# The Affine Yangian of $\mathfrak{gl}_1$ and the Infinitesimal Cherednik Algebras

by

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## Abstract

In the first part of this thesis, we obtain some new results about infinitesimal Cherednik algebras. They have been introduced by Etingof-Gan-Ginzburg in [EGG] as appropriate analogues of the classical Cherednik algebras, corresponding to the reductive groups, rather than the finite ones. Our main result is the realization of those algebras as particular finite W-algebras of associated semisimple Lie algebras with nilpotent 1-block elements. To achieve this, we prove its Poisson counterpart first, which identifies the Poisson infinitesimal Cherednik algebras introduced in [DT] with the Poisson algebras of regular functions on the corresponding Slodowy slices. As a consequence, we obtain some new results about those algebras. We also generalize the classification results of [EGG] from the cases $\mathrm{GL}_n$ and $\mathrm{Sp}_{2n}$ to $\mathrm{SO}_n$.

In the second part of the thesis, we discuss the loop realization of the affine Yangian of $\mathfrak{gl}_1$. Similar objects were recently considered in the work of Maulik-Okounkov on the quantum cohomology theory, see [MO]. We present a purely algebraic realization of these algebras by generators and relations. We discuss some families of their representations. A similarity with the representation theory of the quantum toroidal algebra of $\mathfrak{gl}_1$ is explained by adapting a recent result of Gautam-Toledano Laredo, see [GTL], to the local setting. We also discuss some aspects of those two algebras such as the degeneration isomorphism, a shuffle presentation, and a geometric construction of the Whittaker vectors.

Thesis Supervisor: Pavel Etingof
Title: Professor of Mathematics

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# Contents

# Chapter 1

# Introduction

## 1.1 Continuous Hecke algebras

### 1.1.1 Algebraic distributions

For an affine scheme $X$ of finite type, let $\mathcal{O}(X)$ be the algebra of regular functions on $X$ and $\mathcal{O}(X)^*$ be the dual space, called the space of *algebraic distributions*. Note that $\mathcal{O}(X)^*$ is a module over $\mathcal{O}(X)$: for $f \in \mathcal{O}(X)$, $\mu \in \mathcal{O}(X)^*$ we can define $f \cdot \mu$ by $\langle f \cdot \mu, g\rangle = \langle \mu, fg\rangle$ for all $g \in \mathcal{O}(X)$. For a closed subscheme $Z \subset X$, we say that an algebraic distribution $\mu$ on $X$ is supported on the scheme $Z$ if $\mu$ annihilates the defining ideal $I(Z)$ of $Z$. If $Z$ is reduced, we say that $\mu \in \mathcal{O}(X)^*$ is *set-theoretically supported* on the set $Z$ if $\mu$ annihilates some power of $I(Z)$.

Let $G$ be a reductive algebraic group and $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional algebraic representation of $G$. First note that $\mathcal{O}(G)^*$ is an algebra with respect to the convolution. Moreover, $\delta_{1_G}$ is the unit of this algebra. Next, we consider the semi-direct product $\mathcal{O}(G)^* \ltimes TV$, that is, the algebra generated by $\mu \in \mathcal{O}(G)^*$ and $x \in V$ with the relations

$$x \cdot \mu = \sum_i (v_i^*, gx)\mu \cdot v_i \quad \text{for all } x \in V,\ \mu \in \mathcal{O}(G)^*,$$

where $\{v_i\}$ is a basis of $V$ and $\{v_i^*\}$ the dual basis of $V^*$, while $(v_i^*, gx)\mu$ denotes the product of the regular function $(v_i^*, gx)$ and the distribution $\mu$.

We will denote the vector space of length $N$ columns by $V_N$, so that there are natural actions of $\mathrm{GL}_N, \mathrm{Sp}_N, \mathrm{SO}_N$ on $V_N$. Let us also denote the action of $g \in G$ on $x \in V$ by $x^g$.

### 1.1.2 Continuous Hecke algebras

We recall the definition of the continuous Hecke algebras of $(G, V)$ following [EGG].

Given a reductive algebraic group $G$, its finite dimensional algebraic representation $V$ and a skew-symmetric $G$-equivariant $\mathbb{C}$-linear map $\kappa : V \times V \to \mathcal{O}(G)^*$, we set

$$\mathcal{H}_\kappa(G, V) := \mathcal{O}(G)^* \ltimes TV/([x, y] - \kappa(x, y)|\ x, y \in V).$$

Consider an algebra filtration on $\mathcal{H}_\kappa(G, V)$ by setting $\deg(V) = 1,\ \deg(\mathcal{O}(G)^*) = 0$.

**Definition 1.1.1.** [EGG] We say that $\mathcal{H}_\kappa(G, V)$ *satisfies the PBW property* if the natural surjective map $\mathcal{O}(G)^* \ltimes SV \twoheadrightarrow \mathrm{gr}\ \mathcal{H}_\kappa(G, V)$ is an isomorphism, where $SV$ denotes the symmetric algebra of $V$. We call these $\mathcal{H}_\kappa(G, V)$ the *continuous Hecke algebras* of $(G, V)$.

According to [EGG, Theorem 2.4], $\mathcal{H}_\kappa(G, V)$ satisfies the PBW property if and only if $\kappa$ satisfies the *Jacobi identity*:

$$(z - z^g)\kappa(x, y) + (y - y^g)\kappa(z, x) + (x - x^g)\kappa(y, z) = 0 \text{ for all } x, y, z \in V. \quad (\dagger)$$

Define the closed subscheme $\Phi \subset G$ by the equation $\wedge^3(1 - g\ |_V) = 0$. The set of closed points of $\Phi$ is the set $S = \{g \in G : \mathrm{rk}(1 - g\ |_V) \leq 2\}$. We have:

**Proposition 1.1.1.** *[EGG, Proposition 2.8] If the PBW property holds for $\mathcal{H}_\kappa(G, V)$, then $\kappa(x, y)$ is supported on the scheme $\Phi$ for all $x, y \in V$.*

The classification of all $\kappa$ satisfying ($\dagger$) was obtained in [EGG] for the following two cases:

• for the pairs $(G, \mathfrak{h} \oplus \mathfrak{h}^*)$ with $\mathfrak{h}$ being an irreducible faithful $G$-representation of real or complex type (see [EGG, Theorem 3.5]),

• for the pair $(\mathrm{Sp}_{2n}, V_{2n})$ (see [EGG, Theorem 3.14]).

In general, such a classification is not known at the moment. However, a particular family of those was established in [EGG, Theorem 2.13]:

**Proposition 1.1.2.** *For any* $\tau \in (\mathcal{O}(\mathrm{Ker}\,\rho)^* \otimes \wedge^2 V^*)^G$ *and* $v \in (\mathcal{O}(\Phi)^* \otimes \wedge^2 V^*)^G$, *the pairing* $\kappa_{\tau,v}(x,y) := \tau(x,y) + v((1-g)x, (1-g)y)$ *satisfies the Jacobi identity.*

## 1.2 Infinitesimal Cherednik algebras

### 1.2.1 Infinitesimal Cherednik algebras

For any triple $(\mathfrak{g}, V, \zeta)$ of a Lie algebra $\mathfrak{g}$, its representation $V$ and a $\mathfrak{g}$-equivariant $\mathbb{C}$-bilinear pairing $\zeta : \wedge^2 V \to U(\mathfrak{g})$, we define

$$H_\zeta(\mathfrak{g}, V) := U(\mathfrak{g}) \ltimes TV/([x,y] - \zeta(x,y)|\ x, y \in V).$$

Endow this algebra with a filtration by setting $\deg(V) = 1,\ \deg(\mathfrak{g}) = 0$.

**Definition 1.2.1.** [EGG, Section 4] We call this algebra the *infinitesimal Hecke/Cherednik algebra* of $(\mathfrak{g}, V)$ if it satisfies the *PBW property*, that is, the natural surjective map $U(\mathfrak{g}) \ltimes SV \twoheadrightarrow \mathrm{gr}\, H_\zeta(\mathfrak{g}, V)$ is an isomorphism.

Any such algebra gives rise to a continuous Hecke algebra

$$\mathcal{H}_\zeta(G, V) := \mathcal{O}(G)^* \otimes_{U(\mathfrak{g})} H_\zeta(\mathfrak{g}, V),$$

where $U(\mathfrak{g})$ is identified with the subalgebra $\mathcal{O}(G)^*_{1_G} \subset \mathcal{O}(G)^*$, consisting of all algebraic distributions set-theoretically supported at $1_G \in G$.

In particular, having a full classification of the continuous Hecke algebras of type $(G, V)$ yields a corresponding classification for the infinitesimal Hecke algebras of $(\mathrm{Lie}(G), V)$. The latter classification was determined explicitly for the cases of $(\mathfrak{g}, V) = (\mathfrak{gl}_n, V_n \oplus V_n^*), (\mathfrak{sp}_{2n}, V_{2n})$ in [EGG, Theorem 4.2].

### 1.2.2 Classifications for $\mathfrak{gl}_n$ and $\mathfrak{sp}_{2n}$

For a pair $(\mathfrak{gl}_n, V_n \oplus V_n^*)$, we have the following result (see [EGG, Theorem 4.2]):

**Proposition 1.2.1.** *The PBW property holds for $H_\zeta(\mathfrak{gl}_n)$ if and only if*

$$\zeta(y,y') = 0,\ \zeta(x,x') = 0,\ \zeta(y,x) = \sum_{j=0}^{k} \zeta_j r_j(y,x), \forall\ y,y' \in V_n,\ x,x' \in V_n^*,$$

*for some nonnegative integer $k$ and $\zeta_j \in \mathbb{C}$, where $r_j(y,x) \in U(\mathfrak{gl}_n)$ is the symmetrization of $\alpha_j(y,x) \in S(\mathfrak{gl}_n) \simeq \mathbb{C}[\mathfrak{gl}_n]$ and $\alpha_j(y,x)$ is defined via the expansion*

$$(x, (1-\tau A)^{-1} y) \det(1-\tau A)^{-1} = \sum_{j\geq 0} \alpha_j(y,x)(A)\tau^j, \quad A \in \mathfrak{gl}_n.$$

**Definition 1.2.2.** Define the *length* of such $\zeta$ by $l(\zeta) := \min\{m \in \mathbb{Z}_{\geq -1} | \ \zeta_{\geq m+1} = 0\}$.

*Example* 1.2.1. [EGG, Example 4.7] If $l(\zeta) = 1$ then $H_\zeta(\mathfrak{gl}_n) \cong U(\mathfrak{sl}_{n+1})$. Thus, for an arbitrary $\zeta$, we can regard $H_\zeta(\mathfrak{gl}_n)$ as a deformation of $U(\mathfrak{sl}_{n+1})$.

Let us also recall a similar classification for the pair $(\mathfrak{sp}_{2n}, V_{2n})$. Here we assume that $\mathfrak{sp}_{2n}$ is defined with respect to a symplectic form $\omega$ on $V_{2n}$.

**Proposition 1.2.2.** *The PBW property holds for $H_\zeta(\mathfrak{sp}_{2n})$ if and only if*

$$\zeta(x,y) = \sum_{j=0}^{k} \zeta_j r_{2j}(x,y)$$

*for some nonnegative integer $k$ and $\zeta_j \in \mathbb{C}$, where $r_{2j}(x,y) \in U(\mathfrak{sp}_{2n})$ is the symmetrization of $\beta_{2j}(x,y) \in S(\mathfrak{sp}_{2n}) \simeq \mathbb{C}[\mathfrak{sp}_{2n}]$ and $\beta_{2j}(x,y)$ is defined via the expansion*

$$\omega(x, (1-\tau^2 A^2)^{-1} y) \det(1-\tau A)^{-1} = \sum_{j\geq 0} \beta_{2j}(x,y)(A)\tau^{2j}, \quad A \in \mathfrak{sp}_{2n}.$$

**Definition 1.2.3.** Define the *length* of such $\zeta$ by $l(\zeta) := \min\{m \in \mathbb{Z}_{\geq -1} | \ \zeta_{\geq m+1} = 0\}$.

*Example* 1.2.2. [EGG, Example 4.11] For $\zeta_0 \neq 0$ we have $H_{\zeta_0 r_0}(\mathfrak{sp}_{2n}) \cong U(\mathfrak{sp}_{2n}) \ltimes W_n$, where $W_n$ is the $n$-th Weyl algebra. Thus, $H_\zeta(\mathfrak{sp}_{2n})$ can be regarded as a deformation of $U(\mathfrak{sp}_{2n}) \ltimes W_n$.

## 1.3 The quantum toroidal and the affine Yangian of $\mathfrak{gl}_1$

The quantum toroidal algebra of $\mathfrak{gl}_1$ appeared independently in [SV, FT1] as a certain algebra, which naturally acts on the equivariant K-theory of the Hilbert scheme of points on a plane. It also has a connection to the Hall algebra of an elliptic curve and to the spherical DAHA as established in [SV, S]. Moreover, it admits an interesting realization via the shuffle algebra, which was recently completed in [N1].

On the other hand, there has been a purely algebraic activity around those algebras for the last five years, initiated by Feigin et al. In papers [FFJMM1, FFJMM2], the authors constructed several families of representations by using the formal comultiplication and the aforementioned geometric representations.

Independently, the notion of an affine Yangian of $\mathfrak{gl}_1$ was introduced in [MO]. However, the authors were more interested in the particular family of representations of this algebra (which arise geometrically), rather then in the algebra itself.

## 1.4 Organization of the thesis

• *Chapter 2.* The main results of this chapter are as follows:

∘ Computation of the Shapovalov determinant for $H_\varsigma(\mathfrak{gl}_n)$. This provides a simple criteria for the irreducibility of Verma modules.

∘ Computation of the simplest central element, called the *Casimir* element. We also obtain the formula for its action on the Verma modules.

∘ Classification of finite dimensional representations of $H_\varsigma(\mathfrak{gl}_n)$. Computation of their characters.

∘ Computation of the Poisson center of the Poisson infinitesimal Cherednik algebras.

• *Chapter 3.* The main results of this chapter are as follows:

∘ Identification of the universal infinitesimal Cherednik algebras $H_m(\mathfrak{gl}_n)$, $H_m(\mathfrak{sp}_{2n})$ with the finite $W$-algebras of $\mathfrak{sl}_{n+m}, \mathfrak{sp}_{2n+2m}$ corresponding to 1-block nilpotent elements. We also establish a Poisson analogue of this result.

∘ Some new properties of the infinitesimal Cherednik algebras.

∘ Obtaining some results on the completions of the infinitesimal Cherednik algebras. In particular, we immediately get a proof of the result stated in [Tik3].

• *Chapter 4.* The main results of this chapter are as follows:

∘ Classification results on the continuous Hecke algebras and infinitesimal Cherednik algebras of types $\mathrm{SO}_n$ and $\mathfrak{so}_n$.

∘ Generalization of the results from Chapter 3 to the case of $H_\varsigma(\mathfrak{so}_n)$.

• *Chapter 5.* The main results of this chapter are as follows:

∘ An explicit presentation of the affine Yangian of $\mathfrak{gl}_1$.

∘ Geometric construction of representations for the quantum toroidal and the affine Yangian of $\mathfrak{gl}_1$ via the Gieseker moduli spaces.

∘ Construction of some series of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations. We also relate them to the representations, which arise geometrically.

∘ Description of the limits of both algebras in interest as one of the parameters trivializes.

∘ Construction of a homomorphism $\Upsilon : \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. This is analogous to the relation between a quantum loop algebra $U_q(L\mathfrak{g})$ and a Yangian $Y_h(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$, discovered in [GTL].

∘ Discussion on the shuffle presentation of those algebras, emphasizing two alternative descriptions of their commutative subalgebras.

∘ Establishing an explicit connection between representations from [FFJMM1] and [FHHSY] by introducing a "horizontal realization" of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$.

• *Appendix.* We outline some further generalizations.

# Chapter 2

# Infinitesimal Cherednik algebras

This chapter is based on [DT].

## 2.1 Representations of $H_\zeta(\mathfrak{gl}_n)$

### 2.1.1 Basic notations

Similarly to the representation theory of $\mathfrak{sl}_{n+1}$, we define the Verma module of $H_\zeta(\mathfrak{gl}_n)$ as

$$M(\lambda) = H_\zeta(\mathfrak{gl}_n)/\{H_\zeta(\mathfrak{gl}_n)\cdot \mathfrak{n}^+ + H_\zeta(\mathfrak{gl}_n)(h-\lambda(h))\}_{h\in\mathfrak{h}},$$

where the set of positive root elements $\mathfrak{n}^+$ is spanned by the positive root elements of $\mathfrak{gl}_n$ (i.e., matrix units $e_{ij}$ with $i<j$) and elements of $V$; the set of negative root elements $\mathfrak{n}^-$ is spanned by the negative root elements of $\mathfrak{gl}_n$ (i.e., matrix units $e_{ij}$ with $i>j$) and elements of $V^*$; and the Cartan subalgebra $\mathfrak{h}$ is spanned by diagonal matrices. The highest weight, $\lambda$, is an element of $\mathfrak{h}^*$, and $v_\lambda$ is the corresponding highest-weight vector.

Let us denote the set of positive roots by $\Delta^+$, so that $\Delta^+ = \{e_{ii}^* - e_{jj}^*\}\cup\{e_{kk}^*\}$ for $1\le i<j\le n$, $1\le k\le n$. To denote the positive roots of $\mathfrak{gl}_n$, we use $\Delta^+(\mathfrak{gl}_n)$, and to denote the weights of $y_i$, we use $\Delta^+(V)$. We define $\rho = \frac{1}{2}\sum_{\lambda\in\Delta^+(\mathfrak{gl}_n)}\lambda = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}\right)$, a *quasiroot* to be an integral multiple of an element in $\Delta^+$, and $Q^+$ to be the set of linear combinations of positive roots with nonnegative integer

coefficients. Finally, $U(\mathfrak{n}^-)_\nu$ denotes the $-\nu$ weight-space of $U(\mathfrak{n}^-)$, where $\nu \in Q^+$.

### 2.1.2 The Shapovalov Form

As in the classical representation theory of Lie algebras, the Shapovalov form can be used to investigate the basic structure of Verma modules. Similarly to the classical case, $M(\lambda)$ possesses a maximal proper submodule $\overline{M}(\lambda)$ and has a unique irreducible quotient $L(\lambda) = M(\lambda)/\overline{M}(\lambda)$. Define the Harish-Chandra projection $\mathrm{HC} : H_\zeta(\mathfrak{gl}_n) \to S(\mathfrak{h})$ with respect to the decomposition $H_\zeta(\mathfrak{gl}_n) = (H_\zeta(\mathfrak{gl}_n)\mathfrak{n}^+ + \mathfrak{n}^- H_\zeta(\mathfrak{gl}_n)) \oplus U(\mathfrak{h})$, and let $\sigma : H_\zeta(\mathfrak{gl}_n) \to H_\zeta(\mathfrak{gl}_n)$ be the anti-involution that takes $y_i$ to $x_i$ and $e_{ij}$ to $e_{ji}$.

**Definition 2.1.1.** The *Shapovalov form* $S : H_\zeta(\mathfrak{gl}_n) \times H_\zeta(\mathfrak{gl}_n) \to U(\mathfrak{h}) \cong S(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$ is a bilinear form given by $S(a,b) = \mathrm{HC}(\sigma(a)b)$. The bilinear form $S(\lambda)$ on the Verma module $M(\lambda)$ is defined by $S(\lambda)(u_1 v_\lambda, u_2 v_\lambda) = S(u_1, u_2)(\lambda)$, for $u_1, u_2 \in U(\mathfrak{n}^-)$.

This definition is motivated by the following two properties (compare with [KK]):

**Proposition 2.1.1.** 1. $S(U(\mathfrak{n}^-)_\mu, U(\mathfrak{n}^-)_\nu) = 0$ for $\mu \neq \nu$,
2. $\overline{M}(\lambda) = \ker S(\lambda)$.

Statement 1 of Proposition 2.1.1 reduces $S$ to its restriction to $U(\mathfrak{n}^-)_\nu \times U(\mathfrak{n}^-)_\nu$, which we will denote as $S_\nu$. Statement 2 of Proposition 2.1.1 gives a necessary and sufficient condition for the Verma module $M(\lambda)$ to be irreducible, namely that for any $\nu \in Q^+$, the bilinear form $S_\nu(\lambda)$ is nondegenerate, or equivalently, that $\det S_\nu(\lambda) \neq 0$, where the determinant is computed in any basis; note that this condition is independent of basis. For convenience, we choose the basis $\{f^{\mathbf{m}}\}$, where $\mathbf{m}$ runs over all partitions of $\nu$ into a sum of positive roots and $f^{\mathbf{m}} = \prod f_\alpha^{\mathbf{m}_\alpha}$ with $f_\alpha \in \mathfrak{n}^-$ of weight $-\alpha$. We will use the notation $a \vdash b$ to mean that $(a_1, \ldots, a_n)$ is a partition of $b$ into a sum of $n$ nonnegative integers when $b \in \mathbb{N}$, and $\mathbf{m} \vdash \nu$ to mean that $\mathbf{m}$ is a partition of $\nu$ into a sum of elements of $\Delta^+$ when $\nu \in Q^+$. Then, the basis we will work with is $\{f^{\mathbf{m}}\}_{\mathbf{m} \vdash \nu}$.

Now, we present a formula for the determinant of the Shapovalov form for $H_\zeta(\mathfrak{gl}_n)$ generalizing the classical result presented in [KK]. This formula uses the following result proven in Section 2.1.5: for a deformation $\zeta = \zeta_0 r_0 + \zeta_1 r_1 + \cdots + \zeta_m r_m$, the central element $t_1'$ (introduced in Section 2.1.3) acts on the Verma module $M(\lambda)$ by a constant $P(\lambda) = \sum_{j=0}^{m+1} w_j H_j(\lambda+\rho)$, where $H_j(\lambda) = \sum_{p\vdash j} \prod_{1\le i\le n} \lambda_i^{p_i}$ are the complete symmetric functions (we take $H_0(\lambda) = 1$) and $w_j(\zeta_0, \ldots, \zeta_j)$ are linearly independent linear functions on $\zeta_k$.

Define the Kostant partition function $\tau$ as $\tau(\nu) = \dim U(\mathfrak{n}^-)_\nu$. Then:

**Theorem 2.1.1.** *Up to a nonzero constant factor, the Shapovalov determinant computed in the basis* $\{f^{\mathbf{m}}\}_{\mathbf{m}\vdash\nu}$ *is given by*

$$\det S_\nu(\lambda) = \prod_{\alpha\in\Delta^+(V)} \prod_{k=1}^{\infty} (P(\lambda) - P(\lambda - k\,\alpha))^{\tau(\nu - k\,\alpha)} \times \prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)} \prod_{k=1}^{\infty} ((\lambda+\rho, \alpha) - k)^{\tau(\nu-k\,\alpha)}.$$

*Remark* 2.1.1. For $\zeta = \zeta_0 r_0 + \zeta_1 r_1, \zeta_1 \neq 0$, we get the classical formula from [KK].

*Proof.*

The proof of this theorem is quite similar to the classical case with a few technical details and differences that will be explained below. We begin with the following lemma, which shows that irreducible factors of $\det S_\nu(\lambda)$ must divide $P(\lambda) - P(\lambda - \mu)$ for some $\mu \in Q^+$.

**Lemma 2.1.1.** *Suppose* $\det S_\nu(\lambda) = 0$. *Then, there exists* $\mu \in Q^+\backslash\{0\}$ *such that* $P(\lambda) - P(\lambda - \mu) = 0$.

*Proof.*

Note that $\det S_\nu(\lambda) = 0$ implies that the Verma module $M(\lambda)$ has a critical vector (a vector on which all elements of $\mathfrak{n}^+$ act by 0) of weight $\lambda - \mu$ for some $\mu \in Q^+$ satisfying $0 < \mu < \nu$. Thus, $M(\lambda - \mu)$ is embedded in $M(\lambda)$. Since $t_1'$ acts by constants on both $M(\lambda)$ and $M(\lambda - \mu)$, which can be considered as a submodule of $M(\lambda)$, we get $P(\lambda) = P(\lambda - \mu)$. □

The top term of the Shapovalov determinant $\det S_\nu(\lambda)$ in the basis $\{f^{\mathbf{m}}\}_{\mathbf{m}\vdash\nu}$ comes from the product of diagonal elements, that is, $\prod_{\mathbf{m}\vdash\nu} \prod [\sigma(f_\alpha), f_\alpha]^{\mathbf{m}_\alpha}(\lambda)$. The top term of $[e_{ij}, e_{ji}](\lambda)$ for $i < j$ is $\lambda_i - \lambda_j = (\lambda, \alpha)$ where $\alpha$ is the weight of $e_{ij}$. The following lemma gives the top term of $[y_j, x_j](\lambda)$:

**Lemma 2.1.2.** *The highest term of $[y_j, x_j](\lambda)$ for $\zeta = \zeta_0 r_0 + \cdots + \zeta_m r_m$ equals $\zeta_m \sum_{\mathbf{p}} (\mathbf{p}_j + 1) \prod \lambda_i^{\mathbf{p}_i}$, where the sum is over all partitions $\mathbf{p}$ of $m$ into $n$ summands.*

*Proof.*

From [EGG, Theorem 4.2], we know that the top term of $[y_j, x_j]$ for $\zeta = \zeta_0 r_0 + \zeta_1 r_1 + \cdots + r_m$ is given by the coefficient of $\tau^m$ in $\det(1 - \tau A)^{-1}(x_j, (1 - \tau A)^{-1} y_j)$. Because the set of diagonalizable matrices is dense in $\mathfrak{gl}_n$, we can assume $A$ is a diagonal matrix $A = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ so that
$\det(1 - \tau A)^{-1} = \prod \frac{1}{1 - \tau \lambda_i} = \sum_k \sum_{p \vdash k} \prod_i \lambda_i^{\mathbf{p}_i} \tau^k$ and $x_j (1 - \tau A)^{-1} y_j = 1 + \sum_{k>0} \lambda_j^k \tau^k$. Multiplying these series gives the statement in the lemma. □

Thus, we see that the top term of the determinant computed in the basis $\{f^{\mathbf{m}}\}_{\mathbf{m} \vdash \nu}$, up to a scalar multiple, is of the form

$$\left( \prod_{\alpha \in \Delta^+(\mathfrak{gl}_n)} (\lambda, \alpha)^{\sum_{\mathbf{m}} \mathbf{m}_\alpha} \right) \left( \prod_{\alpha = \mathrm{wt}(y_j) \in \Delta^+(V)} \left( \sum_{\mathbf{p}} (\mathbf{p}_j + 1) \prod \lambda_i^{\mathbf{p}_i} \right)^{\sum_{\mathbf{m}} \mathbf{m}_\alpha} \right).$$

Since $\tau(\mu)$ is the number of partitions of a weight $\mu$, the sum $\sum_{\mathbf{m}} \mathbf{m}_\alpha$ over all partitions $\mathbf{m}$ of $\nu$ with $\alpha$ fixed must equal $\sum_{k=1}^{\infty} \tau(\nu - k\alpha)$, so the expression above simplifies to

$$\left( \prod_{\alpha \in \Delta^+(\mathfrak{gl}_n)} \prod_{k=1}^{\infty} (\lambda, \alpha)^{\tau(\nu - k\alpha)} \right) \left( \prod_{\alpha = \mathrm{wt}(y_j) \in \Delta^+(V)} \prod_{k=1}^{\infty} \left( \sum_{\mathbf{p} \vdash m} (\mathbf{p}_j + 1) \prod \lambda_i^{\mathbf{p}_i} \right)^{\tau(\nu - k\alpha)} \right).$$

This highest term comes from the product of the highest terms of factors of $P(\lambda) - P(\lambda - \mu)$ for various $\mu \in Q^+$.

**Lemma 2.1.3.** 1. For all $\mu \neq k\alpha$, $\alpha \in \Delta^+(\mathfrak{gl}_n)$, $P(\lambda) - P(\lambda - \mu)$ is irreducible as a polynomial in $\lambda$.
2. For $\mu = k\alpha$, $\alpha \in \Delta^+(\mathfrak{gl}_n)$, $\frac{P(\lambda) - P(\lambda - k\alpha)}{(\lambda + \rho, \alpha) - k}$ is irreducible.

If Lemma 2.1.3 is true, then all $\mu$ contributing to the above product must be quasiroots: if $\mu \neq k\alpha$ for some $\alpha \in \Delta^+(\mathfrak{gl}_n)$, the highest term of the irreducible polynomial $P(\lambda) - P(\lambda - \mu)$, $\sum_{p \vdash m} \sum_j \mu_j(\mathbf{p}_j + 1) \prod \lambda_i^{\mathbf{p}_i}$, does not match any factor in the highest term of the Shapovalov determinant unless $\mu$ is a $V$-quasiroot. Moreover, if $\mu = k\alpha$ for $\alpha \in \Delta^+(\mathfrak{gl}_n)$, since $\frac{P(\lambda)-P(\lambda-k\alpha)}{(\lambda+\rho,\alpha)-k}$ is irreducible for $\alpha \in \Delta^+(\mathfrak{gl}_n)$, comparison with the highest term of the determinant shows that only the linear factor $(\lambda+\rho, \alpha) - k$ of $P(\lambda) - P(\lambda - k\alpha)$ appears in the Shapovalov determinant.

*Proof.*

We will prove that $P(\lambda) - P(\lambda - \mu)$ is irreducible for $\mu \neq k\alpha$ ($\alpha \in \Delta^+(\mathfrak{gl}_n)$); similar arguments will show that $\frac{P(\lambda)-P(\lambda-k\alpha)}{(\lambda+\rho,\alpha)-k}$ is irreducible for any $\alpha \in \Delta^+(\mathfrak{gl}_n)$, $k \in \mathbb{N}$.

Consider the parameters $w_i$ as formal variables. Then, we have $P(\lambda) - P(\lambda - \mu) = \sum_{i \geq 0} w_i(H_i(\lambda + \rho) - H_i(\lambda + \rho - \mu))$. We can absorb the $\rho$ vector into the $\lambda$ vector. For this polynomial to be reducible in $w_i$ and $\lambda_j$, the coefficient of $w_1$ should be zero: $H_1(\lambda) - H_1(\lambda - \mu) = H_1(\mu) = 0$. Also, since the coefficient of $w_2$ is linear in $\lambda_j$, it must divide the coefficients of every other $w_i$. In particular, the highest term of $H_2(\lambda) - H_2(\lambda - \mu)$ must divide that of $H_3(\lambda) - H_3(\lambda - \mu)$. The highest term of $H_2(\lambda) - H_2(\lambda - \mu)$ is $\sum_i \lambda_i(\mu_i + \sum_j \mu_j) = (\lambda, \mu)$ and the highest term of $H_3(\lambda) - H_3(\lambda - \mu)$ is given by $H_3'(\lambda)(\mu)$, the evaluation of the gradient $H_3'(\lambda)$ at $\mu$. Since this term is quadratic and is divisible by $(\lambda, \mu)$, we can write $H_3'(\lambda)(\mu) = (\lambda, \mu)(\lambda, \xi)$ for some $\xi \in \mathfrak{h}^*$. Now, let us match coefficients of $\lambda_i \lambda_j$ for $i \neq j$ and of $\lambda_i^2$ on both sides of the equation. By doing so (and using the fact that $\sum \mu_i = 0$), we obtain $\mu_i \xi_j + \mu_j \xi_i = \mu_i + \mu_j$ and $\mu_i \xi_i = 2\mu_i$. Since $\mu_1 + \cdots + \mu_n = 0$ and $\mu \neq 0$, at least two of $\mu_i$ are nonzero, say $\mu_{i_1}$ and $\mu_{i_2}$. From the two equations, we obtain $\mu_{i_1} + \mu_{i_2} = 0$. If $\mu_{i_3} \neq 0$, then by similar arguments, $\mu_{i_1} + \mu_{i_3} = \mu_{i_2} + \mu_{i_3} = \mu_{i_1} + \mu_{i_2} = 0$, which is impossible since $\mu_{i_1}, \mu_{i_2}, \mu_{i_3} \neq 0$. Thus, $P(\lambda) - P(\lambda - \mu)$ is reducible only if exactly two of the $\mu_i$ are nonzero and opposite to each other; that is, $\mu = k\alpha$ for $\alpha \in \Delta^+(\mathfrak{gl}_n)$. □

To prove that the power of each factor in the determinant formula of Theorem 2.1.1 is correct, we use an argument involving the Jantzen filtration, which we define as in

[KK, page 101] (for our purposes, we switch $U(\mathfrak{g})$ to $H_\zeta(\mathfrak{gl}_n)$). The Jantzen filtration is a technique to track the order of zero of a bilinear form's determinant. Instead of working over the complex numbers, we consider the ring of localized polynomials $\mathbb{C}\langle t\rangle = \{\frac{p(t)}{q(t)} \mid p(t), q(t) \in \mathbb{C}[t], q(0) \neq 0\}$. A word-to-word generalization of [KK, Lemma 3.3], proves that the power of $P(\lambda) - P(\lambda - k\alpha)$ for $\alpha \in \Delta^+(V)$ and of $(\lambda + \rho, \alpha) - k$ for $\alpha \in \Delta^+(\mathfrak{gl}_n)$ is given by $\tau(\nu - k\alpha)$, completing the proof of Theorem 2.1.1. □

### 2.1.3 The Casimir Element of $H_\zeta(\mathfrak{gl}_n)$

Let $\Omega_1, \Omega_2, \Omega_3, ..., \Omega_n \in S(\mathfrak{gl}_n^*)$ (which can be identified as elements of $S(\mathfrak{gl}_n)$ under the trace-map) be defined by the power series $\det(t\mathrm{Id} - X) = \sum_{j=0}^{n} (-1)^j t^{n-j} \Omega_j(X)$, and let $\beta_i$ be the image of $\Omega_i$ under the symmetrization map from $S(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$. The center of $U(\mathfrak{gl}_n)$ is a polynomial algebra generated by these $\beta_i$. Define $t_i = \sum_j x_j[\beta_i, y_j]$. According to [T1, Theorems 1.1, 2.1], the center of $H_0(\mathfrak{gl}_n)$ is a polynomial algebra in $\{t_i\}_{1\leq i \leq n}$, and there exist unique (up to a constant) $c_i \in \mathfrak{z}(U(\mathfrak{gl}_n))$ such that the center of $H_\zeta(\mathfrak{gl}_n)$ is a polynomial algebra in $t_i' = t_i + c_i$, $1 \leq i \leq n$.

**Definition 2.1.2.** The *Casimir element* of $H_\zeta(\mathfrak{gl}_n)$ is defined (up to a constant) as $t_1'$.

We will construct the Casimir element of $H_\zeta(\mathfrak{gl}_n)$ and prove that its action on the Verma module $M(\lambda)$ is given by $P(\lambda) = \sum_{j=0}^{m+1} w_j H_j(\lambda + \rho)$, where $w_j$ are linear functions in $\zeta_i$.

### 2.1.4 The center

Let us switch to the approach elaborated in [EGG, Section 4], where all deformations satisfying the PBW property were determined. Define $\delta^{(m)} = (i\partial)^m \delta$ with $\delta$ being a standard delta function at 0, i.e., $\int \delta(\theta)\phi(\theta)d\theta = \phi(0)$. Let $f(z)$ be a polynomial satisfying $f(z) - f(z-1) = \partial^n(z^n \zeta(z))$, where $\zeta(z)$ is the generating series of the deformation parameters: $\zeta(z) = \zeta_0 + \zeta_1 z + \zeta_2 z^2 + \cdots$. Since $f(z)$ is defined up to a constant, we can specify $f(0) = 0$. Recall from [EGG, Section 4.2], that for $\hat{f}(\theta) = \sum_{m\geq 0} f_m \delta^{(m)}(\theta)$,

$$[y,x] = \frac{1}{2\pi^n} \int_{v\in\mathbb{C}^n:|v|=1} (x,(v\otimes\bar{v})y) \int_{-\pi}^{\pi} \left(1-e^{-i\theta}\right) \hat{f}(\theta) e^{i\theta(v\otimes\bar{v})}\, d\theta\, dv.$$

**Theorem 2.1.2.** *Let $g(z) = \sum g_m z^m = \sum \frac{f_m}{(m+1)(m+2)\cdots(m+n-1)} z^m$. The Casimir element of $H_\varsigma(\mathfrak{gl}_n)$ is given by $t_1' = \sum x_j y_j + \mathrm{Res}_{z=0} g(z^{-1}) \det(1-zA)^{-1}\, dz/z$.*

*Proof.*

Define $C' = \mathrm{Res}_{z=0} g(z^{-1}) \det(1-zA)^{-1}\, dz/z$. Let us compute $[y, t_1 + C'] = \sum_j [y,x_j]y_j + [y,C']$. The first summand is:

$$\begin{aligned}\sum_j [y,x_j]y_j &= \frac{1}{2\pi^n} \sum_j \int_{v\in\mathbb{C}^n:|v|=1} \int_{-\pi}^{\pi} (1-e^{-i\theta})\hat{f}(\theta) e^{i\theta(v\otimes\bar{v})} (x_j,(v\otimes\bar{v})y) y_j\, d\theta\, dv \\ &= \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} (1-e^{-i\theta})\hat{f}(\theta) e^{i\theta(v\otimes\bar{v})} \otimes (v\otimes\bar{v})y\, d\theta\, dv.\end{aligned}$$

Following [EGG, Section 4.2], we define

$$F_m(A) = \int_{|v|=1} \langle Av, v\rangle^{m+1}\, dv = \int_{|v|=1} (v\otimes\bar{v})^{m+1}\, dv.$$

There, it was proven that

$$\sum_m f_m F_{m-1}(A) = 2\pi^n \mathrm{Res}_{z=0} g(z^{-1}) \det(1-zA)^{-1} z^{-1} dz = 2\pi^n C'.$$

Thus, we can write

$$C' = \frac{1}{2\pi^n} \sum_m f_m \int_{|v|=1} (v\otimes\bar{v})^m dv = \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} \hat{f}(\theta) e^{i\theta(v\otimes\bar{v})}\, d\theta\, dv,$$

which implies that $[y,C'] = \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} \hat{f}(\theta) [y, e^{i\theta(v\otimes\bar{v})}]\, d\theta\, dv$. Since

$$e^{-i\theta(v\otimes\bar{v})}[y, e^{i\theta(v\otimes\bar{v})}] = e^{-i\theta(v\otimes\bar{v})} y e^{i\theta(v\otimes\bar{v})} - y = e^{-i\theta \mathrm{ad}(v\otimes\bar{v})} y - y = (e^{-i\theta}-1)(v\otimes\bar{v})y,$$

we get $[y,C'] = \frac{1}{2\pi^n} \int_{|v|=1} \int_{-\pi}^{\pi} \hat{f}(\theta) e^{i\theta(v\otimes\bar{v})} (e^{-i\theta}-1)(v\otimes\bar{v})y\, d\theta\, dv$, and so $\sum_i [y,x_i]y_i + [y,C'] = 0$ as desired. By using the anti-involution $\sigma$ defined in the beginning of

Section 2.1.2, this implies $[x, t_1 + C'] = 0$ for any $x \in V^*$, while $[e_{ij}, t_1 + C'] = 0$ by [T1], and hence, $t_1' = t_1 + C'$. □

*Remark* 2.1.2. This proof resembles calculations in [EGG, Section 4]. In particular, [EGG, Proposition 5.3 ] provides a formula for the Casimir element of continuous Cherednik algebras. However, adopting this formula for the specific case of infinitesimal Cherednik algebras is nontrivial and requires the above computations.

### 2.1.5 Action of the Casimir Element on the Verma Module

In this section, we justify our claim that the action of the Casimir element $t_1'$ is given by $P(\lambda) = \sum_{j=0}^{m+1} w_j H_j(\lambda + \rho)$. Obviously, $t_1'$ acts by a scalar on $M(\lambda - \rho)$, which we will denote by $t_1'(\lambda)$. Since $t_1' = \sum x_i y_i + C'$, $C' \in \mathfrak{z}(U(\mathfrak{g})) \cong S(\mathfrak{g})^G$, we see that $t_1'(\lambda) = C'(\lambda)$ where $C'(\lambda)$ denotes the constant by which $C'$ acts on $M(\lambda - \rho)$.

**Theorem 2.1.3.** *Let $w(z)$ be the unique degree $m+1$ polynomial satisfying $f(z) = (2\sinh(\partial/2))^{n-1} z^{n-1} w(z)$. Then $t_1'(\lambda) = \sum_{p \geq 0} w_p H_p(\lambda)$.*

*Proof.*

Because $C'(\lambda)$ is a polynomial in $\lambda$, we can consider a finite-dimensional representation of $U(\mathfrak{gl}_n)$ instead of the Verma module $M(\lambda - \rho)$ of $H_\zeta(\mathfrak{gl}_n)$. For a dominant weight $\lambda - \rho$ (so that the highest weight $\mathfrak{gl}_n$-module $V_{\lambda-\rho}$ is finite dimensional) we define the normalized trace $T(\lambda, \theta) = \mathrm{tr}_{V_{\lambda-\rho}}(e^{i\theta(v \otimes \bar{v})}) / \dim V_{\lambda-\rho}$ for any $v$ satisfying $|v| = 1$ (note that $T(\lambda, \theta)$ does not depend on $v$). To compute $T(\lambda, \theta)$, we will use the Weyl Character formula (see [FH]): $\chi_{\lambda-\rho} = \frac{\sum_{w \in W} (-1)^w e^{w\lambda}}{\sum_{w \in W} (-1)^w e^{w\rho}}$, where $W$ denotes the Weyl group (which is $S_n$ for $\mathfrak{gl}_n$). However, direct substitution of $e^{i\theta(v \otimes \bar{v})}$ into this formula gives zero in the denominator, so instead we compute $\lim_{\epsilon \to 0} \chi_{\lambda-\rho}(e^{i\theta(v \otimes \bar{v}) + \epsilon\mu})$ for a general diagonal matrix $\mu$.

Without loss of generality, we may suppose $v = y_1$, so that

$$v \otimes \bar{v} = q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}\lim_{\epsilon\to 0} \chi_{\lambda-\rho}(e^{i\theta(v\otimes\bar{v})+\epsilon\mu}) &= \lim_{\epsilon\to 0} \frac{\sum_{w\in S_n}(-1)^w e^{\langle w\lambda, i\theta q+\epsilon\mu\rangle}}{\sum_{w\in S_n}(-1)^w e^{\langle w\rho, i\theta q+\epsilon\mu\rangle}} \\ &= \lim_{\epsilon\to 0} \frac{\sum_{w\in S_n}(-1)^w e^{\langle w\lambda, i\theta q+\epsilon\mu\rangle}}{\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}(e^{\langle\alpha/2, i\theta q+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta q+\epsilon\mu\rangle})}.\end{aligned}$$

Partition $\Delta^+(\mathfrak{gl}_n)$ into $\Delta_1 \sqcup \Delta_2 = \Delta^+(\mathfrak{gl}_n)$, where $\Delta_1 = \{e_{11}^* - e_{jj}^* : 1 < j \leq n\}$. For $\alpha \in \Delta_1$,

$$\lim_{\epsilon\to 0}\left(e^{\langle\alpha/2, i\theta q+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta q+\epsilon\mu\rangle}\right) = e^{i\theta/2} - e^{-i\theta/2} = 2i\sin\left(\frac{\theta}{2}\right),$$

so $\lim_{\epsilon\to 0}\prod_{\alpha\in\Delta_1}(e^{\langle\alpha/2, i\theta q+\epsilon\mu\rangle} - e^{-\langle\alpha/2, i\theta q+\epsilon\mu\rangle})^{-1} = \left(2i\sin\left(\frac{\theta}{2}\right)\right)^{1-n}$.

Next, we compute the numerator. We can divide $S_n = \bigsqcup_{1\leq j\leq n} B_j$, where $B_j = \{w \in S_n | w(j) = 1\}$. Note that $B_j = \sigma_j \cdot S_{n-1}$, where $\sigma_j = (12\cdots j)$ and $S_{n-1}$ denotes the subgroup of $S_n$ corresponding to permutations of $\{1, 2, \ldots, j-1, j+1, \ldots, n\}$. We can then write

$$\begin{aligned}\sum_{w\in B_j}(-1)^w e^{\langle w\lambda, i\theta q+\epsilon\mu\rangle} &= \sum_{\sigma\in S_{n-1}}(-1)^{\sigma_j}(-1)^\sigma e^{i\theta\lambda_j} e^{\epsilon\langle\sigma_j\circ\sigma(\lambda),\mu\rangle} \\ &= (-1)^{j-1} e^{i\theta\lambda_j} e^{\epsilon\lambda_j\mu_1} \sum_{\sigma\in S_{n-1}}(-1)^\sigma e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}\end{aligned}$$

where $\widetilde{\lambda_j} = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n)$ and $\tilde{\mu} = (\mu_2, \ldots, \mu_n)$.

Combining the results of the last two paragraphs, we get

$$\lim_{\epsilon\to 0}\frac{\sum_{w\in S_n}(-1)^w e^{\langle w\lambda, i\theta q+\epsilon\mu\rangle}}{\prod_{\alpha\in\Delta^+(\mathfrak{gl}_n)}(e^{\langle\alpha/2, i\theta q+\epsilon\mu\rangle}-e^{-\langle\alpha/2, i\theta q+\epsilon\mu\rangle})}$$
$$=\lim_{\epsilon\to 0}\sum_{1\le j\le n}(-1)^{j-1}\frac{e^{i\theta\lambda_j+\epsilon\lambda_j\mu_1}}{(2i\sin\frac{\theta}{2})^{n-1}}\frac{\sum_{\sigma\in S_{n-1}}(-1)^\sigma e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}}{\prod_{\alpha\in\Delta_2}(e^{\langle\alpha/2, i\theta q+\epsilon\mu\rangle}-e^{-\langle\alpha/2, i\theta q+\epsilon\mu\rangle})}.$$

Using the Weyl character formula again, we see that

$$\frac{\sum_{\sigma\in S_{n-1}}(-1)^\sigma e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}}{\prod_{\alpha\in\Delta_2}(e^{\langle\alpha/2,\epsilon\mu\rangle}-e^{-\langle\alpha/2,\epsilon\mu\rangle})}=\mathrm{tr}_{V_{\tilde{\lambda}_j-\tilde{\rho}}}(e^{\epsilon\tilde{\mu}})$$

where $\tilde{\rho}$ is half the sum of all positive roots of $\mathfrak{gl}_{n-1}$. Thus,

$$\lim_{\epsilon\to 0}\frac{\sum_{\sigma\in S_{n-1}}(-1)^\sigma e^{\epsilon\langle\sigma(\tilde{\lambda}_j),\tilde{\mu}\rangle}}{\prod_{\alpha\in\Delta_2}(e^{\langle\alpha/2, i\theta q+\epsilon\mu\rangle}-e^{-\langle\alpha/2, i\theta q+\epsilon\mu\rangle})}=\mathrm{tr}_{V_{\tilde{\lambda}_j-\tilde{\rho}}}(1)=\dim V_{\tilde{\lambda}_j-\tilde{\rho}}.$$

We substitute to obtain

$$\mathrm{tr}_{V_{\lambda-\rho}}(e^{i\theta(v\otimes\bar{v})})=\sum_{1\le j\le n}(-1)^{j-1}\frac{e^{i\theta\lambda_j}\dim V_{\tilde{\lambda}_j-\tilde{\rho}}}{(2i\sin\frac{\theta}{2})^{n-1}}.$$

Our original goal was to calculate $T(\lambda,\theta)=\mathrm{tr}_{V_{\lambda-\rho}}(e^{i\theta(v\otimes\bar{v})})/\dim V_{\lambda-\rho}$. We obtain

$$T(\lambda,\theta)=\sum_{1\le j\le n}(-1)^{j-1}\frac{e^{i\theta\lambda_j}\dim V_{\tilde{\lambda}_j-\tilde{\rho}}}{(2i\sin\frac{\theta}{2})^{n-1}\dim V_{\lambda-\rho}}.$$

Using the dimension formula (see [FH, Equation 15.17]):

$$\dim V_{\lambda-\rho}=\prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j}{j-i},$$

we get $T(\lambda,\theta)=(2i\sin(\theta/2))^{1-n}(n-1)!\sum_{j=1}^n\frac{e^{i\lambda_j\theta}}{\prod_{k\neq j}(\lambda_j-\lambda_k)}$.

Since $\sum_{j=1}^n\frac{x_j^m}{\prod_{k\neq j}(x_j-x_k)}=H_{m-n+1}(x_1,...,x_n)$, we have

$$T(\lambda,\theta)=(2i\sin(\theta/2))^{1-n}(n-1)!\sum_{p\ge 0}\frac{H_p(\lambda)(i\theta)^{p+n-1}}{(p+n-1)!}.$$

Thus, we get

$$t_1'(\lambda) = C'(\lambda) = \left(\frac{1}{2\pi^n}\int_{|v|=1}\int_{-\pi}^{\pi}\hat{f}(\theta)e^{i\theta(v\otimes\bar{v})}d\theta dv\right)(\lambda) = \frac{1}{(n-1)!}\int_{-\pi}^{\pi}\hat{f}(\theta)T(\lambda,\theta)d\theta$$
$$= \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\sum_{p\geq 0}\frac{H_p(\lambda)(i\theta)^{p+n-1}}{(p+n-1)!}d\theta = \sum_{p\geq 0}w_p' H_p(\lambda),$$

where $w_p' = \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\frac{(i\theta)^{p+n-1}}{(p+n-1)!}d\theta$. Let $w'(z) = \sum w_p' z^p$. We verify that

$$\left(e^{\partial/2} - e^{-\partial/2}\right)^{n-1} z^{n-1} w'(z) = \int_{-\pi}^{\pi}\hat{f}(\theta)\sum_{p\geq 0}(2i\sin(\theta/2))^{1-n}\left(e^{\partial/2} - e^{-\partial/2}\right)^{n-1}\frac{(iz\theta)^{p+n-1}}{(p+n-1)!}\,d\theta$$
$$= \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\left(e^{\partial/2} - e^{-\partial/2}\right)^{n-1} e^{iz\theta}\,d\theta$$
$$= \int_{-\pi}^{\pi}\hat{f}(\theta)(2i\sin(\theta/2))^{1-n}\left(e^{i\theta/2} - e^{-i\theta/2}\right)^{n-1} e^{iz\theta}\,d\theta$$
$$= \int_{-\pi}^{\pi}\hat{f}(\theta)e^{iz\theta}\,d\theta = f(z),$$

and it is easy to see that the polynomial solution to $f(z) = (2\sinh(\partial/2))^{n-1}z^{n-1}w(z)$ is unique. □

### 2.1.6 Finite Dimensional Representations

In this section, we investigate when the irreducible $H_\zeta(\mathfrak{gl}_n)$ representation $L(\lambda)$ is finite dimensional. As in the case of classical Lie algebras, any finite dimensional irreducible representation is isomorphic to $L(\lambda)$ for a unique weight $\lambda$. Theorem 2.1.4 provides a necessary and sufficient condition for $L(\lambda)$ to be finite dimensional. In particular, all such representations have a *rectangular form.*

In Section 2.1.8, we prove that for any allowed *rectangular form* there exists a deformation $\zeta$ such that the representation $L(\lambda)$ of $H_\zeta(\mathfrak{gl}_n)$ has exactly that shape.

### 2.1.7 Rectangular Nature of Irreducible Representations

**Theorem 2.1.4.** (a) The representation $L(\lambda)$ is finite dimensional if and only if $\lambda$ is a dominant $\mathfrak{gl}_n$ weight and $P(\lambda) = P(\lambda - (0, \ldots, 0, \nu_n + 1))$ for some $\nu_n \in \mathbb{N}_0$.

For every $1 \le i \le n-1$ let $k_i \in \mathbb{N}_0$ be the smallest nonnegative integer such that $P(\lambda) = P(\lambda - (0,\ldots,0,k_i+1,0,\ldots,0))$ (we set $k_i = \infty$ if no such nonnegative integer exists). We define parameters $\nu_i = \min(k_i, \lambda_i - \lambda_{i+1})$.

(b) If $L(\lambda)$ is finite dimensional, then as a $\mathfrak{gl}_n$ module it decomposes into

$$L(\lambda) = \bigoplus_{0 \le \lambda - \lambda' \le \nu} V_{\lambda'},$$

where $\nu = (\nu_1, \ldots, \nu_n)$ are the parameters defined above (depending on $\zeta$ and $\lambda$).

*Proof.*

In order for $L(\lambda)$ to be finite dimensional, it is clearly necessary for $\lambda$ to be a dominant $\mathfrak{gl}_n$ weight. Recalling the PBW property and the definition of the Verma module $M(\lambda)$, we see that as a $\mathfrak{gl}_n$ module, $M(\lambda)$ decomposes as

$$M(\lambda) = V_\lambda \oplus (V_\lambda \otimes S_1) \oplus (V_\lambda \otimes S_2) \oplus \cdots, \text{ where } S_k = \mathrm{Sym}^k(x_1, x_2, ..., x_n).$$

We can further decompose each $V_\lambda \otimes S_i$ into irreducible modules of $\mathfrak{gl}_n$; once we do so, we find that $M(\lambda)$ has a simple $\mathfrak{gl}_n$ spectrum. Note that $V_\mu \otimes S_1$ can be decomposed as $V_{\mu - e_{11}^*} \oplus V_{\mu - e_{22}^*} \oplus \cdots \oplus V_{\mu - e_{nn}^*}$ (taking $V_{\mu - e_{ii}^*} = \{0\}$ if $\mu - e_{ii}^*$ is not dominant). We can thus associate each $V_\mu$ for $\mu = \lambda - a_1 e_{11}^* - \cdots - a_n e_{nn}^*$ in the decomposition of $M(\lambda)$ with a lattice point $P_\mu = (-a_1, -a_2, \ldots, -a_n) \in \mathbb{Z}^n$. We draw a directed edge from $P_\mu$ to $P_{\mu'}$ if $V_{\mu'}$ is in the decomposition of $V_\mu \otimes S_1$, and we say $P_{\mu'}$ is *smaller* than $P_\mu$. A key property of this graph is that any $H_\zeta(\mathfrak{gl}_n)$-submodule of $M(\lambda)$ intersecting the module $V_\mu$ must necessarily contain $V_\mu$ and all $V_{\mu'}$ such that $P_{\mu'}$ is reachable from $P_\mu$ by a walk along directed edges. Recall that $L(\lambda) = M(\lambda)/\overline{M}(\lambda)$, where $\overline{M}(\lambda)$ is the maximal proper $H_\zeta(\mathfrak{gl}_n)$-submodule of $M(\lambda)$. The aforementioned property guarantees that as a $\mathfrak{gl}_n$ module, $\overline{M}(\lambda) = \bigoplus_{s \in S} V_s$ for some set $S$ of vertices closed under walks, so that $L(\lambda)$ is finite dimensional if and only if $\bar{S}$ (the complement of $S$) is a finite set.

We now prove part (a). First, suppose that $L(\lambda)$ is finite dimensional. The finiteness of $\bar{S}$ implies the existence of some $l$ such that $(0, \ldots, 0, -l-1) \in S$ (note that $(0,\dots,0) \notin S$). Let $\nu_n$ be the minimal such $l$. We define $S'$ as the set of vertices that can be reached by walking from $(0,\dots,0,-\nu_n-1)$. Because $S' \subseteq S$, the Verma module $M(\lambda)$ must possess a submodule $M(\lambda-(0,\dots,0,\nu_n+1))$. By considering the action of the Casimir element on $M(\lambda)$ and $M(\lambda-(0,\dots,0,\nu_n+1))$, we get $P(\lambda) = P(\lambda-(0,\dots,0,\nu_n+1))$.

Next, suppose that there exists $\nu_n \in \mathbb{N}_0$ such that $P(\lambda) = P(\lambda-(0,\dots,0,\nu_n+1))$. The determinant formula of Theorem 2.1.1 implies that the Verma module $M(\lambda)$ contains the submodule $M(\lambda-(0,\dots,0,\mu))$ for some $\mu \leq \nu_n$. Define $S'$ to be the set of vertices that can be reached by walking from $(0,\dots,0,-\mu)$. Its complement $\bar{S}'$ is finite, since for any vertex $(-a_1,\dots,-a_n)$ of our graph, we have $\lambda_1 - a_1 \geq \lambda_2 - a_2 \geq \cdots \geq \lambda_n - a_n$. Because $\bar{S} \subseteq \bar{S}'$, $\bar{S}$ is finite, finishing the proof of (a). We note that explicitly, $\bar{S}' = \{(-a_1,\dots,-a_n) | 0 \leq a_i \leq \lambda_i - \lambda_{i+1}, 0 \leq a_n \leq \nu_n\}$ and the corresponding finite dimensional quotient is $L'(\lambda) = M(\lambda)/(\sum_{1\leq i\leq n-1} H_\zeta(\mathfrak{gl}_n) e_{i+1,i}^{\lambda_i-\lambda_{i+1}+1} v_\lambda + H_\zeta(\mathfrak{gl}_n) x_n^{\nu_n+1} v_\lambda)$.

Part (b) requires an additional argument. Namely, if $L(\lambda)$ is finite dimensional, then it can also be considered as a lowest weight representation. Let $\bar{b} = (b_1,\dots,b_n) \in \bar{S}$ be the vertex corresponding to the lowest weight of $L(\lambda)$. If the statement of (b) was wrong, there would be a vertex $\bar{e} = (e_1,\dots,e_n) \in S$ with two nonzero coordinates, such that $(e_1,\dots,e_{i-1},e_i+1,e_{i+1},\dots,e_n) \in \bar{S}$ for any $i$. Without loss of generality, suppose $e_1, e_2 \neq 0$. As we can walk along reverse edges from $\bar{b}$ to both points $(e_1+1,e_2,\dots,e_n)$ and $(e_1,e_2+1,e_3,\dots,e_n)$, we can also walk along reverse edges to $\bar{e}$, which is a contradiction. This proves part (b) and explains our terminology "*rectangular form*". □

The decomposition of $L(\lambda)$ as a $\mathfrak{gl}_n$ module provides the character formula for $L(\lambda)$ as the sum of the characters of $\mathfrak{gl}_n$ modules:

$$\chi_{\lambda;\zeta} = \sum_{0\leq\lambda-\lambda'\leq\nu} \frac{\sum_{w\in S_n}(-1)^w e^{w(\lambda'+\rho)}}{\sum_{w\in S_n}(-1)^w e^{w\rho}}. \qquad (*)$$

As in the classical theory, this character allows us to calculate the decomposition of finite dimensional representations into irreducible ones.

*Example* 2.1.1. Let us illustrate the decomposition of $L(\lambda)$ from the proof of Theorem 2.1.4; for clarity, we will work with $\mathfrak{sl}_2$ representations instead of $\mathfrak{gl}_2$ representations. Using the notation of the proof, $S_k = S^k(x_1, x_2) \cong V_k$, the irreducible $\mathfrak{sl}_2$ representation of dimension $k+1$. By the Clebsch-Gordon formula,

$$V_m \otimes V_k \cong V_{m+k} \oplus V_{m+k-2} \oplus \cdots \oplus V_{m+k-2\min(k,m)}.$$

We can use the above formula to draw the graph, representing the decomposition of $L((2,0))$, with $\nu = (0,3)$, into $\mathfrak{sl}_2$ modules. This representation is the quotient of $M((2,0))/H_\zeta(\mathfrak{gl}_2)e_{21}^3 v_\lambda$ by the submodules represented by the shaded areas of the diagram, and $L((2,0)) \cong V_2 \oplus V_3 \oplus V_4 \oplus V_5$ as $\mathfrak{sl}_2$ modules.

*Example* 2.1.2. For $H_\zeta(\mathfrak{gl}_1)$, the irreducible finite dimensional representation $L(\lambda)$, for $\lambda \in \mathbb{C}$, has character $\chi_{\lambda,\zeta} = \sum_{\nu'=0}^{\nu} e^{\lambda-\nu'}$, where $\nu$ is some nonnegative integer. The infinitesimal Cherednik algebras of $\mathfrak{gl}_1$ are generated by elements $e$, $f$, and $h$, satisfying the relations $[h,e]=e$, $[h,f]=-f$, and $[e,f]=\phi(h)$ for some polynomial $\phi$. In literature, these algebras are known as generalized Weyl algebras ([Sm]). In this case, the Casimir element equals $fe+g(h)$, where $g$ satisfies the equation $g(x)-g(x-1)=\phi(x)$. Then, $\nu$ is the smallest nonnegative integer such that $g(\lambda)-g(\lambda-\nu-1)=0$.

*Example* 2.1.3. For $H_\zeta(\mathfrak{gl}_2)$, the irreducible finite dimensional representations are necessarily of the form $L(\lambda)$ with $\lambda = (\lambda_2 + m, \lambda_2)$, where $\lambda_2 \in \mathbb{C},\ m \in \mathbb{N}_0$. The character of $L(\lambda)$ equals

$$\chi_{\lambda;\zeta} = \sum_{(0,0)\le(\nu_1',\nu_2')\le(\nu_1,\nu_2)} \frac{e^{(\lambda_2+m-\nu_1',\lambda_2-\nu_2')} - e^{(\lambda_2-\nu_2'-1,\lambda_2+m-\nu_1'+1)}}{1-e^{(-1,1)}}.$$

Let $f_1(\lambda,\mu) = P(\lambda_2+m+\frac{1}{2},\lambda_2-\frac{1}{2}) - P(\lambda_2+m+\frac{1}{2}-\mu,\lambda_2-\frac{1}{2})$ and $f_2(\lambda,\mu) = P(\lambda_2+m+\frac{1}{2},\lambda_2-\frac{1}{2}) - P(\lambda_2+m+\frac{1}{2},\lambda_2-\mu-\frac{1}{2})$. Again, $\nu_2$ is defined as the minimal nonnegative integer satisfying $f_2(\lambda,\nu_2+1)=0$, while $\nu_1$ is either $m$ or the minimal nonnegative integer satisfying $f_1(\lambda,\nu_1+1)=0$. For instance, if $\zeta = \zeta_0 r_0$ with $\zeta_0 \neq 0$, then $f_2(\lambda,\mu)$ is a multiple of $\mu$, and so the only solution to the equation $f_2(\lambda,\nu_2+1)=0$ is $\nu_2=-1$, which is negative. Thus, $H_{\zeta_0 r_0}(\mathfrak{gl}_2)$ has no finite dimensional irreducible representations. If $\zeta = \zeta_0 r_0 + \zeta_1 r_1$ with $\zeta_1 \neq 0$, $P(\lambda) = \zeta_0(\lambda_1 + \lambda_2) + \zeta_1((\lambda_1+\frac{1}{2})^2 + (\lambda_1+\frac{1}{2})(\lambda_2-\frac{1}{2}) + (\lambda_2-\frac{1}{2})^2)$, so $f_2(\lambda,\mu) = \zeta_1\mu\left(\frac{\zeta_0}{\zeta_1} + \lambda_1 + 2\lambda_2 - \mu\right)$. Thus, $L(\lambda)$ is finite dimensional if and only if $\frac{\zeta_0}{\zeta_1} + \lambda_1 + 2\lambda_2$ is a positive integer. This agrees with the description of finite dimensional representations of $\mathfrak{sl}_3$.

### 2.1.8 Existence of $L(\lambda)$ with a given shape

**Theorem 2.1.5.** *For any $\mathfrak{gl}_n$ dominant weight $\lambda$ and $\nu \in \mathbb{N}_0^n$ such that $\nu_i \leq \lambda_i - \lambda_{i+1}$ for all $1 \leq i \leq n-1$, there exists a deformation $\zeta$, such that the irreducible representation $L(\lambda)$ of $H_\zeta(\mathfrak{gl}_n)$ is finite dimensional and its character is given by (*).*

*Proof.*

Let $\lambda' = \lambda + \rho$. We can write $\lambda'_i = \lambda'_n + k_i$ for $k_1 > \cdots > k_{n-1} > k_n = 0$ (we have strict inequalities because of the shift by $\rho$). Recall that $P(\lambda) = \sum w_m H_m(\lambda')$ for $w_i$ defined as in Theorem 2.1.3. Let $\mu_i = (0,\ldots,\nu_i+1,0,\ldots,0)$. We will find $w_i$ such that $P(\lambda') - P(\lambda'-\mu_i) = 0$, while for all $0 < \mu'_i < \mu_i$, $P(\lambda') - P(\lambda'-\mu'_i) \neq 0$. This implies that there are embeddings of $M(\lambda'-\mu_i)$ into $M(\lambda')$ with an irreducible quotient $L(\lambda') = M(\lambda')/\sum_i M(\lambda'-\mu_i)$, due to Theorem 2.1.4.

