multiplied by \( u_0(y) = e^{-\frac{1}{2}y^2} \) at infinity. Consequently, we need the series to terminate, which requires \( \epsilon_m = 2m + 1 \) for some \( m \). Thus,

\[
E_n = \frac{\hbar \omega}{2} \epsilon_n = \hbar \omega \left( n + \frac{1}{2} \right) \rightarrow \text{HO energy levels} \quad (18-1)
\]

**Quantized energy levels of a harmonic oscillator.** The ground state (zero-point) energy is \( E_0 = \frac{1}{2} \hbar \omega \), the energy levels are equidistant.

*Note.* This feature allows us to identify the HO not only with a particle in potential \( V(x) = \frac{1}{2} m \omega^2 x^2 \), but also with a **system of noninteracting (bosonic) particles**. Therefore, a mode of an electromagnetic field of frequency \( \omega \) can be viewed as a HO with frequency \( \omega \); \( n \) photons in that mode correspond to the \( n \)-th occupied state of the HO. The uncertainty in \( x \) and \( p \) of the HO ground state corresponds to the “vacuum fluctuations” of the electromagnetic field \( \langle x \rangle = 0 \), \( \langle x^2 \rangle \neq 0 \) corresponds to \( \langle E \rangle = 0 \), \( \langle E^2 \rangle \neq 0 \) etc.

For given \( \epsilon = 2n + 1 \), the recursion relation (*)

\[
(m + 1)(m + 2)c_{m+2} = (2m - \epsilon_n + 1)c_m = (2m - 2n)c_m \quad (18-2)
\]

yields

- \( c_2 = -\frac{2n}{12} c_0 \)

- \( c_4 = \frac{4-2n}{34} c_2 = \frac{(4-2n)(0-2n)}{1\cdot2\cdot3\cdot4} c_0 \)

and in general

\[
c_{2k} = (-2)^k \frac{n(n-2) \cdots (n-2k+4)(n-2k+2)}{(2k)!} c_0,
\]

\[
0 \leq 2k \leq n, \ n \text{ even} \quad (18-3)
\]

for the even coefficients.

For the odd coefficients we have

- \( c_3 = \frac{(2-2n)}{23} c_1 = (-2)^{n-1} \frac{1}{23} c_1 \)

- \( c_5 = \frac{6-2n}{45} c_3 = (-2)^{n-3} \frac{3}{45} c_3 = (-2)^2 \frac{(n-3)(n-1)}{1\cdot2\cdot3\cdot4\cdot5} c_1 \)

and in general

\[
c_{2k+1} = (-2)^k \frac{(n-1)(n-3) \cdots (n-2k+3)(n-2k+1)}{(2k+1)!} c_1,
\]

\[
0 \leq 2k + 1 \leq n, \ n \text{ odd} \quad (18-4)
\]
The eigenfunction \( u_n(x) \) for energy level \( n \) with energy \( E_n = \hbar \omega \left(n + \frac{1}{2}\right) \) is given by

\[
 u_n(y) = e^{-\frac{1}{2}y} \sum_{n=0}^{n/2} c_{2k} y^{2k} \quad \text{for even } n, \tag{18-5}
\]

\[
 u_n(y) = e^{-\frac{1}{2}y} \sum_{n=0}^{(n-1)/2} c_{2k} y^{2k+1} \quad \text{for odd } n, \tag{18-6}
\]

The coefficient \( c_0 \) or \( c_1 \) has to be chosen such that the wavefunction is normalized, and \( y \) is related to the position coordinate \( x \) via \( y = \frac{\sqrt{\frac{\hbar}{m\omega}}}{2} x \). The quantity \( \sqrt{\frac{\hbar}{m\omega}} \) has units of length and defines the natural quantum length scale for the harmonic oscillator. Apart from the normalization, the polynomials

\[
 h_n(y) = \sum_{k=0}^{n/2} c_{2k} y^{2k} \tag{18-7}
\]

\[
 h_n(y) = \sum_{k=0}^{(n+1)/2} c_{2k+1} y^{2k+1} \tag{18-8}
\]

are the Hermite polynomials \( H_n(y) \). The Hermite polynomials obey the following relations:

\[
 H_n''(y) - 2y H_n'(y) + 2n H_n(y) = 0 \quad \rightarrow \quad \text{(defining equation)} \tag{18-9}
\]

\[
 H_{n+1}(y) - 2y H_n(y) + 2n H_{n-1}(y) = 0 \tag{18-10}
\]

\[
 H_{n+1}(y) - H_n'(y) + 2y H_n(y) = 0 \tag{18-11}
\]

\[
 \sum_{n=0}^{\infty} H_n(y) \frac{z^n}{n!} = e^{2zy-z^2} \tag{18-12}
\]

\[
 H_n(y) = (-1)^n e^{y^2} \left( \frac{d}{dy} \right)^2 e^{-y^2} \rightarrow \quad \left( \text{alternative definition} \right) \tag{18-13}
\]

