma,\mu\rangle}t),$$

$$n_w\, \mu(x)\, n_w^{-1} = w(\mu)(x),$$

and

$$\gamma^\vee(x)n_\gamma = u_\gamma(x)u_{-\gamma}(-x^{-1})u_\gamma(x). \tag{1.5}$$

We will often make use of the following implications of (1.5). For $xy \neq -1$,

$$u_{-\gamma}(x)u_\gamma(y) = u_\gamma(\frac{y}{1+xy})\gamma^\vee(\frac{1}{1+xy})u_{-\gamma}(\frac{x}{1+xy}) \tag{1.6}$$

and, for $x \neq 0$,

$$u_{-\gamma}(x) = u_\gamma(-x^{-1})\gamma^\vee(-x^{-1})n_\gamma u_\gamma(-x^{-1}). \tag{1.7}$$

Chapter 2

The Subgroup I_{ht} and Vectors $\phi_{U,\psi;w}$

We define a subgroup of G called I_{ht}. This subgroup will play the same role in evaluating the Bessel function as that of the Iwahori subgroup I in evaluating the spherical function.

Definition 2.0.7 Let $S \subset R$ be a choice of positive roots and let $\text{ht} = \text{ht}(S)$ be the height function with respect to S. Recall that if $\gamma \in S$ is

$$\gamma = m_1\alpha_1 + \ldots + m_r\alpha_r$$

where α_i are the simple roots of S and m_i positive integers, then

$$\text{ht}(\gamma) = m_1 + \ldots + m_r.$$

Define $I_{\text{ht}}(S)$ to be the subgroup of G generated by $T(\mathfrak{o})$ and the elements $u_\gamma(x)$

subject to the following conditions:

$$x \in \mathfrak{p}^{\mathrm{ht}\,(-\gamma)} \qquad \text{if } \gamma \in -S \tag{2.1}$$

$$x \in \mathfrak{p}^{1-\mathrm{ht}\,(\gamma)} \qquad \text{if } \gamma \in S.$$

We call such $u_\gamma(x)$ the **unipotent generators** of $I_{\mathrm{ht}}(S)$.

For our specific choice R^+ of positive roots, we let I_{ht} denote $I_{\mathrm{ht}}(R^+)$ and $\mathrm{ht} = \mathrm{ht}(R^+)$ when there is no possibility for confusion.

I_{ht} shares many properties of I. Indeed, I and I_{ht} are isomorphic as groups (Lemma 2.0.12). A key property we will often use is the following.

Lemma 2.0.8 Suppose $u_\alpha(x)$ and $u_\beta(y)$ are unipotent generators of I_{ht}. Then in the commutator relation

$$u_\alpha(x)u_\beta(y) = u_\beta(y)\left(\prod_{i,j>0} u_{i\alpha+j\beta}(c_{i,j;\alpha,\beta}x^i y^j)\right) u_\alpha(x)$$

each element in the product is also a unipotent generator of I_{ht}.

Proof Follows from a direct check of the cases according to the signs of α, β and $i\alpha + j\beta$ along with the assumption that $c_{i,j;\alpha,\beta}$ are integers. □

Lemmas 2.0.9, 2.0.10, and 2.0.11 are proved in exactly the same manner as they are proved for I. They may be proved by writing an element $g \in G$ as

$$g = bn_w u$$

for some $b \in B, w \in W$, and $u \in w^{-1}(U^-) \cap U$. Write u as a product of unipotents, ordered by a convex ordering of $w^{-1}(R^-) \cap R^+$. Starting from the right end of this product, if $u_\gamma \in I$ (or I_{ht}), then apply a step of the explicit Bruhat decomposition (chapter 7.2) if $u_\gamma \notin I$ (or I_{ht}) and proceed to the next unipotent. Otherwise, if $u_\gamma \in I$ (or I_{ht}), then move to the next unipotent. Then end result is an expression in Bn_yI (or Bn_yI_{ht}) for some $y \in W$. We call this the **explicit Iwahori** (or $\boldsymbol{I_{\text{ht}}}$) **decomposition**.

Lemma 2.0.9 *Let $S \subset R$ be a choice of positive roots. Suppose u is an element of the form*

$$u = \prod_{\gamma \in S} u_\gamma(x_\gamma)$$

where the product over S is taken in any order. If $u \in I_{\text{ht}}(R^+)$, then each $u_\gamma(x_\gamma)$ is a unipotent generator of $I_{\text{ht}}(R^+)$.

Lemma 2.0.10 *Let $w \in W$. Then an element $i \in I_{\text{ht}}$ has a unique factorization of the form*

$$i = i_-(w) a i_+(w)$$

where $i_-(w) \in (I_{\text{ht}} \cap w(U^-))$, $a \in T(\mathfrak{o})$, and $i_+(w) \in (I_{\text{ht}} \cap w(U))$.

Lemma 2.0.11 *G is the disjoint union*

$$G = \bigcup_{w \in W} Bn_w I_{\text{ht}}.$$

More precisely,

$$G = \bigcup_{w \in W} Bn_w (I_{\text{ht}} \cap w^{-1}(U^-))$$

and this decomposition is unique. Moreover, if g is of the form

$$g = bn_w u$$

with $b \in B$, $w \in W$, *and* $u \in w^{-1}(U^-)$, *then* $g \in Bn_w(I_{\text{ht}} \cap w^{-1}(U^-))$ *if and only if* $u \in I_{\text{ht}}$.

Let $\rho^\vee$ denote the half-sum of roots in $R^{\vee +}$.

Lemma 2.0.12 *If* $\rho^\vee \in X^\vee$, *then*

$$I_{\text{ht}} = n_{w_\ell}\pi^{\rho^\vee}\, I\, (n_{w_\ell}\pi^{\rho^\vee})^{-1}.$$

Sometimes $\rho^\vee \notin R^{\vee +}$. In that case we extend the field F by adjoining $\pi^{1/2}$ to obtain

$$2\rho^\vee(\pi^{1/2})$$

which serves the same purpose as π^ρ in the above lemma. Either way, I_{ht} is isomorphic to I via an automorphism of G.

Lemma 2.0.12 follows from fact (ii) of the following lemma. Fact (iii) will be needed in the proof of Theorem 5.0.31.

Lemma 2.0.13 *Let* α *be a simple root and* γ *any root in* R.

(i). $\langle \rho, \alpha^\vee \rangle = 1$

(ii). $\langle \rho, \gamma^\vee \rangle = \text{ht}(\gamma^\vee)$

(iii). $\displaystyle\sum_{\beta \in w^{-1}(R^+) \cap R^+} \langle \beta, \gamma^\vee \rangle = \text{ht}(\gamma^\vee) + \text{ht}(w(\gamma^\vee))$

Proof We make use of the invariance of $\langle\, ,\rangle$ under the Weyl group W.

(i). Because $s_\alpha(R^+\backslash\alpha) = R^+\backslash\alpha$, we have

$$\begin{aligned}\langle \rho - \frac{\alpha}{2}, \alpha^\vee\rangle &= \langle s_\alpha(\rho - \frac{\alpha}{2}), s_\alpha(\alpha^\vee)\rangle \\ &= \langle \rho - \frac{\alpha}{2}, -\alpha^\vee\rangle \\ &= -\langle \rho - \frac{\alpha}{2}, \alpha^\vee\rangle.\end{aligned}$$

Therefore $\langle \rho - \frac{\alpha}{2}, \alpha^\vee\rangle = 0$ and $\langle \rho, \alpha^\vee\rangle = \langle \frac{\alpha}{2}, \alpha^\vee\rangle = 1$.

(ii). This follows immediately from (i).

(iii). The statement is equivalent to

$$\langle -\rho + \sum_{\beta \in w^{-1}(R^+)\cap R^+} \beta, \gamma^\vee\rangle = \langle \rho, w(\gamma^\vee)\rangle.$$

But we check that

$$-\rho + \sum_{\beta \in w^{-1}(R^+)\cap R^+} \beta = w^{-1}(\rho),$$

so the identity follows from the invariance of $\langle\, ,\rangle$ under W. □

Now we can define the "I_{ht}-standard basis".

Definition 2.0.14 For $v, w \in W$, define $\phi^v_{U,\psi;w}$ to be the vector in $\mathrm{Ind}_B^G(\chi^v)$ determined by

$$\phi^v_{U,\psi;w}(g) = \phi^v_{U,\psi}(g)\mathbb{1}_{Bn_w I_{\mathrm{ht}}}(g).$$

Define the vector space $V_{U,\psi}(\chi^v) \subset \mathrm{Ind}_B^G(\chi^v)$ to be the space spanned by the $\phi^v_{U,\psi;w}$ for all w.

Just as

$$\phi_K = \sum_{w\in W} \phi_{K;w}$$

it is clear

$$\phi_{U,\psi} = \sum_{w \in W} \phi_{U,\psi;w}$$

by the I_{ht}-decomposition for G (Lemma 2.0.11).

Chapter 3

Defining Integrals

This section explains how we interpret the integral

$$\int_{U^-} \phi_{U,\psi;w}(ua)\psi^{-1}(u)\,du$$

that is not absolutely convergent.

Recall that F has a topology induced by the non-archimdeadean metric and that F^N has the product topology, for a positive integer N. Also recall that U^- is a topological group whose topology is generated by the base of open sets

$$u_\gamma(\mathfrak{p}^n)$$

about the identity element, where $\gamma \in R^-$ and $n > 0$. Furthermore, U^- is homeomorphic to F^N as topological spaces, where $N = |R^+|$.

Definition 3.0.15 If f is homeomorphism from some open subset $\mathcal{O} \subset F^N$ into U^- such that

$$U^- \setminus f(\mathcal{O})$$

has Haar measure 0, then we call f a **parametrization** of U^-.

An element of F^N is described by an N-tuple

$$(t_1, ..., t_N)$$

of elements $t_i \in F$. Recall that a non-zero element t of F is of the form $v\pi^n$, where $v \in \mathfrak{o}^\times, n \in \mathbb{Z}$, and π is a fixed generator of $\mathfrak{p}$. The integer n is called the **valuation** of t and we call v the **unit** of t.

Definition 3.0.16 Let $f : (F^\times)^N \to U^-$ be a parametrization of U^- . Given an N-tuple of integers

$$\overline{n} = (n_1, ..., n_N),$$

define the subset $\boldsymbol{f(\overline{n})} \subset U^-$ to be the image

$$f(\mathfrak{o}^\times \pi^{n_1}, ..., \mathfrak{o}^\times \pi^{n_N}).$$

Let $\mathscr{S}$ denote a finite set of N-tuples of integers. Define the subset $\boldsymbol{f(\mathscr{S})} \subset U^-$ to be

$$f(\mathscr{S}) = \bigcup_{\overline{n} \in \mathscr{S}} f(\overline{n}).$$

Each set $f(\overline{n})$ is compact in U^- and homeomorphic to $(\mathfrak{o}^\times)^N$. We require the parametrizations f to possess the following property.

Definition 3.0.17 Let $f : (F^\times)^N \to U^-$. We say that f is **torus-monomial** with respect to $Bn_{w_\ell}U$ if whenever $f(t_1, ..., t_N)$ is of the form

$$au_1 n_{w_\ell} u_2$$

for some $u_1, u_2 \in U$ and $a \in T$, then a is of the form

$$a = \prod_{\alpha \in D} \alpha^\vee(m_\alpha)$$

where m_α is some monomial in $\mathbb{Q}[t_1, t_1^{-1}, ..., t_N, t_N^{-1}]$.

If a parametrization f is torus-monomial, then the Whittaker vector $\phi_{U,\psi}$ is bounded on $f(\mathscr{I})$, for a fixed $\mathscr{I}$, in the sense that the unramified character $\delta_B^{1/2}\chi$ takes on only finitely many values. As $f(\mathscr{I})$ is compact, it follows that

$$\int_{f(\mathscr{I})} \phi_{U,\psi}(ua)\psi^{-1}(u)\, du$$

is absolutely convergent for for any $a \in T$. Likewise the corresponding integrals for $\rho(a) \cdot \phi_{U,\psi;w}$ are also absolutely convergent.

Definition 3.0.18 Consider a sequence sets of N-tuples $\{\mathscr{I}_k\}_k$. We say that $\mathscr{I} \to \infty$ if for any positive integer M, there exists some k_0 such that for all $k > k_0$, $\mathscr{I}_k$ contains each N-tuple $(n_1, ..., n_N)$ satisfying $|n_i| < M$ for $1 \leq i \leq N$.

For a torus-monomial parametrization f and a sequence $\mathscr{I}_k \to \infty$, define

$$\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w}) = \lim_{k \to \infty} \int_{f(\mathscr{I}_k)} \phi_{U,\psi;w}(ua)\psi^{-1}(u)\, du.$$

We will present, for each $\phi_{U,\psi;w}$, torus-monomial parametrizations f that depend on a reduced word for w_ℓ. We show that the above limit exists and is independent of the choice of reduced word. The parametrizations are constructed in chapters 4 and 7.2.

We then define the Bessel function $W_{U,\psi}(a)$ to be

$$W_{U,\psi}(a) = \sum_{w \in W} \mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w}).$$

Chapter 4

Computing $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w})$ via a Quotient Space

The main result of this section is Lemma 4.0.23 which partially constructs the torus-monomial parametrization used to define $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w})$; the remaining construction is in chapter 7. In this parametrization, Lemma 4.0.23 expresses $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w})$ as a product of two factors. This expression is central to the proof of Theorem 5.0.31.

Recall that $\mathcal{W}$ is defined as an integral over U^-. We break up this integral according to a subgroup H of U^- and its quotient space. We state the following theorem.

Theorem 4.0.19 *Let G be a locally compact group and H a closed subgroup. Then there exists a G-quasi-invariant measure $\mu_{G/H}$ on the quotient space G/H and*

$$\int_G f(g)d\mu(g) = \int_{G/H} \left(\int_H f(gh)d\mu_H \right) d\mu_{G/H}(gH).$$

Proof Chapter 1 of [OV].

For a $w \in W$, we apply this theorem with $G = U^-$ and $H = w^{-1}(U^-) \cap U^-$. The following result characterizes certain coset representatives of $U^-/\big(w^{-1}(U^-) \cap U^-\big)$ we need for Lemma 4.0.23.

Lemma 4.0.20

$$U^- \cap (Bn_w I_{\text{ht}}) = \dot{\bigcup_{\overline{u}}} \overline{u}\,(w^{-1}(U^-) \cap U^- \cap I_{\text{ht}})$$

where the disjoint union is indexed over the elements $\overline{u} \in U^- \cap \big(Bn_w(w^{-1}(U^-) \cap U \cap I_{\text{ht}})\big)$. *That is, we may assume* $\overline{u}$ *are coset representatives of the form*

$$\overline{u} = bn_w i \in U^-$$

for $b \in B$ *and* $i \in w^{-1}(U^-) \cap U \cap I_{\text{ht}}$.

Proof Follows from the Lemmas 2.0.9 and 2.0.11.

Lemma 4.0.21 *Let* $s_n...s_1$ *be a reduced word for* $w \in W$, *where* s_i *is the simple reflection for* $\alpha_i \in D$.*The representative* $\overline{u}$ *of* $U^-/\big(w^{-1}(U^-) \cap U^-\big)$ *may be written as*

$$\overline{u} = \prod_{i=1}^{n} u_{-\alpha_i}(t_i).$$

Furthermore, when $\overline{u}$ *is of the form* $aun_w i$ *with* $a \in T, u \in U$, *and* $i \in w^{-1}(U^-) \cap U$, *then* a *is of the form*

$$\prod_{\alpha \in D} \alpha^\vee(m_\alpha)$$

where m_α *is a monomial in* $t_1, t_1^{-1}, ..., t_n, t_n^{-1}$.

Proof For a fixed $w \in W$, almost every element u of U^- may be written as

$$\left(\prod_{i=1}^{n} u_{-\alpha_i}(t_i)\right)\left(\prod_{\gamma \in w^{-1}(R^-)\cap R^-} u_\gamma(t_\gamma)\right). \tag{4.1}$$

This may be proved by writing $un_w^{-1}n_{w_\ell}$ as

$$u_1 n_{w_\ell} u_2$$

for some $u_1, u_2 \in U$; this can be done for almost every $u \in U^-$. Then we may write

$$u = u_1' n_w \left(\prod_{i=n}^{1} u_{w_{i-1}(\alpha_i)}(t_i)\right) u_2'$$

for some $u_2' \in U^- \cap w^{-1}(U^-)$ and where

$$w_i = s_i \ldots s_1.$$

This in turn may be written as

$$u_1' \left(\prod_{i=n}^{1} n_{\alpha_i} u_{\alpha_i}(t_i)\right) u_2'.$$

Now, starting with $i = n$ and proceeding to $i = 1$, apply the relation

$$n_{\alpha_i} u_{\alpha_i}(t) = u_{\alpha_i}(t^{-1}) \alpha^\vee(t) u_{-\alpha}(t^{-1})$$

and move $u_{\alpha_i}(t^{-1})$ to the left as in Lemma 7.2.1. The resulting expression for u is of the form

$$u = bu'$$

with $b \in B$ and $u' \in U^-$ of the form (4.1). Thus $b = 1$, proving the statement.