Define $P_{mj} = P(\lambda') - P(\lambda'-\mu)$ for $\mu = (0,\ldots,m+1,0,\ldots,0)$ with the $m+1$ at the $j$-th location. We must prove that there exist $w$ such that $P_{\nu_1 1} = \cdots = P_{\nu_n n} = 0$ and $P_{\nu'_1 1},\ldots,P_{\nu'_n n} \neq 0$ for all $0 < \nu'_i < \nu_i$. We can write $P_{mj} = \sum_{i>0} w_i R^i_{mj}$, where

$$R^N_{mj} = \sum_{i_1+\ldots+i_n=N} (\lambda'_n+k_1)^{i_1}\cdots((\lambda'_n+k_j)^{i_j} - (\lambda'_n+k_j-m-1)^{i_j})\cdots(\lambda'_n+k_n)^{i_n}.$$

Note that the condition $P_{kj} = 0$ determines a hyperplane $\Pi_{kj}$ in the space $(w_0, w_1, \ldots)$ ($\Pi_{kj}$ might in fact be the entire space, but the following argument would be unaffected). Hence, the intersection $\bigcap \Pi_{\nu_j j}$ belongs to the union $\bigcup_{j,0<\nu'_j<\nu_j} \Pi_{\nu'_j,j}$ if and only if it belongs to some $\Pi_{\nu'_j,j}$. Thus, it suffices to show that $\{P_{\nu_1 1},\ldots,P_{\nu_n n},P_{\nu'_l l}\}$ are linearly independent as functions of $w_i$ for all $1 \leq l \leq n$ and $0 < \nu'_l < \nu_l$. This condition of linear independence is satisfied if

$$\det\begin{pmatrix} R^1_{\nu_1 1} & R^2_{\nu_1 1} & \cdots & R^{n+1}_{\nu_1 1} \\ R^1_{\nu_2 2} & R^2_{\nu_2 2} & \cdots & R^{n+1}_{\nu_2 2} \\ \vdots & \vdots & \ddots & \vdots \\ R^1_{\nu_n n} & R^2_{\nu_n n} & \cdots & R^{n+1}_{\nu_n n} \\ R^1_{\nu_l' l} & R^2_{\nu_l' l} & \cdots & R^{n+1}_{\nu_l' l} \end{pmatrix} \neq 0.$$

Now we shall prove that using column transformations, we can reduce the above matrix to its evaluation at $\lambda_n' = 0$. We proceed by induction on the column number. The elements of the first column, $R^1_{mj}$, are of degree zero with respect to $\lambda_n'$, so $R^1_{mj} = R^1_{mj}(0)$. Suppose that using column transformations, all columns before column $p$ are reduced to their constant terms. Now, we note that

$$\begin{aligned} \frac{\partial R^p_{mj}(\lambda_n')}{\partial \lambda_n'} &= \frac{\partial}{\partial \lambda_n'}\left(\sum_{i_1+\ldots+i_n=p} (\lambda_n'+k_1)^{i_1}\cdots((\lambda_n'+k_j)^{i_j} - (\lambda_n'+k_j-m-1)^{i_j})\cdots\lambda_n'^{i_n}\right) \\ &= (p+n-1)R^{p-1}_{mj}(\lambda_n'). \end{aligned}$$

Thus, we see that $R^p_{mj} - R^p_{mj}(0)$ is a linear combination of $R^{p-i}_{mj}(0)$, the entries of the other columns:

$$R^p_{mj}(\lambda_n') = \sum_i \frac{1}{i!}\lambda_n'^i \left.\frac{\partial^i R^p_{mj}}{\partial \lambda_n'^i}\right|_{\lambda_n'=0} = \sum_i \binom{p+n-1}{i} R^{p-i}_{mj}(0)\lambda_n'^i.$$

By selecting pivots of $\binom{p+n-1}{i}\lambda_n'^i$, we can eliminate every term except $R^p_{mj}(0)$. By repeating this step, we reduce the matrix to its evaluation at $\lambda_n' = 0$:

$$\det\begin{pmatrix} R^1_{\nu_1 1}(\lambda_n') & R^2_{\nu_1 1}(\lambda_n') & \cdots & R^{n+1}_{\nu_1 1}(\lambda_n') \\ R^1_{\nu_2 2}(\lambda_n') & R^2_{\nu_2 2}(\lambda_n') & \cdots & R^{n+1}_{\nu_2 2}(\lambda_n') \\ \vdots & \vdots & \ddots & \vdots \\ R^1_{\nu_n n}(\lambda_n') & R^2_{\nu_n n}(\lambda_n') & \cdots & R^{n+1}_{\nu_n n}(\lambda_n') \\ R^1_{\nu_l' l}(\lambda_n') & R^2_{\nu_l' l}(\lambda_n') & \cdots & R^{n+1}_{\nu_l' l}(\lambda_n') \end{pmatrix} = \det\begin{pmatrix} R^1_{\nu_1 1}(0) & R^2_{\nu_1 1}(0) & \cdots & R^{n+1}_{\nu_1 1}(0) \\ R^1_{\nu_2 2}(0) & R^2_{\nu_2 2}(0) & \cdots & R^{n+1}_{\nu_2 2}(0) \\ \vdots & \vdots & \ddots & \vdots \\ R^1_{\nu_n n}(0) & R^2_{\nu_n n}(0) & \cdots & R^{n+1}_{\nu_n n}(0) \\ R^1_{\nu_l' l}(0) & R^2_{\nu_l' l}(0) & \cdots & R^{n+1}_{\nu_l' l}(0) \end{pmatrix}.$$

Let us now rewrite $R^N_{mj}(0)$:

$$\begin{aligned} R^N_{mj}(0) &= \sum_{i_1+\ldots+i_n=N} k_1^{i_1}\cdots k_{j-1}^{i_{j-1}}(k_j^{i_j}-(k_j-m-1)^{i_j})k_{j+1}^{i_{j+1}}\cdots k_n^{i_n} \\ &= \sum_{i=0}^{N-1} H'_{N-i-1}\left(k_j^{i+1}-(k_j-m-1)^{i+1}\right) \\ &= \sum_{i=0}^{N-1} H_{N-i-1}\left(k_j^{i+1}-(k_j-m-1)^{i+1}-k_j(k_j^i-(k_j-m-1)^i)\right) \\ &= \sum_{i=0}^{N-1} H_{N-i-1}\left((m+1)(k_j-m-1)^i\right), \end{aligned}$$

where $H_{N-i}=\sum_{i_1+\ldots+i_n=N-i} k_1^{i_1}\cdots k_n^{i_n}$ and $H'_{N-i}=\sum_{i_1+\ldots+\widehat{i_j}+\ldots+i_n=N-i} k_1^{i_1}\cdots\widehat{k_j^{i_j}}\cdots k_n^{i_n}$. The third equality is because $H'_{N-i}=H_{N-i}-k_jH_{N-i-1}$. It is easy to see that the above determinant can be reduced further to

$$\det\begin{pmatrix} \nu_1+1 & (\nu_1+1)(k_1-\nu_1-1) & \cdots & (\nu_1+1)(k_1-\nu_1-1)^n \\ \nu_2+1 & (\nu_2+1)(k_2-\nu_2-1) & \cdots & (\nu_2+1)(k_2-\nu_2-1)^n \\ \vdots & \vdots & \ddots & \cdots \\ \nu_n+1 & (\nu_n+1)(k_n-\nu_n-1) & \cdots & (\nu_n+1)(k_n-\nu_n-1)^n \\ \nu'_l+1 & (\nu'_l+1)(k_l-\nu'_l-1) & \cdots & (\nu'_l+1)(k_l-\nu'_l-1)^n \end{pmatrix} =$$

$$T\cdot\det\begin{pmatrix} 1 & k_1-\nu_1-1 & \cdots & (k_1-\nu_1-1)^n \\ 1 & k_2-\nu_2-1 & \cdots & (k_2-\nu_2-1)^n \\ \vdots & \vdots & \ddots & \cdots \\ 1 & k_n-\nu_n-1 & \cdots & (k_n-\nu_n-1)^n \\ 1 & k_l-\nu'_l-1 & \cdots & (k_l-\nu'_l-1)^n \end{pmatrix},$$

where $T=(\nu_1+1)(\nu_2+1)\cdots(\nu_n+1)(\nu'_l+1)$ and the determinant is $\prod_{i=1}^n(k_l-k_i+\nu_i-\nu'_l)\prod_{1\le i<j\le n}(k_j-k_i+\nu_i-\nu_j)$ by the Vandermonde determinant formula. Now, recalling the conditions $0\le\nu_i\le\lambda_i-\lambda_{i+1}=k_i-k_{i+1}-1$ we get $k_j-k_i+\nu_i-\nu_j<0$ for any $i<j$ and so $\prod_{1\le i<j\le n}(k_j-k_i+\nu_i-\nu_j)$ is nonzero. Similarly, we get $\prod_{i=1}^n(k_l-k_i+\nu_i-\nu'_l)\neq 0$. Hence, the determinant is nonzero, and so $\{P_{\nu_1,1},\ldots,P_{\nu_n,n},P_{\nu'_l,l}\}$ are linearly independent as desired. $\square$

## 2.2 Poisson Infinitesimal Cherednik Algebras

### 2.2.1 Poisson Infinitesimal Cherednik Algebras of $\mathfrak{gl}_n$

Now we will study infinitesimal Cherednik algebras by using their Poisson analogues. The Poisson infinitesimal Cherednik algebras are as natural as $H_\zeta(\mathfrak{gl}_n)$, and their theory goes along the same lines with some simplifications. Although these algebras have not been defined before in the literature, the authors of [EGG] were aware of them, and technical calculations with these algebras are similar to those made in [T1]. This approach provides another proof of Theorem 2.1.2.

Let $\zeta$ be a deformation parameter, $\zeta : V \times V^* \to S(\mathfrak{gl}_n)$. The Poisson infinitesimal Cherednik algebra $H'_\zeta(\mathfrak{gl}_n)$ is defined to be the algebra $S\mathfrak{gl}_n \otimes S(V \oplus V^*)$ with a bracket defined on the generators by:

$$\begin{aligned}
&\{a, b\} = [a, b] \text{ for } a, b \in \mathfrak{gl}_n,\\
&\{g, v\} = g(v) \text{ for } g \in \mathfrak{gl}_n, v \in V \oplus V^*,\\
&\{y, y'\} = \{x, x'\} = 0 \text{ for } y, y' \in V, x, x' \in V^*,\\
&\{y, x\} = \zeta(y, x) \text{ for } y \in V, x \in V^*.
\end{aligned}$$

This bracket extends to a Poisson bracket on $H'_\zeta(\mathfrak{gl}_n)$ if and only if the Jacobi identity $\{\{x, y\}, z\} + \{\{y, z\}, x\} + \{\{z, x\}, y\} = 0$ holds for any $x, y, z \in \mathfrak{gl}_n \ltimes (V \oplus V^*)$. As can be verified by computations analogous to [EGG, Theorem 4.2], the Jacobi identiy holds iff $\zeta = \sum_{j=0}^{k} \zeta_j \mathfrak{r}_j$ where $\zeta_j \in \mathbb{C}$ and $\mathfrak{r}_j$ is the coefficient of $\tau^j$ in the expansion of $(x, (1-\tau A)^{-1} y) \det(1-\tau A)^{-1}$. Actually, we can consider the infinitesimal Cherednik algebras of $\mathfrak{gl}_n$ as quantizations of $H'_\zeta(\mathfrak{gl}_n)$.

*Remark* 2.2.1. Note that

$$\{y_i, x_j\} = \sum \zeta_l \mathfrak{r}_l(y_i, x_j) = \sum \zeta_l \frac{\partial \operatorname{tr}(S^{l+1} A)}{\partial e_{ji}};$$

this follows from

$$\frac{\partial}{\partial B}(\det(1-\tau A)^{-1}) = \frac{\operatorname{tr}(\tau B(1-\tau A)^{-1})}{\det(1-\tau A)}$$

when $B = y_i \otimes x_j$. In fact, if $\{y_i, x_j\} = F_{ji}(A)$, the Jacobi identity implies that $F_{ij}(A) = \frac{\partial F}{\partial e_{ij}}$ for some $GL(n)$ invariant function $F$, and that $\Lambda^2 D_A(F) = 0$, where $D_A$ is the matrix with $(D_A)_{ij} = \frac{\partial}{\partial e_{ij}}$. One can then show that the only $GL(n)$ invariant functions $F$ satisfying this partial differential equation are linear combinations of $\mathrm{tr}(S^l A)$.

Our main goal is to compute explicitly the Poisson center of the algebra $H'_\zeta(\mathfrak{gl}_n)$. As before, we set $\Omega_k$ to be the coefficient of $(-t)^k$ in the expansion of $\det(1 - tA)$, $\tau_k = \sum_{i=1}^n x_i \{\Omega_k, y_i\}$, and $\zeta(z) = \zeta_0 + \zeta_1 z + \zeta_2 z^2 + \cdots$.

**Theorem 2.2.1.** *The Poisson center* $\mathfrak{z}_{\mathrm{Pois}}(H'_\zeta(\mathfrak{gl}_n)) = \mathbb{C}[\tau_1 + c_1, \tau_2 + c_2, \ldots, \tau_n + c_n]$, *where* $(-1)^i c_i$ *is the coefficient of* $t^i$ *in the series*

$$c(t) = \mathrm{Res}_{z=0}\, \zeta(z^{-1}) \frac{\det(1 - tA)}{\det(1 - zA)} \frac{1}{1 - t^{-1}z} \frac{dz}{z}.$$

*Proof.*

First, we claim that $\mathfrak{z}_{\mathrm{Pois}}(H'_0(\mathfrak{gl}_n)) = \mathbb{C}[\tau_1, \ldots, \tau_n]$. The inclusion $\mathbb{C}[\tau_1, \ldots, \tau_n] \subseteq \mathfrak{z}_{\mathrm{Pois}}(H'_0(\mathfrak{gl}_n))$ is straightforward, while the reverse inclusion follows from the structure of the coadjoint action of the Lie group corresponding to $\mathfrak{gl}_n \ltimes (V \oplus V^*)$ (as in the proof of [T1, Theorem 2]).

We prove that the Poisson center of $H'_0(\mathfrak{gl}_n)$ can be lifted to the Poisson center of $H'_\zeta(\mathfrak{gl}_n)$ by verifying that $\tau_i + c_i$ are indeed Poisson central. Since $\tau_k \in \mathfrak{z}_{\mathrm{Pois}}(H_0(\mathfrak{gl}_n))$ and $c_k \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{gl}_n))$, $\tau_k + c_k$ Poisson-commutes with elements of $S(\mathfrak{gl}_n)$. We can define an anti-involution on $H'_\zeta(\mathfrak{gl}_n)$ that acts on basis elements by taking $e_{ij}$ to $e_{ji}$ and $y_i$ to $x_i$. By using the arguments explained in the proof of [T1, Theorem 2], we can show that $\tau_k$ is fixed by this anti-involution, while $c_k$ is also fixed since it lies in $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{gl}_n))$. Applying this anti-involution, we see that it suffices to show that $c_k$ satisfies $\{\tau_k + c_k, y_l\} = 0$ for basis elements $y_l \in V$.

First, notice that if $g \in S(\mathfrak{gl}_n)$, then $\{g, y_l\} = \sum_{i,j=1}^n \frac{\partial g}{\partial e_{ij}} \{e_{ij}, y_l\}$, and together with the equation $\{\{\Omega_k, y_i\}, y_l\} = 0$ (see the proof of [T1, Lemma 2.1]), we get

$$\{\tau_k, y_l\} = \left\{ \sum_{i=1}^n x_i \{\Omega_k, y_i\}, y_l \right\} = -\sum_{i=1}^n \left( \mathrm{Res}_{z=0}\, \zeta(z^{-1}) \frac{\mathrm{tr}(x_i (1 - zA)^{-1} y_l)}{z \det(1 - zA)} dz \right) \{\Omega_k, y_i\}.$$

Thus, we have

$$\{\tau_k + c_k, y_l\} = \sum_{i,j=1}^{n} \frac{\partial c_k}{\partial e_{ij}}\{e_{ij}, y_l\} - \sum_{i=1}^{n}\left(\operatorname{Res}_{z=0}\zeta(z^{-1})\frac{\operatorname{tr}(x_i(1-zA)^{-1}y_l)}{z\det(1-zA)}dz\right)\{\Omega_k, y_i\}.$$

Hence, $\{\tau_k + c_k, y_l\} = 0$ is equivalent to the system of partial differential equations:

$$\sum_{i,j=1}^{n} \frac{\partial c_k}{\partial e_{ij}}\{e_{ij}, y_l\} = \sum_{i=1}^{n}\left(\operatorname{Res}_{z=0}\zeta(z^{-1})\frac{\operatorname{tr}(x_i(1-zA)^{-1}y_l)}{z\det(1-zA)}dz\right)\{\Omega_k, y_i\}.$$

Multiplying both sides by $(-t)^k$ and summing over $k = 1, \ldots, n$, we obtain an equivalent single equation

$$\sum_{i,j=1}^{n} \frac{\partial c(t)}{\partial e_{ij}}\{e_{ij}, y_l\} = \sum_{i=1}^{n}\left(\operatorname{Res}_{z=0}\zeta(z^{-1})\frac{\operatorname{tr}(x_i(1-zA)^{-1}y_l)}{z\det(1-zA)}dz\right)\{\det(1-tA), y_i\}.$$

Since all terms above are $GL(n)$ invariant and diagonalizable matrices are dense in $\mathfrak{gl}_n$, we can set $A = \operatorname{diag}(a_1, \ldots, a_n)$:

$$\begin{aligned}\frac{\partial c(t)}{\partial a_l}y_l &= \left(\operatorname{Res}_{z=0}\frac{\zeta(z^{-1})}{z(1-za_l)\det(1-zA)}dz\right)\{\det(1-tA), y_l\}\\ &= \left(\operatorname{Res}_{z=0}\frac{\zeta(z^{-1})}{z(1-za_l)\det(1-zA)}dz\right)\frac{\partial\det(1-tA)}{\partial a_l}y_l\\ &= -\left(\operatorname{Res}_{z=0}\frac{\zeta(z^{-1})}{z(1-za_l)\det(1-zA)}dz\right)\frac{t\det(1-tA)}{1-ta_l}y_l,\end{aligned}$$

and it is easy to see that $c(t)$ satisfies the above equation. □

*Example* 2.2.1. In particular, $c_1 = \sum_{i=0}^{k}\zeta_i \operatorname{tr} S^{i+1}A$.

*Remark* 2.2.2. Another way of writing the formula for $c_k$ is

$$c_k = \operatorname{Res}_{z=0}\zeta(z^{-1})G_k(z)\frac{dz}{z^2},$$

where $G_k(z) = \sum z^m y_{m,k}(A)$ and $y_{m,k}(A) = \chi\underbrace{(m, 1, \ldots, 1)}_{k}$, the character of an irreducible $\mathfrak{gl}_n$ module corresponding to a hook Young diagram.[1]

[1] This formula follows from the fact that in the Grothendieck ring of finite dimensional $\mathfrak{gl}_n$

### 2.2.2 Passing from Commutative to Noncommutative Algebras

Note that $\{g, y\} \in S(\mathfrak{gl}_n) \otimes V$ for $g \in S(\mathfrak{gl}_n)$ and $y \in V$; we can thus identify $\{g, y\} = \sum_{i=1}^n h_i \otimes y_i \in H'_\zeta(\mathfrak{gl}_n)$ with the element $\sum_{i=1}^n \mathrm{Sym}(h_i) y_i \in H_\zeta(\mathfrak{gl}_n)$.

**Lemma 2.2.1.** *We have*

$$[\operatorname{tr} S^{k+1}A, y] = \left\{ \sum_{j=0}^{k} \frac{(-1)^j}{k+n+1} \binom{k+n+1}{j+1} \operatorname{tr} S^{k+1-j}A, y \right\}.$$

*Proof.*

It is enough to consider the case $y = y_1$. Recall that $\operatorname{tr} S^{k+1}(A)$ can be written as a sum of degree $k+1$ monomials of form $e_{1,i_1} \cdots e_{1,i_{s_1}} e_{2,i_{s_1+1}} \cdots e_{2,i_{s_1+s_2}} \cdots e_{n,i_{s_1+\cdots+s_n}}$ where $s_1 + \cdots + s_n = k+1$ and the sequence $\{i_k\}$ is a permutation of the sequence of $s_1$ ones, $s_2$ twos, and so forth; for conciseness, we will denote the above monomial by $e_{1,i_1} \cdots e_{n,i_{k+1}}$. The only terms of $\operatorname{tr} S^{k+1}A$ that contribute to $[\operatorname{tr} S^{k+1}A, y_1]$ and to $\{\operatorname{tr} S^{k+1}A, y_1\}$ have $s_1 \geq 1$. Since to compute $[\operatorname{tr} S^{k+1}A, y_1]$ we first symmetrize $\operatorname{tr} S^{k+1}A$, we will compute $[\mathrm{Sym}(e_{1,i_1} \cdots e_{n,i_{k+1}}), y_1] - \{\mathrm{Sym}(e_{1,i_1} \cdots e_{n,i_{k+1}}), y_1\}$. For both the Lie bracket and the Poisson bracket, we use Leibniz's rule to compute the bracket, but whereas in the Poisson case we can transfer the resulting elements of $V$ to the right since the Poisson algebra is commutative, in the Lie case when we do so extra terms appear.

Consider a typical term that may appear after we use Leibniz's rule to compute $[\operatorname{tr} S^{k+1}A, y_1]$:

$$\cdots y_{j_0} \cdots e_{j_1 j_0} \cdots e_{j_2 j_1} \cdots e_{j_N j_{N-1}} \cdots$$

When we move $y_{j_0}$ to the right, we get, besides $\cdots e_{j_1 j_0} \cdots e_{j_2 j_1} \cdots e_{j_N j_{N-1}} \cdots y_{j_0}$, additional residual terms like $- \cdots e_{j_2 j_1} \cdots e_{j_N j_{N-1}} \cdots y_{j_1}$ and $\cdots e_{j_3 j_2} \cdots e_{j_N j_{N-1}} \cdots y_{j_2}$, up to $(-1)^N \cdots y_{j_N}$. Without loss of generality, we can consider only the last expression,

representations, $[\bigwedge^k V \otimes S^m V] - [\bigwedge^{k+1} V \otimes S^{m-1} V] + \cdots + (-1)^m [\bigwedge^{k+m} V] = [V_{(m+1,1,\ldots,1)}]$ due to Pieri's formula.

since the others will appear in the smaller chains

$$\cdots y_{j_0} \cdots e_{j_1 j_0} \cdots \widehat{e_{j_2 j_1}} \cdots \widehat{e_{j_3 j_2}} \cdots \widehat{e_{j_N j_{N-1}}}$$

and

$$\cdots y_{j_0} \cdots e_{j_1 j_0} \cdots e_{j_2 j_1} \cdots \widehat{e_{j_3 j_2}} \cdots \widehat{e_{j_N j_{N-1}}},$$

and so forth, with the same coefficients. For notational convenience, we let $z_1$ denote the coefficient of $y_{j_N}$ in the residual term, i.e., the term represented by the ellipsis: $(-1)^N \underbrace{\cdots}_{z_1} y_{j_N}$. Then, $z_1 y_{j_N}$ is a term in the expression $(-1)^N\{z_1 e_{j_N 1}, y_1\}$, which appears in $(-1)^N\{\operatorname{tr} S^{k+1-N}A, y_1\}$. Thus, we can write

$$[\operatorname{tr} S^{k+1}A, y_1] = \left\{\sum_{N=0}^{k}(-1)^N C_N \operatorname{tr} S^{k+1-N}A, y_1\right\}$$

for some coefficients $C_N$.

Next, we compute $C_N$. We first count how many times $z_1 y_{j_N}$ appears in $\{\operatorname{tr} S^{k+1-N}A, y_1\}$. Notice that since $z_1$ is the product of $k-N$ $e_{jl}$'s, we can insert $e_{j_N 1}$ in $k-N+1$ places to obtain $z_2$ such that $\{z_2, y_1\}$ contains $z_1 y_{j_N}$.

Now we compute the coefficient of $z_2$ in $\operatorname{tr} S^{k+1-N}A$. As noted before, $\operatorname{tr} S^{k+1-N}(A)$ can be written as a sum of degree $k+1-N$ monomials of form

$$e_{1,i_1} \cdots e_{1,i_{s_1}} e_{2,i_{s_1+1}} \cdots e_{2,i_{s_1+s_2}} \cdots e_{n,i_{k+1-N}}.$$

Any term that is a permutation of those $k+1-N$ unit matrices will appear in the symmetrization of $\operatorname{tr} S^{k+1-N}A$. We count the number of sequences $i_1, \ldots, i_{k+1-N}$ such that $z_2$ is the product of the elements $e_{1,i_1}, \ldots, e_{n,i_{k+1-N}}$ (in some order); this tells us the multiplicity of $z_2$ in the symmetrization of $\operatorname{tr} S^{k+1-N}A$. Suppose $z_2 = e_{1,i_1} \cdots e_{n,i_{k+1-N}}$ for a certain sequence $i_1, \ldots, i_{k+1-N}$. Then, $z_2 = e_{1,i'_1} \cdots e_{n,i'_{k+1-N}}$ if and only if $i'_{s_1+\cdots+s_{j-1}+1}, \ldots, i'_{s_1+\cdots+s_j}$ is a permutation of $i_{s_1+\cdots+s_{j-1}+1}, \ldots, i_{s_1+\cdots+s_j}$ for all $j$. Thus, $z_2$ appears $s_1!s_2!\cdots s_n!$ times in $\operatorname{tr} S^{k+1-N}A$. Since each term has coefficient $\frac{1}{(k-N+1)!}$ in the symmetrization, $z_2$ appears with coefficient

$$\frac{s_1!s_2!\cdots s_n!}{(k-N+1)!}$$

in the symmetrization of $\operatorname{tr} S^{k+1-N}A$. In conjunction with the previous paragraph, we see that $z_1 y_{j_N}$ appears

$$\frac{s_1!s_2!\cdots s_n!}{(k-N+1)!} \times (k-N+1) = \frac{s_1!s_2!\cdots s_n!}{(k-N)!}$$

times in $\{\operatorname{tr} S^{k+1-N}A, y_1\}$.

It remains to calculate how many times $z_1 y_{j_N}$ appears in $[\operatorname{tr} S^{k+1}A, y_1]$. Recall that $z_1$ is obtained from a term like:

$$\cdots e_{j_0 1} \cdots e_{j_1 j_0} \cdots e_{j_2 j_1} \cdots e_{j_N j_{N-1}} \cdots$$

where the ordered union of the ellipsis equals $z_1$. Thus, $z_1$ comes from terms of the following form: we choose arbitrary numbers $j_0, \ldots, j_{N-1}$, and insert $e_{j_0 1}, e_{j_1 j_0}, \ldots, e_{j_N j_{N-1}}$ into $z_1$. There are

$$\frac{(k+1)(k)\cdots(k+1-N)}{(N+1)!}$$

ways for this choice for any fixed $j_0, \ldots, j_{N-1}$. Any such term $z_3$ appears in $\operatorname{tr} S^{k+1}A$ with coefficient

$$\frac{s_1'!\cdots s_n'!}{(k+1)!}$$

where $s_l'$ is the total number of $e_{li}$'s (for some $i$) in $z_3$, i.e., $s_l$+number of $j_i$'s with $j_i = l$, $0 \leq i < N$.

Combining the results of the last two paragraphs, we see that $\{\operatorname{tr} S^{k+1-N}A, y_1\}$ must appear with coefficient

$$\left(\frac{(k+1)(k)\cdots(k+1-N)}{(N+1)!}\sum\frac{s_1'!\cdots s_n'!}{(k+1)!}\right) \Big/ \frac{s_1!s_2!\cdots s_n!}{(k-N)!} = \frac{1}{(N+1)!}\sum\frac{s_1'!\cdots s_n'!}{s_1!s_2!\cdots s_n!},$$

where the summation is over all length-$N$ sequences $\{j_l\}$ of integers from 1 to $n$. We claim that

$$\frac{\sum s_1'!\cdots s_n'!}{s_1!\cdots s_n!} = (k+n)\cdots(k+n-N+1).$$

To see this, notice that $\frac{\sum s_1'!\cdots s_n'!}{s_1!\cdots s_n!}$ is the coefficient of $t^N$ in the expression

$$N!\prod_{i=1}^{n}\left(1+(s_i+1)t+\frac{(s_i+1)(s_i+2)}{2!}t^2+\cdots\right).$$

The above generating function equals $N!\prod_{i=1}^{n}(1-t)^{-(s_i+1)} = N!(1-t)^{-(k+1-N+n)}$, and the coefficient of $t^N$ in this expression is $(k+n)\cdots(k+n-N+1)$.

Finally, we arrive at the simplified coefficient of $\{\operatorname{tr} S^{k+1-N}A, y_1\}$:

$$C_N = \frac{1}{(N+1)!}\sum\frac{s_1'!\cdots s_n'!}{s_1!s_2!\cdots s_n!} = \frac{(k+n)\cdots(k+n-N+1)}{(N+1)!},$$

as desired. □

Now we will give an alternative proof of Theorem 2.1.2.

*Proof.*

Let $f(z)$ be the polynomial satisfying $f(z)-f(z-1)=\partial^n(z^n\zeta(z))$ and $g(z) = z^{1-n}\frac{1}{\partial^{n-1}}f(z)$ (in the expression for $g(z)$, we discard any negative powers of $z$). Note that if $g(z)=g_{k+1}z^{k+1}+\cdots+g_1z$, then

$$\zeta(z)=\sum_{j=1}^{k+1}\sum_{i=0}^{j-1}\frac{1}{j+n}\binom{j+n}{i+1}(-1)^i g_j z^{j-1-i},\quad \zeta_{j-1}=\sum_{i=0}^{k-j+1}\frac{1}{j+i+n}\binom{j+n+i}{i+1}(-1)^i g_{j+i}.$$

Lemma 2.2.1 allows us to write

$$\left[\sum_{j=1}^{k+1} g_j \operatorname{tr} S^j A, y\right] = \left\{\sum_{j=1}^{k+1}\sum_{i=0}^{j-1}\frac{1}{j+n}\binom{j+n}{i+1}(-1)^i g_j \operatorname{tr} S^{j-i}A, y\right\} =$$

$$\left\{\sum_{j=1}^{k+1}\sum_{i=0}^{k-j+1}\frac{1}{j+i+n}\binom{j+i+n}{i+1}(-1)^i g_{j+i}\operatorname{tr} S^j A, y\right\} = \left\{\sum_{j=1}^{k+1}\zeta_{j-1}\operatorname{tr} S^j A, y\right\}.$$

Hence,

$$[t_1, y] = \sum_{i=1}^{n} [x_i, y] y_i = \sum_{i=1}^{n} \{x_i, y\} y_i = -\left\{ \sum_{j=1}^{k+1} \zeta_{j-1} \operatorname{tr} S^j A, y \right\} = -\left[ \sum_{j=1}^{k+1} g_j \operatorname{tr} S^j A, y \right],$$

where the third equality follows from the fact that $\tau_1 + \sum_{j=1}^{k+1} \zeta_{j-1} \operatorname{tr} S^j A$ is Poisson-central in $H'_\zeta(\mathfrak{gl}_n)$ (see Example 2.2.1). Thus, we get $t_1' = t_1 + C'$, where

$$C' = \sum_{j=1}^{k+1} g_j \operatorname{tr} S^j A = \operatorname{Res}_{z=0} g(z^{-1}) \det(1 - zA)^{-1} z^{-1} dz.$$

□

### 2.2.3 Poisson infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$

Choose a basis $v_j$ of $V$, so that the symplectic form $\omega$ has a form

$$\omega(x, y) = x^T J y,$$

with

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

As before, we study the noncommutative infinitesimal Cherednik algebra $H_\zeta(\mathfrak{sp}_{2n})$ by considering its Poisson analogue $H'_\zeta(\mathfrak{sp}_{2n})$. We define $\sum_{i=0}^{n} \Omega_i z^{2i} = \det(1 - zA)$ and

$$\tau_i = (-1)^{i-1} \sum_{j=1}^{2n} \{\Omega_i, v_j\} v_j^*,$$

where $\{v_j^*\}$ is dual to $\{v_j\}$ (that is, $\omega(v_i, v_j^*) = \delta_{ij}$). When viewed as an element of $\mathbb{C}[\mathfrak{sp}_{2n} \ltimes V]$,

$$\tau_i = -\sum_{j=0}^{i-1} \mathfrak{Q}_j \omega(A^{2i-1-2j}v, v),$$

so $\tau_i$ is $\mathfrak{sp}_{2n}$ invariant and independent of the choice of basis $\{v_i\}$.

**Proposition 2.2.1.** *The Poisson center of $H_0'(\mathfrak{sp}_{2n})$ is $\mathbb{C}[\tau_1, \ldots, \tau_n]$.*

*Proof.*

We will follow a similar approach as in the proof of [T1, Theorem 2.1]. Let $L$ be the Lie algebra $\mathfrak{sp}_{2n} \ltimes V$ and $S$ be the Lie group of $L$. We need to verify that $\mathbb{C}[\tau_1, \ldots, \tau_n] = \mathfrak{z}_{\mathrm{Pois}}(H_0'(\mathfrak{sp}_{2n}))$, the latter being identified with $\mathbb{C}[L^*]^S$. Let $M \subset L$ be the $2n$-dimensional subspace consisting of elements of the form

$$y = \left\{ \begin{pmatrix} 0 & y_{12} & 0 & \cdots & 0 & 0 \\ y_{21} & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & y_{2n-3,2n-2} & 0 & 0 \\ 0 & 0 & y_{2n-2,2n-3} & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & y_{2n-1,2n} \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ y_{2n} \end{pmatrix} \right\},$$

where all the $y$'s belong to $\mathbb{C}$. In what follows, we identify $L^*$ and $L$ via the non-degenerate pairing, so that the coadjoint action of $S$ is on $L$. We use the following two facts proved in [Ka]: first, that the orbit of $M$ under the coadjoint action of $S$ on $L^*$ is dense in $L^*$; and second, that $\mathbb{C}[L^*]^S \cong \mathbb{C}[f_1, \ldots, f_n]$, where

$$f_i|_M(y) = \sigma_{i-1}(y_{2,1}y_{1,2}, y_{3,2}y_{2,3}, \ldots, y_{2n-2,2n-3}y_{2n-3,2n-2})y_{2n-1,2n}y_{2n}^2$$

and $\sigma_j$ is the $j$-th elementary symmetric polynomial. It is straightforward to see that $\tau_i|_M = f_i$, and so $\mathbb{C}[L^*]^S \cong \mathbb{C}[\tau_1, \ldots, \tau_n]$ as desired. $\square$

As before, let $\zeta(z) = \zeta_0 + \zeta_2 z^2 + \zeta_4 z^4 + \cdots$.

**Theorem 2.2.2.** *The Poisson center* $\mathfrak{z}_{\mathrm{Pois}}(H'_\zeta(\mathfrak{sp}_{2n})) = \mathbb{C}[\tau_1+c_1, \tau_2+c_2, \ldots, \tau_n+c_n]$, *where* $(-1)^{i-1}c_i$ *is the coefficient of* $t^{2i}$ *in the series*

$$c(t) = 2\,\mathrm{Res}_{z=0}\,\zeta(z^{-1})\frac{\det(1-tA)}{\det(1-zA)}\frac{z^{-1}}{1-z^2t^{-2}}dz.$$

*Proof.*

Since $c_i \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{sp}_{2n}))$, $\{\tau_i + c_i, g\} = 0$ for any $g \in S(\mathfrak{sp}_{2n})$, and so it suffices to show that $\{\tau_i + c_i, v\} = 0$ for all $v \in V$. By the Jacobi rule,

$$\{\tau_i, v\} = (-1)^{i-1}\sum_j \{\Omega_i, v_j\}\{v_j^*, v\} + (-1)^{i-1}\sum_j \{\{\Omega_i, v_j\}, v\}v_j^*.$$

Thus,

$$\{\tau_i + c_i, v\} = (-1)^{i-1}\sum_j \{\Omega_i, v_j\}\{v_j^*, v\} + (-1)^{i-1}\sum_j \{\{\Omega_i, v_j\}, v\}v_j^* + \{c_i, v\}. \quad (2.1)$$

In the case of $H'_\zeta(\mathfrak{gl}_n)$, $\sum_j \{\{\Omega_i, y_j\}, y\}x_j = 0$ by straightforward application of properties of the determinant. However, for $H'_\zeta(\mathfrak{sp}_{2n})$, $\sum_j \{\{\Omega_i, v_j\}, v\}v_j^* \neq 0$. To calculate this sum, let $B$ be a basis of $\mathfrak{sp}_{2n}$ (for the purposes of this section the specific choice of $B$ is irrelevant). Write

$$\sum_j \{\{\Omega_i, v_j\}, v\}v_j^* = \sum_j \left\{\sum_{e\in B} \frac{\partial \Omega_i}{\partial e} e(v_j), v\right\} v_j^* =$$

$$\sum_j \left(\sum_{e\in B} \frac{\partial \Omega_i}{\partial e}\{e(v_j), v\}v_j^* + \left\{\frac{\partial \Omega_i}{\partial e}, v\right\} e(v_j)v_j^*\right).$$

**Lemma 2.2.2.** *We have*

$$\sum_j \sum_{e\in B} \left\{\frac{\partial \Omega_i}{\partial e}, v\right\} e(v_j)v_j^* = 0.$$

We will prove this lemma in the end of this section.

Using the fact that $\sum_j \{\{\Omega_i, v_j\}, v\}v_j^* = \sum_j \sum_{e\in B} \frac{\partial \Omega_i}{\partial e}\{e(v_j), v\}v_j^*$, we can restrict

(2.1) to diagonal matrices, which are spanned by elements $e_i = \operatorname{diag}(0, \dots, 1, -1, 0, \dots, 0)$ with 1 at the $2i-1$-th coordinate. Thus, the condition $\{\tau_i + c_i, v\} = 0$ is equivalent to the following sum being zero:

$$(-1)^{i-1}\sum_j\sum_k \frac{\partial \Omega_i}{\partial e_k}\{e_k, v_j\}\{v_j^*, v\} + (-1)^{i-1}\sum_k\left(\frac{\partial \Omega_i}{\partial e_k}\{v_{2k-1}, v\}v_{2k} + \frac{\partial \Omega_i}{\partial e_k}\{v_{2k}, v\}v_{2k-1}\right) + \sum_k \frac{\partial c_i}{\partial e_k}\{e_k, v\} = 2(-1)^{i-1}\sum_k \frac{\partial \Omega_i}{\partial e_k}(v_{2k-1}\{v_{2k}, v\} + v_{2k}\{v_{2k-1}, v\}) + \sum_k \frac{\partial c_i}{\partial e_k}\{e_k, v\}.$$

Multiplying the above equation by $(-1)^{i-1}t^{2i}$ and summing over $i$ for $i = 1, \dots, n$, the required condition transforms into:

$$0 = 2\sum_k \frac{\partial \det(1-tA)}{\partial e_k}(v_{2k-1}\{v_{2k}, v\} + v_{2k}\{v_{2k-1}, v\}) + \sum_k \frac{\partial c(t)}{\partial e_k}\{e_k, v\}.$$

It suffices to check this condition for basis vectors $v = v_{2s-1}$ and $v = v_{2s}$. Substituting, we get

$$0 = 2\sum_k \frac{\partial \det(1-tA)}{\partial e_k}(v_{2k-1}\{v_{2k}, v_{2s-1}\} + v_{2k}\{v_{2k-1}, v_{2s-1}\}) + \frac{\partial c(t)}{\partial e_s}v_{2s-1}$$

and

$$0 = 2\sum_k \frac{\partial \det(1-tA)}{\partial e_k}(v_{2k-1}\{v_{2k}, v_{2s}\} + v_{2k}\{v_{2k-1}, v_{2s}\}) - \frac{\partial c(t)}{\partial e_s}v_{2s}.$$

These last two formulas both reduce to

$$\begin{aligned}\frac{\partial c(t)}{\partial e_s} &= -2\frac{\partial \det(1-tA)}{\partial e_s}\{v_{2s}, v_{2s-1}\} \\ &= -2\frac{\partial \det(1-tA)}{\partial e_s}\left(\operatorname{Res}_{z=0}\zeta(z^{-1})\omega(v_{2s}, (1-z^2A^2)^{-1}v_{2s-1})\det(1-zA)^{-1}z^{-1}dz\right) \\ &= 2\operatorname{Res}_{z=0}\zeta(z^{-1})\frac{\partial \det(1-tA)}{\partial e_s}\frac{1}{1-z^2\lambda_s^2}\det(1-zA)^{-1}z^{-1}dz,\end{aligned}$$

and it is straightforward to verify that $c(t)$ satisfies the above equation. □

### 2.2.4 Proof of Lemma 2.2.2

In this section, we will outline the proof of Lemma 2.2.2, which states:

$$\sum_{j=1}^{2n}\sum_{e\in B}\left\{\frac{\partial \Omega_i}{\partial e}, v\right\} e(v_j)v_j^* = 0. \tag{2.2}$$

We use the basis for $V$ defined in Section 2.2.3, in which $\omega$ is represented by the matrix $J$.

Let us multiply (2.2) by $t^{2i}$ and sum over $i$ to get the equivalent assertion that

$$\sum_{j}\sum_{e\in B}\left\{\frac{\partial \det(1-tA)}{\partial e}, v\right\} e(v_j)v_j^* = 0.$$

Since the whole sum is $\mathfrak{sp}_{2n}$-invariant (even though each term considered separately is not), we can look at the restriction of the sum to $\mathfrak{h}$. Thus, this sum equals zero if and only if

$$\left.\sum_{j}\sum_{e\in B}\left\{\frac{\partial \det(1-tA)}{\partial e}, v\right\} e(v_j)v_j^*\right|_{\mathfrak{h}} = 0.$$

We choose the following basis $B$ for $\mathfrak{sp}_{2n}$: $e_{2j-1,2j}$, $e_{2j,2j-1}$, $e_{2j-1,2j-1}-e_{2j,2j}$, for all $1 \le j \le n$, and for all $1 \le k < l \le n$, the elements $e_{2l-1,2k}+e_{2k-1,2l}$, $e_{2l,2k}-e_{2k-1,2l-1}$, $e_{2l-1,2k-1}-e_{2k,2l}$, and $e_{2l,2k-1}+e_{2k,2l-1}$. We observe that for any $1 \le j, j' \le 2n$, there exists a unique basis vector in $B$ that takes $v_j$ to $\pm v_{j'}$; we shall denote this element by $v_{j',j} \in B$. These $v_{j',j}$ are not pairwise distinct since there are basis vectors with two nonzero entries.

Since $\mathrm{Sp}_{2n}$ acts transitively on $V$, we can assume $v = v_1$. Using the above basis, we get

$$\sum_{j}\sum_{e\in B}\left\{\frac{\partial \det(1-tA)}{\partial e}, v_1\right\} e(v_j)v_j^* = \sum_{j,j',k}\frac{\partial^2 \det(1-tA)}{\partial v_{k,1}\partial v_{j',j}} v_{j'}v_k v_j^*(-1)^{\iota_{jj'}},$$

where

$$\iota_{jj'} = \begin{cases} 1 & \text{if } j \equiv j' \bmod 2 \text{ and } j < j', \text{ or if } j' = j \text{ and } j \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

We now restrict to $\mathfrak{h}$. We have $\left.\frac{\partial^2 \det(1-tA)}{\partial v_{k,1}\partial v_{j',j}}\right|_{\mathfrak{h}} \neq 0$ only when the matrices for $v_{k,1}$ and $v_{j',j}$ have nonzero entries on the diagonal, or if $v_{k,1}$ and $v_{j',j}$ have nonzero entries at the $i$-th row $j$-th column and $j$-th row $i$-th column, respectively. This can only happen when $v_{j'}v_k v_j^* = v_1 v_a v_a^*$ for some $a$. We can list all the ways this can happen for $a = 2b$ or $a = 2b-1$ with $b \neq 1$ (keeping in mind that $v_{2b-1}^* = v_{2b}$ and $v_{2b}^* = -v_{2b-1}$):

1. $\frac{\partial^2 \det(1-tA)}{\partial v_{1,1}\partial v_{2b-1,2b-1}} v_1 v_{2b-1} v_{2b}$,
2. $\frac{\partial^2 \det(1-tA)}{\partial v_{1,1}\partial v_{2b,2b}} v_{2b} v_1 v_{2b-1}$,
3. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b-1,1}\partial v_{1,2b-1}} v_1 v_{2b-1} v_{2b}$,
4. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b,1}\partial v_{1,2b}} (-v_1 v_{2b} v_{2b-1})$,
5. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b,1}\partial v_{2b-1,2}} (-v_{2b-1} v_{2b} v_1)$,
6. $\frac{\partial^2 \det(1-tA)}{\partial v_{2b-1,1}\partial v_{2b,2}} v_{2b-1} v_{2b} v_1$.

To calculate the derivatives, let $A_1$ be the 4 by 4 matrix formed by the intersections of the first, second, $2b-1$-th, and $2b$-th rows and columns of $A$, and let $A_2$ be the $2n-4$ by $2n-4$ matrix formed by the intersections of the remaining rows and columns. The space of all such $A_2$ is isomorphic to $\mathfrak{sp}_{2n-4}$, and we denote the Cartan subalgebra of diagonal matrices of this space by $\mathfrak{h}(A_2)$. All six of the above derivatives evaluate to the same polynomial in $\mathfrak{h}(A_2)$ times the corresponding derivative in $\mathfrak{sp}_4$; for instance, $\frac{\partial^2 \det(1-tA)}{\partial v_{1,1}\partial v_{2b-1,2b-1}} = h\frac{\partial^2 \det(1-tA_1)}{\partial v'_{1,1}\partial v'_{3,3}}$ with $v'_{1,1}, v'_{3,3} \in \mathfrak{sp}_4$ and $h \in S(\mathfrak{h}(A_2))[t]$. Thus, we can reduce our problem to $\mathfrak{sp}_4$, and straightforward computations verify (2.2) for $\mathfrak{sp}_4$. Similarly, when $b = 1$ (that is, when the term is of the form $v_1 v_1 v_2$), all computations will reduce to analogous ones in $\mathfrak{sp}_2$.

# Chapter 3

# Infinitesimal Cherednik algebras as W-algebras

This chapter is based on [LT].

## 3.1 Basics

### 3.1.1 Length of the deformation

We start this section by investigating for which deformation parameters $\zeta$ and $\zeta'$, the infinitesimal Cherednik algebras $H_\zeta(\mathfrak{gl}_n), H_{\zeta'}(\mathfrak{gl}_n)$ are isomorphic. Even for $n = 1$ (when $H_\zeta(\mathfrak{gl}_1)$ are simply the *generalized Weyl algebras*), the answer to this question (given in [BJ]) is quite nontrivial. Instead, we will look only for the filtration preserving isomorphisms, where both algebras are endowed with the *$N$-th standard filtration* $\{\mathcal{F}^{(N)}_\bullet\}$. Those are induced from the grading on $T(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)$ with $\deg(\mathfrak{gl}_n) = 2$ and $\deg(V_n \oplus V_n^*) = N$, where $N > l(\zeta)$. For $N \geq \max\{l(\zeta)+1, l(\zeta')+1, 3\}$ we have the following result:

**Lemma 3.1.1.** *(a) $N$-standardly filtered algebras $H_\zeta(\mathfrak{gl}_n)$ and $H_{\zeta'}(\mathfrak{gl}_n)$ are isomorphic if and only if there exist $\lambda \in \mathbb{C}, \theta \in \mathbb{C}^*, s \in \{\pm\}$ such that $\zeta' = \theta\varphi_\lambda(\zeta^s)$, where*

- *$\varphi_\lambda : U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n)$ is an isomorphism defined by $\varphi_\lambda(A) = A + \lambda \cdot \operatorname{tr} A$ for any $A \in \mathfrak{gl}_n$,*

• *for* $\zeta = \zeta_0 r_0 + \zeta_1 r_1 + \zeta_2 r_2 + \ldots$ *we define* $\zeta^- := \zeta_0 r_0 - \zeta_1 r_1 + \zeta_2 r_2 - \ldots,\ \zeta^+ := \zeta$.

*(b) For any length* $m$ *deformation* $\zeta$*, there is a length* $m$ *deformation* $\zeta'$ *with* $\zeta'_m = 1,\ \zeta'_{m-1} = 0$*, such that algebras* $H_\zeta(\mathfrak{gl}_n)$ *and* $H_{\zeta'}(\mathfrak{gl}_n)$ *are isomorphic as filtered algebras.*

Same discussion can be applied to the $\mathfrak{sp}_{2n}$-case. For any $N > 2l(\zeta)$, we introduce the $N$-th standard filtration $\{\mathcal{F}^{(N)}_\bullet\}$ on $H_\zeta(\mathfrak{sp}_{2n})$ by setting $\deg(\mathfrak{sp}_{2n}) = 2,\ \deg(V_{2n}) = N$. The following result is analogous to Lemma 3.1.1:

**Lemma 3.1.2.** *For* $N \geq \max\{2l(\zeta)+1, 2l(\zeta')+1, 3\}$*, the* $N$*-standardly filtered algebras* $H_\zeta(\mathfrak{sp}_{2n})$ *and* $H_{\zeta'}(\mathfrak{sp}_{2n})$ *are isomorphic if and only if there exists* $\theta \in \mathbb{C}^*$ *such that* $\zeta' = \theta\zeta$.

### 3.1.2 Proof of Lemmas 3.1.1, 3.1.2

• *Proof of Lemma 3.1.1(a)*

Let $\phi : H_\zeta(\mathfrak{gl}_n) \xrightarrow{\sim} H_{\zeta'}(\mathfrak{gl}_n)$ be a filtration preserving isomorphism. We have $\phi(1) = 1$, so that $\phi$ is the identity on the 0-th level of the filtration.

Since $\mathcal{F}^{(N)}_2(H_\zeta(\mathfrak{gl}_n)) = \mathcal{F}^{(N)}_2(H_{\zeta'}(\mathfrak{gl}_n)) = U(\mathfrak{gl}_n)_{\leq 1}$, we have $\phi(A) = \psi(A) + \gamma(A),\ \forall A \in \mathfrak{gl}_n$, with $\psi(A) \in \mathfrak{gl}_n, \gamma(A) \in \mathbb{C}$. Then $\phi([A,B]) = [\phi(A), \phi(B)]$ for all $A, B \in \mathfrak{gl}_n$ if and only if $\gamma([A,B]) = 0$ and $\psi$ is an automorphism of the Lie algebra $\mathfrak{gl}_n$. Since $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$, we have $\gamma(A) = \lambda \cdot \operatorname{tr} A$ for some $\lambda \in \mathbb{C}$. For $n \geq 3$, $\operatorname{Aut}(\mathfrak{gl}_n) = \operatorname{Aut}(\mathfrak{sl}_n) \times \operatorname{Aut}(\mathbb{C}) = (\mu_2 \ltimes \mathrm{SL}(n)) \times \mathbb{C}^*$, where $-1 \in \mu_2$ acts on $\mathfrak{sl}_n$ via $\sigma : A \mapsto -A^t$. This determines $\phi$ up to the filtration level $N-1$.

Finally, $\mathcal{F}^{(N)}_N(H_\zeta(\mathfrak{gl}_n)) = \mathcal{F}^{(N)}_N(H_{\zeta'}(\mathfrak{gl}_n)) = V_n \oplus V_n^* \oplus U(\mathfrak{gl}_n)_{\leq N}$. As explained, $\phi_{|U(\mathfrak{gl}_n)}$ is parameterized by $(\epsilon, T, \nu, \lambda) \in (\mu_2 \ltimes \mathrm{SL}(n)) \times \mathbb{C}^* \times \mathbb{C}$ (no $\mu_2$-factor for $n = 1, 2$). Let $I_n \in \mathfrak{gl}_n$ be the identity matrix. Note that $[I_n, y] = y, [I_n, x] = -x, [I_n, A] = 0$ for any $y \in V_n, x \in V_n^*, A \in \mathfrak{gl}_n$. Since $\phi(y) = [\nu \cdot I_n + n\lambda, \phi(y)] = \nu[I_n, \phi(y)],\ \forall y \in V_n$, we get $\nu = \pm 1$.

**Case 1:** $\nu = 1$. Then $\phi(y) \in V_n,\ \phi(x) \in V_n^*\ (\forall y \in V_n, x \in V_n^*)$. Since $V_n \not\simeq V_n^\sigma$ as $\mathfrak{sl}_n$-modules for $n \geq 3$ and $\operatorname{End}_{\mathfrak{sl}_n}(V_n) = \mathbb{C}^*$, we get $\epsilon = 1 \in \mu_2$ (so that $\phi(A) = TAT^{-1},\ \forall A \in \mathfrak{sl}_n$) and there exist $\theta_1, \theta_2 \in \mathbb{C}^*$ such that $\phi(y) = \theta_1 \cdot T(y),\ \phi(x) =$

$\theta_2 \cdot T(x)$ ($\forall y \in V_n, x \in V_n^*$). Hence, we get $\varphi(T,\lambda)(\zeta(y,x)) = \phi([y,x]) = [\phi(y),\phi(x)] = \theta\zeta'(T(y),T(x))$, where $\theta = \theta_1\theta_2$ and isomorphism $\varphi(T,\lambda) : U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n)$ is defined by $A \mapsto TAT^{-1} + \lambda \operatorname{tr} A$, $\forall A \in \mathfrak{gl}_n$.

Thus, $\zeta' = \theta^{-1}\varphi_\lambda(\zeta^+)$ in that case.

**Case 2:** $\nu = -1$. Then $\phi(y) \in V_n^*$, $\phi(x) \in V_n$ ($\forall y \in V_n$, $x \in V_n^*$). Similarly to the above reasoning we get $\epsilon = -1 \in \mu_2, \phi(A) = -TA^tT^{-1} + \lambda \operatorname{tr} A$ ($\forall A \in \mathfrak{gl}_n$), so that there exist $\theta_1, \theta_2 \in \mathbb{C}^*$ such that $\phi(y_i) = \theta_1 \cdot T(x_i)$, $\phi(x_j) = \theta_2 \cdot T(y_j)$. Then $\phi(\zeta(y_i,x_j)) = -\theta_1\theta_2\zeta'(T(y_j),T(x_i))$.

Hence, $\zeta' = -\theta_1^{-1}\theta_2^{-1}\varphi_{-\lambda}(\zeta^-)$ in that case.

Finally, the above arguments also provide isomorphisms $\phi_{\theta,\lambda,s} : H_\zeta(\mathfrak{gl}_n) \xrightarrow{\sim} H_{\theta\varphi_\lambda(\zeta^s)}(\mathfrak{gl}_n)$ for any deformation $\zeta$, constants $\lambda \in \mathbb{C}, \theta \in \mathbb{C}^*$ and a sign $s \in \{\pm\}$.

- *Proof of Lemma 3.1.1(b)*

Let $\zeta$ be a length $m$ deformation. Since $(\theta\zeta)_m = \theta\zeta_m$, we can assume $\zeta_m = 1$. We claim that $\varphi_\lambda(\zeta)_{m-1} = 0$ for $\lambda = -\zeta_{m-1}/(n+m)$, which is equivalent to $\frac{\partial\alpha_m}{\partial I_n} = (n+m)\alpha_{m-1}$. This equality follows from comparing coefficients of $s\tau^m$ in the identity

$$\sum \alpha_i(y,x)(A+sI_n)\tau^i = (1-s\tau)^{-n-1}\sum \alpha_i(y,x)(A)(\tau(1-s\tau)^{-1})^i.$$

- *Proof of Lemma 3.1.2*

Let $\phi : H_\zeta(\mathfrak{sp}_{2n}) \xrightarrow{\sim} H_{\zeta'}(\mathfrak{sp}_{2n})$ be a filtration preserving isomorphism. Being an isomorphism, we have $\phi(1) = 1$, so that $\phi$ is the identity on the 0-th level of the filtration.

Since $\mathcal{F}_2^{(N)}(H_\zeta(\mathfrak{sp}_{2n})) = \mathcal{F}_2^{(N)}(H_{\zeta'}(\mathfrak{sp}_{2n})) = U(\mathfrak{sp}_{2n})_{\leq 1}$, we have $\phi(A) = \psi(A) + \gamma(A)$ for all $A \in \mathfrak{sp}_{2n}$, with $\psi(A) \in \mathfrak{sp}_{2n}, \gamma(A) \in \mathbb{C}$. Then $\phi([A,B]) = [\phi(A),\phi(B)]$ for all $A, B \in \mathfrak{sp}_{2n}$ if and only if $\gamma([A,B]) = 0$ and $\psi$ is an automorphism of the Lie algebra $\mathfrak{sp}_{2n}$. Since $[\mathfrak{sp}_{2n},\mathfrak{sp}_{2n}] = \mathfrak{sp}_{2n}$, we have $\gamma \equiv 0$. Meanwhile, any automorphism of $\mathfrak{sp}_{2n}$ is inner, since $\mathfrak{sp}_{2n}$ is a simple Lie algebra whose Dynkin diagram has no automorphisms. This proves $\phi_{|U(\mathfrak{sp}_{2n})} = \operatorname{Ad}(T)$, $T \in \mathrm{Sp}_{2n}$. Composing with an automorphism $\phi'$ of $H_{\zeta'}(\mathfrak{sp}_{2n})$, defined by $\phi'(A) = \operatorname{Ad}(T^{-1})(A), \phi'(x) = T^{-1}(x)$ ($A \in \mathfrak{sp}_{2n}, x \in V_{2n}$) we can assume $\phi_{|U(\mathfrak{sp}_{2n})} = \mathrm{Id}$.

Recall the element $I'_n = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) \in \mathfrak{sp}_{2n}$. Since $\mathrm{ad}(I'_n)$ has only even eigenvalues on $U(\mathfrak{sp}_{2n})$ and eigenvalues $\pm 1$ on $V_{2n}$, we actually have $\phi(V_{2n}) \subset V_{2n}$. Together with $\mathrm{End}_{\mathfrak{sp}_{2n}}(V_{2n}) = \mathbb{C}^*$ this implies the result.

The converse, that is $H_\zeta(\mathfrak{sp}_{2n}) \cong H_{\theta\zeta}(\mathfrak{sp}_{2n})$ for any $\zeta$ and $\theta \in \mathbb{C}^*$, is obvious.

### 3.1.3 Universal algebras $H_m(\mathfrak{gl}_n)$ and $H_m(\mathfrak{sp}_{2n})$

It is natural to consider a version of the infinitesimal Cherednik algebras with $\zeta_j$ being independent central variables. This motivates the following notion of the universal length $m$ infinitesimal Cherednik algebras.

**Definition 3.1.1.** The *universal length $m$ infinitesimal Cherednik algebra* $H_m(\mathfrak{gl}_n)$ is the quotient of $U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ by the relations

$$[x, x'] = 0,\ [y, y'] = 0,\ [A, x] = A(x),\ [A, y] = A(y),\ [y, x] = \sum_{j=0}^{m-2} \zeta_j r_j(y, x) + r_m(y, x),$$

where $x, x' \in V_n^*,\ y, y' \in V_n,\ A \in \mathfrak{gl}_n$ and $\{\zeta_j\}_{j=0}^{m-2}$ are central. The filtration is induced from the grading on $T(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ with

$$\deg(\mathfrak{gl}_n) = 2,\ \deg(V_n \oplus V_n^*) = m + 1,\ \deg(\zeta_i) = 2(m - i)$$

(the latter is chosen in such a way that $\deg(\zeta_j r_j) = 2m$ for all $j$).

Algebra $H_m(\mathfrak{gl}_n)$ is free over $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$ and $H_m(\mathfrak{gl}_n)/(\zeta_0 - c_0, \ldots, \zeta_{m-2} - c_{m-2})$ is the usual infinitesimal Cherednik algebra $H_{\zeta_c}(\mathfrak{gl}_n)$ with $\zeta_c = c_0 r_0 + \ldots + c_{m-2} r_{m-2} + r_m$. In fact, for odd $m$, $H_m(\mathfrak{gl}_n)$ can be viewed as a universal family of length $m$ infinitesimal Cherednik algebras of $\mathfrak{gl}_n$, while for even $m$, there is an action of $\mathbb{Z}/2\mathbb{Z}$ we should quotient by (this follows from our proof of Lemma 3.1.1).

*Remark* 3.1.1. One can consider all possible quotients

$$U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]/([x, x'], [y, y'], [A, x] - A(x), [A, y] - A(y), [y, x] - \eta(y, x)),$$

with a $\mathfrak{gl}_n$-invariant pairing $\eta : V_n \times V_n^* \to U(\mathfrak{gl}_n)[\zeta_0, \ldots, \zeta_{m-2}]$, such that $\deg(\eta(y, x)) \leq 2m$. Such a quotient satisfies a PBW property if and only if

$$\eta(y,x) = \sum_{i=0}^{m} \eta_i(\zeta_0, \ldots, \zeta_{m-2}) r_i(y, x) \text{ with } \deg(\eta_i(\zeta_0, \ldots, \zeta_{m-2})) \leq 2(m-i)$$

(this is completely analogous to [EGG, Theorem 4.2]).

We define the universal version of $H_\zeta(\mathfrak{sp}_{2n})$ in a similar way:

**Definition 3.1.2.** The *universal length $m$ infinitesimal Cherednik algebra $H_m(\mathfrak{sp}_{2n})$* is defined as

$$H_m(\mathfrak{sp}_{2n}) := U(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})[\zeta_0, \ldots, \zeta_{m-1}]/([A,x]-A(x), [x,y]-\sum_{j=0}^{m-1} \zeta_j r_{2j}(x,y) - r_{2m}(x,y)),$$

where $A \in \mathfrak{sp}_{2n}$, $x, y \in V_{2n}$ and $\{\zeta_i\}_{i=0}^{m-1}$ are central. The filtration is induced from the grading on $T(\mathfrak{sp}_{2n} \oplus V_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$ with $\deg(\mathfrak{sp}_{2n}) = 2$, $\deg(V_{2n}) = 2m+1$ and $\deg(\zeta_i) = 4(m-i)$.

The algebra $H_m(\mathfrak{sp}_{2n})$ is free over $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}]$ and $H_m(\mathfrak{sp}_{2n})/(\zeta_0 - c_0, \ldots, \zeta_{m-1} - c_{m-1})$ is the usual infinitesimal Cherednik algebra $H_{\zeta_c}(\mathfrak{sp}_{2n})$ for $\zeta_c = c_0 r_0 + \ldots + c_{m-1} r_{2(m-1)} + r_{2m}$. In fact, the algebra $H_m(\mathfrak{sp}_{2n})$ can be viewed as a universal family of length $m$ infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$, due to Lemma 3.1.2.

*Remark* 3.1.2. Analogously to Remark 3.1.1, the result of [EGG, Theorem 4.2], generalizes straightforwardly to the case of $\mathfrak{sp}_{2n}$-invariant pairings $\eta : V_{2n} \times V_{2n} \to U(\mathfrak{sp}_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$.

### 3.1.4 Poisson counterparts of $H_m(\mathfrak{g})$

Following Section 2.2, we introduce the Poisson algebras $H_m^{\mathrm{cl}}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sp}_{2n}$. As algebras these are $S(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ (respectively $S(\mathfrak{sp}_{2n} \oplus V_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$) with a Poisson bracket $\{\cdot, \cdot\}$ modeled after the commutator $[\cdot, \cdot]$ from the definition of $H_m(\mathfrak{g})$, so that $\{y, x\} = \alpha_m(y, x) + \sum_{j=0}^{m-2} \zeta_j \alpha_j(y, x)$ (respectively $\{x, y\} = \beta_{2m}(x, y) + \sum_{j=0}^{m-1} \zeta_j \beta_{2j}(x, y)$). Their quotients $H_m^{\mathrm{cl}}(\mathfrak{gl}_n)/(\zeta_0 - c_0, \ldots, \zeta_{m-2} - c_{m-2})$

and $H^{\rm cl}_m(\mathfrak{sp}_{2n})/(\zeta_0 - c_0, \ldots, \zeta_{m-1} - c_{m-1})$, are the Poisson infinitesimal Cherednik algebras $H^{\rm cl}_{\zeta_c}(\mathfrak{gl}_n)$ ($\zeta_c = c_0\alpha_0 + \ldots + c_{m-2}\alpha_{m-2} + \alpha_m$) and $H^{\rm cl}_{\zeta_c}(\mathfrak{sp}_{2n})$ ($\zeta_c = c_0\beta_0 + \ldots + c_{m-1}\beta_{2m-2} + \beta_{2m}$) from Section 2.2.

Let us describe the Poisson centers of the algebras $H^{\rm cl}_m(\mathfrak{gl}_n)$ and $H^{\rm cl}_m(\mathfrak{sp}_{2n})$. For $\mathfrak{g} = \mathfrak{gl}_n$ and $1 \le k \le n$ we define an element $\tau_k \in H^{\rm cl}_m(\mathfrak{g})$ by $\tau_k := \sum_{i=1}^n x_i\{\widetilde{Q}_k, y_i\}$, where $1 + \sum_{j=1}^n \widetilde{Q}_j z^j = \det(1 + zA)$. We set $\zeta(w) := \sum_{i=0}^{m-2} \zeta_i w^i + w^m$ and define $c_i \in S(\mathfrak{gl}_n)$ via

$$c(t) = 1 + \sum_{i=1}^n (-1)^i c_i t^i := \mathrm{Res}_{z=0}\zeta(z^{-1})\frac{\det(1-tA)}{\det(1-zA)}\frac{z^{-1}dz}{1-t^{-1}z}.$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $1 \le k \le n$ we define an element $\tau_k \in H^{\rm cl}_m(\mathfrak{g})$ by $\tau_k := \sum_{i=1}^{2n}\{\widetilde{Q}_k, y_i\}y_i^*$, where $1 + \sum_{j=1}^n \widetilde{Q}_j z^{2j} = \det(1 + zA)$, while $\{y_i\}_{i=1}^{2n}$ and $\{y_i^*\}_{i=1}^{2n}$ are the dual bases of $V_{2n}$, that is $\omega(y_i, y_j^*)$=1. We set $\zeta(w) := \sum_{i=0}^{m-1} \zeta_i w^i + w^m$ and define $c_i \in S(\mathfrak{sp}_{2n})$ via

$$c(t) = 1 + \sum_{i=1}^n c_i t^{2i} := 2\mathrm{Res}_{z=0}\zeta(z^{-2})\frac{\det(1-tA)}{\det(1-zA)}\frac{z^{-1}dz}{1-t^{-2}z^2}.$$

The following result is a consequence of our computations from Section 2.2:

**Theorem 3.1.3.** *We have:*

*(a) The Poisson center* $\mathfrak{z}_{\rm Pois}(H^{\rm cl}_m(\mathfrak{gl}_n))$ *is a polynomial algebra in free generators* $\{\zeta_0, \ldots, \zeta_{m-2}, \tau_1 + c_1, \ldots, \tau_n + c_n\}$;

*(b) The Poisson center* $\mathfrak{z}_{\rm Pois}(H^{\rm cl}_m(\mathfrak{sp}_{2n}))$ *is a polynomial algebra in free generators* $\{\zeta_0, \ldots, \zeta_{m-1}, \tau_1 + c_1, \ldots, \tau_n + c_n\}$.

### 3.1.5 $W$-algebras

Here we recall finite $W$-algebras following [GG].

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and $e \in \mathfrak{g}$ be a nonzero nilpotent element. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form $( \, , \, )$. Let $\chi$ be the element of $\mathfrak{g}^*$ corresponding to $e$ and $\mathfrak{z}_\chi$ be the stabilizer of $\chi$ in $\mathfrak{g}$ (which is the same as the centralizer of $e$ in $\mathfrak{g}$). Fix an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{g}$. Then $\mathfrak{z}_\chi$ is $\mathrm{ad}(h)$-stable and the eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{z}_\chi$ are nonnegative integers.

Consider the $\mathrm{ad}(h)$–weight grading on $\mathfrak{g}$, that is,

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \text{ where } \mathfrak{g}(i) := \{\xi \in \mathfrak{g} | [h, \xi] = i\xi\}.$$

Equip $\mathfrak{g}(-1)$ with the symplectic form $\omega_\chi(\xi, \eta) := \langle \chi, [\xi, \eta] \rangle$. Fix a Lagrangian subspace $l \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} := \bigoplus_{i \le -2} \mathfrak{g}(i) \oplus l \subset \mathfrak{g}$, $\mathfrak{m}' := \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{m}\} \subset U(\mathfrak{g})$.

**Definition 3.1.3.** [P1, GG] By the $W$-algebra associated with $e$ (and $l$), we mean the algebra $U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}')^{\mathrm{ad}\, \mathfrak{m}}$ with multiplication induced from $U(\mathfrak{g})$.

Let $\{F_\bullet^{st}\}$ denote the PBW filtration on $U(\mathfrak{g})$, while $U(\mathfrak{g})(i) := \{x \in U(\mathfrak{g}) | [h, x] = ix\}$. Define $F_k U(\mathfrak{g}) = \sum_{i+2j \le k} (F_j^{st} U(\mathfrak{g}) \cap U(\mathfrak{g})(i))$ and equip $U(\mathfrak{g}, e)$ with the induced filtration, denoted $\{F_\bullet\}$ and referred to as the *Kazhdan* filtration.

One of the key results of [P1, GG] is a description of the associated graded algebra $\mathrm{gr}_{F_\bullet} U(\mathfrak{g}, e)$. Recall that the affine subspace $\mathrm{S} := \chi + (\mathfrak{g}/[\mathfrak{g}, f])^* \subset \mathfrak{g}^*$ is called the *Slodowy slice.* As an affine subspace of $\mathfrak{g}$, the Slodowy slice S coincides with $e + \mathfrak{c}$, where $\mathfrak{c} = \mathrm{Ker}_\mathfrak{g} \mathrm{ad}(f)$. So we can identify $\mathbb{C}[\mathrm{S}] \cong \mathbb{C}[\mathfrak{c}]$ with the symmetric algebra $S(\mathfrak{z}_\chi)$. According to [GG, Section 3], algebra $\mathbb{C}[\mathrm{S}]$ inherits a Poisson structure from $\mathbb{C}[\mathfrak{g}^*]$ and is also graded with $\deg(\mathfrak{z}_\chi \cap \mathfrak{g}(i)) = i + 2$.

