Last time we discussed

- boundary between CM and QM set by resolution of measurement apparatus in phase space
- transition between CM and QM
- Fermat’s principle of stationary time
- Light takes the path where there is no first order change in travel time for nearby paths.

\[
\Delta p \Delta x \gg h \quad \iff \quad \text{CM}
\]
\[
\Delta p \Delta x \sim h \quad \iff \quad \text{QM}
\]
\[
\Delta p \Delta x \ll h \quad \iff \quad \text{forbidden by QM}
\]

How does Fermat’s principle work?

- At any point in space, the total electric field is the sum of all electric field contributions.
- As EM field of wavelength \(\lambda\) (wavevector \(k = \frac{2\pi}{\lambda}\)) travels a distance \(x\), its phase changes by \(\phi = kx\).
- Complex Notation: \(E(x) = E(0)e^{ikx}\) (Electric field is real part of the expression)
Phasor representation

1. Indicate magnitude and phase of light by electric field vector in the complex plane.

2. As light moves along $x$, the vector rotates in the complex plane by $\Delta \phi = k \Delta x$.

Figure III: Phasor description of electric-field vector. The phasor rotates in the complex plane; the electric field is the projection onto the real axis.

$$E_A = E_0$$

$$E_B = E_0 e^{ikl_1} + E_0 e^{ikl_2} + \cdots + E_0 e^{ikl_N} = E_1 + E_2 + \cdots + E_N$$ (3-2)

If the phase varies quickly as we change from one path to the next, we have at point $B$:

Figure IV: Possible light paths with different path lengths.

Figure V: The sum of the phasors corresponding to all possible paths yields the total electric field at the observation point $B$. 
Figure VI: Contribution to electric field $E_B$ at point B from those paths where the phase increases or decreases monotonically averages to zero.

Blocking those paths where the sum of the phasors is close to zero has no effect on electric field at point B. See Figure VI. Phasors associated with paths near $x_0$ have almost constant phase, i.e., point nearly in the same direction in the complex plane. See Figure VII.

Figure VII: Phasors in the vicinity of $x_0$ where the path length has an extremum provide the major contribution to the electric field at B.
By far, the dominant contribution to the total field $E_B$ at point $B$ originates from the paths where the phase is stationary. Other paths interfere destructively with nearby paths, do not contribute to the field at the target, and can therefore be blocked without changing appreciably the total field $B$. A classical path can be defined by a region of stationary phase. The light “explores” all space, but only a small region near the classical path contributes to the total field at the target.

**The same description is valid in QM**

A particle is described by a complex wavefunction $\psi(r)$ that explores all space, but only a small region of stationary phase contributes to the total field at the target. The classical path is the region of space where nearby paths interfere constructively to produce virtually all the wavefunction amplitude at the target. The other regions of space can be blocked without substantially changing the wavefunction at the target.

What is the wavelength $\lambda$ (or wavevector $k = \frac{2\pi}{\lambda}$) to be used to compute the phase $\phi = kx$ for particles? An EM wave of energy $E$ carries momentum $p = \frac{E}{c}$ in the direction of the wavevector $\mathbf{k} = \frac{2\pi}{\lambda} \hat{e}$. A photon of frequency $\nu$ has energy $E = h\nu$ and carries momentum

$$ p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} $$

$$ = \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda} $$

$$ = \hbar k. $$

\[
\begin{align*}
E &= h\nu = \hbar\omega & \Rightarrow & & \text{energy of a photon} \\
p &= \frac{\hbar}{\lambda} \hat{e} = \hbar k & \Rightarrow & & \text{momentum of a photon}
\end{align*}
\]

For light, the classical path no longer exists for distance scales $d \sim \lambda$ or $d \cdot p = d \cdot \frac{h}{\lambda} \sim \lambda \cdot \frac{h}{\lambda} = h$ or

$$ d \cdot p \sim h. $$

(3-4)

This is the same criterion that identifies border between classical and quantum phenomena. Classical path (geometric optics) is recovered in the limit

$$ d \cdot p \gg h \quad \Rightarrow \quad \text{geometric optics.} $$

(3-5)

Wave optics with no geometric path for $d \cdot p \sim h$. We recognize that this is the same criterion for the transition from CM to QM if we identify for particles, just like for photons. $\lambda_{db}$ is the de Broglie wavelength of a particle with momentum $p$.

$$ \lambda_{db} = \frac{\hbar}{p} \iff p = \frac{\hbar}{\lambda_{db}} $$
8.04 Quantum Physics