To see that a is a monomial in the t_i, t_i^{-1}, proceed from $i = n$ to $i = 1$, applying the relation

$$u_{-\alpha_i}(t) = u_{\alpha_i}(t^{-1})\alpha^\vee(t^{-1})n_\alpha u_\alpha(t^{-1})$$

to $\overline{u}$ and move $u_{\alpha_i}(t^{-1})$ to the left using the relation

$$u_\alpha(x)u_{-\alpha}(y) = u_{-\alpha}(y(1+xy)^{-1})\alpha^\vee(1+xy))u_\alpha(x(1+xy)^{-1}).$$

Then by induction it follows that the parameters of a are monomials. □

We now define two terms that arise from the quotient integral of Lemma 4.0.23.

Definition 4.0.22 Let $a = a_0\pi^\mu$ with $a_0 \in T$ and $\mu \in X^\vee$ dominant. Define $\mathcal{W}_H(\chi^v, w)$ and $\mathcal{W}_{U^-/H}(\chi^v, w, a)$ by

$$\mathcal{W}_H(\chi^v, w) = \int_{w^{-1}(U^-)\cap U^-} \phi^v_{U,\psi;w}(n_w u)\, du$$

and

$$\mathcal{W}_{U^-/H}(\chi^v, w, a) = \delta_B^{-1/2}\chi^v(a) \int_{U^-/w^{-1}(U^-)\cap U^-} \delta_B^{1/2}\chi^v(b)\psi^{-1}(a\overline{u}a^{-1})\mathbb{1}_{Bn_wI_{\text{ht}}}(\overline{u})\, d\overline{u}.$$

where $\overline{u} = bn_w i$ according to Lemma 4.0.20 and $d\overline{u}$ is an abbreviation for the quotient space measure $d\mu_{U^-/w^{-1}(U^-)\cap U^-}(\overline{u}\,(w^{-1}(U^-)\cap U^-))$. The torus-monomial parametrization defining $\mathcal{W}_{U^-/H}(\chi^v, w, a)$ is constructed in the proof of Lemma 4.0.23 and in chapter 7.2.

Lemma 4.0.23 *Suppose that $a = a_0\pi^\mu$ with $a_0 \in T(\mathfrak{o})/T$ and $\mu \in X^\vee$ dominant. Then*

$$\mathcal{W}(\rho(a) \cdot \phi^v_{U,\psi;w}) = \mathcal{W}_H(\chi^v, w)\mathcal{W}_{U^-/H}(\chi^v, w, a) \tag{4.2}$$

Proof We have

$$\int_{U^-} \phi^v_{U,\psi;w}(ua)\psi^{-1}(u)\,du = \delta_B^{-1/2}\chi^v(a)\int_{U^-} \phi^v_{U,\psi;w}(u)\psi^{-1}(aua^{-1})\,du.$$

conjugating out from a change of variable in u. Apply Theorem 4.0.19 with $H = w^{-1}(U^-) \cap U^-$ to get

$$\delta_B^{-1/2}\chi^v(a)\int_{U^-/H}\int_H \phi^v_{U,\psi;w}(\overline{u}h)\psi^{-1}(a\overline{u}a^{-1})\psi^{-1}(aha^{-1})\,dh\,d\overline{u}.$$

By the support of $\phi_{U,\psi;w}$, we may assume that $u = \overline{u}h \in Bn_wI_{\mathrm{ht}}$. Lemma 4.0.20 then says we may assume $h \in H \cap I_{\mathrm{ht}}$ and $\overline{u} = bn_wi$ with $i \in w^{-1}(U^-) \cap U \cap I_{\mathrm{ht}}$. Therefore h lies in the kernel of ψ, and the dominance of a implies that aha^{-1} does as well. We obtain

$$\delta_B^{-1/2}\chi^v(a)\int_{U^-/H}\int_H \delta_B^{1/2}\chi(b)\phi^v_{U,\psi;w}(n_wih)\psi^{-1}(a\overline{u}a^{-1})\,dh\,d\overline{u}.$$

Next we move the unipotent generators that comprise i to the right over h. We refer to Lemma 7.2.1 about moving unipotent elements. Make changes in h to cancel the effect of these moves; these changes take the form of the operator B_γ which is discussed in chapter 7.2. Since each new unipotent must be a unipotent generator (Lemma 2.0.8), these changes do not alter the fact that h ranges over $H \cap I_{\mathrm{ht}}$ or that aha^{-1} lies in the kernel of ψ.

We finally obtain

$$\delta_B^{-1/2}\chi^v(a)\int_{U^-/H}\int_H \delta_B^{1/2}\chi(b)\phi^v_{U,\psi;w}(n_w h)\psi^{-1}(a\overline{u}a^{-1})\mathbb{1}_{Bn_wI_{\mathrm{ht}}}(\overline{u})\,dh\,d\overline{u}$$

which factors into the two integrals $\mathcal{W}_H(\chi^v, w)\ \mathcal{W}_{U^-/H}(\chi^v, w, a)$. □

Theorem 7.1.1 gives the following important result needed in chapter 5:

$$\mathcal{W}_H(\chi^v, w) = \prod_{\gamma\in w^{-1}(R^+)\cap R^+} -\pi^{-w(\gamma^\vee)}q^{\mathrm{ht}(w(\gamma^\vee))-\mathrm{ht}(\gamma)-1}. \tag{4.3}$$

Chapter 5

Intertwining Operators Applied to $\phi_{U,\psi;w}$

There exists a unique intertwining map $T_{s_\alpha} : \mathrm{Ind}_B^G(\chi^{s_\alpha v}) \to \mathrm{Ind}_B^G(\chi^v)$. It may be expressed via an integral for appropriate $\phi \in \mathrm{Ind}_B^G(\chi^{s_\alpha v})$. For the purposes of this document, we will let T_{s_α} denote the integral itself.

Definition 5.0.24 For $\alpha \in D$ and ϕ a function on G, define T_{s_α} by

$$T_{s_\alpha}(\phi)(g) = \lim_{m \to \infty} \int_{\mathfrak{p}^{-m}} \phi(n_\alpha u_\alpha(x) g)\, dx.$$

Remark 5.0.25 In the K-fixed side, the intertwining maps T_{s_α} are used to construct the vectors $T_w(\phi_{K;1}^{w^{-1}})$ for each $w \in W$. These vectors are in fact a basis of $V_K(\chi)$ and are instrumental in the proof of the formula for the spherical function. Therefore, we would like to consider the vectors $T_w(\phi_{U,\psi;1}^{w^{-1}}))$ in hopes that they constitute a basis of $V_{U,\psi}(\chi)$. Upon computation, however, it turns out that these vectors may not even lie in $V_{U,\psi}(\chi)$. Nevertheless, we can still obtain a critical result "modulo the kernel of $\mathcal{W}$: we consider a multiple integral determined by $\mathcal{W} \circ T_{s_\alpha}$ over the $|R^+| + 1$-dimensional set $n_\alpha u_\alpha U^-$. The T_{s_α} thus aid in establishing relations among the $\mathcal{W}(\phi_{U,\psi;w})$. To explain this result (Theorem 5.0.31), we first define actual maps between the $V_{U,\psi}(\chi^v)$.

Definition 5.0.26 For $\gamma \in R$, define $c_{\mathcal{W}}(\gamma) \in \mathbb{C}(X^\vee)$ to be

$$\frac{1 - q^{-1}\pi^{\gamma^\vee}}{1 - \pi^{-\gamma^\vee}}.$$

Definition 5.0.27 For $w \in W$ and $\alpha \in D$ such that $\ell(s_\alpha w) = 1 + \ell(w)$, define the map $T'_{s_\alpha} : V_{U,\psi}(\chi^{s_\alpha v}) \to V_{U,\psi}(\chi^v)$ by

$$T'_{s_\alpha}(\phi^{s_\alpha v}_{U,\psi;w}) = \left(c_{\mathcal{W}}(v^{-1}(\alpha)) + \pi^{v^{-1}(\alpha^\vee)}\right)\phi^v_{U,\psi;w} - q^{-1}\pi^{v^{-1}(\alpha^\vee)}\phi^v_{U,\psi;s_\alpha w} \quad (5.1)$$

$$T'_{s_\alpha}(\phi^{s_\alpha v}_{U,\psi;w}) = -\pi^{v^{-1}(\alpha^\vee)}\phi^v_{U,\psi;w} + \left(c_{\mathcal{W}}(v^{-1}(\gamma)) + q^{-1}\pi^{v^{-1}(\alpha^\vee)}\right)\phi^v_{U,\psi;s_\alpha w}. \quad (5.2)$$

For a reduced word of $w = s_n ... s_1$, define T'_w by

$$T'_w = T'_{s_n} \circ ... \circ T'_{s_1}.$$

Lemma 5.0.28 *The maps T'_w are independent of the reduced word of w used to define them.*

Proof The T'_w are essentially multiples of the maps T_{s_α} applied to V_K in the K-fixed case; see Theorem 3.4 of [Cas]. These maps on V_K are independent of the reduced word for w because the intertwining integrals are themselves independent of the reduced word. The same independence follows for the T'_w. The definitions (5.1), (5.2) then determine T'_w on all of $V_{U,\psi}(\chi^v)$.

Recall that Gindikin-Karpelevich formula for the K-fixed vector states that the intertwining map T_{s_α} takes the K-fixed vector $\phi_K^{s_\alpha v}$ to a multiple of ϕ_K^v; there is analogous result for the Whittaker vector.

Theorem 5.0.29 *For $\alpha \in D$,*

$$T_{s_\alpha}(\phi_{U,\psi}^{s_\alpha v}) = T'_{s_\alpha}(\phi_{U,\psi}^{s_\alpha v}) = c_{\mathcal{W}}(v^{-1}(\alpha))\phi_{U,\psi}^v.$$

Proof Recall that the Whittaker vector $\phi_{U,\psi}^v \in \mathrm{Ind}_B^G(\chi^v)$ is supported on the double coset $Bn_{w_\ell}U$ where it is evaluated as

$$\phi_{U,\psi}^v(bn_{w_\ell}u) = \delta_B^{1/2}\chi^v(b)\psi(u).$$

We may assume that $g = bn_w u$ for some $b \in B, w \in W$, and $u \in U$. After a change in x, we get

$$\lim_{m\to\infty} \delta_B^{1/2}\chi^v(b)\int_F \phi_{U,\psi}^{s_\alpha v}(n_\alpha u_\alpha(x)n_w u)\,dx$$

for some $c, d \in F$. Consider three cases for w.

(i). $w = w_\ell$. We have $\phi_{U,\psi}^v(bw_\ell u) = \delta_B^{1/2}\chi^v(b)\psi(u)$. In the argument of $\phi_{U,\psi}^{s_\alpha v}$, apply the relation

$$n_\alpha u_\alpha(x) = u_\alpha(x^{-1})\alpha^\vee(x^{-1})u_{-\alpha}(x^{-1})$$

and move the n_{w_ℓ} to the left over $u_{-\alpha}(x^{-1})$ by conjugation to obtain

$$\delta_B^{1/2}\chi^v(b)\int_F \phi_{U,\psi}^{s_\alpha v}(u_\alpha(x^{-1})\alpha^\vee(x^{-1})n_{w_\ell}u_{-w_\ell(\alpha)}(x^{-1})u).$$

Since $-w_\ell(\alpha)$ is a positive simple root, the integral reduces to

$$\delta_B^{1/2}\chi^v(b)\psi(u)\int_F \delta_B^{1/2}\chi^{s_\alpha v}(\alpha^\vee(x^{-1}))\psi(x^{-1})\,dx.$$

Letting $m\to\infty$, we get $c_{\mathcal{W}}(v^{-1}(\gamma))\delta_B^{1/2}\chi^v(b)\psi(u)$, verifying the theorem in this case.

(ii). $w=s_\alpha w_\ell$. Then $\phi^v_{U,\psi}(bn_wu)=0$, and

$$\begin{aligned}\int_{\mathfrak{p}^{-m}}\phi^{s_\alpha v}_{U,\psi}(n_\alpha u_\alpha(x)bn_\alpha^{-1}n_{w_\ell}u)\,dx &= \int_F \phi^{s_\alpha v}_{U,\psi}(n_{w_\ell}u_{-w_\ell(\alpha)}(d_{\alpha,w_\ell}x)u)\,dx\\ &= \delta_B^{1/2}\chi^v(b)\psi(u)\int_F\psi(d_{\alpha,w_\ell}x)\,dx\\ &= 0\end{aligned}$$

for sufficiently large m.

(iii). $w\neq w_\ell,\ s_\alpha w_\ell$. Then $\phi^{s_\alpha}_{U,\psi}(n_w)=0$. The integral in this case is also 0 because

$$n_\alpha u_{-\alpha}(x)n_w$$

is not in $Bn_{w_\ell}U$ for any value of x, and the final case is proved.

The second equality easily follows from the definition of T'_{s_α} and by pairing up w with $s_\alpha w$ for each $w\in W$.□

This formula will be used to construct the normalized intertwining maps $T_w^{\mathcal{W}}$. Ultimately we show that the set of vectors $T_w(\phi^{w^{-1}}_{U,\psi;1})$ is the desired basis of $V_{U,\psi}(\chi)$.

Definition 5.0.30 Define $T^{\mathcal{W}}_{s_\alpha}:\mathrm{Ind}_B^G(\chi^{s_\alpha v})\to\mathrm{Ind}_B^G(\chi^v)$ by

$$T^{\mathcal{W}}_{s_\alpha}=c_{\mathcal{W}}(v^{-1}(\gamma))^{-1}T'_{s_\alpha}.$$

For a reduced word of $s_n...s_1$ of $w \in W$, define $T_w^{\mathcal{W}} : \mathrm{Ind}_B^G(\chi^{w^{-1}v}) \to \mathrm{Ind}_B^G(\chi^v)$ by

$$T_w^{\mathcal{W}} = T_{s_n}^{\mathcal{W}} \circ ... \circ T_{s_1}^{\mathcal{W}}.$$

We check

$$\begin{aligned} T_{s_\alpha}^{\mathcal{W}}(\phi_{U,\psi}^{s_\alpha v}) &= \phi_{U,\psi}^{v} \\ T_{s_\alpha}^{\mathcal{W}} \circ T_{s_\alpha}^{\mathcal{W}}(\phi_{U,\psi;w}^{v}) &= \phi_{U,\psi;w}^{v} \end{aligned} \tag{5.3}$$

for all $w \in W$. Equation (5.3) implies

$$T_{w_1}^{\mathcal{W}} \circ T_{w_2}^{\mathcal{W}} = T_{w_1 w_2}^{\mathcal{W}}. \tag{5.4}$$

for all $w_1, w_2 \in W$. These facts will help us express $\phi_{U,\psi}$ in terms of the $T_w(\phi_{U,\psi;1}^{w^{-1}})$.

We emphasize that the purpose of the intertwining maps in this paper is to establish relations among $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w})$. The intertwining maps serve the same purpose in [Cas] and [HKP] in the K-fixed side. Casselman first describes the intertwining maps on $V_K(\chi)$ by first directly computing the intertwining integral for $T_{s_\alpha}(\phi_{K;1})$. He then obtains the result for arbitrary w by applying the action of the Iwahori-Hecke algebra (see his proof of Theorem 3.4 in [Cas]). At this point, we do not have recourse to such a tool for the (U,ψ) side. Therefore we prove the analogous result outright for general w. This is the content of the the next theorem.