**Theorem 3.1.4.** *[GG, Theorem 4.1] The filtered algebra $U(\mathfrak{g}, e)$ does not depend on the choice of $l$ (up to a distinguished isomorphism) and $\mathrm{gr}_{F_\bullet} U(\mathfrak{g}, e) \cong \mathbb{C}[\mathrm{S}]$ as graded Poisson algebras.*

### 3.1.6 Additional properties of $W$-algebras

We want to describe some other properties of $U(\mathfrak{g}, e)$.

(a) Let $G$ be the adjoint group of $\mathfrak{g}$. There is a natural action of the group $Q := Z_G(e, h, f)$ on $U(\mathfrak{g}, e)$, due to [GG]. Let $\mathfrak{q}$ stand for the Lie algebra of $Q$. In [P2] Premet constructed a Lie algebra embedding $\mathfrak{q} \overset{\iota}{\hookrightarrow} U(\mathfrak{g}, e)$. The adjoint action of $\mathfrak{q}$ on $U(\mathfrak{g}, e)$ coincides with the differential of the aforementioned $Q$-action.

(b) Restricting the natural map $U(\mathfrak{g})^{\mathrm{ad\,m}} \longrightarrow U(\mathfrak{g}, e)$ to $Z(U(\mathfrak{g}))$, we get an algebra homomorphism $Z(U(\mathfrak{g})) \xrightarrow{\rho} Z(U(\mathfrak{g}, e))$, where $Z(A)$ stands for the center of an algebra $A$. According to the following theorem, $\rho$ is an isomorphism:

**Theorem 3.1.5.** *(a) [P1, Section 6.2] The homomorphism $\rho$ is injective.*

*(b) [P2, footnote to Question 5.1] The homomorphism $\rho$ is surjective.*

## 3.2 Main Theorem

Let us consider $\mathfrak{g} = \mathfrak{sl}_N$ or $\mathfrak{g} = \mathfrak{sp}_{2N}$, and let $e_m \in \mathfrak{g}$ be a 1-*block* nilpotent element of Jordan type $(1, \ldots, 1, m)$ or $(1, \ldots, 1, 2m)$, respectively. We make a particular choice of such $e_m$:

- $e_m = E_{N-m+1,N-m+2} + \ldots + E_{N-1,N}$ in the case of $\mathfrak{sl}_N$, $2 \leq m \leq N$,
- $e_m = E_{N-m+1,N-m+2} + \ldots + E_{N+m-1,N+m}$ in the case of $\mathfrak{sp}_{2N}$, $1 \leq m \leq N$.[1]

Recall the Lie algebra inclusion $\iota : \mathfrak{q} \hookrightarrow U(\mathfrak{g}, e)$ from Section 3.1.6. In our cases:

- For $(\mathfrak{g}, e) = (\mathfrak{sl}_{n+m}, e_m)$, we have $\mathfrak{q} \simeq \mathfrak{gl}_n$. Define $\bar{T} \in U(\mathfrak{sl}_{n+m}, e_m)$ to be the $\iota$-image of the identity matrix $I_n \in \mathfrak{gl}_n$, the latter being identified with

$$T_{n,m} = \mathrm{diag}(\frac{m}{n+m}, \cdots, \frac{m}{n+m}, \frac{-n}{n+m}, \cdots, \frac{-n}{n+m})$$

under the inclusion $\mathfrak{q} \hookrightarrow \mathfrak{sl}_{n+m}$. Let Gr be the induced $\mathrm{ad}(\bar{T})$-weight grading on $U(\mathfrak{sl}_{n+m}, e_m)$, with the $j$-th grading component denoted by $U(\mathfrak{sl}_{n+m}, e_m)_j$.

- For $(\mathfrak{g}, e) = (\mathfrak{sp}_{2n+2m}, e_m)$, we have $\mathfrak{q} \simeq \mathfrak{sp}_{2n}$. Define $\bar{T}' := \iota(I'_n) \in U(\mathfrak{sp}_{2n+2m}, e_m)$, where $I'_n = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) \in \mathfrak{sp}_{2n} \simeq \mathfrak{q}$. Let Gr be the induced $\mathrm{ad}(\bar{T}')$-weight grading on $U(\mathfrak{sp}_{2n+2m}, e_m) = \bigoplus_j U(\mathfrak{sp}_{2n+2m}, e_m)_j$.

**Lemma 3.2.1.** *There is a natural Lie algebra inclusion $\Theta : \mathfrak{gl}_n \ltimes V_n \hookrightarrow U(\mathfrak{sl}_{n+m}, e_m)$ such that $\Theta\,|_{\mathfrak{gl}_n} = \iota\,|_{\mathfrak{gl}_n}$ and $\Theta(V_n) = F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_1$.*

[1] We view $\mathfrak{sp}_{2N}$ as corresponding to the pair $(V_{2N}, \omega_{2N})$, where $\omega_{2N}$ is represented by the skew symmetric *antidiagonal* matrix $J = (J_{ij} := (-1)^j \delta_{i+j}^{2N+1})_{1 \leq i,j \leq 2N}$. In this presentation, $A = (a_{ij}) \in \mathfrak{sp}_{2N}$ if and only if $a_{2N+1-j,2N+1-i} = (-1)^{i+j+1} a_{ij}$ for any $1 \leq i, j \leq 2N$.

*Proof.*

First, choose a Jacobson-Morozov $\mathfrak{sl}_2$-triple $(e_m, h_m, f_m) \subset \mathfrak{sl}_{n+m}$ in a standard way.[2] As a vector space, $\mathfrak{z}_\chi \cong \mathfrak{gl}_n \oplus V_n \oplus V_n^* \oplus \mathbb{C}^{m-1}$ with

$$\mathfrak{gl}_n = \mathfrak{z}_\chi(0) = \mathfrak{q},\ V_n \oplus V_n^* \subset \mathfrak{z}_\chi(m-1),\ \xi_j \in \mathfrak{z}_\chi(2m-2j-2).$$

Here $\mathbb{C}^{m-1}$ has a basis $\{\xi_{m-2-j} = E_{n+1,n+j+2} + \ldots + E_{n+m-j-1,n+m}\}_{j=0}^{m-2}$, $V_n \oplus V_n^*$ is embedded via $y_i \mapsto E_{i,n+m}$, $x_i \mapsto E_{n+1,i}$, while $\mathfrak{gl}_n \cong \mathfrak{sl}_n \oplus \mathbb{C} \cdot I_n$ is embedded in the following way: $\mathfrak{sl}_n \hookrightarrow \mathfrak{sl}_{n+m}$ as a *left-up block*, while $I_n \mapsto T_{n,m}$.

Under the identification $\mathrm{gr}_{F_\bullet} U(\mathfrak{sl}_{n+m}, e_m) \simeq \mathbb{C}[\mathrm{S}] \simeq S(\mathfrak{z}_\chi)$, the induced grading $\mathrm{Gr}'$ on $S(\mathfrak{z}_\chi)$ is the $\mathrm{ad}(T_{n,m})$-weight grading. Together with the above description of $\mathrm{ad}(h_m)$-grading on $\mathfrak{z}_\chi$, this implies that $F_m U(\mathfrak{sl}_{n+m}, e_m)_1 = 0$ and that $F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_1$ coincides with the image of the composition $V_n \hookrightarrow \mathfrak{z}_\chi \hookrightarrow S(\mathfrak{z}_\chi)$. Let $\Theta(y) \in F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_1$ be the element whose image is identified with $y$. We also set $\Theta(A) := \iota(A)$ for $A \in \mathfrak{gl}_n$. Finally, we define $\Theta : \mathfrak{gl}_n \oplus V_n \hookrightarrow U(\mathfrak{sl}_{n+m}, e_m)$ by linearity. We claim that $\Theta$ is a Lie algebra inclusion, that is

$$[\Theta(A), \Theta(B)] = \Theta([A,B]),\ [\Theta(y), \Theta(y')] = 0,\ [\Theta(A), \Theta(y)] = \Theta(A(y))$$

for all $A, B \in \mathfrak{gl}_n, y, y' \in V_n$. The first equality follows from $[\Theta(A), \Theta(B)] = [\iota(A), \iota(B)] = \iota([A,B]) = \Theta([A,B])$. The second one follows from the observation that $[\Theta(y), \Theta(y')] \in F_{2m}U(\mathfrak{g}, e_m)_2$ and the only such element is 0. Similarly, $[\Theta(A), \Theta(y)] \in F_{m+1}U(\mathfrak{g}, e_m)_1$, so that $[\Theta(A), \Theta(y)] = \Theta(y')$ for some $y' \in V_n$. Since $y' = \mathrm{gr}(\Theta(y')) = \mathrm{gr}([\Theta(A), \Theta(y)]) = [A, y] = A(y)$, we get $[\Theta(A), \Theta(y)] = \Theta(A(y))$. □

Our main result is:

**Theorem 3.2.2.** *(a) There is a unique isomorphism* $\bar{\Theta} : H_m(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)$ *of filtered algebras, whose restriction to* $\mathfrak{sl}_n \ltimes V_n \hookrightarrow H_m(\mathfrak{gl}_n)$ *is equal to* $\Theta$ *(we assume* $m \geq 2$*).*

[2] That is we set $h_m := \sum_{j=1}^{m} (m+1-2j)E_{n+j,n+j}$ and $f_m := \sum_{j=1}^{m-1} j(m-j)E_{n+j+1,n+j}$.

*(b) There are exactly two isomorphisms* $\bar{\Theta}_{(1)}, \bar{\Theta}_{(2)} : H_m(\mathfrak{sp}_{2n}) \xrightarrow{\sim} U(\mathfrak{sp}_{2n+2m}, e_m)$ *of filtered algebras such that* $\bar{\Theta}_{(i)} |_{\mathfrak{sp}_{2n}} = \iota |_{\mathfrak{sp}_{2n}}$ *(we assume* $m \geq 1$*). Moreover, we have*

$$\bar{\Theta}_{(2)} \circ \bar{\Theta}_{(1)}^{-1} : y \mapsto -y, A \mapsto A, \zeta_k \mapsto \zeta_k.$$

Let us point out that there is no explicit presentation of $W$-algebras in terms of generators and relations in general. Among few known cases are: (a) $\mathfrak{g} = \mathfrak{gl}_n$, due to [BK1], (b) $\mathfrak{g} \ni e$–the minimal nilpotent, due to [P2, Section 6]. The latter corresponds to $(e_2, \mathfrak{sl}_N)$ and $(e_1, \mathfrak{sp}_{2N})$ in our notation.

*Proof of Theorem 3.2.2.*

(a) Analogously to Lemma 3.2.1, we have an identification $F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_{-1} \simeq V_n^*$. For any $x \in V_n^*$, let $\Theta(x) \in F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_{-1}$ be the element identified with $x \in V_n^*$. The same argument as in the proof of Lemma 3.2.1 implies $[\Theta(A), \Theta(x)] = \Theta(A(x))$.

Let $\{\widetilde{F}_j\}_{j=2}^{n+m}$ be the standard degree $j$ generators of $\mathbb{C}[\mathfrak{sl}_{n+m}]^{\mathrm{SL}_{n+m}} \simeq S(\mathfrak{sl}_{n+m})^{\mathrm{SL}_{n+m}}$ (that is $1 + \sum_{j=2}^{n+m} \widetilde{F}_j(A) z^j = \det(1 + zA)$ for $A \in \mathfrak{sl}_{n+m}$) and $F_j := \mathrm{Sym}(\widetilde{F}_j) \in U(\mathfrak{sl}_{n+m})$ be the free generators of $Z(U(\mathfrak{sl}_{n+m}))$. For all $0 \leq i \leq m-2$ we set $\Theta_i := \rho(F_{m-i}) \in Z(U(\mathfrak{sl}_{n+m}, e_m))$. Then $\mathrm{gr}(\Theta_k) = \widetilde{F}_{m-k|\mathfrak{s}} \equiv \xi_k \mod S(\mathfrak{gl}_n \oplus \bigoplus_{l=k+1}^{m-2} \mathbb{C}\xi_l)$, where $\xi_k$ was defined in the proof of Lemma 3.2.1.

Let $U'$ be a subalgebra of $U(\mathfrak{sl}_{n+m}, e_m)$, generated by $\Theta(\mathfrak{gl}_n)$ and $\{\Theta_k\}_{k=0}^{m-2}$. For all $y \in V_n$, $x \in V_n^*$ we define $W(y, x) := [\Theta(y), \Theta(x)] \in F_{2m}U(\mathfrak{sl}_{n+m}, e_m)_0 \subset U'$. Let us point out that equalities $[\Theta(A), \Theta(x)] = \Theta([A, x])$, $[\Theta(A), \Theta(y)] = \Theta([A, y])$ (for all $A \in \mathfrak{gl}_n, y \in V_n, x \in V_n^*$) imply the $\mathfrak{gl}_n$-invariance of $W : V_n \times V_n^* \to U' \simeq U(\mathfrak{gl}_n)[\Theta_0, \ldots .\Theta_{m-2}]$.

By Theorem 3.1.4, $U(\mathfrak{sl}_{n+m}, e_m)$ has a basis formed by the ordered monomials in

$$\{\Theta(E_{ij}),\ \Theta(y_k),\ \Theta(x_l),\ \Theta_0, \ldots, \Theta_{m-2}\}.$$

In particular, $U(\mathfrak{sl}_{n+m}, e_m) \simeq U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\Theta_0, \ldots, \Theta_{m-2}]/(y \otimes x - x \otimes y - W(y, x))$ satisfies the PBW property. According to Remark 3.1.1, there exist polynomials $\eta_i \in \mathbb{C}[\Theta_0, \ldots, \Theta_{m-2}]$, for $0 \leq i \leq m-2$, such that $W(y, x) = \sum \eta_j r_j(y, x)$

and $\deg(\eta_i(\Theta_0,\ldots,\Theta_{m-2})) \leq 2(m-i)$. As a consequence of the latter condition: $\eta_m, \eta_{m-1} \in \mathbb{C}$. The following claim follows from the main result of the next section:

**Claim 3.2.3.** *(i) The constant $\eta_m$ is nonzero,*
*(ii) The polynomial $\eta_i(\Theta_0,\ldots,\Theta_{m-2})$ contains a nonzero multiple of $\Theta_i$, $\forall i \leq m-2$.*

This claim implies the existence and uniqueness of $\bar{\Theta} : H_m(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)$ with $\bar{\Theta}(y_k) = \Theta(y_k)$ and $\bar{\Theta}(A) = \Theta(A)$ for $A \in \mathfrak{sl}_n$. Moreover, $\bar{\Theta}(x_k) = \eta_m^{-1}\Theta(x_k)$ and $\bar{\Theta}(I_n) = \Theta(I_n) - \frac{n\eta_{m-1}}{(n+m)\eta_m}$[3], while $\bar{\Theta}(\zeta_k) \in \mathbb{C}[\Theta_k,\ldots,\Theta_{m-2}]$.

(b) Choose a Jacobson-Morozov $\mathfrak{sl}_2$-triple $(e_m, h_m, f_m) \subset \mathfrak{sp}_{2n+2m}$ in a standard way.[4] As a vector space, $\mathfrak{z}_\chi \cong \mathfrak{sp}_{2n} \oplus V_{2n} \oplus \mathbb{C}^m$ with $\mathfrak{sp}_{2n} = \mathfrak{z}_\chi(0)$, $V_{2n} = \mathfrak{z}_\chi(2m-1)$ and $\xi_j \in \mathfrak{z}_\chi(4m-4j-2)$. Here $\mathbb{C}^m$ has a basis $\{\xi_{m-k} = E_{n+1,n+2k} + \ldots + E_{n+2m-2k+1,n+2m}\}_{k=1}^m$, $V_{2n}$ is embedded via $(i \leq n)$

$$y_i \mapsto E_{i,n+2m} + (-1)^{n+i+1}E_{n+1,2n+2m+1-i},\ y_{n+i} \mapsto E_{n+2m+i,n+2m} + (-1)^{i+1}E_{n+1,n+1-i},$$

while $\mathfrak{q} = \mathfrak{z}_\chi(0) \simeq \mathfrak{sp}_{2n}$ is embedded in a natural way (via four $n\times n$ *corner blocks*). Recall the grading Gr on $U(\mathfrak{sp}_{2n+2m}, e_m)$. The induced grading Gr$'$ on $\operatorname{gr} U(\mathfrak{sp}_{2n+2m}, e_m)$, is the $\operatorname{ad}(I'_n)$-weight grading on $S(\mathfrak{z}_\chi)$. The operator $\operatorname{ad}(I'_n)$ acts trivially on $\mathbb{C}^m$, with even eigenvalues on $\mathfrak{sp}_{2n}$ and with eigenvalues $\pm 1$ on $V_{2n}^{\pm}$, where $V_{2n}^+$ is spanned by $\{y_i\}_{i\leq n}$, while $V_{2n}^-$ is spanned by $\{y_{n+i}\}_{i\leq n}$.

Analogously to Lemma 3.2.1, we get identifications of $F_{2m+1}U(\mathfrak{sp}_{2n+2m}, e_m)_{\pm 1}$ and $V_{2n}^{\pm}$. For $y \in V_{2n}^{\pm}$, let $\Theta(y)$ be the corresponding element of $F_{2m+1}U(\mathfrak{sp}_{2n+2m}, e_m)_{\pm 1}$, while for $A \in \mathfrak{sp}_{2n}$ we set $\Theta(A) := \iota(A)$. We define $\Theta : \mathfrak{sp}_{2n} \oplus V_{2n} \hookrightarrow U(\mathfrak{sp}_{2n+2m}, e_m)$ by linearity. The same reasoning as in the $\mathfrak{gl}_n$-case proves that $[\Theta(A), \Theta(y)] = \Theta(A(y))$ for any $A \in \mathfrak{sp}_{2n}, y \in V_{2n}$.

Finally, the argument involving the center goes along the same lines, so we can pick central generators $\{\Theta_k\}_{0\leq k\leq m-1}$ such that $\operatorname{gr}(\Theta_k) \equiv \xi_k \mod S(\mathfrak{sp}_{2n} \oplus \mathbb{C}\xi_{k+1} \oplus \ldots \oplus \mathbb{C}\xi_{m-1})$.

[3] The appearance of the constant $\frac{n\eta_{m-1}}{(n+m)\eta_m}$ is explained by the proof of Lemma 3.1.1(b).
[4] That is $h_m := \sum_{j=1}^{2m}(2m+1-2j)E_{n+j,n+j}$ and $f_m := \sum_{j=1}^{2m-1} j(2m-j)E_{n+j+1,n+j}$.

Let $U'$ be the subalgebra of $U(\mathfrak{sp}_{2n+2m}, e_m)$, generated by $\Theta(\mathfrak{sp}_{2n})$ and $\{\Theta_k\}_{k=0}^{m-1}$. For $x, y \in V_{2n}$, we set $W(x,y) := [\Theta(x), \Theta(y)] \in F_{4m}U(\mathfrak{sp}_{2n+2m}, e_m)_{\text{even}} \subset U'$. The map

$$W : V_{2n} \times V_{2n} \to U' \simeq U(\mathfrak{sp}_{2n})[\Theta_0, \ldots, \Theta_{m-1}]$$

is $\mathfrak{sp}_{2n}$-invariant.

Since $U(\mathfrak{sp}_{2n+2m}, e_m) \simeq U(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})[\Theta_0, \ldots, \Theta_{m-1}]/(x \otimes y - y \otimes x - W(x,y))$ satisfies the PBW property, there exist polynomials $\eta_i \in \mathbb{C}[\Theta_0, \ldots, \Theta_{m-1}]$, for $0 \leq i \leq m-1$, such that $W(x,y) = \sum \eta_j r_{2j}(x,y)$ and $\deg(\eta_i(\Theta_0, \ldots, \Theta_{m-1})) \leq 4(m-i)$ (Remark 3.1.2). The following result is analogous to Claim 3.2.3:

**Claim 3.2.4.** *(i) The constant $\eta_m$ is nonzero,*
*(ii) The polynomial $\eta_i(\Theta_0, \ldots, \Theta_{m-1})$ contains a nonzero multiple of $\Theta_i$, $\forall i \leq m-1$.*

This claim implies Theorem 3.2.2(b), where $\bar{\Theta}_{(i)}(y) = \lambda_i \cdot \Theta(y)$ for all $y \in V_{2n}$ and $\lambda_i^2 = \eta_m^{-1}$. □

## 3.3 Poisson analogue of Theorem 3.2.2

To state the main result of this section, let us introduce more notation:

- In the contexts of $(\mathfrak{sl}_{n+m}, e_m)$ and $(\mathfrak{sp}_{2n+2m}, e_m)$, we use $\mathrm{S}_{n,m}$ and $\mathfrak{z}_{n,m}$ instead of S and $\mathfrak{z}_\chi$.

- Let $\bar{\iota} : \mathfrak{gl}_n \oplus V_n \oplus V_n^* \oplus \mathbb{C}^{m-1} \xrightarrow{\sim} \mathfrak{z}_{n,m}$ be the identification from the proof of Lemma 3.2.1.

- Let $\bar{\iota} : \mathfrak{sp}_{2n} \oplus V_{2n} \oplus \mathbb{C}^m \xrightarrow{\sim} \mathfrak{z}_{n,m}$ be the identification from the proof of Theorem 3.2.2(b).

- Define $\bar{\Theta}_k = \mathrm{gr}(\Theta_k) \in S(\mathfrak{z}_{n,m})$ $0 \leq k \leq m-s$, where $s = 1$ for $\mathfrak{sp}_{2N}$ and $s = 2$ for $\mathfrak{sl}_N$.

- We consider the Poisson structure on $S(\mathfrak{z}_{n,m})$ arising from the identification

$$S(\mathfrak{z}_{n,m}) \cong \mathbb{C}[\mathrm{S}_{n,m}].$$

The following theorem can be viewed as a Poisson analogue of Theorem 3.2.2:

**Theorem 3.3.1.** *(a) The formulas*

$$\bar{\Theta}^{\mathrm{cl}}(A) = \bar{\iota}(A),\ \bar{\Theta}^{\mathrm{cl}}(y) = \bar{\iota}(y),\ \bar{\Theta}^{\mathrm{cl}}(x) = \bar{\iota}(x),\ \bar{\Theta}^{\mathrm{cl}}(\zeta_k) = (-1)^{m-k}\bar{\Theta}_k$$

*define an isomorphism* $\bar{\Theta}^{\mathrm{cl}} : H^{\mathrm{cl}}_m(\mathfrak{gl}_n) \xrightarrow{\sim} S(\mathfrak{z}_{n,m}) \simeq \mathbb{C}[\mathrm{S}_{n,m}]$ *of Poisson algebras.*

*(b) The formulas*

$$\bar{\Theta}^{\mathrm{cl}}(A) = \bar{\iota}(A),\ \bar{\Theta}^{\mathrm{cl}}(y) = \bar{\iota}(y)/\sqrt{2},\ \bar{\Theta}^{\mathrm{cl}}(\zeta_k) = \bar{\Theta}_k$$

*define an isomorphism* $\bar{\Theta}^{\mathrm{cl}} : H^{\mathrm{cl}}_m(\mathfrak{sp}_{2n}) \xrightarrow{\sim} S(\mathfrak{z}_{n,m}) \simeq \mathbb{C}[\mathrm{S}_{n,m}]$ *of Poisson algebras.*

Claims 3.2.3 and 3.2.4 follow from this theorem.

*Remark* 3.3.1. An alternative proof of Claims 3.2.3 and 3.2.4 is based on the recent result of [LNS] about the universal Poisson deformation of $\mathrm{S}\cap\mathcal{N}$ (here $\mathcal{N}$ denotes the nilpotent cone of the Lie algebra $\mathfrak{g}$). We find this argument a bit overkilling (besides, it does not provide precise formulas in the Poisson case).

*Proof of Theorem 3.3.1.*

(a) The Poisson algebra $S(\mathfrak{z}_{n,m})$ is equipped both with the Kazhdan grading and the internal grading $\mathrm{Gr}'$. In particular, the same reasoning as in the proof of Theorem 3.2.2(a) implies:

$$\{\bar{\iota}(A), \bar{\iota}(B)\} = \bar{\iota}([A,B]),\ \{\bar{\iota}(A), \bar{\iota}(y)\} = \bar{\iota}(A(y)),\ \{\bar{\iota}(A), \bar{\iota}(x)\} = \bar{\iota}(A(x)).$$

We set $\bar{W}(y,x) := \{\bar{\iota}(y), \bar{\iota}(x)\}$ for all $y \in V_n, x \in V_n^*$. Arguments analogous to those used in the proof of Theorem 3.2.2(a) imply an existence of polynomials $\bar{\eta}_i \in \mathbb{C}[\bar{\Theta}_0, \ldots, \bar{\Theta}_{m-2}]$ such that $\bar{W}(y,x) = \sum_j \bar{\eta}_j \alpha_j(y,x)$ and $\deg(\bar{\eta}_j(\bar{\Theta}_0, \ldots, \bar{\Theta}_{m-2})) = 2(m-j)$.

Combining this with Theorem 3.1.3(a) one gets that

$$\tau_1' = \sum_i x_i y_i + \sum_j \bar{\eta}_j \operatorname{tr} S^{j+1} A$$

is a Poisson-central element of $S(\mathfrak{z}_{n,m}) \cong \mathbb{C}[\mathrm{S}_{n,m}]$.

Let $\bar{\rho} : \mathfrak{z}_{\mathrm{Pois}}(\mathbb{C}[\mathfrak{sl}_{n+m}]) \to \mathfrak{z}_{\mathrm{Pois}}(\mathbb{C}[\mathrm{S}_{n,m}])$ be the restriction homomorphism. The Poisson analogue of Theorem 3.1.5 (which is, actually, much simpler) states that $\bar{\rho}$ is an isomorphism. In particular, $\tau_1' = c\bar{\rho}(\widetilde{F}_{m+1}) + p(\bar{\rho}(\widetilde{F}_2), \ldots, \bar{\rho}(\widetilde{F}_m))$ for some $c \in \mathbb{C}$ and a polynomial $p$.

Note that $\bar{\rho}(\widetilde{F}_i) = \bar{\Theta}_{m-i}$ for all $2 \leq i \leq m$. Let us now express $\bar{\rho}(\widetilde{F}_{m+1})$ via the generators of $S(\mathfrak{z}_{n,m})$. First, we describe explicitly the slice $\mathrm{S}_{n,m}$. It consists of the following elements:

$$\left\{ e_m + \sum_{i,j \leq n} x_{i,j} E_{i,j} + \sum_{i \leq n} u_i E_{i,n+1} + \sum_{i \leq n} v_i E_{n+m,i} + \sum_{k \leq m-1} w_k f_m^k - \frac{\sum_{i \leq n} x_{ii}}{m} \sum_{n < j \leq n+m} E_{jj} \right\},$$

which can be explicitly depicted as follows:

$$\mathrm{S}_{n,m} = \left\{ X = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & u_1 & 0 & 0 & \cdots & 0 \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} & u_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} & u_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \star & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \star & \star & \star & \cdots & 1 \\ v_1 & v_2 & \cdots & v_n & \star & \star & \star & \cdots & \lambda \end{pmatrix} \right\}$$

For $X \in \mathfrak{sl}_{n+m}$ of the above form let us define $X_1 \in \mathfrak{gl}_n$, $X_2 \in \mathfrak{gl}_m$ by

$$X_1 := \sum_{i,j \leq n} x_{i,j} E_{i,j}, \; X_2 := e_m + \sum_{k \leq m-1} w_k f_m^k - \frac{x_{11} + \cdots + x_{nn}}{m} \sum_{n < j \leq n+m} E_{jj},$$

that is, $X_1$ and $X_2$ are the left-up $n \times n$ and right-down $m \times m$ blocks of $X$, respectively.

The following result is straightforward:

**Lemma 3.3.2.** *Let $X, X_1, X_2$ be as above. Then:*

*(i) For $2 \leq k \leq m$ : $\widetilde{F}_k(X) = \operatorname{tr} \Lambda^k(X_1) + \operatorname{tr} \Lambda^{k-1}(X_1) \operatorname{tr} \Lambda^1(X_2) + \ldots + \operatorname{tr} \Lambda^k(X_2)$.*

*(ii) We have* $\widetilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \operatorname{tr}\Lambda^{m+1}(X_1) + \operatorname{tr}\Lambda^m(X_1)\operatorname{tr}\Lambda^1(X_2) + \ldots + \operatorname{tr}\Lambda^{m+1}(X_2)$.

Combining both statements of this lemma with the standard equality

$$\sum_{0\le j\le l} (-1)^j \operatorname{tr} S^{l-j}(X_1)\operatorname{tr}\Lambda^j(X_1) = 0, \qquad \forall l \ge 1, \tag{3.1}$$

we obtain the following result:

**Lemma 3.3.3.** *For any* $X \in \mathrm{S}_{n,m}$ *we have:*

$$\widetilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \sum_{2\le j\le m} (-1)^{m-j}\widetilde{F}_j(X)\operatorname{tr} S^{m+1-j}(X_1) + (-1)^m \operatorname{tr} S^{m+1}(X_1). \tag{3.2}$$

*Proof of Lemma 3.3.3.*

Lemma 3.3.2(i) and equality (1) imply by an induction on $k$:

$$\operatorname{tr}\Lambda^k(X_2) = \widetilde{F}_k(X) - \operatorname{tr} S^1(X_1)\widetilde{F}_{k-1}(X) + \operatorname{tr} S^2(X_1)\widetilde{F}_{k-2}(X) - \ldots + (-1)^k \operatorname{tr} S^k(X_1)\widetilde{F}_0(X),$$

for all $k \le m$, where $\widetilde{F}_1(X) := 0$, $\widetilde{F}_0(X) := 1$.

Those equalities together with Lemma 3.3.2(ii) imply:

$$\widetilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \sum_{0\le j\le m}\sum_{0\le k< m+1-j} (-1)^k \operatorname{tr}\Lambda^{m+1-j-k}(X_1)\operatorname{tr} S^k(X_1)\widetilde{F}_j(X).$$

According to (1), we have $\sum_{k=0}^{m-j}(-1)^k \operatorname{tr}\Lambda^{m+1-j-k}(X_1)\operatorname{tr} S^k(X_1) = (-1)^{m-j}\operatorname{tr} S^{m+1-j}(X_1)$. Recalling our convention $\widetilde{F}_1(X) = 0$, $\widetilde{F}_0(X) = 1$, we get (2). □

Identifying $\mathbb{C}[\mathrm{S}_{n,m}]$ with $S(\mathfrak{z}_{n,m})$ we get

$$\bar{\rho}(\widetilde{F}_{m+1}) = (-1)^m \left( \sum x_i y_i + \operatorname{tr} S^{m+1}A + \sum_{2\le j\le m} (-1)^j \bar{\Theta}_{m-j} \operatorname{tr} S^{m+1-j}A \right). \tag{3.3}$$

Substituting this into $\tau_1' = c\bar{\rho}(\widetilde{F}_{m+1}) + p(\bar{\Theta}_0, \ldots, \bar{\Theta}_{m-2})$ with $\bar{\Theta}_{m-1} := 0$, $\bar{\Theta}_m := 1$, we get

$$p(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-2})=(1-(-1)^m c)\sum_i x_iy_i+\sum_{0\leq j\leq m}(\bar{\eta}_j(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-2})-(-1)^j c\bar{\Theta}_j)\operatorname{tr}S^{j+1}A.$$

Hence $c=(-1)^m$, $p(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-2})=\sum_{0\leq j\leq m}(\bar{\eta}_j(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-2})-(-1)^{m-j}\bar{\Theta}_j)\operatorname{tr}S^{j+1}A$. According to Remark 3.1.1, the last equality is equivalent to

$$\bar{\eta}_m=1,\ \bar{\eta}_{m-1}=0,\ \bar{\eta}_j(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-2})=(-1)^{m-j}\bar{\Theta}_j,\quad\forall\,0\leq j\leq m-2,\ \ p=0.$$

This implies the statement.

(b) Analogously to the previous case and the proof of Theorem 3.2.2(b) we have:

$$\{\bar{\iota}(A),\bar{\iota}(B)\}=\bar{\iota}([A,B]),\ \{\bar{\iota}(A),\bar{\iota}(y)\}=\bar{\iota}(A(y)),\ \{\bar{\iota}(x),\bar{\iota}(y)\}=\sum\bar{\eta}_j\beta_{2j}(x,y),$$

for some $\bar{\eta}_j\in\mathbb{C}[\bar{\Theta}_0,\dots,\bar{\Theta}_{m-1}]$ such that $\deg(\bar{\eta}_j(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-1}))=4(m-j)$.

Due to Theorem 3.1.3(b), we get $\tau_1':=\sum_{i=1}^{2n}\{\widetilde{Q}_1,y_i\}y_i^*-2\sum_j\bar{\eta}_j\operatorname{tr}S^{2j+2}A\in\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{z}_{n,m}))$. In particular, $\tau_1'=c\bar{\rho}(\widetilde{F}_{m+1})+p(\bar{\rho}(\widetilde{F}_1),\dots,\bar{\rho}(\widetilde{F}_m))$ for some $c\in\mathbb{C}$ and a polynomial $p$.

Note that $\bar{\rho}(\widetilde{F}_k)=\bar{\Theta}_{m-k}$ for $1\leq k\leq m$. Let us now express $\bar{\rho}(\widetilde{F}_{m+1})$ via the generators of $S(\mathfrak{z}_{n,m})$. First, we describe explicitly the slice $\mathrm{S}_{n,m}$. It consists of the following elements:

$$\{e_m+\bar{\iota}(X_1)+\sum_{i\leq n}v_iU_{i,n+1}+\sum_{i\leq n}v_{n+i}U_{n+2m+i,n+1}+\sum_{k\leq m}w_kf_m^{2k-1}|X_1\in\mathfrak{sp}_{2n},\ v_i,v_{n+i},w_k\in\mathbb{C}\},$$

where $U_{i,j}:=E_{i,j}+(-1)^{i+j+1}E_{2n+2m+1-j,2n+2m+1-i}\in\mathfrak{sp}_{2n+2m}$. For $X\in\mathfrak{sp}_{2n+2m}$ as above, we define $X_2:=e_m+\sum_{k\leq m}w_kf_m^{2k-1}\in\mathfrak{sp}_{2m}$, viewed as the *centered* $2m\times 2m$ block of $X$.

Analogously to (3.3), we get

$$\bar{\rho}(\widetilde{F}_{m+1})=\frac{1}{4}\sum_{i=1}^{2n}\{\widetilde{Q}_1,y_i\}y_i^*-\operatorname{tr}S^{2m+2}A-\sum_{0\leq j\leq m-1}\bar{\Theta}_j\operatorname{tr}S^{2j+2}A.\qquad(3.4)$$

Comparing the above two formulas for $\tau_1'$, we get the following equality:

$$\sum_{i=1}^{2n}\{\widetilde{Q}_1, y_i\}y_i^* - 2\sum_j \bar{\eta}_j \operatorname{tr} S^{2j+2}A = c\cdot\bar{p}(\widetilde{F}_{m+1}) + p(\bar{\Theta}_0,\dots,\bar{\Theta}_{m-1}).$$

Arguments analogous to those used in part (a) establish

$$c = 4,\ p = 0,\ \bar{\eta}_m = 2,\ \bar{\eta}_j = 2\bar{\Theta}_j,\quad \forall\, j < m.$$

Part (b) follows. □

*Remark* 3.3.2. Recalling the standard convention $U(\mathfrak{g},0) = U(\mathfrak{g})$ and Example 1.2.1, we see that Theorem 3.2.2(a) (as well as Theorem 3.3.1(a)) obviously holds for $m = 1$ with $e_1 := 0 \in \mathfrak{sl}_{n+1}$.

## 3.4 Consequences

In this section we use Theorem 3.2.2 to get some new (and recover some old) results about the algebras of interest. On the $W$-algebra side, we get presentations of $U(\mathfrak{sl}_n, e_m)$ and $U(\mathfrak{sp}_{2n}, e_m)$ via generators and relations (in the latter case there was no presentation known for $m > 1$). We get much more results about the structure and the representation theory of infinitesimal Cherednik algebras using the corresponding results on $W$-algebras.

Also we determine the isomorphism from Theorem 3.2.2(a) basically explicitly.

### 3.4.1 Centers of $H_m(\mathfrak{gl}_n)$ and $H_m(\mathfrak{sp}_{2n})$

We set $s = 2$ for $\mathfrak{g} = \mathfrak{sl}_N$ and $s = 1$ for $\mathfrak{g} = \mathfrak{sp}_{2N}$. Recall the elements $\{\widetilde{F}_i\}_{i=s}^N$, where $\deg(\widetilde{F}_i) = (3-s)i$. These are the free generators of the Poisson center $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{g}))$. The Lie algebra $\mathfrak{q} = \mathfrak{z}_{\mathfrak{g}}(e,h,f)$ from Section 3.1.6 equals $\mathfrak{gl}_n$ for $(\mathfrak{g},e) = (\mathfrak{sl}_{n+m}, e_m)$ and $\mathfrak{sp}_{2n}$ for $(\mathfrak{g},e) = (\mathfrak{sp}_{2n+2m}, e_m)$. Thus $\{\widetilde{Q}_j\}$ from Section 3.1.4 are the free generators of $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{q}))$ and $Q_j := \mathrm{Sym}(\widetilde{Q}_j)$ are the free generators of $Z(U(\mathfrak{q}))$.

The following result is a straightforward generalization of formulas (3.3) and (3.4):

**Proposition 3.4.1.** *There exist* $\{b_i\}_{i=1}^n \in S(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}}[\bar\rho(\widetilde{F}_s),\ldots,\bar\rho(\widetilde{F}_m)]$ *such that:*

$$\bar\rho(\widetilde{F}_{m+i}) \equiv s_{n,m}\tau_i + b_i \mod \mathbb{C}[\bar\rho(\widetilde{F}_s),\ldots,\bar\rho(\widetilde{F}_{m+i-1})], \quad \forall\, 1\le i\le n,$$

*where* $s_{n,m}=(-1)^m$ *for* $\mathfrak{g}=\mathfrak{gl}_n$ *and* $s_{n,m}=1/4$ *for* $\mathfrak{g}=\mathfrak{sp}_{2n}$.

Define $t_k \in H_m(\mathfrak{gl}_n)$ by $t_k := \sum_{i=1}^n x_i[Q_k,y_i]$ and $t_k\in H_m(\mathfrak{sp}_{2n})$ by $t_k := \sum_{i=1}^{2n}[Q_k,y_i]y_i^*$. Combining Proposition 3.4.1, Theorems 3.1.5, 3.2.2 with the identification $\mathrm{gr}(Z(U(\mathfrak{g},e))) = \mathfrak{z}_{\mathrm{Pois}}(\mathbb{C}[\mathrm{S}])$ we get

**Corollary 3.4.2.** *For* $\mathfrak{g}$ *either* $\mathfrak{gl}_n$ *or* $\mathfrak{sp}_{2n}$, *there exist* $C_1,\ldots,C_n\in Z(U(\mathfrak{g}))[\zeta_0,\ldots,\zeta_{m-s}]$, *such that the center* $Z(H_m(\mathfrak{g}))$ *is a polynomial algebra in free generators* $\{\zeta_i\}\cup\{t_j + C_j\}_{j=1}^n$.

Considering the quotient of $H_m(\mathfrak{g})$ by the ideal $(\zeta_0-a_0,\ldots,\zeta_{m-s}-a_{m-s})$ for any $a_i\in\mathbb{C}$, we see that the center of the standard infinitesimal Cherednik algebra $H_a(\mathfrak{g})$ contains a polynomial subalgebra $\mathbb{C}[t_1+c_1,\ldots,t_n+c_n]$ for some $c_j\in Z(U(\mathfrak{g}))$.

As a consequence we also get:

**Corollary 3.4.3.** *We actually have* $Z(H_a(\mathfrak{g}))=\mathbb{C}[t_1+c_1,\ldots,t_n+c_n]$.

For $\mathfrak{g}=\mathfrak{gl}_n$ this is [Tik1, Theorem 1.1].

### 3.4.2 Symplectic leaves of Poisson infinitesimal Cherednik algebras

By Theorem 3.3.1, we get an identification of the full Poisson-central reductions of the algebras $\mathbb{C}[\mathrm{S}_{n,m}]$ and $H_m^{\mathrm{cl}}(\mathfrak{gl}_n)$ or $H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n})$. As an immediate consequence we obtain the following result:

**Proposition 3.4.4.** *Poisson varieties corresponding to arbitrary full central reductions of Poisson infinitesimal Cherednik algebras* $H_\zeta^{\mathrm{cl}}(\mathfrak{g})$ *have finitely many symplectic leaves.*

### 3.4.3 The analogue of Kostant's theorem

As another immediate consequence of Theorem 3.2.2 and discussions from Section 3.4.1, we get a generalization of the following classical result:

**Proposition 3.4.5.** *(a) The infinitesimal Cherednik algebras $H_\zeta(\mathfrak{g})$ are free over their centers.*

*(b) The full central reductions of $\operatorname{gr} H_\zeta(\mathfrak{g})$ are normal, complete intersection integral domains.*

This is [Tik2, Theorem 2.1] for $\mathfrak{g} = \mathfrak{gl}_n$, and [DT, Theorem 8.1] for $\mathfrak{g} = \mathfrak{sp}_{2n}$.

### 3.4.4 The category $\mathcal{O}$ and finite dimensional representations

The categories $\mathcal{O}$ for the finite $W$-algebras were first introduced in [BGK] and were further studied by the first author in [L3]. Namely, recall that we have an embedding $\mathfrak{q} \subset U(\mathfrak{g}, e)$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{q}$ and set $\mathfrak{g}_0 := \mathfrak{z}_\mathfrak{g}(\mathfrak{t})$. Pick an integral element $\theta \in \mathfrak{t}$ such that $\mathfrak{z}_\mathfrak{g}(\theta) = \mathfrak{g}_0$. By definition, the category $\mathcal{O}$ (for $\theta$) consists of all finitely generated $U(\mathfrak{g}, e)$-modules $M$, where the action of $\mathfrak{t}$ is diagonalizable with finite dimensional eigenspaces and, moreover, the set of weights is bounded from above in the sense that there are complex numbers $\alpha_1, \ldots, \alpha_k$ such that for any weight $\lambda$ of $M$ there is $i$ with $\alpha_i - \langle \theta, \lambda \rangle \in \mathbb{Z}_{\leqslant 0}$. The category $\mathcal{O}$ has analogues of Verma modules, $\Delta(N^0)$. Here $N^0$ is an irreducible module over the $W$-algebra $U(\mathfrak{g}_0, e)$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{t}$. In the cases of interest ($(\mathfrak{g}, e) = (\mathfrak{sl}_{n+m}, e_m), (\mathfrak{sp}_{2n+2m}, e_m)$), we have $\mathfrak{g}_0 = \mathfrak{gl}_m \times \mathbb{C}^{n-1}, \mathfrak{g}_0 = \mathfrak{sp}_{2m} \times \mathbb{C}^n$ and $e$ is principal in $\mathfrak{g}_0$. In this case, the $W$-algebra $U(\mathfrak{g}_0, e)$ coincides with the center of $U(\mathfrak{g}_0)$. Therefore $N^0$ is a one-dimensional space, and the set of all possible $N^0$ is identified, via the Harish-Chandra isomorphism, with the quotient $\mathfrak{h}^*/W_0$, where $\mathfrak{h}, W_0$ are a Cartan subalgebra and the Weyl group of $\mathfrak{g}_0$ (we take the quotient with respect to the dot-action of $W_0$ on $\mathfrak{h}^*$). As in the usual BGG category $\mathcal{O}$, each Verma module has a unique irreducible quotient, $L(N^0)$. Moreover, the map $N^0 \mapsto L(N^0)$ is a bijection between the set of finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules, $\mathfrak{h}^*/W_0$, in our case, and the set of irreducible objects in $\mathcal{O}$. We remark that all finite dimensional irreducible modules lie in $\mathcal{O}$.

One can define a formal character for a module $M \in \mathcal{O}$. The characters of Verma modules are easy to compute basically thanks to [BGK, Theorem 4.5(1)]. So to compute the characters of the simples, one needs to determine the multiplicities of the simples in the Vermas. This was done in [L3, Section 4] in the case when $e$ is principal in $\mathfrak{g}_0$. The multiplicities are given by values of certain Kazhdan-Lusztig polynomials at 1 and so are hard to compute, in general. In particular, one cannot classify finite dimensional irreducible modules just using those results.

When $\mathfrak{g} = \mathfrak{sl}_{n+m}$, a classification of the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules was obtained in [BK2]; this result is discussed in the next section. When $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, one can describe the finite dimensional irreducible representations using [L2, Theorem 1.2.2]. Namely, the centralizer of $e$ in $\mathrm{Ad}(\mathfrak{g})$ is connected. So, according to [L2], the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules are in one-to-one correspondence with the primitive ideals $\mathcal{J} \subset U(\mathfrak{g})$ such that the associated variety of $U(\mathfrak{g})/\mathcal{J}$ is $\overline{\mathbb{O}}$, where we write $\mathbb{O}$ for the adjoint orbit of $e$. The set of such primitive ideals is computable (for a fixed central character, those are in one-to-one correspondence with certain left cells in the corresponding integral Weyl group), but we will not need details on that.

One can also describe all $N^0 \in \mathfrak{h}^*/W_0$ such that $\dim L(N^0) < \infty$ when $e$ is principal in $\mathfrak{g}_0$. This is done in [L4, 5.1]. Namely, choose a representative $\lambda \in \mathfrak{h}^*$ of $N^0$ that is *antidominant* for $\mathfrak{g}_0$ meaning that $\langle \alpha^\vee, \lambda \rangle \notin \mathbb{Z}_{>0}$ for any positive root $\alpha$ of $\mathfrak{g}_0$. Then we can consider the irreducible highest weight module $L(\lambda)$ for $\mathfrak{g}$ with highest weight $\lambda - \rho$. Let $\mathcal{J}(\lambda)$ be its annihilator in $U(\mathfrak{g})$, this is a primitive ideal that depends only on $N^0$ and not on the choice of $\lambda$. Then $\dim L(N^0) < \infty$ if and only if the associated variety of $U(\mathfrak{g})/\mathcal{J}(\lambda)$ is $\overline{\mathbb{O}}$. The associated variety is computable thanks to results of [BV]; however this computation requires quite a lot of combinatorics. It seems that one can still give a closed combinatorial answer for $(\mathfrak{sp}_{2n+2m}, e_m)$ similar to that for $(\mathfrak{sl}_{n+m}, e_m)$ but we are not going to elaborate on that.

Now let us discuss the infinitesimal Cherednik algebras. In the $\mathfrak{gl}_n$-case the category $\mathcal{O}$ was defined in [Tik1, Definition 4.1] (see also [EGG, Section 5.2]). Under the isomorphism of Theorem 3.2.2(a), that category $\mathcal{O}$ basically coincides with its

$W$-algebra counterpart. The classification of finite dimensional irreducible modules and the character computation in that case was presented in Chapter 2, but the character formulas for more general simple modules were not known. For the algebras $H_m(\mathfrak{sp}_{2n})$, no category $\mathcal{O}$ was introduced, in general; the case $n = 1$ was discussed in [Kh]. The classification of finite dimensional irreducible modules was not known either.

### 3.4.5 Finite dimensional representations of $H_m(\mathfrak{gl}_n)$

Let us compare classifications of the finite dimensional irreducible representations of $U(\mathfrak{sl}_{n+m}, e_m)$ from [BK2] and $H_a(\mathfrak{gl}_n)$ from Section 2.1.6.

In the notation of [BK2][5], a nilpotent element $e_m \in \mathfrak{gl}_{n+m}$ corresponds to the partition $(1, \ldots, 1, m)$ of $n + m$. Let $S_m$ act on $\mathbb{C}^{n+m}$ by permuting the last $m$ coordinates. According to [BK2, Theorem 7.9], there is a bijection between the irreducible finite dimensional representations of $U(\mathfrak{gl}_{n+m}, e_m)$ and the orbits of the $S_m$-action on $\mathbb{C}^{n+m}$ containing a strictly dominant representative. An element $\bar{\nu} = (\nu_1, \ldots, \nu_{n+m}) \in \mathbb{C}^{n+m}$ is called strictly dominant if $\nu_i - \nu_{i+1}$ is a positive integer for all $1 \leq i \leq n$. The corresponding irreducible $U(\mathfrak{gl}_{n+m}, e_m)$-representation is denoted $L_{\bar{\nu}}$. Viewed as a $\mathfrak{gl}_n$-module (since $\mathfrak{gl}_n = \mathfrak{q} \subset U(\mathfrak{gl}_{n+m}, e_m)$), $L_{\bar{\nu}} = L'_{\bar{\nu}} \oplus \bigoplus_{i \in I} L'_{\eta_i}$, where $L'_\eta$ is the highest weight $\eta$ irreducible $\mathfrak{gl}_n$-module, $\bar{\nu} := (\nu_1, \ldots, \nu_n)$ and $I$ denotes some set of weights $\eta < \bar{\nu}$.

According to Section 2.1.6, the irreducible finite dimensional representations of the infinitesimal Cherednik algebra $H_a(\mathfrak{gl}_n)$ are parameterized by strictly dominant $\mathfrak{gl}_n$-weights $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ (that is $\lambda_i - \lambda_{i+1}$ is a positive integer for every $1 \leq i < n$), for which there exists a positive integer $k$ satisfying $P(\bar{\lambda}) = P(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - k)$. Here $P$ is a degree $m + 1$ polynomial function on the Cartan subalgebra $\mathfrak{h}_n$ of all diagonal matrices of $\mathfrak{gl}_n$, introduced in Section 2.1.2. These two descriptions are intertwined by a natural bijection, sending $\bar{\nu} = (\nu_1, \ldots, \nu_{n+m})$ to $\bar{\lambda} := (\nu_1, \ldots, \nu_n)$, while $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is sent to the class of $(\lambda_1, \ldots, \lambda_n, \nu_{n+1}, \ldots, \nu_{n+m})$ with $\{\nu_{n+1}, \ldots, \nu_{n+m}\} \cup \{\lambda_n\}$ being the set of roots of the polynomial $P(\lambda_1, \ldots, \lambda_{n-1}, t) - P(\bar{\lambda})$.

[5] In the *loc.cit.* $\mathfrak{g} = \mathfrak{gl}_{n+m}$, rather then $\mathfrak{sl}_{n+m}$. This is not crucial since $\mathfrak{gl}_{n+m} = \mathfrak{sl}_{n+m} \oplus \mathbb{C}$.

### 3.4.6 Explicit isomorphism in the case $\mathfrak{g} = \mathfrak{gl}_n$

We compute the images of particular central elements of $H_m(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_{n+m}, e_m)$ under the corresponding Harish-Chandra isomorphisms. Comparison of these images enables us to determine the isomorphism $\bar{\Theta}$ of Theorem 3.2.2(a) explicitly, in the same way as Theorem 3.3.1(a) was deduced.

Let us start from the following commutative diagram:

$$\begin{array}{ccccc} & & U(\mathfrak{sl}_{n+m}, e_m)_0 & \xleftarrow{\;j_{n,m}\;} & Z(U(\mathfrak{sl}_{n+m}, e_m)) \\ & \overset{\pi}{\swarrow} & \Big\downarrow{\scriptstyle \varpi} & & \Big\downarrow{\scriptstyle \varphi^W} \\ U(\mathfrak{sl}_{n+m}, e_m)^0 & \underset{o}{\searrow} & & & \\ & & U(\mathfrak{gl}_n) \otimes U(\mathfrak{sl}_m, e_m) & \xleftarrow{\;j_n \otimes \mathrm{Id}\;} & Z(U(\mathfrak{gl}_n)) \otimes U(\mathfrak{sl}_m, e_m) \end{array}$$

(Diagram 1)

In the above diagram:

- $U(\mathfrak{sl}_{n+m}, e_m)_0$ is the 0-weight component of $U(\mathfrak{sl}_{n+m}, e_m)$ with respect to Gr.
- $U(\mathfrak{sl}_{n+m}, e_m)^0 := U(\mathfrak{sl}_{n+m}, e_m)_0 / (U(\mathfrak{sl}_{n+m}, e_m)_0 \cap U(\mathfrak{sl}_{n+m}, e_m) U(\mathfrak{sl}_{n+m}, e_m)_{>0})$.
- $\pi$ is the quotient map, while $o$ is an isomorphism, constructed in [L3, Theorem 4.1].[6]
- The homomorphism $\varpi$ is defined as $\varpi := o \circ \pi$, making the triangle commutative.
- The homomorphisms $j_{n+m}$, $j_n$ are the natural inclusions.
- The homomorphism $\varphi^W$ is the restriction of $\varpi$ to the center, making the square commutative.
- $U(\mathfrak{sl}_m, e_m) \cong Z(U(\mathfrak{sl}_m, e_m)) \cong Z(U(\mathfrak{sl}_m))$ since $e_m$ is a principal nilpotent of $\mathfrak{sl}_m$.

We have an analogous diagram for the universal infinitesimal Cherednik algebra of $\mathfrak{gl}_n$:

$$\begin{array}{ccccc} & & H_m(\mathfrak{gl}_n)_0 & \xleftarrow{\;j'_{n,m}\;} & Z(H_m(\mathfrak{gl}_n)) \\ & \overset{\pi'}{\swarrow} & \Big\downarrow{\scriptstyle \varpi'} & & \Big\downarrow{\scriptstyle \varphi^H} \\ H_m(\mathfrak{gl}_n)^0 & \underset{o'}{\searrow} & & & \\ & & U(\mathfrak{gl}_n) \otimes \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}] & \xleftarrow{\;j_n \otimes \mathrm{Id}\;} & Z(U(\mathfrak{gl}_n)) \otimes \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}] \end{array}$$

(Diagram 2)

In the above diagram:

[6] Here we actually use the fact that $U(\mathfrak{gl}_n) \otimes U(\mathfrak{sl}_m, e_m)$ is the finite $W$-algebra $U(\mathfrak{gl}_n \oplus \mathfrak{sl}_m, 0 \oplus e_m)$.

• $H_m(\mathfrak{gl}_n)_0$ is the degree 0 component of $H_m(\mathfrak{gl}_n)$ with respect to the grading Gr, defined by $\deg(\mathfrak{gl}_n) = \deg(\zeta_0) = \ldots = \deg(\zeta_{m-2}) = 0,\ \deg(V_n) = 1,\ \deg(V_n^*) = -1$.
• $H_m(\mathfrak{gl}_n)^0$ is the quotient of $H_m(\mathfrak{gl}_n)_0$ by $H_m(\mathfrak{gl}_n)_0 \cap H_m(\mathfrak{gl}_n)H_m(\mathfrak{gl}_n)_{>0}$.[7]
• $\pi'$ denotes the quotient map, $o'$ is the natural isomorphism, $\varpi' := o' \circ \pi'$.
• The inclusion $j'_{n,m}$ is a natural inclusion of the center.
• The homomorphism $\varphi^H$ is the one induced by restricting $\varpi'$ to the center.

The isomorphism $\bar{\Theta}$ of Theorem 3.2.2(a) intertwines the gradings Gr, inducing an isomorphism $\bar{\Theta}^0 : H_m(\mathfrak{gl}_n)^0 \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)^0$. This provides the following commutative diagram:

$$\begin{array}{ccc}
Z(H_m(\mathfrak{gl}_n)) & \xrightarrow{\quad\vartheta\quad} & Z(U(\mathfrak{sl}_{n+m}, e_m)) \\
\Big\downarrow{\scriptstyle \varphi^H} & & \Big\downarrow{\scriptstyle \varphi^W} \\
Z(U(\mathfrak{gl}_n)) \otimes \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}] & \xrightarrow{\quad\underline{\vartheta}\quad} & Z(U(\mathfrak{gl}_n)) \otimes Z(U(\mathfrak{sl}_m))
\end{array}$$

(Diagram 3)

In the above diagram:
• The isomorphism $\vartheta$ is the restriction of the isomorphism $\bar{\Theta}$ to the center.
• The isomorphism $\underline{\vartheta}$ is the restriction of the isomorphism $\bar{\Theta}^0$ to the center.

Let $\mathrm{HC}_N$ denote the Harish-Chandra isomorphism $\mathrm{HC}_N : Z(U(\mathfrak{gl}_N)) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}_N^*]^{S_N,\bullet}$, where $\mathfrak{h}_N \subset \mathfrak{gl}_N$ is the Cartan subalgebra consisting of the diagonal matrices and $(S_N, \bullet)$-action arises from the $\rho_N$-*shifted* $S_N$-action with $\rho_N = (\frac{N-1}{2}, \frac{N-3}{2}, \cdots, \frac{1-N}{2}) \in \mathfrak{h}_N^*$. This isomorphism has the following property: any central element $z \in Z(U(\mathfrak{gl}_N))$ acts on the Verma module $M_{\lambda-\rho_N}$ of $U(\mathfrak{gl}_N)$ via $\mathrm{HC}_N(z)(\lambda)$.

According to Corollary 3.4.2, the center $Z(H_m(\mathfrak{gl}_n))$ is the polynomial algebra in free generators $\{\zeta_0, \ldots, \zeta_{m-2}, t'_1, \ldots, t'_n\}$, where $t'_k = t_k + C_k$. In particular, any central element of Kazhdan degree $2(m+1)$ has the form $ct'_1 + p(\zeta_0, \ldots, \zeta_{m-2})$ for some $c \in \mathbb{C}$ and $p \in \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$.

Recall that $t'_1 = t_1 + C_1$ is the Casimir element, introduced in Chapter 2. We will need to restate the results of Sections 2.1.4–2.1.5 in slightly different terms. We start by recalling the following notation:

[7] It is easy to see that $H_m(\mathfrak{gl}_n)_0 \cap H_m(\mathfrak{gl}_n)H_m(\mathfrak{gl}_n)_{>0}$ is actually a two-sided ideal of $H_m(\mathfrak{gl}_n)_0$.

- the generating series $\zeta(z)=\sum_{i=0}^{m-2}\zeta_i z^i+z^m$ (already introduced in Section 3.1.4),
- a unique degree $m+1$ polynomial $f(z)$ satisfying

$$f(z)-f(z-1)=\partial^n(z^n\zeta(z)) \text{ and } f(0)=0,$$

- a unique degree $m+1$ polynomial $g(z)=\sum_{i=1}^{m+1}g_i z^i$ satisfying

$$\partial^{n-1}(z^{n-1}g(z))=f(z),$$

- a unique degree $m$ polynomial $w(z)=\sum_{i=0}^{m}w_i z^i$ satisfying

$$f(z)=(2\sinh(\partial/2))^{n-1}(z^n w(z)),$$

- the symmetric polynomials $\sigma_i(\lambda_1,\ldots,\lambda_n)$ via

$$(u+\lambda_1)\cdots(u+\lambda_n)=\sum\sigma_i(\lambda_1,\ldots,\lambda_n)u^{n-i},$$

- the symmetric polynomials $h_j(\lambda_1,\ldots,\lambda_n)$ via

$$(1-u\lambda_1)^{-1}\cdots(1-u\lambda_n)^{-1}=\sum h_j(\lambda_1,\ldots,\lambda_n)u^j,$$

- the central element $H_j\in Z(U(\mathfrak{gl}_n))$ which is the symmetrization of $\operatorname{tr}S^j(\cdot)\in\mathbb{C}[\mathfrak{gl}_n]\cong S(\mathfrak{gl}_n)$.

The following theorem summarizes the results on $t_1'$ from Chapter 2:

**Theorem 3.4.6.** *(a) We have* $\varphi^H(t_1')=\sum_{j=1}^{m+1}H_j\otimes g_j$ *(where $g_j$ are viewed as elements of* $\mathbb{C}[\zeta_0,\ldots,\zeta_{m-2}]$*),*
*(b) We have* $(\mathrm{HC}_n\otimes\mathrm{Id})\circ\varphi^H(t_1')=\sum_{j=0}^{m}h_{j+1}\otimes w_j$.

Let $\mathrm{HC}_N'$ denote the Harish-Chandra isomorphism $Z(U(\mathfrak{sl}_N))\xrightarrow{\sim}\mathbb{C}[\bar{\mathfrak{h}}_N^*]^{S_N,\bullet}$, where $\bar{\mathfrak{h}}_N$ is the Cartan subalgebra of $\mathfrak{sl}_N$, consisting of the diagonal matrices, which can be identified with $\{(z_1,\ldots,z_N)\in\mathbb{C}^N|\sum z_i=0\}$. The natural inclusion $\bar{\mathfrak{h}}_N\hookrightarrow\mathfrak{h}_N$

induces the map

$$\mathfrak{h}_N^* \to \bar{\mathfrak{h}}_N^* : \ (\lambda_1, \ldots, \lambda_N) \mapsto (\lambda_1 - \mu, \ldots, \lambda_N - \mu), \ \textit{where} \ \ \mu := \frac{\lambda_1 + \ldots + \lambda_N}{N}.$$

The isomorphisms $\mathrm{HC}'_{n+m}, \mathrm{HC}'_m, \mathrm{HC}_n$ fit into the following commutative diagram:

$$\begin{array}{ccccc}
 & & Z(U(\mathfrak{sl}_{n+m})) & \xrightarrow{\mathrm{HC}'_{n+m}} & \mathbb{C}[\mathbb{C}^{n+m-1}]^{S_{n+m},\bullet} \\
 & \overset{\rho}{\swarrow} & \Big\downarrow \bar{\varphi}^W & & \Big\downarrow \varphi^C \\
Z(U(\mathfrak{sl}_{n+m}, e_m)) & \underset{\varphi^W}{\searrow} & Z(U(\mathfrak{gl}_n)) \otimes Z(U(\mathfrak{sl}_m)) & \xrightarrow{\mathrm{HC}_n \otimes \mathrm{HC}'_m} & \mathbb{C}[\mathbb{C}^n]^{S_n,\bullet} \otimes \mathbb{C}[\mathbb{C}^{m-1}]^{S_m,\bullet}
\end{array}$$

(Diagram 4)

In the above diagram:

- $\rho$ is the isomorphism of Theorem 3.1.5.
- The homomorphism $\bar{\varphi}^W$ is defined as the composition $\bar{\varphi}^W := \varphi^W \circ \rho$.
- The homomorphism $\varphi^C$ arises from an identification $\mathbb{C}^n \times \mathbb{C}^{m-1} \cong \mathbb{C}^{n+m-1}$ defined by

$$(\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_m) \mapsto \left(\lambda_1, \ldots, \lambda_n, \nu_1 - \frac{\lambda_1 + \ldots + \lambda_n}{m}, \ldots, \nu_m - \frac{\lambda_1 + \ldots + \lambda_n}{m}\right).$$

In particular, $\varphi^C$ is injective, so that $\varphi^W$ is injective and, hence, $\varphi^H$ is injective.

Define $\bar{\sigma}_k \in \mathbb{C}[\bar{\mathfrak{h}}_N^*]$ as the restriction of $\sigma_k$ to $\mathbb{C}^{N-1} \hookrightarrow \mathbb{C}^N$. According to Lemma 3.3.3,

$$\varphi^C(\bar{\sigma}_{m+1}) = (-1)^m h_{m+1} \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} h_{m+1-j} \otimes 1 \cdot \varphi^C(\bar{\sigma}_j). \tag{3.5}$$

Define $S_k \in Z(U(\mathfrak{sl}_{n+m}))$ by $S_k := (\mathrm{HC}'_{n+m})^{-1}(\bar{\sigma}_k)$ for all $0 \leq k \leq n+m$, so that $S_0 = 1, \ S_1 = 0$. Similarly, define $T_k \in Z(U(\mathfrak{gl}_n))$ as $T_k := \mathrm{HC}_n^{-1}(h_k)$ for all $k \geq 0$, so that $T_0 = 1$.

Equality (3.5) together with the commutativity of Diagram 4 imply

$$\bar{\varphi}^W(S_{m+1}) = (-1)^m T_{m+1} \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} T_{m+1-j} \otimes 1 \cdot \bar{\varphi}^W(S_j).$$

According to our proof of Theorem 3.2.2(a), we have $\bar{\Theta}(A) = \Theta(A) + s \operatorname{tr} A$ for all $A \in \mathfrak{gl}_n$, where $s = -\frac{\eta_{m-1}}{(n+m)\eta_m}$. In particular, $\underline{\vartheta}^{-1}(X \otimes 1) = \varphi_{-s}(X) \otimes 1$ for all $X \in Z(U(\mathfrak{gl}_n))$, where $\varphi_{-s}$ was defined in Lemma 3.1.1.

As a consequence, we get:

$$\underline{\vartheta}^{-1}(\bar{\varphi}^W(S_{m+1})) = (-1)^m \varphi_{-s}(T_{m+1}) \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} \varphi_{-s}(T_{m+1-j}) \otimes 1 \cdot \underline{\vartheta}^{-1}(\bar{\varphi}^W(S_j)). \tag{3.6}$$

The following identity is straightforward:

**Lemma 3.4.7.** *For any positive integer $i$ and any constant $\delta \in \mathbb{C}$ we have*

$$h_i(\lambda_1 + \delta, \ldots, \lambda_n + \delta) = \sum_{j=0}^{i} \binom{n+i-1}{j} h_{i-j}(\lambda_1, \ldots, \lambda_n)\delta^j.$$

As a result, we get

$$\varphi_{-s}(T_i) = \sum_{j=0}^{i} \binom{n+i-1}{j} (-s)^j T_{i-j}. \tag{3.7}$$

Combining equations (3.6) and (3.7), we get:

$$\underline{\vartheta}^{-1}(\bar{\varphi}^W(S_{m+1})) = (-1)^m T_{m+1} \otimes 1 + (-1)^{m+1} s(n+m) T_m \otimes 1 + \sum_{l=-1}^{m-2} (-1)^l T_{l+1} \otimes 1 \cdot \bar{V}_l, \tag{3.8}$$

where $\bar{V}_l = \underline{\vartheta}^{-1}(\bar{\varphi}^W(V_l))$ and for $0 \leq l \leq m-2$ we have

$$V_l = \sum_{0 \leq j \leq m-l} s^{m-l-j} \binom{n+m-j}{m-l-j} S_j.$$

On the other hand, the commutativity of Diagram 3 implies

$$\underline{\vartheta}^{-1}(\bar{\varphi}^W(S_{m+1})) = \varphi^H(\vartheta^{-1}(\rho(S_{m+1}))).$$

Recall that there exist $c \in \mathbb{C}$, $p \in \mathbb{C}[\zeta_0, \ldots \zeta_{m-2}]$ such that $\vartheta^{-1}(\rho(S_{m+1})) = ct_1' + p$.