Hermite polynomials are real. One can show that

\[
 \int_{-\infty}^{\infty} dy |u_n(y)|^2 = \int_{-\infty}^{\infty} dy |C|^2 e^{-y^2} H_n^2(y) \tag{18-14}
\]

The Hermite polynomials are real. One can show that

\[
 \int_{-\infty}^{\infty} dy e^{-y^2} H_n^2(y) = 2^n n! \sqrt{\pi} \tag{18-15}
\]
Normalization actually requires $\int_{-\infty}^{\infty} dx |u_n(x)|^2 = 1$, but since $x$ and $y = \sqrt{\frac{m \omega}{\hbar}} x$ are related by a constant factor, $\int_{-\infty}^{\infty} dx |u_n(x)|^2 = \sqrt{\frac{\hbar}{m \omega}} \int_{-\infty}^{\infty} dy |u_n(y)|^2$.

\[
\begin{align*}
H_0(y) &= 1 \\
H_1(y) &= 2y \\
H_2(y) &= 4y^2 - 2 \\
H_3(y) &= 8y^3 - 12y \\
H_4(y) &= 16y^4 - 48y^2 + 12 \\
H_5(y) &= 32y^5 - 160y^3 + 120y
\end{align*}
\]  

Consequently, the lowest eigenfunctions look like (Fig. I). The eigenfunctions of the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{HO eigenfunctions.}
\end{figure}

HO look the same in momentum space, since the Hamiltonian is symmetric in $x$ and $p$, and

\[
\begin{align*}
\hat{x} &= x, & \hat{p} &= \frac{\hbar}{i} \frac{\partial}{\partial x} & \text{in position space } \psi(x) \\
\hat{x} &= i\hbar \frac{\partial}{\partial p}, & \hat{p} &= p & \text{in momentum space } \phi(p)
\end{align*}
\]
The harmonic oscillator ground state, being a Gaussian function with no spatial dependence of the complex phase, has minimum uncertainty allowed by the Heisenberg relation:

\[ \Delta x \Delta p = \frac{\hbar}{2} \]

for the ground state \hspace{1cm} (18-24)

\[ \Delta x \Delta p > \frac{\hbar}{2} \]

for any excited state \hspace{1cm} (18-25)

- Show Bose-Einstein condensate expansion
- Thermal cloud
- isotropic expansion for anistropic trap
  \((\frac{p^2}{2m} = \frac{1}{2} kT)\)
- condensate:
  \(\frac{p^2}{2m} \propto \frac{1}{x_0^2} \propto \frac{m\omega}{\hbar} \rightarrow \) anisotropic expansion

**HO: operator method**

There is an elegant and instructive way to derive the HO eigenstates without directly solving the SE. Instead, we use commutation relation between operators. We start by writing the Hamiltonian in dimensionless form

\[
H = \hbar \omega \left[ \frac{p^2}{2m\hbar \omega} + \frac{m\omega}{2\hbar} x^2 \right] \tag{18-26}
\]

\[
= \hbar \omega \left( \frac{p}{\sqrt{2m\hbar \omega}} \right)^2 + \left( \frac{x}{\sqrt{2\hbar/m\omega}} \right)^2 \tag{18-27}
\]

\[
= \hbar \omega \left[ \left( \frac{p}{p_0} \right)^2 + \left( \frac{x}{x_0} \right)^2 \right] \tag{18-28}
\]

with \(p_0^2 = 2m\hbar \omega, x_0^2 = \frac{2\hbar}{m\omega}\). Classically we can write,

\[ H_{cl} = \hbar \omega \left( \frac{x}{x_0} - i \frac{p}{p_0} \right) \left( \frac{x}{x_0} + i \frac{p}{p_0} \right), \tag{18-29} \]