Theorem 5.0.31 *Let $a = a_0\pi^\lambda$ where $a_0 \in T(\mathfrak{o})$ and $\lambda \in X^\vee$ is dominant. Let $\alpha \in D$ and $v \in W$ such that $\ell(s_\alpha v) = 1 + \ell(v)$ and let w be any element of W. Then*

$$c_{\mathcal{W}}(v^{-1}(\gamma))\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;w}^{s_\alpha v}) = \mathcal{W}\big(\rho(a) \cdot T'_{s_\alpha}(\phi_{U,\psi;w}^{s_\alpha v})\big)$$

Proof Consider the Whittaker functional applied to the intertwining integral $\mathcal{W} \circ T_{s_\alpha}\big(\rho(a) \cdot \phi^{s_\alpha v}_{U,\psi;w}\big)$:

$$\int_{U^-} \int_F \phi^{s_\alpha v}_{U,\psi;w}(n_\alpha u_\alpha(x) u)\psi^{-1}(u)\, dx\, du.$$

We will construct a parametrization of this domain of integration that is torus-monomial to evaluate the integral as a convergent limit. Then we will evaluate this integral in two different ways which will give the two sides of the equation in the theorem.

To construct the parametrization, we begin with the parametrization $u_\alpha(x)\overline{u}h$, where $\overline{u}h$ is the parametrization of U^- from chapter 4. We then perform changes of variable to this expression to create a parametrization that is torus-monomial.

First conjugating a to the left and then making a change in the u and x gives

$$\delta_B^{-1/2}(a)\chi^v(a) \int_{U^-} \int_F \phi^{s_\alpha v}_{U,\psi;w}(n_\alpha u_\alpha(x) u)\psi^{-1}(aua^{-1})\, dx\, du.$$

Recall

$$U^- = \bigcup_{y \in W} U^- \cap B n_y I_{\text{ht}}$$

and that the set $n_\alpha u_\alpha(x) B n_y I_{\text{ht}} \in B n_w I_{\text{ht}}$ only if $y = s_\alpha w$ or $y = w$, depending on x. We may therefore restrict the variable u to $B n_{s_\alpha w} I_{\text{ht}}$ and $B n_w I_{\text{ht}}$. We divide the proof according to whether $\ell(s_\alpha w) = 1 + \ell(w)$ or $\ell(s_\alpha w) = -1 + \ell(w)$.

$\boldsymbol{\ell(s_\alpha w) = 1 + \ell(w)}$:

$\boldsymbol{u \in B n_{s_\alpha w} I_{\text{ht}}}$: Consider

$$\delta_B^{-1/2}(a)\chi^v(a) \int_{U^- \cap B n_{s_\alpha w} I_{\text{ht}}} \int_F \phi^{s_\alpha v}_{U,\psi;w}(n_\alpha u_\alpha(x) u)\psi^{-1}(aua^{-1})\, dx\, du.$$

Since $\ell(s_\alpha w) = 1 + \ell(w)$, we have $(s_\alpha w)^{-1}(\alpha) \in R^-$. Write u as $\overline{u}h$ as in chapter 4, with $h \in H = (s_\alpha w)^{-1}(U^-) \cap U^-$ and give $\overline{u}h$ the torus-monomial parametrization. Express $\overline{u}h = bn_\alpha n_w i$. Move b to the left over $n_\alpha u\alpha(x)$ and make a change in x to reverse the effect of this move. As in the proof of Lemma 4.0.23, we get

$$\delta_B^{-1/2}(a)\chi^v(a)\int_{U^-\cap Bn_{s_\alpha w}I_{\text{ht}}} \delta_B^{1/2}\chi^v(b)\int_F \phi_{U,\psi;w}^{s_\alpha v}(n_\alpha u_\alpha(x)n_\alpha n_w h)\psi^{-1}(a\overline{u}a^{-1})\,dx\,du.$$

Re-arranging, we get in the argument of $\phi_{U,\psi;w}^{s_\alpha v}$

$$n_w u_{w^{-1}(-\alpha)}(x)h. \tag{5.5}$$

That $w^{-1}(-\alpha) \in R^-$ implies $x \in \mathfrak{p}^{\text{ht}(w^{-1}(-\alpha))}$. Now $u_{w^{-1}(-\alpha)}(x)h$ ranges over $w^{-1}(U^-)\cap U^- \cap I_{\text{ht}}$. Make the change in x so that (5.5) so that has the torus-monomial parametrization. Integrating with respect to x and h gives

$$\mathcal{W}_H(\chi^{s_\alpha v}, w)$$

and with respect to $\overline{u}$ gives

$$\mathcal{W}_{U^-/H}(\chi^v, s_\alpha w, a).$$

Now (4.3) implies

$$\frac{\mathcal{W}_H(\chi^{s_\alpha v}, w)}{\mathcal{W}_H(\chi^v, s_\alpha w)} = -\frac{\pi^{v^{-1}(\lambda_1)}}{q}q^{M_1}$$

where

$$\lambda_1 = -\sum_{\gamma\in R^+_{w^{-1}w_\ell}} s_\alpha w(\gamma^\vee) + \sum_{\gamma\in R^+_{w^{-1}s_\alpha w_\ell}} s_\alpha w(\gamma^\vee)$$

and

$$M_1 = \sum_{\gamma \in R^+_{w^{-1}w_\ell}} (\operatorname{ht} w(\gamma^\vee) - \operatorname{ht}\gamma) - \sum_{\gamma \in R^+_{w^{-1}s_\alpha w_\ell}} (\operatorname{ht} s_\alpha w(\gamma^\vee) - \operatorname{ht}\gamma).$$

We simplify λ_1 and M_1. The fact $R^+_{w^{-1}s_\alpha w_\ell} = R^+_{w^{-1}w_\ell} \backslash w^{-1}(\alpha)$ shows that $\lambda_1 = \alpha^\vee$. We claim $M_1 = 0$. First we collect the terms involving roots and immediately see they reduce to $-\operatorname{ht} w^{-1}(\alpha)$. Re-expressing the sum of coroots using $w(R^+_{w^{-1}w_\ell}) = R^+_{ww_\ell}$ gives

$$-\operatorname{ht}\alpha^\vee + \sum_{\gamma \in R^+_{ww_\ell}} \operatorname{ht}\gamma^\vee - \operatorname{ht} s_\alpha(\gamma^\vee).$$

This expression reduces to

$$-1 + \sum_{\gamma \in R^+_{ww_\ell}} \langle \alpha, \gamma^\vee \rangle$$

which is equal to $\operatorname{ht} w^{-1}(\alpha)$ by Lemma 2.0.13. Thus $M_1 = 0$.

Putting together these results with Lemma 4.0.23, we see the contribution from integrating over these regions is

$$-\frac{\pi^{v^{-1}(\alpha^\vee)}}{q} \mathcal{W}(\rho(a) \cdot \phi^v_{U,\psi;s_\alpha w}).$$

$\boldsymbol{u \in Bn_w I_{\mathrm{ht}}}$:

We now express u as $\overline{u}h$, with $h \in H = w^{-1}(U^-) \cap U^-$. By reasoning similar to that of the previous case, we consider

$$\delta_B^{-1/2}(a)\chi^v(a) \int_{U^- \cap Bn_w I_{\mathrm{ht}}} \delta_B^{1/2} \chi^v(b) \int_F \phi^{s_\alpha v}_{U,\psi;w}(n_\alpha u_\alpha(x) n_w h) \psi^{-1}(a\overline{u}a^{-1})\, dx\, du.$$

Apply the relation

$$u_{-\alpha}(x) n_\alpha = u_\alpha(x^{-1}) \alpha^\vee(x^{-1}) u_{-\alpha}(x^{-1})$$

and re-arrange to obtain

$$\delta_B^{-1/2}(a)\chi^v(a)\int_{U^-\cap Bn_wI_{\mathrm{ht}}}\delta_B^{1/2}\chi^v(b)\int_F\delta_B^{1/2}\chi^{s_\alpha v}(\alpha^\vee(x^{-1}))\phi_{U,\psi;w}(n_wu_{-w^{-1}(\alpha)}(x^{-1})h).$$

That $-w^{-1}(\alpha)\in R^-$ implies $x\in\mathfrak{p}^{\mathrm{ht}(w^{-1}(\alpha))}$. Make a change in h

$$h\mapsto u_{-w^{-1}(\alpha)}h.$$

Integrating with respect to x and h gives

$$(1-q^{-1})v^{-1}\left(\frac{\pi^{-\alpha^\vee\mathrm{ht}(w^{-1}(\alpha))}}{1-\pi^{-\alpha^\vee}}\right)\mathcal{W}_H(\chi^{s_\alpha v},w)$$

while with respect to $\overline{u}$ gives

$$\mathcal{W}_{U^-/H}(\chi^v,w,a).$$

Now

$$\frac{\mathcal{W}_H(\chi^{s_\alpha v},w))}{\mathcal{W}_H(\chi^v,w)}=\pi^{v^{-1}(\lambda_2)}$$

where

$$\lambda_2=\sum_{\gamma\in R^+_{w^{-1}w_\ell}}-s_\alpha(w(\gamma^\vee))+w(\gamma^\vee).$$

Re-express this sum as before to get

$$\sum_{\gamma\in R^+_{ww_\ell}}\langle\alpha,\gamma^\vee\rangle\alpha^\vee$$

which is $(1+\operatorname{ht} w^{-1}(\alpha))\alpha^\vee$ by Lemma 2.0.13. Putting together these results yields

$$c_{\mathcal{W}}(v^{-1}(\gamma)) + \pi^{v^{-1}(\alpha^\vee)}\mathcal{W}(\rho(a)\cdot\phi^v_{U,\psi;w}).$$

The two cases together show the total contribution is

$$(c_\alpha(\chi^v) + \pi^{v^{-1}(\alpha^\vee)})\mathcal{W}(\rho(a)\cdot\phi^v_{U,\psi;w}) - q^{-1}\pi^{v^{-1}(\alpha^\vee)}\mathcal{W}(\rho(a)\cdot\phi^v_{U,\psi;w})$$

which is $\mathcal{W}\big(\rho(a)\cdot T'_{s_\alpha}(\phi^{s_\alpha v}_{U,\psi;w})\big)$.

$\boldsymbol{\ell(s_\alpha w) = -1 + \ell(w)}$:

$\boldsymbol{u \in B n_{s_\alpha w} I_{\mathrm{ht}}}$:

By the reasoning above, we consider

$$\delta_B^{-1/2}\chi(a)\int_{U^-\cap B n_{s_\alpha w} I_{\mathrm{ht}}} \delta_B^{1/2}\chi(b)\int_F \phi^{s_\alpha v}_{U,\psi;w}(n_\alpha u_\alpha(x) n_{s_\alpha w} h)\psi(a\overline{u}a^{-1})\,dx\,du. \tag{5.6}$$

That $-w^{-1}(\alpha)\in R^+$ implies $x \in \mathfrak{p}^{1-\mathrm{ht}(w^{-1}(\alpha))}$. The parametrization of h is of the form

$$h = u_{w^{-1}(\alpha)}(t)h' \tag{5.7}$$

with $t\in\mathfrak{p}^{\mathrm{ht} w^{-1}(\alpha)}$ and h' a parametrization of $w^{-1}(U^-)\cap U^-$. Apply to

$$n_\alpha u_\alpha(x) u_{-\alpha}(t) n_{s_\alpha w} h' \tag{5.8}$$

the relation

$$u_\alpha(x)u_{-\alpha}(t) = u_{-\alpha}(\frac{t}{1+xt})\alpha^\vee\big((1+xt)^{-1}\big)u_\alpha(\frac{x}{1+xt})$$

and the fact that $1 + xt_{w^{-1}(\alpha)} \in \mathfrak{o}^\times$ allows us to write the argument as

$$u_{-\alpha}(\frac{x}{1+xt})n_w h'.$$

using the left U-invariance of $\phi^{s_\alpha v}_{U,\psi;w}$. The integrand is now independent of t which, in the limit $\mathscr{I} \to \infty$, ranges over $\mathfrak{p}^{\mathrm{ht}(w^{-1}(\alpha))}$. Applying the explicit Bruhat decomposition to h' yields

$$u_{-\alpha}(\frac{x}{1+xt})a'_0 \pi^\lambda n_{w_\ell} u'$$

for some $\pi^\lambda \in X^\vee$ and $a'_0 \in T(\mathfrak{o})$. From chapter 7, we may assume that each parameter t_γ of h' has valuation $\mathrm{ht}(\gamma)$. This, along with $x \in \mathfrak{p}^{1-\mathrm{ht}(w^{-1}(\alpha))}$, implies that

$$\pi^{-\lambda} u_{-\alpha}(\frac{x}{1+xt})\pi^\lambda \in U^-(\mathfrak{o}).$$

This implies, after one more re-arranging, that $\phi^{s_\alpha v}_{U,\psi;w}$ is then independent of x as well. Thus the integral is absolutely convergent. Integrating over x, t, h' and $\overline{u}$ and letting $m, \mathscr{I} \to \infty$ yields, respectively,

$$m(\mathfrak{p}^{1-\mathrm{ht}w^{-1}(\alpha)})m(\mathfrak{p}^{\mathrm{ht}w^{-1}(\alpha)})\mathcal{W}_H(\chi^{s_\alpha v}, w))\mathcal{W}_{U^-/H}(\chi^v, s_\alpha w, a).$$

Combining

$$m(\mathfrak{p}^{1-\mathrm{ht}w^{-1}(\alpha)})m(\mathfrak{p}^{\mathrm{ht}w^{-1}(\alpha)}) = q^{-1}$$

and the previous calculation

$$\frac{\mathcal{W}_H(\chi^{s_\alpha v}, w))}{\mathcal{W}_H(\chi^v, s_\alpha w)} = -q\pi^{v^{-1}(\alpha^\vee)}$$

gives us the contribution

$$-\pi^{v^{-1}(\alpha^\vee)}\mathcal{W}(\rho(a)\cdot\phi^{v}_{U,\psi;w}).$$

$\boldsymbol{u \in Bn_w I_{\mathrm{ht}}}$:

By the reasoning above, we consider

$$\delta_B^{-1/2}\chi(a)\int_{U^-\cap Bn_w I_{\mathrm{ht}}}\delta_B^{1/2}\chi(b)\int_F \phi^{s_\alpha v}_{U,\psi;w}(n_\alpha u_\alpha(x)n_w h)\psi(a\overline{u}a^{-1})\,dx\,du. \tag{5.9}$$

The relation

$$n_\alpha u_\alpha(x) = u_\alpha(x^{-1})\alpha^\vee(x^{-1})u_{-\alpha}(x^{-1})$$

implies $x^{-1}\in\mathfrak{p}^{1-\mathrm{ht}(w^{-1}(\alpha))}$, as $-w^{-1}(\alpha)\in R^+$, yielding

$$\delta_B^{-1/2}(a)\chi^v(a)\int_{U^-\cap Bn_{s_\alpha w}I_{\mathrm{ht}}}\delta_B^{1/2}\chi^v(b)\int_F \delta_B^{1/2}\chi^{s_\alpha v}(\alpha^\vee(x^{-1}))\phi_{U,\psi;w}(u_{-\alpha}(x^{-1})n_w h).$$

The same argument of the previous case shows that $\phi^{s_\alpha v}_{U,\psi;w}$ is independent of x. Integrating as before gives

$$(1-q^{-1})v^{-1}\left(\frac{\pi^{-(1-\mathrm{ht}(w^{-1}(\alpha))\alpha^\vee)}}{1-\pi^{-\alpha^\vee}}\right)\mathcal{W}_H(\chi^{s_\alpha v},w)\mathcal{W}_{U^-/H}(\chi^v,w,a),$$

as in the case $\ell(s_\alpha w)=1+\ell(w), u\in Bn_w I_{\mathrm{ht}}$. This expression is then

$$\left(c_{\mathcal{W}}(v^{-1}(\gamma))+q^{-1}\pi^{v^{-1}(\alpha^\vee)}\right)\mathcal{W}(\rho(a)\cdot\phi^{v}_{U,\psi;s_\alpha w}).$$

The two cases together show

$$\mathcal{W}(\rho(a)T_{s_\alpha}(\phi_{U,\psi;s_\alpha w})) = -\pi^{\alpha^\vee}\mathcal{W}(\rho(a)\phi_{U,\psi;w}) + (c'_\alpha + \frac{\pi^{\alpha^\vee}}{q})\mathcal{W}(\rho(a)\phi_{U,\psi;s_\alpha w})$$

which is $\mathcal{W}\big(\rho(a)\cdot T'_{s_\alpha}(\phi^{s_\alpha v}_{U,\psi;w})\big)$.

We now perform the same integration in a different order to obtain the right side of the equation in the theorem statement.