As $\varphi^H(\zeta_i) = 1 \otimes \zeta_i$ and $\varphi^H(t_1') = \sum_{j=0}^m T_{j+1} \otimes w_j$ (by Theorem 3.4.6(b)), we get

$$\varphi^H(\vartheta^{-1}(\rho(S_{m+1}))) = 1 \otimes p(\zeta_0, \dots, \zeta_{m-2}) + \sum_{0 \le j \le m} T_{j+1} \otimes cw_j. \tag{3.9}$$

Since $w_m = 1, w_{m-1} = \frac{n+m}{2}$, the comparison of (3.8) and (3.9) yields:

- The coefficients of $T_{m+1}$ must coincide, so that $(-1)^m = cw_m \Rightarrow c = (-1)^m$.
- The coefficients of $T_m$ must coincide, so $cw_{m-1} = (-1)^{m+1}(n+m)s \Rightarrow s = -1/2$.
- The coefficients of $T_{j+1}$ must coincide for all $j \ge 0$, so that

$$w_j = (-1)^{m-j}\bar{V}_j \Rightarrow \vartheta(w_j) = (-1)^{m-j}\rho(V_j).$$

Recall that $\bar{\eta}_m = 1$, and so $\eta_m = \bar{\eta}_m = 1$. As a result $s = -\frac{\eta_{m-1}}{n+m}$, so that $\eta_{m-1} = \frac{n+m}{2}$.

The above discussion can be summarized as follows:

**Theorem 3.4.8.** *Let* $\bar{\Theta} : H_m(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)$ *be the isomorphism from Theorem 3.2.2(a). Then* $\bar{\Theta}(A) = \Theta(A) - \frac{1}{2}\operatorname{tr} A$, $\bar{\Theta}(y) = \Theta(y)$, $\bar{\Theta}(x) = \Theta(x)$, *while* $\bar{\Theta}\mid_{\mathbb{C}[\zeta_0,\dots,\zeta_{m-2}]}$ *is uniquely determined by* $\bar{\Theta}(w_j) = (-1)^{m-j}\rho(V_j)$ *for all* $0 \le j \le m-2$.

### 3.4.7 Higher central elements

It was conjectured in [DT, Remark 6.1], that the action of central elements $t_i' = t_i + c_i \in Z(H_m(\mathfrak{gl}_n))$ on the Verma modules of $H_a(\mathfrak{gl}_n)$ should be obtained from the corresponding formulas at the the Poisson level (see Theorem 3.1.3) via a *basis change* $\zeta(z) \rightsquigarrow w(z)$ and a $\rho_n$*-shift*. Actually, that is not true. However, we can choose another set of generators $u_i \in Z(H_m(\mathfrak{gl}_n))$, whose action is given by formulas similar to those of Theorem 3.1.3.

Let us define:

- central elements $u_i \in Z(H_m(\mathfrak{gl}_n))$ by $u_i := \underline{\vartheta}^{-1}(\rho(S_{m+i}))$ for all $0 \le i \le n$,
- the generating polynomial $\widetilde{u}(t) := \sum_{i=0}^n (-1)^i u_i t^i$,
- the generating polynomial $S(z) := \sum_{i=0}^n (-1)^i \underline{\vartheta}^{-1}(\bar{\varphi}^W(S_{m-i}))z^i \in \mathbb{C}[\zeta_0, \dots, \zeta_{m-2}; z]$.

The following result is proved using the arguments of the Section 3.4.6:

**Theorem 3.4.9.** *We have:*

$$(\mathrm{HC}_n \otimes \mathrm{Id}) \circ \varphi^H(\widetilde{u}(t)) = (\varphi_{1/2} \otimes \mathrm{Id}) \left( \mathrm{Res}_{z=0}\, S(z^{-1}) \prod_{1 \le i \le n} \frac{1 - t\lambda_i}{1 - z\lambda_i} \frac{z^{-1} dz}{1 - t^{-1} z} \right).$$

# 3.5 Completions

## 3.5.1 Completions of graded deformations of Poisson algebras

We first recall the machinery of completions, elaborated in [L7]. Let $Y$ be an affine Poisson scheme equipped with a $\mathbb{C}^*$-action, such that the Poisson bracket has degree $-2$. Let $\mathcal{A}_\hbar$ be an associative flat graded $\mathbb{C}[\hbar]$-algebra (where $\deg(\hbar) = 1$) such that $[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subset \hbar^2 \mathcal{A}_\hbar$ and $\mathbb{C}[Y] = \mathcal{A}_\hbar/(\hbar)$ as a graded Poisson algebra. Pick a point $x \in Y$ and let $I_x \subset \mathbb{C}[Y]$ be the maximal ideal of $x$, while $\widetilde{I}_x$ will denote its inverse image in $\mathcal{A}_\hbar$.

**Definition 3.5.1.** The completion of $\mathcal{A}_\hbar$ at $x \in Y$ is by definition $\mathcal{A}_\hbar^{\wedge_x} := \varprojlim \mathcal{A}_\hbar / \widetilde{I}_x^n$.

This is a complete topological $\mathbb{C}[[\hbar]]$-algebra, flat over $\mathbb{C}[[\hbar]]$, such that $\mathcal{A}_\hbar^{\wedge_x}/(\hbar) = \mathbb{C}[Y]^{\wedge_x}$. Our main motivation for considering this construction is the decomposition theorem, generalizing the corresponding classical result at the Poisson level:

**Proposition 3.5.1.** *[K, Theorem 2.3] The formal completion $\widehat{Y}_x$ of $Y$ at $x \in Y$ admits a product decomposition $\widehat{Y}_x = \mathcal{Z}_x \times \widehat{Y}_x^s$, where $Y^s$ is the symplectic leaf of $Y$ containing $x$ and $\mathcal{Z}_x$ is a local formal Poisson scheme.*

Fix a maximal symplectic subspace $V \subset T_x^* Y$. One can choose an embedding $V \overset{i}{\hookrightarrow} \widetilde{I}_x^{\wedge_x}$ such that $[i(u), i(v)] = \hbar^2 \omega(u, v)$ and composition $V \overset{i}{\hookrightarrow} \widetilde{I}_x^{\wedge_x} \twoheadrightarrow T_x^* Y$ is the identity map. Finally, we define $W_\hbar(V) := T(V)[h]/(u \otimes v - v \otimes u - \hbar^2 \omega(u, v))$, which is graded by setting $\deg(V) = 1,\ \deg(\hbar) = 1$ (the *homogenized Weyl algebra*). Then we have:

**Theorem 3.5.2.** *[L7, Sect. 2.1][Decomposition theorem] There is a splitting*

$$\mathcal{A}_\hbar^{\wedge_x} \cong W_\hbar(V)^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\underline{\mathcal{A}}'_\hbar,$$

*where $\underline{\mathcal{A}}'_\hbar$ is the centralizer of $V$ in $\mathcal{A}_\hbar^{\wedge_x}$.*

*Remark* 3.5.1. Recall that a filtered algebra $\{F_i(B)\}_{i\geq 0}$ is called a *filtered deformation* of $Y$ if $\mathrm{gr}_{F_\bullet} B \cong \mathbb{C}[Y]$ as Poisson graded algebras. Given such $B$, we set $\mathcal{A}_\hbar := \mathrm{Rees}_\hbar(B)$ (the Rees algebra of the filtered algebra $B$), which naturally satisfies all the above conditions.

This remark provides the following interesting examples of $\mathcal{A}_\hbar$:

- *The homogenized Weyl algebra.*

  Algebra $W_\hbar(V)$ from above is obtained via the Rees construction from the usual Weyl algebra. In the case $V = V_n \oplus V_n^*$ with a natural symplectic form, we denote $W_\hbar(V)$ just by $W_{\hbar,n}$.

- *The homogenized universal enveloping algebra.*

  For any graded Lie algebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with a Lie bracket of degree $-2$, we define

$$U_\hbar(\mathfrak{g}) := T(\mathfrak{g})[\hbar]/(x\otimes y - y\otimes x - \hbar^2[x,y] | x, y \in \mathfrak{g}),$$

graded by setting $\deg(\mathfrak{g}_i) = i,\ \deg(\hbar) = 1$.

- *The homogenized universal infinitesimal Cherednik algebra of $\mathfrak{gl}_n$.*

  Define $H_{\hbar,m}(\mathfrak{gl}_n)$ as a quotient $H_{\hbar,m}(\mathfrak{gl}_n) := U_\hbar(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \dots, \zeta_{m-2}]/J$, where

$$J = \left([x,x'], [y,y'], [A,x] - \hbar^2 A(x), [A,y] - \hbar^2 A(y), [y,x] - \hbar^2(\sum_{j=0}^{m-2} \zeta_j r_j(y,x) + r_m(y,x))\right).$$

This algebra is graded by setting $\deg(V_n \oplus V_n^*) = m+1,\ \deg(\zeta_i) = 2(m-i)$.

- *The homogenized universal infinitesimal Cherednik algebra of $\mathfrak{sp}_{2n}$.*

  Define $H_{\hbar,m}(\mathfrak{sp}_{2n})$ as a quotient $H_{\hbar,m}(\mathfrak{sp}_{2n}) := U_\hbar(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})[\zeta_0, \dots, \zeta_{m-1}]/J$, where

$$J=\left([A,y]-\hbar^2A(y),[x,y]-\hbar^2(\sum_{j=0}^{m-1}\zeta_j r_{2j}(x,y)+r_{2m}(x,y))|A\in\mathfrak{sp}_{2n},\ x,y\in V_{2n}\right).$$

This algebra is graded by setting $\deg(V_{2n})=2m+1,\ \deg(\zeta_i)=4(m-i)$.

- *The homogenized $W$-algebra.*

The homogenized $W$-algebra, associated to $(\mathfrak{g},e)$ is defined by

$$U_\hbar(\mathfrak{g},e):=(U_\hbar(\mathfrak{g})/U_\hbar(\mathfrak{g})\mathfrak{m}')^{\mathrm{ad}\,\mathfrak{m}}.$$

There are many interesting contexts in which Theorem 3.5.2 proved to be a useful tool. Among such let us mention Rational Cherednik algebras ([BE]), Symplectic Reflection algebras ([L5]) and $W$-algebras ([L1, L7]).

Actually, combining results of [L7] with Theorem 3.2.2, we get isomorphisms

$$\Psi_m: H_{\hbar,m}(\mathfrak{gl}_n)^{\wedge_v}\xrightarrow{\sim}H_{\hbar,m+1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_v},\tag{3.10}$$

$$\Upsilon_m: H_{\hbar,m}(\mathfrak{sp}_{2n})^{\wedge_v}\xrightarrow{\sim}H_{\hbar,m+1}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,2n}^{\wedge_v},\tag{3.11}$$

where $v\in V_n$ (respectively $v\in V_{2n}$) is a nonzero element and $m\geq 1$.

These decompositions can be viewed as *quantizations* of their Poisson versions:

$$\Psi_m^{\mathrm{cl}}: H_m^{\mathrm{cl}}(\mathfrak{gl}_n)^{\wedge_v}\xrightarrow{\sim}H_{m+1}^{\mathrm{cl}}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}}W_n^{\mathrm{cl},\wedge_v},\tag{3.12}$$

$$\Upsilon_m^{\mathrm{cl}}: H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n})^{\wedge_v}\xrightarrow{\sim}H_{m+1}^{\mathrm{cl}}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}}W_{2n}^{\mathrm{cl},\wedge_v},\tag{3.13}$$

where $W_n^{\mathrm{cl}}\simeq\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ with $\{x_i,x_j\}=\{y_i,y_j\}=0,\{x_i,y_j\}=\delta_i^j$.

Isomorphisms (3.10) and (3.11) are not unique and, what is worse, are inexplicit.

Let us point out that localizing at other points of $\mathfrak{gl}_n\times V_n\times V_n^*$ (respectively $\mathfrak{sp}_{2n}\times V_{2n}$) yields other decomposition isomorphisms. In particular, one gets [Tik3, Theorem 3.1][8] as follows:

*Remark* 3.5.2. For $n = 1, m > 0$, consider $e' := e_m + E_{1,2n+2} \in \mathrm{S}_{1,m} \subset \mathfrak{sp}_{2m+2}$, which is a subregular nilpotent element of $\mathfrak{sp}_{2m+2}$. Above arguments yield a decomposition isomorphism

$$H_{\hbar,m}(\mathfrak{sp}_2)^{\wedge_{E_{12}}} \xrightarrow{\sim} U_\hbar(\mathfrak{sp}_{2m+2}, e')^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} W_{\hbar,1}^{\wedge_0}. \qquad (\clubsuit)$$

The full central reduction of ($\clubsuit$) provides an isomorphism of [Tik3, Theorem 3.1].[9]

In the next section, we establish explicitly suitably modified versions of (3.10) and (3.11) for the cases $m = -1,\ 0$, which do not follow from the above arguments. In particular, the reader will get a flavor of what the formulas look like.

### 3.5.2 Decompositions (3.10) and (3.11) for $m = -1,\ 0$

• Decomposition isomorphism $H_{\hbar,-1}(\mathfrak{gl}_n)^{\wedge_v} \cong H'_{\hbar,0}(\mathfrak{gl}_{n-1})^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} W_{\hbar,n}^{\wedge_v}$.

Here $H'_{\hbar,0}(\mathfrak{gl}_{n-1})$ is defined similarly to $H_{\hbar,0}(\mathfrak{gl}_{n-1})$ with an additional central parameter $\zeta_0$ and the main relation being $[y,x] = \hbar^2\zeta_0 r_0(y,x)$, while $H_{\hbar,-1}(\mathfrak{gl}_n) := U_\hbar(\mathfrak{gl}_n \ltimes (V_n \oplus V_n^*))$.

**Notation:** We use $y_k,\ x_l,\ e_{k,l}$ when referring to the elements of $H_{\hbar,-1}(\mathfrak{gl}_n)$ and capital $Y_i,\ X_j,\ E_{i,j}$ when referring to the elements of $H'_{\hbar,0}(\mathfrak{gl}_{n-1})$. We also use indices $1 \leq k, l \leq n$ and $1 \leq i, j, i', j' < n$ to distinguish between $\leq n$ and $< n$. Finally, set $v_n := (0, \ldots, 0, 1) \in V_n$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 3.5.3.** *Formulas*

$$\Psi_{-1}(y_k) = z_k,\ \Psi_{-1}(e_{n,k}) = z_n\partial_k,\ \Psi_{-1}(e_{i,j}) = E_{i,j} + z_i\partial_j,\ \Psi_{-1}(x_j) = X_j,$$

$$\Psi_{-1}(e_{i,n}) = z_n^{-1}Y_i - \sum_{j<n} z_n^{-1}z_jE_{i,j} + z_i\partial_n,\ \Psi_{-1}(x_n) = -z_n^{-1}\zeta_0 - \sum_{p<n} z_n^{-1}z_pX_p$$

*define the isomorphism* $\Psi_{-1} : H_{\hbar,-1}(\mathfrak{gl}_n)^{\wedge_{v_n}} \xrightarrow{\sim} H'_{\hbar,0}(\mathfrak{gl}_{n-1})^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} W_{\hbar,n}^{\wedge_{v_n}}$.

[8] This result is stated in [Tik3]. However, its proof in the *loc. cit.* is computationally wrong.

[9] To be precise, we use an isomorphism of the $W$-algebra $U(\mathfrak{sp}_{2m+2}, e')$ and the non-commutative deformation of Crawley-Boevey and Holland of type $D_{m+2}$ Kleinian singularity.

Its proof is straightforward and is left to an interested reader (most of the verifications are the same as those carried out in the proof of Lemma 3.5.4 below).

• Decomposition isomorphism $H_{\hbar,0}(\mathfrak{gl}_n)^{\wedge_v} \cong H'_{\hbar,1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W^{\wedge_v}_{\hbar,n}$.

Here $H'_{\hbar,1}(\mathfrak{gl}_{n-1})$ is an algebra defined similarly to $H_{\hbar,1}(\mathfrak{gl}_{n-1})$ with an additional central parameter $\zeta_0$ and the main relation being $[y,x] = \hbar^2(\zeta_0 r_0(y,x) + r_1(y,x))$. We follow analogous conventions as for variables $y_k,\ x_l,\ e_{k,l},\ Y_i,\ X_j,\ E_{i,j}$ and indices $i,j,i',j',k,l$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 3.5.4.** *Formulas*

$$\Psi_0(y_k) = z_k,\ \Psi_0(e_{n,k}) = z_n\partial_k,\ \Psi_0(e_{i,j}) = E_{i,j}+z_i\partial_j,\ \Psi_0(e_{i,n}) = z_n^{-1}Y_i - \sum_{j<n} z_n^{-1}z_jE_{i,j}+z_i\partial_n,$$

$$\Psi_0(x_j) = -\partial_j + X_j,\ \Psi_0(x_n) = -\partial_n - \sum_{i<n} z_n^{-1}z_iX_i - z_n^{-1}(\zeta_0 + \sum_{i<n} E_{i,i})$$

*define the isomorphism* $\Psi_0 : H_{\hbar,0}(\mathfrak{gl}_n)^{\wedge_{v_n}} \xrightarrow{\sim} H'_{\hbar,1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W^{\wedge_{v_n}}_{\hbar,n}$.

*Proof.*

These formulas provide a homomorphism $H_{\hbar,0}(\mathfrak{gl}_n)^{\wedge_{v_n}} \longrightarrow H'_{\hbar,1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W^{\wedge_{v_n}}_{\hbar,n}$ if and only if $\Psi_0$ preserves all the defining relations of $H_{\hbar,0}(\mathfrak{gl}_n)$. This is quite straightforward and we present only the most complicated verifications, leaving the rest to an interested reader.

∘ Verification of $[\Psi_0(e_{i,n}),\Psi_0(e_{i',j'})] = -\hbar^2\delta^i_{j'}\Psi_0(e_{i',n})$:

$$[\Psi_0(e_{i,n}),\Psi_0(e_{i',j'})] = [z_n^{-1}Y_i - \sum_{p<n} z_n^{-1}z_pE_{i,p} + z_i\partial_n,\ E_{i',j'} + z_{i'}\partial_{j'}] =$$

$$\hbar^2(-\delta^i_{j'}z_n^{-1}Y_{i'} - z_n^{-1}z_{i'}E_{i,j'} + \delta^i_{j'}\sum_{p<n} z_n^{-1}z_pE_{i',p} + z_n^{-1}z_{i'}E_{i,j'} - \delta^i_{j'}z_{i'}\partial_n) = -\hbar^2\delta^i_{j'}\Psi_0(e_{i',n}).$$

∘ Verification of $[\Psi_0(e_{i,n}),\Psi_0(x_j)] = -\hbar^2\delta^j_i\Psi_0(x_n)$:

$$[\Psi_0(e_{i,n}),\Psi_0(x_j)] = [z_n^{-1}Y_i - \sum_{q<n} z_n^{-1}z_qE_{i,q} + z_i\partial_n, -\partial_j + X_j] =$$

$$-\hbar^2 z_n^{-1}E_{i,j} + \delta_i^j\hbar^2\partial_n + \delta_i^j\hbar^2\sum_{q<n} z_n^{-1}z_qX_q + z_n^{-1}[Y_i, X_j] =$$

$$-\hbar^2 z_n^{-1}E_{i,j} + \delta_i^j\hbar^2(\partial_n + \sum_{q<n} z_n^{-1}z_qX_q) + \hbar^2 z_n^{-1}(E_{i,j} + \delta_i^j\sum_{i<n} E_{i,i} + \delta_i^j\zeta_0) = -\delta_i^j\hbar^2\Psi_0(x_n).$$

∘ Verification of $[\Psi_0(e_{i,n}), \Psi_0(x_n)] = 0$:

$$[\Psi_0(e_{i,n}), \Psi_0(x_n)] = [z_n^{-1}Y_i - \sum_{p<n} z_n^{-1}z_pE_{i,p} + z_i\partial_n, -\partial_n - \sum_{j<n} z_n^{-1}z_jX_j - z_n^{-1}(\zeta_0 + \sum_{j<n} E_{j,j})] =$$

$$\hbar^2(\sum_{p<n} z_n^{-2}z_pE_{i,p} - z_n^{-2}Y_i + z_iz_n^{-2}\zeta_0 + z_iz_n^{-2}\sum_{j<n} E_{j,j} + z_n^{-2}Y_i - \sum_{j<n} z_jz_n^{-2}[Y_i, X_j]) = 0.$$

Once homomorphism $\Psi_0$ is established, it is easy to check that the map

$$z_k \mapsto y_k,\ \partial_k \mapsto y_n^{-1}e_{n,k},\ E_{i,j} \mapsto e_{i,j} - y_iy_n^{-1}e_{n,j}, X_j \mapsto x_j + y_n^{-1}e_{n,j},$$

$$Y_i \mapsto \sum y_k(e_{i,k} - y_iy_n^{-1}e_{n,k}),\ \zeta_0 \mapsto -\sum y_kx_k - \sum e_{k,k}$$

provides the inverse to $\Psi_0$. This completes the proof of the lemma. □

• Decomposition isomorphism $H_{\hbar,-1}(\mathfrak{sp}_{2n})^{\wedge_v} \cong H'_{\hbar,0}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,2n}^{\wedge_v}$.

Here $H'_{\hbar,0}(\mathfrak{sp}_{2n-2})$ is defined similarly to $H_{\hbar,0}(\mathfrak{sp}_{2n-2})$ with an additional central parameter $\zeta_0$ and the main relation being $[x,y] = \hbar^2\zeta_0r_0(x,y)$, while $H_{\hbar,-1}(\mathfrak{sp}_{2n}) := U_\hbar(\mathfrak{sp}_{2n} \ltimes V_{2n})$.

**Notation:** We use $y_k,\ u_{k,l} := e_{k,l} + (-1)^{k+l+1}e_{2n+1-l,2n+1-k}$ when referring to the elements of $H_{\hbar,-1}(\mathfrak{sp}_{2n})$ and $Y_i,\ U_{i,j} := E_{i,j} + (-1)^{i+j+1}E_{2n-1-j,2n-1-i}$ when referring to the elements of $H'_{\hbar,0}(\mathfrak{sp}_{2n-2})$. Note that $\{u_{k,l}\}_{k,l\geq 1}^{k+l\leq 2n+1}$ is a basis of $\mathfrak{sp}_{2n}$, while $\{U_{i,j}\}_{i,j\geq 1}^{i+j\leq 2n-1}$ is a basis of $\mathfrak{sp}_{2n-2}$. We use indices $1 \leq k,l \leq 2n$ and $1 \leq i,j \leq 2n-2$. Finally, set $v_1 := (1,0,\ldots,0) \in V_{2n}$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 3.5.5.** *Define* $\psi_1(u_{k,l}) := z_k\partial_l + (-1)^{k+l+1}z_{2n+1-l}\partial_{2n+1-k}$ *for all* $k,l$. *We also define* $\psi_0(u_{1,k}) = 0,\ \psi_0(u_{i+1,1}) = Y_i,\ \psi_0(u_{i+1,j+1}) = U_{i,j},\ \psi_0(u_{2n,1}) = \zeta_0$. *Then* *formulas* $\Upsilon_{-1}(y_k) = z_k,\ \Upsilon_{-1}(u_{k,l}) = \psi_0(u_{k,l}) + \psi_1(u_{k,l})$ *give rise to the isomorphism*

$$\Upsilon_{-1} : H_{\hbar,-1}(\mathfrak{sp}_{2n})^{\wedge_{v_1}} \xrightarrow{\sim} H'_{\hbar,0}(\mathfrak{sp}_{2n-2})^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} W_{\hbar,2n}^{\wedge_{v_1}}.$$

The proof of this lemma is straightforward and is left to an interested reader.

- Finally, we have the case of $\mathfrak{g} = \mathfrak{sp}_{2n},\ m = 0$.

  There is also a decomposition isomorphism

$$\Upsilon_0 : H_{\hbar,0}(\mathfrak{sp}_{2n})^{\wedge_v} \xrightarrow{\sim} H_{\hbar,1}(\mathfrak{sp}_{2n-2})^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} W_{\hbar,2n}^{\wedge_v}.$$

This isomorphism can be made explicit, but we find the formulas quite heavy and unrevealing, so we leave them to an interested reader.

# Chapter 4

# Generalization to the $\mathrm{SO}_N$ case

This chapter is based on [T1].

## 4.1 Classification results

Our first result is a full classification of all $\kappa$ from Section 1.1.2 satisfying (†) for the case of $(\mathrm{SO}_N, V_N)$, which is similar to [EGG, Theorem 3.14] for $(\mathrm{Sp}_{2n}, V_{2n})$. But it turns out that the subscheme $\Phi \subset \mathrm{SO}_N$ from Section 1.1.2 is not reduced in this case and so we need a more detailed argument.

**Theorem 4.1.1.** *The PBW property holds for $\mathcal{H}_\kappa(\mathrm{SO}_N, V_N)$ if and only if there exists an $\mathrm{SO}_N$-invariant distribution $c \in \mathcal{O}(S)^*$ such that $\kappa(x,y) = ((g-g^{-1})x, y)c$ for all $x, y \in V_N$.*

The proof of this theorem is presented in Section 4.2.

To formulate our classification of infinitesimal Cherednik algebras $H_\kappa(\mathfrak{so}_N, V_N)$ we define:

- $\gamma_{2j+1}(x,y) \in S(\mathfrak{so}_N) \simeq \mathbb{C}[\mathfrak{so}_N]$ by

$$(x, A(1+\tau^2A^2)^{-1}y)\det(1+\tau^2A^2)^{-1/2} = \sum_{j\geq 0}\gamma_{2j+1}(x,y)(A)\tau^{2j}, \quad A \in \mathfrak{so}_N,$$

- $r_{2j+1}(x,y) \in U(\mathfrak{so}_N)$ to be the symmetrization of $\gamma_{2j+1}(x,y) \in S(\mathfrak{so}_N)$.

The following theorem is proved in Section 4.3:

**Theorem 4.1.2.** *The PBW property holds for $H_\kappa(\mathfrak{so}_N, V_N)$ if and only if there exists a non-negative integer $k$ and parameters $\zeta_0, \ldots, \zeta_k \in \mathbb{C}$ such that $\kappa = \sum_{j=0}^k \zeta_j r_{2j+1}$.*

We denote the corresponding algebra by $H_\zeta(\mathfrak{so}_N, V_N)$ for $\kappa$ of the above form.

*Remark* 4.1.1. (a) For $\zeta_0 \neq 0$ we have $H_{\zeta_0 r_1}(\mathfrak{so}_N, V_N) \simeq U(\mathfrak{so}_{N+1})$. Thus, for an arbitrary $\zeta$ we can regard $H_\zeta(\mathfrak{so}_N, V_N)$ as a deformation of $U(\mathfrak{so}_{N+1})$.
(b) Theorem 4.1.2 does not hold for $N = 2$, since only half of the infinitesimal Hecke algebras are of the form given in the theorem (algebras $H_\kappa(\mathfrak{so}_2, V_2)$ are the same as $H_{\kappa'}(\mathfrak{gl}_1, V_1 \oplus V_1^*)$).

## 4.2 Proof of Theorem 4.1.1

• *Sufficiency.*

Given any $c \in (\mathcal{O}(S)^*)^{\mathrm{SO}_N}$, the formula $\kappa(x, y) := ((g - g^{-1})x, y)c$ defines a skew-symmetric $\mathrm{SO}_N$-equivariant pairing $\kappa : V_N \times V_N \to \mathcal{O}(\mathrm{SO}_N)^*$. For $x, y, z \in V_N$ and $g \in \mathrm{SO}_N$ we define

$$h(x, y, z; g) := (z - z^g)(x^g - x^{g^{-1}}, y) + (y - y^g)(z^g - z^{g^{-1}}, x) + (x - x^g)(y^g - y^{g^{-1}}, z).$$

**Lemma 4.2.1.** *We have $h(x, y, z; g) = 0$ for all $x, y, z \in V_N$ and $g \in S$.*

*Proof.*

For any $g \in S$ consider the decomposition $V = V^g \oplus (V^g)^\perp$, where $V^g := \mathrm{Ker}(1-g)$ is a codimension $\leq 2$ subspace of $V$. If either of the vectors $x, y, z$ belongs to $V^g$, then all the three summands are zero and the result follows. Thus, we can assume $x, y, z \in (V^g)^\perp$. Without loss of generality, we can assume that $z = \alpha x + \beta y$ with $\alpha, \beta \in \mathbb{C}$, since dim $(V^g)^\perp \leq 2$. Then

$$h(x, y, z; g) = \alpha \left((x - x^g)(x^g - x^{g^{-1}}, y) + (x - x^g)(y^g - y^{g^{-1}}, x) + (y - y^g)(x^g - x^{g^{-1}}, x)\right)$$

$$+\beta \left((y - y^g)(x^g - x^{g^{-1}}, y) + (y - y^g)(y^g - y^{g^{-1}}, x) + (x - x^g)(y^g - y^{g^{-1}}, y)\right).$$

Clearly, $(x^g - x^{g^{-1}}, x) = (x^g, x) - (x, x^g) = 0$ and $(x^g - x^{g^{-1}}, y) = -(y^g - y^{g^{-1}}, x)$, so that the first sum is zero. Likewise, the second sum is zero. The result follows. $\square$

Since $c$ is scheme-theoretically supported on $S$ and $h(x, y, z; g) = 0$ for all $x, y, z \in V_N, g \in S$, we get $h(x, y, z; g)c = 0$ and so (†) holds.

- *Necessity.*

Let $I \subset \mathbb{C}[\mathrm{SO}_N]$ be the defining ideal of $\Phi$, that is, $I$ is generated by $3 \times 3$ determinants of $1 - g$. Consider a closed subscheme $\bar{\Phi} \subset \mathfrak{so}_N$, defined by the ideal $\bar{I} := (\wedge^3 A) \subset \mathbb{C}[\mathfrak{so}_N]$.

Define $E := \mathrm{Rad}(I)/I$ and $\bar{E} := \mathrm{Rad}(\bar{I})/\bar{I}$. Notice that $\bar{E} \simeq E$, since $\Phi$ is reduced in the formal neighborhood of any point $g \neq 1$, while the exponential map defines an isomorphism of formal completions $\exp : \bar{\Phi}^{\wedge_0} \xrightarrow{\sim} \Phi^{\wedge_1}$.

On the other hand, we have a short exact sequence of $\mathrm{SO}_N$-modules

$$0 \to E \to \mathcal{O}(\Phi) \to \mathcal{O}(S) \to 0,$$

inducing the following short exact sequence of vector spaces

$$0 \to (\wedge^2 V_N^* \otimes \mathcal{O}(S)^*)^{\mathrm{SO}_N} \xrightarrow{\phi} (\wedge^2 V_N^* \otimes \mathcal{O}(\Phi)^*)^{\mathrm{SO}_N} \xrightarrow{\psi} (\wedge^2 V_N^* \otimes E^*)^{\mathrm{SO}_N} \to 0. \qquad (\natural)$$

It is easy to deduce the necessity for $\kappa \in \mathrm{Im}(\phi)$ by utilizing the arguments from the proof of [EGG, Theorem 3.14(ii)]. Combining this observation with Proposition 1.1.1 and an isomorphism $E \simeq \bar{E}$, it suffices to prove the following result:

**Lemma 4.2.2.** *(a) The space $(\wedge^2 V_N^* \otimes \bar{E}^*)^{\mathrm{SO}_N}$ is either zero or one-dimensional.*
*(b) If $(\wedge^2 V_N^* \otimes \bar{E}^*)^{\mathrm{SO}_N} \neq 0$, there exists $\kappa' \in (\wedge^2 V_N^* \otimes \mathcal{O}(\bar{\Phi})^*)^{\mathrm{SO}_N}$ not satisfying (†).*[1]

Notice that the adjoint action of $\mathrm{SO}_N$ on $\mathfrak{so}_N$ extends to the action of $\mathrm{GL}_N$ by $g.A = gAg^t$ for $A \in \mathfrak{so}_N, g \in \mathrm{GL}_N$. This endows $\mathbb{C}[\mathfrak{so}_N]$ with a structure of a $\mathrm{GL}_N$-module and both $\bar{I}, \mathrm{Rad}(\bar{I})$ are $\mathrm{GL}_N$-invariant. The following fact was communicated to us by Steven Sam:

[1] So that any element of $(\wedge^2 V_N^* \otimes \mathcal{O}(\Phi)^*)^{\mathrm{SO}_N}$ satisfying (†) should be in the image of $\phi$.

**Claim 4.2.3.** *As* $\mathfrak{gl}_N$*-representations* $\bar{E} \simeq \wedge^4 V_N$.

Let us first deduce Lemma 4.2.2 from this Claim.

*Proof of Lemma 4.2.2.*

(a) The following facts are well-known (see [FH, Theorems 19.2, 19.14]):

∘ the $\mathfrak{so}_{2n+1}$-representations $\{\wedge^i V_{2n+1}\}_{i=0}^n$ are irreducible and pairwise non-isomorphic,

∘ the $\mathfrak{so}_{2n}$-representation $\wedge^n V_{2n}$ decomposes as $\wedge^n V_{2n} \simeq \wedge^n_+ V_{2n} \oplus \wedge^n_- V_{2n}$, and $\mathfrak{so}_{2n}$-representations $\{\wedge^0 V_{2n}, \dots, \wedge^{n-1} V_{2n}, \wedge^n_+ V_{2n}, \wedge^n_- V_{2n}\}$ are irreducible and pairwise non-isomorphic.

Combining these facts with Claim 4.2.3 and an isomorphism $\wedge^k V_N \simeq \wedge^{N-k} V_N^*$, we get

$$(\wedge^2 V_{2n+1}^* \otimes \bar{E}^*)^{\mathrm{SO}_{2n+1}} = 0, \quad \text{while} \quad \dim((\wedge^2 V_{2n}^* \otimes \bar{E}^*)^{\mathrm{SO}_{2n}}) = \begin{cases} 1\,, & n = 3 \\ 0\,, & n \neq 3 \end{cases}.$$

(b) For $N = 6$, any nonzero element of $(\wedge^2 V_6^* \otimes \bar{E}^*)^{\mathrm{SO}_6}$ corresponds to the composition

$$\wedge^2 V_6 \underset{\varphi}{\xrightarrow{\sim}} \wedge^4 V_6^* \simeq \bar{E}^*.$$

Let $M_4 \subset \mathbb{C}[\mathfrak{so}_N]_2$ be the subspace spanned by the Pfaffians of all $4 \times 4$ principal minors. This subspace is $\mathrm{GL}_6$-invariant and $M_4 \simeq \wedge^4 V_6$ as $\mathfrak{gl}_6$-representations. Claim 4.2.3 and simplicity of the spectrum of the $\mathfrak{gl}_6$-module $\mathbb{C}[\mathfrak{so}_6]$ (see Theorem 4.2.5 below) imply $M_4 \subset \mathrm{Rad}(\bar{I})$ and $M_4 \cap \bar{I} = 0$. It follows that $M_4$ corresponds to the copy of $\wedge^4 V_6 \subset \mathrm{Rad}(\bar{I})/\bar{I}$ from Claim 4.2.3.

Choose an orthonormal basis $\{y_i\}_{i=1}^6$ of $V_6$, so that any element $A \in \mathfrak{so}_6$ is skew-symmetric with respect to this basis. We denote the corresponding Pfaffian by $\mathrm{Pf}_{\widehat{i,j}}$ (with a correctly chosen sign).[2] We define $\kappa'(y_i \otimes y_j) \in U(\mathfrak{so}_6)$ to be the symmetrization of $\mathrm{Pf}_{\widehat{i,j}}$. Identifying $U(\mathfrak{so}_6)$ with $S(\mathfrak{so}_6)$ as $\mathfrak{so}_6$-modules, we easily see that $\kappa' : \wedge^2 V_6 \to U(\mathfrak{so}_6)$ is $\mathfrak{so}_6$-invariant.

[2] To make a compatible choice of signs, define $\mathrm{Pf}_{\widehat{i,j}}$ as the derivative of the total Pfaffian Pf along $E_{ij} - E_{ji}$.

However, $\kappa'$ does not satisfy the Jacobi identity. Indeed, let us define $\bar{\kappa}' : V_6 \otimes V_6 \to S(\mathfrak{so}_6)$ by $\bar{\kappa}'(y_i \otimes y_j) = \mathrm{Pf}_{\widehat{i,j}}$. Then for any three different indices $i, j, k$, the corresponding expressions $\{P_{\widehat{i,j}}, x_k\}, \{P_{\widehat{j,k}}, x_i\}, \{P_{\widehat{k,i}}, x_j\}$ coincide up to a sign and are nonzero. So their sum is also non-zero, implying that (†) fails for $\kappa'$. □

• *Proof of Claim 4.2.3*

○ *Step 1: Description of* $\mathrm{Rad}(\bar{I})$.

Let $\mathrm{Pf}_{ijkl} \in \mathbb{C}[\mathfrak{so}_N]_2$ be the Pfaffians of the principal $4 \times 4$ minors corresponding to the rows/columns $\#i, j, k, l$. It is clear that $\mathrm{Pf}_{ijkl}$ vanish at rank $\leq 2$ matrices and so $\mathrm{Pf}_{ijkl} \in \mathrm{Rad}(\bar{I})$. A beautiful classical result states that those elements generate $\mathrm{Rad}(\bar{I})$, in fact:

**Theorem 4.2.4.** *[We, Theorem 6.4.1(b)] The ideal* $\mathrm{Rad}(\bar{I})$ *is generated by* $\{\mathrm{Pf}_{ijkl}\}$.

○ *Step 2: Decomposition of* $\mathbb{C}[\mathfrak{so}_N]$ *as a* $\mathfrak{gl}_N$*-module.*

Let $T$ be the set of all length $\leq N$ Young diagrams $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$. There is a natural bijection between $T$ and the set of all irreducible finite dimensional polynomial $\mathfrak{gl}_N$-representations. For $\lambda \in T$, we denote the corresponding irreducible $\mathfrak{gl}_N$-representation by $L_\lambda$. Let $T^e$ be the subset of $T$ consisting of all Young diagrams with even columns.

The following result describes the decomposition of $\mathbb{C}[\mathfrak{so}_N]$ into irreducibles:

**Theorem 4.2.5.** *[AF, Theorem 2.5] As* $\mathfrak{gl}_N$*-representations*

$$\mathbb{C}[\mathfrak{so}_N] \simeq S(\wedge^2 V_N) \simeq \bigoplus_{\lambda \in T^e} L_\lambda.$$

For any $\lambda \in T^e$, let $\mathfrak{J}_\lambda \subset \mathbb{C}[\mathfrak{so}_N]$ be the ideal generated by $L_\lambda \subset \mathbb{C}[\mathfrak{so}_N]$, while $T^e_\lambda \subset T^e$ be the subset of the diagrams containing $\lambda$. The arguments of [AF] (see also [D, Theorem 5.1]) imply that $\mathfrak{J}_\lambda \simeq \bigoplus_{\mu \in T^e_\lambda} L_\mu$ as $\mathfrak{gl}_N$-modules.

○ *Step 3:* $\mathrm{Rad}(\bar{I})$ *and* $\bar{I}$ *as* $\mathfrak{gl}_N$*-representations.*

Since the subspace $M_4 \subset \mathbb{C}[\mathfrak{so}_N]$, spanned by $\mathrm{Pf}_{ijkl}$, is $\mathfrak{gl}_N$-invariant and is isomorphic to $\wedge^4 V_N$, the results of the previous steps imply that $\mathrm{Rad}(\bar{I}) \simeq \bigoplus_{\mu \in T^e_{(1^4)}} L_\mu$ as $\mathfrak{gl}_N$-modules.

Let $N_3 \subset \mathbb{C}[\mathfrak{so}_N]_3$ be the subspace spanned by the determinants of all $3\times 3$ minors. This is a $\mathfrak{gl}_N$-invariant subspace.

**Lemma 4.2.6.** *We have $N_3 \simeq L_{(2^2,1^2)} \oplus L_{(1^6)}$ as $\mathfrak{gl}_N$-representations.*

*Proof.*

According to Step 1, we have $\mathbb{C}[\mathfrak{so}_N]_3 \simeq L_{(1^6)} \oplus L_{(2^2,1^2)} \oplus L_{(3^2)}$. Since the space of $3 \times 3$ minors identically vanishes when $N = 2$, and the Schur functor (3,3) does not, it rules $L_{(3^2)}$ out. Also, the space of $3 \times 3$ minors is nonzero for $N = 4$, while the Schur functor $(1^6)$ vanishes, so $N_3 \not\simeq L_{(1^6)}$. Since partition $(1^6)$ corresponds to the subspace $M_6 \subset \mathbb{C}[\mathfrak{so}_N]$ spanned by $6 \times 6$ Pfaffians, it suffices to prove that $M_6 \subset N_3$. The latter is sufficient to verify for $N = 6$, that is, the Pfaffian Pf of a $6 \times 6$ matrix is a linear combination of its $3 \times 3$ determinants.[3]

Let $\det^{pqs}_{ijk}$ be the determinant of the $3 \times 3$ minor, obtained by intersecting rows $\#i, j, k$ and columns $\#p, q, s$. The following identity is straightforward:

$$-4\,\mathrm{Pf} = -\det^{456}_{123}+\det^{356}_{124}-\det^{346}_{125}+\det^{345}_{126}-\det^{256}_{134}+\det^{246}_{135}-\det^{245}_{136}-\det^{236}_{145}+\det^{235}_{146}-\det^{234}_{156}.$$

This completes the proof of the lemma. □

The results of Step 2 imply that $\bar{I} \simeq \bigoplus_{\mu \in T^e_{(2^2,1^2)} \cup T^e_{(1^6)}} L_\mu$ as $\mathfrak{gl}_N$-modules.

Claim 4.2.3 follows from the aforementioned descriptions of $\mathfrak{gl}_N$-modules $\bar{I}$ and $\mathrm{Rad}(\bar{I})$. ■

## 4.3 Proof of Theorem 4.1.2

Let us introduce some notation:

- $K := \mathrm{SO}_N(\mathbb{R})$ (the maximal compact subgroup of $G = \mathrm{SO}_N(\mathbb{C})$),

[3] The conceptual proof of this fact is as follows. Note that determinants of $3\times 3$ minors of $A \in \mathfrak{so}_6$ are just the matrix elements of $\wedge^3 A$, and $\wedge^3 A$ acts on $\wedge^3 V_6 = \wedge^3_+ V_6 \oplus \wedge^3_- V_6$. It is easy to see that the trace of $\wedge^3 A$ on $\wedge^3_+ V_6$ is nonzero. This provides a cubic invariant for $\mathfrak{so}_6$, which is unique up to scaling (multiple of Pf).

- $s_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \cdots & 0 \\ \sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \in K, \quad \theta \in [-\pi, \pi],$
- $S_\theta := \{g s_\theta g^{-1} | g \in K\} \subset K,$
- $S_{\mathbb{R}} := S \cap K = \bigcup_{\theta\in[0,\pi]} S_\theta$[4], so that $S_{\mathbb{R}}/K$ gets identified with $S^1/\mathbb{Z}_2$.

According to Theorem 4.1.1, there exists a $\mathbb{Z}_2$-invariant $c \in \mathcal{O}_0(S^1)^*$, which is a linear combination of the delta-function $\delta_0$ (at $0 \in S^1$) and its even derivatives $\delta_0^{(2k)}$, such that[5]

$$\kappa(x,y) = \int_{-\pi}^{\pi} c(\theta) \left( \int_{S_\theta} ((g - g^{-1})x, y)\, dg \right) d\theta \quad \text{for all } x, y \in V_N.$$

For $g \in S_{\mathbb{R}}$ we define a 2-dimensional subspace $V_g \subset V_N$ by $V_g := \mathrm{Im}(1-g)$. To evaluate the above integral, choose length 1 orthogonal vectors $p, q \in V_g$ such that the restriction of $g$ to $V_g$ is given by the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in the basis $\{p, q\}$.

Let us define $J_{p,q} := q \otimes p^t - p \otimes q^t \in \mathfrak{so}_N(\mathbb{R})$. We have:

- $((g - g^{-1})x, y) = 2\sin\theta \cdot (x, J_{p,q}y),$
- $g = \exp(\theta J_{p,q})$, since $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \exp\left(\theta \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$

As a result, we get:[6]

$$\kappa(x,y) = \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} (x, J_{p,q}y) \left( \int_{-\pi}^{\pi} 2c(\theta)\sin\theta \cdot e^{\theta J_{p,q}}\, d\theta \right) dqdp, \qquad (4.1)$$

where $S^{N-1}$ is the unit sphere in $\mathbb{R}^N$ centered at the origin and $S^{N-2}(p)$ is the unit sphere in $\mathbb{R}^{N-1}(p) \subset \mathbb{R}^N$, the hyperplane orthogonal to the line passing through $p$

[4] Note that $S_\theta$ and $S_{-\theta}$ coincide for $N \geq 3$. That explains why $\theta \in [0, \pi]$ instead of $\theta \in [-\pi, \pi]$.

[5] Here we integrate over the whole circle $S^1$ instead of $S^1/\mathbb{Z}_2$, but we require $c(\theta) = c(-\theta)$.

[6] Generally speaking, the integration should be taken over the Grassmannian $G_2(\mathbb{R}^N)$. However, it is easier to integrate over the Stiefel manifold $V_2(\mathbb{R}^N)$, which is a principal O(2)-bundle over $G_2(\mathbb{R}^N)$.

and the origin.

Since $c(\theta)$ is an arbitrary linear combination of the delta-function and its even derivatives, the above integral is a linear combination of the following integrals:

$$\int_{p\in S^{N-1}}\int_{q\in S^{N-2}(p)}(x,J_{p,q}y)\cdot J_{p,q}^{2k+1}\;dq\;dp,\quad k\geq 0.$$

This is a standard integral (see [EGG, Section 4.2] for the analogous calculations). Identifying $U(\mathfrak{so}_N)$ with $S(\mathfrak{so}_N)$ via the symmetrization map, it suffices to compute the integral

$$I_{m;x,y}(A)=\int_{p\in S^{N-1}}\int_{q\in S^{N-2}(p)}(x,J_{p,q}y)\cdot \operatorname{tr}(AJ_{p,q})^m\;dq\;dp,\quad A\in\mathfrak{so}_N(\mathbb{R}).$$

To compute this expression we introduce

$$F_m(A):=\int_{p\in S^{N-1}}\int_{q\in S^{N-2}(p)}\operatorname{tr}(AJ_{p,q})^{m+1}\;dq\;dp=\int_{p\in S^{N-1}}\int_{q\in S^{N-2}(p)}(2(Aq,p))^{m+1}\;dq\;dp,$$

so that the former integral can be expressed in the following way:

$$dF_m(A)(x\otimes y^t-y\otimes x^t)=-2(m+1)I_{m;x,y}(A).$$

Now we compute $F_m(A)$. Notice that

$$G_m(A,\zeta):=\int_{p\in\mathbb{R}^N}\int_{q\in\mathbb{R}^{N-1}(p)}(2(Aq,p))^{m+1}e^{-\zeta(p,p)-\zeta(q,q)}\;dq\;dp=$$

$$\int_0^\infty\int_0^\infty e^{-\zeta r_1^2-\zeta r_2^2}\int_{|p|=r_1}\int_{|q|=r_2}(2(Aq,p))^{m+1}\;dq\;dp\;dr_2\;dr_1=$$

$$\int_0^\infty\int_0^\infty e^{-\zeta r_1^2-\zeta r_2^2}r_1^{m+N}r_2^{m+N-1}\;dr_2dr_1\cdot F_m(A)=K_{m+N}(\zeta)K_{m+N-1}(\zeta)F_m(A),$$

where

$$K_l(\zeta):=\int_0^\infty e^{-\zeta r^2}r^l\;dr=\begin{cases}\frac{k!}{2\zeta^{k+1}}, & l=2k+1\\ \frac{(2k-1)!!\sqrt{\pi}}{2^{k+1}\zeta^{k+1/2}}, & l=2k\end{cases}.$$

As a result, we get

$$G_m(A,\zeta) = \frac{\sqrt{\pi}(m+N-1)!}{2^{m+N+1}\zeta^{m+N+1/2}} F_m(A).$$

On the other hand, we have:

$$\sum_{m=-1}^{\infty} \frac{1}{(m+1)!} G_m(A,\zeta) = \int_{p\in\mathbb{R}^N} \int_{q\in\mathbb{R}^{N-1}(p)} e^{2(Aq,p)} e^{-\zeta(p,p)-\zeta(q,q)}\, dq\, dp =$$

$$\int_{p\in\mathbb{R}^N} e^{-\zeta(p,p)} \int_{q\in\mathbb{R}^{N-1}(p)} e^{-2(q,Ap)-\zeta(q,q)}\, dq\, dp \overset{q':=q+\frac{Ap}{\zeta}}{=}$$

$$\int_{p\in\mathbb{R}^N} e^{-\zeta(p,p)} \int_{q'\in\mathbb{R}^{N-1}(p)} e^{-\zeta(q',q')} e^{\frac{1}{\zeta}(Ap,Ap)}\, dq'\, dp = \int_{p\in\mathbb{R}^N} e^{-\zeta(p,p)+\frac{1}{\zeta}(Ap,Ap)}\, dp \cdot (\pi/\zeta)^{\frac{N-1}{2}} =$$

$$(\pi/\zeta)^{\frac{N-1}{2}} \int_{p\in\mathbb{R}^N} e^{((-\zeta-\frac{1}{\zeta}A^2)p,p)}\, dp = \frac{\pi^{N-\frac{1}{2}}}{\zeta^{\frac{N-1}{2}}} \det\left(\zeta + \frac{1}{\zeta}A^2\right)^{-1/2} = \frac{\pi^{N-\frac{1}{2}}}{\zeta^{N-\frac{1}{2}}} \det(1+\zeta^{-2}A^2)^{-1/2}.$$

Hence, $F_m(A)$ is equal to a constant times the coefficient of $\tau^{m+1}$ in $\det(1+\tau^2A^2)^{-1/2}$, expanded as a power series in $\tau$. Differentiating $\det(1+\tau^2A^2)^{-1/2}$ along $B \in \mathfrak{so}_N$, we get

$$\frac{\partial}{\partial B}\left(\det(1+\tau^2A^2)^{-1/2}\right) = -\frac{\tau^2\,\mathrm{tr}(BA(1+\tau^2A^2)^{-1})}{\det(1+\tau^2A^2)^{1/2}}.$$

Setting $B = x\otimes y^t - y\otimes x^t$ yields $2\tau^2(x, A(1+\tau^2A^2)^{-1}y)\det(1+\tau^2A^2)^{-1/2}$ as desired.
■

## 4.4 The Poisson center of algebras $H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N)$

Analogously to the cases of $\mathfrak{gl}_n$ and $\mathfrak{sp}_{2n}$, we introduce the Poisson algebras $H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N)$, where $\zeta = (\zeta_0, \ldots, \zeta_k)$ is a deformation parameter. As algebras these are $S(\mathfrak{so}_N \oplus V_N)$ with a Poisson bracket $\{\cdot,\cdot\}$ modeled after the commutator $[\cdot,\cdot]$ of $H_\zeta(\mathfrak{so}_N, V_N)$, that is, $\{x,y\} = \sum_j \zeta_j \gamma_{2j+1}(x,y)$. We prefer the following short formula for $\{\cdot,\cdot\}$ :

$V_N \times V_N \to \mathbb{C}[\mathfrak{so}_N] \simeq S(\mathfrak{so}_N)$:

$$\{x,y\} = \mathrm{Res}_{z=0}\, \zeta(z^{-2})(x, A(1+z^2A^2)^{-1}y)\det(1+z^2A^2)^{-1/2}z^{-1}dz,\ \ \forall\, x,y \in V_N, A \in \mathfrak{so}_N, \tag{4.2}$$

where $\zeta(z) := \sum_{i\geq 0}\zeta_i z^i$ is the generating function of the deformation parameters.

In fact, we can view algebras $H_\zeta(\mathfrak{so}_N, V_N)$ as *quantizations* of the algebras $H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N)$. The latter algebras still carry some important information. The main result of this section is a computation of the Poisson center $\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N))$.

Let us first recall the corresponding result in the non-deformed case ($\zeta = 0$), when the corresponding algebra is just $S(\mathfrak{so}_N \ltimes V_N)$ with a Lie-Poisson bracket. To state the result we introduce some more notation:

- Define $p_i(A) \in \mathbb{C}$ via $\det(I_N + tA) = \sum_{j=0}^N t^j p_j(A)$ for $A \in \mathfrak{gl}_N$.
- Define $b_i(A) \in \mathfrak{gl}_N$ via $b_0(A) = I_N,\ b_k(A) = \sum_{j=0}^k (-1)^j p_j(A) A^{k-j}$ for $k > 0$.
- Define $\mathfrak{a}_N := \mathfrak{so}_N \ltimes V_N$; we identify $\mathfrak{a}_N^*$ with $\mathfrak{a}_N$ via the natural pairing.
- Define $\psi_k : \mathfrak{a}_N^* \to \mathbb{C}$ by $\psi_k(A,v) = (v, b_{2k}(A)v)$ for $A \in \mathfrak{so}_N,\ v \in V_N,\ k \geq 0$.
- If $N = 2n+1$, $\psi_n$ is actually the square of a polynomial function $\widehat{\psi}_n$, which can be realized explicitly as the Pfaffian of the matrix $\begin{pmatrix} A & v \\ -v^t & 0 \end{pmatrix} \in \mathfrak{so}_{2n+2}$.
- Identifying $\mathbb{C}[\mathfrak{a}_N^*] \simeq S(\mathfrak{a}_N)$, let $\tau_k \in S(\mathfrak{a}_N)$ (respectively $\widehat{\tau}_{n+1} \in S(\mathfrak{a}_{2n+1})$) be the elements corresponding to $\psi_{k-1}$ (respectively $\widehat{\psi}_n$).

The following result is due to [R, Sections 3.7, 3.8]:

**Proposition 4.4.1.** *We have:*

*(a)* $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{a}_{2n}))$ *is a polynomial algebra in free generators* $\{\tau_1, \ldots, \tau_n\}$*;*

*(b)* $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{a}_{2n+1}))$ *is a polynomial algebra in free generators* $\{\tau_1, \ldots, \tau_n, \widehat{\tau}_{n+1}\}$.

Similarly to the cases of $\mathfrak{gl}_n, \mathfrak{sp}_{2n}$, this result can be generalized for arbitrary deformations $\zeta$. In fact, for any deformation parameter $\zeta = (\zeta_0, \ldots, \zeta_k)$ the Poisson center $\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_N, V_N))$ is still a polynomial algebra in $\lceil \frac{N+1}{2} \rceil$ generators. This is established in the following theorem:

**Theorem 4.4.2.** *Define* $c_i \in \mathbb{C}[\mathfrak{so}_N]^{SO_N} \simeq \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{so}_N))$ *via* $\sum_i (-1)^i c_i t^{2i} = c(t)$, *where*

$$c(t) := \mathrm{Res}_{z=0}\ \zeta(z^{-2})\frac{\det(1+t^2A^2)^{1/2}}{\det(1+z^2A^2)^{1/2}}\frac{z^{-1}dz}{1-t^{-2}z^2}.$$

*(a)* $\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_{2n}, V_{2n}))$ *is a polynomial algebra in free generators* $\{\tau_1+c_1, \ldots, \tau_n+c_n\}$*;*

*(b)* $\mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_\zeta(\mathfrak{so}_{2n+1}, V_{2n+1}))$ *is a polynomial algebra in free generators* $\{\tau_1+c_1, \ldots, \tau_n+c_n, \widehat{\tau}_{n+1}\}$.

Let us introduce some more notation before proceeding to the proof:

- Let $\{x_i\}_{i=1}^N$ be a basis of $V_N$ such that $(x_i, x_j) = \delta^j_{N+1-i}$.
- Let $J = (J_{ij})_{i,j=1}^N$ be the corresponding *anti-diagonal* symmetric matrix, i.e., $J_{ij} = \delta^j_{N+1-i}$. Notice that $A = (a_{ij}) \in \mathfrak{so}_N$ if and only if $a_{ij} = -a_{N+1-j,N+1-i}$ for all $i, j$.
- Let $\mathfrak{h}_N$ be the Cartan subalgebra of $\mathfrak{so}_N$ consisting of the diagonal matrices.
- Define $e_{(i,j)} := E_{i,j} - E_{N+1-j,N+1-i} \in \mathfrak{so}_N$ for $i, j \leq N$ (so $e_{(i,N+1-i)} = 0\ \forall i$).
- We set $e_i := e_{(i,i)}$ for $1 \leq i \leq n := \lfloor \frac{N}{2} \rfloor$, so that $\{e_i\}_{i=1}^n$ form a basis of $\mathfrak{h}_N$.
- Define $\sigma_i \in \mathbb{C}[z_1, \ldots, z_l]^{S_l}$ via $\prod_{i=1}^l (1+tz_i) = \sum_{i=0}^l t^i \sigma_i(z_1, \ldots, z_l)$.

*Proof of Theorem 4.4.2.*

We shall show that the elements $\tau_i + c_i$ (and $\widehat{\tau}_{n+1}$ for $N = 2n+1$) are Poisson central. Combined with Proposition 4.4.1 this clearly implies the result by a deformation argument. Since $\{\tau_i, \mathfrak{so}_N\} = 0$ for $\zeta = 0$, we still have $\{\tau_i, \mathfrak{so}_N\} = 0$ for arbitrary $\zeta$. This implies $\{\tau_i + c_i, \mathfrak{so}_N\} = 0$ as $c_i \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{so}_N))$. Therefore we just need to verify

$$\{c_i, x_q\} = -\{\tau_i, x_q\} \quad \text{for all } 1 \leq q \leq N. \tag{4.3}$$

Using $\psi_s(A, v) = (v, b_{2s}(A)v) = \sum_{k,l=1}^N x_k x_l b_{2s}(A)_{N+1-k,l}$, we get:

$$\{\tau_{s+1}, x_q\} =$$

$$\sum_{k,l}\{b_{2s}(A)_{N+1-k,l}, x_q\}x_k x_l + \sum_{k,l} b_{2s}(A)_{N+1-k,l}\{x_k, x_q\}x_l + \sum_{k,l} b_{2s}(A)_{N+1-k,l}x_k\{x_l, x_q\}.$$

The first summand is zero due to Proposition 4.4.1.

On the other hand, $AJ + JA^t = 0$ implies $(A^{2j})_{N+1-k,l} = (A^{2j})_{N+1-l,k}$ and

$p_{2j+1}(A) = 0$ for all $j \geq 0$. Hence,

$$b_{2s}(A) = A^{2s} + p_2(A)A^{2s-2} + p_4(A)A^{2s-4} + \ldots + p_{2s}(A),\ b_{2s}(A)_{n+1-k,l} = b_{2s}(A)_{n+1-l,k}.$$

Combining this with $\{c_{s+1}, x_q\} = \sum_{p \neq N+1-q} \frac{\partial c_{s+1}}{\partial e_{(p,q)}} x_p$, we see that (4.3) is equivalent to:

$$\frac{\partial c_{s+1}}{\partial e_{(p,q)}} = -2 \sum_l b_{2s}(A)_{N+1-p,l} \operatorname{Res}_{z=0} \zeta(z^{-2}) \frac{(x_l, A(1+z^2A^2)^{-1}x_q)}{\det(1+z^2A^2)^{1/2}} \frac{dz}{z} \quad \text{for all } p, q \leq N. \tag{4.4}$$

Because both sides of (4.4) are $\mathrm{SO}_N$-invariant, it suffices to verify (4.4) for $A \in \mathfrak{h}_N$, that is, for

- $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)$ in the case $N = 2n$,
- $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1)$ in the case $N = 2n+1$.

For $p \neq q$, both sides of (4.4) are zero. For $p = q \leq n$, the only nonzero summand on the right hand side of (4.4) is the one corresponding to $l = N+1-q$. In this case:

$$b_{2s}(A)_{N+1-q,N+1-q} = \lambda_q^{2s} - \sigma_1(\lambda_1^2, \ldots, \lambda_n^2)\lambda_q^{2s-2} + \ldots + (-1)^s \sigma_s(\lambda_1^2, \ldots, \lambda_n^2)$$

$$= (-1)^s \frac{\partial \sigma_{s+1}(\lambda_1^2, \ldots, \lambda_n^2)}{\partial \lambda_q^2},$$

while $(x_{N+1-q}, A(1+z^2A^2)^{-1}x_q) = \frac{\lambda_q}{1+z^2\lambda_q^2}$ and $\det(1+z^2A^2)^{1/2} = \prod_{i=1}^n (1+z^2\lambda_i^2)$. For $p = q > \lceil \frac{N+1}{2} \rceil$, we get the same equalities with $\lambda_i \leftrightarrow -\lambda_i$. As a result, (4.4) is equivalent to:

$$\frac{\partial c_{s+1}(\lambda_1, \ldots, \lambda_n)}{\partial \lambda_q^2} = (-1)^{s+1} \frac{\partial \sigma_{s+1}(\lambda_1^2, \ldots, \lambda_n^2)}{\partial \lambda_q^2} \operatorname{Res}_{z=0} \zeta(z^{-2}) \frac{z^{-1}dz}{(1+z^2\lambda_q^2)\prod_{i=1}^n (1+z^2\lambda_i^2)}.$$

We thus need to verify the following identities for $c(t)$:

$$\frac{\partial c(t)}{\partial \lambda_q^2} = \frac{\partial \prod_{i=1}^n (1+t^2\lambda_i^2)}{\partial \lambda_q^2} \operatorname{Res}_{z=0} \frac{\zeta(z^{-2})z^{-1}dz}{(1+z^2\lambda_q^2)\prod_{i=1}^n (1+z^2\lambda_i^2)}. \tag{4.5}$$

This is a straightforward verification and we leave it to an interested reader. This proves that $\tau_i + c_i \in \mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_{\zeta}(\mathfrak{so}_N, V_N))$ for all $1 \le i \le n$. For $N = 2n+1$, we also get a Poisson-central element $\tau_{n+1} + c_{n+1}$. Since $c_{n+1} = 0$, we have

$$\widehat{\tau}^2_{n+1} = \tau_{n+1} \in \mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_{\zeta}(\mathfrak{so}_{2n+1}, V_{2n+1})) \Rightarrow \widehat{\tau}_{n+1} \in \mathfrak{z}_{\mathrm{Pois}}(H^{\mathrm{cl}}_{\zeta}(\mathfrak{so}_{2n+1}, V_{2n+1})).$$

This completes the proof of the theorem. ■

**Definition 4.4.1.** The element $\tau'_1 = \tau_1 + c_1$ is called the *Poisson Casimir element* of $H^{\mathrm{cl}}_{\zeta}(\mathfrak{so}_N, V_N)$.

As a straightforward consequence of Theorem 4.4.2, we get:

**Corollary 4.4.3.** *We have* $\tau'_1 = \tau_1 + \sum_{j=0}^{k}(-1)^{j+1}\zeta_j \operatorname{tr} S^{2j+2}A$.

## 4.5 The key isomorphism

### 4.5.1 Algebras $H_m(\mathfrak{so}_N, V_N)$

Analogously to Section 3.1.3, we introduce the universal infinitesimal Hecke algebras of $(\mathfrak{so}_N, V_N)$:

**Definition 4.5.1.** Define the *universal length $m$ infinitesimal Hecke algebra* $H_m(\mathfrak{so}_N, V_N)$ as the quotient $H_m(\mathfrak{so}_N, V_N) := U(\mathfrak{so}_N) \ltimes T(V_N)[\zeta_0, \ldots, \zeta_{m-1}]/J$, where

$$J = ([A, x] - A(x), [x, y] - \sum_{j=0}^{m-1} \zeta_j r_{2j+1}(x, y) - r_{2m+1}(x, y)).$$

Here $A \in \mathfrak{so}_N$, $x, y \in V_N$ and $\{\zeta_i\}_{i=0}^{m-1}$ are central. The filtration is induced from the grading on $T(\mathfrak{so}_N \oplus V_N)[\zeta_0, \ldots, \zeta_{m-1}]$ with $\deg(\mathfrak{so}_N) = 2$, $\deg(V_N) = 2m+2$ and $\deg(\zeta_i) = 4(m-i)$.

The algebra $H_m(\mathfrak{so}_N, V_N)$ is free over $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}]$ and $H_m(\mathfrak{so}_N, V_N)/(\zeta_i - c_i)_{i=0}^{m-1}$ is the infinitesimal Hecke algebra $H_{\zeta_c}(\mathfrak{so}_N, V_N)$ for $\zeta_c = c_0 r_1 + \ldots + c_{m-1} r_{2m-1} + r_{2m+1}$.

*Remark* 4.5.1. For an $\mathfrak{so}_N$-equivariant pairing $\eta : \wedge^2 V_N \to U(\mathfrak{so}_N)[\zeta_0, \ldots, \zeta_{m-1}]$ such that $\deg(\eta(x,y)) \leq 4m+2$, the algebra

$$U(\mathfrak{so}_N) \ltimes T(V_N)[\zeta_0, \ldots, \zeta_{m-1}]/([A,x] - A(x), [x,y] - \eta(x,y))$$

satisfies the PBW property if and only if $\eta(x,y) = \sum_{i=0}^{m} \eta_i r_{2i+1}(x,y)$ with $\eta_i \in \mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}]$ degree $\leq 4(m-i)$ polynomials (compare to Theorem 3.3.1).

### 4.5.2 Isomorphisms $\bar{\Theta}$ and $\bar{\Theta}^{\mathrm{cl}}$

The main goal of this section it to establish an abstract isomorphism between the algebras $H_m(\mathfrak{so}_N, V_N)$ and the $W$-algebras $U(\mathfrak{so}_{N+2m+1}, e_m)$, where $e_m \in \mathfrak{so}_{N+2m+1}$ is a nilpotent element of the Jordan type $(1^N, 2m+1)$. We make a particular choice of such an element:[7]

- $e_m := \sum_{j=1}^{m} E_{N+j,N+j+1} - \sum_{j=1}^{m} E_{N+m+j,N+m+j+1}$.

Recall the Lie algebra inclusion $\iota : \mathfrak{q} \hookrightarrow U(\mathfrak{g}, e)$ from Section 3.1.6, where $\mathfrak{q} := \mathfrak{z}_{\mathfrak{g}}(e,h,f)$. For $(\mathfrak{g}, e) = (\mathfrak{so}_{N+2m+1}, e_m)$ we have $\mathfrak{q} \simeq \mathfrak{so}_N$. We will also denote the corresponding centralizer of $e_m \in \mathfrak{so}_{N+2m+1}$ and the Slodowy slice by $\mathfrak{z}_{N,m}$ and $\mathrm{S}_{N,m}$, respectively.

**Theorem 4.5.1.** *For $m \geq 1$, there is a unique isomorphism*

$$\bar{\Theta} : H_m(\mathfrak{so}_N, V_N) \xrightarrow{\sim} U(\mathfrak{so}_{N+2m+1}, e_m)$$

*of filtered algebras such that $\bar{\Theta}\mid_{\mathfrak{so}_N} = \iota\mid_{\mathfrak{so}_N}$.*

*Sketch of the proof.*

Notice that $\mathfrak{z}_{N,m} \simeq \mathfrak{so}_N \oplus V_N \oplus \mathbb{C}^m$ as vector spaces, where $\mathfrak{so}_N \simeq \mathfrak{q} = \mathfrak{z}_{N,m}(0)$, $V_N \subset \mathfrak{z}_{N,m}(2m)$ and $\mathbb{C}^m$ has a basis $\{\xi_0, \ldots, \xi_{m-1}\}$ with $\xi_i \in \mathfrak{z}_{N,m}(4m-4i-2)$. Here

[7] In this section, we view $\mathfrak{so}_N$ as corresponding to the pair $(V_N, (\cdot,\cdot))$, where $(\cdot,\cdot)$ is represented by the symmetric matrix $J' = (J'_{ij})$ with $J'_{ij} = \delta_i^j, J'_{i,N+k} = J'_{N+k,i} = 0, J'_{N+k,N+l} = \delta_{k+l}^{2m+2}$, $\forall\, i,j \leq N,\ k,l \leq 2m+1$.