however, since in QM \(\hat{p}\) and \(\hat{x}\) do not commute, we have

\[
\left( \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0} \right) \left( \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0} \right) = \left( \frac{\hat{x}}{x_0} \right)^2 + \left( \frac{\hat{p}}{p_0} \right)^2 \tag{18-30}
\]

\[
= \left( \frac{\hat{x}}{x_0} \right)^2 + \left( \frac{\hat{p}}{p_0} \right)^2 + i \frac{\hat{x}}{x_0 p_0} [\hat{x}, \hat{p}]. \tag{18-31}
\]
Using the commutator $[\hat{x}, \hat{p}] = i\hbar \frac{\partial}{\partial p} p - p i \hbar \frac{\partial}{\partial p} = i\hbar$ and $\frac{1}{\lambda_0 p_0} = \sqrt{\frac{m\omega}{2\hbar}} \sqrt{\frac{1}{2m\hbar}} = \frac{1}{2\hbar}$, we have

$$\hat{H} = \hbar \omega \left( \frac{x}{x_0} \right)^2 + \left( \frac{\hat{p}}{p_0} \right)^2$$  \hspace{1cm} (18-32)

$$= \hbar \omega \left( \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0} \right) \left( \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0} \right) - i \frac{1}{x_0 p_0} [\hat{x}, \hat{p}]$$ \hspace{1cm} (18-33)

$$= \hbar \omega \left( \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0} \right) \left( \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0} \right) + \frac{1}{2}$$ \hspace{1cm} (18-34)

We can define a new, non-Hermitian operator by

$$\hat{a} := \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0}$$ \hspace{1cm} (18-35)

Consequently, the Hermitian conjugate operator is

$$\hat{a}^\dagger = \left( \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0} \right)^\dagger = \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0}$$ \hspace{1cm} (18-36)

since $\hat{p}^\dagger = \hat{p}$, $\hat{x}^\dagger = \hat{x}$

**Note.** The Hermitian conjugate operator $O^\dagger$ of any operator is defined by the relation

$$\int_{-\infty}^{\infty} dx \psi_2^*(x) O^\dagger \psi_1(x) = \int_{-\infty}^{\infty} dx \left( O \psi_2(x) \right)^* \psi_1(x)$$ \hspace{1cm} (18-37)

for any well-behaved wavefunctions $\psi_1(x), \psi_2(x)$.

Consequently, for any operator $\hat{O} = c_1 \hat{O}_1 + c_2 \hat{O}_2$, where $c_1$ and $c_2$ are complex numbers,

$$\hat{O}^\dagger = \left( c_1 \hat{O}_1 + c_2 \hat{O}_2 \right)^\dagger = c_1^* \hat{O}_1^\dagger + c_2^* \hat{O}_2^\dagger$$ \hspace{1cm} (18-38)

and for any operator $\hat{O} = \hat{O}_1 \hat{O}_2$ we have $\hat{O}^\dagger = \left( \hat{O}_1 \hat{O}_2 \right)^\dagger = \hat{O}_2^\dagger \hat{O}_1^\dagger$

**Proof.** See problem set. \hfill $\square$

Using the operators $\hat{a}, \hat{a}^\dagger$, we can write the Hamiltonian for the HO in the particularly simple form

$$\hat{H} = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$ \hspace{1cm} (18-39)
Rather than being explicitly defined in terms of $\hat{x}$, $\hat{p}$, ..., and operator can be defined through its commutation relations with other operators. Let us look at $\hat{a}$, $\hat{a}^\dagger$:

$$\left[ \hat{a}, \hat{a}^\dagger \right] = \left[ \frac{\hat{x}}{x_0} + i \frac{\hat{p}}{p_0}, \frac{\hat{x}}{x_0} - i \frac{\hat{p}}{p_0} \right] = i \left[ \frac{\hat{p}}{p_0}, \frac{\hat{x}}{x_0} \right] - i \left[ \frac{\hat{x}}{x_0}, \frac{\hat{p}}{p_0} \right]$$

$$= i \frac{1}{p_0 x_0} (\left[ \hat{p}, \hat{x} \right] - \left[ \hat{x}, \hat{p} \right])$$

$$= i \frac{2}{\hbar} \left[ \hat{p}, \hat{x} \right]$$

$$= \frac{\hbar}{\hbar} i$$

$$= 1,$$ (18-45)

where we have used $\left[ \hat{x}, \hat{x} \right] = 0 = \left[ \hat{p}, \hat{p} \right]$. So we have

$$\left[ \hat{a}, \hat{a}^\dagger \right] = \left[ \hat{a}, \hat{a} \right] = 0$$