For $\phi \in \mathrm{Ind}_B^G(\chi^v)$, write $\mathcal{W}(T_{s_\alpha}(\phi))$ as the double integral

$$\int_{U^-}\int_F \phi(u_{-\alpha}(x)n_\alpha u)\psi^{-1}(u)dx\,du.$$

The relation

$$u_{-\alpha}(x)n_\alpha = u_\alpha(x^{-1})\alpha^\vee(x^{-1})u_{-\alpha}(x^{-1})$$

and the change of variable

$$u \mapsto u_{-\alpha}(-x^{-1})u$$

yields

$$\left(\int_F \delta_B^{1/2}\chi^v(\alpha^\vee(x^{-1}))\psi^{-1}(u_{-\alpha}(x^{-1})\,dx\right)\left(\int_{U^-}\phi(u)\psi^{-1}(u)\,du\right).$$

The integral with respect to x equal to $c_{\mathcal{W}}(v^{-1}(\alpha))$ and with respect to u is $\mathcal{W}(\phi)$.

□

Corollary 5.0.32 *For $w_1, w_2 \in W$ and a as in the lemma,*

$$\mathcal{W}(\rho(a)\cdot T^{\mathcal{W}}_{w_1}(\phi^{w_1^{-1}}_{U,\psi;w_2})) = \mathcal{W}(\rho(a)\cdot\phi^{w_1^{-1}}_{U,\psi;w_2})).$$

Proof This follows from straightforward induction.

Chapter 6

Evaluation of the Bessel Function

This chapter evaluates the Bessel function $W_{U,\psi}(a)$ at torus elements a of dominant coweight. Lemma 6.0.33 expresses the Whittaker in the basis $T'_w(\phi^{w^{-1}}_{U,\psi;1}), w \in W$. We use Corollary 7.1.2 from chapter 7. The arguments in this section closely follow those in [HKP].

It follows from the definition of T'_w that $T'_w(\phi^{w^{-1}}_{U,\psi;1})$ is a linear combination

$$T'_w(\phi^{w^{-1}}_{U,\psi;1}) = \sum_{v \leq w} a_{vw} \phi_{U,\psi;v}$$

such that the coefficient a_{ww} is nonzero. In particular,

$$a_{ww} = \prod_{\gamma \in w(R^-) \cap R^+} -c_{\mathcal{W}}(\gamma)^{-1} q^{-1} \pi^{\gamma^\vee}. \tag{6.1}$$

Therefore, there is some triangular, invertible matrix which expresses the change between the basis the $\phi_{U,\psi;w}$ and the $T'_w(\phi^{w^{-1}}_{U,\psi;1})$.

Let's express $\phi_{U,\psi}$ in this basis.

Lemma 6.0.33

$$\phi_{U,\psi} = \sum_{w \in W} w(\prod_{\gamma \in R^+} \frac{1 - q\pi^{\gamma^\vee}}{1 - \pi^{\gamma^\vee}}) T'_w(\phi^{w^{-1}}_{U,\psi;1}).$$

Proof Set

$$\phi_{U,\psi} = \sum_{w \in W} d_w T'_w(\phi^{w^{-1}}_{U,\psi;1}) \tag{6.2}$$

for some constants d_w. Recall

$$\phi_{U,\psi} = \sum_{w \in W} \phi_{U,\psi;w}.$$

Equating the coefficients of $\phi_{U,\psi;w_\ell}$ in both expressions yields

$$d_{w_\ell} a_{w_\ell w_\ell} = 1.$$

Therefore

$$\begin{aligned} d_{w_\ell} &= \prod_{\gamma \in R^+} -c_{\mathcal{W}}(\gamma) q\pi^{-\gamma^\vee} \\ &= \prod_{\gamma \in R^+} \frac{1 - q\pi^{-\gamma^\vee}}{1 - \pi^{-\gamma^\vee}}. \end{aligned}$$

Now the d_w satisfy

$$d_{w_1 w_2} = w_1(d_{w_2}).$$

To see this, apply T'_{w_1} to (6.2) and then apply the Weyl group action w_1 to recover vectors in $V_{U,\psi}(\chi)$. From the expression for d_{w_ℓ} it follows

$$d_w = w(\prod_{\gamma \in R^+} \frac{1 - q\pi^{\gamma^\vee}}{1 - \pi^{\gamma^\vee}}).$$

by taking $w_2 = w, w_1 = w_\ell w^{-1}$. □

Theorem 6.0.34 *Let $a = a_0\pi^\mu$ with $a \in T/T(\mathfrak{o})$ and $\mu \in X^\vee$ be dominant. Then*

$$\mathcal{W}(\rho(a)\cdot\phi_{U,\psi}) = \delta_B^{1/2}(\pi^{-\mu})\sum_{w\in W} w(\prod_{\gamma\in R^+}\frac{1-q^{-1}\pi^{-\gamma^\vee}}{1-\pi^{\gamma^\vee}})\pi^{w(\mu)}$$

Proof Apply $\rho(a)$ to the expression for $\phi_{U,\psi}$ in Lemma 6.0.33. The fact that $\mathcal{W}\circ T'_w = \mathcal{W}$ implies

$$\mathcal{W}(\rho(a)\cdot\phi_{U,\psi}) = \sum_{w\in W} w(\prod_{\gamma\in R^+}\frac{1-q\pi^{\gamma^\vee}}{1-\pi^{\gamma^\vee}})\mathcal{W}(\rho(a)\cdot\phi_{U,\psi;1}^{w^{-1}}).$$

By Theorem 7.1.1

$$\mathcal{W}(\rho(a)\cdot\phi_{U,\psi;1}^{w^{-1}}) = \delta_B^{1/2}(\pi^{-\mu})w(\pi^\mu\prod_{\gamma\in R^+}-q^{-1}\pi^{-\gamma^\vee}).$$

These results together provide the formula. □

Chapter 7

Computing $\mathcal{W}_H(\chi, w)$

7.1 Statement of Theorem

Recall that $\mathcal{W}_H(\chi, w)$ is defined by

$$\mathcal{W}_H(\chi, w) = \int_{w^{-1}(U^-)\cap U^-} \phi_{U,\psi;w}(n_w u)\, du. \tag{7.1}$$

This chapter is devoted to proving the following theorem, which was used in chapter 5, and its corollary, which was used in chapter 6.

Theorem 7.1.1

$$\mathcal{W}_H(\chi, w) = \prod_{\gamma \in w(R^+)\cap R^+} -\pi^{-\gamma^\vee} q^{\mathrm{ht}(w(\gamma^\vee))-\mathrm{ht}(\gamma)-1}.$$

Corollary 7.1.2

$$\mathcal{W}(\phi_{U,\psi;1}) = \prod_{\gamma \in R^+} -q^{-1}\pi^{-\gamma^\vee}$$

Proof Apply the theorem with $w = 1$ and use the support of $\phi_{U,\psi;1}$ plus the fact

that

$$\sum_{\gamma\in R^+} \mathrm{ht}(\gamma^\vee) = \sum_{\gamma\in R^+} \mathrm{ht}(\gamma)$$

for any root system R.

Our approach to proving Theorem 7.1.1 consists of the following steps. We first parametrize $w^{-1}(U^-)\cap U^-$ using unipotent root elements ordered by a convex ordering of the roots in $w^{-1}(R^-)\cap R^-$. After an "explicit Bruhat decomposition" in section 7.2, we express elements of $n_w\big(w^{-1}(U^-)\cap U^-\big)$ as the product $u'_+ a n_{w_\ell} u_+$, with $u_+, u'_+ \in U$ and $a \in T$. We make this decomposition because the definition of the Whittaker vector requires its argument to be in this factored form. This Bruhat decomposition then determines a sequence of changes of variable that simplify the torus element a. Under this change, $\chi(a)$ depends on only the valuations of the integration variables. These changes determine the torus-monomial parametrization. We thus partition the domain of integration into cells for which each integration variable has a fixed valuation (each cell is of the form $f(\overline{n})$ in the notation of chapter 3). Integration over such a cell then becomes an integral over a product of unit groups $\mathfrak{o}^\times$. We show that this integral is zero for every cell except one.

To perform the integration over each cell, we introduce in section 7.4 coordinates in the unit variables which we call "nesting coordinates" that make the integrand easier to handle. In particular, they isolate sub-integrals of the following forms:

$$\int_{\mathfrak{o}^\times} \psi(u\pi^{-n})du = 0,\ n > 1 \tag{7.2}$$

$$\int_{\mathfrak{o}} \psi(z\pi^{-1})\, dz = 0 \tag{7.3}$$

These are are two basic ways to prove an integral of ψ vanishes. Section 7.6 identifies key terms in the integrand with combinatorial objects in the root system R. After analyzing these objects, we can finally carry out the integration in section 7.5.

7.2 Explicit Bruhat Decomposition

Let u be an element of U^-. Let N be the the number of roots in R^-. After a sequence of N steps, we express u as $u'_N a_N n_{w_\ell} u_N$ with $u_N, u'_N \in U$ and $a_N \in T$, provided that u is sufficiently generic. Any element of U^- can be written as

$$u = \prod_{\gamma \in R^-} u_\gamma(t_\gamma) \tag{7.4}$$

for some $t_\gamma \in F$. In general, the roots γ may be taken in any order. For our purposes, we always choose convex orderings; the convex ordering ensures that, among other things, the group operations we apply are well-defined and will eventually terminate.

Recall that a convex ordering of the roots in R^- is equivalent to a reduced decomposition w_ℓ by the following correspondence. Let N be the number of roots in R^- and let $\{\alpha_N, ..., \alpha_2, \alpha_1\}$ be a sequence of (possibly non-distinct) simple roots in D^+ that corresponds to a reduced decomposition of w_ℓ:

$$w_\ell = s_{\alpha_N} ... s_{\alpha_2} s_{\alpha_1}.$$

Define w_j by

$$w_j = s_{\alpha_j} ... s_{\alpha_1}$$

with $w_0 = 1$. For $i \geq 1$, define γ_i by

$$\gamma_i = -w_{i-1}^{-1}(\alpha_i).$$

Accordingly, we say that γ_i *maps to the simple root* α_i. The ordering determined by $\gamma_{i+1} \prec \gamma_i$ is a convex ordering of R^-. We will achieve the Bruhat decomposition $u^- = u'_N a_N n_{w^{-1}w_\ell} u_N$ via a sequence of steps. Our initial expression for u is (7.4). After the jth step the expression for u appears as

$$u = u'_j a_j n_{w_j^{-1}} \left(\prod_{\gamma \prec \gamma_j} u_{w_j(\gamma)}(f_{\gamma,j}) \right) u_j \tag{7.5}$$

where

$$u_j = \prod_{i=j}^{1} u_{\alpha_i}(f^{-1}_{\gamma_{i+1},i}),$$

$$a_j = \lambda_j(-1) \prod_{i=1}^{j} \gamma_i^\vee (f^{-1}_{\gamma_{i+1},i}),$$

u'_j is some element of U, λ_j is some element of $X^\vee$, and $f_{\gamma_i,j} \in F(t_{\gamma_1}, ..., t_{\gamma_i})$ are rational expressions. The expression (7.5) reduces to (7.4) when $j = 0$, and attains the final form when $j = N$.

We show what each step of the Bruhat decomposition entails. Assume that we have expression (7.5) after the jth step. As $-w_j(\gamma_{j+1})$ is the simple root α_j, apply relation (1.7) to $u_{-\alpha_{j+1}}$ in the product over roots $\gamma \prec \gamma_j$ to obtain

$$\left(\prod_{\gamma \prec \gamma_{j+1}} u_{w_j(\gamma)}(f_{\gamma,j}) \right) u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j}) \overline{\alpha^\vee_{j+1}}(f_{j+1}) n_{\alpha_{j+1}} u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j}) \tag{7.6}$$

The rightmost factor $u_{\alpha_{j+1}}(f^{-1}_{j+1})$ in (7.6) will remain in this position and contribute to u_{j+1}. The other unipotent factor $u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j})$, however, will contribute to u'_{j+1}. Therefore we have to make sure that we can "move" this unipotent to the left; Corollary (7.2.2) provides this result. We postpone the proof and assume for now

that

$$\left(\prod_{\gamma\prec\gamma_{j+1}} u_{w_j(\gamma)}(f_{\gamma,j})\right) u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j})\overline{\alpha^\vee_{j+1}}(f_{\gamma_{j+1},j})n_{\alpha_{j+1}}$$

is equal to

$$u'\overline{\alpha^\vee_{j+1}}(f_{j+1})n_{\alpha_{j+1}}\left(\prod_{\gamma\prec\gamma_{j+1}} u_{w_{j+1}(\gamma)}(f_{\gamma,j+1})\right) \tag{7.7}$$

for some rational functions $f_{\gamma,j+1}$ and $u' \in w_j(U)$. Therefore this u' may join the unipotent element u'_j and we set u'_{j+1} to be

$$u'_{j+1} = u'_j a_j n_{w_j} u' n^{-1}_{w_j} a^{-1}_j.$$

in addition to setting

$$a_{j+1} = a_j n_{w_j} \alpha^\vee_{j+1}(f_{\gamma_{j+1},j}) n^{-1}_{w_j}$$

and

$$u_{j+1} = u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j})u_j.$$

This completes the $j+1$-th of the Bruhat decomposition.

The following technical lemma about moving unipotents in general is needed to prove the result assumed above.

Lemma 7.2.1 *Let $\gamma_N \prec ... \prec \gamma_1$ be a convex ordering of $S \subset R$. Suppose we have a product consisting of u_γ in this order with other unipotent elements $u_\beta = u_{\beta_{i,m}}$ interspersed between them:*

$$u_{\gamma_N}u_{\gamma_{M-1}}...u_{\gamma_{n+1}}u_{\beta_{n,1}}u_{\beta_{n,2}}...u_{\gamma_n}u_{\beta_{n-1,1}}u_{\beta_{n-1,2}}...u_{\beta_{1,1}}u_{\beta_{1,2}}...u_{\gamma_1} \tag{7.8}$$

The $\beta_{i,m}$ may be any roots in R subject to the following conditions:

$$(i) \text{ If } \beta_{i,m} \in S, \text{ then } \beta_{i,m} = \gamma_j \text{ for some } \gamma_j \prec \gamma_i.$$

$$(ii) \text{ If } \beta_{i,m} \in S^-, \text{ then } \beta_{i,m} = -\gamma_j \text{ for some } \gamma_j \succeq \gamma_i.$$

Then we may re-write the above product as

$$u' \prod_{i=N}^{1} u_\gamma \tag{7.9}$$

for some $u' \in U(S^-)$ and with parameters of the u_γ possibly altered.

Proof We obtain the expression (7.9) by sequentially "moving" the u_β "to the left", "passing over" the other elements u_γ in the product. We show each move must end in one of two ways: either we reach the far left end of the product, where the u_β contributes to the u' factor; or we reach a u_{γ_j} with $\gamma_j = \beta$, in which case the u_β is absorbed into u_{γ_j} by the relation

$$u_{\gamma_j}(x)u_\beta(y) = u_{\gamma_j}(x+y), \tag{7.10}$$

for whatever parameters x, y that these unipotent elements have. It is in this manner that the parameters of the u_γ may be altered, as referred to in the lemma. The element we choose to move will always be the "leftmost" of the u_β, in particular $u_{\beta n,1}$ in (7.8). Of course, during this move we may create "new" unipotents by the commutator relations, which we label as $u_{\beta_{i,m}}$ appropriately. We will show that these new unipotents also satisfy conditions (i) and (ii). Thus, we show that conditions (i) and (ii) on products in the same form as (7.8) are invariant under moving the leftmost u_β to the left.

First, suppose $\beta_{n,1} \in S$. Then by condition (i), $\beta_{n,1} = \gamma_j \prec \gamma_n$. We move $u_{\beta_{n,1}}$ to the left and absorb it into u_{γ_j} as in (7.10). Now, if we have γ_i between γ_j and γ_n such that γ_j and γ_i span an irreducible rank 2 root system within R^-, then $u_{\beta_{n,1}}$ creates new unipotents as it moves to the left over u_{γ_i}, with roots in this rank 2 system, appearing in a block immediately adjacent to u_{γ_i} on its left. The convex property of the ordering implies that each of these roots must be equal to some γ_k with $\gamma_j \prec \gamma_k \prec \gamma_i$. Thus new unipotents of this move satisfy condition (i) (in this case, any new unipotent must lie in S so condition (ii) does not apply).