$\xi_{m-j} = e_m^{2j-1} \in \mathfrak{so}_N$ for $1 \le j \le m$, $V_N$ is embedded via $x_i \mapsto E_{i,N+2m+1} - E_{N+1,i}$, while $\mathfrak{so}_N$ is embedded as a top-left $N \times N$ block of $\mathfrak{so}_{N+2m+1}$.

Let us recall that one of the key ingredients in the proof of Theorem 3.2.2 was an additional $\mathbb{Z}$-grading Gr on the corresponding $W$-algebras.[8] In both cases of $(\mathfrak{sl}_{n+m}, e_m), (\mathfrak{sp}_{2n+2m}, e_m)$ such a grading was induced from the weight-decomposition with respect to $\mathrm{ad}(\iota(h)),\ h \in \mathfrak{q}$.

If $N = 2n$ same argument works for $\mathfrak{g} = \mathfrak{so}_{N+2m+1}$ as well. Namely, consider $h \in \mathfrak{q} \simeq \mathfrak{so}_{2n}$ to be the diagonal matrix $I_n' := \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. The operator $\mathrm{ad}(\iota(I_n'))$ acts on $\mathfrak{z}_{N,m}$ with zero eigenvalues on $\mathbb{C}^m$, with even eigenvalues on $\mathfrak{so}_N$, and with eigenvalues $\{\pm 1\}$ on $V_N$.

However, there is no appropriate $h \in \mathfrak{q}$ in the case of $N = 2n+1$. Instead, such a grading originates from the adjoint action of the element

$$\mathrm{g}_0 := (\underbrace{-1, \ldots, -1}_{N}, \underbrace{1, \ldots, 1}_{2m+1}) \in \mathrm{O}(N + 2m + 1).$$

This element defines a $\mathbb{Z}_2$-grading on $U(\mathfrak{so}_{N+2m+1})$ and further a $\mathbb{Z}_2$-grading Gr on the $W$-algebra $U(\mathfrak{so}_{N+2m+1}, e_m)$. The induced $\mathbb{Z}_2$-grading Gr′ on $\mathrm{gr}\, U(\mathfrak{so}_{N+2m+1}, e_m) \simeq S(\mathfrak{z}_{N,m})$ satisfies the desired properties: $\deg(\mathbb{C}^m) = 0,\ \deg(\mathfrak{so}_N) = 0,\ \deg(V_N) = 1$.

Therefore the algebra $U(\mathfrak{so}_{N+2m+1}, e_m)$ is equipped both with a Kazhdan filtration and a $\mathbb{Z}_2$-grading Gr. Moreover, the corresponding isomorphism at the Poisson level is established in Theorem 4.5.2. Now the proof proceeds along the same lines as in the $\mathfrak{sp}_{2n}$ case. ■

Let us introduce some more notation:

• Let $\bar{\iota} : \mathfrak{so}_N \oplus V_N \oplus \mathbb{C}^m \xrightarrow{\sim} \mathfrak{z}_{N,m}$ denote the isomorphism from the proof of Theorem 4.5.1.

• Let $H_m^{\mathrm{cl}}(\mathfrak{so}_N, V_N)$ be the Poisson counterpart of $H_m(\mathfrak{so}_N, V_N)$ (compare to algebras $H_\varsigma^{\mathrm{cl}}(\mathfrak{so}_N, V_N)$).

• Define $P_j \in \mathbb{C}[\mathfrak{so}_{N+2m+1}]$ by $\det(I_{N+2m+1} + tA) = \sum_{j=0}^{N+2m+1} P_j(A) t^j$.

• Define $\{\bar{\Theta}_i\}_{i=0}^{m-1} \in S(\mathfrak{z}_{N,m}) \simeq \mathbb{C}[\mathrm{S}_{N,m}]$ by $\bar{\Theta}_i := P_{2(m-i)|\mathrm{S}_{N,m}}$.

[8] Actually, as exhibited by the case of $\mathfrak{sp}_{2n+2m}$, it suffices to have a $\mathbb{Z}_2$-grading.

The following result can be considered as a Poisson version of Theorem 4.5.1:

**Theorem 4.5.2.** *The formulas*

$$\bar{\Theta}^{\mathrm{cl}}(A) = \bar{\iota}(A),\ \bar{\Theta}^{\mathrm{cl}}(y) = \frac{(-1)^{\frac{m}{2}}}{2}\cdot\bar{\iota}(y),\ \bar{\Theta}^{\mathrm{cl}}(\zeta_k) = (-1)^{m-j}\bar{\Theta}_k$$

*define an isomorphism* $\bar{\Theta}^{\mathrm{cl}} : H_m^{\mathrm{cl}}(\mathfrak{so}_N, V_N)\xrightarrow{\sim} S(\mathfrak{z}_{N,m}) \simeq \mathbb{C}[\mathrm{S}_{N,m}]$ *of Poisson algebras.*

The proof of this theorem proceeds along the same lines as for $\mathfrak{sp}_{2n}$ case.

Let us now deduce a few corollaries for the infinitesimal Hecke algebras of $(\mathfrak{so}_N, V_N)$.

**Corollary 4.5.3.** *Poisson varieties corresponding to arbitrary full central reductions of Poisson infinitesimal Hecke algebras* $H_\zeta^{\mathrm{cl}}(\mathfrak{so}_N, V_N)$ *have finitely many symplectic leaves.*

**Corollary 4.5.4.** *(a) The center* $Z(H_\zeta(\mathfrak{so}_N, V_N))$ *is a polynomial algebra in* $\lfloor\frac{N+1}{2}\rfloor$ *generators.*
*(b) The infinitesimal Hecke algebra* $H_\zeta(\mathfrak{so}_N, V_N)$ *is free over its center* $Z(H_\zeta(\mathfrak{so}_N, V_N))$.
*(c) Full central reductions of* $\mathrm{gr}\, H_\zeta(\mathfrak{so}_N, V_N)$ *are normal, complete intersection integral domains.*

Finally, one can define the appropriate category $\mathcal{O}$ in the same fashion this was done for $\mathfrak{sp}_{2n}$.

## 4.6 The Casimir element

In this section we determine the first nontrivial central element of the algebras $H_\zeta(\mathfrak{so}_N, V_N)$. In the non-deformed case $\zeta = 0$ we have $t_1 := (v, v) \in Z(H_0(\mathfrak{so}_N, V_N))$. Similarly to Corollary 4.4.3, this element can be deformed to a central element of $H_\zeta(\mathfrak{so}_N, V_N)$ by adding an element of $Z(U(\mathfrak{so}_N))$.

In order to formulate the result, we introduce some more notation:

- Define $\omega_s := \frac{\pi^{1/2}(s+N-1)!}{2^{s+N+1}}$ and $\mu_s := \pi^{N-\frac{1}{2}}(s+1)!\omega_s^{-1},\ \nu_s := -\frac{\mu_s}{s+1}$.
- Define a sequence $\{a_j\}_{j=0}^m$ recursively via $\zeta_j = 2\nu_{2j+1}\sum_{l=1}^{m+1-j}(-1)^{l+1}\binom{2j+2l}{2l-1}a_{j+l-1}$.

- Define a sequence $\{g_j\}_{j=1}^{m+1}$ via $g_j = 2\mu_{2j-1}(-2a_{j-1} + \sum_{l=1}^{m+1-j}(-1)^{l+1}\binom{2j+2l}{2l}a_{j+l-1})$.
- Define a polynomial $g(z) := \sum_{j=1}^{m+1} g_j z^j$.
- Define $A(z)(x,y) := (x, A(1+z^2A^2)^{-1}y)\det(1+z^2A^2)^{-1/2}, B(z) := \det(1+z^2A^2)^{-1/2}$.
- Let $[z^m]f(z)$ denote the coefficient of $z^m$ in the series $f(z)$.
- Define $C \in Z(U(\mathfrak{so}_N))$ to be the symmetrization of $\mathrm{Res}_{z=0}\, g(z^{-2})\det(1+z^2A^2)^{-1/2}z^{-1}dz$.

Then we have:

**Theorem 4.6.1.** *The element $t_1' := t_1 + C$ is a central element of $H_\zeta(\mathfrak{so}_N, V_N)$.*

**Definition 4.6.1.** We call $t_1' = t_1 + C$ the *Casimir element* of $H_\zeta(\mathfrak{so}_N, V_N)$.

*Remark* 4.6.1. The same formula provides a central element of the algebra $H_m(\mathfrak{so}_N, V_N)$, where $C \in Z(U(\mathfrak{so}_N))[\zeta_0, \ldots, \zeta_{m-1}]$.

Theorem 4.6.1 can be used to establish explicitly the isomorphism $\bar{\Theta}$ of Theorem 4.5.1 in the same way as this has been achieved in Section 3.4.6 for the $\mathfrak{gl}_n$ case.

*Proof of Theorem 4.6.1.*

Commutativity of $t_1'$ with $\mathfrak{so}_N$ follows from the following argument:

$$[t_1, \mathfrak{so}_N] = 0 \in H_0(\mathfrak{so}_N, V_N) \Rightarrow [t_1, \mathfrak{so}_N] = 0 \in H_\zeta(\mathfrak{so}_N, V_N) \Rightarrow [t_1', \mathfrak{so}_N] = 0 \in H_\zeta(\mathfrak{so}_N, V_N).$$

Let us now verify $[t_1 + C, x] = 0$ for any $x \in V_N$. Identifying $U(\mathfrak{so}_N)$ with $S(\mathfrak{so}_N)$ via the symmetrization map and recalling (4.2), we get:

$$[\sum x_i^2, x] = \sum_i x_i \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} (x_i, J_{p,q}x)\left(\int_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} d\theta\right) dqdp +$$

$$\sum_i \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} \left(\int_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} d\theta\right)(x_i, J_{p,q}x)x_i dqdp.$$

Since $\sum_i x_i(x_i, J_{p,q}x) = J_{p,q}x$ and $ve^{\theta J_{p,q}} = e^{\theta J_{p,q}}(\cos\theta \cdot v - \sin\theta \cdot J_{p,q}v)$ for $v \in V_N$, we have

$$[t_1, x] = \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} \int_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}}(\sin\theta\cdot x + (1+\cos\theta)\cdot J_{p,q}x)d\theta dqdp. \quad (4.6)$$

The right hand side of (4.6) can be written as $[x, C']$, where

$$C' := \int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} \left( \int_{-\pi}^{\pi} c(\theta)(-2-2\cos\theta)e^{\theta J_{p,q}} d\theta \right) dqdp.$$

Thus, it suffices to prove $C' = C$.

The following has been established during the proof of Theorem 4.1.2:

$$\int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} J_{p,q}^s dqdp = F_{s-1} = \mu_{s-1}[z^s]B(z), \tag{4.7}$$

$$\int_{p\in S^{N-1}} \int_{q\in S^{N-2}(p)} (x, J_{p,q}y) J_{p,q}^s dqdp = I_{s;x,y} = \nu_s[z^{s-1}]A(z)(x,y). \tag{4.8}$$

Let $c(\theta) = c_0\delta_0 + c_2\delta_0'' + c_4\delta_0^{(4)} + \dots$ be the distribution from (4.1), where $\delta_0^{(k)}$ is the $k$-th derivative of the delta-function. Since

$$\int_{-\pi}^{\pi} 2c(\theta)\sin\theta e^{\theta J_{p,q}} d\theta = 2\sum_{j\geq 1} c_j \sum_{l=1}^{\lfloor \frac{j+1}{2} \rfloor} (-1)^{l+1} \binom{j}{2l-1} J_{p,q}^{j-2l+1},$$

formulas (4.1) and (4.8) imply

$$[x, y] = \mathrm{Res}_{z=0}\, \bar{\zeta}(z^{-2})A(z)(x,y)z^{-1}dz,$$

where $\bar{\zeta}(z^{-2}) = \sum_{j\geq 0} \bar{\zeta}_j z^{-2j}$ and $\bar{\zeta}_j = 2\nu_{2j+1}\sum_{l\geq 1}(-1)^{l+1}\binom{2j+2l}{2l-1}c_{2j+2l}$.

Comparing with $[x, y] = \mathrm{Res}_{z=0}\, \zeta(z^{-2})A(z)(x,y)z^{-1}dz$, we get $\bar{\zeta}(z^{-2}) = \zeta(z^{-2})$ and so $c_{2s+2} = a_s$, where $a_{>m} := 0$. On the other hand,

$$\int_{-\pi}^{\pi} c(\theta)(-2\cos\theta - 2)e^{\theta J_{p,q}} d\theta = 2\sum_{j\geq 0} c_j \left( -2J_{p,q}^j + \sum_{l=1}^{\lfloor j/2 \rfloor} (-1)^{l+1} \binom{j}{2l} J_{p,q}^{j-2l} \right).$$

Combining this equality with (4.7), we find:

$$C' = \mathrm{Res}_{z=0}\, g(z^{-2})B(z)z^{-1}dz = C.$$

This completes the proof of the theorem. ■

# Chapter 5

# The affine Yangian of $\mathfrak{gl}_1$

This chapter is based on [T2].

## 5.1 Basic definitions

In this section we define the algebras $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$.

### 5.1.1 The toroidal algebra of $\mathfrak{gl}_1$

Let $\{q_i\}_{i=1}^3$ be complex parameters satisfying $q_1q_2q_3=1, q_i\neq 1$.

The toroidal algebra of $\mathfrak{gl}_1$, denoted $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, is an associative unital $\mathbb{C}$-algebra generated by $\{e_i, f_i, \psi_j^\pm, \psi_0^{\pm -1} | i\in\mathbb{Z}, j\in\mathbb{Z}_+\}$ ($\mathbb{Z}_+ := \{n\in\mathbb{Z}|n\geq 0\}$) with the following defining relations:

$$\psi_0^\pm\cdot\psi_0^{\pm -1}=\psi_0^{\pm -1}\cdot\psi_0^\pm=1,\ \ [\psi^\pm(z),\psi^\pm(w)]=0,\ \ [\psi^+(z),\psi^-(w)]=0, \tag{T0}$$

$$e(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w)=-e(w)e(z)(w-q_1z)(w-q_2z)(w-q_3z), \tag{T1}$$

$$f(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z)=-f(w)f(z)(z-q_1w)(z-q_2w)(z-q_3w), \tag{T2}$$

$$[e(z),f(w)]=\frac{\delta(z/w)}{(1-q_1)(1-q_2)(1-q_3)}(\psi^+(w)-\psi^-(z)), \tag{T3}$$

$$\psi^\pm(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w)=-e(w)\psi^\pm(z)(w-q_1z)(w-q_2z)(w-q_3z), \tag{T4}$$

$$\psi^{\pm}(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) = -f(w)\psi^{\pm}(z)(z-q_1w)(z-q_2w)(z-q_3w), \quad \text{(T5)}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1},[e_{i_2+1},e_{i_3-1}]] = 0, \;\; \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1},[f_{i_2+1},f_{i_3-1}]] = 0, \quad \text{(T6)}$$

where these generating series are defined as follows:

$$e(z) := \sum_{i=-\infty}^{\infty} e_i z^{-i}, \;\; f(z) := \sum_{i=-\infty}^{\infty} f_i z^{-i}, \;\; \psi^{\pm}(z) := \sum_{j\geq 0} \psi_j^{\pm} z^{\mp j}, \;\; \delta(z) := \sum_{i=-\infty}^{\infty} z^i.$$

*Remark* 5.1.1. (a) The relations (T0)-(T5) should be viewed as collections of termwise relations, which can be recovered by evaluating the coefficients of $z^k w^l$ on both sides of the equalities.

(b) The algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ differs from the Ding-Iohara algebra, considered in [FT1], by an additional relation (T6). However, it is a correct object to consider as will be explained later.

### 5.1.2 Elements $t_i \in \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$

We introduce the generators $\{t_i\}$ instead of $\{\psi_j^{\pm}\}$, similarly to the case of a quantum affine algebra. The main advantage is a simplification of (T4)-(T6).

Choose $\{t_j\}_{\pm j>0} \subset \mathbb{C}[\psi_0^{\pm^{-1}}, \psi_0^{\pm}, \psi_1^{\pm}, \ldots]$ as the elements satisfying the following identities:

$$\psi^{\pm}(z) = \psi_0^{\pm} \cdot \exp\left(\mp \sum_{\pm m>0} \frac{\beta_m}{m} t_m z^{-m}\right),$$

where $\beta_m := (1-q_1^m)(1-q_2^m)(1-q_3^m)$. We assume $\beta_m \neq 0$, i.e., $q_1, q_2, q_3$ are not roots of 1. This choice of $t_i$ is motivated by the following two results.

**Proposition 5.1.1.** *The relations* (T4,T5) *are equivalent to* $[\psi_0^{\pm}, e_j] = 0 = [\psi_0^{\pm}, f_j]$ *together with*

(T4t) $[t_i, e_j] = e_{i+j}$ *for* $i \neq 0, j \in \mathbb{Z}$.

(T5t) $[t_i, f_j] = -f_{i+j}$ *for* $i \neq 0, j \in \mathbb{Z}$.

The proof of this proposition follows formally from the identity:

$$\ln\left(\frac{(z-q_1^{-1}w)(z-q_2^{-1}w)(z-q_3^{-1}w)}{(z-q_1w)(z-q_2w)(z-q_3w)}\right)=\sum_{m>0}-\frac{\beta_m}{m}\cdot\frac{w^m}{z^m}.$$

**Proposition 5.1.2.** *If relations* (T4t, T5t) *hold, then* (T6) *is equivalent to its particular case*

$$[e_0,[e_1,e_{-1}]]=0,\quad [f_0,[f_1,f_{-1}]]=0. \tag{T6t}$$

The algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ also satisfies a natural triangular decomposition. Let $\ddot{U}^-,\ddot{U}^0,\ddot{U}^+$ be the subalgebras of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $\{f_i\}$, $\{\psi_j^\pm,(\psi_0^\pm)^{-1}\}$, $\{e_i\}$.

**Proposition 5.1.3.** *(a) (Triangular decomposition for $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$) The multiplication map $m:\ddot{U}^-\otimes\ddot{U}^0\otimes\ddot{U}^+\to\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ is an isomorphism of vector spaces.*
*(b) The subalgebras $\ddot{U}^-,\ddot{U}^+,\ddot{U}^0$ are generated by $\{f_i\}$, $\{e_i\}$, $\{\psi_j^\pm,(\psi_0^\pm)^{-1}\}$ with the defining relations (T2, T6), (T1, T6), and (T0), respectively.*

The proof is standard. Consider an associative algebra $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by $e_i,f_i,\psi_j^\pm,(\psi_0^\pm)^{-1}$ subject to the relations (T0, T3, T4, T5). We define the subalgebras $\ddot{V}^-,\ddot{V}^0,\ddot{V}^+$ of $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ in the same way. Let $I^\pm$ be the two-sided ideal of $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by the quadratic and cubic relations in $e_i$ and $f_i$ arising from (T1, T2, T6). Explicitly, $I^+$ is generated by

$$A_{i,j}=e_{i+3}e_j-\sigma_1e_{i+2}e_{j+1}+\sigma_2e_{i+1}e_{j+2}-e_ie_{j+3}-e_je_{i+3}+\sigma_2e_{j+1}e_{i+2}-\sigma_1e_{j+2}e_{i+1}+e_{j+3}e_i,$$

$$B_{i_1,i_2,i_3}=\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1},[e_{i_2+1},e_{i_3-1}]].$$

We also let $J^\pm$ stay for the corresponding two-sided ideals of $\ddot{V}^\pm$. Proposition 5.1.3 follows from:

**Lemma 5.1.4.** *(a) (Triangular decomposition for $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$) The multiplication map $m:\ddot{V}^-\otimes\ddot{V}^0\otimes\ddot{V}^+\to\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ is an isomorphism of vector spaces.*
*(b) The subalgebras $\ddot{V}^-,\ddot{V}^+$ are free associative algebra in $\{f_i\}$, $\{e_i\}$, respectively. The subalgebra $\ddot{V}^0$ is generated by $\psi_j^\pm,(\psi_0^\pm)^{-1}$ with the defining relations (T0).*
*(c) We have $I^+=m(\ddot{V}^-\otimes\ddot{V}^0\otimes J^+)$ and $I^-=m(J^-\otimes\ddot{V}^0\otimes\ddot{V}^+)$.*

*Proof of Lemma 5.1.4.*

Part (a) is standard. Part (b) follows immediately from (a).

Part (c) is equivalent to $\ddot{V}^{-}\ddot{V}^{0}J^{+}$ being a two-sided ideal of $\ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. Using the equality $\ddot{V}^{-}\ddot{V}^{0}\ddot{V}^{+} = \ddot{V}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, we reduce to showing

$$[A_{i,j}, t_r], [B_{i_1,i_2,i_3}, t_r], [A_{i,j}, f_r], [B_{i_1,i_2,i_3}, f_r] \in \ddot{V}^{0}J^{+}.$$

Relation (T4t) implies that the first two commutators are just the linear combinations of $A_{i',j'}$ and $B_{i'_1,i'_2,i'_3}$. Also $[A_{i,j}, f_r] = 0$ (it is a sum of two quadratic expressions from (T4)).

To prove $[B_{i_1,i_2,i_3}, f_r] \in \ddot{V}^{0}J^{+}$ we work with the generating series. The relation (T3) implies

$$\beta_1 \cdot [e(z_1)e(z_2)e(z_3), f(w)] = \delta\left(\frac{z_1}{w}\right)\psi(z_1)e(z_2)e(z_3)+$$

$$\delta\left(\frac{z_2}{w}\right)\psi(z_2)e(z_1)e(z_3)\rho(z_2,z_1) + \delta\left(\frac{z_3}{w}\right)\psi(z_3)e(z_1)e(z_2)\rho(z_3,z_1)\rho(z_3,z_2).$$

where $\rho(x,y) := -\frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(y-q_1x)(y-q_2x)(y-q_3x)}$ and $\psi(z) = \psi^{+}(z) - \psi^{-}(z)$. Hence, we have

$$\left[\mathrm{Sym}_{\mathfrak{S}_3}\left\{\left(\frac{z_2}{z_1}+\frac{z_2}{z_3}-\frac{z_1}{z_2}-\frac{z_3}{z_2}\right)e(z_1)e(z_2)e(z_3)\right\}, f(w)\right] =$$

$$\beta_1^{-1}\left(\delta(z_1/w)\psi(z_1)C_1(z_2,z_3) + \delta(z_2/w)\psi(z_2)C_2(z_3,z_1) + \delta(z_3/w)\psi(z_3)C_3(z_1,z_2)\right),$$

where $C_1(z_2,z_3) = e(z_2)e(z_3)C_{123} + e(z_3)e(z_2)C_{132}$ and

$$C_{123} = \left(\frac{z_2}{z_1}+\frac{z_2}{z_3}-\frac{z_1}{z_2}-\frac{z_3}{z_2}\right)+$$

$$\rho(z_1,z_2)\left(\frac{z_1}{z_2}+\frac{z_1}{z_3}-\frac{z_2}{z_1}-\frac{z_3}{z_1}\right) + \rho(z_1,z_2)\rho(z_1,z_3)\left(\frac{z_3}{z_1}+\frac{z_3}{z_2}-\frac{z_1}{z_3}-\frac{z_2}{z_3}\right),$$

$$C_{132} = \left(\frac{z_3}{z_1}+\frac{z_3}{z_2}-\frac{z_1}{z_3}-\frac{z_2}{z_3}\right)+$$

$$\rho(z_1,z_3)\left(\frac{z_1}{z_2}+\frac{z_1}{z_3}-\frac{z_2}{z_1}-\frac{z_3}{z_1}\right) + \rho(z_1,z_2)\rho(z_1,z_3)\left(\frac{z_2}{z_1}+\frac{z_2}{z_3}-\frac{z_1}{z_2}-\frac{z_3}{z_2}\right).$$

The equality $C_{132} = -\rho(z_3, z_2)C_{123}$ implies actually that $C_1(z_2, z_3)$ is proportional to the generating function of $A_{i,j}$. Same results apply to $C_2(z_3, z_1), C_3(z_1, z_2)$. This yields the inclusion $[B_{i_1,i_2,i_3}, f_r] \in \ddot{V}^0 J^+$ for any $i_1, i_2, i_3, r \in \mathbb{Z}$. □

### 5.1.3 The affine Yangian of $\mathfrak{gl}_1$

Let $h_1, h_2, h_3$ be complex parameters satisfying $h_1 + h_2 + h_3 = 0$.

The affine Yangian of $\mathfrak{gl}_1$, denoted $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$, is an associative unital $\mathbb{C}$-algebra generated by $\{e_j, f_j, \psi_j\}_{j \in \mathbb{Z}_+}$ with the following defining relations (here $i, j \in \mathbb{Z}_+$):

(Y0) $[\psi_i, \psi_j] = 0$,

(Y1) $([e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}]) + \sigma_2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = \sigma_3\{e_i, e_j\}$,

(Y2) $([f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}]) + \sigma_2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = -\sigma_3\{f_i, f_j\}$,

(Y3) $[e_i, f_j] = \psi_{i+j}$,

(Y4) $([\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}]) + \sigma_2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = \sigma_3\{\psi_i, e_j\}$,

(Y4′) $[\psi_0, e_j] = 0, \quad [\psi_1, e_j] = 0, \quad [\psi_2, e_j] = 2e_j$,

(Y5) $([\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}]) + \sigma_2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = -\sigma_3\{\psi_i, f_j\}$,

(Y5′) $[\psi_0, f_j] = 0, \quad [\psi_1, f_j] = 0, \quad [\psi_2, f_j] = -2f_j$,

(Y6) For $i_1, i_2, i_3 \in \mathbb{Z}_+ :$ $\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0, \quad \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0$,

where $\sigma_1 := h_1 + h_2 + h_3 = 0,\ \sigma_2 := h_1h_2 + h_1h_3 + h_2h_3,\ \sigma_3 := h_1h_2h_3$ and $\{a, b\} := ab + ba$.

### 5.1.4 Generating series for $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$

Let us introduce the generating series:

$$e(z) := \sum_{j\geq 0} e_j z^{-j-1}, \quad f(z) := \sum_{j\geq 0} f_j z^{-j-1}, \quad \psi(z) := 1 + \sigma_3 \sum_{j\geq 0} \psi_j z^{-j-1}.$$

Define $\ddot{Y}^{\geq 0}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ and $\ddot{Y}^{\leq 0}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ as the subalgebras of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ generated by $e_j, \psi_j$ and $f_j, \psi_j$, respectively. Let us consider the homomorphisms

$$\sigma^+ : \ddot{Y}^{\geq 0}_{h_1,h_2,h_3}(\mathfrak{gl}_1) \to \ddot{Y}^{\geq 0}_{h_1,h_2,h_3}(\mathfrak{gl}_1), \; \sigma^- : \ddot{Y}^{\leq 0}_{h_1,h_2,h_3}(\mathfrak{gl}_1) \to \ddot{Y}^{\leq 0}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$$

defined on the generators by $\psi_j \mapsto \psi_j, \; e_j \mapsto e_{j+1}$ (respectively $\psi_j \mapsto \psi_j, \; f_j \mapsto f_{j+1}$). Let

$$\mu : \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)^{\otimes 2} \to \ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$$

be the multiplication. The following result is straightforward:

**Proposition 5.1.5.** *Let us introduce* $P^a(z,w) := (z-w-h_1)(z-w-h_2)(z-w-h_3)$. *Then:*

*(a) The relation (Y0) is equivalent to*

$$[\psi(z), \psi(w)] = 0.$$

*(b) The relation (Y1) is equivalent to*

$$\partial_z^3 \mu(P^a(z, \sigma^+_{(2)}) e(z) \otimes e_j + P^a(\sigma^+_{(1)}, z) e_j \otimes e(z)) = 0 \quad \forall j \in \mathbb{Z}_+.$$

*(c) The relation (Y2) is equivalent to*

$$\partial_z^3 \mu(P^a(\sigma^-_{(2)}, z) f(z) \otimes f_j + P^a(z, \sigma^-_{(1)}) f_j \otimes f(z)) = 0 \quad \forall j \in \mathbb{Z}_+.$$

*(d) The relation (Y3) is equivalent to*

$$\sigma_3 \cdot (w - z)[e(z), f(w)] = \psi(z) - \psi(w).$$

*(e) The relations (Y4)+(Y4′) are equivalent to*

$$P^a(z, \sigma^+)\psi(z)e_j + P^a(\sigma^+, z)e_j\psi(z) = 0 \quad \forall j \in \mathbb{Z}_+.$$

*(f) The relations (Y5)+(Y5′) are equivalent to*

$$P^a(\sigma^-, z)\psi(z)f_j + P^a(z, \sigma^-)f_j\psi(z) = 0 \quad \forall j \in \mathbb{Z}_+.$$

## 5.2 Representation theory via the Hilbert scheme

### 5.2.1 Correspondences and fixed points for $(\mathbb{A}^2)^{[n]}$

We set $X = \mathbb{A}^2$ in this section.

Let $X^{[n]}$ be the Hilbert scheme of $n$ points in $X$. Its $\mathbb{C}$-points are the codimension $n$ ideals $J \subset \mathbb{C}[x,y]$. Let $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ be the Nakajima-Grojnowski correspondence. For $i > 0$, the correspondence $P[i] \subset \coprod_n X^{[n]} \times X^{[n+i]}$ consists of all pairs of ideals $(J_1, J_2)$ of $\mathbb{C}[x,y]$ of codimension $n$, $n+i$ respectively, such that $J_2 \subset J_1$ and the factor $J_1/J_2$ is supported at a single point. It is known that $P[1]$ is a smooth variety. Let $L$ be the tautological line bundle on $P[1]$ whose fiber at a point $(J_1, J_2) \in P[1]$ equals $J_1/J_2$. There are natural projections $\mathbf{p}, \mathbf{q}$ from $P[1]$ to $X^{[n]}$ and $X^{[n+1]}$, correspondingly.

Consider a natural action of $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ on each $X^{[n]}$ induced from the one on $X$ given by the formula $(t_1, t_2)(x, y) = (t_1 \cdot x, t_2 \cdot y)$. The set $(X^{[n]})^{\mathbb{T}}$ of $\mathbb{T}$-fixed points in $X^{[n]}$ is finite and is in bijection with size $n$ Young diagrams. For a size $n$ Young diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$, the corresponding ideal $J_\lambda \in (X^{[n]})^{\mathbb{T}}$ is given by $J_\lambda = \mathbb{C}[x,y] \cdot (\mathbb{C}x^{\lambda_1}y^0 \oplus \cdots \oplus \mathbb{C}x^{\lambda_k}y^{k-1} \oplus \mathbb{C}y^k)$.

**Notation:** For a Young diagram $\lambda$, let $\lambda^*$ be the conjugate diagram and define $|\lambda| := \sum \lambda_i$. For a box $\square$ with the coordinates $(i,j)$, we define $a_\lambda(\square) := \lambda_j - i$, $l_\lambda(\square) := \lambda^*_i - j$. We denote the diagram obtained from $\lambda$ by adding a box to its $j$-th row by $\lambda + \square_j$ or simply by $\lambda + j$.

### 5.2.2 Geometric $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action I

We recall the key theorem from [FT1] (see also [SV]).

Let $'M$ be the direct sum of equivariant (complexified) $K$-groups: $'M = \bigoplus_n K^{\mathbb{T}}(X^{[n]})$. It is a module over $K^{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathbb{T}] = \mathbb{C}[t_1, t_2]$. We define

$$M := {'M} \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt})) = {'M} \otimes_{\mathbb{C}[t_1,t_2]} \mathbb{C}(t_1, t_2).$$

It has a natural grading: $M = \bigoplus_n M_n$, $M_n = K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt}))$.

According to the localization theorem, restriction to the $\mathbb{T}$-fixed point set induces an isomorphism

$$K^{\mathbb{T}}(X^{[n]}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt})) \xrightarrow{\sim} K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt})).$$

The structure sheaves $\{\lambda\}$ of the $\mathbb{T}$-fixed points $J_\lambda$ (defined in Section 5.2.1) form a basis in $\bigoplus_n K^{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{K^{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}}(\mathrm{pt}))$. Since embedding of a point $J_\lambda$ into $X^{[|\lambda|]}$ is a proper morphism, the direct image in the equivariant $K$-theory is well defined, and we denote by $[\lambda] \in M$ the direct image of the structure sheaf $\{\lambda\}$. The set $\{[\lambda]\}$ forms a basis of $M$.

Let $\mathfrak{F}$ be the *tautological vector bundle* on $X^{[n]}$, whose fiber $\mathfrak{F}_{|J}$ is naturally identified with the quotient $\mathbb{C}[x,y]/J$. Consider the generating series $\mathbf{a}(z),\ \mathbf{c}(z) \in M(z)$ defined as follows:

$$\mathbf{a}(z) := \Lambda^{\bullet}_{-1/z}(\mathfrak{F}) = \sum_{i\geq 0} [\Lambda^i(\mathfrak{F})](-1/z)^i,$$

$$\mathbf{c}(z) := \mathbf{a}(zt_1)\mathbf{a}(zt_2)\mathbf{a}(zt_3)\mathbf{a}(zt_1^{-1})^{-1}\mathbf{a}(zt_2^{-1})^{-1}\mathbf{a}(zt_3^{-1})^{-1}, \text{ where } t_3 := t_1^{-1}t_2^{-1}.$$

Finally, we define the linear operators $e_i, f_i, \psi_j^{\pm}$ ($i \in \mathbb{Z}, j \in \mathbb{Z}_+$) on $M$:

$$e_i = \mathbf{q}_*(L^{\otimes i} \otimes \mathbf{p}^*) : \ M_n \to M_{n+1}, \tag{5.1}$$

$$f_i = \mathbf{p}_*(L^{\otimes (i-1)} \otimes \mathbf{q}^*) : \ M_n \to M_{n-1}, \tag{5.2}$$

$$\psi^{\pm}(z)_{|M_n} = \sum_{r=0}^{\infty} \psi_r^{\pm} z^{\mp r} := \left(-\frac{1-t_3z^{-1}}{1-z^{-1}}\mathbf{c}(z)\right)^{\pm} \in M_n[[z^{\mp 1}]], \tag{5.3}$$

where $\gamma(z)^{\pm}$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

**Theorem 5.2.1.** *The operators $e_i, f_i, \psi_j^{\pm}$, defined in (5.1)-(5.3), satisfy the relations (T0)-(T6) with the parameters $q_i = t_i, 1 \leq i \leq 3$. This endows $M$ with the structure of a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation.*

This theorem is proved in [FT1] modulo a straightforward verification of (T6) (see [FFJMM1]).

### 5.2.3 Geometric $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action I

We provide a cohomological analogue of Theorem 5.2.1.

Let $'V$ be the direct sum of equivariant (complexified) cohomology: $'V = \bigoplus_n H^\bullet_{\mathbb{T}}(X^{[n]})$. It is a module over $H^\bullet_{\mathbb{T}}(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}] = \mathbb{C}[s_1, s_2]$, where $\mathfrak{t}$ is the Lie algebra of $\mathbb{T}$. We define

$$V := \,'V \otimes_{H^\bullet_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}}(\mathrm{pt})) = \,'V \otimes_{\mathbb{C}[s_1,s_2]} \mathbb{C}(s_1, s_2).$$

It has a natural grading: $V = \bigoplus_n V_n$, $V_n = H^\bullet_{\mathbb{T}}(X^{[n]}) \otimes_{H^\bullet_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}}(\mathrm{pt}))$. According to the localization theorem, restriction to the $\mathbb{T}$-fixed point set induces an isomorphism

$$H^\bullet_{\mathbb{T}}(X^{[n]}) \otimes_{H^\bullet_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}}(\mathrm{pt})) \xrightarrow{\sim} H^\bullet_{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{H^\bullet_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}}(\mathrm{pt})).$$

The fundamental cycles $[\lambda]$ of the $\mathbb{T}$-fixed points $J_\lambda$ form a basis in

$$\bigoplus_n H^\bullet_{\mathbb{T}}((X^{[n]})^{\mathbb{T}}) \otimes_{H^\bullet_{\mathbb{T}}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}}(\mathrm{pt})).$$

Since embedding of a point $J_\lambda$ into $X^{[|\lambda|]}$ is a proper morphism, the direct image in the equivariant cohomology is well defined, and we will denote by $[\lambda] \in V_{|\lambda|}$ the direct image of the fundamental cycle of the point $J_\lambda$. The set $\{[\lambda]\}$ forms a basis of $V$.

We introduce the generating series $\mathbf{C}(z) \in V[[z^{-1}]]$ as follows:

$$\mathbf{C}(z) := \left( \frac{\mathrm{ch}(\mathfrak{F}t_1^{-1}, -z^{-1})\mathrm{ch}(\mathfrak{F}t_2^{-1}, -z^{-1})\mathrm{ch}(\mathfrak{F}t_3^{-1}, -z^{-1})}{\mathrm{ch}(\mathfrak{F}t_1, -z^{-1})\mathrm{ch}(\mathfrak{F}t_2, -z^{-1})\mathrm{ch}(\mathfrak{F}t_3, -z^{-1})} \right)^+,$$

where $\mathrm{ch}(F, \bullet)$ denotes the Chern polynomial of $F$. We also set $s_3 := -s_1 - s_2$.

Finally, we define the linear operators $e_j, f_j, \psi_j$ $(j \in \mathbb{Z}_+)$ on $V$:

$$e_j = \mathbf{q}_*(c_1(L)^j \cdot \mathbf{p}^*): \ V_n \to V_{n+1}, \tag{1'}$$

$$f_j = \mathbf{p}_*(c_1(L)^j \cdot \mathbf{q}^*): \ V_n \to V_{n-1}, \tag{2'}$$

$$\psi(z)_{|V_n} = 1 + s_1 s_2 s_3 \sum_{r=0}^{\infty} \psi_r z^{-r-1} := ((1 - s_3/z)\mathbf{C}(z))^+ \in V_n[[z^{-1}]]. \qquad (3')$$

**Theorem 5.2.2.** *The operators $e_j, f_j, \psi_j$, defined in (1')-(3'), satisfy the relations (Y0)-(Y6) with the parameters $h_i = s_i, 1 \le i \le 3$. This endows $V$ with the structure of a $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation.*

Let us compute the matrix coefficients of $e_j, f_j, \psi_j$ in the fixed point basis.

**Lemma 5.2.3.** *Consider the fixed point basis $\{[\lambda]\}$ of $V$.*

*(a) The only nonzero matrix coefficients of the operators $e_k, f_k$ are as follows:*

$$e_{k[\lambda-i,\lambda]} = \frac{((\lambda_i - 1)s_1 + (i-1)s_2)^k}{(s_1+s_2)((\lambda_1 - \lambda_i + 1)s_1 + (1-i)s_2)} \cdot \prod_{j \ge 1} \frac{(\lambda_j - \lambda_i + 1)s_1 + (j-i+1)s_2}{(\lambda_{j+1} - \lambda_i + 1)s_1 + (j-i+1)s_2},$$

$$f_{k[\lambda+i,\lambda]} = \frac{(\lambda_i s_1 + (i-1)s_2)^k((\lambda_i - \lambda_1 + 1)s_1 + i s_2)}{s_1+s_2} \cdot \prod_{j \ge 1} \frac{(\lambda_i - \lambda_{j+1} + 1)s_1 + (i-j)s_2}{(\lambda_i - \lambda_j + 1)s_1 + (i-j)s_2}.$$

*(b) The eigenvalue of $\psi(z)$ applied to $[\lambda]$ equals*

$$\left(\left(1 - \frac{s_3}{z}\right) \prod_{\square \in \lambda} \frac{(1 - \frac{\chi(\square)-s_1}{z})(1 - \frac{\chi(\square)-s_2}{z})(1 - \frac{\chi(\square)-s_3}{z})}{(1 - \frac{\chi(\square)+s_1}{z})(1 - \frac{\chi(\square)+s_2}{z})(1 - \frac{\chi(\square)+s_3}{z})}\right)^+,$$

*where $\chi(\square_{i,j}) = (i-1)s_1 + (j-1)s_2$ for a box $\square_{i,j}$ staying in the $j$-th row and $i$-th column.*

This lemma is a *cohomological analogue* of [FT1, Lemma 3.1, Proposition 3.1]. Using this result, proof of Theorem 5.2.2 reduces to a routine verification of the relations (Y0)-(Y6) in the fixed point basis. The only non-trivial relation is actually (Y3). A similar issue in the $K$-theory case was resolved by [FT1, Lemma 4.1]. We conclude this section by proving an analogous result.

**Lemma 5.2.4.** *Let us consider the linear operator $\phi_{i,j} := [e_i, f_j]$ acting on $V$.*

*(a) The operator $\phi_{i,j}$ is diagonalizable in the fixed point basis $\{[\lambda]\}$ of $V$.*

*(b) For any Young diagram $\lambda$, we have $\phi_{i,j}([\lambda]) = \gamma_{i+j|_\lambda} \cdot [\lambda]$, where*

$$\gamma_{m|_\lambda} = s_1^{-2}\sum_{i=1}^{k} y_i^m \prod_{1\le j\le k-1}^{j\neq i} \frac{(y_i-y_j+s_2)(y_j-y_i+s_1+s_2)}{(y_i-y_j)(y_j-y_i+s_1)} \cdot \frac{y_i+s_1+(2-k)s_2}{-y_i+(k-1)s_2}$$
$$-s_1^{-2}\sum_{i=1}^{k}(y_i+s_1)^m \prod_{1\le j\le k-1}^{j\neq i} \frac{(y_j-y_i+s_2)(y_i-y_j+s_1+s_2)}{(y_j-y_i)(y_i-y_j+s_1)} \cdot \frac{y_i+2s_1+(2-k)s_2}{-y_i-s_1+(k-1)s_2}. \tag{$\sharp$}$$

*Here $y_i := (\lambda_i - 1)s_1 + (i-1)s_2$ and $k$ is a positive integer such that $\lambda_{k-1} = 0$.*

*(c) For any Young diagram $\lambda$, we have:*

$$\gamma_{0|_\lambda} = -1/s_1s_2, \quad \gamma_{1|_\lambda} = 0, \quad \gamma_{2|_\lambda} = 2|\lambda|.$$

*Proof.*

Parts (a) and (b) follow from Lemma 5.2.3(a) by straightforward calculations.

Let us now prove (c). First we observe that for $m \ge 0$, the expression for $\gamma_{m|_\lambda}$ in ($\sharp$) is a rational function with simple poles at $y_i = y_j$, $y_i = y_j + s_1$, $y_i = (k-1)s_2$, $y_i = -s_1 + (k-1)s_2$. But an easy counting of residues shows that there are actually no poles and the resulting expression is an element of $\mathbb{C}(s_1, s_2)[y_1, y_2, \ldots]$. Let us now consider each of the cases $m = 0, 1, 2$.

∘ *Case 1: $m = 0$.*

Since $\gamma_{0|_\lambda}$ is a polynomial in $y_i$ of degree $\le 0$, it should be just an element of $\mathbb{C}(s_1, s_2)$ independent of $\lambda$. Evaluating at the empty diagram, we find $\gamma_{0|_\lambda} = \gamma_{0|_\emptyset} = -1/s_1s_2$.

∘ *Case 2: $m = 1$.*

First note that $\gamma_{1|_\lambda}$ is a polynomial in $y_i$ of degree $\le 1$. Further for any $i_0$ the limit of the expression ($\sharp$) for $r = 1$ as $y_{i_0} \to \infty$ while $y_j$ are fixed for all $j \ne i_0$, is finite. Thus $\gamma_{1|_\lambda}$ is actually a polynomial of degree 0, that is, an element of $\mathbb{C}(s_1, s_2)$ independent of $\lambda$. Evaluating at the empty diagram, we find $\gamma_{1|_\lambda} = \gamma_{1|_\emptyset} = 0$.

∘ *Case 3: $m = 2$.*

Recall that $\gamma_{2|_\lambda}$ is a polynomial in $y_i$ of degree $\leq 2$. However, arguments similar to those used in the previous case show that it is a degree $\leq 1$ polynomial in $y_i$ over $\mathbb{C}(s_1, s_2)$. Let us compute the principal linear part of this polynomial.

The coefficient of $y_{i_0}$ equals the limit $\lim_{\xi\to\infty} \frac{1}{\xi}\gamma_{2|_\lambda}$ as $y_j$ is fixed for $j \neq i_0$ and $y_{i_0} = \xi \to \infty$. Formula ($\sharp$) implies that this limit is equal to $\frac{2}{s_1}$. Therefore, there exists a $\lambda$-independent $F(s_1, s_2) \in \mathbb{C}(s_1, s_2)$ such that $\gamma_{2|_\lambda} = \frac{2}{s_1}(\widetilde{y}_1 + \widetilde{y}_2 + \ldots) + F(s_1, s_2) = 2|\lambda| + F(s_1, s_2)$, where $\widetilde{y}_i = y_i - ((i-1)s_2 - s_1)$[1]. Evaluating at the empty Young diagram, we find $F(s_1, s_2) = 0$. The equality $\gamma_{2|_\lambda} = 2|\lambda|$ follows. $\square$

Arguments similar to those from [FT1] prove $\gamma_{m|_\lambda} = \psi_{m|_\lambda}$.

*Remark* 5.2.1. Comparing (3′) with Lemma 2.3(b), we find the next $\psi$-coefficient:

$$\psi_{3|_\lambda} = 6\sum_{\square\in\lambda}\chi(\square) + 2(s_1+s_2)|\lambda|.$$

In particular, $\frac{1}{6}(\psi_3 + s_3\psi_2)$ corresponds to the cup product with $c_1(\mathfrak{F})$. This operator was first studied by M. Lehn. It is also related to the Laplace-Beltrami operator (see [Na2, Section 4]).

[1] Note that for any Young diagram $\lambda$, the sequence $\{\widetilde{y}_i\}$ stabilizes to 0 as $i \to \infty$, unlike $\{y_i\}$.

## 5.3 Representation theory via the Gieseker space

The Hilbert scheme $(\mathbb{A}^2)^{[n]}$ can be viewed as the first member of the family of the Gieseker moduli spaces $M(r,n)$, corresponding to $r=1$. The purpose of this section is to generalize the results of Section 2 to the case of higher rank $r$.

### 5.3.1 Correspondences and fixed points for $M(r,n)$

We recall some basics on $M(r,n)$.

Let $M(r,n)$ be the Gieseker framed moduli space of torsion free sheaves on $\mathbb{P}^2$ of rank $r$ and $c_2=n$. Its $\mathbb{C}$-points are the isomorphism classes of pairs $\{(E,\Phi)\}$, where $E$ is a torsion free sheaf on $\mathbb{P}^2$ of rank $r$ and $c_2(E)=n$ which is locally free in a neighborhood of the line $l_\infty=\{(0:z_1:z_2)\}\subset\mathbb{P}^2$, while $\Phi: E\mid_{l_\infty}\xrightarrow{\sim}\mathcal{O}_{l_\infty}^{\oplus r}$ (called a *framing at infinity*).

This space has an alternative quiver description (see [Na1, Ch. 2] for details):

$$M(r,n)=\mathcal{M}(r,n)/GL_n(\mathbb{C}),\ \mathcal{M}(r,n)=\{(B_1,B_2,i,j)|[B_1,B_2]+ij=0\}^s,$$

where $B_1,B_2\in\mathrm{End}(\mathbb{C}^n), i\in\mathrm{Hom}(\mathbb{C}^r,\mathbb{C}^n), j\in\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)$, the $GL_n(\mathbb{C})$-action is given by $g\cdot(B_1,B_2,i,j)=(gB_1g^{-1},gB_2g^{-1},gi,jg^{-1})$, while the superscript $s$ symbolizes the stability condition

"there is no proper subspace $S\subsetneq\mathbb{C}^n$ which contains $\mathrm{Im}\ i$ and is $B_1,B_2-$invariant".

Consider a natural action of $\mathbb{T}_r=(\mathbb{C}^*)^2\times(\mathbb{C}^*)^r$ on $M(r,n)$, where $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$ via $(t_1,t_2)([z_0:z_1:z_2])=[z_0:t_1z_1:t_2z_2]$, while $(\mathbb{C}^*)^r$ acts by rescaling the framing isomorphism. The set $M(r,n)^{\mathbb{T}_r}$ of $\mathbb{T}_r$-fixed points in $M(r,n)$ is finite and is in bijection with *r-partitions* of $n$, collections of $r$ Young diagrams $(\lambda^1,\ldots,\lambda^r)$ satisfying $|\lambda^1|+\ldots+|\lambda^r|=n$ (see [NY, Proposition 2.9]). For an $r$-partition $\bar{\lambda}=(\lambda^1,\ldots,\lambda^r)\vdash n$, the corresponding point $\xi_{\bar{\lambda}}\in M(r,n)^{\mathbb{T}_r}$ is given by $E_{\bar{\lambda}}=J_{\lambda^1}\oplus\cdots\oplus J_{\lambda^r}$, where $\Phi$ is given by a sum of natural inclusions $J_{\lambda^j|_{l_\infty}}\hookrightarrow\mathcal{O}_{l_\infty}$.

Let us recall the *Hecke correspondences*, generalizing the correspondence $P[1]$ from

Section 2. Consider $\mathcal{M}(r;n,n+1) \subset \mathcal{M}(r,n) \times \mathcal{M}(r,n+1)$ consisting of pairs of tuples $\{(B_1^{(k)}, B_2^{(k)}, i^{(k)}, j^{(k)})\}$ for $k = n, n+1$, such that there exists $\xi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ satisfying

$$\xi B_1^{(n+1)} = B_1^{(n)}\xi,\ \xi B_2^{(n+1)} = B_2^{(n)}\xi,\ \xi i^{(n+1)} = i^{(n)},\ j^{(n+1)} = j^{(n)}\xi.$$

The stability condition implies $\xi$ is surjective. Therefore $S := \operatorname{Ker}\ \xi \subset \mathbb{C}^{n+1}$ is a 1-dimensional subspace of Ker $j^{(n+1)}$ invariant with respect to $B_1^{(n+1)}, B_2^{(n+1)}$. This provides an identification of $\mathcal{M}(r;n,n+1)$ with pairs of $(B_1^{(n+1)}, B_2^{(n+1)}, i^{(n+1)}, j^{(n+1)}) \in \mathcal{M}(r,n+1)$ and a 1-dimensional subspace $S \subset \mathbb{C}^{n+1}$ satisfying the above conditions. Define the Hecke correspondence

$$M(r;n,n+1) \subset M(r,n) \times M(r,n+1) = \mathcal{M}(r,n) \times \mathcal{M}(r,n+1)/GL_n(\mathbb{C}) \times GL_{n+1}(\mathbb{C})$$

to be the image of $\mathcal{M}(r;n,n+1)$. The set $M(r;n,n+1)^{\mathbb{T}_r}$ of $\mathbb{T}_r$-fixed points in $M(r;n,n+1)$ is in bijection with $r$-partitions $\bar{\lambda} \vdash n, \bar{\mu} \vdash n+1$ such that $\lambda^j \subseteq \mu^j$ for $1 \leq j \leq r$; the corresponding fixed point will be denoted by $\xi_{\bar{\lambda},\bar{\mu}}$. We refer the reader to [Na3, Section 5.1] for more details.

Let $L_r$ be the *tautological* line bundle on $M(r;n,n+1)$, $\mathfrak{F}_r$ be the *tautological* rank $n$ vector bundle on $M(r,n)$. There are natural projections $\mathbf{p}_r, \mathbf{q}_r$ from $M(r;n,n+1)$ to $M(r,n)$ and $M(r,n+1)$, correspondingly. Our further computations are based on the following well-known result:

**Proposition 5.3.1.** *(a) The variety $M(r;n,n+1)$ is smooth of complex dimension $2rn+r+1$.*

*(b) The $\mathbb{T}_r$-character of the tangent space to $M(r,n)$ at the $\mathbb{T}_r$-fixed point $\xi_{\bar{\lambda}}$ equals*

$$T_{\bar{\lambda}} = \sum_{a,b=1}^{r} \left( \sum_{\square \in \lambda^a} t_1^{-a_{\lambda^b}(\square)} t_2^{l_{\lambda^a}(\square)+1} \frac{\chi_b}{\chi_a} + \sum_{\square \in \lambda^b} t_1^{a_{\lambda^a}(\square)+1} t_2^{-l_{\lambda^b}(\square)} \frac{\chi_b}{\chi_a} \right).$$

*(c) The $\mathbb{T}_r$-character of the fiber of the normal bundle of $M(r;n,n+1)$ at $\xi_{\bar{\lambda},\bar{\mu}}$ equals*

$$N_{\bar{\lambda},\bar{\mu}} = -t_1t_2 + \sum_{a,b=1}^{r} \left( \sum_{\square \in \lambda^a} t_1^{-a_{\lambda^b}(\square)} t_2^{l_{\mu^a}(\square)+1} \frac{\chi_b}{\chi_a} + \sum_{\square \in \mu^b} t_1^{a_{\mu^a}(\square)+1} t_2^{-l_{\lambda^b}(\square)} \frac{\chi_b}{\chi_a} \right).$$

### 5.3.2 Geometric $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action II

We generalize Theorem 5.2.1 for a higher rank $r$.

Let $'M^r$ be the direct sum of equivariant $K$-groups: $'M^r = \bigoplus_n K^{\mathbb{T}_r}(M(r,n))$. It is a module over $K^{\mathbb{T}_r}(\mathrm{pt}) = \mathbb{C}[\mathbb{T}_r] = \mathbb{C}[t_1, t_2, \chi_1, \ldots, \chi_r]$. We define

$$M^r := {}'M^r \otimes_{K^{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt})) = {}'M^r \otimes_{\mathbb{C}[t_1,t_2,\chi_1,\ldots,\chi_r]} \mathbb{C}(t_1, t_2, \chi_1, \ldots, \chi_r).$$

It has a natural grading: $M^r = \bigoplus_n M^r_n$, $M^r_n = K^{\mathbb{T}_r}(M(r,n)) \otimes_{K^{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt}))$. According to the localization theorem, restriction to the $\mathbb{T}_r$-fixed point set induces an isomorphism

$$K^{\mathbb{T}_r}(M(r,n)) \otimes_{K^{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt})) \xrightarrow{\sim} K^{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r}) \otimes_{K^{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt})).$$

The structure sheaves $\{\bar{\lambda}\}$ of the $\mathbb{T}_r$-fixed points $\xi_{\bar{\lambda}}$ (defined in Section 5.3.1) form a basis in $\bigoplus_n K^{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r}) \otimes_{K^{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(K^{\mathbb{T}_r}(\mathrm{pt}))$. Since embedding of a point $\xi_{\bar{\lambda}}$ into $M(r, |\bar{\lambda}|)$ is a proper morphism, the direct image in the equivariant $K$-theory is well defined, and we denote by $[\bar{\lambda}] \in M^r_{|\bar{\lambda}|}$ the direct image of the structure sheaf $\{\bar{\lambda}\}$. The set $\{[\bar{\lambda}]\}$ forms a basis of $M^r$.

Consider the generating series $\mathbf{a}_r(z)$, $\mathbf{c}_r(z) \in M^r(z)$ defined as follows:

$$\mathbf{a}_r(z) := \Lambda^{\bullet}_{-1/z}(\mathfrak{F}_r) = \sum_{i \geq 0} [\Lambda^i(\mathfrak{F}_r)](-1/z)^i,$$

$$\mathbf{c}_r(z) := \mathbf{a}_r(zt_1)\mathbf{a}_r(zt_2)\mathbf{a}_r(zt_3)\mathbf{a}_r(zt_1^{-1})^{-1}\mathbf{a}_r(zt_2^{-1})^{-1}\mathbf{a}_r(zt_3^{-1})^{-1}.$$

Finally, we define the linear operators $e_i, f_i, \psi^{\pm}_j$ $(i \in \mathbb{Z}, j \in \mathbb{Z}_+)$ on $M^r$:

$$e_i = \mathbf{q}_{r*}(L_r^{\otimes i} \otimes \mathbf{p}_r^*): \ M^r_n \to M^r_{n+1}, \tag{5.4}$$

$$f_i = \mathbf{p}_{r*}(L_r^{\otimes(i-r)} \otimes \mathbf{q}_r^*): \ M^r_n \to M^r_{n-1}, \tag{5.5}$$

$$\psi^{\pm}(z)_{|M_n^r} = \sum_{r=0}^{\infty} \psi_r^{\pm} z^{\mp r} := \left( (-1)^r t_1 t_2 \chi_1 \dots \chi_r \prod_{a=1}^{r} \frac{1 - t_1 t_2 \chi_a z}{1 - \chi_a z} \cdot \mathbf{c}_r(z) \right)^{\pm} \in M_n^r[[z^{\mp 1}]]. \tag{5.6}$$

**Theorem 5.3.2.** *The operators $e_i, f_i, \psi_j^{\pm}$, defined in (5.4)-(5.6), satisfy the relations (T0)-(T6) with the parameters $q_i = t_i, 1 \le i \le 3$. This endows $M^r$ with the structure of a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation.*

Let us compute the matrix coefficients of those operators in the fixed point basis.

**Lemma 5.3.3.** *Consider the fixed point basis $\{[\bar{\lambda}]\}$ of $M^r$. Define $\chi_k^{(a)} := t_1^{\lambda_k^a - 1} t_2^{k-1} \chi_a^{-1}$.*
*(a) The only nonzero matrix coefficients of the operators $e_p, f_p$ are as follows:*

$$e_{p[\bar{\lambda}-\square_j^l, \bar{\lambda}]} = \frac{(\chi_j^{(l)})^p}{1 - t_1^{-1} t_2^{-1}} \cdot \prod_{a=1}^{r} \prod_{k=1}^{\infty} \frac{1 - t_1 t_2 \chi_k^{(a)}/\chi_j^{(l)}}{1 - t_1 \chi_k^{(a)}/\chi_j^{(l)}},$$

$$f_{p[\bar{\lambda}+\square_j^l, \bar{\lambda}]} = \frac{(t_1 \chi_j^{(l)})^{p-r}}{1 - t_1^{-1} t_2^{-1}} \cdot \prod_{a=1}^{r} \prod_{k=1}^{\infty} \frac{1 - t_1 t_2 \chi_j^{(l)}/\chi_k^{(a)}}{1 - t_1 \chi_j^{(l)}/\chi_k^{(a)}},$$

*where $\bar{\lambda} \pm \square_j^l$ denotes the $r$-partition obtained from $\bar{\lambda}$ by adding/erasing a box in $j$-th row of $\lambda^l$.*
*(b) The eigenvalue of $\psi^{\pm}(z)$ applied to $[\bar{\lambda}]$ equals*

$$\left( T \prod_{a=1}^{r} \frac{1 - t_3 \chi_a^{-1}/z}{1 - \chi_a^{-1}/z} \prod_{a=1}^{r} \prod_{\square \in \lambda^a} \frac{(1 - t_1^{-1}\chi(\square)/z)(1 - t_2^{-1}\chi(\square)/z)(1 - t_3^{-1}\chi(\square)/z)}{(1 - t_1\chi(\square)/z)(1 - t_2\chi(\square)/z)(1 - t_3\chi(\square)/z)} \right)^{\pm},$$

*where $T = (-1)^r t_1^{r+1} t_2^{r+1} \chi_1 \dots \chi_r$ and $\chi(\square_{i,j}^a) = t_1^{i-1} t_2^{j-1} \chi_a^{-1}$ for a box $\square_{i,j}^a$ staying in the $j$-th row and $i$-th column of $\lambda^a$.*

This lemma allows to prove Theorem 5.3.2 just by a straightforward verification of the relations (T0)-(T6) in the fixed point basis. The only nontrivial relation (T3) can be verified analogously to the case of $(\mathbb{A}^2)^{[n]}$. We will sketch the proof at the end of this section.

### 5.3.3 Geometric $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action II

We generalize Theorem 5.2.2 for a higher rank $r$.

Let $'V^r$ be the direct sum of equivariant (complexified) cohomology: $'V^r = \bigoplus_n H^\bullet_{\mathbb{T}_r}(M(r,n))$. It is a module over $H^\bullet_{\mathbb{T}_r}(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}_r] = \mathbb{C}[s_1, s_2, x_1, \ldots, x_r]$, where $\mathfrak{t}_r = \mathrm{Lie}(\mathbb{T}_r)$. We define

$$V^r := {}'V^r \otimes_{H^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}_r}(\mathrm{pt})) = {}'V^r \otimes_{\mathbb{C}[s_1,s_2,x_1,\ldots,x_r]} \mathbb{C}(s_1, s_2, x_1, \ldots, x_r).$$

It has a natural grading: $V^r = \bigoplus_n V^r_n$, $V^r_n = H^\bullet_{\mathbb{T}_r}(M(r,n)) \otimes_{H^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}_r}(\mathrm{pt}))$. According to the localization theorem, restriction to the $\mathbb{T}_r$-fixed point set induces an isomorphism

$$H^\bullet_{\mathbb{T}_r}(M(r,n)) \otimes_{H^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}_r}(\mathrm{pt})) \xrightarrow{\sim} H^\bullet_{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r}) \otimes_{H^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}_r}(\mathrm{pt})).$$

The fundamental cycles $[\bar{\lambda}]$ of the $\mathbb{T}_r$-fixed points $\xi_{\bar{\lambda}}$ form a basis in the direct sum $\bigoplus_n H^\bullet_{\mathbb{T}_r}(M(r,n)^{\mathbb{T}_r}) \otimes_{H^\bullet_{\mathbb{T}_r}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{\mathbb{T}_r}(\mathrm{pt}))$. Since embedding of a point $\xi_{\bar{\lambda}}$ into $M(r, |\bar{\lambda}|)$ is a proper morphism, the direct image in the equivariant cohomology is well defined, and we will denote by $[\bar{\lambda}] \in V^r_{|\bar{\lambda}|}$ the direct image of the fundamental cycle of the point $\xi_{\bar{\lambda}}$. The set $\{[\bar{\lambda}]\}$ forms a basis of $V^r$.

We introduce the generating series $\mathbf{C}_r(z) \in V^r[[z^{-1}]]$ as follows:

$$\mathbf{C}_r(z) := \left( \frac{\mathrm{ch}(\mathfrak{F}_r t_1^{-1}, -z^{-1}) \mathrm{ch}(\mathfrak{F}_r t_2^{-1}, -z^{-1}) \mathrm{ch}(\mathfrak{F}_r t_3^{-1}, -z^{-1})}{\mathrm{ch}(\mathfrak{F}_r t_1, -z^{-1}) \mathrm{ch}(\mathfrak{F}_r t_2, -z^{-1}) \mathrm{ch}(\mathfrak{F}_r t_3, -z^{-1})} \right)^+ .$$

Finally, we define the linear operators $e_j, f_j, \psi_j$ ($j \in \mathbb{Z}_+$) on $V^r$:

$$e_j = \mathbf{q}_{r*}(c_1(L_r)^j \cdot \mathbf{p}_r^*) : \; V^r_n \to V^r_{n+1}, \tag{4'}$$

$$f_j = (-1)^{r-1} \mathbf{p}_{r*}(c_1(L_r)^j \cdot \mathbf{q}_r^*) : \; V^r_n \to V^r_{n-1}, \tag{5'}$$

$$\psi(z)_{|V^r_n} = 1 + s_1 s_2 s_3 \sum_{r=0}^{\infty} \psi_r z^{-r-1} := \left( \prod_{a=1}^{r} \frac{1 + \frac{x_a - s_3}{z}}{1 + \frac{x_a}{z}} \cdot \mathbf{C}_r(z) \right)^+ \in V^r_n[[z^{-1}]]. \tag{6'}$$

**Theorem 5.3.4.** *The operators $e_j, f_j, \psi_j$, defined in (4')-(6'), satisfy the relations (Y0)-(Y6) with the parameters $h_i = s_i, 1 \leq i \leq 3$. This endows $V^r$ with the structure of a $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation.*

Let us compute the matrix coefficients of those operators in the fixed point basis.

**Lemma 5.3.5.** *Consider the fixed point basis $\{[\bar{\lambda}]\}$ of $V^r$. Define $x_k^{(a)} := (\lambda_k^a - 1)s_1 + (k-1)s_2 - x_a$.*

*(a) The only nonzero matrix coefficients of the operators $e_p, f_p$ are as follows:*

$$e_{p[\bar{\lambda}-\square_j^l,\bar{\lambda}]} = \frac{(x_j^{(l)})^p}{s_1+s_2} \cdot \prod_{a=1}^{r}\prod_{k=1}^{\infty} \frac{s_1+s_2+x_k^{(a)}-x_j^{(l)}}{s_1+x_k^{(a)}-x_j^{(l)}},$$

$$f_{p[\bar{\lambda}+\square_j^l,\bar{\lambda}]} = (-1)^{r-1}\frac{(s_1+x_j^{(l)})^p}{s_1+s_2} \cdot \prod_{a=1}^{r}\prod_{k=1}^{\infty} \frac{s_1+s_2+x_j^{(l)}-x_k^{(a)}}{s_1+x_j^{(l)}-x_k^{(a)}}.$$

*(b) The eigenvalue of $\psi(z)$ applied to $[\bar{\lambda}]$ equals*

$$\left(\prod_{a=1}^{r}\frac{1+\frac{x_a-s_3}{z}}{1+\frac{x_a}{z}} \cdot \prod_{a=1}^{r}\prod_{\square\in\lambda^a} \frac{(1-\frac{\chi(\square)-s_1}{z})(1-\frac{\chi(\square)-s_2}{z})(1-\frac{\chi(\square)-s_3}{z})}{(1-\frac{\chi(\square)+s_1}{z})(1-\frac{\chi(\square)+s_2}{z})(1-\frac{\chi(\square)+s_3}{z})}\right)^+,$$

*where $\chi(\square_{i,j}^a) = (i-1)s_1 + (j-1)s_2 - x_a$.*

This lemma allows to prove Theorem 5.3.4 just by a straightforward verification of the relations (Y0)-(Y6) in the fixed point basis. The only nontrivial relation is actually (Y3). Its proof is based on the statement analogous to Lemma 5.2.4.

**Corollary 5.3.6.** *We have*

$$\psi(z)_{|\bar{\lambda}} = 1 - \frac{rs_3}{z} + \frac{s_3\sum x_j + \binom{r}{2}s_3^2}{z^2} + \frac{2\sigma_3|\bar{\lambda}| - s_3\sum x_j^2 - (r-1)s_3^2\sum x_j - \binom{r}{3}s_3^3}{z^3} + o(z^{-3}).$$

### 5.3.4 Sketch of the proof of Theorem 5.3.2

The purpose of this section is to outline the main computation required to carry out verifications of (T0)-(T6) in the proof of Theorem 5.3.2.

The verification of relations (T0, T1, T2, T6t) is straightforward just by using the formulas for the matrix coefficients from Lemma 5.3.3(a). It is also easy to see that the operators $[e_i, f_j]$ are diagonalizable in the fixed point basis and depend on $i+j$ only: $[e_i, f_j]([\bar{\lambda}]) = \gamma_{i+j|\bar{\lambda}} \cdot [\bar{\lambda}]$.

