Atomic Physics II (8.422) Spring 2005
Ketterle / Chuang

Problem Set 10 REVISED

Due Wednesday, May 4 (just in time for Cinco de Mayo)

1. **Decoherence and the Operator Sum Representation**

Our study of decoherence has centered around the master equation, but in class we have seen that alternatives such as the quantum monte-carlo wavefunction technique are also valid. Here is another approach, known as the "operator sum representation", which is suitable for input-output treatments where you want to know what happens due to a system-environment interaction only after a specific time.

We assume that we can write the initial system-plus-environment as a product state, $\rho_S \otimes \rho_E$. This total system evolves according to some unitary transform U. The final state of the system S, ρ', which we find by tracing over the environment E, can be given by a *quantum operation*, $\mathcal{E}(\rho_S)$.

$$\rho' \equiv \mathcal{E}(\rho_S) = tr_E\left[U(\rho_S \otimes \rho_E)U^\dagger\right] \tag{1}$$

Let $|e_k\rangle$ be an orthonormal basis for the environment, and $\rho_E = |e_0\rangle\langle e_0|$ be the initial state of the environment. We can then rewrite equation (1) as

$$\mathcal{E}(\rho_S) = \sum_k \langle e_k|U\ (\rho_S \otimes |e_0\rangle\langle e_0|)\ U^\dagger|e_k\rangle \tag{2}$$

$$= \sum_k E_k\, \rho_S\, E_k^\dagger \tag{3}$$

where the "operation elements" given by $E_k \equiv \langle e_k|U|e_0\rangle$, are subject to the condition that $\sum_k E_k^\dagger E_k = I$. This is known as the *operator sum representation* (OSR). Some of the physics implied by this model are very insightful.

(a) Suppose we have a single qubit system interacting with a single qubit environment through the transform

$$U = P_0^S \otimes I + P_1^S \otimes \sigma_x^E \tag{4}$$

where $P_0 \equiv |0\rangle\langle 0|$ and $P_1 \equiv |1\rangle\langle 1|$ are projectors acting on the system and σ_x^E is the Pauli matrix acting on the environment. Give the quantum operation for this process, in the OSR, assuming the environment starts in the state $|0\rangle$ (i.e., find E_0 and E_1). (The transform U represents a controlled-NOT gate where ρ_S is the control qubit.)

Bloch Sphere Deformation: There is an elegant geometric method for picturing quantum operations on a single qubit. This method allows one to get an intuitive feel for the behavior of quantum operations in terms of their action on the Bloch sphere. Recall that the state of a single qubit can always be written in the Bloch representation,

$$\rho_S = \frac{I + \vec{r}\cdot\vec{\sigma}}{2} \doteq \frac{1}{2}\begin{bmatrix} 1 + r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{bmatrix} \tag{5}$$

where $\vec{r}$ is the three-component Bloch vector. The effect of quantum operation is to map each point on the Bloch sphere onto a point on some other surface, i.e., to *deform* the Bloch sphere.

$$(r_x, r_y, r_z) \rightarrow (r'_x, r'_y, r'_z) \tag{6}$$

(b) Suppose a projective measurement is performed on a single qubit in the basis $|+\rangle$,$|-\rangle$, where $|\pm\rangle \equiv (|0\rangle \pm |1\rangle)/\sqrt{2}$. In the event that we are ignorant of the result of the measurement, the density matrix evolves according to the equation

$$\rho_S \rightarrow \mathcal{E}(\rho_S) = |+\rangle\langle +|\rho_S|+\rangle\langle +| \;+\; |-\rangle\langle -|\rho_S|-\rangle\langle -| \tag{7}$$

Find the transformation of the Bloch sphere, and sketch it.

Amplitude Damping: An important application of quantum operations is the description of energy dissipation.

(c) Suppose we have a single optical mode containing the quantum state $a|0\rangle + b|1\rangle$, impinging on a beamsplitter. The beamsplitter couples this mode to another optical mode (the "environment") which starts with zero photons, according to the unitary operator $B = \exp\left[\theta(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger)\right]$. The output of the beamsplitter is

$$B|0\rangle_E(a|0\rangle + b|1\rangle) = a|00\rangle + b\,\cos\theta\,|01\rangle + b\,\sin\theta\,|10\rangle \tag{8}$$

Tracing over the "environment", find the operation elements E_0 and E_1 for describing this situation in the OSR, defining $\gamma \equiv \sin^2\theta$.