(18-47)

As will be elaborated on in 8.06, this defines a commutation relation for bosonic (quasi)-particles, i.e. particles whose wavefunction is symmetric under the exchange of two particles. For the commutators with the Hamiltonian, we have

$$\left[ \hat{H}, \hat{a} \right] = \hbar \omega \left[ \hat{a}^\dagger \hat{a}, \hat{a} \right]$$

$$= \hbar \omega (\hat{a}^\dagger \hat{a} \hat{a} - \hat{a} \hat{a}^\dagger \hat{a})$$

$$= \hbar \omega (\hat{a}^\dagger \hat{a} \hat{a} - (1 + \hat{a}^\dagger \hat{a}) \hat{a})$$

$$= -\hbar \omega \hat{a}$$

(18-51)

and

$$\left[ \hat{H}, \hat{a}^\dagger \right] = \hbar \omega \left[ \hat{a}^\dagger \hat{a}, \hat{a}^\dagger \right]$$

$$= \hbar \omega (\hat{a}^\dagger \hat{a} \hat{a}^\dagger - \hat{a} \hat{a}^\dagger \hat{a})$$

$$= \hbar \omega (\hat{a}^\dagger \hat{a}^\dagger \hat{a} - \hat{a}^\dagger (\hat{a}^\dagger - 1))$$

$$= \hbar \omega \hat{a}^\dagger$$

(18-55)

We are now in the situation to calculate the spectrum of eigenenergies of the HO simply using those commutation relations. Let us first note that since $\hat{H}$ is quadratic in $x$ and $p$, all eigenvalues must be positive:

$$\langle E \rangle = \langle \hat{H} \rangle = \langle T \rangle + \langle V \rangle$$

$$= \frac{1}{2m} \int dp \phi^*(p)p^2\phi(p) + \frac{1}{2}m\omega^2 \int dx \psi^*(x)x^2\psi(x) > 0$$

(18-57)
for any wavefunction $\psi(x)$ and its Fourier transform $\phi(p)$. Before determining the eigenspectrum, let us define a convenient notation.

**State vector notation (Dirac notation)**

We have already argued that a physical state, (i.e., a physical system whose initial conditions have been prepared to the maximum extent allowed by QM), is described by a vector in an abstract vector space (Hilbert space), and that a wavefunction in position space is only one possible representation of the state. Alternatively, the state can be described in the momentum representation (wavefunction in momentum space), or by specifying the expansion coefficients when expanding the basis of energy eigenstates. Using a notation introduced by Paul Dirac, one of the creators of QM, we write the state as

$$|\psi\rangle$$

and define

$$\langle \phi|\psi \rangle := \int dx \phi^*(x)\psi(x)$$

for any two states $|\psi\rangle, |\phi\rangle$ whose wavefunctions are by $\psi(x), \phi(x)$. Dirac introduced

$$\langle \phi| \psi \rangle$$

So $|\psi\rangle$ is called a “ket”, and $\langle \phi| a "bra"$. You can think of the “bra” as the transpose of the “ket” vector

$$\begin{pmatrix} w_1 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix} = c\text{-}number \text{ (complex number)}, \quad (18-61)$$

but possibly for infinite-dimensional vectors. In this sandwich or Dirac notation, the expectation value of any operator $\hat{A}$ is given by

$$\langle \hat{A} \rangle = \langle \psi|\hat{A}|\psi \rangle = \int dx \psi^*(x)\hat{A}\psi(x). \quad (18-62)$$

In Dirac notation,

$$\langle \psi|\phi \rangle = \int dx \psi^*(x)\phi(x) = \left( \int dx \phi^*(x)\psi(x) \right)^* = \langle \phi|\psi \rangle^* \quad (18-63)$$

An operator $\hat{A}$ acting on a state produces another state, symbolically

$$\hat{A} |\psi\rangle = |\hat{A}\psi\rangle \quad (18-64)$$

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Consequently,
\[
\langle \phi | \hat{A} | \psi \rangle = \langle \phi | \hat{A} \psi \rangle = \int dx \phi^*(x) (\hat{A} \psi(x)) = \int dx \phi^*(x) \hat{A} \psi(x)
\]  
(18-65)