Next, we introduce series of operators $\phi^{\pm}(z) = \sum_{i=0}^{\infty} \phi_i^{\pm} z^{\mp i}$, diagonalizable in the fixed point basis and satisfying the equation

$$[e(z), f(w)] = \frac{\delta(z/w)}{(1-t_1)(1-t_2)(1-t_3)}(\phi^+(w) - \phi^-(z)).$$

Actually, this determines $\phi_{>0}^{\pm}$ and $\phi_0^+ - \phi_0^-$ uniquely. Our next goal is to specify $\phi_0^{\pm}$.

**Lemma 5.3.7.** *We have*

$$\gamma_{0|\bar{\lambda}} = (-1)^{r-1}\chi_1 \dots \chi_r \frac{t_1 t_2 - t_1^{r+1} t_2^{r+1}}{(1-t_1)(1-t_2)(1-t_3)},$$

$$\gamma_{1|\bar{\lambda}} = (-1)^r \chi_1 \dots \chi_r t_1^{r+1} t_2^{r+1} \left( \sum_{a=1}^{r} \chi_a^{-1} - \sum_{a=1}^{r} \sum_{\square \in \lambda^a} \chi(\square) \right).$$

*Proof.*

Fix positive integers $L_a > \lambda_1^{a*}$. Applying Lemma 5.3.3(a), we find:

$$\gamma_{s|\bar{\lambda}} = \sum_{l=1}^{r} \sum_{j=1}^{L_l} \frac{t_1^2 t_2^2}{(1-t_1)^2} (\chi_j^{(l)})^{s-r} \cdot \frac{\chi_j^{(l)}(1 - \chi_j^{(l)} t_2^{1-L_l} t_1 \chi_l)}{\chi_j^{(l)} - t_2^{L_l} \chi_l^{-1}} \cdot \prod_{k \neq j}^{k \leq L_l} \frac{(\chi_j^{(l)} - t_1 t_2 \chi_k^{(l)})(\chi_k^{(l)} - t_2 \chi_j^{(l)})}{(\chi_j^{(l)} - t_1 \chi_k^{(l)})(\chi_k^{(l)} - \chi_j^{(l)})} \times$$
$$\prod_{a \neq l} \left( \frac{\chi_j^{(l)}(1 - \chi_j^{(l)} t_2^{1-L_a} t_1 \chi_a)}{\chi_j^{(l)} - t_2^{L_a} \chi_a^{-1}} \cdot \prod_{k=1}^{L_a} \frac{(\chi_j^{(l)} - t_1 t_2 \chi_k^{(a)})(\chi_k^{(a)} - t_2 \chi_j^{(l)})}{(\chi_j^{(l)} - t_1 \chi_k^{(a)})(\chi_k^{(a)} - \chi_j^{(l)})} \right) -$$
$$\sum_{l=1}^{r} \sum_{j=1}^{L_l} \frac{t_1^2 t_2^2}{(1-t_1)^2} (t_1 \chi_j^{(l)})^{s-r} \cdot \frac{\chi_j^{(l)}(1 - \chi_j^{(l)} t_2^{1-L_l} t_1^2 \chi_l)}{\chi_j^{(l)} - t_2^{L_l} t_1^{-1} \chi_l^{-1}} \cdot \prod_{k \neq j}^{k \leq L_l} \frac{(\chi_k^{(l)} - t_1 t_2 \chi_j^{(l)})(\chi_j^{(l)} - t_2 \chi_k^{(l)})}{(\chi_k^{(l)} - t_1 \chi_j^{(l)})(\chi_j^{(l)} - \chi_k^{(l)})} \times$$
$$\prod_{a \neq l} \left( \frac{\chi_j^{(l)}(1 - \chi_j^{(l)} t_2^{1-L_a} t_1^2 \chi_a)}{\chi_j^{(l)} - t_2^{L_a} t_1^{-1} \chi_a^{-1}} \cdot \prod_{k=1}^{L_a} \frac{(\chi_k^{(a)} - t_1 t_2 \chi_j^{(l)})(\chi_j^{(l)} - t_2 \chi_k^{(a)})}{(\chi_k^{(a)} - t_1 \chi_j^{(l)})(\chi_j^{(l)} - \chi_k^{(a)})} \right), \quad (\heartsuit)$$

where $\chi_k^{(a)} = t_1^{\lambda_k^a - 1} t_2^{k-1} \chi_a^{-1}$. The result does not depend on the choice of $\{L_a\}$.

(i) For $s = 0$, the right hand side of $(\heartsuit)$ is a degree 0 rational function in the variables $\chi_k^{(a)}$. It is easy to see that it has no poles, in fact. Therefore, it is an element of $\mathbb{C}(t_1, t_2, \chi_1, \dots, \chi_r)$ independent of $\bar{\lambda}$. It suffices to compute its value at the empty $r$-partition $\bar{\emptyset}$. For $\bar{\lambda} = \bar{\emptyset}$, we can choose $L_1 = \dots = L_r = 1$, while $\chi_k^{(a)} = t_1^{-1} t_2^{k-1} \chi_a^{-1}$

for $k \geq 1$. Applying ($\heartsuit$), we get

$$\gamma_{0|_{\bar\lambda}} = \gamma_{0|_{\bar\emptyset}} = -\frac{t_1^2t_2^2}{(1-t_1)^2}\sum_{l=1}^{r} t_1^{-r}\frac{1-t_1}{t_1^{-1}\chi_l^{-1}-t_2t_1^{-1}\chi_l^{-1}}\prod_{a\neq l}\frac{(t_1^{-1}\chi_a^{-1}-t_2\chi_l^{-1})(1-t_1\chi_a\chi_l^{-1})}{(t_1^{-1}\chi_l^{-1}-t_1^{-1}\chi_a^{-1})(t_1^{-1}\chi_a^{-1}-\chi_l^{-1})} =$$

$$\frac{(-1)^r t_1^2t_2^2}{(1-t_1)(1-t_2)}\chi_1\dots\chi_r\sum_{l=1}^{r}\prod_{a\neq l}\frac{\chi_l-t_1t_2\chi_a}{\chi_l-\chi_a} = \frac{(-1)^r t_1^2t_2^2}{(1-t_1)(1-t_2)}\chi_1\dots\chi_r\frac{1-t_1^rt_2^r}{1-t_1t_2},$$

where we used the identity $\sum_{l=1}^r\prod_{a\neq l}\frac{\chi_l-u\chi_a}{\chi_l-\chi_a}=\frac{1-u^r}{1-u}$. The first result follows.

(ii) For $s=1$, the right hand side of ($\heartsuit$) is a degree 1 rational function in the variables $\chi_k^{(a)}$. It is easy to see that it has no poles, actually. Therefore, it is a linear function. Its leading term equals $(-1)^r\chi_1\dots\chi_r\frac{t_1^{r+2}t_2^{r+1}}{1-t_1}\cdot\sum_{l=1}^r\sum_{j=1}^{L_l}\chi_j^{(l)}$. Hence, we have:

$$\gamma_{1|_{\bar\lambda}} = (-1)^r\chi_1\dots\chi_r\frac{t_1^{r+2}t_2^{r+1}}{1-t_1}\cdot\sum_{l=1}^{r}\sum_{j=1}^{\infty}\widetilde{\chi}_j^{(l)}+C$$

for a constant $C\in\mathbb{C}(t_1,t_2,\chi_1,\dots,\chi_r)$ independent of $\bar\lambda$, where $\widetilde{\chi}_j^{(l)} := \chi_j^{(l)}-t_1^{-1}t_2^{j-1}\chi_a^{-1}$.

Note that $\sum_{a=1}^r\sum_{\square\in\lambda^a}\chi(\square)=\sum_{a=1}^r\sum_{j=1}^{\lambda_1^{a*}}t_2^{j-1}(1+t_1+\ldots+t_1^{\lambda_j^a-1})\chi_a^{-1}=\sum_{l=1}^r\sum_j\frac{-t_1}{1-t_1}\widetilde{\chi}_j^{(l)}$. On the other hand, $C=\gamma_{1|_{\bar\emptyset}}$. Applying ($\heartsuit$), we get

$$C=\gamma_{1|_{\bar\emptyset}} = -\frac{t_1^2t_2^2}{(1-t_1)^2}\sum_{l=1}^{r} t_1^{-r}\frac{\chi_l^{-1}(1-t_1)}{t_1^{-1}\chi_l^{-1}-t_2t_1^{-1}\chi_l^{-1}}\prod_{a\neq l}\frac{(t_1^{-1}\chi_a^{-1}-t_2\chi_l^{-1})(1-t_1\chi_a\chi_l^{-1})}{(t_1^{-1}\chi_l^{-1}-t_1^{-1}\chi_a^{-1})(t_1^{-1}\chi_a^{-1}-\chi_l^{-1})} =$$

$$\frac{(-1)^r t_1^2t_2^2}{(1-t_1)(1-t_2)}\chi_1\dots\chi_r\sum_{l=1}^{r}\frac{1}{\chi_l}\prod_{a\neq l}\frac{\chi_l-t_1t_2\chi_a}{\chi_l-\chi_a} = \frac{(-1)^r t_1^{r+1}t_2^{r+1}}{(1-t_1)(1-t_2)}\chi_1\dots\chi_r\sum_{l=1}^{r}\chi_l^{-1},$$

where we used $\sum_{l=1}^r\frac{1}{\chi_l}\prod_{a\neq l}\frac{\chi_l-u\chi_a}{\chi_l-\chi_a}=u^{r-1}\sum_{l=1}^r\chi_l^{-1}$. The second result follows. □

Due to the first equality of this lemma, we can set

$$\phi_0^+ := (-1)^r t_1^{r+1}t_2^{r+1}\chi_1\dots\chi_r,\ \phi_0^- := (-1)^r t_1t_2\chi_1\dots\chi_r.$$

Next, we claim that $\phi^\pm(z)$ satisfy the following relations:

$$\phi^\pm(z)e(w)(z-q_1w)(z-q_2w)(z-q_3w) = -e(w)\phi^\pm(z)(w-q_1z)(w-q_2z)(w-q_3z) \quad (5.7)$$

$$\phi^{\pm}(z)f(w)(w-q_1z)(w-q_2z)(w-q_3z) = -f(w)\phi^{\pm}(z)(z-q_1w)(z-q_2w)(z-q_3w) \quad (5.8)$$

The proof is based on straightforward computations in the fixed point basis.

Finally, relation (5.7) implies the following identity:

$$\phi^+(z)_{|\vec{\lambda}+\square^l_j} = \phi^+(z)_{|\vec{\lambda}} \cdot \frac{(1-t_1^{-1}\chi(\square^l_j)/z)(1-t_2^{-1}\chi(\square^l_j)/z)(1-t_3^{-1}\chi(\square^l_j)/z)}{(1-t_1\chi(\square^l_j)/z)(1-t_2\chi(\square^l_j)/z)(1-t_3\chi(\square^l_j)/z)}.$$

Therefore,

$$\phi^+(z)_{|\vec{\lambda}} = \phi^+(z)_{|\vec{\emptyset}} \cdot c_r(z)^+_{|\vec{\lambda}}.$$

Applying formula (♡) once again, we get:

$$\phi^+(z)_{|\vec{\emptyset}} = (\phi_0^- + \sum_{i\geq 0}(1-t_1)(1-t_2)(1-t_3)\gamma_i z^{-i})_{|\vec{\emptyset}} =$$

$$(-1)^r t_1t_2\chi_1\dots\chi_r + t_1t_2(1-t_1t_2)\chi_1\dots\chi_r \sum_{l=1}^{r} \frac{1}{1-\chi_l^{-1}z^{-1}} \prod_{a\neq l} \frac{\chi_l - t_1t_2\chi_a}{\chi_a-\chi_l} =$$

$$(-1)^r t_1t_2\chi_1\dots\chi_r \prod_{l=1}^{r} \left(\frac{1-t_1t_2\chi_l z}{1-\chi_l z}\right)^+,$$

where we used the identity

$$1-(1-u)\sum_{l=1}^{r}\frac{1}{1-1/(\chi_l z)}\prod_{a\neq l}\frac{\chi_l-u\chi_a}{\chi_l-\chi_a} = \prod_{l=1}^{r}\frac{u-\frac{1}{\chi_l z}}{1-\frac{1}{\chi_l z}}.$$

This proves $\phi^+(z)=\psi^+(z)$. The same arguments prove $\phi^-(z)=\psi^-(z)$. The relation (T3) follows. On the other hand, relations (5.7) and (5.8) imply that the relations (T4,T5) also hold.

This completes the proof of Theorem 5.3.2.

### 5.3.5 Sketch of the proof of Theorem 5.3.4

The proof of the cohomological counterpart of the previous result is completely analogous and is parallel to the proof of Lemma 5.2.4.

The verification of the relations (Y0, Y1, Y2, Y6) is straightforward. To verify the remaining relations, we follow the same pattern as above. It is easy to check that $[e_i, f_j]$ is diagonalizable in the fixed point basis and depends on $i+j$ only: $[e_i, f_j]([\bar{\lambda}]) = \gamma_{i+j|\bar{\lambda}} \cdot [\bar{\lambda}]$.

**Lemma 5.3.8.** *We have* $\gamma_{0|\bar{\lambda}} = \frac{-r}{s_1 s_2}$, $\gamma_{1|\bar{\lambda}} = \frac{1}{s_1 s_2}\left(\sum_{a=1}^r x_a - \binom{r}{2}(s_1+s_2)\right)$,

$$\gamma_{2|\bar{\lambda}} = 2|\bar{\lambda}| - \frac{1}{s_1 s_2}\left(\sum_{a=1}^{r} x_a^2 - (r-1)(s_1+s_2)\sum_{a=1}^{r} x_a + \binom{r}{3}(s_1+s_2)^2\right).$$

*Proof.*

Applying Lemma 5.3.5(a), we find:

$$\gamma_{s|\bar{\lambda}} =$$

$$\sum_{l=1}^{r}\sum_{j=1}^{L_l} \frac{1}{s_1^2}(x_j^{(l)})^s \cdot \frac{x_j^{(l)} + (1-L_l)s_2 + s_1 + x_l}{-x_j^{(l)} + L_l s_2 - x_l} \cdot \prod_{k\neq j}^{k\leq L_l} \frac{(x_j^{(l)} - x_k^{(l)} - s_1 - s_2)(x_k^{(l)} - x_j^{(l)} - s_2)}{(x_j^{(l)} - x_k^{(l)} - s_1)(x_k^{(l)} - x_j^{(l)})} \times$$
$$\prod_{a\neq l}\left(\frac{x_j^{(l)} + (1-L_a)s_2 + s_1 + x_a}{x_j^{(l)} - L_a s_2 + x_a} \cdot \prod_{k=1}^{L_a} \frac{(x_j^{(l)} - x_k^{(a)} - s_1 - s_2)(x_k^{(a)} - x_j^{(l)} - s_2)}{(x_j^{(l)} - x_k^{(a)} - s_1)(x_k^{(a)} - x_j^{(l)})}\right) -$$
$$\sum_{l=1}^{r}\sum_{j=1}^{L_l} \frac{1}{s_1^2}(x_j^{(l)} + s_1)^s \cdot \frac{x_j^{(l)} + (1-L_l)s_2 + 2s_1 + x_l}{-x_j^{(l)} + L_l s_2 - s_1 - x_l} \cdot \prod_{k\neq j}^{k\leq L_l} \frac{(x_k^{(l)} - x_j^{(l)} - s_1 - s_2)(x_j^{(l)} - x_k^{(l)} - s_2)}{(x_k^{(l)} - x_j^{(l)} - s_1)(x_j^{(l)} - x_k^{(l)})} \times$$
$$\prod_{a\neq l}\left(\frac{x_j^{(l)} + (1-L_a)s_2 + 2s_1 + x_a}{x_j^{(l)} - L_a s_2 + s_1 + x_a} \cdot \prod_{k=1}^{L_a} \frac{(x_k^{(a)} - x_j^{(l)} - s_1 - s_2)(x_j^{(l)} - x_k^{(a)} - s_2)}{(x_k^{(a)} - x_j^{(l)} - s_1)(x_j^{(l)} - x_k^{(a)})}\right), \qquad (\spadesuit)$$

where $x_k^{(a)} = (\lambda_k^a - 1)s_1 + (k-1)s_2 - x_a$ as before. The right hand side of ($\spadesuit$) is a degree $s$ rational function in the variables $x_k^{(a)}$. Actually, it is easy to see that it has no poles for $s \geq 0$.

(i) For $s = 0$, we therefore get an element of $\mathbb{C}(s_1, s_2, x_1, \ldots, x_l)$ independent of $\bar{\lambda}$. Using ($\spadesuit$) once again, we get $\gamma_{0|\bar{\lambda}} = \gamma_{0|\bar{\emptyset}} = -r/s_1 s_2$.

(ii) For $s = 1$, we therefore get a linear function. But its leading term is zero, in fact. So $\gamma_{1|\bar{\lambda}} = \gamma_{1|\bar{\emptyset}}$. Using ($\spadesuit$) once again, we get $\gamma_{1|\bar{\emptyset}} = \frac{1}{s_1 s_2}\left(\sum_{a=1}^r x_a - \binom{r}{2}(s_1+s_2)\right)$.

(iii) For $s = 2$, we therefore get a quadratic function. But its leading quadratic part is zero, in fact. So $\gamma_{2|\bar{\lambda}}$ is a linear function. Similarly to the proof of Lemma 5.2.4, we find that the leading linear part is actually $\frac{2}{s_1}\sum_{a=1}^{r}\sum_{k=1}^{\infty}\widetilde{x}_k^{(a)}=2|\bar{\lambda}|$, where $\widetilde{x}_k^{(a)}:=x_k^{(a)}-(-s_1+(k-1)s_2-x_a)$. Hence, $\gamma_{2|_{\bar{\lambda}}}=2|\bar{\lambda}|+\gamma_{2|_{\bar{\emptyset}}}$. Applying (♠) once again, we get the last formula. □

Using this lemma together with computations in the fixed point basis, it is straightforward to check that $\{\phi_i, e_i, f_i\}_{i\in\mathbb{Z}_+}$ satisfy the relations (Y4,Y4′,Y5,Y5′). This in turn implies

$$\phi(z)_{|_{\bar{\lambda}+\square_j^l}}=\phi(z)_{|_{\bar{\lambda}}}\cdot\frac{(z-\chi(\square_j^l)+s_1)(z-\chi(\square_j^l)+s_2)(z-\chi(\square_j^l)+s_3)}{(z-\chi(\square_j^l)-s_1)(z-\chi(\square_j^l)-s_2)(z-\chi(\square_j^l)-s_3)},$$

where $\phi(z):=1+\sigma_3\sum_{i\geq 0}\phi_i z^{-i-1}$. Therefore, $\phi(z)_{|_{\bar{\lambda}}}=\phi(z)_{|_{\bar{\emptyset}}}\cdot C_r(z)^+_{|_{\bar{\lambda}}}$. Applying (♠), we get:

$$\phi(z)_{|_{\bar{\emptyset}}}=1-\frac{\sigma_3}{s_1s_2}\sum_{i\geq 0}\sum_{l=1}^{r}(-x_l)^i z^{-i-1}\prod_{a\neq l}\frac{x_l-x_a-s_1-s_2}{x_l-x_a}=$$

$$1-s_3\sum_{l=1}^{r}\frac{1}{z+x_l}\prod_{a\neq l}\frac{x_l-x_a-s_1-s_2}{x_l-x_a}.$$

It remains to use the identity $1+u\sum_{l=1}^{r}\frac{1}{z+x_l}\prod_{a\neq l}\frac{x_l-x_a-u}{x_l-x_a}=\prod_{j=1}^{r}\frac{z+x_j+u}{z+x_j}$.

This proves $\phi(z)=\psi(z)$. The relations (Y3-Y5′) follow. Theorem 5.3.4 is proved.

## 5.4 Some representations of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$

In this section, we recall several families of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations from [FFJMM1, FFJMM2] and establish their analogues for the case of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. This should be viewed as an analogy between the representation theory of $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$.

### 5.4.1 Vector representations

We start from the simplest representations $V(u)$ and $V^a(u)$.

The main building block of all constructions is the family of *vector representations* of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, whose basis is parametrized by $\mathbb{Z}$ (see [FFJMM1, Proposition 3.1]).

**Proposition 5.4.1** (Vector representation of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}^*$, let $V(u)$ be a $\mathbb{C}$-vector space with the basis $\{[u]_j\}_{j\in\mathbb{Z}}$. The following formulas define $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action on $V(u)$:*

$$e(z)[u]_i = (1-q_1)^{-1}\delta(q_1^i u/z)\cdot[u]_{i+1},$$

$$f(z)[u]_i = (q_1^{-1}-1)^{-1}\delta(q_1^{i-1}u/z)\cdot[u]_{i-1},$$

$$\psi^{\pm}(z)[u]_i = \left(\frac{(z-q_1^i q_2 u)(z-q_1^i q_3 u)}{(z-q_1^i u)(z-q_1^{i-1}u)}\right)^{\pm}\cdot[u]_i.$$

Analogously to that, we define a family of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ *vector representations*:

**Proposition 5.4.2** (Vector representation of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}$, let ${}^aV(u)$ be a $\mathbb{C}$-vector space with the basis $\{[u]_j\}_{j\in\mathbb{Z}}$. The following formulas define $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action on ${}^aV(u)$:*

$$e(z)[u]_i = \frac{1}{h_1 z}\delta^+((ih_1+u)/z)[u]_{i+1} = \left(\frac{1}{h_1(z-u-ih_1)}\right)^+\cdot[u]_{i+1},$$

$$f(z)[u]_i = -\frac{1}{h_1 z}\delta^+(((i-1)h_1+u)/z)[u]_{i-1} = \left(\frac{-1}{h_1(z-u-(i-1)h_1)}\right)^+\cdot[u]_{i-1},$$

$$\psi(z)[u]_i = \left(\frac{(z-(ih_1+h_2+u))(z-(ih_1+h_3+u))}{(z-(ih_1+u))(z-((i-1)h_1+u))}\right)^+\cdot[u]_i,$$

*where $\delta^+(w) := 1+w+w^2+\ldots = (\frac{1}{1-w})^+$.*

### 5.4.2 Fock representations

Next, we introduce a family of Fock modules $F(u)$ and ${}^aF(u)$.

A more interesting family of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations, whose basis is parametrized by all Young diagrams $\{\lambda\}$, was established in [FFJMM1, Theorem 4.3, Corollary 4.4].

**Proposition 5.4.3** (Fock representation of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}^*$, let $F(u)$ be a $\mathbb{C}$-vector space with the basis $\{|\lambda\rangle\}$. The following formulas define $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-action on $F(u)$:*

$$e(z)|\lambda\rangle = \sum_{i\geq 1}\prod_{j=1}^{i-1} \frac{(1-q_1^{\lambda_i-\lambda_j}q_2^{i-j-1})(1-q_1^{\lambda_i-\lambda_j+1}q_2^{i-j+1})}{(1-q_1^{\lambda_i-\lambda_j}q_2^{i-j})(1-q_1^{\lambda_i-\lambda_j+1}q_2^{i-j})}\cdot\frac{\delta(q_1^{\lambda_i}q_2^{i-1}u/z)}{1-q_1}\cdot|\lambda+i\rangle,$$

$$f(z)|\lambda\rangle = \sum_{i\geq 1}\frac{1-q_1^{\lambda_{i+1}-\lambda_i}}{1-q_1^{\lambda_{i+1}-\lambda_i+1}q_2}\prod_{j=i+1}^{\infty}\frac{(1-q_1^{\lambda_j-\lambda_i+1}q_2^{j-i+1})(1-q_1^{\lambda_{j+1}-\lambda_i}q_2^{j-i})}{(1-q_1^{\lambda_{j+1}-\lambda_i+1}q_2^{j-i+1})(1-q_1^{\lambda_j-\lambda_i}q_2^{j-i})}\times$$
$$\frac{\delta(q_1^{\lambda_i-1}q_2^{i-1}u/z)}{q_1^{-1}-1}\cdot|\lambda-i\rangle,$$

$$\psi^{\pm}(z)|\lambda\rangle = \left(\frac{z-q_1^{\lambda_1-1}q_2^{-1}u}{z-q_1^{\lambda_1}u}\prod_{i=1}^{\infty}\frac{(z-q_1^{\lambda_i}q_2^{i}u)(z-q_1^{\lambda_{i+1}-1}q_2^{i-1}u)}{(z-q_1^{\lambda_{i+1}}q_2^{i}u)(z-q_1^{\lambda_i-1}q_2^{i-1}u)}\right)^{\pm}\cdot|\lambda\rangle.$$

*Remark* 5.4.1. The Fock module $F(u)$ was originally constructed from $V(u)$ by using the semi-infinite wedge construction and the coproduct structure on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ defined by:

$$\Delta: e(z)\mapsto e(z)\otimes 1+\psi^-(z)\otimes e(z),\, f(z)\mapsto f(z)\otimes\psi^+(z)+1\otimes f(z),\, \psi^{\pm}(z)\mapsto\psi^{\pm}(z)\otimes\psi^{\pm}(z).$$

Let us also recall the relation between $F(u)$ and $M$ from Theorem 5.2.1.

*Remark* 5.4.2. (a) According to [FFJMM1, Corollary 4.5], there exist constants $\{c_\lambda\}$ such that the map $[\lambda]\mapsto c_\lambda|\lambda\rangle$ establishes an isomorphism $M\xrightarrow{\sim}F(1)$ of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations.

(b) Let $\phi_u$ be the *shift automorphism* of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ defined on the generators by

$$e_r\mapsto u^r\cdot e_r,\quad f_r\mapsto u^r\cdot f_r,\quad \psi_j^{\pm}\mapsto u^{\pm j}\cdot\psi_j^{\pm},\ r\in\mathbb{Z}, j\in\mathbb{Z}_+.$$

Then the modules $F(u)$ and $V(u)$ are obtained from $F(1)$ and $V(1)$ via a $\phi_u$-twist.

This construction also has an analogue in the $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-case.

**Proposition 5.4.4** (Fock representation of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$). *For $u \in \mathbb{C}$, let ${}^aF(u)$ be a $\mathbb{C}$-vector space with the basis $\{|\lambda\rangle\}$. The following formulas define $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-action on ${}^aF(u)$:*

$$e(z)|\lambda\rangle = \frac{1}{h_1 z}\sum_{i\geq 1}\prod_{j=1}^{i-1}\frac{((\lambda_i-\lambda_j)h_1+(i-j-1)h_2)((\lambda_i-\lambda_j+1)h_1+(i-j+1)h_2)}{((\lambda_i-\lambda_j)h_1+(i-j)h_2)((\lambda_i-\lambda_j+1)h_1+(i-j)h_2)}\times$$

$$\delta^+\left(\frac{\lambda_i h_1+(i-1)h_2+u}{z}\right)\cdot|\lambda+i\rangle,$$

$$f(z)|\lambda\rangle = -\frac{1}{h_1 z}\sum_{i\geq 1}\prod_{j=i+1}^{\infty}\frac{((\lambda_j-\lambda_i+1)h_1+(j-i+1)h_2)((\lambda_{j+1}-\lambda_i)h_1+(j-i)h_2)}{((\lambda_{j+1}-\lambda_i+1)h_1+(j-i+1)h_2)((\lambda_j-\lambda_i)h_1+(j-i)h_2)}\times$$

$$\frac{(\lambda_{i+1}-\lambda_i)h_1}{(\lambda_{i+1}-\lambda_i+1)h_1+h_2}\delta^+\left(\frac{(\lambda_i-1)h_1+(i-1)h_2+u}{z}\right)\cdot|\lambda-i\rangle,$$

$$\psi(z)|\lambda\rangle = \left(\prod_{i=1}^{\infty}\frac{(z-(\lambda_i h_1+ih_2+u))(z-((\lambda_{i+1}-1)h_1+(i-1)h_2+u))}{(z-(\lambda_{i+1}h_1+ih_2+u))(z-((\lambda_i-1)h_1+(i-1)h_2+u))}\right)^+\cdot$$

$$\left(\frac{z-((\lambda_1-1)h_1-h_2+u)}{z-(\lambda_1 h_1+u)}\right)^+\cdot|\lambda\rangle.$$

The proof of this proposition follows from the following lemma:

**Lemma 5.4.5.** *(a) For $u\in\mathbb{C}$, there exists the* shift automorphism *$\phi^a_u$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ such that $\phi^a_u:\ e(z)\mapsto e(z-u),\ f(z)\mapsto f(z-u),\ \psi(z)\mapsto\psi(z-u)$.*
*(b) The Fock representation ${}^aF(u)$ is obtained from ${}^aF(0)$ via a twist by $\phi^a_u$.*
*(c) There exist constants $\{c^a_\lambda\}$ such that the map $[\lambda]\mapsto c^a_\lambda|\lambda\rangle$ establishes an isomorphism $V\xrightarrow{\sim}{}^aF(0)$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations, where $V$ is from Theorem 5.2.2.*

*Proof.*

Parts (a) and (b) are straightforward.

We define $c^a_\lambda$ by the following formula:

$$c^a_\lambda = \prod_{i\geq 1}\prod_{p=0}^{\lambda_i-1}(-ph_1+h_2)\cdot\prod_{i\geq 2}\prod_{j=1}^{i-1}\prod_{p=1}^{\lambda_i}\frac{(p-\lambda_j)h_1+(i-j)h_2}{(p-\lambda_j)h_1+(i-j+1)h_2}.$$

It is a routine verification to check that the map $[\lambda]\mapsto c^a_\lambda|\lambda\rangle$ intertwines the formulas for the matrix coefficients of $e_j, f_j, \psi_j$ from Lemma 5.2.3 and Proposition 5.4.4. $\square$

**Definition 5.4.1.** We say that a representation $U$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ has central charge $(c_0, c_1)$ $(c_i \in \mathbb{C})$ if central elements $\psi_i$ act on $U$ as multiplications by $c_i$ for $i = 0, 1$.

Thus ${}^aV(u)$ has central charge $\left(0, \frac{1}{h_1}\right)$, while ${}^aF(u)$ has central charge $\left(-\frac{1}{h_1h_2}, -\frac{u}{h_1h_2}\right)$.

### 5.4.3 The tensor product of Fock modules $F(u)$

In this section, we express the representation $M^r$ from Section 3 as the appropriate tensor product of Fock modules $F(u)$.

Let $\Delta$ be the *formal* comultiplication on $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ from Remark 5.4.1. This is not a comultiplication in the usual sense, since $\Delta(e_i)$ and $\Delta(f_i)$ contain infinite sums. However, for all modules of our concern, these formulas make sense. Recall the $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation $M^r$, constructed in Theorem 5.3.2. Let $\kappa$ be the automorphism of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, defined on the generators by $\kappa(e_i) = e_i$, $\kappa(f_i) = T^{-1}f_i$, $\kappa(\psi_i^\pm) = T^{-1}\psi_i^\pm$, where $T = (t_1t_2)^{r+1}\chi_1\cdots\chi_r$. Let $\bar{M}^r$ be the $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation, obtained from $M^r$ via a twist by $\kappa$.

**Theorem 5.4.6.** *There exists a unique collection of constants $c_{\bar{\lambda}} \in \mathbb{C}(t_1, t_2, \chi_1, \ldots, \chi_r)$ with $c_{\bar{\emptyset}} = 1$ such that the map $[\bar{\lambda}] = [(\lambda^1, \ldots, \lambda^r)] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ establishes an isomorphism $\bar{M}^r \xrightarrow{\sim} F(\chi_1) \otimes \cdots \otimes F(\chi_r)$ of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations.*

Let us first explain why the formal coproduct $\Delta$ endows the tensor product $F(\chi_1)\otimes F(\chi_2)$ (the case $r > 2$ is completely analogous) with a structure of a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module. In order to make sense of the *formal* coproduct in this setting, note that

$$e(z)|\lambda\rangle = \sum_{\square} a_{\lambda,\square}\delta\left(\frac{\chi(\square)}{z}\right)|\lambda+\square\rangle,\ \ f(z)|\lambda\rangle = \sum_{\square} b_{\lambda,\square}\delta\left(\frac{\chi(\square)}{z}\right)|\lambda-\square\rangle,$$

where $a_{\lambda,\square}, b_{\lambda,\square} \in \mathbb{C}(t_1, t_2, \chi_1, \ldots, \chi_r)$, the first sum is over $\square \notin \lambda$ such that $\lambda + \square$ is a Young diagram, while the second sum is over $\square \in \lambda$ such that $\lambda - \square$ is a Young diagram.

According to the coproduct formula, we have

$$\Delta(e(z))(|\lambda^1\rangle \otimes |\lambda^2\rangle) = e(z)(|\lambda^1\rangle) \otimes |\lambda^2\rangle + \psi^-(z)(|\lambda^1\rangle) \otimes e(z)(|\lambda^2\rangle).$$

The first summand is well defined. To make sense of the second summand we use the formula

$$g(z)\delta(a/z) = g(a)\delta(a/z). \tag{5.9}$$

Recall that $\psi^\pm(z)(|\lambda\rangle) = \gamma_\lambda(z)^\pm \cdot |\lambda\rangle$, where $\gamma_\lambda(z)$ is a rational function in $z$ depending on $\lambda$. Combining this with (5.9), we rewrite

$$\psi^-(z)(|\lambda^1\rangle) \otimes e(z)(|\lambda^2\rangle) = \sum_{\square} a_{\lambda^2,\square}\gamma_{\lambda^1}(\chi(\square))\delta\left(\frac{\chi(\square)}{z}\right) \cdot |\lambda^1\rangle \otimes |\lambda^2 + \square\rangle.$$

Analogously we make sense of the formula for the action of $f_i$ on $F(\chi_1) \otimes F(\chi_2)$. Finally, the formula $\Delta(\psi^\pm(z)) = \psi^\pm(z) \otimes \psi^\pm(z)$ provides a well-defined action of $\psi_i^\pm$ on $F(\chi_1) \otimes F(\chi_2)$.

*Proof of Theorem 5.4.6.*

Due to Remark 5.4.2, we identify $F(\chi_j) \simeq M^{\phi_{\chi_j}}$, the twist of $M$ by the shift automorphism $\phi_{\chi_j}$. For any $r$-partition $\bar{\lambda} = (\lambda^1, \ldots, \lambda^r)$, Lemma 5.3.3(b) implies that the eigenvalue of $\psi^\pm(z)$ on $[\bar{\lambda}] \in \bar{M}^r$ equals the eigenvalue of $\psi^\pm(z)$ on $|\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle \in F(\chi_1) \otimes \cdots \otimes F(\chi_r)$. Hence, for any constants $c_{\bar{\lambda}}$ the map $[\bar{\lambda}] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ intertwines actions of $\{\psi_j^\pm\}_{j \geq 0}$.

Consider constants $c_{\bar{\lambda}}$ defined by $c_{\bar{\emptyset}} = 1$ and $c_{\bar{\lambda}+\square_j^l}/c_{\bar{\lambda}} = d_{\bar{\lambda},\square_j^l}$, where

$$d_{\bar{\lambda},\square_j^l} := (-1)^{l-1}t_1^{-1}t_2^{-1} \cdot \prod_{a=l+1}^{r}\prod_{k=1}^{\infty} \frac{\chi_j^{(l)} - \chi_k^{(a)}}{\chi_j^{(l)} - t_2\chi_k^{(a)}} \cdot \prod_{a=1}^{l-1}\prod_{k=1}^{\infty} \frac{\chi_j^{(l)} - t_1^{-1}t_2^{-1}\chi_k^{(a)}}{\chi_j^{(l)} - t_1^{-1}\chi_k^{(a)}}. \tag{5.10}$$

Here $\chi_p^{(m)} = t_1^{\lambda_p^m - 1}t_2^{p-1}\chi_m^{-1}$ and $\bar{\lambda} + \square_j^k$ denotes the $r$-partition obtained from $\bar{\lambda}$ by adding a box to the $j$-th row of $\lambda^k$. Note that $\chi_{p+1}^{(m)} = t_2\chi_p^{(m)}$ for $p \geq |n|$ and so the infinite products of (5.10) are actually finite. It is straightforward to check that $c_{\bar{\lambda}}$ are well-defined, that is, $d_{\bar{\lambda},\square_j^k}$ satisfy $d_{\bar{\lambda}+\square_j^k,\square_a^l}d_{\bar{\lambda},\square_j^k} = d_{\bar{\lambda}+\square_a^l,\square_j^k}d_{\bar{\lambda},\square_a^l}$. Using

Lemma 5.3.3(a), it is straightforward to check that the map $[\bar{\lambda}] \mapsto c_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ intertwines actions of $e_i$ and $f_i$ as well. The result follows. □

### 5.4.4 The tensor product of Fock modules ${}^aF(u)$

In this section, we express the representation $V^r$ from Section 3 as the appropriate tensor product of Fock modules ${}^aF(u)$. To formulate the result, we need to define the tensor product $W_1 \otimes W_2$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations $W_i$.

The action of $\psi(z)$ on $W_1 \otimes W_2$ is defined via the comultiplication $\Delta(\psi(z)) = \psi(z) \otimes \psi(z)$, that is, $\psi(z)(w^1 \otimes w^2) = \psi(z)(w^1) \otimes \psi(z)(w^2)\ \forall\ w^1 \in W_1, w^2 \in W_2$. To define the action of $e(z), f(z)$ on $W_1 \otimes W_2$, we should restrict to a particular class of representations. A $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation $W$ is called *admissible* if there exists a basis $\{w_\alpha\}_{\alpha\in I}$ of $W$ such that

∘ $e(z)(w_\alpha) = \sum_{\alpha'\in I} \frac{c_{\alpha,\alpha'}}{z}\delta^+(\lambda_{\alpha,\alpha'}/z)w_{\alpha'}, f(z)(w_\alpha) = \sum_{\alpha''\in I} \frac{d_{\alpha,\alpha''}}{z}\delta^+(\lambda_{\alpha'',\alpha}/z)w_{\alpha''}$ for some $c_{\alpha,\alpha'}, d_{\alpha,\alpha''}, \lambda_{\alpha,\alpha'} \in \mathbb{C}$. For each $\alpha$, both sums have only finite number of nonzero summands.

∘ $\psi(z)(w_\alpha) = \gamma_W(\alpha, z)^+ \cdot w_\alpha$ for a rational function $\gamma_W(\alpha, \bullet)$ defined by

$$\gamma_W(\alpha, \bullet) = 1 + \sigma_3 \sum_{\alpha''\in I} \frac{d_{\alpha,\alpha''}c_{\alpha'',\alpha}}{z - \lambda_{\alpha'',\alpha}} - \sigma_3 \sum_{\alpha'\in I} \frac{c_{\alpha,\alpha'}d_{\alpha',\alpha}}{z - \lambda_{\alpha,\alpha'}}.$$

*Example* 5.4.1. The modules ${}^aV(u)$ and ${}^aF(u)$ are admissible.

Let $W_1, W_2$ be admissible $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations with the corresponding bases $\{w^1_\alpha\}_{\alpha\in I}$ and $\{w^2_\beta\}_{\beta\in J}$. Consider the operator series $e(z), f(z)$ on $W_1 \otimes W_2$ defined by

$$e(z)(w^1_\alpha \otimes w^2_\beta) := \sum_{\alpha'\in I} \frac{c^1_{\alpha,\alpha'}}{z}\delta^+\left(\frac{\lambda^1_{\alpha,\alpha'}}{z}\right) w^1_{\alpha'} \otimes w^2_\beta + \sum_{\beta'\in J} \frac{c^2_{\beta,\beta'}\gamma_{W_1}(\alpha, \lambda^2_{\beta,\beta'})}{z}\delta^+\left(\frac{\lambda^2_{\beta,\beta'}}{z}\right) w^1_\alpha \otimes w^2_{\beta'}.$$

$$f(z)(w^1_\alpha \otimes w^2_\beta) := \sum_{\beta'\in J} \frac{d^2_{\beta,\beta''}}{z}\delta^+(\frac{\lambda^2_{\beta'',\beta}}{z}) w^1_\alpha \otimes w^2_{\beta''} + \sum_{\alpha''\in I} \frac{d^1_{\alpha,\alpha''}\gamma_{W_2}(\beta, \lambda^1_{\alpha'',\alpha})}{z}\delta^+(\frac{\lambda^1_{\alpha'',\alpha}}{z}) w^1_{\alpha''} \otimes w^2_\beta.$$

*Remark* 5.4.3. Those formulas are well-defined only if for any $\beta'$ such that $c^2_{\beta,\beta'} \neq 0$, the function $\gamma_{W_1}(\alpha, z)$ is regular at $z = \lambda^2_{\beta,\beta'}$ for any $\alpha \in I$, and similarly for the summand with $\gamma_{W_2}(\beta, z)$.

We can depict those by

$$\Delta(e(z)) = e(z) \otimes 1 + \psi(\bullet) \otimes e(z), \ \Delta(f(z)) = f(z) \otimes \psi(\bullet) + 1 \otimes f(z),$$

where $\psi(\bullet)$ indicates that we plug in the argument of the corresponding $\delta^+$-function.

The following is straightforward:

**Lemma 5.4.7.** *If $W_1$ and $W_2$ are admissible $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations and the assumptions of Remark 5.4.3 hold, then the above formulas define an action of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on $W_1 \otimes W_2$.*

More importantly, it might be possible to define an action of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on a submodule or a factor-module of $W_1 \otimes W_2$, even when the assumptions of Remark 5.4.3 fail.

**Lemma 5.4.8.** *Let $S$ be a subset of $I \times J$ such that $e(z)(w^1_\alpha \otimes w^2_\beta), f(z)(w^1_\alpha \otimes w^2_\beta)$ are well-defined (in the sense of Remark 5.4.3) for any $(\alpha, \beta) \in S$ and satisfy one of the following conditions:*

*(a) For any $(\alpha, \beta) \in S, (\alpha', \beta') \notin S$, $w^1_{\alpha'} \otimes w^2_{\beta'}$ does not appear in $e(z)(w^1_\alpha \otimes w^2_\beta), f(z)(w^1_\alpha \otimes w^2_\beta)$.*

*(b) For any $(\alpha, \beta) \in S, (\alpha', \beta') \notin S$, $w^1_\alpha \otimes w^2_\beta$ does not appear in $e(z)(w^1_{\alpha'} \otimes w^2_{\beta'}), f(z)(w^1_{\alpha'} \otimes w^2_{\beta'})$.*

*Then the above formulas define an action of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on the space with a basis $\{w^1_\alpha \otimes w^2_\beta\}_{(\alpha,\beta) \in S}$.*

Now we are ready to state the main result of this subsection:

**Theorem 5.4.9.** *There exists a unique collection of constants $c^a_{\bar{\lambda}} \in \mathbb{C}(s_1, s_2, x_1, \dots, x_r)$ with $c^a_{\bar{\emptyset}} = 1$ such that the map $[\bar{\lambda}] = [(\lambda^1, \dots, \lambda^r)] \mapsto c^a_{\bar{\lambda}} \cdot |\lambda^1\rangle \otimes \cdots \otimes |\lambda^r\rangle$ establishes an isomorphism $V^r \xrightarrow{\sim} {}^aF(x_1) \otimes \cdots \otimes {}^aF(x_r)$ of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representations.*

*Remark* 5.4.4. As $V \simeq V^1$, we have $V^r \simeq V^1(x_1) \otimes \cdots \otimes V^1(x_r)$. In other words, the representation of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ on the sum of equivariant cohomology groups of $M(r, n)$ is a tensor product of $r$ copies of such representations for $(\mathbb{A}^2)^{[m]}$.

### 5.4.5 Other series of representations

We recall some other series of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representations from [FFJMM1, FFJMM2]. All of them admit a straightforward modification to the $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-case. Those have the same bases, while the matrix coefficients of $e(z), f(z), \psi(z)$ in these bases are modified as follows:

$$1-q_1^i q_2^j q_3^k u/z \rightsquigarrow ih_1+jh_2+kh_3+u-z,\ \delta(q_1^i q_2^j q_3^k u/z) \rightsquigarrow \pm\frac{1}{z}\delta^+((ih_1+jh_2+kh_3+u)/z),$$

where the latter sign is "+" for $e(z)$ and "−" for $f(z)$.

• *Representation* $W^N(u)$.

Consider the tensor product $V^N(u) := V(u)\otimes V(uq_3^{-1})\otimes V(uq_3^{-2})\otimes\cdots\otimes V(uq_3^{1-N})$. Define $\mathcal{P}^N := \{\lambda = (\lambda_1,\ldots,\lambda_N) \in \mathbb{Z}^N | \lambda_1 \geq \cdots \geq \lambda_N\}, \mathcal{P}^{N,+} := \{\lambda \in \mathcal{P}^N | \lambda_N \geq 0\}$. Let $W^N(u) \subset V^N(u)$ be the subspace spanned by

$$[u]_\lambda := [u]_{\lambda_1} \otimes [uq_3^{-1}]_{\lambda_2-1} \otimes \cdots \otimes [uq_3^{1-N}]_{\lambda_N-N+1} \text{ for } \lambda \in \mathcal{P}^N.$$

According to [FFJMM1, Lemma 3.7], $W^N(u)$ is a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-submodule of $V^N(u)$. The subspace $W^{N,+}(u) \subset W^N(u)$ corresponding to $\mathcal{P}^{N,+}$ is not a submodule. However, its limit as $N \to \infty$ is well-defined (after an appropriate renormalization) and coincides with the Fock module $F(u)$.

• *Representation* $G^{k,r}_{\mathbf{a}}$.

Let $q_1, q_2$ be in the $(r,k)$-resonance condition: $q_1^a q_2^b = 1$ iff $a = (1-r)c, b = (k+1)c$ for some $c \in \mathbb{Z}$ (assume $k \geq 1, r \geq 2$). In this case the action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on $W^N(u)$ is ill-defined. Consider the set of $(k,r)$-*admissible partitions*

$$S^{k,r,N} := \{\lambda \in \mathcal{P}^N | \lambda_i - \lambda_{i+k} \geq r\ \forall\ i \leq N-k\}.$$

Let $W^{k,r,N}(u)$ be the subspace of $W^N(u)$, corresponding to the subset $S^{k,r,N} \subset \mathcal{P}^N$. According to [FFJMM1, Lemma 6.2], the comultiplication rule makes $W^{k,r,N}(u)$ into a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module. We think of it as "a submodule of $W^N(u)$ or even $V^N(u)$"

even though none of them has a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module structure.

Moreover, one can define an action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on the corresponding limit of $W^{k,r,N}(u)$ as $N \to \infty$. Let us fix a sequence of non-negative integers $\mathbf{a} = (a_1, \ldots, a_k)$ satisfying $\sum_{i=1}^k a_i = r$. Define

$$\mathcal{P}_{\mathbf{a}}^{k,r} := \{(\lambda_1 \geq \lambda_2 \geq \cdots) | \lambda_j - \lambda_{j+k} \geq r \;\forall\, j \geq 1, \lambda_j = \lambda_j^0 \;\forall\, j \gg 0\},$$

where we set $\lambda_{\mu k+i+1}^0 := -\mu r - \sum_{j=1}^i a_j$ for $0 \leq i \leq k-1$. The above limit construction provides an action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on the space $G_{\mathbf{a}}^{k,r}$ parametrized by $\lambda \in \mathcal{P}_{\mathbf{a}}^{k,r}$, see [FFJMM1, Theorem 6.5]

• *Representation* $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$.

Let us consider the tensor product of Fock representations. If $q_1, q_2, u_1, \ldots, u_n$ are *generic* ($q_1^a q_2^b u_1^{c_1} \cdots u_n^{c_n} = 1$ iff $a = b = c_1 = \ldots = c_n = 0$), then the tensor product $F(u_1) \otimes \cdots \otimes F(u_n)$ is well-defined. Consider the resonance case $u_i = u_{i+1} q_1^{a_i+1} q_2^{b_i+1}$ for some $a_i, b_i \geq 0, 1 \leq i \leq n-1$.

Let $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u) \subset F(u_1) \otimes \cdots \otimes F(u_n)$ be the subspace spanned by $|\lambda^1, \ldots, \lambda^n\rangle := [u_1]_{\lambda^1} \otimes \cdots \otimes [u_n]_{\lambda^n}$, where Young diagrams $\lambda^1, \ldots, \lambda^n$ satisfy $\lambda_s^i \geq \lambda_{s+b_i}^{i+1} - a_i$ for $i \leq n-1, s \geq 1$. According to [FFJMM2, Proposition 3.3], the comultiplication rule makes $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ into a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module for generic $q_1, q_2, u$. Moreover, it is an irreducible *lowest weight* module.

• *Representation* $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}(u)$.

Assume further that $q_1, q_2$ are not generic: there exist $p, p' \geq 1$ such that $q_1^a q_2^b = 1$ iff $a = p'c, b = pc$ for some $c \in \mathbb{Z}$. We require that $a_n := p' - 1 - \sum_{i=1}^{n-1}(a_i + 1), b_n := p - 1 - \sum_{i=1}^{n-1}(b_i + 1)$ are non-negative. In this case, the action of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ on $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is ill-defined.

Consider a subspace $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}(u) \subset F(u_1) \otimes \cdots \otimes F(u_n)$ spanned by $|\lambda^1, \ldots, \lambda^n\rangle$ satisfying the same conditions $\lambda_s^i \geq \lambda_{s+b_i}^{i+1} - a_i$, but with $i \leq n$, where $\lambda^{n+1} := \lambda^1$. The comultiplication rule makes it into a $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module, due to [FFJMM2, Proposition 3.7]. We think of $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}(u)$ as "a subquotient of $F(u_1) \otimes \cdots \otimes F(u_n)$". Their characters coincide with the characters of the $\mathcal{W}_n$-minimal series of $\mathfrak{sl}_n$-type, according to the main result of [FFJMM2].

### 5.4.6 The categories $\mathcal{O}$

We conclude this section by introducing the appropriate categories $\mathcal{O}$ both for the quantum toroidal and the affine Yangian of $\mathfrak{gl}_1$.

• Category $\mathcal{O}$ for $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$.

As we will see in the next sections it is convenient to work with the quotient algebra $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1) := \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(\psi_0^- - (\psi_0^+)^{-1})$, rather than $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ itself. The algebra $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ is graded by $\deg(e_i) = -1, \deg(f_i) = 1, \deg(\psi_j^\pm) = 0$.

**Definition 5.4.2.** A $\mathbb{Z}$-graded $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-module $L$ is in the category $\mathcal{O}$ if
(i) for any $v \in L$ there exists $N \in \mathbb{Z}$ such that $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)_{\geq N}(v) = 0$,
(ii) all graded components $L_k$ are finite dimensional (module $L$ is of *finite type*).

We say that $L$ is a highest weight module if there exists $v_0 \in L$ generating $L$ and such that $f_i(v_0) = 0,\ \psi_j^\pm(v_0) = p_j^\pm \cdot v_0,\ \forall\, i \in \mathbb{Z}, j \in \mathbb{Z}_+$, for some $p_j^\pm \in \mathbb{C}$, $p_0^+ \cdot p_0^- = 1$. To such a collection $\{p_j^\pm\}$, we associate two series $p^\pm(z) := \sum_{j\geq 0} p_j^\pm z^{\mp j} \in \mathbb{C}[[z^{\mp 1}]]$. Given any two series $p^+(z), p^-(z)$ satisfying $p_0^+ \cdot p_0^- = 1$, there is a universal highest weight representation $M_{p^+,p^-}$, which may be defined as the quotient of $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ by the left-ideal generated by $\{f_i\} \cup \{\psi_j^\pm - p_j^\pm\}$. By a standard argument $M_{p^+,p^-}$ has a unique irreducible quotient $V_{p^+,p^-}$.

The module $V_{p^+,p^-}$ obviously satisfies the condition (i) from the definition of the category $\mathcal{O}$. Our next result provides a criteria for $V_{p^+,p^-}$ to satisfy (ii) (i.e., to be in the category $\mathcal{O}$).

**Proposition 5.4.10.** *The module $V_{p^+,p^-}$ is of finite type iff there exists a rational function $P(z)$ such that $p^\pm(z) = P(z)^\pm$ and $P(0)P(\infty) = 1$.*

*Proof.*

The proof is standard and is based on the arguments from [CP]. Define constants $\{p_i\}_{i\in\mathbb{Z}}$ as $p_i^+$ (for $i > 0$), $-p_{-i}^-$ (for $i < 0$), and $p_0^+ - p_0^-$ (for $i = 0$). To prove the "only if" part we choose indices $k \in \mathbb{Z}, l \in \mathbb{Z}_+$ such that $\{e_k(v_0), \ldots, e_{k+l}(v_0)\}$ span the degree $-1$ component $(V_{p^+,p^-})_{-1}$, while this fails for the collection $\{e_k(v_0), \ldots, e_{k+l-1}(v_0)\}$. As a result, there are complex numbers $a_0, \ldots, a_l \in \mathbb{C}, a_l \neq 0$, such that $a_0 e_k(v_0) + a_1 e_{k+1}(v_0) + \ldots + a_l e_{k+l}(v_0) = 0$. Applying the operator $f_{r-k}$ to this identity and using the equality $f_i e_j(v_0) = -\beta_1^{-1} p_{i+j} \cdot v_0$ (due to (T3)), we get $a_0 p_r + a_1 p_{r+1} + \ldots + a_l p_{r+l} = 0$ for all $r \in \mathbb{Z}$. Therefore, the collection $\{p_i\}_{i \in \mathbb{Z}}$ satisfies a simple recurrence relation. Solving this recurrence relation and using the conditions $p_0 = p_0^+ - p_0^-, p_0^- = (p_0^+)^{-1}$, we immediately see that $p^\pm(z)$ are extension in $z^{\mp 1}$ of the same rational function.

To prove the "if" direction, the same arguments show $\dim(V_{p^+,p^-})_{-1} < \infty$. Combining this with the relation (T1) a simple induction argument implies $\dim(V_{p^+,p^-})_{-l} < \infty$ for any $l > 0$. $\square$

- Category $\mathcal{O}$ for $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$.

The algebra $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is graded by $\deg(e_j) = -1, \deg(f_j) = 1, \deg(\psi_j) = 0$.

**Definition 5.4.3.** A $\mathbb{Z}$-graded $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-module $L$ is in the category $\mathcal{O}$ if
(i) for any $v \in L$ there exists $N \in \mathbb{Z}$ such that $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)_{\geq N}(v) = 0$,
(ii) all graded components $L_k$ are finite dimensional (module $L$ is of *finite type*).

We say that $L$ is a highest weight module if there exists $v_0 \in L$ generating $L$ and such that $f_j(v_0) = 0, \psi_j(v_0) = p_j \cdot v_0, \forall\ j \in \mathbb{Z}_+$, for some $p_j \in \mathbb{C}$. Set $p(z) := 1 + \sum_{j\geq 0} p_j z^{-j-1} \in \mathbb{C}[[z^{-1}]]$. For any $\{p_j\}$, there is a universal highest weight $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$-representation $M_p$, which may be defined as the quotient of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ by the left-ideal generated by $\{f_j\} \cup \{\psi_j - p_j\}$. It has a unique irreducible quotient $V_p$. The following is analogous to Proposition 5.4.10:

**Proposition 5.4.11.** *The module $V_p$ is in the category $\mathcal{O}$ iff there exists a rational function $P(z)$ such that $p(z) = P(z)^+$ and $P(\infty) = 1$.*

## 5.5 Limit algebras

### 5.5.1 Algebras $\mathfrak{d}_h$ and $\bar{\mathfrak{d}}_h$

We recall the algebra of $h$-difference operators on $\mathbb{C}$.

For a formal variable $h$, let $\mathfrak{d}_h$ be an associative algebra over $\mathbb{C}[[h]]$ topologically generated by $Z^{\pm 1}, D^{\pm 1}$ subject to the following relations

$$Z{\cdot}Z^{-1} = Z^{-1}{\cdot}Z = 1,\ D{\cdot}D^{-1} = D^{-1}{\cdot}D = 1,\ D{\cdot}Z = qZ{\cdot}D,\ \text{where } q = \exp(h) \in \mathbb{C}[[h]].$$

We will view $\mathfrak{d}_h$ as a Lie algebra with the natural commutator-Lie bracket $[\cdot,\cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{d}} \in C^2(\mathfrak{d}_h, \mathbb{C}[[h]])$:

$$\phi_{\mathfrak{d}}(Z^a D^j, Z^b D^{-j'}) = \begin{cases} 0 & j \neq j' \text{ or } j = j' = 0 \\ \sum_{i=-j}^{-1} q^{ai+b(j+i)} & j = j' > 0 \\ -\sum_{i=j}^{-1} q^{bi+a(-j+i)} & j = j' < 0 \end{cases}.$$

This endows $\bar{\mathfrak{d}}_h = \mathfrak{d}_h \oplus \mathbb{C}[[h]] \cdot c_{\mathfrak{d}}$ with a structure of a Lie algebra.

### 5.5.2 Algebras $\mathfrak{D}_h$ and $\overline{\mathfrak{D}}_h$

We recall the algebra of $q$-difference operators on $\mathbb{C}^*$.

For a formal variable $h$, let $\mathfrak{D}_h$ be an associative algebra over $\mathbb{C}[[h]]$ topologically generated by $x, \partial^{\pm 1}$ subject to the following defining relations

$$\partial \cdot \partial^{-1} = \partial^{-1} \cdot \partial = 1,\ \partial \cdot x = (x+h) \cdot \partial.$$

We will view $\mathfrak{D}_h$ as a Lie algebra with the natural commutator-Lie bracket $[\cdot,\cdot]$. It is easy to check that the following formula defines a 2-cocycle $\phi_{\mathfrak{D}} \in C^2(\mathfrak{D}_h, \mathbb{C}[[h]])$:

$$\phi_{\mathfrak{D}}(f(x)\partial^r, g(x)\partial^{-s}) = \begin{cases} 0 & r \neq s \text{ or } r = s = 0 \\ \sum_{l=-r}^{-1} f(lh)g((l+r)h) & r = s > 0 \\ -\sum_{l=r}^{-1} g(lh)f((l-r)h) & r = s < 0 \end{cases}.$$

This endows $\overline{\mathfrak{D}}_h = \mathfrak{D}_h \oplus \mathbb{C}[[h]] \cdot c_{\mathfrak{D}}$ with a structure of a Lie algebra.

### 5.5.3 The isomorphism $\Upsilon_0$

We construct an isomorphism of the completions of $\bar{\mathfrak{d}}_h$ and $\overline{\mathfrak{D}}_h$.

First we introduce the appropriate completions of the algebras $\bar{\mathfrak{d}}_h, \overline{\mathfrak{D}}_h$:

∘ $\widehat{\bar{\mathfrak{d}}}_h$ is the completion of $\bar{\mathfrak{d}}_h$ with respect to the powers of the two-sided ideal $J_{\mathfrak{d}} = (Z-1, q-1)$;

∘ $\widehat{\overline{\mathfrak{D}}}_h$ is the completion of $\overline{\mathfrak{D}}_h$ with respect to the powers of the two-sided ideal $J_{\mathfrak{D}} = (x, h)$.

In other words, we have:

$$\widehat{\bar{\mathfrak{d}}}_h := \varprojlim \bar{\mathfrak{d}}_h / \bar{\mathfrak{d}}_h \cdot (Z-1, q-1)^j,\ \widehat{\overline{\mathfrak{D}}}_h := \varprojlim \overline{\mathfrak{D}}_h / \overline{\mathfrak{D}}_h \cdot (x, h)^j.$$

*Remark* 5.5.1. Taking completions of $\mathfrak{d}_h$ and $\mathfrak{D}_h$ with respect to the ideals $J_{\mathfrak{d}}$ and $J_{\mathfrak{D}}$ commutes with taking central extensions with respect to the 2-cocycles $\phi_{\mathfrak{d}}$ and $\phi_{\mathfrak{D}}$.

The following result is straightforward:

**Proposition 5.5.1.** *There exists an isomorphism* $\Upsilon_0 : \widehat{\bar{\mathfrak{d}}}_h \xrightarrow{\sim} \widehat{\overline{\mathfrak{D}}}_h$, *defined on the generators by*

$$D^{\pm 1} \mapsto \partial^{\pm 1},\ Z^{\pm 1} \mapsto e^{\pm x},\ c_{\mathfrak{d}} \mapsto c_{\mathfrak{D}}.$$

*Remark* 5.5.2. Specializing $h$ to a complex parameter $h_0 \in \mathbb{C}$, that is taking factor by $(h - h_0)$, we get the classical $\mathbb{C}$-algebras of difference operators $\mathfrak{D}_{h_0}$ and $\mathfrak{d}_{h_0}$. However, one can not define their completions as above and, moreover, completions of their central extensions.

### 5.5.4 The renormalized algebra $\ddot{U}'_h(\mathfrak{gl}_1)$

We introduce the *limit algebra* $\ddot{U}'_h(\mathfrak{gl}_1)$.

Throughout this section, we let $h_1, h_2$ be formal variables and set $h_3 := -h_1 - h_2$. We define $q_i := \exp(h_i) \in \mathbb{C}[[h_1, h_2]]$ for $i = 1, 2, 3$. First we introduce a *formal* analogue of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. While the relations (T0, T1, T2, T4t, T5t, T6t) are well defined over $\mathbb{C}[[h_1, h_2]]$, we need to change (T3) in an appropriate way. This will also lead to *renormalizations* of (T4t, T5t).

*Remark* 5.5.3. This is analogous to the classical relation between $U_q(\widehat{\mathfrak{g}})$ and $U_q(L\mathfrak{g})$.

We start by *renormalizing* (T3) to the following form:

$$[e_i, f_j] = (\psi^+_{i+j} - \psi^-_{-i-j})/(1-q_3). \tag{T3$'$}$$

This procedure is called renormalization, since for the case of complex parameters $q_i \neq 1$, this algebra is obtained from $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ just by *rescaling* $e_i$ by $1-q_1$ and $f_i$ by $1-q_2$.

Next, we write $\psi^{\pm}(z)$ as $\psi^{\pm}(z) = \exp(\frac{h_3}{2}\kappa_{\pm})\cdot\exp(\pm(1-q_3)\sum_{\pm m>0} H_m z^{-m})$. Then $\kappa_{\pm}$ are central elements and the relations (T4t, T5t) get modified to:

$$[H_m, e_i] = -\frac{(1-q_1^m)(1-q_2^m)(1-q_3^m)}{m(1-q_3)} e_{i+m},\ [H_m, f_i] = \frac{(1-q_1^m)(1-q_2^m)(1-q_3^m)}{m(1-q_3)} f_{i+m}.$$

These relations are well-defined in the formal setting since $\frac{(1-q_1^m)(1-q_2^m)(1-q_3^m)}{1-q_3} \in \mathbb{C}[[h_1, h_2]]$. Note that the right hand side of the relation (T3$'$) also makes sense. The corresponding algebra over $\mathbb{C}[[h_1, h_2]]$ topologically generated by $\{e_i, f_i, \kappa_{\pm}, H_m\}$ will be denoted by $\ddot{U}_{h_2,h_3}(\mathfrak{gl}_1)$. We also introduce $\ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1) := \ddot{U}_{h_2,h_3}(\mathfrak{gl}_1)/(\kappa_+ + \kappa_-)$. Finally, we define

$$\ddot{U}'_h(\mathfrak{gl}_1) := \ddot{U}_{h,h_3}(\mathfrak{gl}_1)/(\kappa_+ + \kappa_-, h_3).$$

It is an algebra over $\mathbb{C}[[h]]$ topologically generated by $\{e_j, f_j, \kappa, H_m\}$ subject to (T1, T2, T6) and the following four relations:

$$[H_k, H_m] = 0,\ \kappa \text{ is central}, \tag{T0L}$$

$$[e_i, f_j] = \begin{cases} H_{i+j} & i+j \neq 0 \\ -\kappa & i+j = 0 \end{cases}, \tag{T3L}$$

$$[H_m, e_i] = -(1-q^m)(1-q^{-m})e_{i+m}, \tag{T4tL}$$

$$[H_m, f_i] = (1-q^m)(1-q^{-m})f_{i+m}, \tag{T5tL}$$

where $q := \exp(h) \in \mathbb{C}[[h]]$. Now we are ready to relate $\ddot{U}'_h(\mathfrak{gl}_1)$ to $\bar{\mathfrak{d}}_h$.

**Proposition 5.5.2.** *There exists a homomorphism $\theta_m : \ddot{U}'_h(\mathfrak{gl}_1) \to U(\bar{\mathfrak{d}}_h)$ such that*

$$\theta_m(e_i) = Z^i D,\ \theta_m(f_i) = -D^{-1}Z^i,\ \theta_m(H_k) = -(1-q^{-k})Z^k - q^{-k}c_{\mathfrak{d}},\ \theta_m(\kappa) = c_{\mathfrak{d}}.$$

*Proof.*

It suffices to check that

$$\widetilde{e}_i := Z^i D, \widetilde{f}_i := -D^{-1}Z^i, \widetilde{H}_k := -(1-q^{-k})Z^k - q^{-k}c_{\mathfrak{d}}, \widetilde{\kappa} = c_{\mathfrak{d}}$$

satisfy the defining relations of $\ddot{U}'_h(\mathfrak{gl}_1)$. The only nontrivial relations are (T1,T3L,T4tL, T6t).

- For $i, j \in \mathbb{Z}$, we have $[\widetilde{e}_i, \widetilde{e}_j] = [Z^i D, Z^j D] = (q^j - q^i)\cdot Z^{i+j}D^2$. (T1) follows:

$$[\widetilde{e}_{n+3}, \widetilde{e}_m] - (1+q+q^{-1})[\widetilde{e}_{n+2}, \widetilde{e}_{m+1}] + (1+q+q^{-1})[\widetilde{e}_{n+1}, \widetilde{e}_{m+2}] - [\widetilde{e}_n, \widetilde{e}_{m+3}] = 0.$$

- The relation (T3L) follows from the following identity:

$$[\widetilde{e}_i, \widetilde{f}_j] = -[Z^i D, D^{-1}Z^j] = (-1+q^{-i-j})Z^{i+j} - q^{-i-j}c_{\mathfrak{d}}.$$

- The relation (T4tL) follows from the following identity:

$$[\widetilde{H}_m, \widetilde{e}_i] = -(1-q^{-m})[Z^m, Z^i D] = -(1-q^{-m})(1-q^m)Z^{i+m}D = -(1-q^m)(1-q^{-m})\widetilde{e}_{i+m}.$$

- The relation (T6t) follows from $[D,[ZD, Z^{-1}D]] = [D, (q^{-1}-q)D + q^{-1}c_{\mathfrak{d}}] = 0.$ □

The image of $\theta_m$ is easy to describe.

**Lemma 5.5.3.** *Let $\bar{\mathfrak{d}}^0_h \subset \bar{\mathfrak{d}}_h$ be a free $\mathbb{C}[[h]]$-submodule generated by*

$$\{c_{\mathfrak{d}}, h\cdot Z^k D^0, h^{j-1}Z^i D^j, h^{j-1}Z^i D^{-j} | k \neq 0, j > 0\}.$$

*Then $\bar{\mathfrak{d}}^0_h$ is a Lie subalgebra of $\bar{\mathfrak{d}}_h$ and* $\mathrm{Im}(\theta_m) \subset U(\bar{\mathfrak{d}}^0_h)$.