Spontaneous emission in a two-level system is also conveniently modelled as *amplitude damping* in the OSR. To see this, we begin with the Jaynes–Cummings interaction for a single atom and single photon, in the limit of a single quantum exchanged,

$$H = -\begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{bmatrix}, \tag{9}$$

(the basis states are $|00\rangle$, $|01\rangle$, $|10\rangle$, from left to right and top to bottom, where the left label corresponds to the field/environment, and the right one to the atom/system. The unitary evolution given by $U = e^{-iHt}$ is

$$\begin{aligned} U &= e^{-i\delta t}|00\rangle\langle 00| + (\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t)|01\rangle\langle 01| \\ &\quad + (\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t)|10\rangle\langle 10| - i\frac{g}{\Omega}\sin\Omega t\left(|01\rangle\langle 10| + |10\rangle\langle 01|\right), \end{aligned} \tag{10}$$

where $\Omega = \sqrt{g^2+\delta^2}$ is the Rabi frequency.

(d) Set the detuning δ to zero and assume the field is initially $|0\rangle$; give the quantum operation elements (for the two-level atom) E_0 and E_1, resulting from taking the *partial trace* over the field, that is $E_0 = \langle 0|U|0\rangle$ and $E_1 = \langle 1|U|0\rangle$.

(e) For the general single qubit state $\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$ show that amplitude damping leads to

$$\mathcal{E}_{\rm AD}(\rho) = \sum_k E_k \rho E_k^\dagger = \begin{bmatrix} 1-(1-\gamma)(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}. \tag{11}$$

Give γ in terms of g and t.

(f) Describe and sketch the deformation of the Bloch sphere.

Phase damping: Phase damping is an important decoherence mechanism, described by the operation elements

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix} \tag{12}$$

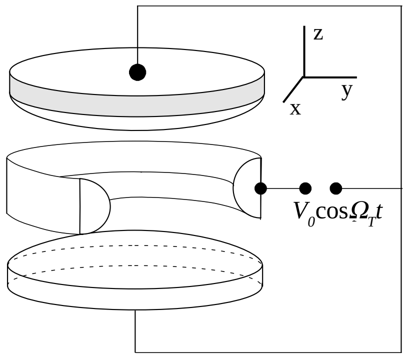
Phase damping is often referred to as a T_2 relaxation process, where $e^{-t/2T_2} = \sqrt{1-\lambda}$.

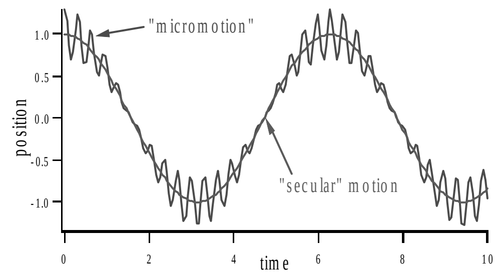
(g) Find and sketch the transformation of the Bloch sphere for phase damping.

2. **Electrodynamics of the Spherical Paul Trap**

In class, we studied the behavior of a charged atom confined in an ideal one-dimensional parabolic potential; here, we consider a realistic physical apparatus and show how the ideal model arises from it.

The real geometry of an ion trap involves a variety of electrodes, which in their simplest instance, are configured as shown here (the "spherical Paul trap"):





The central electrode has minimum radius r_0 and is known as the "ring." The outer two electrodes, with minimum separation $2z_0$, are the "endcaps." For perfectly hyperbolic electrodes, the potential between the electrodes is

$$V = V_0 \cos(\Omega_T t) \left(\frac{x^2 + y^2 - 2z^2}{d_0^2} \right) , \tag{13}$$

where $d_0 = \sqrt{r_0^2 + 2z_0^2}$.

(a) Write down the equations of motion for a particle of charge Q in the potential, in the z direction and in the radial direction r.

(b) Decompose the complete z_{tot} motion into a slow motion z (the "secular motion") and a small-amplitude high frequency motion z_μ at frequency Ω_T, such that $z_{tot} = z + z_\mu$, with $z_\mu \ll z$. Assume $\ddot{z}_\mu \gg \ddot{z}$. Give equations of motion for z_μ and z.

(c) Average over one period of Ω_T to obtain the oscillation frequency ω_z of the secular motion, and the depth $\overline{D_z}$ of the average potential seen by the slow motion. This "pseudopotential" is the potential used in the ideal analysis. Derive the radial pseudopotential as well.

(d) An important dimensionless parameter describing the trap behavior is the Mathieu q parameter, defined as

$$q = \frac{-8QV_0}{md_0^2\Omega_T^2} \tag{14}$$

for the z motion. The trap is stable for $q < 0.908$; suppose $q = 0.2$ (a typical operating point). For $d_0 = 1$ mm, $\Omega_T = 2\pi \times 10$ MHz, and a singly charged ^{88}Sr atom, give numerical values for ω_z, $\overline{D_z}$, and the temperature corresponding to the well depth.