On the other hand, suppose $\beta_{n,1} \in S^-$, so $-\beta_{n,1} = \gamma_j \succ \gamma_n$. Then we move $u_{\beta_{n,1}}$ to the left end of the product. Condition (ii) ensures that this move does not create torus elements by relation (1.6). Now, if we have $\gamma_i \prec \gamma_n$ with $-\gamma_j$ and γ_i spanning an irreducible rank 2 root system of R, then $u_{\beta_{n,1}}$ creates new unipotents as it moves to the left over u_{γ_i}. To see that the roots in this rank 2 system satisfy conditions (i) and (ii), consider the sequence $\{-\gamma_j, ..., -\gamma_2, -\gamma_1, \gamma_N, \gamma_{N-1}, ..., \gamma_{j+1}\}$ which is a convexly-ordered positive system. This proves the invariance of (i) and (ii) under moving $u_{\beta_{n,1}}$ to the left.

Now, we claim that iteratively moving the leftmost u_β to the left will eventually yield a product in which no u_β appear, that is, a product of the form (7.9). This is proved by induction on the position of the leftmost unipotent $u_{\beta_{n,1}}$. with $n = N$ as the base case. □

Corollary 7.2.2 *In the previous notation for the explicit Bruhat decomposition, the expression*

$$\left(\prod_{\gamma \prec \gamma_{j+1}} u_\gamma(f_{\gamma,j})\right) u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j}) \tag{7.11}$$

may be re-written as

$$u'\left(\prod_{\gamma\prec\gamma_{j+1}} u_\gamma(f_{\gamma,j+1})\right) \tag{7.12}$$

for some $u' \in w_j(U)$ *and* $f_{\gamma_i,j+1} \in F(t_{\gamma_i}, \ldots, t_{\gamma_1})$.

Proof Apply the lemma with $S = w_j(R^-)$. □

The explicit Bruhat decomposition shows how to express $u \in U^-$ as $u'_N a_N w_\ell u_N$. Recall, however, the argument of the Whittaker vector in (7.1) is $n_w u$ for $u \in U^- \cap w^{-1}(U^-)$. We can view $n_w u$ as a j-th step in an explicit Bruhat decomposition, with $j = \ell(w)$ and $f_{\gamma,j} = t_\gamma$. Thus we can use the same procedure to express $n_w u$ in the form $u'_N a_N n_{w_\ell} u_N$. We therefore assume $w = 1$ from now on, as the general case translates easily from this one.

Using

$$u_N = \prod_{j=N-1}^{0} u_{\alpha_{j+1}}(f^{-1}_{\gamma_{j+1},j}), \tag{7.13}$$

we write that integral for $\mathcal{W}(\phi_{U,\psi;1})$ as

$$\int_{U^-} \phi_{U,\psi;1}(u)\, du = \int_{U^-} \chi(a_N)\left(\prod_{0\le j<N} \psi(f^{-1}_{\gamma_{j+1},j})\right) 1_{BI_{\mathrm{ht}}}(u)\, du. \tag{7.14}$$

We now describe changes of variable in the t_γ that render the rational expressions $f_{\gamma_{j+1},j}$ into monomials.

Changes of Variable From Moving u_γ. Given $\gamma \in R^-$ and a convex order $\prec$ on R^- , consider the product

$$\left(\prod_{\gamma'\prec\gamma} u_{\gamma'}(t_{\gamma'})\right) u_{-\gamma}(t_\gamma^{-1}).$$

Move the unipotent $u_{-\gamma}$ and then all new unipotents u_β to the left as described in Lemma 7.2.1. The u_β may get absorbed by a u_{γ_i} via

$$u_{\gamma_i}(t_{\gamma_i})u_\beta(x) = u_{\gamma_i}(t_{\gamma_i} + x),$$

where $\beta = \gamma_i$ and x is some polynomial in the parameters. In this case, we make the change of variable

$$t_{\gamma_i} \mapsto t_{\gamma_i} - x.$$

The order of these changes is determined by when they occur in moving the new unipotents to the left by always moving the leftmost u_β. We call the set of all changes of variable that occur in this manner the **changes of variable from moving $\boldsymbol{u_\gamma}$**. We denote the application of these changes by the operator $\boldsymbol{B_\gamma}$. We also let $\boldsymbol{B_{\succeq\gamma}}$ and $\boldsymbol{B_{\preceq\gamma}}$ denote the changes of variable from moving $u_{\gamma'}$ for $\gamma' \succeq \gamma$ and $\gamma' \preceq \gamma$, respectively. The order of these changes is determined by the convex ordering, starting from the most succeeding root. The B signifies that the changes are derived from the explicit Bruhat decomposition.

The $B_{\preceq\gamma_1}$ simplify the elements a_N and u_N so that there parameters are monomials in t_γ. The next lemma describes these monomials.

Lemma 7.2.3 *In the notation of the explicit Bruhat decomposition, let $m_{\gamma,j}$ denote $B_{\succ\gamma_{j+1}}(f_{\gamma,j})$ and $m_{\gamma_{j+1}}$ to denote $m_{\gamma_{j+1},j}$. Then*

$$m_{\gamma,j} = t_\gamma \prod_{\beta \succ \gamma_{j+1}} m_\beta^{\langle \gamma, \beta^\vee \rangle}.$$

In addition, $B_{\preceq\gamma_1}(f_{\gamma_{j+1},j}) = B_{\succ\gamma_{j+1}}(f_{\gamma_{j+1},j})$.

Proof Follows from the explicit Bruhat decomposition.

On the other hand, the changes $B_{\preceq\gamma_1}$ complicate the support condition $1_{BI_{\text{ht}}}(u)$ in the integrand of (7.14). Namely, in the original coordinates (7.4) of u, the statement $u \in BI_{\text{ht}}$ is equivalent to

$$t_\gamma \in \mathfrak{p}^{\text{ht}\gamma} \tag{7.15}$$

for all $\gamma \in R^-$. After the changes of variable from moving u_γ, the conditions (7.15) appear as

$$B_{\preceq\gamma_1}(t_\gamma) \in \mathfrak{p}^{\text{ht}\gamma}.$$

We now partition the domain of integration F^N into cells on which $B_{\preceq\gamma_1}(\chi(a_N))$ is constant. Recall F^N is parametrized by

$$\bar{t} = (t_{\gamma_N}, ..., t_{\gamma_1}), \quad t_\gamma \in F.$$

An element $t \in F$ is of the form

$$t = \upsilon\pi^n$$

where $u \in \mathfrak{o}^\times$ and $n \in \mathbb{Z}$ is $v(t)$, the valuation of t.

Definition 7.2.4 Let $\overline{n}$ denote an N-tuple of integers

$$\overline{n} = (n_N, ..., n_1).$$

Call the subset of F^N such that $v(t_{\gamma_i}) = n_i$ for each i the **cell** $C(\overline{n})$. On $C(\overline{n})$, $t_{\gamma_i} = \upsilon_{\gamma_i}\pi^{n_i}$ with n_i fixed and υ_{γ_i} free to range over $\mathfrak{o}^\times$.

Since χ is unramified, $\chi(B_{\preceq\gamma_1}(a_N))$ depends only on $\overline{n}$: let $\chi(\overline{n})$ denote the value of $\chi(B_{\preceq\gamma_1}(a_N))$ on the cell $C(\overline{n})$. Clearly F^N is the disjoint union of $C(\overline{n})$ as $\overline{n}$ ranges over all N-tuples. Therefore we consider the integral $\mathcal{I}(\overline{n})$ over $C(\overline{n})$:

$$\mathcal{I}(\overline{n}) = \chi(\overline{n}) \int_{C(\overline{n})} \left(\prod_{\gamma<0} \psi(m_\gamma^{-1}) \right) \left(\prod_{\gamma<0} 1_{\mathfrak{p}^{\mathrm{ht}\gamma}}(B_{\preceq\gamma_1}(t_\gamma)) \right) \prod_{\gamma<0} dt_\gamma. \tag{7.16}$$

We will show that this integral is 0 for every $\overline{n}$ except one, namely $\overline{n} = (\mathrm{ht}(\gamma_N), ..., \mathrm{ht}(\gamma_1))$.

7.3 Paths in a Root System

We first examine the structure of these functions by defining combinatorial objects which we call "paths in a root system". This definition is derived from moving the unipotent element $u_{-\beta}$ to the left and thus will be used to describe the functions $B_\beta(t_\gamma)$

Definition 7.3.1 Let $S \subset R$ be a positive system. Let P denote the 3-tuple of the following sequences. For an integer $\ell(P) > 0$, let $\{a_{\ell(P)}, ..., a_1\}$ and $\{b_{\ell(P)}, ..., b_1\}$ be sequences of integers in $\{\pm1, \pm2, \pm3\}$ and $\{P_{\ell(P)}, ..., P_1, P_0\}$ a sequence of roots in S. Define $P_{j,i}$ to be the subsequence $\{P_j, ..., P_i\}$. Define $\Sigma(P_{1,0})$ to be

$$a_1 P_1 + b_1 P_0$$

and, for $1 < k \leq \ell(P)$, define $\Sigma(P_{k,0})$ to be

$$\Sigma(P_{k,0}) = a_k P_k + b_k \Sigma(P_{k-1,0}).$$

If $\Sigma(P_{k,0}) \in R$ for all $1 \leq k \leq \ell(P)$, and if $b_1 < 0$ with all other $a_i, b_i > 0$, then say P is a path in S. Furthermore, given a convex ordering $\prec$ of S, if $\Sigma(P) \prec P_{\ell(P)}$ and $P_{i+1} \prec P_i$ for $0 \leq i < \ell(P)$, then say ***P* respects the convex ordering $\prec$ of *S***.

Lemma 7.3.2 *Let P be a path in R^- that respects the convex ordering. Then for $k > 0$,*

$$\Sigma(P_{k,0}) \prec \Sigma(P_{k+1,0}).$$

Proof Follows from convex ordering.

Definition 7.3.3 For a path P in R, define $c(P) \in \mathbb{Z}$ to be

$$c(P) = \prod_{i=0}^{\ell(P)-1} c_{P_{i+1},\Sigma(P_{i,0});a_{i+1},b_{i+1}}.$$

Furthermore, define $t(P_{1,0})$ to be

$$t(P_{1,0}) = t_{P_1}^{a_1} t_{P_0}^{b_1}$$

and $t(P_{k,0})$ to be

$$t(P_{k,0}) = t_{P_k}^{a_k} t(P_{k-1,0})^{b_k}$$

with $t(P) = t(P_{\ell(P),0})$.

Remark 7.3.4 With these definitions, we can characterize the elements in each change of variable from moving u_{-P_0}. Each change is of the form

$$t_{\Sigma(P)} \mapsto t_{\Sigma(P)} - c(P)t(P)$$

where P is some path that respects $\prec$ of R^- beginning at P_0 and which sums to $\Sigma(P)$.

Definition 7.3.5 Let P and Q be two paths in R. We say that $Q \in \mathcal{T}$ **branches off** P if $Q_0 = P_0$ and $\Sigma(Q) = P_i$ for some $i \neq 0$. We call such a pair of paths (P, Q)

a **branching pair**, and that P_i is the **absorbing root** for this branching (as this root corresponds to a unipotent absorbing another unipotent). We give the following rooted tree structure on paths in R. Let $\mathcal{T}$ be a rooted tree and label the vertices of $\mathcal{T}$ by paths in R^-, not necessarily distinct. We call $\mathcal{T}$ a **rooted tree of paths in R^-** if the following property holds for each vertex v of $\mathcal{T}$: the path of a child vertex of v branches off the path P of v such that the paths of the child vertices have distinct absorbing roots in P.

Definition 7.3.6 Let (P, Q) be a branching pair of paths in R. Let k be the greatest index such that $P_i = Q_i$ for all $i \leq k$. We say that

$$\begin{aligned} P \prec Q \quad &\text{if} \quad P_{k+1} \prec Q_{k+1}, \\ Q \prec P \quad &\text{if} \quad Q_{k+1} \prec P_{k+1}. \end{aligned}$$

Definition 7.3.7 For a path P in R^-, define the operator $\boldsymbol{B_P}$ inductively by

$$B_P(t_\gamma) = t_\gamma - \sum_{\Sigma(Q)=\gamma,\, Q_0=P_0} \mathbb{1}(P \prec Q) c(Q) B_Q(t(Q))$$

and extend this action to $\mathbb{Q}(t_{\gamma_1}, ..., t_{\gamma_N})$.

Lemma 7.3.8

$$B_\beta(t_\gamma) = t_\gamma - \sum_{\Sigma(P)=\gamma,\, P_0=\beta} c(P) B_P(t(P)).$$

7.4 Nesting Coordinates

The integral (7.16) is actually an integral over $(\mathfrak{o}^\times)^N$:

$$\mathcal{I}(\overline{n}) = \chi(\overline{n}) \left(\prod_{i=1}^{N} |\pi^{n_i}|\right) \int_{(\mathfrak{o}^\times)^N} \left(\prod_{\gamma<0} \psi(m_\gamma^{-1})\right) \left(\prod_{\gamma<0} \mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(B_{\preceq\gamma_1}(t_\gamma))\right) \prod_{\gamma<0} dv_\gamma \tag{7.17}$$

with t_γ restricted to $C(\overline{n})$. We will make a change of variables among the units v_γ that is suited to the structure of the m_γ and $B_{\preceq\gamma_1}(t_\gamma)$. This change will reveal "nesting phenomena" (7.19) among the m_γ which will make the integral over $(\mathfrak{o}^\times)^N$ easier to evaluate.

Changes of Unit Variable for Nesting Coordinates. Recall that $m(\gamma, \alpha)$ denotes the multiplicity of the simple root α as a summand of γ. For a root $\gamma_j \in R^-$ and another root $\gamma \prec \gamma_j$, call the set of changes of variable

$$v_\gamma \mapsto v_\gamma v_{\gamma_j}^{m(w_{j-1}(\gamma),\alpha_j)}.$$

for all $\gamma \prec \gamma_j$ the **changes of unit variable for γ_j**. We denote the application of these changes by the operator $\boldsymbol{N}_{\gamma_j}$. We let $\boldsymbol{N}_{\succ\gamma}$ denote the changes for $\gamma' \succ \gamma$, starting from γ_1. The N signifies that these changes create the nesting coordinates.

Now $N_{\succ\gamma_{j+1}}(m_{\gamma,j})$ is a power of the uniformizer π times a product of unit variables. We describe exactly which unit variables appear in this coefficient.

Lemma 7.4.1 *Let $v(\gamma, \alpha)$ be*

$$v(\gamma, \alpha) = \prod_{\beta\succ\gamma,\, \beta \text{ maps to } \alpha} v_\beta,$$

with an empty product defined to be 1. *Then the unit variable coefficient of* $N_{\succ\gamma_{j+1}}(m_{\gamma,j})$ *is*

$$v_\gamma \prod_{\alpha\in D^-} v(\gamma_{j+1},\alpha)^{m(w_j(\gamma),\alpha)}.$$

Proof The proof is by induction. We note that the lemma is true for all $m_{\gamma,0}$ in which case $m_{\gamma,0} = t_\gamma$. Fix $0 \geq j < N$ and assume it is true for all $N_{\succ\gamma_{j+1}}(m_{\gamma,j})$ for $\gamma \preceq \gamma_{j+1}$. Now apply $N_{\gamma_{j+1}}$ to see that the unit variable coefficient of $N_{\succeq\gamma_{j+1}}(m_{\gamma,j})$ for $\gamma \prec \gamma_{j+1}$ is

$$v_\gamma[v_{\gamma_{j+1}}v(\gamma_{j+1},\alpha_{j+1})]^{m(w_j(\gamma),\alpha_{j+1})} \prod_{\alpha\in D\backslash\alpha_{j+1}} v(\gamma_{j+1},\alpha)^{m(w_j(\gamma),\alpha)} \tag{7.18}$$

The $j+1$-th step in the Bruhat decomposition is to conjugate $u_{w_j(\gamma)}(N_{\succeq\gamma_{j+1}}(m_{\gamma,j}))$ by $\overline{\alpha_{j+1}^\vee}(N_{\succ\gamma_{j+1}}(m_{\gamma_{j+1}}))$. Using the induction hypothesis on the unit coefficients of $N_{\succ\gamma_{j+1}}(m_{\gamma_{j+1}})$, this conjugation has the effect (in terms of units) of multiplying (7.18) by

$$[v_{\gamma_{j+1}}v(\gamma_{j+1},\alpha_{j+1})]^{\langle w_j(\gamma),\alpha_{j+1}^\vee\rangle}.$$

Now

$$m(w_j(\gamma),\alpha_{j+1}) - m(w_{j+1}(\gamma),\alpha_{j+1}) = \langle w_j(\gamma),\alpha_{j+1}^\vee\rangle$$

and

$$v(\gamma_{j+1},\alpha)^{m(w_j(\gamma),\alpha)} = v(\gamma_{j+2},\alpha)^{m(w_{j+1}(\gamma),\alpha)}$$

for all $\alpha \neq \alpha_{j+1}$. (In the case $j = N-1$, we let

$$v(\gamma_{N+1},\alpha) = \prod_{\beta\in R^-,\,\beta\text{ maps to }\alpha} v_\beta.)$$

These facts combined with (7.18) proves the induction step. □

Corollary 7.4.2 *Let $\gamma \in R^-$ map to $\alpha \in D^-$. Then the unit variable coefficient of $N_{\succ\gamma}(m_\gamma)$ is*

$$\prod_{\beta \succeq \gamma, \beta \text{ maps to } \alpha} v_\beta.$$

Now we may apply $N_{\preceq\gamma_1}$ to the integrand of (7.17) to get

$$\prod_{\gamma<0} \psi(N_{\preceq\gamma_1}(m_\gamma^{-1})) = \prod_{\alpha \in D^-} \psi(v_{\alpha,1}^{-1}(\pi^{n_{\alpha,1}} + v_{\alpha,2}^{-1}(\pi^{n_{\alpha,2}} + \ldots + v_{\alpha,m_\alpha}^{-1}\pi^{n_{\alpha,m_\alpha}})\ldots)) \quad (7.19)$$

where we have re-indexed the v_γ as $v_{\alpha,i}$ such that γ is the i-th most succeeding root that maps to α, and $n_{\alpha,i}$ are integers depending on $\overline{n}$. This is the nesting phenomenon referred to above.