Actually, we have the following result:

**Theorem 5.5.4.** *The homomorphism $\theta_m$ provides an isomorphism $\ddot{U}'_h(\mathfrak{gl}_1)\xrightarrow{\sim}U(\bar{\mathfrak{d}}^0_h)$.*

Note that all the defining relations of $\ddot{U}'_h(\mathfrak{gl}_1)$ are of Lie-type. Hence, $\ddot{U}'_h(\mathfrak{gl}_1)$ is an enveloping algebra of the Lie algebra generated by $e_i, f_i, \kappa, H_m$ with the aforementioned defining relations. Thus, Theorem 5.5.4 provides a presentation of the Lie algebra $\bar{\mathfrak{d}}^0_h$ by generators and relations.

Actually, we have a more general result:

**Theorem 5.5.5.** *If $h_0 \in \mathbb{C}\backslash\{\mathbb{Q}\cdot\pi i\}$, then $\theta_m$ induces an isomorphism of the $\mathbb{C}$-algebras: $\ddot{U}'_{h_0}(\mathfrak{gl}_1)\xrightarrow{\sim}U(\bar{\mathfrak{d}}'_{h_0})$, where $\bar{\mathfrak{d}}'_{h_0}\subset\bar{\mathfrak{d}}_{h_0}$ is a Lie subalgebra spanned by $c_{\mathfrak{d}}$ and $\{Z^iD^j\}_{(i,j)\neq(0,0)}$.*

### 5.5.5 The renormalized algebra $\ddot{Y}'_h(\mathfrak{gl}_1)$

We introduce the *limit algebra* $\ddot{Y}'_h(\mathfrak{gl}_1)$

Analogously to the previous section, we let $h_1, h_2$ be formal variables and set $h_3 := -h_1-h_2$. We view $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ as a *formal* version of the corresponding algebra introduced in Section 1.3. In other words, $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ is an associative algebra over $\mathbb{C}[[h_1,h_2]]$ topologically generated by $\{e_j, f_j, \psi_j\}_{j\in\mathbb{Z}_+}$ subject to the relations (Y0)-(Y6).

We will actually need a *homogenized version* of this algebra. Let $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$ be an associative algebra over $\mathbb{C}[[h_1,h_2]]$ defined similarly to $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ with the following few changes:

$$[\psi_2, e_i] = -2h_1h_2e_i,\ \ [\psi_2, f_i] = 2h_1h_2f_i.$$

The specializations of algebras $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ at $h_k\in\mathbb{C}^*$ are isomorphic. However, $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$ is a $\mathbb{Z}_+$-graded algebra with $\deg(e_i)=i, \deg(f_i)=i, \deg(\psi_i)=i, \deg(h_k)=1$.

We define $\ddot{Y}'_h(\mathfrak{gl}_1)$ by

$$\ddot{Y}'_h(\mathfrak{gl}_1) := \ddot{Y}'_{h,h_3}(\mathfrak{gl}_1)/(h_3).$$

It is an algebra over $\mathbb{C}[[h]]$. The following result is straightforward:

**Proposition 5.5.6.** *There exists a homomorphism $\theta_a : \ddot{Y}'_h(\mathfrak{gl}_1) \to U(\overline{\mathfrak{D}}_h)$ such that*

$$\theta_a(e_j) = x^j\partial,\ \theta_a(f_j) = -\partial^{-1}x^j,\ \theta_a(\psi_j) = (x-h)^j - x^j - (-h)^j c_{\mathfrak{D}}.$$

The image of $\theta_a$ is easy to describe.

**Lemma 5.5.7.** *Let $\overline{\mathfrak{D}}_h^0 \subset \overline{\mathfrak{D}}_h$ be a free $\mathbb{C}[[h]]$-submodule generated by*

$$\{c_{\mathfrak{D}}, h\cdot x^i\partial^0, h^{j-1}x^i\partial^j, h^{j-1}x^i\partial^{-j} | i \geq 0, j > 0\}.$$

*Then $\overline{\mathfrak{D}}_h^0$ is a Lie subalgebra of $\overline{\mathfrak{D}}_h$ and $\mathrm{Im}(\theta_a) \subset U(\overline{\mathfrak{D}}_h^0)$.*

Actually, we have the following result:

**Theorem 5.5.8.** *The homomorphism $\theta_a$ provides an isomorphism $\theta_a : \ddot{Y}'_h(\mathfrak{gl}_1) \xrightarrow{\sim} U(\overline{\mathfrak{D}}_h^0)$.*

Note that all the defining relations of $\ddot{Y}'_h(\mathfrak{gl}_1)$ are of Lie-type. Hence, $\ddot{Y}'_h(\mathfrak{gl}_1)$ is an enveloping algebra of the Lie algebra generated by $e_j, f_j, \psi_j$ with the aforementioned defining relations. Thus, Theorem 5.5.8 provides a presentation of the Lie algebra $\overline{\mathfrak{D}}_h^0$ by generators and relations.

Actually, we have a more general result:

**Theorem 5.5.9.** *For $h_i \in \mathbb{C}^*$, $\theta_a$ induces an isomorphism $\theta_a : \ddot{Y}'_{h_0}(\mathfrak{gl}_1) \xrightarrow{\sim} U(\overline{\mathfrak{D}}_{h_0})$.*

### 5.5.6 Proof of Theorem 5.5.5

We prove that $\theta_m$ is an isomorphism of $\mathbb{C}$-algebras for $h_0 \notin \mathbb{Q}\cdot\pi i$.

As mentioned in Section 5.4, all the defining relations of the algebra $\ddot{U}'_{h_0}(\mathfrak{gl}_1)$ are of Lie-type. Therefore, it is the universal enveloping algebra of the Lie algebra $\ddot{u}'_{h_0}$ generated by $\{e_i, f_i, H_m, \kappa\}$ with the same defining relations. Moreover, $\ddot{u}'_{h_0}$ is a $\mathbb{C}\cdot\kappa$–central extension of the Lie-algebra $\ddot{u}_{h_0}$ generated by $\{e_i, f_i, H_m\}$ with the following defining relations:

$$[H_k, H_l] = 0, \tag{u0}$$

$$[e_{i+3}, e_j] - (1+q+q^{-1})[e_{i+2}, e_{j+1}] + (1+q+q^{-1})[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] = 0, \tag{u1}$$

$$[f_{i+3}, f_j] - (1+q+q^{-1})[f_{i+2}, f_{j+1}] + (1+q+q^{-1})[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] = 0, \quad \text{(u2)}$$

$$[e_i, f_j] = H_{i+j} \text{ for } j \neq -i, \quad [e_i, f_{-i}] = 0, \quad \text{(u3)}$$

$$[H_m, e_i] = -(1-q^m)(1-q^{-m})e_{i+m}, \quad \text{(u4)}$$

$$[H_m, f_i] = (1-q^m)(1-q^{-m})f_{i+m}, \quad \text{(u5)}$$

$$[e_0, [e_1, e_{-1}]] = 0, \quad [f_0, [f_1, f_{-1}]] = 0, \quad \text{(u6)}$$

where $q := e^{h_0} \in \mathbb{C}^*$. Note that $h_0 \notin \mathbb{Q} \cdot \pi i$ iff $q \neq \sqrt{1}$ (not a root of 1).

Hence, it suffices to check that the corresponding homomorphism $\theta_m : \ddot{u}_{h_0} \to \mathfrak{d}'_{h_0}$ defined by

$$\theta_m : e_i \mapsto Z^i D, \ f_i \mapsto -D^{-1}Z^i, \ H_k \mapsto (q^{-k}-1)Z^k$$

is an isomorphism of the $\mathbb{C}$-Lie algebras for $q \neq \sqrt{1}$. The surjectivity of $\theta_m$ is clear.

The Lie algebra $\ddot{u}_{h_0}$ is $\mathbb{Z}^2$-graded, where we set

$$\deg(e_i) = (i, 1), \ \deg(f_i) = (i, -1), \ \deg(H_k) = (k, 0).$$

The Lie algebra $\mathfrak{d}'_{h_0}$ is also $\mathbb{Z}^2$-graded, where we set $\deg(Z^i D^j) = (i, j)$. Moreover, $\theta_m$ intertwines those $\mathbb{Z}^2$-gradings. Since $\dim(\mathfrak{d}'_{h_0})_{i,j} = 1$ for $(i,j) \neq (0,0)$, it suffices to show that $\dim(\ddot{u}_{h_0})_{i,j} \leq 1$. This statement is clear for $j = 0$. In the remaining part we prove it for $j > 0$.

Let $\ddot{u}^{\geq 0}_{h_0}$ be the Lie algebra, generated by $\{e_i, H_m\}$ with the defining relations (u0,u1,u4,u6). It suffices to prove that $\dim(\ddot{u}^{\geq 0}_{h_0})_{i,j} \leq 1$ for $j > 0$. We prove this by an induction on $j$.

- *Case* $j = 1$.

  It is clear that $(\ddot{u}^{\geq 0}_{h_0})_{N,1}$ is spanned by $e_N$.

- *Case* $j = 2$.

  It is clear that $(\ddot{u}^{\geq 0}_{h_0})_{N,2}$ is spanned by $\{[e_{i_1}, e_{i_2}]\}_{i_1+i_2=N}$. However, (u1) implies

$$[e_{i+2+k}, e_{i+1-k}] = \frac{q^{k+1}-q^{-k}}{q-1}[e_{i+2}, e_{i+1}], \ [e_{i+2+k}, e_{i-k}] = \frac{q^{k+2}-q^{-k}}{q^2-1}[e_{i+2}, e_i].$$

These formulas can be unified in the following way:

$$[e_{i_1}, e_{i_2}] = \frac{q^{i_2} - q^{i_1}}{q^{i_1+i_2} - 1}[e_0, e_{i_1+i_2}] \text{ if } i_1 + i_2 \neq 0,\ [e_i, e_{-i}] = \frac{q^{i+1} - q^{1-i}}{q^2 - 1}[e_1, e_{-1}]. \quad (5.11)$$

Therefore, $(\ddot{u}_{\bar{h}_0}^{\geq 0})_{N,2}$ is either spanned by $[e_0, e_N]$ (if $N \neq 0$) or $[e_1, e_{-1}]$ (if $N = 0$).

• *Case* $j = 3$.

Let us introduce the following common notation:

$$[a_1; a_2; \ldots; a_n]_n := [a_1, [a_2, [\ldots [a_{n-1}, a_n]]]].$$

The space $(\ddot{u}_{\bar{h}_0}^{\geq 0})_{N,3}$ is spanned by $\{[e_{i_1}; e_{i_2}; e_{i_3}]\}_{i_1+i_2+i_3=N}$. Using the automorphism $\pi$ of the Lie algebra $\ddot{u}_{\bar{h}_0}^{\geq 0}$, defined by $e_i \mapsto e_{i+1}, H_m \mapsto H_m$, we can assume $i_1, i_2, i_3 > 0$. Together with the case $j = 2$, it suffices to show that $[e_k; e_0; e_l]$ is a multiple of $[e_0; e_0; e_{k+l}]$ for any $k, l > 0$.

Define $\mathbf{h}_m := -\frac{mH_m}{(1-q^m)(1-q^{-m})}$ for $m \neq 0$. Then $\mathrm{ad}(\mathbf{h}_m)(e_i) = e_{i+m}$. Therefore:

$$\mathrm{ad}(\mathbf{h}_1)([e_k; e_0; e_l]) = [e_{k+1}; e_0; e_l] + [e_k; e_1; e_l] + [e_k; e_0; e_{l+1}].$$

Assuming $[e_k; e_0; e_l]$ is a multiple of $[e_0; e_0; e_{k+l}]$, we get $[e_{k+1}; e_0; e_l]$ is a linear combination of $[e_0; e_0; e_{k+l+1}]$ and $[e_1; e_0; e_{k+l}]$ (we use (5.11) there). It remains to consider $k = 1$ case.

We will prove by an induction on $N > 1$ that $[e_1; e_0; e_{N-1}] = \frac{(q^{N-1}-q^2)(q^{N-1}-1)}{(q^N-1)^2}[e_0; e_0; e_N]$. This is equivalent to $[e_1; e_0; e_{N-1}]$ being a multiple of $[e_0; e_0; e_N]$, since we can recover the constant $\lambda_{N,3} := \frac{(q^{N-1}-q^2)(q^{N-1}-1)}{(q^N-1)^2}$ by comparing the images $\theta_m([e_1; e_0; e_{N-1}])$ and $\theta_m([e_0; e_0; e_N])$.

∘ *Case* $N = 2$.

Recall that the relation (u6) combined with (u4) imply

$$\mathrm{Sym}[e_{i_1}; e_{i_2+1}; e_{i_3-1}] = 0 \ \ \forall\, i_1, i_2, i_3 \in \mathbb{Z}. \quad (\mathrm{u}6')$$

Plugging $i_1 = 1, i_2 = 1, i_3 = 0$, we get $[e_1; e_2; e_{-1}] + [e_1; e_1; e_0] + [e_0; e_2; e_0] = 0$.

Combining this equality with (5.11), we get

$$[e_0;e_0;e_2] = -\frac{(q+1)^2}{q}[e_1;e_0;e_1] \Longrightarrow [e_1;e_0;e_1] = \lambda_{2,3}[e_0;e_0;e_2].$$

∘ *Case* $N=3$.

Plugging $i_1=1, i_2=2, i_3=0$ into (u6′), we get

$$[e_1;e_3;e_{-1}] + [e_2;e_2;e_{-1}] + [e_2;e_1;e_0] + [e_0;e_3;e_0] + [e_0;e_2;e_1] = 0.$$

Applying (5.11), we get:

$$-(q+2+q^{-1})[e_2;e_0;e_1] - (q+q^{-1})[e_1;e_0;e_2] - (1+\frac{q^2-q}{q^3-1})[e_0;e_0;e_3] = 0.$$

On the other hand, applying $\mathrm{ad}(\mathbf{h}_1)$ to $(q+1)^2[e_1;e_0;e_1] + q[e_0;e_0;e_2] = 0$ (case $N=2$), we get

$$(q+1)^2[e_2;e_0;e_1] + (q^2+3q+1)[e_1;e_0;e_2] + (q - \frac{q^2-q^3}{q^3-1})[e_0;e_0;e_3] = 0.$$

These two linear combinations of $[e_2;e_0;e_1], [e_1;e_0;e_2], [e_0;e_0;e_3]$ are not proportional for $q \neq \sqrt{1}$. Therefore, we can eliminate $[e_2;e_0;e_1]$, which proves that $[e_1;e_0;e_2]$ is a multiple of $[e_0;e_0;e_3]$.

∘ *Case* $N=k+2, k>1$.

By an induction assumption $[e_1;e_0;e_k] - \lambda_{k+1,3}[e_0;e_0;e_{k+1}] = 0$. Applying $\mathrm{ad}(\mathbf{h}_1)$, we get

$$([e_2;e_0;e_k]+[e_1;e_1;e_k]+[e_1;e_0;e_{k+1}]) - \lambda_{k+1,3}([e_1;e_0;e_{k+1}]+[e_0;e_1;e_{k+1}]+[e_0;e_0;e_{k+2}]) = 0.$$

Note also that

$$(\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2))([e_{i_1};e_{i_2};e_{i_3}]) = [e_{i_1+1};e_{i_2+1};e_{i_3}] + [e_{i_1+1};e_{i_2};e_{i_3+1}] + [e_{i_1};e_{i_2+1};e_{i_3+1}].$$

By an induction assumption $[e_1;e_0;e_{k-1}] = \lambda_{k,3}[e_0;e_0;e_k]$. Applying $\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2)$, we get

$$([e_2;e_1;e_{k-1}]+[e_2;e_0;e_k]+[e_1;e_1;e_k])-\lambda_{k,3}([e_1;e_1;e_k]+[e_1;e_0;e_{k+1}]+[e_0;e_1;e_{k+1}])=0.$$

Applying (5.11), we get two linear combinations of $[e_2;e_0;e_k]$, $[e_1;e_0;e_{k+1}]$, $[e_0;e_0;e_{k+2}]$ being zero. It is a routine verification to check that they are not proportional for $q \neq \sqrt{1}$. Therefore, we can eliminate $[e_2;e_0;e_k]$, which proves that $[e_1;e_0;e_{k+1}]$ is a multiple of $[e_0;e_0;e_{k+2}]$.

• *Case* $j = n > 3$.

Analogously to the previous case, it suffices to show that $[e_1;e_0;\ldots;e_0;e_{N-1}]_n$ is a multiple of $[e_0;\ldots;e_0;e_N]_n$. This is equivalent to

$$[e_1;\ldots;e_0;e_{N-1}]_n = \lambda_{N,n}\cdot[e_0;\ldots;e_0;e_N]_n,\ \lambda_{N,n} = \frac{(q^{N-1}-1)^{n-2}(q^{N-1}-q^{n-1})}{(q^N-1)^{n-1}},$$

the constant being computed by comparing the images under $\theta_m$.

We will need the following *multiple* counterpart of (u6) (follows from Proposition 5.7.5 below):

$$[e_0;e_1;e_0;\ldots;e_0;e_{-1}]_n = 0. \qquad \text{(u7n)}$$

This equality together with the relation (u4) implies

$$\mathrm{Sym}[e_{i_1};e_{i_2+1};e_{i_3};\ldots;e_{i_{n-1}};e_{i_n-1}]_n = 0\ \ \forall\, i_1,\ldots,i_n\in\mathbb{Z}. \qquad \text{(u7'n)}$$

Now we proceed to the proof of the aforementioned result by an induction on $N$.

∘ *Case* $N = 2$.

Applying $\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2)$ to (u7n), we get

$$[e_1,\mathrm{ad}(\mathbf{h}_1)[e_1;e_0;\ldots;e_{-1}]_{n-1}] + [e_0;(\mathrm{ad}(\mathbf{h}_1)^2-\mathrm{ad}(\mathbf{h}_2))[e_1;e_0;\ldots;e_{-1}]_{n-1}] = 0.$$

By the induction assumption for $n$, LHS has a form $a_n\cdot[e_1;e_0;\ldots;e_1]_n + b_n\cdot[e_0;\ldots;e_0;e_2]_n$. Computing the images under $\theta_m$, we find $a_n = \frac{(-1)^{n-3}(q^n-1)^2}{q^{n-1}(q-1)} \neq 0$ for $q \neq \sqrt{1}$.

∘ *Case* $N = 3$.

Applying $\mathrm{ad}(\mathbf{h}_1)$ to $[e_1;\ldots;e_0;e_1]_n - \lambda_{2,n}[e_0;\ldots;e_0;e_2]_n = 0$, we get

$$[e_2;e_0;\ldots;e_1]_n + [e_1, \mathrm{ad}(\mathbf{h}_1)[e_0;\ldots;e_1]_{n-1}] -$$

$$\lambda_{2,n}[e_1;\ldots;e_0;e_2]_n - \lambda_{2,n}[e_0, \mathrm{ad}(\mathbf{h}_1)[e_0,\ldots;e_2]_{n-1}] = 0.$$

Applying the induction step for $j = n-1$, this equation can be simplified to

$$[e_2;e_0;\ldots;e_0;e_1]_n + c_n \cdot [e_1;e_0;\ldots;e_0;e_2]_n + d_n \cdot [e_0;e_0;\ldots;e_0;e_3]_n = 0.$$

Computing the images under $\theta_m$, one gets $c_n = \frac{(q-1)^{n-2}}{(q^2-1)^{n-1}}(q^n + 2q^{n-1} - 2q - 1)$.

On the other hand, applying $\mathrm{ad}(\mathbf{h}_1)\,\mathrm{ad}(\mathbf{h}_2) - \mathrm{ad}(\mathbf{h}_3)$ to (u7n), we get

$$[e_2; \mathrm{ad}(\mathbf{h}_1)[e_1;\ldots;e_{-1}]_{n-1}] + [e_1, \mathrm{ad}(\mathbf{h}_2)[e_1,\ldots,e_{-1}]_{n-1}] +$$

$$[e_0; (\mathrm{ad}(\mathbf{h}_1)\,\mathrm{ad}(\mathbf{h}_2) - \mathrm{ad}(\mathbf{h}_3))[e_1,\ldots,e_{-1}]_{n-1}] = 0.$$

Applying the induction step for $j = n-1$, this equation can be simplified to

$$a'_n \cdot [e_2;e_0;\ldots;e_0;e_1]_n + c'_n \cdot [e_1;e_0;\ldots;e_0;e_2]_n + d'_n \cdot [e_0;e_0;\ldots;e_0;e_3]_n = 0.$$

By computing the images under $\theta_m$, one gets the following formulas

$$a'_n = \frac{(q^{n-1}-1)^2}{(q-1)^2},\; c'_n = \frac{(-1)^n(q-1)^{n-4}(q^{n-1}-1)^2(q^{n-1}+1)}{q^{n-2}(q+1)(q^2-1)^{n-2}}.$$

It remains to notice that $c'_n \neq a'_n c_n$ for $q \neq \sqrt{1}$. Therefore, eliminating $[e_2;e_0;\ldots;e_0;e_1]_n$, we see that $[e_1;e_0;\ldots;e_0;e_2]_n$ is a multiple of $[e_0;e_0;\ldots;e_0;e_3]_n$.

∘ *Case* $N = k+2, k > 1$.

By the induction: $[e_1;e_0;\ldots;e_k]_n - \lambda_{k+1,n}[e_0;\ldots;e_0;e_{k+1}]_n = 0$. Applying $\mathrm{ad}(\mathbf{h}_1)$, we get

$$[e_2;e_0;\ldots;e_k]_n + [e_1, \mathrm{ad}(\mathbf{h}_1)[e_0;\ldots;e_k]_{n-1}] -$$

$$\lambda_{k+1,n}([e_1;e_0;\ldots;e_{k+1}]_n + [e_0, \mathrm{ad}(\mathbf{h}_1)[e_0;\ldots;e_{k+1}]_{n-1}]) = 0.$$

By an induction assumption on length $j < n$ commutators, this equality can be simplified to

$$[e_2; e_0; \ldots; e_k]_n + v_n \cdot [e_1; e_0; \ldots; e_{k+1}]_n + w_n \cdot [e_0; e_0; \ldots; e_{k+2}]_n = 0.$$

By computing the images under $\theta_m$, we find $v_n = \frac{(q^k-1)^{n-2}(q^{n+k}-2q^{k+1}-2q^{n-1}+q^n+q^k+1)}{(q^{k+1}-1)^{n-1}(q-1)}$.

On the other hand, by an induction assumption:

$$[e_1; \ldots; e_0; e_{k-1}]_n - \lambda_{k,n}[e_0; \ldots; e_0; e_k]_n = 0.$$

Applying $\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2)$, we get

$$[e_2, \mathrm{ad}(\mathbf{h}_1)[e_0; \ldots; e_{k-1}]_{n-1}] + [e_1, (\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2))[e_0; \ldots; e_{k-1}]_{n-1}] -$$

$$\lambda_{k,n}([e_1, \mathrm{ad}(\mathbf{h}_1)[e_0; \ldots; e_k]_{n-1}] + [e_0, (\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2))[e_0; \ldots; e_k]_{n-1}]) = 0.$$

By an induction assumption on length $j < n$ commutators, this equality can be simplified to

$$u'_n \cdot [e_2; e_0; \ldots; e_k]_n + v'_n \cdot [e_1; e_0; \ldots; e_{k+1}]_n + w'_n \cdot [e_0; e_0; \ldots; e_{k+2}]_n = 0.$$

By computing the images under $\theta_m$, we find

$$u'_n = \frac{(q^{n-1}-1)(q^{k-1}-1)^{n-2}}{(q-1)(q^k-1)^{n-2}},\ v'_n = \frac{(q^{k-1}-1)^{n-2}(q^{n-1}-1)}{(q^{k+1}-1)^{n-2}(q-1)}\left(\frac{q^{n-1}-q}{q^2-1} - \frac{q^{k-1}-q^{n-1}}{q^k-1}\right).$$

It remains to notice that $v'_n \neq u'_n v_n$ for $q \neq \sqrt{1}$. Therefore, eliminating $[e_2; e_0; \ldots; e_0; e_k]_n$, we see that $[e_1; e_0; \ldots; e_0; e_{k+1}]_n$ is a multiple of $[e_0; e_0; \ldots; e_0; e_{k+2}]_n$.

This completes the proof of $\dim(\ddot{u}_{h_0})_{i,j} \leq 1$ for $j > 0$. The case $j < 0$ is analogous.

### 5.5.7 Proof of Theorem 5.5.9

We prove that $\theta_a$ is an isomorphism of $\mathbb{C}$-algebras for $h_0 \neq 0$.

As mentioned in Section 5.5, all the defining relations of the algebra $\ddot{Y}'_{h_0}(\mathfrak{gl}_1)$

are of Lie-type. Therefore, it is the universal enveloping algebra of the Lie algebra $\ddot{y}'_{h_0}$ generated by $\{e_j, f_j, \psi_j\}$ with the same defining relations. Moreover, $\ddot{y}'_{h_0}$ is a 1-dimensional central extension of the Lie-algebra $\ddot{y}_{h_0}$ generated by $\{e_j, f_j, \psi_{j+1}\}$ with the following defining relations:

$$[\psi_k, \psi_l] = 0, \tag{y0}$$

$$[e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] - h_0^2([e_{i+1}, e_j] - [e_i, e_{j+1}]) = 0, \tag{y1}$$

$$[f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] - h_0^2([f_{i+1}, f_j] - [f_i, f_{j+1}]) = 0, \tag{y2}$$

$$[e_0, f_0] = 0, \ [e_i, f_j] = \psi_{i+j} \text{ if } i + j > 0, \tag{y3}$$

$$[\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] - h_0^2([\psi_{i+1}, e_j] - [\psi_i, e_{j+1}]) = 0, \tag{y4}$$

$$[\psi_1, e_j] = 0, \ [\psi_2, e_j] = 2h_0^2 e_j, \tag{y4$'$}$$

$$[\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] - h_0^2([\psi_{i+1}, f_j] - [\psi_i, f_{j+1}]) = 0, \tag{y5}$$

$$[\psi_1, f_j] = 0, \ [\psi_2, f_j] = -2h_0^2 f_j, \tag{y5$'$}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[e_{i_1}, [e_{i_2}, e_{i_3+1}]] = 0, \ \ \mathrm{Sym}_{\mathfrak{S}_3}[f_{i_1}, [f_{i_2}, f_{i_3+1}]] = 0. \tag{y6}$$

Hence, it suffices to check that the corresponding homomorphism $\theta_a : \ddot{y}_{h_0} \to \mathfrak{D}_{h_0}$ defined by

$$\theta_a : e_j \mapsto x^j\partial, \ f_j \mapsto -\partial^{-1}x^j, \ \psi_{j+1} \mapsto ((x - h_0)^{j+1} - x^{j+1})\partial^0,$$

is an isomorphism of the $\mathbb{C}$-Lie algebras for $h_0 \neq 0$. The surjectivity of $\theta_a$ is clear.

The Lie algebra $\ddot{y}_{h_0}$ is $\mathbb{Z}$-graded, where we set

$$\deg_2(e_j) = 1, \ \deg_2(f_j) = -1, \ \deg_2(\psi_j) = 0.$$

It is also $\mathbb{Z}_+$-filtered with $\deg_1(e_j) = j, \deg_1(f_j) = j, \deg_1(\psi_{j+1}) = j$. The Lie algebra $\mathfrak{D}_{h_0}$ is also $\mathbb{Z}$-graded with $\deg_2(x^i\partial^j) = j$ and $\mathbb{Z}_+$-filtered with $\deg_1(x^i\partial^j) = i$. Moreover, $\theta_a$ intertwines those gradings and filtrations. Note that $\dim(\mathfrak{D}_{h_0})_{\leq i,j} = \dim(\mathfrak{D}_{h_0})_{\leq i-1,j} + 1$. Let $\ddot{y}_{h_0}^{\geq 0}$ be the Lie algebra generated by $\{e_j, \psi_k\}$ with the defining relations (y0,y1,y4,y4′,y6). The result follows from the following inequality: $\dim(\ddot{y}_{h_0}^{\geq 0})_{\leq i,j} - \dim(\ddot{y}_{h_0}^{\geq 0})_{\leq i-1,j} \leq 1$.

Consider a subspace $V_{\leq i,n} \subset (\ddot{y}_{h_0}^{\geq 0})_{\leq i,n}$ spanned by $\{[e_{i_1};\ldots;e_{i_n}] | i_1+\ldots+i_n \leq i+n-1\}$. The above inequality follows from the following result, which we prove by an induction on $j$:

$$\dim V_{\leq i,j} - \dim V_{\leq i-1,j} \leq 1. \qquad (\diamondsuit_{i,j})$$

Note that the relations (y4,y4′) imply the following result:

$$[\psi_k, e_j] - k(k-1)h_0^2 e_{j+k-2} \in V_{<j+k-2,1}.$$

• *Case* $j = 1, 2$.

The subspace $V_{\leq i,1}$ is spanned by $\{e_0, e_1, \ldots, e_i\}$. The inequality $(\diamondsuit_{i,1})$ follows.

The subspace $V_{\leq N,2}$ is spanned by $\{[e_{i_1}, e_{i_2}] | i_1 + i_2 \leq N+1\}$. The relation (y1) implies:

$$[e_{i+2+k}, e_{i+1-k}] - (2k+1)[e_{i+2}, e_{i+1}] \in V_{\leq 2i+1,2}, \ [e_{i+2+k}, e_{i-k}] - (k+1)[e_{i+2}, e_i] \in V_{\leq 2i,2}.$$

These formulas can be unified in the following way:

$$[e_i, e_j] - \frac{j-i}{i+j}[e_0, e_{i+j}] \in V_{\leq i+j-2,2}. \qquad (5.12)$$

Hence, $V_{\leq i,2}/V_{\leq i-1,2}$ is spanned by the image of $[e_0, e_{i+1}]$. The inequality $(\diamondsuit_{i,2})$ follows.

• *Case* $j = 3$.

Our goal is to show that $[e_{i_1}; e_{i_2}; e_{i_3}]$ is a multiple of $[e_0; e_0; e_{i_1+i_2+i_3}]$ modulo $V_{\leq i_1+i_2+i_3-3,3}$, which will be denoted by $[e_{i_1}; e_{i_2}; e_{i_3}] \sim [e_0; e_0; e_{i_1+i_2+i_3}]$. By $(\diamondsuit_{\bullet,2})$, we can assume $i_2 = 0$.

To proceed, we introduce the elements $\mathbf{h}_1, \mathbf{h}_2 \in \ddot{y}_{h_0}^{\geq 0}$ by $\mathbf{h}_1 := \frac{\psi_3}{6h_0^2}, \mathbf{h}_2 := \frac{\psi_4 + h_0^2\psi_2}{12h_0^2}$. According to (y4,y4′), we have $[\mathbf{h}_1, e_j] = e_{j+1}, [\mathbf{h}_2, \psi_j] = e_{j+2}$. Same reasoning as in Appendix B.1 shows that applying $\mathrm{ad}(\mathbf{h}_1)$ to $[e_k; e_0; e_l] \sim [e_0; e_0; e_{k+l}]$ implies $[e_{k+1}; e_0; e_l] \sim [e_0; e_0; e_{k+l+1}]$. Therefore, it remains to prove $[e_1; e_0; e_{N-1}] \sim [e_1; e_0; e_N]$.

Computing the images of both commutators under $\theta_a$, we see that $[e_1; e_0; e_{N-1}] \equiv \beta_{N,3}[e_0; e_0; e_N]$ for $\beta_{N,3} = \frac{N-4}{N}$. We write $a \equiv b$ if $a - b \in V_{\leq i-1,j}$ for $a, b \in V_{\leq i,j}$.

∘ *Case* $N = 1, 2$.

We have $[e_1; e_0; e_0] = 0 = [e_0; e_0; e_1]$. Applying $\mathrm{ad}(\mathbf{h}_1)$ to this, we get $[e_1; e_0; e_1] = -[e_0; e_0; e_2]$.

∘ *Cases* $N = k + 1, k > 1$.

By an induction assumption: $[e_1; e_0; e_{k-1}] \equiv \beta_{k,3}[e_0; e_0; e_k]$. Applying $\mathrm{ad}(\mathbf{h}_1)$, we get

$$[e_2; e_0; e_{k-1}] + [e_1; e_1; e_{k-1}] + [e_1; e_0; e_k] \equiv \beta_{k,3}([e_1; e_0; e_k] + [e_0; e_1; e_k] + [e_0; e_0; e_{k+1}]).$$

Applying (5.12), we get $[e_2; e_0; e_{k-1}] + \frac{k+2}{k}[e_1; e_0; e_k] \sim [e_0; e_0; e_{k+1}]$. On the other hand, we have $[e_1; e_0; e_{k-2}] \equiv \beta_{k-1,3}[e_0; e_0; e_{k-1}]$. Applying $\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2)$, we get

$$[e_2; e_1; e_{k-2}] + [e_2; e_0; e_{k-1}] + [e_1; e_1; e_{k-1}] \equiv \beta_{k-1,3}([e_1; e_1; e_{k-1}] + [e_1; e_0; e_k] + [e_0; e_1; e_k]).$$

Applying (5.12), we get $\frac{2(k-2)}{k-1}[e_2; e_0; e_{k-1}] + \frac{8-k}{k}[e_1; e_0; e_k] \sim [e_0; e_0; e_{k+1}]$. Comparing those two linear combinations of $[e_2; e_0; e_{k-1}], [e_1; e_0; e_k]$, we get $[e_1; e_0; e_k] \sim [e_0; e_0; e_{k+1}]$, unless $k = 3$. We will consider this particular case in the greater generality below.

• *Case* $j > 3$.

Analogously to the $j = 3$ case it suffices to show that $[e_1; e_0; \ldots; e_{N-1}]_n \sim [e_0; \ldots; e_0; e_N]_n$. This is equivalent to

$$[e_1; \ldots; e_0; e_{N-1}]_n \equiv \beta_{N,n} \cdot [e_0; \ldots; e_0; e_N]_n, \ \beta_{N,n} = \frac{N - 2n + 2}{N},$$

the constant being computed by comparing the images under $\theta_a$.

We will need the following *multiple* counterpart of (y6) (follows from Proposition 5.7.10 below):

$$[e_0; \ldots; e_0; e_{n-2}]_n = 0. \qquad \text{(y7n)}$$

Now we proceed to the proof of the aforementioned result by an induction on $N$.

∘ *Case* $N \leq n-1$.

If $N < n-1$, then $[e_0; \ldots; e_0; e_{N-1}]_{n-1} = 0 = [e_0; \ldots; e_0; e_N]_n$. Applying $\mathrm{ad}(\mathbf{h}_1)$ to $[e_0; \ldots; e_0; e_{n-2}]_n = 0$, we get $[e_1; \ldots; e_0; e_{n-2}]_n \sim [e_0; \ldots; e_0; e_{n-1}]_n$.

∘ *Case* $N = k+1, k > n-2$.

Applying $\mathrm{ad}(\mathbf{h}_1)$ to $[e_1; \ldots; e_0; e_{k-1}]_n \equiv \beta_{k,n}[e_0; \ldots; e_0; e_k]_n$, we get

$$[e_2; \ldots; e_0; e_{k-1}] + [e_1, \mathrm{ad}(\mathbf{h}_1)[e_0; \ldots; e_{k-1}]_{n-1}] \equiv$$

$$\beta_{k,n}([e_1; \ldots; e_0; e_k]_n + [e_0, \mathrm{ad}(\mathbf{h}_1)[e_0; \ldots; e_0; e_k]_{n-1}]).$$

By an induction assumption for $j = n-1$, this is equivalent to

$$[e_2; \ldots; e_0; e_{k-1}]_n + \frac{(n-2)k - (n-1)(n-4)}{k}[e_1; \ldots; e_0; e_k]_n \equiv [e_0; \ldots; e_0; e_{k+1}]_n.$$

Applying $\mathrm{ad}(\mathbf{h}_1)^2 - \mathrm{ad}(\mathbf{h}_2)$ to $[e_1; \ldots; e_0; e_{k-2}]_n \equiv \beta_{k-1,n}[e_0; \ldots; e_0; e_{k-1}]_n$ together with an induction assumption for $j = n-1$, we get

$$P[e_2; \ldots; e_0; e_{k-1}]_n + Q[e_1; \ldots; e_0; e_k]_n \equiv [e_0; \ldots; e_0; e_{k+1}]_n,$$

with $P = \frac{(n-1)(k-n+1)}{k-1}, Q = \frac{n-1}{2k(k-1)}(k^2(n-4) - k(2n^2-13n+12) + (n^3-9n^2+18n-8))$. Comparing those two linear combinations, we get $[e_1; \ldots; e_0; e_k]_n \sim [e_0; \ldots; e_{k+1}]_n$ for $k \neq n$.

It remains to consider the case $k = n$. Choose $\mathbf{h}_3 \equiv \frac{\psi_5}{20h_0^2}$ such that $[\mathbf{h}_3, e_j] = e_{j+3}$. Applying $\mathrm{ad}(\mathbf{h}_1)\,\mathrm{ad}(\mathbf{h}_2) - \mathrm{ad}(\mathbf{h}_3)$ to $[e_0; \ldots; e_0; e_{n-2}]_n = 0$, we get

$$[e_2; \ldots; e_0; e_{n-1}]_n + \frac{2}{n}[e_1; \ldots; e_0; e_n]_n \sim [e_0; \ldots; e_0; e_{n+1}]_n.$$

This equivalence together with the previous one implies

$$[e_1; \ldots; e_0; e_n]_n \sim [e_0; \ldots; e_0; e_{n+1}]_n.$$

## 5.6 The homomorphism $\Upsilon$

We construct a homomorphism $\Upsilon : \ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}'_{h_2,h_3}(\mathfrak{gl}_1)$, which induces an inclusion (but not an isomorphism as it was in [GTL]) of appropriate completions. We also construct compatible maps $\mathrm{ch}_r : M^r \to \widehat{V^r}$.

### 5.6.1 Construction of $\Upsilon$

We follow notation of [GTL].

Let $\widehat{\ddot{Y}}'_{h_2,h_3}(\mathfrak{gl}_1)$ denote the completion of $\ddot{Y}_{h_2,h_3}(\mathfrak{gl}_1)$ with respect to the $\mathbb{Z}_+$-grading on it. To state the main result, we introduce the following notation:

- Define $\psi(z) := 1 - h_3 \sum_{i\geq 0} \psi_i z^{-i-1} \in \ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)[[z^{-1}]]$ (agrees with that from Section 5.1.4).
- Define $k_i \in \mathbb{C}[\psi_0, \psi_1, \psi_2, \ldots]$ by $\sum_{i\geq 0} k_i z^{-i-1} =: k(z) = \ln(\psi(z))$.
- Define the *(inverse) Borel transform*

$$B : z^{-1}\mathbb{C}[[z^{-1}]] \to \mathbb{C}[[w]] \text{ by } \sum_{i=0}^{\infty} \frac{a_i}{z^{i+1}} \mapsto \sum_{i=0}^{\infty} \frac{a_i}{i!} w^i.$$

- Define $B(w) \in h_3 \ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)[[z^{-1}]]$ to be the inverse Borel transform of $k(z)$.
- Define a function $G(v) := \log\left(\frac{v}{e^{v/2} - e^{-v/2}}\right) \in v\mathbb{Q}[[v]]$.
- Define $\gamma(v) := -B(-\partial_v) G'(v) \in \widehat{\ddot{Y}}'_{h_2,h_3}(\mathfrak{gl}_1)[[v]]$.
- Define $g(v) := \sum_{i\geq 0} g_i v^i \in \widehat{\ddot{Y}}'_{h_2,h_3}(\mathfrak{gl}_1)[[v]]$ by $g(v) := \left(\frac{h_3}{q_3 - 1}\right)^{1/2} \exp\left(\frac{\gamma(v)}{2}\right)$.

The identity $B(\log(1 - \gamma/z)) = (1 - e^{\gamma w})/w$ immediately implies the following result:

**Corollary 5.6.1.** *The conditions of Proposition 5.1.5(e,f) are equivalent to*

$$[B(w), e_j] = \frac{\sum_{i=1}^{3}(e^{h_i w} - e^{-h_i w})}{w}\, e^{w\sigma^+} e_j,\ [B(w), f_j] = \frac{\sum_{i=1}^{3}(e^{-h_i w} - e^{h_i w})}{w}\, e^{w\sigma^-} f_j.$$

Now we are ready to state the main result:

**Theorem 5.6.2.** *There exists an algebra homomorphism*

$$\Upsilon : \ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1) \to \widehat{\ddot{Y}}'_{h_2,h_3}(\mathfrak{gl}_1)$$

*defined on the generators by*

$$H_m \mapsto \frac{B(m)}{1-q_3},\ e_k \mapsto e^{k\sigma^+} g(\sigma^+)e_0,\ f_k \mapsto e^{k\sigma^-} g(\sigma^-)f_0,\ \kappa \mapsto -\psi_0.$$

*Proof.*

We need to verify that $\Upsilon$ is compatible with the defining relations of $\ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1)$.

• The relation (T0) is obviously preserved by $\Upsilon$.

• According to Corollary 5.6.1, we have

$$[B(m), e_j] = \frac{q_1^m + q_2^m + q_3^m - q_1^{-m} - q_2^{-m} - q_3^{-m}}{m}\ e^{m\sigma^+} e_j = -\frac{\beta_m}{m}\ e^{m\sigma^+} e_j,$$

$$[B(m), f_j] = -\frac{q_1^m + q_2^m + q_3^m - q_1^{-m} - q_2^{-m} - q_3^{-m}}{m} e^{m\sigma^-} f_j = \frac{\beta_m}{m}\ e^{m\sigma^-} f_j.$$

This implies the compatibility of (T4t,T5t) with $\Upsilon$.

• The verification of (T1,T2,T3) is completely analogous to the corresponding computations from [GTL, Ch. 3,4].

• The verification of the cubic relation (T6t) is implicit. Set $E := [\Upsilon(e_0), [\Upsilon(e_1), \Upsilon(e_{-1})]]$. We will see (Proposition 5.6.8 below) that $E$ acts trivially on $V^r$ for all $r$. Note that $V^r$ are $\mathbb{Z}_+$-graded modules of $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$. In particular, the degree $k$ component $E_k$ of $E$ acts trivially on $V^r$. But we will see (Section 6.4 below) that the action of $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$ on $\bigoplus_r V^r$ is faithful. This implies $E_k = 0$ for all $k$, and $E = 0$. The proof of $[\Upsilon(f_0), [\Upsilon(f_1), \Upsilon(f_{-1})]] = 0$ is analogous. □

### 5.6.2 The limit $h_3 = 0$

We verify that the specialization of $\Upsilon$ at $h_3 = 0$ is induced by $\Upsilon_0$.

Recall that we have isomorphisms

$$\ddot{U}'_{h,h_3}(\mathfrak{gl}_1)/(h_3) \xrightarrow{\sim} U(\bar{\mathfrak{d}}^0_h) \quad \text{and} \quad \ddot{Y}'_{h,h_3}(\mathfrak{gl}_1)/(h_3) \xrightarrow{\sim} U(\overline{\mathfrak{D}}^0_h).$$

Our next result evaluates the specialization of $\Upsilon$ at $h_3 = 0$.

**Proposition 5.6.3.** *The homomorphism* $\Upsilon_{|h_3=0} : U(\bar{\mathfrak{d}}^0_h) \to \widehat{U(\overline{\mathfrak{D}}^0_h)}$ *is induced by* $\Upsilon_0$.

*Proof.*

We verify the statement by computing the images of the generators under $\Upsilon_{|h_3=0}$. We have:

- $\Upsilon_{|h_3=0}(c_{\mathfrak{d}}) = c_{\mathfrak{D}}$.
- $\Upsilon_{|h_3=0}((q^{-k}-1)Z^k - q^{-k}c_{\mathfrak{d}}) = \sum_{i\geq 0}((x-h)^i - x^i - (-h)^i c_{\mathfrak{D}})\frac{k^i}{i!} = (q^{-k}-1)e^{kx} - q^{-k}c_{\mathfrak{D}}$.
- $\Upsilon_{|h_3=0}(Z^k D) = \sum_{i\geq 0} \frac{k^i}{i!} \cdot x^i \partial = e^{kx}\partial$.
- $\Upsilon_{|h_3=0}(-D^{-1}Z^k) = -\sum_{i\geq 0} \frac{k^i}{i!}\partial^{-1} \cdot x^i = -\partial^{-1}e^{kx}$.

The result follows. □

### 5.6.3 The elliptic Hall algebra

We recall a notion of the elliptic Hall algebra studied in [BS, S, SV].

We will need the following notation:

- We set $(\mathbb{Z}^2)^* := \mathbb{Z}^2 \backslash \{(0,0)\}$.
- We set $(\mathbb{Z}^2)^+ := \{(a,b) | a > 0 \text{ or } a = 0, b > 0\}$, $(\mathbb{Z}^2)^- := -(\mathbb{Z}^2)^+$.
- For any $\mathbf{x} = (a,b) \in (\mathbb{Z}^2)^*$, we define $\deg(\mathbf{x}) := \gcd(a,b)$.
- For any $\mathbf{x} \in (\mathbb{Z}^2)^*$, we define $\epsilon_{\mathbf{x}} := 1$ if $\mathbf{x} \in (\mathbb{Z}^2)^+$ and $\epsilon_{\mathbf{x}} := -1$ if $\mathbf{x} \in (\mathbb{Z}^2)^-$.
- For a pair of non-collinear $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}^2)^*$, we set $\epsilon_{\mathbf{x},\mathbf{y}} := \mathrm{sign}(\det(\mathbf{x},\mathbf{y})) \in \{\pm 1\}$.
- For non-collinear $\mathbf{x}, \mathbf{y} \in (\mathbb{Z}^2)^*$, we denote the triangle with vertices $\{(0,0), \mathbf{x}, \mathbf{x}+\mathbf{y}\}$ by $\triangle_{\mathbf{x},\mathbf{y}}$.
- We define $\alpha_n := -\beta_n/n = \frac{(1-q_1^{-n})(1-q_2^{-n})(1-q_3^{-n})}{n}$.
- We say that $\triangle_{\mathbf{x},\mathbf{y}}$ is *empty* if there are no lattice points inside this triangle.

Following [BS], we define (central extension of) the elliptic Hall algebra $\widetilde{\mathcal{E}}$ to be the associative algebra generated by $\{u_{\mathbf{x}}, \kappa_{\mathbf{y}} | \mathbf{x} \in (\mathbb{Z}^2)^*, \mathbf{y} \in \mathbb{Z}^2\}$ with the following defining relations:

$$\kappa_{\mathbf{x}}\kappa_{\mathbf{y}} = \kappa_{\mathbf{x}+\mathbf{y}}, \ \kappa_{0,0} = 1, \tag{E0}$$

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \delta_{\mathbf{x},-\mathbf{y}} \cdot \frac{\kappa_{\mathbf{x}} - \kappa_{\mathbf{x}}^{-1}}{\alpha_{\deg(\mathbf{x})}} \text{ if } \mathbf{x}, \mathbf{y} \text{ are collinear}, \tag{E1}$$

$$[u_{\mathbf{y}}, u_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}}\kappa_{\alpha(\mathbf{x},\mathbf{y})}\frac{\theta_{\mathbf{x}+\mathbf{y}}}{\alpha_1} \text{ if } \triangle_{\mathbf{x},\mathbf{y}} \text{ is empty and } \deg(\mathbf{x}) = 1, \tag{E2}$$

where the elements $\theta_{\mathbf{x}}$ are defined via

$$\sum_{n\geq 0} \theta_{n\mathbf{x}_0} x^n = \exp\left(\sum_{r>0} \alpha_r u_{r\mathbf{x}_0} x^r\right) \text{ if } \deg(\mathbf{x}_0) = 1, \tag{E3}$$

while $\alpha(\mathbf{x}, \mathbf{y})$ is defined by

$$\alpha(\mathbf{x}, \mathbf{y}) = \begin{cases} \epsilon_{\mathbf{x}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))/2, & \epsilon_{\mathbf{x},\mathbf{y}} = 1 \\ \epsilon_{\mathbf{y}}(\epsilon_{\mathbf{x}}\mathbf{x} + \epsilon_{\mathbf{y}}\mathbf{y} - \epsilon_{\mathbf{x}+\mathbf{y}}(\mathbf{x}+\mathbf{y}))/2, & \epsilon_{\mathbf{x},\mathbf{y}} = -1 \end{cases}. \tag{E4}$$

This algebra is closely related to the toroidal algebras of $\mathfrak{gl}_1$:

**Theorem 5.6.4.** *[S] There is an isomorphism* $\Xi : \widetilde{\mathcal{E}}/(\kappa_{0,1}-1) \xrightarrow{\sim} \ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ *defined on the generators by*

$$u_{1,i} \mapsto e_i, \ u_{-1,i} \mapsto f_i, \ \theta_{0,j} \mapsto \psi_j^+/\psi_0^+, \ \theta_{0,-j} \mapsto \psi_j^-/\psi_0^-, \ \kappa_{a,b} \mapsto (\psi_0^+)^a \ , \quad j > 0.$$

*Remark* 5.6.1. This theorem has been proved in [S] only for $\mathcal{E} := \widetilde{\mathcal{E}}/(\kappa_{\mathbf{y}} - 1)_{\mathbf{y}\in\mathbb{Z}^2}$, but the above generalization is straightforward. The quotient algebra $\mathcal{E}$ is the spherical Hall algebra of an elliptic curve over $\mathbb{F}_q$.

This result provides distinguished elements $\{u_{\mathbf{x}} | \mathbf{x} \in (\mathbb{Z}^2)^*\}$ of $\ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1)$. As $h_3 \to 0$, their images $\bar{u}_{\mathbf{x}}$ coincide with the natural generators of $U(\bar{\mathfrak{d}}^0_{h_2})$.

**Lemma 5.6.5.** *The* $\theta_m$*-images of* $\bar{u}_{k,l}$ *are given by the following formulas:*

$$\bar{u}_{0,r} \mapsto \operatorname{sign}(r)\frac{1-q_2^{-1}}{1-q_2^r}(1-q_2)\left(Z^r - \frac{1-q_2}{1-q_2^r}c_{\mathfrak{d}}\right), \tag{5.13}$$

$$\bar{u}_{k,l} \mapsto q_2^{f(k,l)} \frac{1-q_2^{-1}}{1-q_2^{-d}}(1-q_2)^{k-1} Z^l D^k,\ \bar{u}_{-k,-l} \mapsto -q_2^{-f(k,l)} \frac{1-q_2}{1-q_2^{d}}(1-q_2^{-1})^{k-1} D^{-k} Z^{-l}, \tag{5.14}$$

*where $k > 0$, $r \neq 0$, $d := \gcd(k,l)$ and $f(k,l) := \frac{kl-k-l-d+2}{2}$ is the (signed) number of lattice points inside the triangle with vertices $\{(0,0),(0,l),(k,l)\}$.*

*Proof.*

Let us first observe that in the limit $h_3 \to 0$, the relation (E2) becomes

$$[\bar{u}_{\mathbf{y}}, \bar{u}_{\mathbf{x}}] = \epsilon_{\mathbf{x},\mathbf{y}} \kappa_{\alpha(\mathbf{x},\mathbf{y})} \frac{\alpha_{\deg(\mathbf{x}+\mathbf{y})}}{\alpha_1} \bar{u}_{\mathbf{x}+\mathbf{y}} \text{ if } \triangle_{\mathbf{x},\mathbf{y}} \text{ is empty and } \deg(\mathbf{x}) = 1. \tag{E2$'$}$$

This formula immediately implies (5.13), since we have $\bar{u}_{0,r} = \mathrm{sign}(r)\frac{\alpha_1}{\alpha_r}[\bar{u}_{-1,0}, \bar{u}_{1,r}]$.

Formula (5.14) will be proved by an induction on $k$; we will consider only the case $k > 0$. Case $k = 1$ is trivial. Given $(k,l) \in \mathbb{Z}_{>1} \times \mathbb{Z}$, choose unique $\mathbf{x} = (k_1, l_1), \mathbf{y} = (k_2, l_2), 0 < k_1, k_2 < k$, such that $\mathbf{x}+\mathbf{y} = (k,l)$, $\epsilon_{\mathbf{x},\mathbf{y}} = 1$, $\deg(\mathbf{x}) = \deg(\mathbf{y}) = 1$ and $\triangle_{\mathbf{x},\mathbf{y}}$ is empty. The formula (E2$'$) together the an induction assumption yield:

$$\theta_m(\bar{u}_{k,l}) = \frac{(1-q_2)(1-q_2^{-1})}{(1-q_2^d)(1-q_2^{-d})} q_2^{f(k_1,l_1)+f(k_2,l_2)} (q_2^{k_2 l_1} - q_2^{k_1 l_2})(1-q_2)^{k_1+k_2-2} Z^{l_1+l_2} D^{k_1+k_2}.$$

By our assumptions on $\mathbf{x}, \mathbf{y}$ and the Pick's formula, we get $q_2^{k_2 l_1} - q_2^{k_1 l_2} = q_2^{k_2 l_1}(1-q_2^d)$. It remains to use the equality $f(k_1,l_1) + f(k_2,l_2) + k_2 l_1 = f(k_1+k_2, l_1+l_2) = f(k,l)$, which obviously follows from the combinatorial meaning of $f$. □

### 5.6.4 Flatness of the deformations

We prove the following result:

**Theorem 5.6.6.** *(a) The algebra $\ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1)$ is a flat deformation of $\ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1)/(h_3) \simeq U(\bar{\mathfrak{d}}^0_{h_2})$.*

*(b) The algebra $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$ is a flat deformation of $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)/(h_3) \simeq U(\overline{\mathfrak{D}}^0_{h_2})$.*

As an immediate consequence of this theorem and Proposition 5.6.3, we get:

**Corollary 5.6.7.** *The homomorphism $\Upsilon$ is injective.*

*Remark* 5.6.2. We do not get an isomorphism of the appropriate completions (as it was in [GTL]), since the limit homomorphism $\Upsilon_{|_{h_3=0}}$ does not extend to an isomorphism of completions.

To prove Theorem 5.6.6 it suffices to provide a faithful $U(\bar{\mathfrak{d}}^0_{h_2})$-representation (respectively $U(\overline{\mathfrak{D}}^0_{h_2})$-representation) which admits a flat deformation to a representation of $\ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1)$ (respectively $\ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1)$). To make use of the representations constructed in the previous sections, we should work over the localization ring $R$ of $\mathbb{C}[[h_2,h_3]]$ by the homogeneous polynomials in $h_2, h_3$. Therefore, we will switch to the extension algebras

$$\ddot{U}'_R(\mathfrak{gl}_1) := \ddot{U}'_{h_2,h_3}(\mathfrak{gl}_1) \otimes_{\mathbb{C}[[h_2,h_3]]} R,\ \ddot{Y}'_R(\mathfrak{gl}_1) := \ddot{Y}'_{h_2,h_3}(\mathfrak{gl}_1) \otimes_{\mathbb{C}[[h_2,h_3]]} R.$$

Let $\mathfrak{gl}_\infty$ be a Lie algebra of matrices $A = \sum_{i,j\in\mathbb{Z}} a_{i,j}E_{i,j}$ such that $a_{i,j} = 0$ for $|i-j| \gg 0$. Let $\mathfrak{gl}_{\infty,\kappa} = \mathfrak{gl}_\infty \oplus \mathbb{C}\cdot\kappa$ be the central extension of this Lie algebra by the 2-cocycle

$$\phi_{\mathfrak{gl}}\left(\sum a_{i,j}E_{i,j}, \sum b_{i,j}E_{i,j}\right) = \sum_{i\le 0<j} a_{i,j}b_{j,i} - \sum_{j\le 0<i} a_{i,j}b_{j,i}.$$

For any $u \in \mathbb{C}^*$, consider the homomorphism $\tau_u : U(\bar{\mathfrak{d}}^0_{h_2})_R \to U(\mathfrak{gl}_{\infty,\kappa})_R$ such that

$$Z^mD \mapsto -\sum_i u^m q_2^{im} E_{i+1,i},\ D^{-1}Z^m \mapsto -\sum_i u^m q_2^{im} E_{i,i+1},$$

$$Z^m \mapsto -\sum_i u^m q_2^{im} E_{i,i} - \frac{u^m - q_2^{-m}}{1-q_2^{-m}}\kappa,\ c_{\mathfrak{d}} \mapsto -\kappa.$$

Let $\varpi_u : \ddot{U}'_R(\mathfrak{gl}_1) \to U(\mathfrak{gl}_{\infty,\kappa})_R$ be the composition of $\ddot{U}'_R(\mathfrak{gl}_1) \to U(\bar{\mathfrak{d}}^0_{h_2})_R$ and $\tau_u$. Then

$$\varpi_u(e(z)) = -\sum_i E_{i+1,i}\delta(q_2^i u/z),\ \varpi_u(f(z)) = \sum_i E_{i,i+1}\delta(q_2^i u/z).$$

Let $V_\infty$ be the basic representation of $\mathfrak{gl}_{\infty,\kappa}$. It is realized on $\Lambda^{\infty/2}\mathbb{C}^\infty$ with the highest weight vector $v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots$. Comparing the formulas for the Fock module $F(u)$ with those for the $\mathfrak{gl}_{\infty,\kappa}$-action on $V_\infty$, we see that $F(u)$ degenerates to the module $\tau_u^*(V_\infty)$.

It remains to prove that the module $\bigoplus_n \bigoplus_{u_1,\ldots,u_n} \tau_{u_1}^*(V_\infty) \otimes \cdots \otimes \tau_{u_n}^*(V_\infty)$ is a faithful representation of $U(\mathfrak{gl}_{\infty,\kappa})$. To prove this, we consider a further degeneration as $h_2 \to 0$. The algebra $\bar{\mathfrak{d}}_0$ is a central extension of the commutative Lie algebra with the basis $\{Z^k D^l\}$. Note that $\tau_u$ degenerates as well to provide a homomorphism $\tau_{u,0} : U(\bar{\mathfrak{d}}_0^0)_R \to U(\mathfrak{gl}_{\infty,\kappa})_R$ defined by $Z^k D^l \mapsto -u^k \sum_i E_{i+l,i},\ c_{\mathfrak{d}} \mapsto -\kappa$. The image of this homomorphism is just $U(\mathfrak{h})_R$, where $\mathfrak{h}$ is the Heisenberg algebra. Clearly

$$\bigoplus_n \bigoplus_{u_1,\ldots,u_n} \tau_{u_1,0}^*(V_\infty) \otimes \cdots \otimes \tau_{u_n,0}^*(V_\infty)$$

is a faithful representation of $U(\mathfrak{h})_R$. This completes the proof.

For the $\ddot{Y}'_R(\mathfrak{gl}_1)$ case, we use the homomorphism $\varsigma_u : U(\overline{\mathfrak{D}}_{h_2}^0)_R \to U(\mathfrak{gl}_{\infty,\kappa})_R$ defined by

$$x^n\partial \mapsto -\sum_i (u+ih_2)^n E_{i+1,i}, \partial^{-1}x^n \mapsto \sum_i (u+ih_2)^n E_{i,i+1}, x^n \mapsto \sum_i (u+ih_2)^E_{i,i} + c_n\kappa, c_{\mathfrak{D}} \mapsto \kappa,$$

where $c_n$ are determined recursively from $\binom{n}{1}h_2c_{n-1} - \binom{n}{2}h_2^2c_{n-2} + \ldots + (-1)^{n+1}h_2^n c_0 + (-h_2)^n - u^n = 0$.

### 5.6.5 The linear map $\mathrm{ch}_r$

Recall the representations $M^r$ and $V^r$ of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ from Section 3, which are defined over the fields $\mathbb{C}(\chi_1,\ldots,\chi_r,q_1,q_2,q_3)$ and $\mathbb{C}(x_1,\ldots,x_r,h_1,h_2,h_3)$ respectively. Let us denote the corresponding representations of $\ddot{U}'_R(\mathfrak{gl}_1)$ and $\ddot{Y}'_R(\mathfrak{gl}_1)$ by $M_R^r$ and $V_R^r$ (here we set $\chi_i = \exp(x_i)$ similarly to $q_i = \exp(h_i)$). Both representations have a basis parametrized by $r$-partitions $\{\bar{\lambda}\}$. The following result is straightforward:

**Proposition 5.6.8.** *There exists a unique collection of constants $b_{\bar{\lambda}} \in R$ such that $b_{\bar{\emptyset}} = 1$ and the linear map $\mathrm{ch}_r : M_R^r \to \widehat{V}_R^r$ defined by $[\bar{\lambda}] \mapsto b_{\bar{\lambda}} \cdot [\bar{\lambda}]$ satisfies*

$$\mathrm{ch}_r(Xv) = \Upsilon(X)\mathrm{ch}_r(v),\ \forall\ X \in \ddot{U}'_R(\mathfrak{gl}_1), v \in M_R^r.$$

## 5.7 Small shuffle algebras $S^m$ and $S^a$

We introduce the *small multiplicative* and *additive shuffle algebras*. We explain their relation to $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ and $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$. We also discuss their interesting commutative subalgebras.

### 5.7.1 The shuffle algebra $S^m$

We introduce the small multiplicative shuffle algebra $S^m$.

Let us consider a $\mathbb{Z}_+$-graded $\mathbb{C}$-vector space $\mathbb{S}^m = \bigoplus_{n\geq 0} \mathbb{S}^m_n$, where $\mathbb{S}^m_n$ consists of rational functions $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)^2}$ with $f \in \mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]^{\mathfrak{S}_n}$ and $\Delta(x_1,\ldots,x_n) := \prod_{1\leq i<j\leq n}(x_i - x_j)$.

Define the star-product $\overset{m}{\star} : \mathbb{S}^m_i \times \mathbb{S}^m_j \to \mathbb{S}^m_{i+j}$ by

$$(F \overset{m}{\star} G)(x_1,\ldots,x_{i+j}) := \mathrm{Sym}_{\mathfrak{S}_{n+m}}\left(F(x_1,\ldots,x_i)G(x_{i+1},\ldots,x_{i+j})\prod_{k\leq i}^{l>i}\omega^m(x_l,x_k)\right)$$

with

$$\omega^m(x,y) := \frac{(x-q_1y)(x-q_2y)(x-q_3y)}{(x-y)^3}.$$

This endows $\mathbb{S}^m$ with a structure of an associative unital $\mathbb{C}$-algebra.

We say that an element $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)^2} \in \mathbb{S}^m$ satisfies the *wheel condition* if $f(x_1,\ldots,x_n) = 0$ for any $\{x_1,\ldots,x_n\} \subset \mathbb{C}$ such that $x_1/x_2 = q_i, x_2/x_3 = q_j, i \neq j$. Let $S^m \subset \mathbb{S}^m$ be a $\mathbb{Z}_+$-graded subspace, consisting of all such elements. The subspace $S^m$ is closed with respect to $\overset{m}{\star}$.

**Definition 5.7.1.** The algebra $(S^m, \overset{m}{\star})$ is called the *small multiplicative shuffle algebra*.

Recall that $q_1, q_2, q_3$ are *generic* if $q_1^a q_2^b q_3^c = 1 \iff a = b = c$. We have the following result:

**Theorem 5.7.1.** *[N1, Proposition 3.5] The algebra $S^m$ is generated by $S^m_1$ for generic $q_1, q_2, q_3$.*

The connection of the shuffle algebra $S^m$ to the Hall algebra $\widetilde{\mathcal{E}}$ was established in [SV]:

**Proposition 5.7.2.** *[SV] The map $u_{1,i} \mapsto x_1^i$ extends to an injective homomorphism $\widetilde{\mathcal{E}}^+ \to S^m$, where $\widetilde{\mathcal{E}}^+$ is the subalgebra of $\widetilde{\mathcal{E}}$ generated by $\{u_{i,j}\}_{i>0}$.*

Combining this result with Theorems 5.7.1 and 5.6.4, we get:

**Theorem 5.7.3.** *The algebras $\widetilde{\mathcal{E}}^+, \ddot{U}^+_{q_1,q_2,q_3}(\mathfrak{gl}_1), S^m$ are isomorphic.*

### 5.7.2 Commutative subalgebra $\mathcal{A}^m \subset S^m$

We recall an interesting subalgebra $\mathcal{A}^m$.

Following [FHHSY], we introduce an important $\mathbb{Z}_+$-graded subspace $\mathcal{A}^m = \bigoplus_{n\geq 0} \mathcal{A}^m_n$ of $S^m$. Its degree $n$ component is defined by

$$\mathcal{A}^m_n = \{F \in S^m_n | \partial^{(0,k)} F = \partial^{(\infty,k)} F \ \ \forall\, 0 \leq k \leq n\},$$

where

$$\partial^{(0,k)} F := \lim_{\xi \to 0} F(x_1, \ldots, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n),\ \partial^{(\infty,k)} F := \lim_{\xi \to \infty} F(x_1, \ldots, \xi \cdot x_{n-k+1}, \ldots, \xi \cdot x_n).$$

This subspace satisfies the following properties:

**Theorem 5.7.4.** *[FHHSY, Section 2] We have:*
*(a) Suppose $F \in S^m_n$ and $\partial^{(\infty,k)} F$ exist for all $1 \leq k \leq n$, then $F \in \mathcal{A}^m_n$.*
*(b) The subspace $\mathcal{A}^m \subset S^m$ is $\overset{m}{\star}$-commutative.*
*(c) $\mathcal{A}^m$ is $\overset{m}{\star}$-closed and it is a polynomial algebra in $\{K^m_j\}_{j\geq 1}$ with $K^m_j \in S^m_j$ defined by:*

$$K^m_1(x_1) = x_1^0,\ K^m_2(x_1, x_2) = \frac{(x_1 - q_1 x_2)(x_2 - q_1 x_1)}{(x_1 - x_2)^2},\ K^m_n(x_1, \ldots, x_n) = \prod_{i<j} K^m_2(x_i, x_j).$$

*Remark* 5.7.1. The aforementioned elements $K^m_j$ played a crucial role in [FT1]. They were used to construct an action of the Heisenberg algebra on the vector space $M$ from Section 2.2.

Our next result provides an alternative choice for generators of the algebra $\mathcal{A}^m$, expressed explicitly via $S_1^m$. We use the following notation: $[P,Q]_m = P \overset{m}{\star} Q - Q \overset{m}{\star} P$ for $P, Q \in S^m$.

**Proposition 5.7.5.** *The algebra $\mathcal{A}^m$ is a polynomial algebra in the generators $\{L_j^m\}_{j\geq 1}$ defined by*

$$L_1^m(x_1) = x_1^0 \quad \text{and} \quad L_j^m = \underbrace{[x^1,[x^0,[x^0,\ldots,[x^0,x^{-1}]_m\ldots]_m]_m]_m}_{j\ \text{factors}} \in S_j^m \quad \text{for } j \geq 2.$$

Proposition 5.7.5 follows from Theorem 5.7.4(c) and the following two lemmas.