### Table III.1: $d \rightarrow$ resolution of (optical) instrument, e.g., slit size.

<table>
<thead>
<tr>
<th>$d \gg \lambda_{\text{photon}}$</th>
<th>geometric optics</th>
<th>$d \gg \lambda_{\text{dB}}$</th>
<th>classical mechanics</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d \sim \lambda_{\text{photon}}$</td>
<td>wave optics</td>
<td>$d \sim \lambda_{\text{dB}}$</td>
<td>QM (wave mechanics)</td>
</tr>
<tr>
<td>$d \gg \lambda \iff d \cdot p \gg h$</td>
<td></td>
<td>$d \sim \lambda \iff d \cdot p \sim h$</td>
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</tbody>
</table>

For $d \gg \lambda_{\text{dB}}$, or equivalently $d \cdot p \gg h$, a classical path can be defined, and the object behaves like a moving classical particle.

The light path of geometric optics can be derived from Fermat’s principle of least time. This principle of least time, or more accurately, of stationary light phase, arises within the more general framework of wave optics from constructive interference of nearby paths. Similarly, the classical path of a particle arises from constructive interference of deBroglie waves along nearby paths. The transverse “size” of the geometric optics ray is defined by the region of constructive interference, or equivalently, by diffraction. Similarly, the transverse size of the classical path of a particle is set by diffraction of deBroglie waves. The transverse size of the classical path is not the size of the particle. In the presence of constrictions with $d \sim \lambda$, no classical path can be defined. Wave optics (wave mechanics) is more general than geometric optics, CM. There are instances where a particle travels from $A$ to $B$, but no classical path can be defined, e.g., double slit, grating, Fresnel lenses. While the concept of stationary phase (of a classical path) cannot be always applied, the more general concept of constructive and destructive interference, to determine whether or not a particle is likely to be found in a certain region of space, is always valid.

To describe particle motion in QM, we need:

1. the concepts of a wavefunction $\Psi(r, t)$
   - interference: $\Psi(r, t) = \Psi_1(r, t) + \Psi_2(r, t)$
   - probability for the particle to be found in a small volume $d^3r$ near $r$: $|\Psi(r, t)|^2 d^3r$

2. an equation that determines how the wavefunction $\psi(r, t)$ evolves in space and time, the Schrödinger equation.

CM arises as the geometrics optics limit of wave mechanics.

### Diffraction and the Heisenberg uncertainty relation

Constructive interference for $\phi_0 \ll 2\pi$, $\phi_0 = \frac{\phi}{\lambda} \cdot 2\pi = \frac{2\pi d}{\lambda} \sin \theta = kd \sin \theta$. Estimate angular spread from location of first minimum, appearing for $\phi_0 = 2\pi$.  

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Figure VIII: Attempt to localize wave to within $\Delta x = d$ by passing it through slit. Far-field diffraction pattern: $I = I_0 \left( \frac{\sin \left( \frac{\pi d}{\lambda} \sin \theta \right)}{\frac{\pi}{d} \sin \theta} \right)^2$.

Figure IX: Path length difference results in variation of phasor angle for waves emerging from different points along slit $\Rightarrow$ diffraction pattern.
Quantum Physics

\[
\sin \theta_{\text{min}} = \frac{\lambda}{d}
\]

\(\theta_{\text{min}}\) is the half-angle for the first minimum. For \(d \gg \lambda\), the angular spread \(\theta_{\text{min}}\) is small and a classical path can be defined. For a wave of wavelength \(\lambda\) and momentum \(p = \frac{h}{\lambda}\) (electromagnetics wave or particle) the uncertainty in the \(x\)-direction is

\[
\Delta p_x \geq p \sin \theta_{\text{min}} = \frac{h}{\lambda} \cdot \frac{\lambda}{d} = \frac{h}{d}
\]  

(3-6)

The uncertainty in position due to filtering by the slit is \(\Delta x = d\). It follows \(\Delta p \Delta x \geq h\). A more exact definition of uncertainties leads to

\[
\Delta x \Delta p \geq \frac{h}{4\pi} = \frac{\hbar}{2} : \text{Heisenberg uncertainty relation}
\]

\(\Delta x\) : uncertainty in position of object along \(x\)  
\(\Delta p\) : uncertainty in \(x-momentum\)  

(3-7)

(3-8)

(3-9)

From our discussion of the analogy between wave mechanics and light, it follows that the Heisenberg uncertainty relation also holds for particles if the relation \(p = \hbar k = \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda} = \frac{h}{\lambda}\) also applies to particles, i.e., if a particle of momentum \(p\) has an associated deBroglie wavelength,

\[
\lambda_{\text{db}} = \frac{h}{p}
\]  