The rational functions $B_{\preceq\gamma_1}(\gamma)$ are also suited to $N_{\preceq\gamma_1}$; Lemma ?? describes how.

Lemma 7.4.3 *Suppose $\gamma_j \in R^-$ maps to a simple root α. Let β be the next root succeeding γ_j that also maps to α, if such a root exists. Then for any $\gamma \prec \beta$,*

$$N_{\gamma_j} B_{\prec\beta}(t_\gamma) = v_{\gamma_j}^{m(w_{j-1}(\gamma),\alpha)} B_{\prec\beta}(t_\gamma)'$$

where $B_{\prec\beta}(t_\gamma)'$ denotes a function independent of the unit variable v_{γ_j} (if β does not exist, then $\prec \beta$ is interpreted as $\in R^-$). In other words, v_{γ_j} completely factors out of $N_{\gamma_j} B_{\prec\beta}(t_\gamma)$.

Proof For $\gamma \prec \beta$, v_{γ_j} completely factors out of $N_{\gamma_j}(t_\gamma)$ to the power $m(w_{j-1}(\gamma), \alpha)$; here we have used that $m(w_{j-1}(\gamma), -\alpha) = 0$ for any $\gamma_j \prec \gamma \prec \beta$. Furthermore, for any path P in R that respects $\prec$ with $P_0 \prec \beta$, then v_{γ_j} also factors out of $t(P)$ to the power $m(w_{j-1}(\Sigma(P), \alpha)$. The lemma follows from these facts and the definition of $B_{\prec\beta}$. □

7.5 Final Integration

This section proves Theorem 7.1.1 using the main result of section 7.6. The proof motivates a "subvariable" z of a unit variable υ_γ which is treated in that section.

For an N-tuple $\overline{n}$, consider the integral $\mathcal{I}(\overline{n})$ given by

$$\chi(\overline{n})\left(\prod_{i=1}^{N}|\pi^{n_i}|\right)\int_{(o^\times)^N}\left(\prod_{\gamma\in R^-}\psi(N_R\text{-}m_\gamma^{-1})\right)\left(\prod_{\gamma\in R^-}\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(N_R\text{-}B_{\preceq\gamma_1}(t_\gamma))\right)\prod_{\gamma\in R^-}d\upsilon_\gamma. \tag{7.20}$$

Lemma 7.5.1 *Let $\overline{n}=(n_N,\ldots,n_1)$ be an N-tuple of integers. Suppose that $n_j=\mathrm{ht}(\gamma_j)$ for some j. Then*

$$\prod_{\gamma\in R^-}\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(B_{\gamma_j}(t_\gamma))=\prod_{\gamma\in R^-}\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(t_\gamma)$$

for t_γ restricted to $C(\overline{n})$.

Proof The operator B_{γ_j} is a sequence of changes of the form

$$t_\gamma\mapsto t_\gamma-c(P)t(P)$$

where P is a path in R^- with $\Sigma(P)=\gamma$ and $P_0=\gamma_j$. We claim each such change leaves invariant the product

$$\prod_{\gamma\in R^-}\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(t_\gamma).$$

Indeed, apply the change for one such P to get

$$\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(t_\gamma-c(P)t(P))\prod_{\gamma'\in R^-,\neq\gamma}\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma')}}(t_{\gamma'}).$$

The conditions on the $t_{\gamma'}$ along with the assumption that the valuation of t_{γ_j} is $\mathrm{ht}(\gamma_j)$ imply

$$t(P) \in \mathfrak{p}^{\mathrm{ht}(\gamma)},$$

as $\Sigma(P) = \gamma$. Since $c(P)$ is an integer, we get

$$t_\gamma - c(P)t(P) \equiv t_\gamma \mod \mathfrak{p}^{\mathrm{ht}(\gamma)}$$

and the desired invariance is proved. □

Definition 7.5.2 *Given an N-tuple $\overline{n}$, let $\boldsymbol{j} = \boldsymbol{j(\overline{n})}$, $1 \leq j \leq N$, be the greatest index such that for all $k < j$, $n_k = \mathrm{ht}(\gamma_k)$.*

Successively applying Lemma 7.5.1 gives

$$\prod_{\gamma \in R^-} \mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(B_{\preceq \gamma_1}(t_\gamma)) = \prod_{\gamma \in R^-} \mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(B_{\preceq \gamma_j}(t_\gamma))$$

The condition $\mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma_j)}}(N_{R^-}B_{\preceq \gamma_j}(t_{\gamma_j}))$ is simply

$$t_{\gamma_j} \in \mathfrak{p}^{\mathrm{ht}(\gamma_j)},$$

so we may assume $n_j \geq \mathrm{ht}(\gamma_j)$. We claim that (7.20) is 0 unless $n_j = \mathrm{ht}(\gamma_j)$.

Assume $n_j > \mathrm{ht}(\gamma_j)$. Recall the re-labeling of γ_j as (α, i), where γ_j is the i-th most succeeding root that maps to α. We integrate $v_{\alpha,i}$ over $\mathfrak{o}^\times$. If no root preceding γ_j maps to α, then from (7.19) we need to consider

$$v_{\alpha,i}^{-1}\pi^{n_{\alpha,i}};$$

otherwise, we have

$$v_{\alpha,i}^{-1}(\pi^{n_{\alpha,i}} + v_{\alpha,i+1}^{-1}v\pi^{n})$$

where $n_{\alpha,i}$ is the negative of the valuation of m_γ, and v and n are independent of both $v_{\alpha,i}$ and $v_{\alpha,i+1}$. We claim

$$n_{\alpha,i} = \text{ht}(\gamma_j) - n_j - 1$$

given our condition on j. This may be proved using the form of the m_γ (Lemma 7.2.3) and induction similar to that in the proof of Lemma 7.4.1. By Lemma 7.4.3, each support conditions $\mathbb{1}_{\mathfrak{p}^{\text{ht}(\gamma)}}(N_R\text{-}B_{\preceq\gamma_j}(t_\gamma))$ is independent of $v_{\alpha,i}$. Thus the integral of $v_{\alpha,i}$ over $\mathfrak{o}^\times$ in the first case is 0 by (7.2) because $n_j > \text{ht}(\gamma_j)$; and likewise in the second case is 0 unless $m = n_{\alpha,i}$ and $1 + v_{\alpha,i+1}^{-1}v \in \mathfrak{p}^{n_j-\text{ht}(\gamma_j)}$. In the second case, $v_{\alpha,i+1}$ must be of the form

$$v_{\alpha,i+1}^{-1} = -v^{-1} + z\pi^{n_j-\text{ht}(\gamma_j)} \tag{7.21}$$

for any $z \in \mathfrak{o}$. This is how the subvariable z of $v_{\alpha,i+1}$ is determined. We would like to integrate z over $\mathfrak{o}$ to get 0 by (7.3), but the support conditions may depend on z. We assume that the support conditions are independent of z which is proved in Section 7.6. Thus we apply (7.3) to see that the integral is 0 in the second case as well. This proves that $\mathcal{I}(n)$ is 0 unless

$$\overline{n} = (\text{ht}(\gamma_N), ..., \text{ht}(\gamma_1)).$$

7.6 The Support Conditions Are Independent of z

In the notation of section 7.5, we show that the product

$$\prod_{\gamma\in R^-} \mathbb{1}_{\mathfrak{p}^{\mathrm{ht}(\gamma)}}(N_{R^-}B_{\preceq\gamma_j}(t_\gamma)) \tag{7.22}$$

restricted to $C(\overline{n})$, where $j = j(\overline{n})$, is independent of the subvariable z of $v_{\alpha,i+1}$. In the following, a primed function f' will denote the result of applying N_{R^-} to f and factoring out the maximal power of $v_{\alpha,i+1}$ (which may be 0); in particular, primed functions will be independent of $v_{\alpha,i+1}$.

Our analysis will depend on the type of the root system R. We first examine the possible levels of paths in R^-.

Lemma 7.6.1 *Let P be a path in R^- with $P_0 = \gamma_j$. Consider a collection of paths of any level from P such that no two branching paths share the same junction. Let m be the number of all junctions. Then m is less than the multiplicity of α_j in $w_j(\Sigma(P))$.*

Proof Apply w_j to the roots in paths in this set to express $w_j(\Sigma(P)) \in R^-$ as a sum of other roots in R^- with α_j appearing at least $m+1$ times in the sum, one from each junction and one from P_0 itself. □

Type *A* In type A, simple roots occur with multiplicity at most one. By Lemma 7.6.1, there are only level 0 paths and thus Lemma 7.3.8 gives

$$B_{\preceq\gamma_j}(t_\gamma) = B_{\prec\gamma_j}(t_\gamma) - \sum_{\Sigma(P)=\gamma, P_0=\gamma_j} c(P)B_{\prec\gamma_j}(t(P)) \in \mathfrak{p}^{\mathrm{ht}(\gamma)}. \tag{7.23}$$

Apply N_{R^-} to (7.23). The unit variable $v_{\alpha,i+1}$ factors out of $N_{R^-}B_{\prec\gamma_j}(t_\gamma)$ to the power 1, and out of $N_{R^-}B_{\prec\gamma_j}(t(P))$ to the power 0 by Lemma 7.4.3. Also for such P,

$$B_{\prec\gamma_j}(t(P)) = B_{\preceq\gamma_j}(t(P))$$

because there are no level 1 paths. Dividing $N_{R^-}B_{\prec\gamma_j}(t_\gamma)$ by $v_{\alpha,i+1}$ gives the condition

$$B_{\prec\gamma_j}(t_\gamma)' - v_{\alpha,i+1}^{-1} \sum_{\Sigma(P)=\gamma, P_0=\gamma_j} c(P)B_{\preceq\gamma_j}(t(P))' \in \mathfrak{p}^{\mathrm{ht}(\gamma)}. \tag{7.24}$$

By (7.22),

$$B_{\preceq\gamma_j}(t(P))' \in \mathfrak{p}^{\mathrm{ht}(\gamma)+\mathrm{ht}(\gamma_j)-n_j}.$$

Then substituting (7.21) into (7.24) shows that (7.24) is independent of z, using $c(P) \in \mathbb{Z}$.

Type D The roots in $D_r, r \geq 4$, are of the form

$$\pm\varepsilon_i \pm \varepsilon_j,$$

where $i \neq j$ and where $\{\varepsilon_i\}_{1\leq i\leq r}$ is a set of r orthonormal vectors. We let R^- consist of the of the roots

$$\varepsilon_i \pm \varepsilon_j$$

where $i > j$, with simple roots D^-

$$D^- = \{\varepsilon_{i+1} - \varepsilon_i\}_{1\leq i<r} \bigcup \varepsilon_2 + \varepsilon_1.$$

The roots in D^- that may appear with multiplicity 2 are $\{\varepsilon_{i+1} - \varepsilon_i\}_{1<i<r-1}$ while

all other appear with multiplicity at most 1. When α_j is a root of the latter type, the analysis done in type A suffices to prove the z independence; when it is of the former, then any path has at most one junction. We analyze the terms arising from such paths and show that, while those terms containing z dependence may not be in $\mathfrak{p}^{\mathrm{ht}(\gamma)}$ individually, the sum of them is. This sum is equation (7.31).

Suppose that P is a path in R and Q branches off of P. We write the root sequence for P as

$$\{P_{\mathrm{prec}}, \Sigma Q, P_{\mathrm{succ}}\}$$

where P_{prec} and P_{succ} denote the subsequences of P's sequence that precede and succeed ΣQ, respectively. Not that P_{succ} is a path in R but P_{prec} is not necessarily one. We make an abuse of notation and define $t(P_{\mathrm{prec}})$ to be

$$t(P_{\mathrm{prec}}) = \frac{t(P)}{t_{\Sigma(Q)}t(P_{\mathrm{succ}})}.$$

As for type A, we now determine how each condition

$$N_R\text{-}B_{\preceq\gamma_j}(t_\gamma) \in \mathfrak{p}^{\mathrm{ht}(\gamma)}$$

depends on $v_{\alpha,i}$ and thus on z. From Lemma 7.3.8,

$$\begin{aligned} B_{\preceq\gamma_j}(t_\gamma) = B_{\prec\gamma_j}(t_\gamma) - \sum_P c(P) B_{\prec\gamma_j}(t(P_{\mathrm{prec}}))\times \\ \left(B_{\prec\gamma_j}(t_{\Sigma(Q)}) - \sum_Q \mathbb{1}(P \prec Q) c(Q) B_{\prec\gamma_j}(t(Q))\right) B_{\prec\gamma_j}(t(P_{\mathrm{succ}})). \end{aligned} \tag{7.25}$$

Apply N_R- to (7.25). Now $v_{\alpha,i+1}$ factors out of $N_R\text{-}B_{\prec\gamma_j}(t_\gamma)$ to the power 2; out of $N_R\text{-}B_{\prec\gamma_j}(t_{\Sigma(Q)})$ to the power 1; and out of $N_R\text{-}B_{\prec\gamma_j}(t(P_{\mathrm{prec}}))$, $N_R\text{-}B_{\prec\gamma_j}(t(P_{\mathrm{succ}}))$, and N_R- $B_{\prec\gamma_j}(t(Q))$ to the power 0. Then

$$\upsilon_{\alpha,i+1}^{-2} N_R\text{-} B_{\preceq\gamma_j}(t_\gamma) = B_{\prec\gamma_j}(t_\gamma)' - \sum_P \upsilon_{\alpha,i+1}^{-1} c(P) B_{\prec\gamma_j}(t(P_{\text{prec}}))' \times \tag{7.26}$$
$$\left(B_{\prec\gamma_j}(t_{\Sigma(Q)})' - \upsilon_{\alpha,i+1}^{-1} \sum_Q \mathbb{1}(P \prec Q) c(Q) B_{\prec\gamma_j}(t(Q))' \right) B_{\prec\gamma_j}(t(P_{\text{succ}}))'.$$

Now

$$B_{\prec\gamma_j}(t_{\Sigma(Q)})' - \upsilon_{\alpha,i+1}^{-1} \sum_Q c(Q) B_{\prec\gamma_j}(t(Q))' = \upsilon_{\alpha,i}^{-1} N_R\text{-} B_{\preceq\gamma_j}(t_{\Sigma(Q)}). \tag{7.27}$$

Therefore we re-write the right side of (7.26) as

$$B_{\prec\gamma_j}(t_\gamma)' - \sum_P \Big[\upsilon_{\alpha,i+1}^{-1} c(P) B_{\prec\gamma_j}(t(P_{\text{prec}}))' \left(\upsilon_{\alpha,i}^{-1} N_R\text{-} B_{\preceq\gamma_j}(t_{\Sigma(Q)}) \right) B_{\prec\gamma_j}(t(P_{\text{succ}}))' \tag{7.28}$$
$$+ \upsilon_{\alpha,i+1}^{-2} c(P) B_{\prec\gamma_j}(t(P_{\text{prec}}))' B_{\prec\gamma_j}(t(P_{\text{succ}}))' \sum_Q \mathbb{1}(Q \prec P) c(Q) B_{\prec\gamma_j}(t(Q))' \Big].$$