**Lemma 5.7.6.** *The elements $L_j^m$ belong to $\mathcal{A}^m$.*

**Lemma 5.7.7.** *The elements $\{L_j^m\}_{j\geq 1}$ are algebraically independent.*

*Proof of Lemma 5.7.6.*

According to Theorem 5.7.4(a), it suffices to show that $\partial^{(\infty,k)} L_n^m$ exist for all $k$. Note that:

$$L_n^m = \mathrm{Sym}_{\mathfrak{S}_n}\left\{\left(\sum_{l=0}^{n-2}(-1)^l\binom{n-2}{l}\frac{x_1}{x_{n-l}} - \sum_{l=0}^{n-2}(-1)^l\binom{n-2}{l}\frac{x_n}{x_{n-1-l}}\right)\prod_{i<j}\omega^m(x_i,x_j)\right\}. \tag{5.15}$$

Our goal is to show that the RHS of (5.15) has a finite limit as $x_{n-k+1} \mapsto \xi \cdot x_{n-k+1}, \ldots, x_n \mapsto \xi \cdot x_n$ with $\xi \to \infty$. Note that $\frac{x_{\sigma(i)}}{x_{\sigma(j)}}$ has a finite limit as $\xi \to \infty$, unless $\sigma(j) \leq n-k < \sigma(i)$, in which case it has a linear growth. On the other hand, $\omega^m(x_i,x_j)$ has a finite limit as $\xi \to \infty$. Moreover: $\omega^m(\xi \cdot x, y) = 1 + O(\xi^{-1})$, $\omega^m(y, \xi \cdot x) = 1 + O(\xi^{-1})$ as $\xi \to \infty$. This reduces to proving $A_1 - A_2 = 0$, where $A_1, A_2$ are given by

$$A_1 = \sum_{s=n-k+1}^{n}\ \sum_{\sigma\in\mathfrak{S}_n}^{\sigma(1)=s}\ \sum_{l:\sigma(n-l)\leq n-k}\frac{(-1)^l}{n!}\binom{n-2}{l}\frac{x_s}{x_{\sigma(n-l)}}\prod_{i<j}^{j\leq n-k}\omega_\sigma^m(x_i,x_j)\prod_{i<j}^{n-k<i}\omega_\sigma^m(x_i,x_j),$$

$$A_2 = \sum_{s=n-k+1}^{n}\ \sum_{\sigma\in\mathfrak{S}_n}^{\sigma(n)=s}\ \sum_{l:\sigma(n-l-1)\leq n-k}\frac{(-1)^l}{n!}\binom{n-2}{l}\frac{x_s}{x_{\sigma(n-l-1)}}\prod_{i<j}^{j\leq n-k}\omega_\sigma^m(x_i,x_j)\prod_{i<j}^{n-k<i}\omega_\sigma^m(x_i,x_j).$$

Here we set $\omega_\sigma^m(x_i,x_j)=\omega^m(x_i,x_j)$ if $\sigma^{-1}(i)<\sigma^{-1}(j)$ and $\omega_\sigma^m(x_i,x_j)=\omega^m(x_j,x_i)$ otherwise.

If $k=1$, then $s=n$ in both sums and the map $(\sigma,l)\mapsto(\sigma',l)$ with $\sigma'(i):=\sigma(i+1)$ establishes a bijection between the equal summands in $A_1$ and $A_2$, so that $A_1-A_2=0$.

For $k>1$, there is no such bijection. Instead, we show $A_1=0$ (equality $A_2=0$ is analogous). Let us group the summands in $A_1$ according to $s$, $\sigma(n-l)$ and also the ordering of $\{x_1,\ldots,x_{n-k}\}$ and $\{x_{n-k+1},\ldots,x_n\}$, which are given by elements $\sigma_1\in\mathfrak{S}_{n-k}$ and $\sigma_2\in\mathfrak{S}_k$. Define

$$\omega^m_{\sigma_1,\sigma_2}(x_1;\ldots;x_n):=\prod_{i<j}^{j\leq n-k}\omega^m_{\sigma_1}(x_i,x_j)\cdot\prod_{i<j}^{n-k<i}\omega^m_{\sigma_2}(x_i,x_j).$$

Then $A_1$ can be written in the form

$$A_1=\frac{1}{n!}\sum_{t\leq n-k}\sum_{\sigma_1\in\mathfrak{S}_{n-k}}\sum_{\sigma_2\in\mathfrak{S}_k}A_{t,\sigma_1,\sigma_2}\frac{x_{\sigma_2(1)}}{x_t}\omega^m_{\sigma_1,\sigma_2}(x_1;\ldots;x_n),\ A_{t,\sigma_1,\sigma_2}\in\mathbb{Z}.$$

We claim that all $A_{t,\sigma_1,\sigma_2}$ are zero. As an example, we compute $A_{t,1_{n-k},1_k}$:

$$A_{t,1_{n-k},1_k}=\sum_{l=n-k-t}^{n-t-1}(-1)^l\binom{n-2}{l}\binom{l}{n-k-t}\binom{n-l-2}{t-1}=$$

$$\frac{(-1)^{n-k-t}(n-2)!}{(t-1)!(k-1)!(n-k-t)!}(1-1)^{k-1}=0\text{ as }k>1.$$

Analogously $A_{t,\sigma_1,\sigma_2}=0$ for any $t,\sigma_1,\sigma_2$. Hence, $A_1=0$ and the result follows. □

*Proof of Lemma 5.7.7.*

The elements $L_j^m$ correspond to nonzero multiples of $\theta_{j,0}$ via $S^m\simeq\widetilde{\mathcal{E}}^+$. An algebraic independence of $\{\theta_{j,0}\}_{j>0}$ follows from an analogue of Proposition 5.1.3(b) applied to $\ddot{\mathrm{U}}_{q_1,q_2,q_3}$. □

### 5.7.3 The shuffle algebra $S^a$

We introduce an analogous additive shuffle algebra $S^a$.

Let us consider a $\mathbb{Z}_+$-graded $\mathbb{C}$-vector space $\mathbb{S}^a = \bigoplus_{n\geq 0} \mathbb{S}^a_n$, where $\mathbb{S}^a_n$ consists of rational functions $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)^2}$ with $f \in \mathbb{C}[x_1,\ldots,x_n]^{\mathfrak{S}_n}$. Define the star-product $\overset{a}{\star} : \mathbb{S}^a_i \times \mathbb{S}^a_j \to \mathbb{S}^a_{i+j}$ by

$$(F \overset{a}{\star} G)(x_1,\ldots,x_{i+j}) := \mathrm{Sym}_{\mathfrak{S}_{n+m}} \left( F(x_1,\ldots,x_i)G(x_{i+1},\ldots,x_{i+j}) \prod_{k\leq i}^{l>i} \omega^a(x_l,x_k) \right)$$

with

$$\omega^a(x,y) := \frac{(x-y-h_1)(x-y-h_2)(x-y-h_3)}{(x-y)^3}.$$

This endows $\mathbb{S}^a$ with a structure of an associative unital $\mathbb{C}$-algebra.

We say that an element $\frac{f(x_1,\ldots,x_n)}{\Delta(x_1,\ldots,x_n)^2} \in \mathbb{S}^a$ satisfies the *wheel condition* if $f(x_1,\ldots,x_n) = 0$ for any $\{x_1,\ldots,x_n\} \subset \mathbb{C}$ such that $x_1 - x_2 = h_i, x_2 - x_3 = h_j, i \neq j$. Let $S^a \subset \mathbb{S}^a$ be a $\mathbb{Z}_+$-graded subspace, consisting of all such elements. The subspace $S^a$ is closed with respect to $\overset{a}{\star}$.

**Definition 5.7.2.** The algebra $(S^a, \overset{a}{\star})$ is called the *small additive shuffle algebra.*

The following result is proved analogously to Theorem 5.7.1.

**Theorem 5.7.8.** *For generic* $h_1, h_2, h_3$ $(ah_1 + bh_2 + ch_3 = 0 \Longleftrightarrow a = b = c)$*, the map* $e_i \mapsto x_1^i$ *extends to an isomorphism* $\ddot{Y}^+_{h_1,h_2,h_3}(\mathfrak{gl}_1) \xrightarrow{\sim} S^a$*. In particular,* $S^a$ *is generated by* $S^a_1$*.*

### 5.7.4 Commutative subalgebra $\mathcal{A}^a \subset S^a$

We construct an *additive version* of $\mathcal{A}^m$.

Let us introduce a $\mathbb{Z}_+$-graded subspace $\mathcal{A}^a = \bigoplus_{n\geq 0} \mathcal{A}^a_n$ of $S^a$. Its degree $n$ component $\mathcal{A}^a_n$ consists of those $F \in S^a_n$ such that the limit

$$\partial^{(\infty,k)} F := \lim_{\xi\to\infty} F(x_1,\ldots,x_{n-k},x_{n-k+1}+\xi,\ldots,x_n+\xi)$$

exists for every $1 \leq k \leq n$. The following is an *additive counterpart* of Theorem 5.7.4:

**Theorem 5.7.9.** *We have:*

*(a) The subspace $\mathcal{A}^a \subset S^a$ is $\overset{a}{\star}$-commutative.*

*(b) $\mathcal{A}^a$ is $\overset{a}{\star}$-closed and it is a polynomial algebra in $\{K_j^a\}_{j\geq 1}$ with $K_j^a \in S_j^a$ defined by:*

$$K_1^a(x_1) = x_1^0,\ K_2^a(x_1, x_2) = \frac{(x_1 - x_2 - h_1)(x_2 - x_1 - h_1)}{(x_1 - x_2)^2},\ K_n^a(x_1, \dots, x_n) = \prod_{i<j} K_2^a(x_i, x_j).$$

Analogously to Proposition 5.7.5, the commutative subalgebra $\mathcal{A}^a$ admits an alternative set of generators expressed via $S_1^a$. Define $[P, Q]_a := P \overset{a}{\star} Q - Q \overset{a}{\star} P$ for $P, Q \in S^a$.

**Proposition 5.7.10.** *The algebra $\mathcal{A}^a$ is a polynomial algebra in the generators $\{L_j^a\}_{j\geq 1}$ defined by*

$$L_1^a(x_1) = x_1^0 \ \text{ and } \ L_j^a = \underbrace{[x^0, [x^0, \dots, [x^0, x^{j-1}]_a \cdots]_a]_a}_{j \text{ factors}} \in S_j^a \ \text{ for } j \geq 2.$$

## 5.8 The horizontal realization of $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$

The goal of this section is to introduce the "horizontal realization" of the algebra $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. This allows to define the tensor product structure on the whole $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-category $\mathcal{O}$. It also provides a natural framework for the generalization of [FT1, Section 7] to K-theory/cohomology of $M(r,n)$. We prove that the natural vectors $v_r^K, v_r^H$ in the appropriate completions of the modules $M^r, V^r$ are eigenvectors with respect to a particular family of operators.

### 5.8.1 The horizontal realization via $\widetilde{\mathcal{E}}$

We introduce a new realization of $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$.

Recall the distinguished collection of elements $\{u_{\mathbf{x}}, \kappa_{\mathbf{x}}\} \subset \ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ from Theorem 5.6.4. Note that there is a natural $\mathrm{SL}_2(\mathbb{Z})$-action on $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(\psi_0^+ - 1) \simeq \widetilde{\mathcal{E}}/(\kappa_{\mathbf{x}} - 1)_{\mathbf{x}\in\mathbb{Z}^2}$. In particular, we have a natural automorphism of $\widetilde{\mathcal{E}}/(\kappa_{\mathbf{x}} - 1)_{\mathbf{x}\in\mathbb{Z}^2}$ induced by $u_{k,l} \mapsto u_{-l,k}$. Though there is no such automorphism for $\widetilde{\mathcal{E}}/(\kappa_{0,1} - 1)$, we still have a nice presentation of this algebra in terms of the generators $\{u_{i,\pm 1}, u_{j,0}, \kappa_{1,0}\}$ rather than $\{u_{\pm 1,i}, u_{0,j}, \kappa_{1,0}\}$.

To formulate the main result, we need to introduce a modification of the algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, which we denote by $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$. The algebra $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$, is an associative unital $\mathbb{C}$-algebra generated by $\{\widetilde{e}_i, \widetilde{f}_i, \widetilde{\psi}_j^{\pm}, \gamma^{\pm 1/2} | i \in \mathbb{Z}, j > 0\}$ with the following defining relations:

$$\widetilde{\psi}^{\pm}(z)\widetilde{\psi}^{\pm}(w) = \widetilde{\psi}^{\pm}(w)\widetilde{\psi}^{\pm}(z),\ g(\gamma^{-1}w/z)\widetilde{\psi}^{+}(z)\widetilde{\psi}^{-}(w) = g(\gamma w/z)\widetilde{\psi}^{-}(w)\widetilde{\psi}^{+}(z), \tag{TT0}$$

$$\widetilde{e}(z)\widetilde{e}(w) = g(z/w)\widetilde{e}(w)\widetilde{e}(z), \tag{TT1}$$

$$\widetilde{f}(z)\widetilde{f}(w) = g(w/z)\widetilde{f}(w)\widetilde{f}(z), \tag{TT2}$$

$$(1-q_1)(1-q_2)(1-q_3)\cdot[\widetilde{e}(z), \widetilde{f}(w)] = \delta(\gamma^{-1}z/w)\widetilde{\psi}^{+}(\gamma^{1/2}w) - \delta(\gamma z/w)\widetilde{\psi}^{-}(\gamma^{-1/2}w), \tag{TT3}$$

$$\widetilde{\psi}^{\pm}(z)\widetilde{e}(w) = g(\gamma^{\pm 1/2}z/w)\widetilde{e}(w)\widetilde{\psi}^{\pm}(z), \tag{TT4}$$

$$\widetilde{\psi}^{\pm}(z)\widetilde{f}(w) = g(\gamma^{\pm 1/2}w/z)\widetilde{f}(w)\widetilde{\psi}^{\pm}(z), \tag{TT5}$$

$$\mathrm{Sym}_{\mathfrak{S}_3}[\widetilde{e}_{i_1}, [\widetilde{e}_{i_2+1}, \widetilde{e}_{i_3-1}]] = 0, \ \ \mathrm{Sym}_{\mathfrak{S}_3}[\widetilde{f}_{i_1}, [\widetilde{f}_{i_2+1}, \widetilde{f}_{i_3-1}]] = 0, \tag{TT6}$$

where $g(y) := \frac{(1-q_1y)(1-q_2y)(1-q_3y)}{(1-q_1^{-1}y)(1-q_2^{-1}y)(1-q_3^{-1}y)}$. Note that $g(y) = g(y^{-1})^{-1}$.

The following result is analogous to Theorem 5.6.4:

**Theorem 5.8.1.** *There is an isomorphism* $\Xi_h : \widetilde{\mathcal{E}}[\kappa_{1,0}^{\pm 1/2}]/(\kappa_{0,1}-1) \xrightarrow{\sim} \ddot{\mathbf{U}}_{q_1,q_2,q_3}$ *defined on the generators by*

$$\kappa_{1,0}^{\pm 1/2} \mapsto \gamma^{\pm 1/2}, \ \theta_{\mp j,0} \mapsto \widetilde{\psi}_j^{\pm}, \ u_{-i,1} \mapsto \gamma^{|i|/2}\widetilde{e}_i, \ u_{-i,-1} \mapsto \gamma^{-|i|/2}\widetilde{f}_i, \ i \in \mathbb{Z}, j > 0.$$

Analogously to $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, there is a similar coproduct $\Delta_h$ on the algebra $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$:

$$\Delta_h(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}, \ \Delta_h(\widetilde{\psi}^{\pm}(z)) = \widetilde{\psi}^{\pm}(\gamma_{(2)}^{\pm 1/2}z) \otimes \widetilde{\psi}^{\pm}(\gamma_{(1)}^{\mp 1/2}z),$$

$$\Delta_h(\widetilde{e}(z)) = \widetilde{e}(z)\otimes 1 + \widetilde{\psi}^{-}(\gamma_{(1)}^{1/2}z)\otimes\widetilde{e}(\gamma_{(1)}z), \ \Delta_h(\widetilde{f}(z)) = 1\otimes\widetilde{f}(z) + \widetilde{f}(\gamma_{(2)}z)\otimes\widetilde{\psi}^{+}(\gamma_{(2)}^{1/2}z),$$

where $\gamma_{(1)}^{\pm 1/2} = \gamma^{\pm 1/2} \otimes 1, \ \gamma_{(2)}^{\pm 1/2} = 1 \otimes \gamma^{\pm 1/2}$ (see [DI]).

According to Theorems 5.6.4 and 5.8.1, the algebras $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)[(\psi_0^+)^{\pm 1/2}]$ and $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$ are isomorphic. In particular, we view $\Delta_h$ as a "horizontal coproduct" on the algebra $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. It provides a tensor product structure on the category $\mathcal{O}$ from Section 4.6. For two $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-modules $L_1, L_2$ we denote the corresponding tensor product by $L_1 \underset{h}{\otimes} L_2$.

### 5.8.2 Modules $V(u), F(u)$ in the horizontal realization

Let us describe the action of the currents $\widetilde{e}(z), \widetilde{f}(z), \widetilde{\psi}^{\pm}(z)$ on the Fock module $F(u)$. Consider the Heisenberg Lie algebra $\mathfrak{h}$ over $\mathbb{C}$ with the generators $\{a_n\}_{n\in\mathbb{Z}}$ and the relations

$$[a_m, a_n] = m(1-q_1^{|m|})/(1-q_2^{-|m|})\delta_{m,-n}a_0.$$

Let $\mathfrak{h}^{\geq 0}$ be the subalgebra generated by $\{a_n\}_{n\geq 0}$ and $\mathcal{F} := \mathrm{Ind}_{\mathfrak{h}^{\geq 0}}^{\mathfrak{h}}\mathbb{C}$ be the Fock $\mathfrak{h}$-representation.

Since the elements $\{\theta_{j,0}\} \subset \widetilde{\mathcal{E}}$ form a Heisenberg Lie algebra and the highest weight vector $|\emptyset\rangle \in F(u)$ is annihilated by $\{\theta_{j,0}\}_{j<0}$, we see that $F(u) \simeq \mathcal{F}$ as modules over the subalgebra generated by $\widetilde{\psi}_j^{\pm}$. Together with the relations (TT4,TT5), we get the following result:

**Proposition 5.8.2.** *Identifying $F(u) \simeq \mathcal{F}$, the action of $\widetilde{e}(z), \widetilde{f}(z), \widetilde{\psi}^{\pm}(z)$ is given by*

$$\rho_c(\gamma^{\pm 1/2}) = q_3^{\pm 1/4},\ \rho_c(\widetilde{\psi}^{\pm}(z)) = \exp\left(\mp \sum_{n>0} \frac{1-q_2^{\mp n}}{n}(1-q_3^n)q_3^{-n/4}a_{\pm n}z^{\mp n}\right),$$

$$\rho_c(\widetilde{e}(z)) = c\exp\left(\sum_{n>0}\frac{1-q_2^n}{n}a_{-n}z^n\right)\exp\left(-\sum_{n>0}\frac{1-q_2^{-n}}{n}a_n z^{-n}\right),$$

$$\rho_c(\widetilde{f}(z)) = c^{-1}\exp\left(-\sum_{n>0}\frac{1-q_2^n}{n}q_3^{n/2}a_{-n}z^n\right)\exp\left(\sum_{n>0}\frac{1-q_2^{-n}}{n}q_3^{n/2}a_n z^{-n}\right),$$

*where $c = (1-q_3)u$.*

These $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$-representations $\{\rho_c\}$ were first considered in [FHHSY]. As we just explained, they correspond to the $\ddot{U}'_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-modules $\{F(u)\}$ under an identification of those two algebras. Similarly one checks that action of currents $\widetilde{e}(z), \widetilde{f}(z), \widetilde{\psi}^{\pm}(z)$ on the vector representation $V(u)$ coincides with the formulas for the $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$-representations $\pi_c$ considered in [FHHSY].

### 5.8.3 The matrix coefficient realization of $\mathcal{A}^m$

We provide a new interpretation of $\mathcal{A}^m$.

For a $\ddot{\mathbf{U}}_{q_1,q_2,q_3}$-module $L$ and two vectors $v_1, v_2 \in L$, we define

$$m_{v_1,v_2}(z_1,\ldots,z_n) := \langle v_1|\widetilde{e}(z_1)\ldots\widetilde{e}(z_n)|v_2\rangle \cdot \prod_{i<j}\omega^m(z_i,z_j) \in \mathbb{C}[[z_1^{\pm 1},\cdots,z_n^{\pm 1}]].$$

The relation (TT1) implies that $m_{v_1,v_2}(z_1,\ldots,z_n)$ is $\mathfrak{S}_n$–symmetric.

**Proposition 5.8.3.** *For $L = \rho_c$ and $v_1 = v_2 = 1$, we have*

$$m_{1,1}(z_1, \ldots, z_n) = (-q_3)^{-n(n-1)/2} c^n \prod_{i<j} \frac{(z_i - q_3 z_j)(z_j - q_3 z_i)}{(z_i - z_j)^2}.$$

*Proof.*

For $n > 0$, we have

$$\exp(u \cdot a_n) \exp(v \cdot a_{-n}) = \exp(v \cdot a_{-n}) \exp(u \cdot a_n) \exp(uv \cdot n(1 - q_1^n)/(1 - q_2^{-n})).$$

Therefore

$$\rho_c(\widetilde{e}(z_i))\rho_c(\widetilde{e}(z_j)) = :\rho_c(\widetilde{e}(z_i))\rho_c(\widetilde{e}(z_j)): \cdot \prod_{n>0} \exp\left(-\frac{(1-q_1^n)(1-q_2^n)}{n}(z_j/z_i)^n\right).$$

It remains to use the equality $\prod_{n>0} \exp\left(-\frac{(1-q_1^n)(1-q_2^n)}{n}(z_j/z_i)^n\right) = \frac{(z_i - z_j)(z_i - q_1 q_2 z_j)}{(z_i - q_1 z_j)(z_i - q_2 z_j)}$. □

In the case of $\rho_{c_1,\ldots,c_n} := \rho_{c_1} \underset{h}{\otimes} \cdots \underset{h}{\otimes} \rho_{c_m}$ we have the following result:

**Proposition 5.8.4.** *Set $\bar{\mathbf{1}} := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \in \rho_{c_1,\ldots,c_n}$. Then $m_{\bar{\mathbf{1}},\bar{\mathbf{1}}}(z_1, \ldots, z_n) \in \mathcal{A}^m$.*

*Proof.*

Combining the formulas of Proposition 5.8.2 with formulas for $\Delta_h$, we get

$$m_{\bar{\mathbf{1}},\bar{\mathbf{1}}}(z_1, \ldots, z_n) = \sum_f c_{f(1)} \cdots c_{f(n)} \prod_{i<j} \omega^m(z_i, z_j) \prod_{i<j} W_f(z_i, z_j),$$

where the sum is over all maps $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ and $W_f(z_i, z_j)$ is 1 (if $f(i) > f(j)$), is $\frac{(z_i - z_j)(z_i - q_1 q_2 z_j)}{(z_i - q_1 z_j)(z_i - q_2 z_j)}$ (if $f(i) = f(j)$) and is $g(z_i/z_j)$ (if $f(i) < f(j)$). The claim follows. □

This realization of $\mathcal{A}^m$ will be important in [FT2].

*Remark* 5.8.1. Same construction applied to $\pi_{c_1} \underset{h}{\otimes} \cdots \underset{h}{\otimes} \pi_{c_n}$ realizes the corresponding matrix coefficients as the classical Macdonald difference operators, see [FHHSY, Proposition A.10].

### 5.8.4 The Whittaker vector in the K-theory case

Let $\widehat{M^r}$ be the completion of $M^r$ with respect to a natural grading. Consider the *Whittaker vector* $v_r^K := \sum_{n\geq 0}\left[\mathcal{O}_{M(r,n)}\right] \in \widehat{M^r}$. To state our main result, we introduce a family of the elements $\{K_i^{(m;j)}\}_{i>0} \in S_i^m$ by

$$K_i^{(m;j)}(x_1,\ldots,x_i) := K_i^m(x_1,\ldots,x_i)x_1^j\cdots x_i^j = \prod_{a<b}\frac{(x_a-q_1x_b)(x_b-q_1x_a)}{(x_a-x_b)^2}\prod_s x_s^j.$$

Let $\{K_i^{(m;j)}\}_{i<0}$ be analogous elements in the opposite algebra $(S^m)^{\mathrm{opp}}$. The name "Whittaker" is motivated by the following result:

**Theorem 5.8.5.** *The vector $v_r^K$ is an eigenvector with respect to $\{K_{-n}^{(m;j)}|0\leq j\leq r, n>0\}$. More precisely: $K_{-n}^{(m;j)}(v_r^K) = C_{j,-n}\cdot v_r^K$, where*

$$C_{0,-n} = (-1)^{n(n+1)/2+nr-n}(t_1t_2\chi_1\cdots\chi_r)^n\frac{t_1^{n(n-1)/2}}{(1-t_1)^n(1-t_2)(1-t_2^2)\cdots(1-t_2^n)},$$

$$C_{1,-n} = \ldots = C_{r-1,-n} = 0,\ C_{r,-n} = \frac{(-t_1t_2)^{n(n+1)/2}}{(1-t_1)^n(1-t_2)(1-t_2^2)\ldots(1-t_2^n)}.$$

*Remark* 5.8.2. Proposition 5.7.5 implies that the subalgebra of $(S^m)^{\mathrm{opp}}$ generated by $\{K_{-n}^{(m;j)}\}_{0\leq j\leq r}^{n>0}$ corresponds to the subalgebra of $\ddot{U}^-_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ generated by

$$\{f_j,[f_{j+1},f_{j-1}],[f_{j+1},[f_j,f_{j-1}]],\ldots\}_{j=0}^r.$$

### 5.8.5 The Whittaker vector in the cohomology case

Let $\widehat{V^r}$ be the completion of $V^r$ with respect to a natural grading. Consider the *Whittaker vector* $v_r^H := \sum_{n\geq 0}[M(r,n)] \in \widehat{V^r}$. To state our main result, we introduce a family of the elements $\{K_i^{(a;j)}\}_{i>0}^{j\geq 0} \in S_i^a$ by

$$K_i^{(a;j)}(x_1,\ldots,x_i) := K_i^a(x_1,\ldots,x_i)x_1^j\cdots x_i^j = \prod_{a<b}\frac{(x_a-x_b-h_1)(x_b-x_a-h_1)}{(x_a-x_b)^2}\prod_s x_s^j.$$

Let $\{K_i^{(a;j)}\}_{i<0}$ be analogous elements in the opposite algebra $(S^a)^{\mathrm{opp}}$. The name "Whittaker" is motivated by the following result:

**Theorem 5.8.6.** *The vector $v_r^H$ is an eigenvector with respect to $\{K_{-n}^{(a;j)}|0 \leq j \leq r, n > 0\}$. More precisely: $K_{-n}^{(a;j)}(v_r^H) = D_{j,-n} \cdot v_r^H$, where $D_{r,-n}$ is a degree $n$ polynomial in $x_a$ and*

$$D_{0,-n} = \ldots = D_{r-2,-n} = 0,\ D_{r-1,-n} = \frac{(-1)^{n(n+1)/2+nr-n}}{n! s_1^n s_2^n},\ D_{r,-1} = \frac{(-1)^{r+1}}{s_1 s_2} \sum_{a=1}^{r} x_a.$$

*Remark* 5.8.3. Proposition 5.7.10 implies that the subalgebra of $(S^a)^{\mathrm{opp}}$ generated by $\{K_{-n}^{(a;j)}\}_{0\leq j\leq r}^{n>0}$ corresponds to the subalgebra of $\ddot{Y}^{-}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ generated by

$$\{f_j, [f_j, f_{j+1}], [f_j, [f_j, f_{j+2}]], \ldots\}_{j=0}^{r}.$$

### 5.8.6 Sketch of the proof of Theorem 5.8.5

According to the fixed point formula, we have

$$v_r^K = \sum_{\bar{\lambda}} a_{\bar{\lambda}} \cdot [\bar{\lambda}],\ a_{\bar{\lambda}} = \prod_{w \in T_{\bar{\lambda}} M(r,|\bar{\lambda}|)} (1-w)^{-1}.$$

Hence, we need to show that for any $r$-partition $\bar{\lambda}$ the following equality holds:

$$C_{j,-n} = \sum_{\bar{\lambda}'} \frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}} \cdot K_{-n}^{(m;j)}{}_{|[\bar{\lambda}',\bar{\lambda}]}, \tag{5.16}$$

where the sum is over all $r$-partitions $\bar{\lambda}'$ such that $\bar{\lambda} \subset \bar{\lambda}'$ and $|\bar{\lambda}'| = |\bar{\lambda}| + n$.

For such a pair of $r$-partitions $(\bar{\lambda}, \bar{\lambda}')$, define a collection of positive integers

$$j_{1,1} \leq j_{1,2} \leq \ldots \leq j_{1,l_1},\ j_{2,1} \leq j_{2,2} \leq \ldots \leq j_{2,l_2},\ \ldots,\ j_{r,1} \leq j_{r,2} \leq \ldots \leq j_{r,l_r},\ \sum_{i=1}^{r} l_i = n \tag{5.17}$$

via the following equality:

$$\bar{\lambda}' = \bar{\lambda} + \square^1_{j_{1,1}} + \ldots + \square^1_{j_{1,l_1}} + \square^2_{j_{2,1}} + \ldots + \square^2_{j_{2,l_2}} + \ldots + \square^r_{j_{r,1}} + \ldots + \square^r_{j_{r,l_r}}.$$

We also introduce the sequence of $r$-partitions $\bar{\lambda} = \bar{\lambda}^{[0]} \subset \bar{\lambda}^{[1]} \subset \cdots \subset \bar{\lambda}^{[n]} = \bar{\lambda}'$, where

$\bar{\lambda}^{[r]}$ is obtained from $\bar{\lambda}$ by adding the first $q$ boxes from above. For $1 \le q \le n$, the $q$-th box from above has a form $\square^{s_q}_{j_{s_q}, i_q}$. We denote its character by $\chi(q)$.

For any $F \in (S^m_n)^{\mathrm{opp}}$, we get the following formula for the matrix coefficient $F_{|[\bar{\lambda}', \bar{\lambda}]}$:

$$F_{|[\bar{\lambda}', \bar{\lambda}]} = \frac{F(\chi(1), \dots, \chi(n))}{\prod_{a<b} \omega^m(\chi(a), \chi(b))} \cdot \prod_{q=1}^{n} f_{0|[\bar{\lambda}^{[q]}, \bar{\lambda}^{[q-1]}]}.$$

In particular, we have

$$K^{(m;j)}_{-n}{}_{|[\bar{\lambda}', \bar{\lambda}]} = \prod_{1 \le a < b \le n} \frac{(\chi(a) - \chi(b))(\chi(b) - t_1 \chi(a))}{(\chi(a) - t_2 \chi(b))(\chi(a) - t_3 \chi(b))} \cdot \prod_{q=1}^{n} \chi(q)^j \cdot \prod_{q=1}^{n} f_{0|[\bar{\lambda}^{[q]}, \bar{\lambda}^{[q-1]}]}.$$

As an immediate consequence of this formula, we get $K^{(m;j)}_{-n}{}_{|[\bar{\lambda}', \bar{\lambda}]} = 0$ if $\bar{\lambda}' \backslash \bar{\lambda}$ contains two boxes in the same row of its $i$-th component, $1 \le i \le r$. Therefore, the sum in (5.16) should be taken only over those collections $\{j_{1,1}, \dots, j_{r,l_r}\}$ from (5.17) which satisfy strict inequalities.

We also split $\frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}}$ into the product over consequent pairs: $\frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}} = \prod_{q=1}^{n} \frac{a_{\bar{\lambda}^{[q]}}}{a_{\bar{\lambda}^{[q-1]}}}$. According to the Bott-Lefschetz fixed point formula, we have

$$\frac{a_{\bar{\lambda}^{[q]}}}{a_{\bar{\lambda}^{[q-1]}}} \cdot f_{0|[\bar{\lambda}^{[q]}, \bar{\lambda}^{[q-1]}]} = e_{-r|[\bar{\lambda}^{[q-1]}, \bar{\lambda}^{[q]}]}.$$

For a pair of two $r$-partitions $(\bar{\mu}, \bar{\mu}')$ such that $\bar{\mu}' = \bar{\mu} + \square^l_j$, the matrix coefficient $e_{-r|[\bar{\mu}, \bar{\mu}']}$ was computed in Lemma 5.3.3(a):

$$e_{-r|[\bar{\mu}, \bar{\mu}']} = -\frac{t_1^{1-r} t_2}{1 - t_1} \prod_{a=1}^{r} \frac{1}{\chi^{(l)}_j - t_1^{-1} t_2^{L_a} \chi_a^{-1}} \prod_{(a,k) \ne (l,j)}^{k \le L_a} \frac{\chi^{(l)}_j - t_2 \chi^{(a)}_k}{\chi^{(l)}_j - \chi^{(a)}_k},$$

where $L_a$ is chosen to satisfy $L_a \ge \mu^{a*}_1 + 1$.

Combining these formulas together, we finally get

$$\frac{a_{\bar{\lambda}'}}{a_{\bar{\lambda}}} \cdot K^{(m;j)}_{-n}{}_{|[\bar{\lambda}', \bar{\lambda}]} = \prod_{1 \le q \le n} \left\{ \frac{(-t_1 t_2)^q}{1 - t_1} \cdot \prod_{a=1}^{r} \frac{1}{\chi(q) - t_2^{L_a} \chi_a^{-1}} \cdot \prod \frac{\chi(q) - t_1 t_2 \chi^{(a)}_k}{\chi(q) - t_1 \chi^{(a)}_k} \cdot \chi(q)^j \right\},$$

the second product is over pairs $(a, k) \notin \{(1, j_{1,1}), \dots, (r, j_{r,l_r})\}$, $k \le L_a$ with $L_a \ge$

$\lambda_1^{a*} + n$.

Let us denote the RHS of this equality by $C_{\mathbf{j}}$, where $\mathbf{j} = \{j_{1,1}, \ldots, j_{r,l_r}\}$ is determined by (5.17). Note that $C_{\mathbf{j}} = 0$ if the corresponding $r$-partition $\bar{\lambda}'$ fails to be a collection of $r$ Young diagrams. Hence, (5.16) reduces to $C_{j,-n} = \sum C_{\mathbf{j}}$, the sum over all $\mathbf{j}$ from (5.17) with strict inequalities.

It is easy to check that the sum $\sum C_{\mathbf{j}}$ has no poles for $j \geq 0$. Together with the degree computation, we see that it is independent of $\bar{\lambda}$ for $0 \leq j \leq r$. Thus $v_r^K$ is indeed an eigenvector with respect to $K_{-n}^{(m;j)}$. To compute its eigenvalue, we evaluate $\sum C_{\mathbf{j}}$ at $\bar{\lambda} = \bar{\emptyset}$. This sum is actually over all partitions $(l_1, \ldots, l_r)$ of $n$ with $j_{a,b} = b$. The total sum equals

$$\frac{(-t_1t_2)^{n(n+1)/2}}{(1-t_1)^n} \sum_{l_1+\ldots+l_r=n} \prod_{a,b=1}^{r} \frac{1}{(\chi_b^{-1} - t_2^{l_a}\chi_a^{-1}) \cdots (t_2^{l_b-1}\chi_b^{-1} - t_2^{l_a}\chi_a^{-1})} \prod_{b=1}^{r} (t_2^{l_b(l_b-1)/2}\chi_b^{-l_b})^j =$$

$$\frac{(-t_1t_2)^{\frac{n(n+1)}{2}}(\chi_1\cdots\chi_r)^n}{(1-t_1)^n} \sum_{l_1+\ldots+l_r=n} \prod_{a,b=1}^{r} \frac{1}{(\chi_a - t_2^{l_a}\chi_b) \cdots (\chi_a - t_2^{l_a-l_b+1}\chi_b)} \prod_{b=1}^{r} (t_2^{-\frac{l_b(l_b-1)}{2}}\chi_b^{l_b})^{r-j}.$$

It is straightforward to check that for $0 \leq j \leq r-1$, the sum from the above equality is a rational function in $\chi_a$ with no poles. Together with the degree estimate, we see that it is independent of $\{\chi_a\}$. To compute this constant we let $\chi_1 \to \infty$. Then the only nonzero contribution comes from the collection $(l_1, l_2, \ldots, l_r) = (n, 0, \ldots, 0)$ and the result equals $C_{j,-n}$.

For $j = r$, the whole expression above has no poles and is of total degree $\leq 0$; therefore, it is independent of $\chi_a$. To compute this constant we let $\chi_1 \to \infty$. The only nonzero contributions come from those $(l_1, \ldots, l_r)$ with $l_1 = 0$. For those we let $\chi_2 \to \infty$, etc. The result follows by straightforward computations.

### 5.8.7 Sketch of the proof of Theorem 5.8.6

According to the fixed point formula, we have

$$v_r^H = \sum_{\bar{\lambda}} b_{\bar{\lambda}} \cdot [\bar{\lambda}], \; b_{\bar{\lambda}} = \prod_{w \in T_{\bar{\lambda}} M(r,|\bar{\lambda}|)} w^{-1}.$$

Hence, we need to show that for any $r$-partition $\bar{\lambda}$ the following equality holds:

$$D_{j,-n} = \sum_{\bar{\lambda}'} \frac{b_{\bar{\lambda}'}}{b_{\bar{\lambda}}} \cdot K^{(a;j)}_{-n}{}_{|[\bar{\lambda}',\bar{\lambda}]}, \tag{5.18}$$

where the sum is over all $r$-partitions $\bar{\lambda}'$ such that $\bar{\lambda} \subset \bar{\lambda}'$ and $|\bar{\lambda}'| = |\bar{\lambda}| + n$.

Analogously to the K-theoretical case, we have:

$$\frac{b_{\bar{\lambda}'}}{b_{\bar{\lambda}}} \cdot K^{(a;j)}_{-n}{}_{|[\bar{\lambda}',\bar{\lambda}]} = \prod_{1\le q\le n} \left\{ \frac{(-1)^{q+r}\chi(q)^j}{s_1} \cdot \prod_{a=1}^{r} \frac{1}{\chi(q) - L_a s_2 + x_a} \prod \frac{\chi(q) - x_k^{(a)} - s_1 - s_2}{\chi(q) - x_k^{(a)} - s_1} \right\},$$

the second product is over pairs $(a,k) \notin \{(1, j_{1,1}), \ldots, (r, j_{r,l_r})\}$, $k \le L_a$ with $L_a \ge \lambda_1^{a*} + n$.

Let us denote the RHS of this equality by $D_{\mathrm{j}}$. Then $\sum_{\mathrm{j}} D_{\mathrm{j}}$ is a rational function in $x_k^{(a)}$ with no poles for $j \ge 0$. The degree estimate implies that for $j \le r$ it is independent of $x_k^{(a)}$. Thus $v_r^H$ is indeed an eigenvector with respect to $\{K^{(a;j)}_{-n}\}^{n>0}_{0\le j\le r}$. To compute its eigenvalue, we evaluate at $\bar{\lambda} = \bar{\emptyset}$. This sum equals

$$\frac{(-1)^{n(n+1)/2+rn}}{s_1^n} \sum_{l_1+\ldots+l_r=n} \left\{ \prod_{a,b=1}^{r} \prod_{k=1}^{l_b} (x_a - x_b - (l_a - k + 1)s_2)^{-1} \prod_{b=1}^{r} \prod_{k=1}^{l_b} ((k-1)s_2 - x_b)^j \right\}.$$

It is straightforward to check that this sum is a rational function in $x_a$ with no poles. Together with the degree estimate for $j \le r-1$, we see that it is independent of $x_a$. To compute this constant we let $x_1 \to \infty$. If $j \le r-2$, then all summands tend to 0. For $j = r-1$ the only nonzero contribution (equal to $D_{r-1,-n}$) comes from the collection $(l_1, l_2, \ldots, l_r) = (n, 0, \ldots, 0)$.

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# Appendix A

# Future work

The goal of this appendix is to outline the generalization of the results from Chapter 5 to the case of the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ and the affine Yangian $\ddot{Y}_{h,u}(\mathfrak{sl}_n)$. We will restrict only to the case of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ and present only the first steps, while the full version will appear in [FT2].

## A.1 Quantum toroidal $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ and the big shuffle algebras

- **The quantum toroidal of $\mathfrak{sl}_n$**

Here we recall the toroidal algebras following [GKV, VV].

Let $q, d \in \mathbb{C}^*$ be two parameters. We set $[n] := \{0, 1, \ldots, n-1\}$, $[n]^\times := [n]\backslash\{0\}$, the former viewed as a set of mod $n$ residues. Let $g_m(z) := \frac{q^m z-1}{z-q^m}$. Define $\{a_{i,j}, m_{i,j} | i, j \in [n]\}$ by

$$a_{i,i} = 2,\ a_{i,i\pm1} = -1,\ m_{i,i\pm1} = \mp1,\ \text{and } a_{i,j} = m_{i,j} = 0 \text{ otherwise.}$$

The quantum toroidal algebra of $\mathfrak{sl}_n$, denoted $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, is the unital associative algebra, generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm1/2} | i \in [n], k \in \mathbb{Z}\}$ with the following defining relations:

$$\psi_{i,0}\cdot\psi_{i,0}^{-1}=\psi_{i,0}^{-1}\cdot\psi_{i,0}=1,\ \ [\psi_i^{\pm}(z),\psi_j^{\pm}(w)]=0,\ \gamma^{\pm 1/2}-\text{central},\tag{T0}$$

$$g_{a_{i,j}}(\gamma^{-1}d^{m_{i,j}}z/w)\psi_i^+(z)\psi_j^-(w)=g_{a_{i,j}}(\gamma d^{m_{i,j}}z/w)\psi_j^-(w)\psi_i^+(z),\tag{T1}$$

$$e_i(z)e_j(w)=g_{a_{i,j}}(d^{m_{i,j}}z/w)e_j(w)e_i(z),\tag{T2}$$

$$f_i(z)f_j(w)=g_{-a_{i,j}}(d^{m_{i,j}}z/w)f_j(w)f_i(z),\tag{T3}$$

$$(q-q^{-1})[e_i(z),f_j(w)]=\delta_{i,j}\left(\delta(\gamma w/z)\psi_i^+(\gamma^{1/2}w)-\delta(\gamma z/w)\psi_i^-(\gamma^{1/2}z)\right),\tag{T4}$$

$$\psi_i^{\pm}(z)e_j(w)=g_{a_{i,j}}(\gamma^{\pm 1/2}d^{m_{i,j}}z/w)e_j(w)\psi_i^{\pm}(z),\tag{T5}$$

$$\psi_i^{\pm}(z)f_j(w)=g_{-a_{i,j}}(\gamma^{\mp 1/2}d^{m_{i,j}}z/w)f_j(w)\psi_i^{\pm}(z),\tag{T6}$$

$$\{e_i(z_1)e_i(z_2)e_{i\pm1}(z_3)-(q+q^{-1})e_i(z_1)e_{i\pm1}(z_3)e_i(z_2)+e_{i\pm1}(z_3)e_i(z_1)e_i(z_2)\}+\{z_1\leftrightarrow z_2\}=0,\tag{T7.1}$$

$$\{f_i(z_1)f_i(z_2)f_{i\pm1}(z_3)-(q+q^{-1})f_i(z_1)f_{i\pm1}(z_3)f_i(z_2)+f_{i\pm1}(z_3)f_i(z_1)f_i(z_2)\}+\{z_1\leftrightarrow z_2\}=0,\tag{T7.2}$$

where the generating series are defined as follows:

$$e_i(z):=\sum_{k=-\infty}^{\infty}e_{i,k}z^{-k},\ \ f_i(z):=\sum_{k=-\infty}^{\infty}f_{i,k}z^{-k},\ \ \psi_i^{\pm}(z):=\psi_{i,0}^{\pm1}+\sum_{\pm j>0}\psi_{i,j}z^{\mp j}.$$

It will be convenient to work with another "Cartan" generators $\{h_{i,j}\}$, instead of $\{\psi_{i,k}\}$. Define $h_{i,\pm k}\in\mathbb{C}[\psi_{i,0}^{\pm1},\psi_{i,\pm1},\psi_{i,\pm2},\ldots]$ via

$$\psi_i^{\pm}(z)=\psi_{i,0}^{\pm1}\exp\left(\pm(q-q^{-1})\sum_{k>0}h_{i,\pm k}z^{\mp k}\right).$$

Then the relations (T5,T6) are equivalent to the following:

$$\psi_{i,0}e_{j,l}=q^{a_{i,j}}e_{j,l}\psi_{i,0},\ [h_{i,k},e_{j,l}]=d^{-km_{i,j}}\gamma^{-|k|/2}\frac{[ka_{i,j}]}{k}e_{j,l+k},\tag{T5$'$}$$

$$\psi_{i,0}f_{j,l}=q^{-a_{i,j}}f_{j,l}\psi_{i,0},\ [h_{i,k},e_{j,l}]=-d^{-km_{i,j}}\gamma^{|k|/2}\frac{[ka_{i,j}]}{k}f_{j,l+k},\tag{T6$'$}$$

where $[m] := (q^m - q^{-m})/(q - q^{-1})$.

Let $\ddot{U}^-, \ddot{U}^0, \ddot{U}^+$ be the subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,j}\}$, $\{f_{i,j}\}$, and $\{\psi_{i,j}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2}\}$.

**Proposition A.1.1.** *[H](Triangular decomposition) The multiplication map $m$ : $\ddot{U}^- \otimes \ddot{U}^0 \otimes \ddot{U}^+ \to \ddot{U}_{q,d}(\mathfrak{sl}_n)$ is an isomorphism of vector spaces.*

Finally, $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ is $\mathbb{Z}^n$-graded by $\deg(e_{i,k}) = 1_i, \deg(f_{i,k}) = -1_i, \deg(\ddot{U}^0) = 0$.

• **Horizontal and vertical $\dot{U}_q(\mathfrak{gl}_n)$**

We recall two important subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

Following [VV], we introduce the *vertical* and *horizontal* copies of the quantum affine algebra of $\mathfrak{sl}_n$, denoted $\dot{U}_q(\mathfrak{sl}_n)$, inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. Consider a subalgebra $\ddot{U}^{(1)}$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,k}, f_{i,k}, \psi_{i,k}, \psi_{i,0}^{-1}, \gamma^{\pm 1/2} | i \in [n]^\times\}$. This algebra is isomorphic to $\dot{U}_q(\mathfrak{sl}_n)$, realized via the "new Drinfeld presentation". Let $\ddot{U}^{(2)}$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ generated by $\{e_{i,0}, f_{i,0}, \psi_{i,0}^{\pm 1} | i \in [n]\}$. This algebra is also isomorphic to $\dot{U}_q(\mathfrak{sl}_n)$, realized via the classical Drinfeld-Jimbo presentation.

We will need a slight upgrade of this construction, which provides two copies of the quantum affine algebra of $\mathfrak{gl}_n$, rather than $\mathfrak{sl}_n$, inside $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. For every $r \neq 0$, choose $\{c_{i,r} | i \in [n]\}$ to be a nontrivial solution of the following system of linear equations[1]:

$$\sum_{i=0}^{n-1} c_{i,r}[r a_{i,j}] d^{-r m_{i,j}} = 0, \ j \in [n]^\times.$$

Let $\mathfrak{h}^{(1)}$ be the subspace of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ spanned by $h_r^{(1)} := \sum_{i=0}^{n-1} c_{i,r} h_{i,r}, \ r \neq 0$. Note that $\mathfrak{h}^{(1)}$ is well-defined and commutes with $\ddot{U}^{(1)}$, due to (T5′, T6′). Moreover, $\mathfrak{h}^{(1)}$ is isomorphic to the Heisenberg Lie algebra[2]. Let $\ddot{U}^v$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, generated by $\ddot{U}^{(1)}$ and $\mathfrak{h}^{(1)}$. The above discussions imply that $\ddot{U}^v \simeq \dot{U}_q(\mathfrak{gl}_n)$, the quantum affine algebra of $\mathfrak{gl}_n$.

Our next goal is to provide a *horizontal* copy of $\dot{U}_q(\mathfrak{gl}_n)$, containing $\ddot{U}^{(2)}$. We are not aware of the explicit formulas. Instead, we will use the following beautiful result, which was communicated to us by Boris Feigin:

[1] It is easy to see that the space of solutions of this system is 1-dimensional for $q \neq \sqrt{1}$.

[2] With the central charge being a function in $\gamma^{\pm 1/2}$.

**Theorem A.1.2.** *[M] There exists an automorphism $\pi$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ such that*

$$\pi(\ddot{U}^{(1)}) = \ddot{U}^{(2)},\ \pi(\ddot{U}^{(2)}) = \ddot{U}^{(1)}.$$

Let us define $\mathfrak{h}^{(2)} := \pi(\mathfrak{h}^{(1)})$ and let $\ddot{U}^h$ be the subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, generated by $\ddot{U}^{(2)}$ and $\mathfrak{h}^{(2)}$. Then $\ddot{U}^h = \pi(\ddot{U}^v)$ and it is isomorphic to $\dot{U}_q(\mathfrak{gl}_n)$.

- **Big shuffle algebras**

We introduce the *big shuffle algebra* $S$ (of $A^{\mathrm{tor}}_{n-1}$-type).

Let us consider a $\mathbb{Z}^n_+$-graded $\mathbb{C}$-vector space

$$\mathbb{S} = \bigoplus_{k_1,\dots,k_n \geq 0} \mathbb{S}_{k_1,\dots,k_n},$$

where $\mathbb{S}_{k_1,\dots,k_n}$ consists of $\prod \mathfrak{S}_{k_i}$-symmetric rational functions in the variables $\{x_{i,j}\}_{i\in[n]}^{1\leq j\leq k_i}$. We also fix an $n\times n$ matrix of rational functions $\Omega = (\omega_{i,j})_{i,j\in[n]} \in \mathrm{Mat}_{n\times n}(\mathbb{C}(z))$ by setting $\omega_{i,j}(z) := \frac{p_{i,j}z - q_{i,j}}{z-1}$, where the constants $p_{i,j}, q_{i,j}$ are given as follows:

$$q_{i,i-1} = qd^{-1}, q_{i,i} = q^{-2}, q_{i,i+1} = q, p_{i,i-1} = 1, p_{i,i} = 1, p_{i,i+1} = d^{-1}$$

and we set $p_{i,j} = q_{i,j} = 1$ otherwise.

We introduce a bilinear operation $\star$ on $\mathbb{S}$.

**Definition A.1.1.** For $f \in \mathbb{S}_{k_1,\dots,k_n}, g \in \mathbb{S}_{l_1,\dots,l_n}$ we define $f \star g \in \mathbb{S}_{k_1+l_1,\dots,k_n+l_n}$ by

$$(f\star g)(x_{1,1},\dots,x_{1,k_1+l_1};\dots;x_{n,1},\dots,x_{n,k_n+l_n}) := \mathrm{Sym}_{\prod \mathfrak{S}_{k_i+l_i}}$$

$$\left( f(x_{1,1},\dots,x_{1,k_1};\dots,x_{n,k_n}) g(x_{1,k_1+1},\dots,x_{1,k_1+l_1};\dots,x_{n,k_n+l_n}) \cdot \prod_{i,i'\in I} \prod_{j\leq k_i}^{k_{i'}<j'} \omega_{i,i'}\left(\frac{x_{i,j}}{x_{i',j'}}\right)\right).$$

This endows $\mathbb{S}$ with a structure of an associative unital algebra with the unit $1 \in \mathbb{S}_{0,\dots,0}$. Similarly to the *small* shuffle algebra $\mathbb{S}^m$, we consider a particular subspace of $\mathbb{S}$ given by the *pole* and *wheel* conditions. Let us introduce these.

**Definition A.1.2.** Define the subspace $S'_{k_1,\dots,k_n} \subset \mathbb{S}_{k_1,\dots,k_n}$ by

$$S'_{k_1,\dots,k_n} := \left\{ F = \frac{f(x_{1,1},\dots,x_{n,k_n})}{\prod_{i\in[n]}\prod_{j\le k_i}^{j'\le k_{i+1}}(x_{i,j}-x_{i+1,j'})} : f \in (\mathbb{C}[x_{i,j}]_{i\in[n]}^{j\le k_i})^{\prod \mathfrak{S}_{k_i}} \right\}.$$

We say that $f \in \mathbb{S}$ satisfies the pole conditions if $f$ belongs to $S' := \bigoplus_{k_1,\dots,k_n\ge 0} S'_{k_1,\dots,k_n}$.

It is easy to see that $f \star g \in S'\ \forall\ f,g \in S'$. Next, we introduce wheel conditions:

**Definition A.1.3.** We say that $f$ satisfies wheel conditions if $f(x_{1,1},\dots,x_{n,k_n}) = 0$ for any collection of $x_{1,1},\dots,x_{n,k_n} \in \mathbb{C}$ such that

$$x_{i,j_1}/x_{i\pm1,l} = qd^{\pm1}, x_{i\pm1,l}/x_{i,j_2} = qd^{\mp1},\ i\in[n], j_1,j_2 \le k_i, l\le k_{i\pm1}.$$

Let $S_{k_1,\dots,k_n} \subset S'_{k_1,\dots,k_n}$ be the space of all such elements, and set $S := \bigoplus_{k_1,\dots,k_n\ge0} S_{k_1,\dots,k_n}$.

The following is straightforward:

**Lemma A.1.3.** *The subspace $S \subset \mathbb{S}$ is $\star$-closed.*

**Definition A.1.4.** The algebra $(S,\star)$ is called the big shuffle algebra (of $A_{n-1}^{\mathrm{tor}}$-type).

• **The relation between $S$ and $\ddot{U}^+$**

We recall the interplay between the above algebras.

It is a standard fact that $\ddot{U}^+$ is generated by $\{e_{i,j}\}_{i\in[n]}^{j\in\mathbb{Z}}$ with the defining relations (T2, T7.1). The following theorem is straightforward:

**Proposition A.1.4.** *The map $e_{i,j} \mapsto x_{i,1}^j$ extends to a homomorphism $\Psi : \ddot{U}^+ \to \mathbb{S}$.*

Note that the image of $\Psi$ is a subalgebra of $\mathbb{S}$ generated by $\mathbb{S}_{1_i}, i\in[n]$. In particular, it belongs to $S$. The following result is essentially due to Andrei Negut:

**Theorem A.1.5.** *[N2] The homomorphism $\Psi$ provides an isomorphism of algebras $\Psi : \ddot{U}^+ \xrightarrow{\sim} S$.*

*Remark* A.1.1. In the *loc.cit.* this statement is proved for the case $d = 1$.[3]

[3] The algebra $\mathcal{A}^+$ from [N2] is isomorphic to our algebra $S$ with $d=1$ via the map

$$F(x_{1,1},\dots,x_{n,k_n}) \mapsto q^{-\sum\frac{k_i(k_i-1)}{2}} F(z_{1,1},\dots,z_{n,k_n}) \cdot \prod_{i\in[n]}\prod_{j<j'} \frac{z_{i,j}-z_{i,j'}}{q^{-1}z_{i,j}-qz_{i,j}} \cdot \prod_{i\in[n]}\prod_{j,j'} \frac{z_{i,j}-z_{i+1,j'}}{z_{i,j}-qz_{i+1,j'}}.$$

## A.2 Subalgebras $\mathcal{A}(s_1, \ldots, s_n)$

We construct a family of commutative subalgebras of $S$, analogous to $\mathcal{A}^m$ from Chapter 5. Let us first introduce the following notation: for integer numbers $a \le b$, we define $l_i$ to be the number of integers from $\{a, a+1, \ldots, b-1, b\}$ that are congruent to $i$ modulo $n$. We define

$$F_\xi^{(a,b)} := F(\xi \cdot x_{1,1}, \ldots, \xi \cdot x_{1,l_1}, x_{1,l_1+1}, \ldots, x_{1,k_1}; \ldots; \xi \cdot x_{n,1}, \ldots, \xi \cdot x_{n,l_n}, x_{n,l_n+1}, \ldots, x_{n,k_n})$$

with $l_i$ defined as above. We also use mod $n$ cyclic conventions everywhere.

**Definition A.2.1.** For $s_1, \ldots, s_n \in \mathbb{C}^*$, consider a $\mathbb{Z}_+^n$-graded subspace $\mathcal{A}(s_1, \ldots, s_n)$ of $S$, whose degree $(k_1, \ldots, k_n)$ component is defined by

$$\mathcal{A}(s_1, \ldots, s_n)_{k_1,\ldots,k_n} = \left\{ F \in S_{k_1,\ldots,k_n} \mid \partial^{(\infty;a,b)} F = \prod_{i=a}^{b} s_i \cdot \partial^{(0;a,b)} F \ \ \forall\, 1 \le a \le b \le \sum k_j \right\},$$

where $\partial^{(\infty;a,b)} F := \lim_{\xi \to \infty} F_\xi^{a,b}$, $\partial^{(0;a,b)} F := \lim_{\xi \to 0} F_\xi^{a,b}$.

We will be only interested in the case when $\prod s_j = 1$. The following result is straightforward:

**Lemma A.2.1.** *For any $k \in \mathbb{Z}_+, \mu \in \mathbb{C}$, we define $F_{k,\mu} \in S_{k,\ldots,k}$ by*

$$F_{k,\mu} = \frac{\prod_{i\in[n]} \prod_{j \ne j' \le k} (x_{i,j} - q^{-2} x_{i,j'}) \cdot \prod_{i \in [n]} (\prod_{l=1}^{i} s_l \prod_{j=1}^{k} x_{i,j} - \mu \prod_{j=1}^{k} x_{i+1,j})}{\prod_{i \in [n]} \prod_{j,j' \le k} (x_{i,j} - x_{i+1,j'})}.$$

*If $s_1 \cdots s_n = 1$, then $F_{k,\mu} \in \mathcal{A}(s_1, \ldots, s_n)$.*

A collection $\{s_i\} \subset \mathbb{C}^*$ satisfying $s_1 \cdots s_n = 1$ is called *generic* if the equality $s_1^{\alpha_1} \cdots s_n^{\alpha_n} = 1$ implies $\alpha_1 = \ldots = \alpha_n$. It turns out that elements $F_{k,\mu}$ generate $\mathcal{A}(s_1, \ldots, s_n)$ for *generic* $\{s_i\}$.

**Theorem A.2.2.** *For generic $\{s_i\}$ satisfying $\prod s_i = 1$, the algebra $\mathcal{A}(s_1, \ldots, s_n)$ is generated by $\{F_{k,\mu} | k \in \mathbb{Z}_+, \mu \in \mathbb{C}\}$. Moreover, $\mathcal{A}(s_1, \ldots, s_n)$ is a polynomial algebra in free generators $\{F_{k,\mu_l} : k \in \mathbb{Z}_+, l \in [n]\}$ for pair-wise distinct $\mu_0, \ldots, \mu_{n-1} \in \mathbb{C}$.*

As an immediate consequence of this theorem we get:

**Corollary A.2.3.** *Let $\mathcal{A}_0 \subset S$ be a subalgebra generated by the elements $K_r \in S_{r,\dots,r}$ defined by*

$$K_r := \frac{\prod_{i\in[n]}\prod_{j\neq j'\le k}(x_{i,j}-q^{-2}x_{i,j'})\cdot\prod_{i\in[n]}\prod_{j=1}^{k}x_{i,j}}{\prod_{i\in[n]}\prod_{j,j'\le k}(x_{i,j}-x_{i+1,j'})}.$$

*Then $\mathcal{A}_0$ is a polynomial algebra in $K_i$ and $\mathcal{A} \subset \mathcal{A}(s_1,\dots,s_n)$ for any $s_1\dots s_n = 1$.*

Define $\mathcal{A}$ to be the subalgebra of $S$ generated by $\{x^0_{i,1}\}$. We conclude this section with the following result:

**Proposition A.2.4.** *The subalgebra $\mathcal{A}_0$ centralizes $\mathcal{A}$.*

*Proof.*

It suffices to show that the commutator $[K_r, x^0_{1,1}] \in S_{r+1,r\dots,r}$ is actually 0. This commutator has the form $\frac{f}{\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-x_{i+1,j'})}$ where $f$ is a degree $nr^2+2r$ polynomial in $\{x_{i,j}\}$.

First, note that $f$ is divisible by

$$\prod_{i\in[n]^\times}\prod_{j=1}^{r}x_{i,j}\cdot\prod_{i\in[n]^\times}\prod_{j\neq j'\le r}(x_{i,j}-q^{-2}x_{i,j'}).$$

We also claim that $f$ is divisible by $\prod_{j=1}^{r+1}x_{1,j}$ and $\prod_{j\neq j'\le r+1}(x_{1,j}-q^{-2}x_{1,j'})$.

To prove the first claim, we note that substituting $x_{1,r+1}=0$ we get

$$(K_r \star x^0_{1,1})_{|x_{1,r+1}=0} = d^{-r}K_r(x_{1,1},\dots,x_{1,r};\dots;x_{n,1},\dots,x_{n,r}),$$

$$(x^0_{1,1}\star K_r)_{|x_{1,r+1}=0} = d^{-r}K_r(x_{1,1},\dots,x_{1,r};\dots;x_{n,1},\dots,x_{n,r}),$$

so that $f$ is divisible by $x_{1,r+1}$. But $f$ is symmetric in $\{x_{1,j}\}$ and the result follows.

To prove the second claim, we note that substituting $x_{1,r+1}=q^{-2}x_{1,r}$, we get that both $(K_r\star x^0_{1,1})_{|_{x_{1,r+1}=q^{-2}x_{1,r}}}$ and $(x^0_{1,1}\star K_r)_{|_{x_{1,r+1}=q^{-2}x_{1,r}}}$ are equal to

$$K_r(x_{1,1},\dots,x_{1,r};\dots,x_{n,r})\cdot\prod_{j=1}^{r}\frac{(x_{1,j}-q^{-4}x_{1,r})(x_{2,j}-d^{-1}q^{-1}x_{1,r})(d^{-1}x_{0,j}-q^{-1}x_{1,r})}{(x_{1,j}-q^{-2}x_{1,r})(x_{2,j}-q^{-2}x_{1,r})(x_{0,j}-q^{-2}x_{1,r})}.$$

Therefore, $f$ is divisible by $x_{1,r+1} - q^{-2}x_{1,r}$. But $f$ is symmetric in $\{x_{1,j}\}$ and the result follows.

Hence, $f$ is a polynomial divisible by $\prod_{i\in[n]}\prod_{j,j'}(x_{i,j}-q^{-2}x_{i,j'})\prod_{i\in[n]}\prod_j x_{i,j}$. Since the degree of the latter product is $nr^2+2r+1$ and $\deg(f)=nr^2+2r$, we get $f=0$. This completes the proof. □

## A.3 Degeneration

In this section we study the limit of the algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ as $q\to 1$. We use this to prove some results about the algebras $\mathcal{A}_0, \mathcal{A}(s_1,\ldots,s_n)$ by considering their limit cases as $q\to 1$.

Note that all the defining relations (T0-T7.2) become of Lie-type in this limit. Therefore, $\ddot{U}_{1,d}(\mathfrak{sl}_n)\simeq U(\ddot{u}_{d,n})$ for a Lie algebra $\ddot{u}_{d,n}$ generated by $\{e_{i,j}, f_{i,j}, h_{i,j} | i\in [n], j\in\mathbb{Z}\}$ with the defining relations read from (T0)-(T7.2).

Consider an associative algebra $\mathcal{L}_n := \mathrm{Mat}_n\otimes\mathbb{C}\langle Z^{\pm1}, D^{\pm1}\rangle/(DZ-d^{-n}ZD)$. We will view $\mathcal{L}_n$ as a Lie algebra with a natural commutator-Lie bracket. Let $\mathcal{L}'_n\subset\mathcal{L}_n$ be a Lie subalgebra spanned by $\sum_{i,j\in\mathbb{Z}} A_{i,j}Z^iD^j$ with $\mathrm{tr}(A_{0,0})=0$. Finally, let $\bar{\mathcal{L}}'_n$ be the central extension of $\mathcal{L}'_n$ with respect to the 2-cocycle

$$\phi(A\otimes Z^{r_1}D^{s_1}, B\otimes Z^{r_2}D^{s_2}) = \delta_{r_1,-r_2}\delta_{s_1,-s_2}d^{-nr_2s_1}\,\mathrm{tr}(AB)s_1.$$

The following result is straightforward:

**Lemma A.3.1.** *There is a homomorphism of Lie algebras $\theta:\ddot{u}_{d,n}\to\bar{\mathcal{L}}'_n$ such that*

$$\theta:\ e_{i,j}\mapsto E_{i,i+1}\otimes Z^jd^{-ij},\ f_{i,j}\mapsto E_{i+1,i}\otimes Z^jd^{-ij},\ h_{i,j}\mapsto (E_{i,i}-E_{i+1,i+1})\otimes Z^jd^{-ij}\ \ i\neq 0,$$

$$\theta:\ e_{0,j}\mapsto E_{n,1}\otimes DZ^j,\ f_{0,j}\mapsto E_{1,n}\otimes Z^jD^{-1},\ h_{0,j}\mapsto (d^{-nj}E_{n,n}-E_{1,1})\otimes Z^j+c\delta_{j,0}.$$

It is clear that $\theta$ is surjective. Actually we have:

**Theorem A.3.2.** *The homomorphism $\theta$ is actually an isomorphism.*

Let us consider the images of some subalgebras of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ under the above degeneration. The following result is straightforward:

**Proposition A.3.3.** *As $q \to 1$, we have the following degenerations:*
*(a) $\ddot{U}^{(1)}$ degenerates to $U(\bigoplus_{m\in\mathbb{Z}} \mathfrak{sl}_n \otimes Z^m)$;*
*(b) $\ddot{U}^{(2)}$ degenerates to $U(\bigoplus_{m\in\mathbb{Z}} \mathfrak{sl}_n \otimes D^m \oplus \mathbb{C}\cdot c)$;*
*(c) $\mathfrak{h}^{(1)}$ degenerates to $\bigoplus_{m\neq 0} \mathbb{C}\cdot I_n \otimes Z^m$;*
*(d) $\mathfrak{h}^{(2)}$ degenerates to $\bigoplus_{m\neq 0} \mathbb{C}\cdot I_n \otimes D^m$;*
*(e) $\Psi^{-1}(\mathcal{A})$ degenerates to $U(\mathfrak{n}_+ \oplus \bigoplus_{m>0} \mathfrak{sl}_n \otimes D^m)$.*

This proposition together with the results of the previous section yield:

**Theorem A.3.4.** *For generic $s_1, \ldots, s_n$ satisfying $\prod s_j = 1$, we have*

$$\mathcal{A}(s_1, \ldots, s_n) \subset \Psi(\dot{U}^h).$$

It turns out that the subalgebras $\mathcal{A}(s_1, \ldots, s_n)$ are related to the Bethe-ansatz problem. In particular, let $M(\lambda)$ be a Verma module of $\dot{U}_q(\mathfrak{gl}_n)$ at the *critical level.* We further identify $\dot{U}_q(\mathfrak{gl}_n)$ with $M(\lambda)$. According to Theorem A.3.4, the subalgebra $\mathcal{A}(s_1, \ldots, s_n)$ belongs to $\Psi(\dot{U}^h) \simeq \dot{U}_q(\mathfrak{gl}_n)$. But according to Proposition A.3.3, we actually have $\mathcal{A}(s_1, \ldots, s_n) \subset \dot{U}^+(\mathfrak{gl}_n)$. The main result is

**Theorem A.3.5.** *[FT2] If $s_i = q^{2(\lambda_i - 1)}$, then the subspace $\mathcal{A}(s_1, \ldots, s_n)$ corresponds to the subspace $M(\lambda)^{\mathrm{crit}}$ of critical vectors in $M(\lambda)$.*

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