(3-10)

A classical path is defined for constrictions \(d \gg \lambda_{\text{db}}\). For macroscopic objects:

\[
m = 10^{-3} \text{ kg}
\]

(3-11)

\[
v = 1 \text{ mm/s}
\]

(3-12)

\[
p = 10^{-6} \text{ kg} \cdot \text{m/s}
\]

(3-13)

\[
\lambda_{\text{db}} = \frac{6 \times 10^{-34}}{10^{-6}} \text{ J} \cdot \text{s} \cdot \text{kg} \cdot \text{m/s} \rightarrow \text{is exceedingly small.}
\]

(3-14)

**The Heisenberg uncertainty relation**

The fact that the attempt to localize a particle in space leads to a spread in its momentum distribution, is due to a wave property, namely diffraction. Localization of the particle within \(d\) removes paths that would otherwise produce destructive interference at angles \(\theta \neq 0\).

After a single realization of the experiment: We observe a particle at \(x_1\).
The particle must have had momentum $p_x = p \sin \theta$ with $\sin \theta \approx \tan \theta \approx \frac{x}{L}$. Since the uncertainty in $x$ is $d$ and $L$ can be made arbitrarily large, it seems that we can violate the Heisenberg uncertainty $\Delta p_x \Delta x = p_x^2 d$ if $L$ is sufficiently large. This is true, but it does not violate the Heisenberg uncertainty that refers only to predicting the outcome of a measurement, not postdicting it. Your measurement of the first realization of the experiment (first particle) does not help you in any way predict the $x$-momentum of the next particle passing through the slit.

**The Heisenberg uncertainty relation and Fourier decomposition**

For a real wave inside a box that vanishes at the walls of the box $f(x = 0) = f(x = L) = 0$ we can write

$$f(x) = \sum_{n=1}^{\infty} c_n \sin k_n x,$$

where $k_n = \frac{2\pi}{L} n$ to satisfy the boundary condition and the $c_n$ are suitably chosen. For convenience, we often choose periodic boundary conditions $\psi(x = 0) = \psi(x = L)$ and for a complex function $\psi(x)$ we can write

$$\psi(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x},$$

with $k_n = \frac{2\pi}{L} n$ to satisfy boundary conditions $\psi(0) = \psi(L)$, and the expansion coefficients $c_n$ are again uniquely defined, but now complex numbers.

From a wave optics viewpoint, this is an expansion in terms of plane waves

$$f(x) = e^{ik_n x},$$

that have definite wavevector $k_n$, and therefore, definite momentum $p_n = \hbar k_n$, so that we can write

$$f_n = e^{ip_n x/\hbar},$$

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Figure XI: A real function \( f(x) \) defined within a box of length \( L \) can be decomposed into a Fourier series.

with

\[
p_n = n\hbar \frac{2\pi}{L} = n\hbar k_1 = np_1, \tag{3-19}
\]

\[
p_1 = \frac{\hbar 2\pi}{L}. \tag{3-20}
\]

The periodic boundary conditions permit only states of discrete momentum \( p_n = \pm |n|\hbar \frac{2\pi}{L} \). A wave packet \( \psi(x) \) in space is synthesized from Fourier components that interfere constructively within the wavepacket, and destructively everywhere else. If we let the box size \( L \to \infty \), the Fourier series becomes a Fourier integral,

\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{\phi}(k) e^{ikx} \tag{3-21}
\]

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\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{\phi}(k) e^{ikx} \tag{3-22}
\]

where

\[
c_n = c(k_n) \to \tilde{\phi}(k) \tag{3-23}
\]

\[
\sum_n = \sum_{k_n} \to \int dk \tag{3-24}
\]

and the normalization factor \( \frac{1}{\sqrt{2\pi}} \) has been introduced to make some expressions that we will appear later look more symmetric. Since the boundary conditions are removed.
to $\pm \infty$, the wavevector $k$ and momentum $p = \hbar k$ of the plane wave are now continuous variables, and no longer restricted to discrete values. In terms of momentum $p = \hbar k$, we can also write:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{\phi}(k) e^{ikx}$$  \hspace{1cm} (3-25)$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \phi(p) e^{ipx/\hbar}$$  \hspace{1cm} (3-26)$$

with $\phi(p) = \frac{1}{\sqrt{\hbar}} \tilde{\phi}(k)$. 

Figure XII: Pictorial Fourier decomposition. Waves of different wavelengths with appropriate phases are introduced to yield constructive interference in the region where the function $\psi(x)$ is large, and destructive interference in the region where it vanishes.