By (7.22), we may assume

$$B_{\prec\gamma_j}(t(P_{\text{prec}}))' \Big(\upsilon_{\alpha,i}^{-1} N_R\text{-} B_{\preceq\gamma_j}(t_{\Sigma(Q)}) \Big) B_{\prec\gamma_j}(t(P_{\text{succ}}))' \in \mathfrak{p}^{\text{ht}(\gamma)+\text{ht}(\gamma_j)-n_j}$$

using

$$B_{\prec\gamma_j}(t(P_{\text{prec}})) = B_{\preceq\gamma_j}(t(P_{\text{prec}})),$$
$$B_{\prec\gamma_j} t(P_{\text{succ}}) = B_{\preceq\gamma_j} t(P_{\text{succ}}).$$

Applying (7.21) to the first $v_{\alpha,i}^{-1}$ in (7.28), this equation simplifies modulo $\mathfrak{p}^{\mathrm{ht}(\gamma)}$ to

$$B_{\prec\gamma_j}(t_\gamma)' - \sum_P \Big[v^{-1}c(P)B_{\prec\gamma_j}(t(P_{\mathrm{prec}}))'\Big(v_{\alpha,i}^{-1}N_R - B_{\preceq\gamma_j}(t_{\Sigma(Q)})\Big)B_{\prec\gamma_j}(t(P_{\mathrm{succ}}))' \tag{7.29}$$
$$+ v_{\alpha,i+1}^{-2}c(P)B_{\prec\gamma_j}(t(P_{\mathrm{prec}}))'B_{\prec\gamma_j}(t(P_{\mathrm{succ}}))'\sum_Q \mathbb{1}(Q \prec P)c(Q)B_{\prec\gamma_j}(t(Q))'\Big]$$

Now, using (7.27), we collect the z-dependent terms in (7.29) modulo $\mathfrak{p}^{\mathrm{ht}(\gamma)}$ to obtain

$$v^{-1}z\pi^{n_j-\mathrm{ht}(\gamma_j)}B_{\prec\gamma_j}\Big(\sum_{(P,Q)\text{ respects }\prec} \epsilon(P,Q)c(P)c(Q)t(P_{\mathrm{prec}})t(Q)t(P_{\mathrm{succ}})\Big)' \tag{7.30}$$

where we have the following definitions. We say that (P, Q) is a **branching pair** if P and Q are paths in R with Q branching off P, and that (P, Q) respects the convex ordering if both P and Q respect the convex ordering. Define

$$\boldsymbol{\epsilon(\mathcal{T})} = \begin{cases} -1 : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \\ \;\;\,1 : \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)}, \end{cases}$$

$$\boldsymbol{c(P,Q)} = c(P)c(Q)$$

and

$$\boldsymbol{t(P,Q)} = t(P_{\mathrm{prec}})t(Q)t(P_{\mathrm{succ}}).$$

The sum in (7.30) is then

$$\sum_{\mathcal{T}\text{ respects }\prec} \epsilon(\mathcal{T})c(\mathcal{T})t(\mathcal{T}). \tag{7.31}$$

We show that this sum is 0 by constructing an involution among the terms in the sum, pairing up each term with one that is its negative. This involution is determined roughly as follows. Each term in the sum arises from a branching pair (P, Q). Essentially, we "transfer" a certain root from one path in this pair to the other, thus constructing a branching pair $(P, Q)^{\mathcal{R}}$.

We first describe a result for the structure constants. For simply-laced root systems R with $\alpha, \beta \in R$, we set $c_{\alpha,\beta} = c_{\alpha,\beta;1,1}$ following notation of [Spr]. As usual, if either α, β or $\alpha+\beta \notin R$, then we set $c_{\alpha,\beta} = 0$. The next lemma states two standard relations among the structure constants.

Lemma 7.6.2 *Let R be a simply-laced root system and let α, β, γ be linearly independent roots in R. Then*

(*i*) $c_{\alpha,\beta} = -c_{\beta,\alpha}$ (***anti-commutativity***)

(*ii*) $c_{\alpha,\beta}c_{\alpha+\beta,\gamma} + c_{\beta,\gamma}c_{\beta+\gamma,\alpha} + c_{\gamma,\alpha}c_{\gamma+\alpha,\beta} = 0$ (***Jacobi relation***)

Proof [Spr]. □

We apply relation (*ii*) to structure constants of the branching pair (P, Q).

Lemma 7.6.3 *Given a branching pair (P, Q), let*

$$\alpha = \min(Q)$$
$$\beta = \Sigma(Q) - \min(Q)$$
$$\gamma = \Sigma(P_{\text{succ}}).$$

Then exactly one of the sums $\beta + \gamma$, $\gamma + \alpha$ is in R, and

$$c_{\alpha,\beta}\, c_{\alpha+\beta,\gamma} = \begin{cases} c_{\beta,\gamma}\, c_{\alpha,\beta+\gamma} & : \beta + \gamma \in R \\ -c_{\alpha,\gamma}\, c_{\beta,\gamma+\alpha} & : \gamma + \alpha \in R \end{cases}$$

Proof That exactly one of the sums $\beta+\gamma$, $\gamma+\alpha$ is in R follows from a straightforward check using the characterization of roots in type D. Therefore, exactly one of the three terms in the Jacobi relation is 0, depending on which sum is not in R, leaving the stated equation between two remaining terms. □

Operations on branching pairs. We construct a branching pair (P', Q') out of P and Q by re-arranging the roots in the paths; $t(P', Q') = t(P, Q)$ because all the relevant roots remain unchanged, and $c(P', Q')$ is equal to $c(P, Q)$ up to sign. There are four operations we consider.

Commute. Suppose that $\min(P_{\text{succ}}) \neq P_0$. If $\Sigma(Q) + \min(P_{\text{succ}}) \notin R$, then set

$$(P, Q)^{\text{comm}} = (\{P_{\text{prec}}, \min(P_{\text{succ}}), \Sigma(Q), P_{\text{succ}} \setminus \min(P_{\text{succ}})\}, \quad Q).$$

Clearly $c(P, Q)^{\text{comm}}$ is equal to $c(P, Q)$.

Transfer to the preceding sequence. In the case $\Sigma(Q) - \min(Q) + \Sigma(P_{\text{succ}}) \in R$, transfer $\min(Q)$ to P_{prec}; set

$$(P, Q)^{\text{prec}} = (\{P_{\text{prec}}, \min(Q), \Sigma(Q) - \min(Q), P_{\text{succ}}\}, \quad Q \setminus \min(Q)).$$

Then it follows from Lemma 7.6.3 that $c(P, Q)^{\text{prec}} = c(P, Q)$.

Transfer to the succeeding sequence. In the case $\min(Q) + \Sigma(P_{\text{prec}}) \in R$, transfer $\min(Q)$ to P_{succ}; set

$$(P, Q)^{\text{succ}} = (\{P_{\text{succ}}, \Sigma(Q) - \min(Q), \min(Q), P_{\text{succ}}\}, \quad Q \setminus \min(Q))$$

Again from Lemma 7.6.3, $c(P,Q)^{\text{succ}} = -c(P,Q)$.

Switch. We set

$$(P,Q)^{\text{switch}} = (\{P_{\text{prec}}, \Sigma(P_{\text{succ}}), Q\},\quad P_{\text{succ}})$$

Then $c(P,Q)^{\text{switch}} = c(P,Q)$; indeed, we get one -1 by changing

$$c_{\Sigma(Q),\Sigma(P_{\text{succ}})} \text{ to } c_{\Sigma(P_{\text{succ}}),\Sigma(Q)}$$

and another -1 from the ϵ factor.

The initial input of $\mathcal{R}$ is a branching pair (P,Q) that respects the convex ordering. It generates a sequence of intermediate branching pairs that do not respect the ordering until a pair that does respect the ordering is obtained. This pair $(P,Q)^{\mathcal{R}}$ is the final output and satisfies $c(P,Q) = -c(P,Q)^{\mathcal{R}}$ and $t(P,Q) = t(P,Q)^{\mathcal{R}}$.

Root Transfer Algorithm $\mathcal{R}$ for Simply-Laced Root Systems

Step 1. Given a branching pair (P,Q),

1a. If $\min(Q) \prec \min(P_{\text{succ}})$,

1ai. If $\Sigma Q - \min(Q) + \Sigma P_{\text{succ}} \in R$, then transfer $\min(Q)$ to P_{prec}. With the resulting pair, go to Step 1.

1aii. If $\min(Q) + \Sigma P_{\text{succ}} \in R$, then transfer $\min(Q)$ to P_{succ}. With the resulting pair, go to to Step 2.

1b. If $\min(P_{\text{succ}}) \prec \min(Q)$,

1bi. If $\Sigma Q + \Sigma P_{\text{succ}} - \min(P_{\text{succ}}) \in R$, then apply the commute operation to $\min(P_{\text{succ}})$ and ΣQ. With the resulting pair, go to Step 1.

1bii. If $\min(P_{\text{succ}}) + \Sigma Q \in R$, then transfer $\min(P_{\text{succ}})$ to Q, i.e., apply the inverse of the transfer to the succeeding sequence operation to $\min(P_{\text{succ}})$. With the resulting pair, go to to Step 2.

Step 2. Given a branching pair (P, Q),

2a. If either (P, Q) or $(P, Q)^{\text{switch}}$ respects the convex ordering, then output that pair.

2b. If $\max(P_{\text{prec}}) + \Sigma P_{\text{succ}} \in R$, then apply the inverse of the commute operation to the roots $\max(P_{\text{prec}})$ and ΣQ in P. With the resulting pair, go to Step 2.

2c. If $\max(P_{\text{prec}}) + \Sigma Q \in R$, then apply the inverse of the transfer to preceding sequence operation to $\max(P_{\text{prec}})$. With the resulting pair, go to Step 2.

Lemma 7.6.4 *The root transfer algorithm $\mathcal{R}$ for simply-laced R is well-defined and is an involution on branching pairs (P, Q). Furthermore, $t(P, Q) = t(P, Q)^{\mathcal{R}}$.*

Proof First observe that during the steps of the algorithm no roots are added to or removed from the union of the root sequences of Q, P_{prec}, and P_{succ}. Therefore $t(P, Q) = t(P, Q)^{\mathcal{R}}$.

We claim each intermediate pair (P, Q) possesses the following properties:

(*i*) Q respects the convex ordering and P does not, but P absent $\Sigma(Q)$ does respect the convex ordering, so that

$$\max(P_{\text{prec}}) \prec \min(P_{\text{succ}}),$$
$$\max(P_{\text{prec}}) \prec \min(Q).$$

(*ii*)

$$\Sigma(P_{\text{succ}}) \prec \min(P_{\text{succ}}), \qquad \Sigma(Q) \prec \min(P_{\text{succ}}),$$
$$\Sigma(P_{\text{succ}}) \prec \min(Q), \qquad \Sigma(Q) \prec \min(Q). \tag{7.32}$$

Property (*ii*) also holds for pairs that do respect the convex ordering.

We will use the fact that in simply-laced root systems R, for any $\alpha, \beta \in R$, the quantity $\langle \alpha, \beta^\vee \rangle$ lies in $\{0, \pm 1, \pm 2\}$ and satisfies the following conditions:

$$\alpha + \beta \in R \iff \langle \alpha, \beta^\vee \rangle = -1$$
$$\alpha = \beta \iff \langle \alpha, \beta^\vee \rangle = 2. \tag{7.33}$$

First we prove that $\min(Q) \neq \min(P_{\text{succ}})$ assuming that P_{succ} and Q respect the convex ordering. We may assume that $\alpha_j = \varepsilon_{i+1} - \varepsilon_i$ for a fixed $0 < i < r-1$, where $P_0 = \gamma_j$. Recall that the integer sequences for any path in D_r are of the form $\overline{a} = \{1, ..., 1, 1\}$, $\overline{b} = \{1, ..., 1, 1, -1\}$. Then $w_{j-1}(\Sigma(Q))$ and $w_{j-1}(\Sigma(P_{\text{succ}}))$ must be of one of the following forms:

$$\{\varepsilon_m - \varepsilon_{i+1}, \, \varepsilon_i - \varepsilon_{m'}, \, \varepsilon_m - \varepsilon_{m'}, \, \varepsilon_k + \varepsilon_{k'}, -\varepsilon_{i+1} + \varepsilon_i\}$$

for some $m > i+1, i > m', k \geq i+1, k > k'$. In addition, $\Sigma Q + \Sigma P_{\text{succ}} \in D_r$ since P is a path. We directly check the possibilities listed to see that $\min(Q)$ cannot equal $\min(P_{\text{succ}})$; thus 1a and 1b are exhaustive. Lemma 7.3.2 applied to the initial input implies that steps 1ai and 1bi always generate intermediate pairs that possess properties (*i*) and (*ii*). Likewise we check that steps 1aii and 1bii preserve property (*ii*).

We claim that the algorithm must eventually enter step 2. For if it did not, then at some point in the algorithm, we have a pair (P, Q) at step 1 such that either P_{succ} or Q consists of the single root P_0 while the other has root sequence

$$\{\beta_n, ..., \beta_1, P_0\}$$

with $n > 0$ such that $\beta_i + P_0 \notin R$ and $\beta_1 \prec P_0, \beta_{i+1} \prec \beta_i$ for all i. By construction,

$$(\beta_n + ... + \beta_1 - P_0) + (-P_0) \in R$$

and thus

$$\langle \beta_n + ... + \beta_1 - P_0, -P_0 \rangle = -1.$$

The assumption that the algorithm never enters step 2 implies that

$$\langle \beta_i, -P_0 \rangle = 0$$

for each $i > 0$. Then $\langle P_0, P_0 \rangle = 2$ gives a contradiction. This proves that the algorithm must eventually enter step 2.

Now, let (P, Q) be an intermediate pair at step 2. If P_{prec} is empty, then $\Sigma P = \Sigma Q + \Sigma P_{\text{succ}}$. Therefore either

$$\Sigma P_{\text{succ}} \prec \Sigma P \prec \Sigma Q \quad \text{or} \quad \Sigma Q \prec \Sigma P \prec \Sigma P_{\text{succ}}$$

by the convex property. We get from these two possibilities that either (P, Q) or $(P, Q)^{\text{switch}}$, respectively, respects the ordering by (ii).

If P_{prec} is not empty, then $\max(P_{\text{prec}})$ succeeds at least one of $\Sigma Q, \Sigma P_{\text{succ}}$; otherwise, the convex property would imply the initial pair did not respect the convex ordering. If $\max(P_{\text{prec}})$ succeeds only one of the two roots, then by (ii) either (P, Q) or $(P, Q)^{\text{switch}}$ respects the convex ordering. If $\max(P_{\text{prec}})$ succeeds both, then neither (P, Q) nor $(P, Q)^{\text{switch}}$ respects the convex ordering and the pair enters step 2b or 2c, which we check are exhaustive. As before, we also see that 2b and 2c preserve (i) and (ii).

The algorithm terminates because 2a and 2b decrease the number of roots in P_{prec}, and we proved that the algorithm terminates when P_{prec} is empty (though P_{prec} being empty is not necessary for termination). We must show that $(P, Q) \neq (P, Q)^{\mathcal{R}}$. The root transferred at step 1aii or 1bii, call it γ, ensures that the pair output at step 2a is distinct from the intitial output pair, except possibly in the case that P and Q have identical roots at the places succeeding γ. But in that case, we get that

$$\gamma + 2\beta \in R$$

for some root β, a contradiction in simply-laced R (For non-simply laced R, there is a modification of the algorithm at this point).

To check that the algorithm is an involution, see that step 2b and 2c or inverse to step 1ai and 1bi, and the root transferred at 1aii or 1bii is the same root transferred there by applying $\mathcal{R}$ to $(P, Q)^{\mathcal{R}}$.