The Hermitian conjugate operator \( \hat{A}^\dagger \) is defined by
\[
\langle \phi | \hat{A}^\dagger | \psi \rangle = \int dx (A\phi(x))^* \psi(x) = \left( \int dx \psi^*(x) (A\phi(x)) \right)^*
\]
\[= \langle \psi | A\phi \rangle^* \]  
(18-66)
\[= \langle A\phi | \psi \rangle \]  
(18-67)

In Dirac notation, the orthonormality condition for eigenstates \( |n\rangle, |m\rangle \) reads
\[
\langle n | m \rangle = \int dx u_n^*(x) u_m(x) = \delta_{nm},
\]  
(18-69)

where the expansion coefficients are
\[
c_n = \int dx u_n^*(x) \psi(x) = \langle n | \psi \rangle.
\]  
(18-70)

A bracket like \( \langle a | b \rangle \) is a complex number, but a ketbra like \( |b \rangle \langle a | \) is an operator since acting on a state it produces another state
\[
\langle a | b \rangle \langle \psi | = \langle a | \psi \rangle.
\]  
(18-71)

One can show that the sum over all eigenstates \( \sum_n |n\rangle \langle n| \) of a Hermitian operator is the unity operator
\[
\sum_n |n\rangle \langle n| = \hat{1},
\]  
(18-72)

and
\[
|\psi\rangle = \hat{1} |\psi\rangle = \sum_n |n\rangle \langle n| \psi\rangle = \sum c_n |n\rangle \rightarrow \begin{array}{c}
\text{expansion into} \\
\text{eigenstates}
\end{array}
\]
(18-73)

Back to the operator treatment of the HO: Let us assume that we have found an energy eigenstate with eigenenergy \( E \) and let us denote that state by \( |E\rangle \). Let us define a new state \( |\psi\rangle \) by having the operator \( \hat{a} \) act on \( |E\rangle \), \( |\psi\rangle := \hat{a} |E\rangle \). What happens if we act with the Hamiltonian \( |\psi\rangle \)?
\[
\hat{H} |\psi\rangle = \hat{H} \hat{a} |E\rangle
\]  
(18-74)
\[= \left( \left[ \hat{H}, \hat{a} \right] + \hat{a} \hat{H} \right) |E\rangle \]  
(18-75)
\[= (-\hbar \omega \hat{a} + \hat{a} E) |E\rangle \]  
(18-76)
\[= (E - \hbar \omega) \hat{a} |E\rangle \]  
(18-77)
\[= (E - \hbar \omega) |\psi\rangle \]  
(18-78)
Here we have used the previously calculated result $[\hat{H}, \hat{a}] = -\hbar \omega \hat{a}$ for the commutator, and the fact that complex (here real) numbers commute with everything. The above formula signifies that $|\psi\rangle$ is also an energy eigenstate, but with lower energy $E = -\hbar \omega$. Since starting from any eigenstate $|E\rangle$ we can repeat the procedure any number of times,

$$\hat{a}^n |E\rangle = |E - n\hbar\omega\rangle,$$

(18-79)

and the eigenenergy has to remain positive, (we have shown $\langle \psi | \hat{H} | \psi \rangle > 0$ for any state), there must exist a state $|0\rangle$ such that

$$\hat{a} |0\rangle = 0$$

(18-80)

i.e., a state whose energy cannot be lowered further.

*Note.* It is important to distinguish between $|0\rangle$ (lowest energy eigenstate, vector in Hilbert space) and 0 (zero of the Hilbert space, vector of zero length).

*Note.* Nothing implies that the state $|0\rangle$ has zero energy. in fact,

$$\hat{H} |0\rangle = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) |0\rangle = \frac{1}{2} \hbar \omega^2 |0\rangle,$$

(18-81)

so the ground state has eigenenergy $E_0 = \frac{1}{2} \hbar \omega$, this is the zero-point energy. In the context of identifying a HO at frequency $\omega$ with an electromagnetic mode at frequency $\omega$, the ground state $|0\rangle$ is also called the vacuum (ground state has no excitations, photon number is zero): the vacuum has finite vacuum energy $E_0 = \frac{1}{2} \hbar \omega_0$.

What happens if $\hat{a}^\dagger$ acts on ground state? Let us define

$$|1\rangle := \hat{a}^\dagger |0\rangle.$$

(18-82)

The tilde $\sim$ is there to remind us that this state is not necessarily normalized, even if $|0\rangle$ is chosen to be normalized.