Type E In E_6, the simple root $\varepsilon_4 - \varepsilon_3$ may appear with multiplicity three. Therefore for there are two types of rooted trees of paths in R^- that begin at $\varepsilon_4 - \varepsilon_3$: those with two children of the root, and those with one child of the child of the root. In the support conditions, we add and subtract terms as we did in type D and collect the z dependent terms. The powers of z that appear are z and z^2. There is a different vertex-relation coefficient each power of z. There is an equivalence relation on the set of rooted trees that respect the convex ordering. We say $\mathcal{T}_1 \sim \mathcal{T}_2$ if $\mathcal{T}_1$ can be obtained by applying a sequence of root transfer operators $\mathcal{R}_e$, where e is an edge of a rooted tree. The sum of these terms weighted by the structure constants and vertex-relation coefficients is 0. In contrast to Type D, where the number of trees in an equivalence class is always 2, this number for Type E can vary, with possibilities including 4, 5, 8, 9, or 16. The strategy is to characterize all possible equivalence classes and show the sum over each class is 0. The equivalence class for a tree with m edges is described by an m-regular graph. Below we list the vetex relation coefficients vr_1 and vr_2 in E_6.

$$vr_1(\mathcal{T}) = \begin{cases} -2 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,2)} \\ 1 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1,2)} \prec \mathcal{T}^{(1)} \\ 1 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,2)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \\ 1 & : \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \text{ and } \mathcal{T}^{(1,2)} \prec \mathcal{T}^{(1)} \\ -2 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1,1,1)} \\ 1 & : \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1,1,1)} \\ 0 & : \text{otherwise} \end{cases}$$

$$
vr_2(\mathcal{T}) = \begin{cases} -1 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,2)} \\ 1 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1,2)} \prec \mathcal{T}^{(1)} \\ 1 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,2)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \\ -1 & : \mathcal{T}^{(1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \\ 1 & : \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1,1,1)} \\ -1 & : \mathcal{T}^{(1,1,1)} \prec \mathcal{T}^{(1,1)} \text{ and } \mathcal{T}^{(1,1)} \prec \mathcal{T}^{(1)} \\ 0 & : \text{otherwise} \end{cases}
$$

Root Transfer Algorithm for Type B

The root transfer algorithm for Type B extends that for Type D with some modifications. As in type D, we use the characterization of roots in B_r. Recall that the positive roots of B_r consists of the vectors $\varepsilon_j \pm \varepsilon_i$ and ε_i, $j > i \geq 0$ with simple roots $\varepsilon_{i+1} - \varepsilon_i$, $i \geq 0$, and ε_0. The coefficient sequences for paths may now include ± 2. We describe the points in the type D algorithm that are modified.

The first modification deals with inputs (P, Q) for which the algorithm outputs two pairs instead of one. That is, in type D we could partition the sum into sets of two terms such that the two terms add to 0. But in type B, we partition the sum into sets of two or three terms, such that the terms in a triple add to 0 as well. The following conditions are to be placed at the indicated spots in the algorithm for type D. Here, (P_1, Q_1) always denotes a pair where $\min(Q_1)$ if of the form ε_i and has coefficient 2 in Q_1; (P_2, Q_2) a pair where $\min(P_{2\,\text{succ}})$ of the form ε_i and has coefficient 2 in $P_{2\,\text{succ}}$; and (P_3, Q_3) a pair where $\min(P_{3\,\text{succ}}) = \min(Q_3)$ is of the form ε_i and has coefficient 1 in both paths.

1aiii. Given a branching pair (P_1, Q_1), if $\min(Q_1) \prec \min(P_{1\,\text{succ}})$ and $\min(Q_1)$ has coefficient 2 in Q_1, then transfer $\min(Q_1)$ to $P_{1\,\text{succ}}$ to make the pair (P_2, Q_2), where $Q_2 = Q_1 \backslash \min(Q_1)$ and $P_2{}'_{\text{succ}} = \{\min(Q_1), P_{1\,\text{succ}}\}$ but with $\min(Q_1)$ having coefficient 2 in $P_{2\,\text{succ}}$. In addition, transfer one copy of $\min(Q_1)$ from Q_1 to $P_{1\,\text{succ}}$ to make (P_3, Q_3), where $Q_3 = Q_1$ but with $\min(Q_1)$ having coefficient 1, and $P_3 = P_2$ but with $\min(Q_1)$ having coefficient 1 also. With these two pairs, go to Step 2.

1biii. Given a branching pair (P_2, Q_2), if $\min(P_{2\,\text{succ}}) \prec \min(Q_2)$ and $\min(P_{2\,\text{succ}})$ has coefficient 2 in $P_{2\,\text{succ}}$, then transfer $\min(P_{2\,\text{succ}})$ to Q_2 to make the pair (P_1, Q_1), where $Q_1 = \{\min(P_{2\,\text{succ}}), Q_2\}$ but with $\min(P_{2\,\text{succ}})$ having coefficient 2 in Q_1; and where $P_1{}'_{\text{succ}} = P_{2\,\text{succ}} \backslash \min(P_{2\,\text{succ}})\}$. In addition, transfer one copy of $\min(P_{2\,\text{succ}})$ from $\min(P_{2\,\text{succ}})$ to Q_2 to make (P_3, Q_3), where $Q_3 = Q_1$ but with $\min(P_{2\,\text{succ}})$ having coefficient 1, and $P_3 = P_2$ but with $\min(P_{2\,\text{succ}})$ having coefficient 1 also. With these two pairs, go to Step 2.

1c. Given a branching pair (P_3, Q_3), if $\min(P_{3\,\text{succ}}) = \min(Q_3) = \gamma$, then γ must have coefficient 1 in both paths. Construct two pairs (P_1, Q_1) and (P_2, Q_2) out of (P, Q). Here $Q_1 = Q_3$ but with γ having coefficient 2 in Q_1, and $P_1{}'_{\text{succ}} = P_{3\,\text{succ}} \backslash \gamma$. Furthermore $Q_2 = Q_3 \backslash \gamma$ and $P_{2\,\text{succ}} = P_{3\,\text{succ}}$ but with γ having coefficient 2. With these two pairs, go to step 2.

The second modification deals with terms in the sum that do not correspond to pairs of paths, but to paths with a 2 in the **b** sequence. Specifically, let S be a path with root sequence

$$\{\ldots, S_{n+1}, S_n, \ldots,\}$$

and coefficient $b_n = 2$ such that $\Sigma S_{0,n}$ is of the form ε_i and $\Sigma S_{0,n}$ is of the form

$\varepsilon_j + \varepsilon_i$. β_{n+1} is of the form $\varepsilon_j - \varepsilon_i$. Let (P, Q) be a branching pair with

$$P_{\text{prec}} = \{..., S_{n+3}, S_{n+2}\}$$
$$Q = \{S_{n+1}, S_n, ...\}$$
$$P_{\text{succ}} = \{S_n, S_{n-1}, ...\}$$

$2c(s) = c(P, Q)$. Apply Step 2 of the algorithm for type D to (P, Q). The output corresponds to the term that cancels the term that corresponds to S.

The third modification deals with a path S such that $S_0 = \varepsilon_0$, $b_1 = -2$, and $S_1 = \varepsilon_j + \varepsilon_0$. Let (P, Q) be the branching pair

$$P_{\text{prec}} = \{..., S_3, S_2\}$$
$$Q = \{S_1, S_0\}$$
$$P_{\text{succ}} = \{S_0\}$$

Apply Step 2 of the algorithm for type D to (P, Q). The output corresponds to the term that cancels the term that corresponds to S.

Type C The root transfer algorithm for Type C also extends that for Type D with some modifications. We use the characterization of roots in C_r. Recall that the positive roots of C_r consists of the vectors $\varepsilon_j \pm \varepsilon_i$ $j \geq i \geq 0$ with simple roots $\varepsilon_{i+1} - \varepsilon_i$, $i \geq 0$, and $2\varepsilon_0$. The coefficient sequences for paths may now include ± 2. We describe the points in the type D algorithm that are modified.

The first modification, as in Type B, deals with pairs (P, Q) such that $\min(Q) = \min(P_{\text{succ}}) = \gamma$. In this case, γ is of the form $\varepsilon_j - \varepsilon_i$, $j > i$.

1c. Given a branching pair (P, Q), if $\min(Q) = \min(P_{\text{succ}}) = \gamma$, then construct the pair (P', Q') such that

$$P'_{\text{prec}} = \{P_{\text{prec}}, \gamma\}$$
$$Q' = Q \backslash \gamma$$
$$P'_{\text{succ}} = P_{\text{succ}} \backslash \gamma$$

with γ having coefficient 2 in P'. With this pair, go to Step 1.

2b. Given a branching pair (P, Q), if $\max(P_{\text{prec}}$ has coefficient 2 in P, then construct the pair (P', Q') such that

$$P'_{\text{prec}} = P_{\text{prec}} \backslash \gamma$$
$$Q' = \{\gamma, Q\}$$
$$P'_{\text{succ}} = \{\gamma, P_{\text{succ}}\}$$

with γ having coefficient 1 in Q and P.

The second modification deals with pairs (P, Q) such that $\min(\min(Q), \min(P_{\text{succ}})) = \varepsilon_j - \varepsilon_i$ and $\Sigma Q + \Sigma P_{\text{succ}} = \varepsilon_j + \varepsilon_i$. In this case, both conditions of 1ai and 1aii (or 1bi and 1bii) may hold; therefore go to the following 1aii (or 1biii).

1aiii. Given a branching pair (P_1, Q_1), if $\Sigma Q_1 - \min(Q_1) + \Sigma P_{1\,\text{succ}} \in R$ and $\min(Q_1) + \Sigma P_{1\,\text{succ}} \in R$, then construct the following pairs (P_2, Q_2) and (P_3, Q_3). Here,

$$P_{2\,\text{prec}} = P_{1\,\text{prec}}$$
$$Q_2 = Q_1 \setminus \min(Q_1)$$
$$P_{2\,\text{succ}} = \{\min(Q_1), P_{1\,\text{succ}}\}$$

and

$$P_{3\,\text{prec}} = \{\min(Q_1), P_{1\,\text{prec}}\}$$
$$Q_3 = Q_1 \setminus \min(Q_1)$$
$$P_{3\,\text{succ}} = P_{1\,\text{succ}}.$$

With (P_2, Q_2) go to Step 2, and with (P_3, Q_3) go to Step 1.

1biii. Given a branching pair (P_2, Q_2), if $\Sigma Q_2 + \Sigma P_{2\,\text{succ}} - \min(P_{2\,\text{succ}}) \in R$ and $\min(P_{2\,\text{succ}})) + \Sigma Q_2 \in R$, then construct the following pairs (P_1, Q_1) and (P_3, Q_3). Here,

$$P_{1\,\text{prec}} = P_{2\,\text{prec}}$$
$$Q_1 = \{\min(P_{2\,\text{succ}}), Q_2\}$$
$$P_{1\,\text{succ}} = P_{2\,\text{succ}} \setminus \min(P_{2\,\text{succ}})$$

and

$$P_{3\,\text{prec}} = \{P_{2\,\text{prec}}, \min(P_{2\,\text{succ}})\}$$
$$Q_3 = Q_2$$
$$P_{3\,\text{succ}} = P_{2\,\text{succ}} \setminus \min(P_{2\,\text{succ}}).$$

With (P_1, Q_1), go to Step 2. With (P_3, Q_3) go to Step 1.

2c. Given a branching pair (P_3, Q_3), if $\max(P_{3\,\text{prec}}) + \Sigma Q_3 \in R$ and $\max(P_{3\,\text{prec}}) + \Sigma P_{3\,\text{succ}} \in R$, then construct two pairs (P_1, Q_1) and (P_2, Q_2) such that

$$P_{1\,\text{prec}} = P_{3\,\text{prec}} \setminus \max(P_{3\,\text{prec}})$$
$$Q_1 = \{\max(P_{3\,\text{prec}}), Q_3\}$$
$$P_{1\,\text{succ}} = P_{3\,\text{succ}}$$

and

$$P_{2\,\text{prec}} = P_{3\,\text{prec}} \setminus \max(P_{3\,\text{prec}})$$
$$Q_2 = Q_3$$
$$P_{2\,\text{succ}} = \{\max(P_{3\,\text{prec}}), P_{3\,\text{succ}}\}.$$

With (P_1, Q_1) and (P_2, Q_2), go to Step 2.

The third modification deals with terms that correspond to a single path instead of a pair. Let S be a path such that $\Sigma S_{0,n} = \varepsilon_j - \varepsilon_i$ and $\Sigma_{0,n+1} = 2\varepsilon_j$. Construct the pair (P, Q) such that

$$P_{\text{prec}} = \{...S_{n+3}, S_{n+2}\}$$
$$Q = \{S_{n+1}, S_n\}$$
$$P_{\text{succ}} = \{S_n, S_{n-1}, ...\}$$

Apply Step 2 of the algorithm to (P, Q), The output then corresponds to the term in the sum that cancels the term corresponding to S.

Chapter 8

The Matrix Coefficient $\mathcal{S}(\rho(g) \cdot \phi_{U,\psi})$

We present some results about the matrix coefficient $\mathcal{S}(\rho(g) \cdot \phi_{U,\psi})$. This coefficient on the (U, ψ) side is analogous to $\mathcal{W}(\rho(g) \cdot \phi_K)$ on the K-fixed side in that it shares a similar evaluation at torus elements. A full treatment should appear in a separate document.

Recall that $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;1})$ is a monomial in $\mathbb{C}(X^\vee)$; this was the main result of chapter 7. We next show that $\mathcal{S}(\rho(a) \cdot \phi_{U,\psi;1})$ and $\mathcal{S}(\rho(a) \cdot \phi_{K;1})$ are also a monomials. We compare the calculation of $\mathcal{S}(\rho(a) \cdot \phi_{K;1})$ to its analogue $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;1})$.

Theorem 8.0.5 *Let $a = a_0\pi^\lambda$ with $a_0 \in T(\mathfrak{o})$ and $\lambda \in X^\vee$ dominant. Then*

$$\int_K \phi_{U,\psi;1}(ka)\, dk = (1 - q^{-1})^d \delta_B^{-1/2} \chi(a) \prod_{\gamma \in R^+} -q^{-1}\pi^{-\gamma^\vee}$$

and

$$\int_K \phi_{K;1}(ka)\, dk = (1 - q^{-1})^d \delta_B^{-1/2} \chi(a) q^{-|R^+|}$$

where d is the dimension of $X^\vee$.

Proof The supports of $\phi_{U,\psi;1}$ and $\phi_{K;1}$ are BI_{ht} and BI, respectively; in addition, $BI_{\mathrm{ht}} \subset BI$. We thus determine when $k\pi^\lambda \in BI$. We have, from the Iwahori decomposition of K,

$$K = \bigcup_w B(\mathfrak{o}) n_w (w^{-1}(U^-) \cap U^- \cap I)(w^{-1}(U^-) \cap U \cap I).$$

The dominance of λ implies

$$\pi^{-\lambda}\big(w^{-1}(U^-) \cap U \cap I)\pi^\lambda \in I.$$

Therefore, after conjugating by π^λ, we are left to determine when

$$\big(n_w(w^{-1}(U^-) \cap U^-)\big) \cap BI$$

is non-empty. Lemma 8.0.6 says that this set is non-empty only if $w = 1$. The integral for $\mathcal{S}(\rho(a) \cdot \phi_{U,\psi;1})$ now follows from the result for $\mathcal{W}(\rho(a) \cdot \phi_{U,\psi;1})$ of chapter 7. The result for $\mathcal{S}(\rho(a) \cdot \phi_{K;1})$, however, now follows simply because the measure of BI is $(1 - q^{-1})^d q^{-|R^+|}$. In this case of $\phi_{K,1}$, there is only one non-zero value of χ as k ranges over K; contrast to the case of $\phi_{U,\psi;1}$, where χ takes on infinitely many values as k ranges over K, just as when u ranges over U^- in chapter 7.

Lemma 8.0.6 *Suppose that the set*

$$\big(n_w(w^{-1}(U^-) \cap U^-)\big) \cap Bn_vI$$

is non-empty for some $v, w \in W$. Then $\ell(v) \geq \ell(w)$.

Proof First let

$$s_n ... s_2 s_1 = w^{-1} w_\ell$$

be a reduced word for $w^{-1}w_\ell$, where s_i are (possibly non-distinct) simple reflections with respect to R^+. Then set

$$y_i = s_i ... s_2 s_1.$$

Apply the Iwahori decomposition for G to an element in the set

$$n_w(w^{-1}(U^-) \cap U^-).$$

This Iwahori decomposition may be achieved using the steps in the explicit Bruhat decomposition of Section 7.2. It follows that if such an element is in Bn_vI, then v is of the form

$$v = w \prod_{i=n}^{1} (y_i y_{i-1}^{-1})^{\epsilon_i}$$

for some choice of $\epsilon_i \in \{0, 1\}$. This implies that $\ell(v) \geq \ell(w) + k$, where k is the number of ϵ_i that equal 1.

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